LATTICES RELATED TO CONWAY'S CONSTRUCTION

Cátia Sofia Peniche Lente Dinis Dias

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ORIENTADOR (A/ES): *Richard Nowakowski*
Carlos Pereira dos Santos
Sandra Maria Vinagre

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Abstract

In 2002 and 2012 some results on the lattice structure associated with Conway's construction were proved. It was also shown that the set of games born by day n is a complete distributive lattice and that this structure is maintained with a not-empty initial set, provided that it is self-generated.

This work deepens the sufficient condition for distributivity. The first known example of non-distributive modular lattice from a Conway's construction is given. The main result is the *Representation Theorem with Games*, which states that complete lattices, finite and infinite, can emerge on the first day of a Conway's construction for some initial set. Finally, the transfinite construction is analyzed: a *Convergence Theorem* for Conway's construction is proved, and a condition that establishes whether the class of games born in the days before a certain ordinal is a lattice is presented.

Reticulados Relacionados com a Construção de Conway

Resumo

Em 2002 e 2012 foram provados alguns resultados sobre a estrutura de reticulado associada à construção de Conway. Foi também mostrado que o conjunto dos jogos nascidos até ao dia n é um reticulado distributivo completo e que essa estrutura é mantida considerando um conjunto inicial não vazio, desde que seja auto-gerado.

Neste trabalho é aprofundada a condição suficiente de distributividade e é dado o primeiro exemplo conhecido de reticulado modular não distributivo proveniente de uma construção tipo Conway. O principal resultado é o *Teorema de Representação com Jogos* que estabelece que reticulados completos, finitos e infinitos, podem emergir no primeiro dia de uma construção de Conway para certo conjunto inicial. Finalmente, é analisada a construção transfinita: é provado um *Teorema de Convergência* para a construção de Conway, e é apresentada uma condição que estabelece se o conjunto dos jogos nascidos em dias anteriores a um certo ordinal é um reticulado.

To Ricardo, Inês and Gabriel.

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Contents

Introduction

This work focuses on lattices that appear in Combinatorial Game Theory. Although the existence of an order relation in the set of combinatorial games is known, results on the lattice structure are recent. In the early 2000's, Calistrate, Paulhus, and Wolfe published a paper that states that the games born by day n form a distributive lattice. Some more results on the lattice of day n's structure, such like bounds on the number of elements and levels, have been introduced since, but there are still several open questions on this issue. In particular, in 2012, Albert and Nowakowski introduced a generalized Conway's construction with an initial set of games, and proved that the set of children of a set of games is a complete lattice. In the same paper, a condition on the initial set is given that guarantees the lattice's distributivity. This condition is rather strong, what motivates the study of weakener ones. More, Nowakowski conjectured that every modular lattice was also distributive lattice. This proved to be wrong, and, in this text, an example of a non-distributive modular lattice is given. More, the main contribution of this work is to prove that every complete lattice can be generated from a set of combinatorial games, and show how to do it. Also, Conway's transfinite construction is studied, for which it is shown that Conway's construction with an initial set coincides, at some point, with the usual Conway's construction. More, it is shown in which conditions the set of games born in days before a certain ordinal is a lattice.

This text is structured in 5 chapters. Chapter 1 contains background required on orders, lattices, well-ordered sets, and ordinal numbers, and some specific results needed in the remaining chapters. Chapter 2, is an introduction to Combinatorial Game Theory. Combinatorial Game Theory exists as an independent subject from the later twentieth century, and studies two-player games with no hidden information and no chance devices. Not only does its results allow players to understand (and win) games, but it intersects with other branches of Mathematics. Also, connections with several other subjects, like Artificial Intelligence and Economics, have been made.

In Chapter 3, the lattice structure of the set of games born by day n is introduced. It is shown that it is a distributive lattice, and the operations of join and meet are defined. Also, it is shown that the set of all short games is not a lattice, since it lacks the join of two fuzzy elements. Next, Conway's construction with an initial set of games is introduced, and the resulting set's structure is studied. It is shown that the set of children of any set of games is a complete lattice, and that, if the initial set is self-generated, then the lattice is distributive.

Chapters 4 and 5 contain the main contributions of this work. In Chapter 4, another condition on the initial set of games is given that implies the distributivity of the lattice of children. When trying to characterize sets that yield a distributive lattice of children, Nowakowski's conjecture that every modular lattice is distributive is disproved. Although the meant characterization is not obtained, an example of a non-distributive, modular lattice of children is given.

In Chapter 5, a *Representation Theorem with Games* is proved. After establishing a first case, allowing the generation of all finite lattices, the result is extended to the infinite case. This theorem states that for a well chosen S (allowing the axiom of choice) every complete lattice is isomorphic to the lattice of children, excluding the top and bottom elements. Also, Conway's transfinite construction is studied. A *Convergence Theorem* is proved, that states that Conway's construction with any initial set corresponds with Conway's construction, that is, for every generating set S, there is an ordinal γ such that $\mathcal{L}^{\gamma}(S) = \mathcal{L}^{\gamma}(\emptyset)$, and it is shown in which conditions the class of games born in days before a certain ordinal is a lattice.

After the main chapters, some directions for future work are given.

In Appendix A, the application used to compute and draw the lattice of children of a given set of games is presented.

Chapter 1

Preliminaries

In this chapter some basic definitions on orders, lattices, well-ordered sets, and ordinal numbers are presented, along with some results used in following chapters. Most of them are usually found in introductory courses on algebra or algebraic structures, and set theory, and their proofs do not require any advanced tools on the subject. The proofs of more advanced results needed can be found in the referred bibliography.

For a more detailed approach on orders and lattices, the reader can consult [15], [11], [24], or any other book of the many available on the subject. For results on well-ordered sets and ordinal numbers, the reader can consult [19], [23], or any other book on set theory.

1.1 Orders

Definition 1.1.1. Let P be a nonempty set. A *partial order* on P is a binary relation $\leqslant$ on P that is reflexive, antisymmetric, and transitive, that is, for all $x, y, z \in P$,

1. $x \leqslant x$;
2. $x \leqslant y$ and $y \leqslant x \Rightarrow x = y$;
3. $x \leqslant y$ and $y \leqslant z \Rightarrow x \leqslant z$.

Observation 1.1.2. A nonempty set equipped with a partial order, $(P, \leqslant)$, is called a *partially ordered set* or *poset*. For simplicity, when the relation $\leqslant$ is understood, a poset $(P, \leqslant)$ is denoted only by P.

Let P be a poset. If $x, y \in P$, then:

$$\begin{array}{lll} y \geqslant x & \text{means that} & x \leqslant y, \\ x < y & \text{means that} & x \leqslant y \text{ but } x \neq y \\ y > x & \text{means that} & x < y. \end{array}$$

Also, x, y are *comparable* if $x \leqslant y$ or $y \leqslant x$. If neither $x \leqslant y$ nor $y \leqslant x$, then x, y are called *incomparable*[1], and it is denoted by $x \parallel y$.

Definition 1.1.3. Let P be a poset. An element $y \in P$ *covers* $x \in P$, $x \prec y$, if

1. $x < y$, and
2. $x \leqslant z \leqslant y \Rightarrow z = x$ or $z = y$.

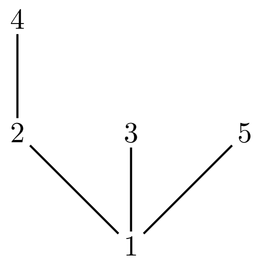
If P is a finite poset, then the covering relation uniquely determines the order on P.

Proposition 1.1.4. *Let P be a finite poset and $x, y \in P$. Then, $x \leqslant y$ if and only if there is a finite sequence of elements $p_1, p_2 \ldots, p_n \in P$ such that*

$$x \prec p_1 \prec p_2 \prec \ldots \prec p_n \prec y.$$

This fact allows finite posets to be represented by a diagram, called *Hasse diagram*. Roughly, a Hasse diagram is a graph whose nodes are labeled with the elements of the poset and whose edges indicate the covering relation.

Example 1.1.5. Consider $P = \{1, 2, 3, 4, 5\}$ ordered under division. A Hasse diagram for this poset is:



Definition 1.1.6. Let P be a poset. A non-empty subset S of P is a *chain* in P if every two elements of S are comparable, and it is an *antichain* in P if every two elements of S are incomparable.

Example 1.1.7. In Example 1.1.5, $\{1, 2, 4\}$ is a chain and $\{2, 3, 5\}$ is an antichain.

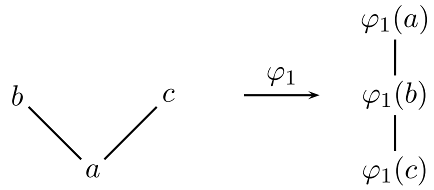
Definition 1.1.8. Let P and Q be posets. A map $\varphi : P \longrightarrow Q$ is:

1. an *order-preserving map* if, for every $x, y \in P$, $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$;
2. an *order-embedding* if, for every $x, y \in P$, $x \leqslant y \Leftrightarrow \varphi(x) \leqslant \varphi(y)$;
3. an *order-isomorphism* if it is a surjective order-embedding.

[1]Usually, combinatorial game theory researchers use the term *fuzzy*.

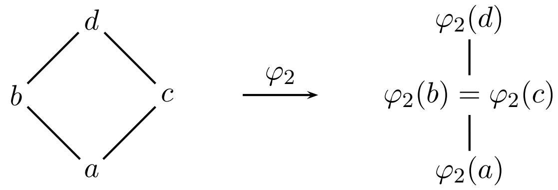
Example 1.1.9.

1. The map φ_1 given by



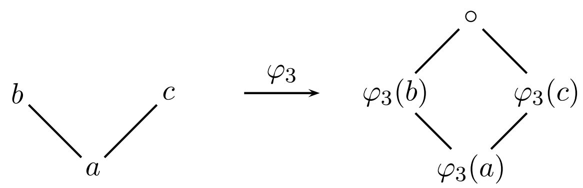
   is not an order-preserving map.

2. The map φ_2 given by



   is an order-preserving map, but is not an order-embedding.

3. The map φ_3 given by

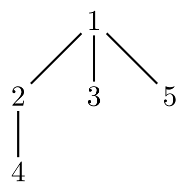


   is an order-embedding, but not an order-isomorphism.

Let $P = (P, \leqslant)$ be a poset. The relation $\geqslant$ is also a binary relation on P that is reflexive, antisymmetric and transitive. Thus, $(P, \geqslant)$ is also a poset, called the *dual* of $(P, \leqslant)$, usually denoted by P^∂.

Observe that if P is finite, the Hasse diagram of P^∂ can be obtained by turning upside down P's diagram.

Example 1.1.10. A Hasse diagram for the dual poset of Example 1.1.5 is:



To each statement about a poset P, there is a corresponding one about P^∂: if Φ is a statement about orders, the dual of Φ, Φ^∂, is the statement obtained from Φ replacing all occurrences of $\leqslant$ by $\geqslant$.

An important and widely used fact about dual sentences is the following Duality Principle:

Theorem 1.1.11 (Duality Principle)**.** *Let Φ be a statement about ordered sets. If Φ is true in all orders, then its dual, Φ^∂, is also true in all orders.*

Let P be a poset. If an element $\top$ exists such that $x \leqslant \top$ for every $x \in P$, it is unique, and is called the *top element* of P. Dually, if an element $\bot$ exists such that $\bot \leqslant x$ for every $x \in P$, it is also unique, and is called the *bottom element* of P.

Example 1.1.12. The poset in Example 1.1.5 has bottom element 1, but has no top element.

If a poset P lacks a top element, it is possible to adjoin one: consider an element, say, $\top$, such that $\top \notin P$, let

$$P_\top = P \cup \{\top\},$$

and define $\leqslant$ on $P_\top$ by

$$x \leqslant y \text{ if and only if } y = \top \text{ or } x \leqslant y \text{ in } P.$$

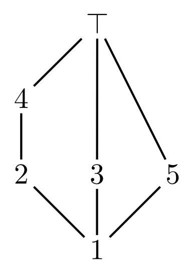
Analogously, it is possible to adjoin a bottom element, by considering an element $\bot \notin P$, let

$$P_\bot = P \cup \{\bot\},$$

and $\leqslant$ on $P_\bot$ by

$$x \leqslant y \text{ if and only if } x = \bot \text{ or } x \leqslant y \text{ in } P.$$

Example 1.1.13. For Example 1.1.5, $P_\top = \{1, 2, 3, 4, 5\} \cup \{\top\}$ has Hasse diagram:



Definition 1.1.14. Let P be a poset with $\bot$. An element $x \in P$ is an *atom* of P if $\bot \prec x$.

Example 1.1.15. In Example 1.1.5, the elements 2, 3, and 5 are atoms.

Definition 1.1.16. Let P be a poset and $S \subseteq P$.

1. An element $m \in S$ is a *maximal element* of S if

$$m \leqslant x \text{ and } x \in S \Rightarrow x = m.$$

   If S has a top element, it is called the maximum element of S.

2. An element $m \in S$ is a *minimal element* of S if

$$x \leqslant m \text{ and } x \in S \Rightarrow x = m.$$

   If S has a bottom element, it is called the minimum element of S.

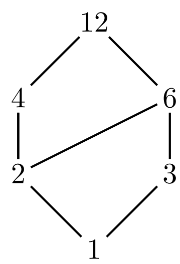
1.2 Lattices

Definition 1.2.1. Let P be a poset and $S \subseteq P$. An element $p \in P$ is an *upper bound* of S if $x \leqslant p$ for every $x \in S$. Dually, $p \in P$ is a *lower bound* of S if $p \leqslant x$ for every $x \in S$.

Denote by S^u the set of all upper bounds of S, and by S^l the set of all lower bounds of S. If S^u has a least element, it is called the *least upper bound* or *supremum* of S, and is denoted by $\bigvee S$. Dually, if S^l has a greater element, it is called the *greatest lower bound* or *infimum* of S, and is denoted by $\bigwedge S$. When S is finite, say $S = \{s_1, \ldots, s_n\}$, $\bigvee S$ and $\bigwedge S$ are usually denoted by $s_1 \vee \cdots \vee s_n$ and $s_1 \wedge \cdots \wedge s_n$, respectively.

Definition 1.2.2. A poset P is a *lattice* if $x \vee y$ and $x \wedge y$ exist, for every $x, y \in P$.

Example 1.2.3. The poset in Example 1.1.5 is not a lattice (for example, $2 \vee 3$ doesn't exist). However, the set $P = \{1, 2, 3, 4, 6, 12\}$ ordered under division, is a lattice:



Definition 1.2.4. A lattice P is *complete* if $\bigvee S$ and $\bigwedge S$ exist, for every $S \subseteq P$.

Example 1.2.5.

1. Every finite lattice is complete.

2. Let S be a set. The poset $(2^S, \subseteq)$ is a complete lattice, with, for $X \subseteq 2^S$, $\bigvee X = \bigcup X$ and $\bigwedge X = \bigcap X$.

3. The poset $(\mathbb{Z}, \leqslant)$ is a lattice, but it is not complete, since $\bigvee \mathbb{Z}$ and $\bigwedge \mathbb{Z}$ don't exist.

If P is a lattice, then $\vee$ and $\wedge$ are binary operations on P, called, respectively, *join* and *meet*. It is possible to describe the lattice using these operations:

Theorem 1.2.6.

1. *Let $L = (L, \leqslant)$ be a lattice. Then, for every $x, y, z \in L$,*

   (L1) $x \vee x = x$ *and* $x \wedge x = x$;

   (L2) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$;

   (L3) $(x \vee y) \vee z = x \vee (y \vee z)$ *and* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

   (L4) $x \vee (x \wedge y) = x$ *and* $x \wedge (x \vee y) = x$.

2. *Let L be a nonempty set equipped with two binary operations $\vee$ and $\wedge$ satisfying* $(L1) - (L4)$. *Then $L = (L, \leqslant)$, where, for every $x, y \in L$,*

$$\begin{aligned} x \leqslant y \quad &\Leftrightarrow \quad x \vee y = y \\ (&\Leftrightarrow \quad x \wedge y = x), \end{aligned}$$

   is a lattice.

If L and M are lattices, an order-preserving map $\varphi : L \longrightarrow M$ may not preserve joins and meets (see, for example, φ_2 from Example 1.1.9).

Definition 1.2.7. Let L and M be lattices. A map $\varphi : L \longrightarrow M$ is:

1. a *join-homomorphism* if, for every $x, y \in L$, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ (and dually, a meet-homomorphism if, for every $x, y \in L$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$);

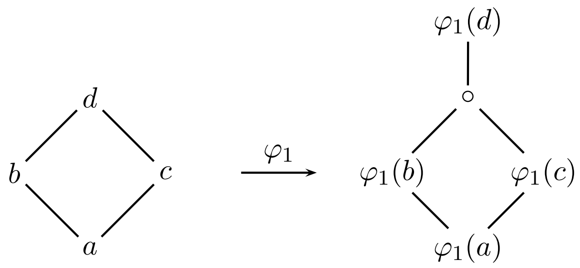
2. a *join-embedding* if it is a one-to-one join-homomorphism;

3. a *lattice-homomorphism* if it is a join- and meet-homorphism;

4. a *lattice-isomorphism* if it is a bijective lattice-homorphism.

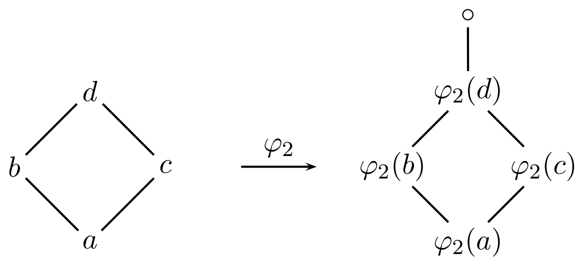
Example 1.2.8.

1. The map φ_1 given by



is a meet-homomorphism, but is not a join-homomorphism.

2. The map φ_2 given by



is a lattice-homomorphism, but is not a lattice-isomorphism.

A useful known fact is that φ is a lattice-isomorphism if and only if it is an order-isomorphism, so, in general, it is usual to write that φ is an isomorphism.

There are several kinds of special lattices that have additional properties. In this work, the following ones have special interest:

Definition 1.2.9. Let L be a lattice.

1. L is *distributive* if it verifies the distributive law, that is, for every $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

2. L is *modular* if it verifies the modular law, that is, for every $x, y, z \in L$, $x \leqslant z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z$.

Observe that, if a lattice is distributive, then it is modular.

Example 1.2.10.

1. The lattice in Example 1.2.3 is modular and distributive.

2. The *diamond* lattice M_3, in Figure 1.1, is modular but not distributive, since $a \wedge (b \vee c) = a \wedge 1 = a$ but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$.

3. The *pentagon* lattice N_5, in Figure 1.2, is not modular and not distributive, since $a \leqslant c$ but $a \wedge (b \vee c) = a \wedge 1 = a$ and $(a \wedge b) \vee c = 0 \vee c = c$.

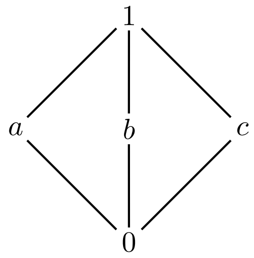


Figure 1.1: M_3

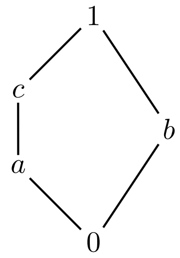


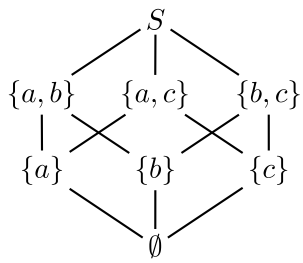
Figure 1.2: N_5

There are several ways to characterize distributivity and modularity. A well-known and widely used one is the characterization by forbidden sublattices.

Definition 1.2.11. Let L be a lattice. A nonempty subset S of L is a *sublattice* if, for every $x, y \in L$,

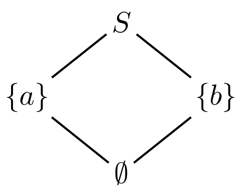
$$x, y \in S \Rightarrow x \vee y, x \wedge y \in S.$$

Example 1.2.12. Let $S = \{a, b, c\}$, and consider the lattice $(2^S, \subseteq)$.



It is easy to verify that $\{\emptyset, \{c\}, \{a, b\}, S\}$ is a sublattice of 2^S.

The set $X = \{\emptyset, \{a\}, \{b\}, S\}$ is not a sublattice of 2^S, since $\{a\} \vee \{b\} = \{a, b\} \notin X$. Observe, however, that $(X, \subseteq)$ is a lattice.



Theorem 1.2.13 ($M_3 - N_5$ Theorem)**.** *Let L be a lattice.*

1. *L is modular if and only if does not contain a sublattice isomorphic to N_5.*

2. L is distributive if and only of does not contain a sublattice isomorphic to either M_3 or N_5.

In some cases, it is possible to obtain stronger results. In particular, an important one for this work is the following.

Proposition 1.2.14 ([14]). *Let L be a finite modular lattice. Then L is distributive if and only if it does not contain a cover-preserving diamond, that is, a sublattice isomorphic to M_3 in which $0 \prec a, b, c \prec 1$.*

Definition 1.2.15. Let L be a lattice. An element $x \in L$ is *join-irreducible* if

$$x \neq \perp \text{ and } x = y \vee z \Rightarrow x = y \text{ or } x = z.$$

Dually, x is *meet-irreducible* if $x \neq \top$ and $x = y \wedge z \Rightarrow x = y$ or $x = z$.

Example 1.2.16. In Example 1.2.3, the elements 2, 3, and 4 are join-irreducible.

A widely studied subject in lattice theory is how to represent a lattice. There are several interesting results, being one of them the Birkhoff's Representation Theorem for distributive lattices. This theorem states that any finite distributive lattice is isomorphic to a lattice of finite sets, in which the operations of join and meet correspond to union and intersection, respectively. The construction of this lattice depends heavily on some properties related to join-irreducible elements, namely, that in a finite lattice, every element is the join of join-irreducible elements. For an arbitrary lattice, this decomposition may not be unique (for example, in the lattice M_3, a, b, and c are join-irreducible, and $1 = a \vee b = a \vee c$), but if the lattice is distributive, it is unique.

There are several generalizations of this theorem, obtained from weakening the conditions on the lattice. In this work, the results used are the following theorem and corollary.

Theorem 1.2.17 ([20]). *Let $(L, \leqslant)$ be a finite lattice, M its set of meet-irreducibles and consider the lattice $(2^M, \subseteq)$. Then,*

$$\varphi : L \to 2^M,$$
$$\varphi(a) = \{y \in M : y \not\geqslant a\}$$

is a join-embedding of $(L, \leqslant)$ into $(2^M, \subseteq)$, that is,

$$\varphi(a \vee b) = \varphi(a) \cup \varphi(b).$$

Proof. See Theorem 2.6, page 179 of [20]. □

Corollary 1.2.18. *In Theorem 1.2.17's conditions, considering the range of φ, $R(\varphi)$, finite family of sets closed for unions, $\varphi : L \to R(\varphi) \subseteq 2^M$ is an isomorphism such that $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$.*

Proof. See page 176 of [20]. □

1.3 Ordinal Numbers

Definition 1.3.1. Let W be a poset. W is a *well-ordered set* if every nonempty subset of W contains a least element.

Example 1.3.2. The poset $(\mathbb{N}, \leqslant)$ is well-ordered.

Observe that a well-ordered set is necessarily a chain. The opposite is not true: not every chain is a well-ordered set. For example, $\mathbb{Z}$, with the usual order, is a chain, but it is not well-ordered.

If two well-ordered sets are isomorphic, they are said to have the same order-type.

Definition 1.3.3. Let T be a set. T is *transitive* if every element of T is a subset of T.

Example 1.3.4. The set $T = \{\emptyset, \{\emptyset\}\}$ is transitive.

Definition 1.3.5. Let α be a set. α is an *ordinal (number)* if:

1. α is transitive, and
2. $(\alpha, \in)$ is well-ordered.

Ordinals are usually denoted by lower case greek letters and the class of all ordinals is denoted by $\mathbb{O}rd$.

Observe that, using an argument similar to Russell's paradox[2], it is possible to conclude that $\mathbb{O}rd$ is a *proper class*, that is, a class that is not a set. More, every unlimited part of $\mathbb{O}rd$ is a proper class, and if A is a proper class and $f : A \longrightarrow B$ is an embedding, then B must also be a proper class.

The following facts are easy to obtain.

Proposition 1.3.6.

1. *$0 = \emptyset \in \mathbb{O}rd$.*
2. *If $\alpha \in \mathbb{O}rd$ and $\beta \in \alpha$, then $\beta \in \mathbb{O}rd$.*
3. *If $\alpha, \beta \in \mathbb{O}rd$ and $\alpha \subsetneqq \beta$, then $\alpha \in \beta$.*
4. *If $\alpha, \beta \in \mathbb{O}rd$, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

In $\mathbb{O}rd$, it is possible to define an order relation by

$$\alpha < \beta \text{ if and only if } \alpha \in \beta,$$

for $\alpha, \beta \in \mathbb{O}rd$. Then, the previous proposition yields that:

[2]**Russell's Paradox**: Consider the set S whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Does S belong to S? If S belongs to S, then S is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then S belongs to S. In either case, it is a contradiction, so one must conclude that S is not a set.

1. $(\mathbb{O}rd, <)$ is a chain.

2. For each $\alpha \in \mathbb{O}rd$, $\alpha = \{\beta : \beta < \alpha\}$.

3. If C is a nonempty class of ordinals, then $\bigcap C \in \mathbb{O}rd$ and $\bigcap C = \inf C$.

4. If S is a nonempty set of ordinals, then $\bigcup S \in \mathbb{O}rd$ and $\bigcup S = \sup S$.

As a consequence, it is possible to prove that the definition of ordinals provide order-types of well-ordered sets.

Theorem 1.3.7 ([19]). *Every well-ordered set is isomorphic to a unique ordinal number.*

More, for every α, $\alpha \cup \{\alpha\} \in \mathbb{O}rd$ and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$. Thus, define $\alpha + 1$, the *successor* of α, by

$$\alpha + 1 = \alpha \cup \{\alpha\}.$$

Definition 1.3.8. Let $\alpha \in \mathbb{O}rd$. α is a *successor ordinal* if $\alpha = \beta + 1$, for some $\beta \in \mathbb{O}rd$. If α is not a successor ordinal, α is said to be a *limit ordinal*.

Observe that, if α is a limit ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\}$. Also, 0 is considered a limit ordinal, defining $\sup \emptyset = 0$.

The least nonzero limit ordinal is usually denoted by ω. The ordinals less than ω are called finite ordinals or natural numbers:

$$\begin{aligned} 0 &= \emptyset \\ 1 &= 0 + 1 = \{\emptyset\} = \{0\} \\ 2 &= 1 + 1 = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\ 3 &= 2 + 1 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\ &\vdots \end{aligned}$$

The results needed in Chapter 5 concern a special kind of ordinals. To obtain it, some facts about ordinal arithmetic are required.

For ordinal numbers, define addition in the following way:

1. $\alpha + 0 = \alpha$,

2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ for all β,

3. $\alpha + \beta = \lim_{\xi \to \beta} (\alpha + \xi)$ for all limit $\beta > 0$.

Observe that ordinal sums are associative, but not commutative: for example, $1 + \omega = \omega \neq \omega + 1$.

Definition 1.3.9. Let $\alpha \in \mathbb{O}rd$. α is *additively indecomposable* if it cannot be decomposed into the sum of two smaller ordinals, that is,

$$\beta, \xi < \alpha \Rightarrow \beta + \xi < \alpha.$$

Call $\mathbb{H}$ the proper class of additively indecomposable ordinals (German "Hauptzahl").

Proposition 1.3.10. *The proper class $\mathbb{H}$ is unbounded.*

Proof. See [23], page 48. $\square$

Finally, the Zermelo's Well-Ordering Theorem (equivalent to the Axiom of Choice):

Theorem 1.3.11 (Zermelo's Well-Ordering Theorem)**.** *Every set S can be well-ordered, i.e., there is $\alpha \in \mathbb{O}rd$ and a one-to-one correspondence (enumeration) $\Psi : S \mapsto \{\xi \in \mathbb{O}rd : \xi < \alpha\}$.*

Proof. See [19], page 48. $\square$

The set $\{\xi \in \mathbb{O}rd : \xi < \alpha\}$ is usually called *initial segment* of $\mathbb{O}rd$ (given by α), and denoted by $\text{seg}(\alpha)$. A sequence of elements of S indexed in ordinals of some initial segment, a_ξ, is usually called an *enumeration* of S.

Chapter 2

Combinatorial Games

In this chapter some basic definitions and results on Combinatorial Game Theory (CGT) are presented. CGT studies two-player games with no hidden information and no chance devices. It was founded as an independent branch of Mathematics in the 1970's/1980's, with the publication of *On Numbers and Games* by John Conway and *Winning Ways* by Elwyn Berlekamp, John Conway, and Richard Guy. Beyond its many interesting results that can help players to win games, it has a rich algebraic structure that is not yet completely studied, and connections with several other subjects, like Artificial Intelligence and Economics, have been made.

For better understanding, some concepts are accompanied with examples (the chosen game is DOMINEERING[1]). Every result presented can be found in [3, 5, 10], and the reader can see [13] for a complete survey.

2.1 Basic Concepts

Definition 2.1.1. A *combinatorial game* is a game which satisfies the following conditions:

1. There are two players who take turns moving alternately;
2. No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the details of the game state at all times;
3. The rules of a combinatorial game ensure that, even ignoring the alternating move condition, there is no infinite sequence of legal moves;
4. The winner is determined on the basis of who made the last move.

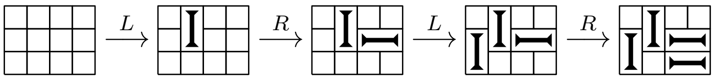
[1] DOMINEERING is a two-player game played in a grid. The two players take turns placing domino pieces on the grid. Each piece covers two adjacent squares. One player, Left, covers vertically adjacent squares, and the other player, Right, covers horizontally adjacent squares. The game ends when one of the players can not make a move, and the last move determines the winner.

Under *normal play*, the last player to move wins, while in *misère play*, the last player loses.

In this work, games are always assumed to be played under normal play.

The two players of a combinatorial game are usually called Left (or L, black or vertical) and Right (or R, white or horizontal).

Example 2.1.2. A game of DOMINEERING in a 3×4 grid, with Left playing first:



Under normal play, Right wins. Under misère play, Left wins.

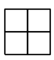
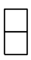
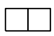
In CGT it is assumed that both players play perfectly. Although it is not an easy concept to define, we can at least interpret it in the following way: if a player can force a win, then he makes a move that allows him to win.

Theorem 2.1.3 (Fundamental Theorem of CGT, [3])**.** *Let G be a combinatorial game. Then, either the first player or the second player to move can force a win, but not both.*

Proof. By induction, the game obtained after the first move is either a win for second player moving first or a win for first player moving second. If any legal first move belongs to the latter category, first player can force a win by choosing it. Otherwise, second player can force a win. □

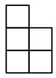
As a consequence of this theorem, there are just four possible outcomes for a combinatorial games: the first player to move wins, the second player to move wins, Left can win regardless of who plays first, or Right can win regardless of who plays first. So, four *outcome classes* are defined:

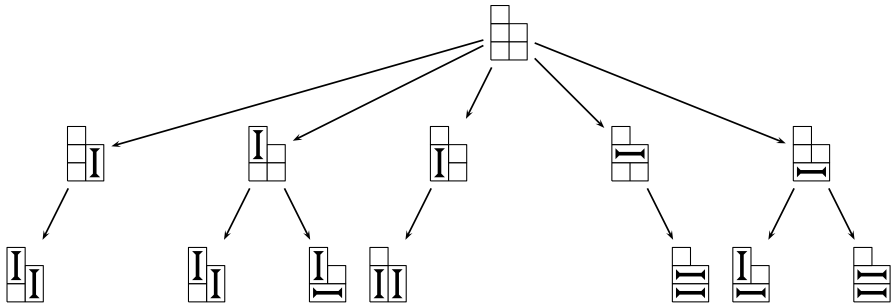
Class	**Name**	**Definition**
$\mathcal{N}$	fuzzy	The $\mathcal{N}$ext player can force a win
$\mathcal{P}$	zero	The $\mathcal{P}$revious player can force a win
$\mathcal{L}$	positive	$\mathcal{L}$eft can force a win regardless of who plays first
$\mathcal{R}$	negative	$\mathcal{R}$ight can force a win regardless of who plays first

Example 2.1.4.  $\in \mathcal{N}$,  $\in \mathcal{P}$,  $\in \mathcal{L}$,  $\in \mathcal{R}$.

Suppose that a particular game G is given. If it is Left's turn to play, he can choose from a certain set of moves determined by the rules of the game. The elements of this set are called the *left options* of G. The *right options* of G are defined similarly. The *options* of G is the union of the left and right options of G.

We can draw a game tree of a game using its options, in a rather natural way.

Example 2.1.5. The game tree of  is:



Game trees can be drawn with more or less details, that is, game trees of options can be omitted and often some options are not presented (for symmetry reasons, or because another option exists at least as good).

2.2 Basic Definitions

In this section some concepts on games are introduced, in a more mathematical way. All the results presented here can be found in [3].

Definition 2.2.1. A *combinatorial game* G is a pair of sets of combinatorial games

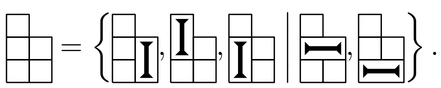
$$G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\},$$

where $\mathcal{G}^L$ is the set of left options and $\mathcal{G}^R$ is the set of right options of G.

A typical element of $\mathcal{G}^L$ is denoted by G^L and a typical element of $\mathcal{G}^R$ is denoted by G^R.

Observation 2.2.2. Observe that this definition, as well as the next ones, is recursive, with base case $\mathcal{G}^L = \mathcal{G}^R = \emptyset$. The game $\{\emptyset \mid \emptyset\} = \{\ \mid\ \}$ is called *zero* and is denoted by 0.

Example 2.2.3. The game in Example 2.1.5 can be represented in the form



Definition 2.2.4. Let G be a game. G is a *short game* if it has a form $\left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$ such that $\mathcal{G}^L$ and $\mathcal{G}^R$ are finite sets of short games. Otherwise, the game is a *long game*.

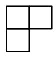
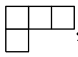
Definition 2.2.5. Let G be a game. G's *followers* are G and all the games that can be reached by all the possible sequences of moves from G.

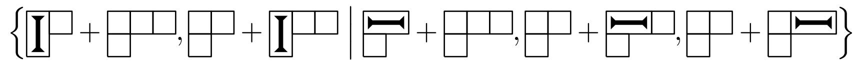
When playing combinatorial games, it is easy to observe that, in many situations, independent components appear. This leads to the following recursive definition.

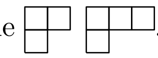
Definition 2.2.6. Let G and H be games. The *sum* $G + H$ is the game

$$G + H = \{\mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R\},$$

where, for a set of games $\mathcal{S}$, $G + \mathcal{S} = \{G + S : S \in \mathcal{S}\}$.

Example 2.2.7. If $G =$  and $H =$ , then

$G+H =$ .

Observe that these correspond to the options of the game .

It is known that this operation is commutative, associative and has an identity (see [3]).

Theorem 2.2.8. *Let G, H and J be games. Then:*

1. $G + 0 = G$;
2. $G + H = H + G$;
3. $(G + H) + J = G + (H + J)$.

Proof.

1. By definition, $G+0 = \{\mathcal{G}^L \mid \mathcal{G}^R\}+\{\ \mid\ \} = \{\mathcal{G}^L + 0, G + \emptyset \mid \mathcal{G}^R + 0, G + \emptyset\}$. But $G + \emptyset = \emptyset$ and, by induction, $\mathcal{G}^L + 0 = \mathcal{G}^L$ and $\mathcal{G}^R + 0 = \mathcal{G}^R$, so
$$G + 0 = \{\mathcal{G}^L \mid \mathcal{G}^R\} = G.$$
2. By definition, $G + H = \{\mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R\}$. By induction,
$$\begin{aligned} G^L + H &= H + G^L,\\ G + H^L &= H^L + G,\\ G^R + H &= H + G^R,\\ G + H^R &= H^R + G, \end{aligned}$$
so
$$G + H = \{H + \mathcal{G}^L, \mathcal{H}^L + G \mid H + G^R, \mathcal{H}^R + G\} = H + G.$$

3. A typical left option of $(G+H)+J$ is

$$
\begin{aligned}
&[(G+H)+J]^L = \\
&\quad = \{(G+H)^L + J, (G+H)+\mathcal{J}^L\} \\
&\quad = \{(\mathcal{G}^L + H)+J, (G+\mathcal{H}^L)+J, (G+H)+\mathcal{J}^L\} \\
&\quad = \{\mathcal{G}^L + (H+J), G+(\mathcal{H}^L+J), G+(H+\mathcal{J}^L)\} \text{ (by induction)} \\
&\quad = \{\mathcal{G}^L + (H+J), G+(H+J)^L\} \\
&\quad = [G+(H+J)]^L.
\end{aligned}
$$

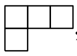
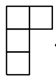
Similarly, $[(G+H)+J]^R = [G+(H+J)]^R$, so $(G+H)+J = G+(H+J)$.

□

It is also natural to think about inverses.

Definition 2.2.9. Let G be a game. The *negative* of G is the game

$$-G = \{-\mathcal{G}^R \mid -\mathcal{G}^L\}.$$

Example 2.2.10. For $G =$ , then $-G =$ .

To verify that the game $-G$ is the inverse of G, it is necessary to formalize the meaning of equality of games.

Definition 2.2.11. Let G and H be games. $G = H$ if, for all games X, $G+X$ and $H+X$ have the same outcome.

It is easy to verify that "=" is an equivalence relation.

Next theorem is known as the Fundamental Theorem of Normal Play (see [3]).

Theorem 2.2.12 (Fundamental Theorem of Normal Play)**.** *Let G be a game. Then $G = 0$ if and only if $G \in \mathcal{P}$.*

Proof. Suppose that $G = 0$. Then, by definition, for all games X, $G+X$ and $0+X$ have the same outcome. In particular, for $X = 0$, the games G and 0 have the same outcome. Since $0 \in \mathcal{P}$ (the first player has no legal move available, so he loses), then $G \in \mathcal{P}$.

Conversely, suppose that $G \in \mathcal{P}$, and let X be a game. To prove that $G = 0$, we must show that $G+X$ has the same outcome as X.

If Left can win moving second on X, he can also win moving second on $G+X$: whenever Right plays on G, Left responds on G, and whenever Right plays on X, Left responds on X. Since $G \in \mathcal{P}$, Left can win moving second on G, so he can win moving second on $G+X$.

If Left can win moving first on X, he can also win moving first on $G+X$: Left, playing first, moves on X, and then proceeds like in the previous case.

Using symmetric arguments, if Right wins X, he also wins $G+X$. Therefore, $G+X$ has the same outcome as X. □

This means that the equivalence class of 0 is exactly $\mathcal{P}$. The following corollary is a direct consequence of this theorem.

Corollary 2.2.13. *Let G be a game. Then $G + (-G) = 0$.*

Solving equations involving games is possible, using the following result.

Proposition 2.2.14. *Let G, H and J be games. Then $G = H$ if and only if $G + J = H + J$.*

Proof. Suppose that $G = H$. Let X be a game and consider

$$X' = J + X.$$

By definition of $G = H$, the games $G + X'$ and $H + X'$ have the same outcome, that is, $G+(J+X)$ and $H+(J+X)$ have the same outcome. But, by Theorem 2.2.8, $G+(J+X) = (G+J)+X$ and $H+(J+X) = (H+J)+X$, so $G + J = H + J$.

Conversely, suppose that $G + J = H + J$. Then, using ($\Rightarrow$), $(G + J) + (-J) = (H + J) + (-J)$. So, using Theorem 2.2.8 and Corollary 2.2.13,

$$G = G + (J - J) = (G + J) - J = (H + J) - J = H + (J - J) = H.$$

$\square$

The following corollary gives a constructive way to prove that two games are equal.

Corollary 2.2.15. *Let G and H be games. Then $G = H$ if and only if $G - H = 0$.*

This means that, to prove that two games G and H are equal, it suffices to play the game $G - H$ and verify that $G - H \in \mathcal{P}$.

Following, the definition of inequality of games and some related results.

Definition 2.2.16. Let G and H be games.

1. $G \geqslant H$ if, for all games X, Left wins $G+X$ whenever Left wins $H+X$;
2. $G \leqslant H$ if, for all games X, Right wins $G + X$ whenever Right wins $H + X$.

Observe that, if $G \geqslant H$ then replacing H by G can never be bad for Left, no matter the context.

It isn't difficult to prove that $G \geqslant H$ if and only if $H \leqslant G$.

The following results are an important tool in proving inequalities, giving a constructive way to do it.

Proposition 2.2.17. *Let G be a game. The following are equivalent:*

1. *$G \geqslant 0$;*

2. *Left wins moving second on G (that is, $G \in \mathcal{P}$ or $G \in \mathcal{L}$);*

3. *For all games X, if Left wins moving second on X, then Left wins moving second on $G + X$;*

4. *For all games X, if Left wins moving first on X, then Left wins moving first on $G + X$.*

Proof. Let G be a game.

To prove that $(2 \Rightarrow 3)$, suppose that Left wins moving second on G, and, for any game X, Left wins moving second on X. Then Left can win moving second on $G + X$ by responding to Right's moves on one component with a winning move on the same component.

Taking $X = 0$, it is immediate that $(3 \Rightarrow 2)$.

The proof of $(2 \Rightarrow 4)$ is analogous to the proof of $(2 \Rightarrow 3)$.

To prove that $(4 \Rightarrow 2)$, suppose the equivalent condition: if Left does not win moving first on $G + X$ then Left does not win moving first on X. Since $G - G = 0$, Left does not win moving first on $G - G$, so Left does not win moving first on $-G$, that is, Right can not win moving first on G. Therefore, Left wins moving second on G.

It is immediate that $(1 \Rightarrow 3, 4)$. □

Proposition 2.2.18. *Let G, H and J be games. Then $G \geqslant H$ if and only if $G + J \geqslant H + J$.*

Proof. Suppose that $G \geqslant H$. Let X be a game, and consider $X' = J + X$. By definition of $G \geqslant H$, Left wins $G + X'$ whenever Left wins $H + X'$, that is, Left wins $(G+J)+X$ whenever Left wins $(H+J)+X$, so $G+J \geqslant H+J$.

Conversely, suppose that $G + J \geqslant H + J$. Then, using $(\Rightarrow)$,

$$(G + J) + (-J) \geqslant (H + J) + (-J).$$

So,

$$G = G + (J - J) = (G + J) - J \geqslant (H + J) - J = H + (J - J) = H.$$

□

Proposition 2.2.19. *Let G and H be games. Then $G \geqslant H$ if and only if Left wins moving second on $G - H$.*

Proof. Taking $J = -H$ in Proposition 2.2.18,

$$G \geqslant H \Leftrightarrow G - H \geqslant H - H \Leftrightarrow G - H \geqslant 0.$$

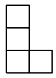
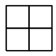
□

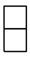
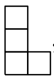
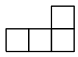
This means that comparing G and H is equivalent to determining the outcome of $G - H$:

$G - H \in \mathcal{L}$	if and only if	$G > H$
$G - H \in \mathcal{P}$	if and only if	$G = H$
$G - H \in \mathcal{R}$	if and only if	$G < H$
$G - H \in \mathcal{N}$	if and only if	$G \parallel H$

where $G > H$ means that $G \geqslant H$ and $G \neq H$, and $G \parallel H$ means that neither $G \geqslant H$ nor $G \leqslant H$. More, writing $G \mathrel{\|\!\!\triangleright} H$ if $G \parallel H$ or $G > H$,

$G \geqslant H$	$G - H \in \mathcal{L} \cup \mathcal{P}$	Left wins moving second on $G - H$
$G \mathrel{\|\!\!\triangleright} H$	$G - H \in \mathcal{L} \cup \mathcal{N}$	Left wins moving first on $G - H$
$G \leqslant H$	$G - H \in \mathcal{R} \cup \mathcal{P}$	Right wins moving second on $G - H$
$G \mathrel{\triangleleft\!\!\|} H$	$G - H \in \mathcal{R} \cup \mathcal{N}$	Right wins moving first on $G - H$

Example 2.2.20.  > 0 and  $\parallel 0$.

Also,  $>$ , because Left wins   regardless of who plays first.

An important result for this work is the following theorem.

Theorem 2.2.21. *The relation $\geqslant$ is a partial order on the set of combinatorial games.*

Proof. Let G, H and J be games.

Suppose that $G \geqslant H$ and $H \geqslant J$. Then Left wins moving second on $G - H$ and on $H - J$, so Left wins moving second on $(G - H) + (H - J)$ by responding to Right's moves on each summand with a winning move on the same summand. Since $(G - H) + (H - J) = G - J$, then $G \geqslant J$, and $\geqslant$ is transitive.

Since $G - G = 0$, Left wins moving second on $G - G$, so $G \geqslant G$ and $\geqslant$ is reflexive.

Suppose that $G \geqslant H$ and $H \geqslant G$, that is, for all games X, Left wins $G + X$ whenever Left wins $H + X$ and Left wins $H + X$ whenever Left wins $G + X$. But then $G + X$ and $H + X$ have the same outcome, so $G = H$, and $\geqslant$ is antisymmetric. $\square$

A consequence from "=" being an equivalence relation is that a game G can be replaced by any other in its equivalence class. More, in G's equivalence class, a unique game with game tree with fewer nodes exists, which is called G's canonical form. This game can be obtained from G by removing dominated options and bypassing reversible options (which will be defined).

Theorem 2.2.22 (Reduction to Canonical Form)**.** *Let G be a short game, and suppose $G = \{A, B, C, \dots \mid \dots\}$.*

1. *If $A \leqslant B$, then*
$$G = \{B, C, \dots \mid \dots\}$$
(A is dominated *by B for Left).*

2. *If, for some right option A^R of A, $A^R \leqslant G$, then*
$$G = \{A_1, A_2, \dots, B, C, \dots \mid \dots\},$$
where $A_1, A_2, \dots$ are the left options of A^R (A reverses through *A^R to the left options of A^R).*

Proof.

1. Let $G' = \{B, C, \dots \mid \dots\}$ and consider the game
$$G - G' = \{A, B, C, \dots \mid \dots\} + \{\dots \mid -B, -C, \dots\}.$$
The only first move that doesn't have a corresponding negative in the other component (leaving 0) is Left's move to A. But if Left moves to A, Right can respond to $-B$, leaving $A - B \leqslant 0$, so Right wins. Therefore, $G - G' \in \mathcal{P}$, that is, $G - G' = 0$.

2. Let $G' = \{A_1, A_2, \dots, B, C, \dots \mid \dots\}$ and consider the game
$$G - G' = \{A, B, C, \dots \mid \dots\} + \{\dots \mid -A_1, -A_2, \dots, -B, -C, \dots\}.$$
Any first move different from Left's move to A or Right's move to one of $-A_1, -A_2, \dots$ has a corresponding negative in the other component, leaving 0.

   If Right moves to one $-A_1, -A_2, \dots$, say, to A_1, leaving $G - A_1$, then Left can win because, since exists A^R such that $G - A^R \geqslant 0$ (that is, Left can win playing second on $G - A^R$), Left can win, after Right plays to $G - A_1$, moving to $G - A^R$.

   If Left moves to A, leaving $A - G'$, Right can win by moving to $A^R - G'$: to this move, Left can only respond to some $A_i - G'$ (and Right then moves to $A_i - A_i = 0$ and wins) or to some $A^R - G^R$, which is an option from $A^R - G \leqslant 0$, so Right wins playing second.

   Hence, $G - G' = 0$.

□

If $A^{RL} = \emptyset$, the option A of G is said to *reverse out*.

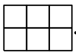
Observation 2.2.23. Analogously, let $G = \{\ldots \mid W, X, Y \ldots\}$ be a short game.

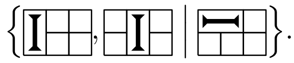
If $W \geqslant X$, then $G = \{\ldots \mid X, Y, \ldots\}$ (W is dominated by X for Right); and if, for some left option W^L of W, $W^L \geqslant G$, then

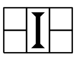
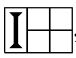
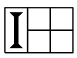
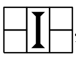
$$G = \{\ldots \mid W_1, W_2, \ldots, X, Y, \ldots\},$$

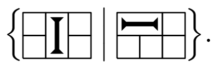
where $W_1, W_2, \ldots$ are the right options of W^L (W reverses through W^L to the right options of W^L).

In other words, if two moves are available but one is better than the other for that player, the worst can be removed from the options of the game, and is said to be dominated.

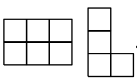
Example 2.2.24. Consider $G =$ . Then

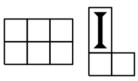
$G =$ .

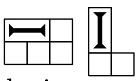
The option  is better for Left than , so  is dominated by , and

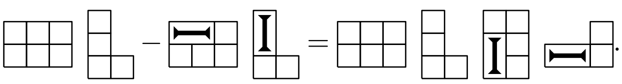
$G =$ .

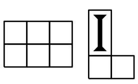
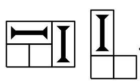
The concept of reversibility is more elaborated. In fact (for Left), if, for some right option A^R of A, $A^R \leqslant G$, then A^R would be better for Right than every right option of G. So, if Left moves to A, it is certain that Right will move to A^R, so Left's move to A can be replaced by all the left options of A^R, and A is reversible.

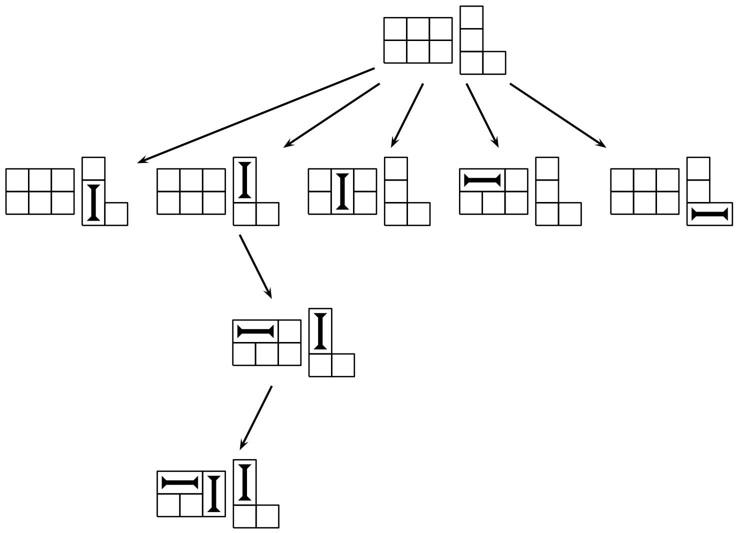
Example 2.2.25. Consider $G =$ . The game



is a left option of G, and has  as a right option. This is better for Right than G because Left, playing second, wins:



Therefore,  can be replaced by its left options, which are .



Definition 2.2.26. Let G be a short game. G is in *canonical form* if G and all G's followers have no dominated or reversible options.

Theorem 2.2.27 (Unicity of the Canonical Form)**.** *Let G and H be short games. If G and H are in canonical form and $G = H$, then $\mathcal{G}^L = \mathcal{H}^L$ and $\mathcal{G}^R = \mathcal{H}^R$.*

Proof. Suppose that G and H are in canonical form, that is, G, H, and all their followers have no dominated or reversible options.

Since $G = H$, Left wins moving second on $G - H$. In particular, for each G^R, Left has a winning move from $G^R - H$. That move must be to $G^R - H^R$: if it was to $G^{RL} - H$, Left would win playing second on $G^{RL} - H$, that is, $G^{RL} \geqslant H = G$, and G would have a reversible move. So $G^R - H^R \geqslant 0$, that is, $G^R \geqslant H^R$.

Analogously, for each H^R, some $G^{R'}$ exists such that $H^R \geqslant G^{R'}$, so $G^R \geqslant H^R \geqslant G^{R'}$. Since G has no dominated options, $G^R = G^{R'}$. Hence $G^R = H^R$, and $\mathcal{G}^R \subseteq \mathcal{H}^R$.

Symmetrically, $\mathcal{H}^R \subseteq \mathcal{G}^R$, and $\mathcal{G}^R = \mathcal{H}^R$.

Analogously, $\mathcal{G}^L = \mathcal{H}^L$. $\square$

2.3 Conway's Construction of Numbers and Games

In [10], Conway gives an axiomatic construction of numbers and ordinal arithmetic. This construction is made by a recursive process, based in a transfinite sequence of days, using a notation based on Dedekind cuts. Also, using this process and notation, he constructs the set of all combinatorial games. More, with this notation, it is possible to see that some games are numbers, some games are infinitesimals, and some games are neither numbers nor infinitesimals.

Definition 2.3.1. Let G be a game. The *birthday* of G, $\mathrm{b}(G)$, is defined recursively by

$$\mathrm{b}(G) = \left\{\mathrm{b}(G^L), \mathrm{b}(G^R) \;\middle|\; \right\},$$

where $\left\{\mathcal{G}^L \;\middle|\; \mathcal{G}^R\right\}$ is a simplest form of G, that is, another form $\left\{\mathcal{G}'^L \;\middle|\; \mathcal{G}'^R\right\}$ doesn't exists with

$$\sup\{\mathrm{b}(H) : H \in \mathcal{G}'^L \cup \mathcal{G}'^R\} < \sup\{\mathrm{b}(H) : H \in \mathcal{G}^L \cup \mathcal{G}^R\}.$$

A game G is said to be *born by* day n if $\mathrm{b}(G) \leqslant n$.

Observation 2.3.2. A game G is short if its birthday is finite, that is, if $\mathrm{b}(G) < \omega$. Long games are born on day ω or after.

It is easy to see that, for a short game $G = \left\{\mathcal{G}^L \;\middle|\; \mathcal{G}^R\right\}$ in canonical form, $\mathrm{b}(G) = 1 + \max\{\mathrm{b}(H) : H \in \mathcal{G}^L \cup \mathcal{G}^R\}$.

On day 0, $\mathcal{G}^L = \mathcal{G}^R = \emptyset$, so the only game born by day 0 is 0.

On day 1, 0 already exists, so $\mathcal{G}^L, \mathcal{G}^R \subseteq \{0\}$. Therefore, there are four games born by day 1: 0, $\{0 \mid \}$, $\{ \mid 0\}$, and $\{0 \mid 0\}$.

The game $\{0 \mid \}$ is a game where Left has available one option, to a game of value 0, and Right has no options available. This game is naturally called 1. Analogously, $\{ \mid 0\}$ is called -1. The game $\{0 \mid 0\}$ is a game where both players have the option of moving to 0, and is called $*$.

Observe that

$$\begin{array}{rcl} 1 & = & \{0 \mid \} \in \mathcal{L}, \\ -1 & = & \{ \mid 0\} \in \mathcal{R}, \\ * & = & \{0 \mid 0\} \in \mathcal{N}, \end{array}$$

that is, $1 > 0$, $-1 < 0$, and $* \parallel 0$.

On day 2, 22 games are born; on day 3, 1474 games are born, and on day 4, somewhere between 3 trillion and 10^{434} games are born (see [30]). The exact number of games born by day $n < \omega$ or even a good asymptotic formula is not known.

2.4 Numbers, Infinitesimals, and Others

In this section, some examples of short games are presented.

Definition 2.4.1. Let G be a game in canonical form. G is a *number* if all G^L, G^R are numbers and $G^L < G^R$ for every $G^L \in \mathcal{G}^L$, $G^R \in \mathcal{G}^R$.

It is easy to see that 0, 1, and -1 are numbers and that $*$ is not a number.

Integers are defined in a rather natural way. In fact, on day n, with $n \geqslant 1$, the game

$$n = \{n - 1 \mid \}$$

is born. In this game, Left has one move to the game of value $n-1$, while Right has no move. By definition, it is easy to see that it is a number, and that $n > 0$. Also,

$$-n = \{\ \mid -n+1\}$$

is a number, born on day n, $-n < 0$, and $n + (-n) = 0$.

On day 2, several other games are born. One of the new games is $G = \{0 \mid 1\}$. By definition, G is a number. Also, it is easy to see that $0 < G < 1$ and $G + G = 1$, so it is natural to say that $G = \frac{1}{2}$. More generally,

Definition 2.4.2. Let $j, m \in \mathbb{N}$, with m odd. The number $\frac{m}{2^j}$ is defined by

$$\frac{m}{2^j} = \left\{ \frac{m-1}{2^j} \,\middle|\, \frac{m+1}{2^j} \right\}.$$

If a game is not in canonical form, it may not be immediate to recognize which number it is. An important tool to solve this problem is given by the next theorem.

Theorem 2.4.3 (Simplest Number)**.** *Let G be a short game, and suppose that $G = \{a \mid b\}$ is a number. If a and b are also numbers, then $G = x$ where x is given by:*

1. *If there is (are) integer(s) n such that $a < n < b$, then x is the one that is smallest in absolute value;*

2. *Otherwise, x is the number of the form $\frac{i}{2^j}$ between a and b for which j is minimal.*

Proof. Let $G = \{a \mid b\}$, with a, b numbers, and suppose that $G > 0$ (if $G < 0$ the arguments are analogous).

Consider the game $G - x$.

Suppose that x is given by condition 1. Left playing first can only move to $a - x < 0$, so he loses. Right playing first can move to $b - x > 0$, and loses, or to $G - x + 1$. To this move, Left can respond moving to $a - x + 1$. But, since $a \geqslant x - 1$, Left wins.

Otherwise, suppose that x is given by condition 2, that is,

$$G - x = G + \left\{ -\frac{i+1}{2^j} \,\middle|\, -\frac{i-1}{2^j} \right\},$$

with j minimal.

Left playing first can move in either component. If he moves to $a - x$, using the previous argument, he loses. If he moves to $G - \frac{i+1}{2^j}$, Right can respond to $b - \frac{i+1}{2^j}$. But $b \leqslant \frac{i+1}{2^j}$ for, if not, j wouldn't be minimal, and so Right wins.

Right playing first can move in either component. If he moves to $b - x$, using the previous argument, he loses. If he moves to $G - \frac{i-1}{2^j}$, Left can respond to $a - \frac{i-1}{2^j}$. But $a \geqslant \frac{i-1}{2^j}$ for, if not, j wouldn't be minimal, and so Left wins.

Either way, $G - x = 0$. □

If $a < b$, the number given by this theorem is usually called the simplest number between a and b, and it is, in fact, the number strictly between a and b with the least birthday.

Example 2.4.4. Using Theorem 2.4.3, it is easy to see that $\{-2 \mid 5\} = 0$, $\left\{\frac{1}{2} \mid 3\right\} = 1$, $\left\{\frac{1}{8} \mid \frac{5}{8}\right\} = \frac{1}{2}$, and $\left\{\frac{1}{2} \mid \frac{7}{8}\right\} = \frac{3}{4}$.

An important result concerning numbers and sums of games is that it is usually reasonable to avoid playing on numbers.

Theorem 2.4.5 (Weak Number Avoidance)**.** *Let G and H be games such that H is a number and G is not. If $G + H \mathrel{|\!\!\triangleright} 0$, then $G^L + H \geqslant 0$ for some G^L.*

Proof. Suppose that $G + H \mathrel{|\!\!\triangleright} 0$, that is, that Left can win moving first on $G + H$.

Suppose that the winning move is on H, H^L. Then $G + H^L \geqslant 0$. Since G is not a number, $G \neq -H^L$, so $G + H^L > 0$, and Left wins moving first on $G + H^L$. By induction, $G^L + H^L \geqslant 0$. But $H > H^L$, so $G^L + H \geqslant 0$. □

Another two important games born by day 2 are

$$\{0 \mid *\} = \uparrow \text{ and } \{* \mid 0\} = \downarrow,$$

which are not numbers. It is easy to see that $\uparrow > 0$, and that $\uparrow < x$, for all positive numbers x.

Definition 2.4.6. Let G be a game. G is *infinitesimal* if $-x < G < x$, for all positive numbers x.

So, $\uparrow$ and $\downarrow$ are infinitesimal. More, using Theorem 2.4.5, it is possible to prove that $*$ is also infinitesimal.

The game $\uparrow* = \uparrow + * = \{0, * \mid 0\}$ is also an infinitesimal, born by day 2. After day 2, more infinitesimals of this kind appear. In fact, the following pattern arises:

$$\begin{array}{ll}
\uparrow = \{0 \mid *\} & \uparrow* = \{0, * \mid 0\} \\
\Uparrow = \{0 \mid \uparrow*\} & \Uparrow* = \{0 \mid \uparrow\} \\
3 \cdot \uparrow = \{0 \mid \Uparrow*\} & 3 \cdot \uparrow* = \{0 \mid \Uparrow\} \\
4 \cdot \uparrow = \{0 \mid 3 \cdot *\} & 4 \cdot \uparrow* = \{0 \mid 3 \cdot \uparrow\} \\
\vdots & \vdots
\end{array}$$

where, for an integer n and a game G, $n \cdot G$ denotes

$$\underbrace{G + \cdots + G}_{n}.$$

Another example of games that are infinitesimals are impartial games. An *impartial game* is a game where left and right options are the same for the game and all its followers. Such is the case of *nimbers* (or stars):

$$*k = \{0, *, \ldots, *(k-1) \mid 0, *, \ldots, *(k-1)\}.$$

It is known that all impartial games are nimbers (Sprague-Grundy Theory, see [16, 28]), and how to determine which one it is: for a set $S \subseteq \mathbb{N}$, define the minimum excluded value of S, $\text{mex}(S)$, as the least integer which is not included in S. The nim-value of an impartial game G, denoted by $\mathfrak{G}(G)$, is given by

$$\mathfrak{G}(G) = \text{mex}(\{\mathfrak{G}(H) : H \text{ is an option of } G\}).$$

The value of an impartial game G is the nimber $*\mathfrak{G}(G)$.

The Infinite Sprague-Grundy Theorem applies also to long games where transfinite nimbers $*\alpha$, with $\alpha \in \mathbb{O}rd$, are games such that the left and right options are all the games $*\beta$ where $\beta \in \mathbb{O}rd$ and $\beta < \alpha$ (see [5]).

The nim-sum of two nonnegative integers is the exclusive or (XOR), written $\oplus$, of their binary representations. It can also be described as adding the numbers in binary without carrying. One important result about impartial games is that if $G = H + J$, then $\mathfrak{G}(G) = \mathfrak{G}(H) \oplus \mathfrak{G}(J)$ (see [3, 5, 10]).

A combinatorial game G is *all-small* if either $G = \{ \mid \}$ or every follower of G is either $\{ \mid \}$ or has both left and right options. The class of all-small games extends the class of impartial games, that is, every impartial game is all-small, and it is widely studied (see, for example, [9]).

There are also games that are neither numbers nor infinitesimals. One particular case of such a game is the switch

$$\{x \mid -x\} = \pm x.$$

Switches are games of the form $\{y \mid z\}$, with y, z numbers, and $y > z$, and are also an example of a *hot game*, that is, a game where each player has a strong interest in making the next move, for it will improve his position.

Chapter 3

Lattices of Games

As it was seen in the previous chapter, the set of combinatorial games equipped with the relation $\leqslant$ is a poset. In this chapter, it is seen that, in some cases, a lattice structure is defined, and some properties are presented. Also, Conway's construction with an initial set of games is introduced, and the resulting set's structure is studied.

The following definitions are fundamental for the remaining chapters.

Definition 3.0.7. Let S be a set of games. The set of *children* of S, $\mathcal{L}(S)$, is the set of games

$$\mathcal{L}(S) = \Big\{G : G = \{\mathcal{G}^L \mid \mathcal{G}^R\},\ \text{where } \mathcal{G}^L, \mathcal{G}^R \subseteq S\Big\}.$$

In Conway's construction, games born by day n are the children of the games previously born. A generalized construction for any generating set S can be defined:

Definition 3.0.8. Let S be a set of games. Define

$$\begin{aligned} \mathcal{L}^0(S) &= \mathcal{L}(S) \\ \mathcal{L}^\gamma(S) &= \mathcal{L}(\{G \in \mathcal{L}^\delta(S) : \delta < \gamma \text{ and } \delta, \gamma \in \mathbb{O}rd\}). \end{aligned}$$

Next, two cases will be considered: $S = \emptyset$ and $S \neq \emptyset$.

3.1 Case $\mathbf{S} = \emptyset$

When $S = \emptyset$ and $\gamma \in \mathbb{O}rd$, $\mathcal{L}^\gamma(S)$ corresponds to the usual Conway's construction. When considering just short games, γ is a finite ordinal and, to simplify the notation, for an integer n, the set of games born by day n will be denote by $\mathcal{L}^n$. That is, in the short case,

$$\begin{aligned} \mathcal{L}^0 &= \mathcal{L}(\emptyset) = \{0\} \\ \mathcal{L}^n &= \mathcal{L}(\mathcal{L}^{n-1}) = \left\{\{\mathcal{G}^L \mid \mathcal{G}^R\} : \mathcal{G}^L, \mathcal{G}^R \subseteq \mathcal{L}^{n-1}\right\}. \end{aligned}$$

Observe that, by induction, $\mathcal{L}^{n-1} \subseteq \mathcal{L}^n$.

3.1.1 Games Born by Day 2

In the previous chapter, it was observed that $\mathcal{L}^1 = \{-1, 0, *, 1\}$. More, $-1 < 0 < 1$, $0 \parallel *$, and, since $*$ is infinitesimal, $-1 < * < 1$. So, it is possible to draw $\mathcal{L}^1$'s Hasse diagram.

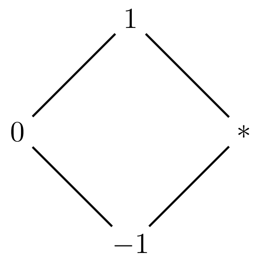


Figure 3.1: $\mathcal{L}^1$

From Figure 3.1, it is immediate to conclude that $\mathcal{L}^1$ is a distributive lattice.

On day 2, 256 games can be constructed, but not all are different. One significant reduction of the number of games that are born by day n can be obtained just by observing that if two comparable options are available to one player, one of them is dominated. So, the left and right options for games born by day n can be obtained from day $n-1$'s antichains. In particular, for day 2, day 1's antichains are:

$$\{1\}, \{0\}, \{*\}, \{0, *\}, \{-1\}, \text{ and } \{\emptyset\}.$$

Therefore, by day 2, at most 36 different games are born.

In order to obtain $\mathcal{L}^n$, a table can be constructed, using antichains. For each player, the antichains are usually arranged so that player's preferred options appear first.

<table>
<tr><th></th><th colspan="7">Right</th></tr>
<tr><th></th><th></th><th>-1</th><th>$0, *$</th><th>0</th><th>$*$</th><th>1</th><th>$\emptyset$</th></tr>
<tr><th rowspan="6">Left</th><td>1</td><td>± 1</td><td>$1 \mid 0, *$</td><td>$1 \mid 0$</td><td>$1 \mid *$</td><td>$1*$</td><td>2</td></tr>
<tr><td>$0, *$</td><td>$0, * \mid -1$</td><td>$*2$</td><td>$\uparrow *$</td><td rowspan="2">$\uparrow$</td><td rowspan="2">$\frac{1}{2}$</td><td rowspan="2">1</td></tr>
<tr><td>0</td><td>$0 \mid -1$</td><td>$\downarrow *$</td><td>$*$</td></tr>
<tr><td>$*$</td><td>$* \mid -1$</td><td colspan="2">$\downarrow$</td><td colspan="3" rowspan="3">0</td></tr>
<tr><td>-1</td><td>$-1*$</td><td colspan="2">$-\frac{1}{2}$</td></tr>
<tr><td>$\emptyset$</td><td>-2</td><td colspan="2">-1</td></tr>
</table>

Table 3.1: $\mathcal{L}^2$ antichains table

In [17] and [18], Richard Guy presented the following Hasse diagram of $\mathcal{L}^2$:

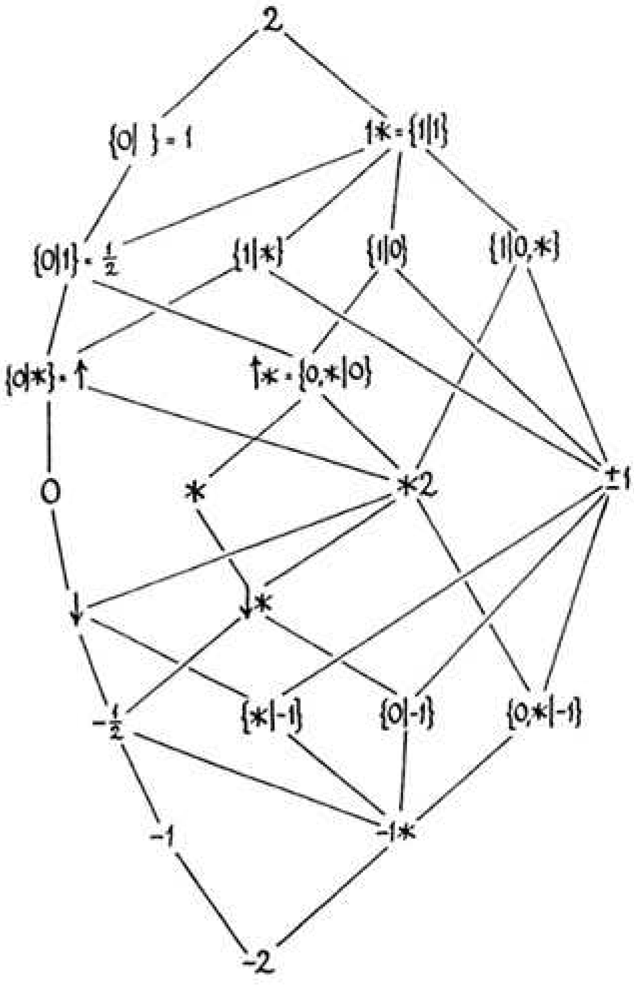


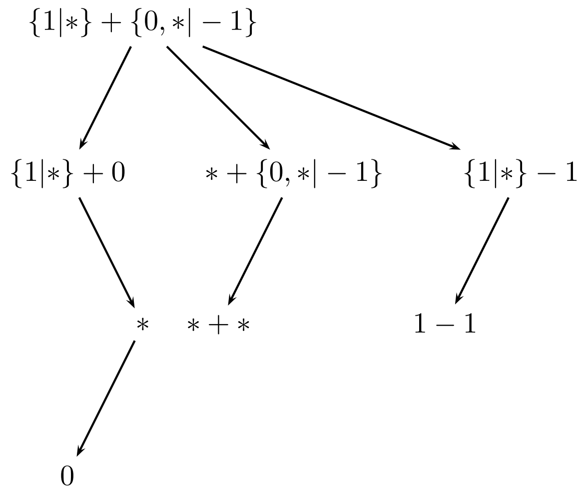
Figure 3.2: $\mathcal{L}^2$'s diagram drawn by Richard Guy in [18]

This diagram contains a mistake, corrected in [6], in 2002. In Guy's diagram, the game $\{1 \mid 0, *\}$ is incomparable with both $\{1 \mid *\}$ and $\{1 \mid 0\}$, but, in fact,

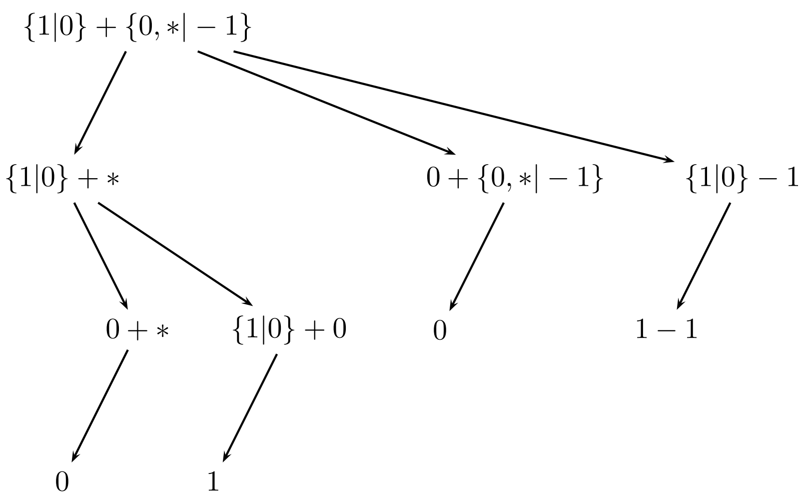
$$\{1 \mid 0, *\} < \{1 \mid *\}, \{1 \mid 0\}.$$

Consider $G = \{1 \mid *\} + \{0, * \mid -1\}$. Left, playing first, can play in the second component, moving to $\{1 \mid *\} + 0$. To this move, Right can only respond playing to $* \parallel 0$, so he loses. Right, playing first, can move either in the first component or in the second. If he moves in the first component, leaving $* + \{0, * \mid -1\}$, Left can respond moving to $* + * = 0$, and wins.

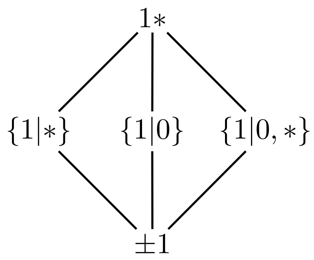
If he moves in the second component, leaving $\{1 \mid *\} - 1$, Left can respond moving to $1 - 1 = 0$, and also wins. Therefore, $G > 0$, so $\{1 \mid 0, *\} < \{1 \mid *\}$.



Consider now $H = \{1 \mid 0\} + \{0, * \mid -1\}$. Left, playing first, can play in the second component, moving to $\{1 \mid 0\} + *$. To this move, Right can respond either playing to $0 + * \parallel 0$ or to $\{1 \mid 0\} + 0 \parallel 0$. Either way, he loses. Right, playing first, can move either in the first component or in the second. If he moves in the first component, leaving $0 + \{0, * \mid -1\}$, Left can respond moving to 0, and wins. If he moves in the second component, leaving $\{1 \mid 0\} - 1$, Left can respond moving to $1 - 1 = 0$, and also wins. Therefore, $H > 0$, so $\{1 \mid 0, *\} < \{1 \mid 0\}$.



Guy's diagram led to the idea that L^2 was not distributive, since it contained



as a sublattice.

The correct Hasse diagram of $\mathcal{L}^2$ is now widely known (see Figure 3.3), and it is also not difficult to see that it is, in fact, a distributive lattice.

This arises the question of what happens in general, that is, is $\mathcal{L}^n$ a distributive lattice, for any $n \in \mathbb{N}$?

3.1.2 Games Born by Day n

The ordered structure of the games born by day n was first analyzed in [6], when it was proved that the set of games born by day n is a distributive lattice. In order to study this structure, it is necessary to define the meet and join of two games, based on the following principles:

Proposition 3.1.1 (Identity Principle)**.** *Let G be a game. If $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$, then, for every $G^L \in \mathcal{G}^L$ and $G^R \in \mathcal{G}^R$, $G^L \mathrel{\lhd\!\!\mid} G$ and $G^R \mathrel{\mid\!\!\rhd} G$.*

Proof. Suppose that, for some $G^L \in \mathcal{G}^L$, $G^L \geqslant G$, that is, Left wins playing second on $G^L - G$. But then Left has a winning move playing first on $G - G$, so $G \mathrel{\mid\!\!\rhd} G$, which is a contradiction. □

Proposition 3.1.2 (Gift Horse Principle)**.** *Let G and H be games. If $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$ and $H \mathrel{\lhd\!\!\mid} G$, then $\left\{\mathcal{G}^L, H \mid \mathcal{G}^R\right\} = G$.*

Proof. Consider the game $\left\{\mathcal{G}^L, H \mid \mathcal{G}^R\right\} - G$. The only move that doesn't have a corresponding negative is Left's move to $H - G$. Since $H \mathrel{\lhd\!\!\mid} G$, Right has a winning move in $H - G$, so $\left\{\mathcal{G}^L, H \mid \mathcal{G}^R\right\} - G = 0$. □

The Identity Principle states that, for a given form of a game G, its options cannot be better, to each player, than G itself. More, the Gift Horse Principle states that it is possible to add to a player's options any other, provided that is is not better, for that player, than G itself.

The remaining principle is the Greediness Principle, which basically states that it is always better to have more options and to have an opponent with fewer options.

Proposition 3.1.3 (Greediness Principle)**.** *Let G and H be games, $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$ and $H = \left\{H^L \mid \mathcal{H}^R\right\}$. If $\mathcal{H}^L \subseteq \mathcal{G}^L$ and $\mathcal{G}^R \subseteq \mathcal{H}^R$ then $G \geqslant H$.*

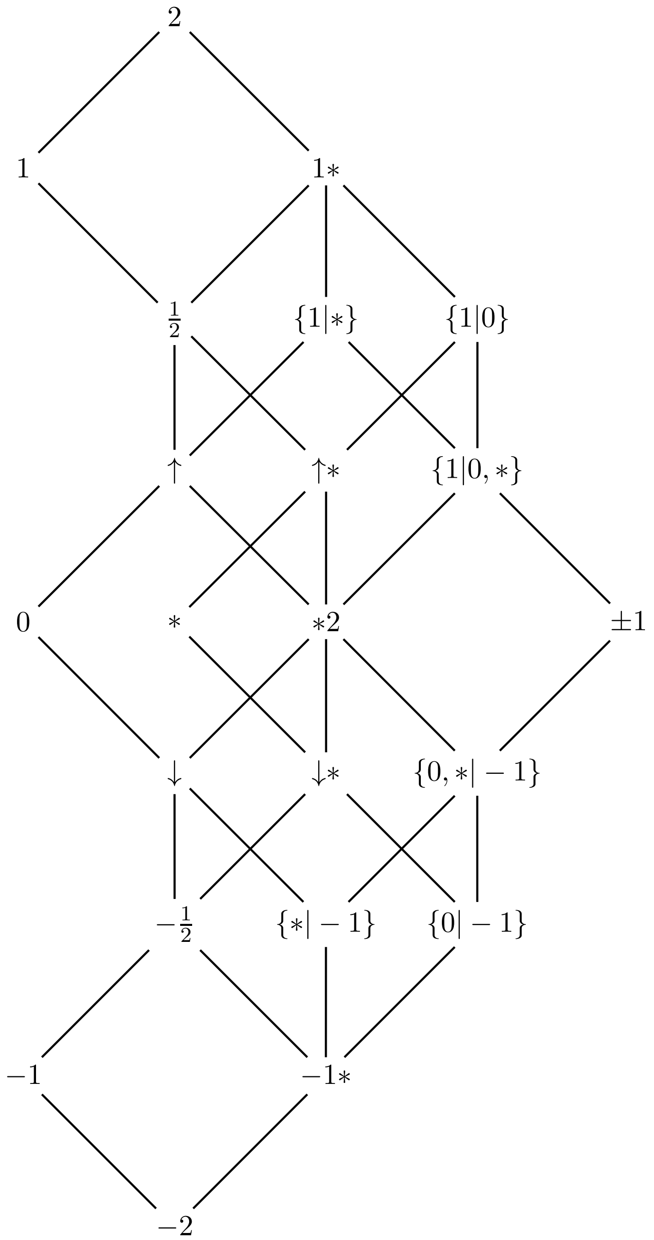


Figure 3.3: $\mathcal{L}^2$

Proof. Right, playing first on $G - H$, can not win. If he plays in the first component, leaving $G^R - H$, and since $\mathcal{G}^R \subseteq \mathcal{H}^R$, $G^R \Vdash\!\triangleright\, H$, so Left wins. If he plays in the second component, leaving $G - H^L$, and since $\mathcal{H}^L \subseteq \mathcal{G}^L$, $G \Vdash\!\triangleright\, H^L$, Left wins. $\square$

If, instead of canonical forms, complete forms (defined below) are considered, these three principles provide a direct way of comparing games. Of course that, sometimes, it can not be easy to obtain the complete form of a game, but, if it is available, it solves completely the problem of comparing games.

Definition 3.1.4. Let G be a game. For $n \in \mathbb{N}$, G is in its day n *complete form* if $G = \{\lfloor G \rfloor \mid \lceil G \rceil\}$, where $\lfloor G \rfloor = \{H \in \mathcal{L}^{n-1} : H \lhd\!\!| \ G\}$ is the floor and $\lceil G \rceil = \{H \in \mathcal{L}^{n-1} : H \ |\!\!\rhd G\}$ is the ceiling of G.

Observe that this form of G is well defined, due to Propositions 3.1.1 and 3.1.2.

Example 3.1.5. Figure 3.4 shows day 2 complete form of every element of $\mathcal{L}^2$.

Calistrate, Paulus and Wolfe, in [6], successfully defined the meet and join of two games. Their idea was to define these operations using complete forms, and unions and intersections of sets (in Figure 3.4, it is possible to observe the relations between the complete forms of any two elements and their meet and join). To carry out their idea, the following result it is also necessary.

Proposition 3.1.6. *Let G and H be games. If $H = \left\{\mathcal{H}^L \mid \mathcal{H}^R\right\} \geqslant G$ then, for every $H^R \in \mathcal{H}^R$, $H^R \ |\!\!\rhd G$.*

Proof. If a H^R exists such that $H^R \leqslant G$, then $H^R - G \leqslant 0$, that is, Right wins moving second on $H^R - G$. But then, Right can win moving first on $H - G$, that is, $H \lhd\!\!| \ G$. ☐

Theorem 3.1.7 (Calistrate, Paulus, Wolfe, [6])**.** *The set of games born on day n, $\mathcal{L}^n$, is a lattice.*

Proof. Let $G, H \in \mathcal{L}^n$ and define

$$\begin{aligned} G \vee H &= \{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\}, \\ G \wedge H &= \{\lfloor G \rfloor \cap \lfloor H \rfloor \mid \lceil G \rceil \cup \lceil H \rceil\}. \end{aligned}$$

To prove that $G \vee H$ is, in fact, the join of G and H, it is necessary to see that:

- $G \vee H \geqslant G, H$; and
- for every $X \in \mathcal{L}^n$, if $X \geqslant G, H$, then $X \geqslant G \vee H$.

Since $\lfloor G \rfloor, \lfloor H \rfloor \subseteq \lfloor G \rfloor \cup \lfloor H \rfloor$ and $\lceil G \rceil \cap \lceil H \rceil \subseteq \lceil G \rceil, \lceil H \rceil$, using the Greediness Principle (Proposition 3.1.3), it is immediate that $G \vee H \geqslant G, H$.

Suppose now that $X \in \mathcal{L}^n$ is such that $X \geqslant G, H$, and consider the game $X - G \vee H$. Right, playing first, can either move to some $X^R - G \vee H$ or

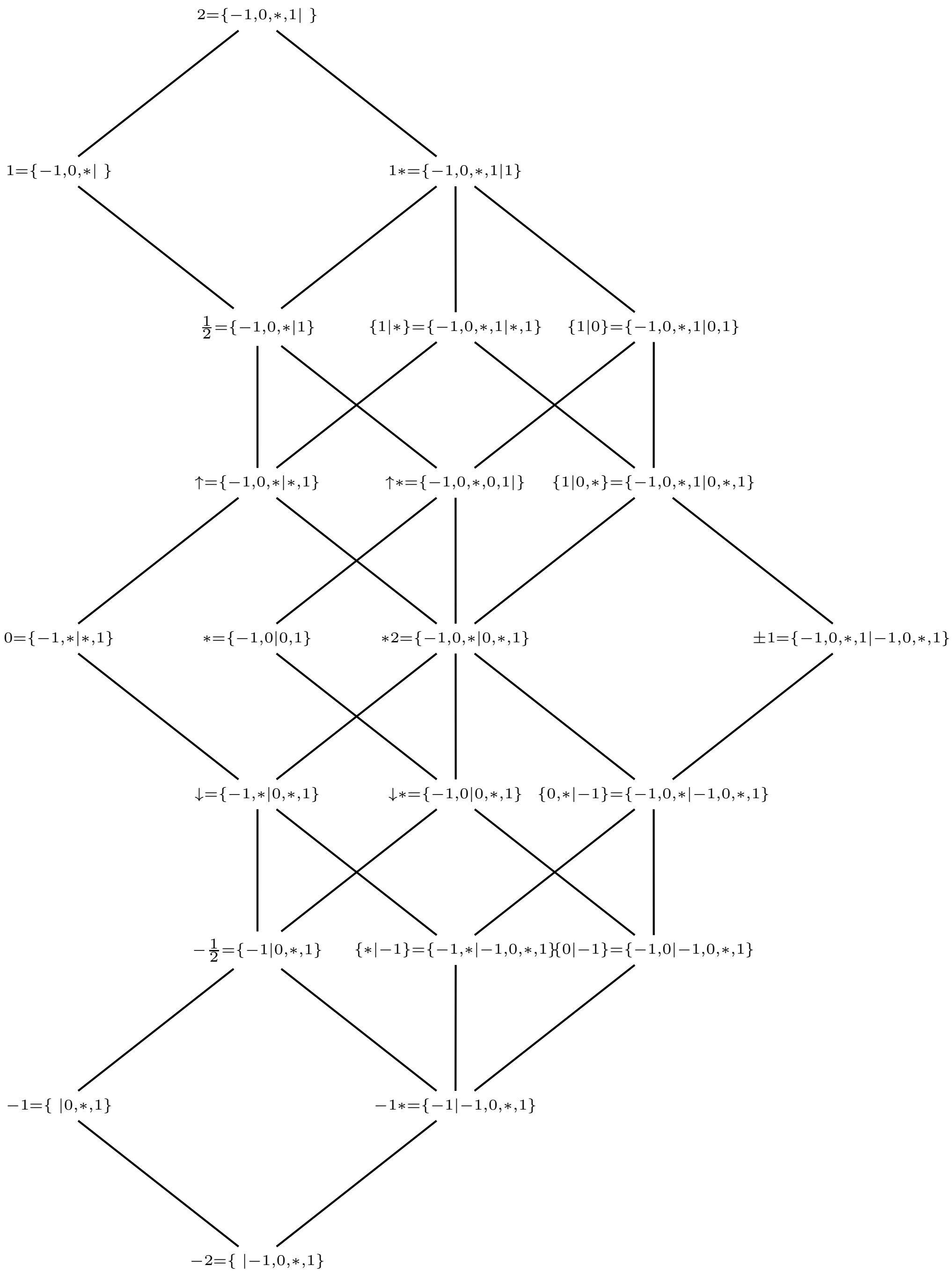


Figure 3.4: $\mathcal{L}^2$'s elements complete forms

to some $X-(G\vee H)^R$. In the first case, by Proposition 3.1.6, $X^R \mathrel{\Vert\!\!\rhd} G, H$, so, $\mathcal{X}^R \subseteq \lceil G\rceil \cap \lceil H\rceil$, and Left can win playing first on $X^R - G \vee H$. In the second case, since $(G \vee H)^R \in \lfloor G \rfloor \cup \lfloor H \rfloor$, $X - (G \vee H)^R$ is either an option of $X - G$ or of $X - H$. In either case, Left can win moving second, so Right loses.

Analogously, $G \wedge H$ is the meet of G and H, so $\mathcal{L}^n$ is a lattice. $\square$

To prove that $\mathcal{L}^n$ is distributive, two lemmas will be used.

Lemma 3.1.8. *Let G and H be games, $G, H \in \mathcal{L}^n$. Then*

$$\begin{aligned} \lfloor G \vee H \rfloor &= \lfloor G \rfloor \cup \lfloor H \rfloor \text{ and} \\ \lceil G \wedge H \rceil &= \lceil G \rceil \cup \lceil H \rceil. \end{aligned}$$

Proof. If $X \in \lfloor G \vee H \rfloor$, then $X \in \mathcal{L}^{n-1}$ and $X \lhd\!\!| \; G \vee H$. Since $X \in \mathcal{L}^n$ (because, in Conway's construction, $k < n$ implies that $\mathcal{L}^k \subseteq \mathcal{L}^n$), if $X \notin \lfloor G \rfloor \cup \lfloor H \rfloor$, then $X \geqslant G$ and $X \geqslant H$. But then $X \geqslant G \vee H$, which is a contradiction.

Reciprocally, if $X \in \lfloor G \rfloor \cup \lfloor H \rfloor$, then $X \in \mathcal{L}^{n-1}$, and $X \lhd\!\!| \; G$ or $X \lhd\!\!| \; H$. If $X \notin \lfloor G \vee H \rfloor$, then $X \geqslant G \vee H \geqslant G, H$, which is a contradiction.

The other equality is similar. $\square$

Lemma 3.1.9. *Let G and H be games, and let $\{\mathcal{G}^L \mid \mathcal{G}^R\}$ and $\{\mathcal{H}^L \mid \mathcal{H}^R\}$ be any form of G and H. Then*

$$\begin{aligned} G \vee H &= \left\{\mathcal{G}^L \cup \mathcal{H}^L \mid \lceil G \rceil \cap \lceil H \rceil\right\} \text{ and} \\ G \wedge H &= \left\{\lfloor G \rfloor \cap \lfloor H \rfloor \mid \mathcal{G}^R \cup \mathcal{H}^R\right\}. \end{aligned}$$

Proof. To prove that $G \vee H = \left\{\mathcal{G}^L \cup \mathcal{H}^L \mid \lceil G \rceil \cap \lceil H \rceil\right\}$, consider the game $G \vee H - \left\{\mathcal{G}^L \cup \mathcal{H}^L \mid \lceil G \rceil \cap \lceil H \rceil\right\}$, that is,

$$\{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\} + \left\{-(\lceil G \rceil \cap \lceil H \rceil) \mid -(\mathcal{G}^L \cup \mathcal{H}^L)\right\}.$$

Right, playing first, can move in either component. If he moves in the first component, Left has a correspondent move available in the second component, so he loses. Suppose, without loss of generality, that he moves in the second component, to some

$$\{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\} - G^L.$$

By the Identity Principle, 3.1.1, $G - G^L \;|\!\!\rhd\; 0$ and, so, Left has a winning move in

$$\{\lfloor G \rfloor \mid \lceil G \rceil\} - G^L.$$

If this move is in $\{\lfloor G \rfloor \mid \lceil G \rceil\}$, it is also available for Left in

$$\{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\} - G^L.$$

If it is in G^L, then $\{\lfloor G \rfloor \mid \lceil G \rceil\} - G^{LR} \geqslant 0$ for some G^{LR}. By the Greediness Principle, 3.1.3,

$$\{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\} \geqslant \{\lfloor G \rfloor \mid \lceil G \rceil\},$$

so Left's move to $\{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\} - G^{LR}$ is a winning move. Either way, Right loses.

Left, playing first, can move in either component. If he moves in the second component, Right has a correspondent move available in the first component, so he loses. Suppose, without loss of generality, that he moves in the first component, to some

$$X + \big\{-(\lceil G \rceil \cap \lceil H \rceil) \bigm| -(\mathcal{G}^L \cup \mathcal{H}^L)\big\},$$

with $X \in \lfloor G \rfloor$. Since $G = \big\{\mathcal{G}^L \bigm| \lceil G \rceil\big\}$ and $X \lhd\!\!\mid G$, Right has a winning move in

$$X + \big\{-\lceil G \rceil \bigm| -\mathcal{G}^L\big\}.$$

If this move is in $\big\{-\lceil G \rceil \bigm| -\mathcal{G}^L\big\}$, it is also available for Right in

$$X + \big\{-(\lceil G \rceil \cap \lceil H \rceil) \bigm| -(\mathcal{G}^L \cup \mathcal{H}^L)\big\}.$$

If it is in X, then $X^R + \big\{-\lceil G \rceil \bigm| -\mathcal{G}^L\big\} \leqslant 0$, for some X^R. By the Greediness Principle, 3.1.3,

$$\big\{\mathcal{G}^L \bigm| \lceil G \rceil\big\} \leqslant \big\{\mathcal{G}^L \cup \mathcal{H}^L \bigm| \lceil G \rceil \cap \lceil H \rceil\big\},$$

so Right's move to $X^R + \big\{-(\lceil G \rceil \cap \lceil H \rceil) \bigm| -(\mathcal{G}^L \cup \mathcal{H}^L)\big\}$ is a winning move. Either way, Left loses, and therefore $G \vee H = \big\{\mathcal{G}^L \cup \mathcal{H}^L \bigm| \lceil G \rceil \cap \lceil H \rceil\big\}$.

A similar argument yields the other equality. $\square$

Theorem 3.1.10 (Calistrate, Paulus, Wolfe, [6])**.** *The lattice of games born on day n, $\mathcal{L}^n$, is distributive.*

Proof. Let $G, H, J \in \mathcal{L}^n$. Then,

$$\begin{aligned}
&G \wedge (H \vee J) = \\
&= \{\lfloor G \rfloor \cap \lfloor H \vee J \rfloor \mid \lceil G \rceil \cup \lceil H \vee J \rceil\} \text{ (by definition)} \\
&= \{\lfloor G \rfloor \cap \lfloor H \vee J \rfloor \mid \lceil G \rceil \cup (\lceil H \rceil \cap \lceil J \rceil)\} \text{ (Lemma 3.1.9)} \\
&= \{\lfloor G \rfloor \cap (\lfloor H \rfloor \cup \lfloor J \rfloor) \mid \lceil G \rceil \cup (\lceil H \rceil \cap \lceil J \rceil)\} \text{ (Lemma 3.1.8)} \\
&= \{(\lfloor G \rfloor \cap \lfloor H \rfloor) \cup (\lfloor G \rfloor \cap \lfloor J \rfloor) \mid (\lceil G \rceil \cup \lceil H \rceil) \cap (\lceil G \rceil \cup \lceil J \rceil)\} \text{ (de Morgan)} \\
&= \big\{(G \wedge H)^L \cup (G \wedge J)^L \bigm| \lceil G \wedge H \rceil \cap (\lceil G \wedge J \rceil\big\} \text{ (Lemma 3.1.9)} \\
&= (G \wedge H) \vee (G \wedge J) \text{ (by definition).}
\end{aligned}$$

$\square$

Observation 3.1.11. Observe that, if long games are considered, all proofs are analogous. Therefore, for Conway's construction, $\mathcal{L}^{\omega}(\emptyset)$ is a distributive lattice.

Another interesting result from the same paper is the following:

Theorem 3.1.12 (Calistrate, Paulus, Wolfe, [6])**.** *The collection of short games is not a lattice.*

Proof. Let

$$\mathcal{L} = \bigcup_{n \geqslant 0} \mathcal{L}^n$$

be the collection of all short games.

Let $G, H \in \mathcal{L}$, $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$, $H = \left\{\mathcal{H}^L \mid \mathcal{H}^R\right\}$, and $G \parallel H$. Suppose that the join of G and H in $\mathcal{L}$ exists, say, $S = \left\{\mathcal{S}^L \mid \mathcal{S}^R\right\}$, where $\mathcal{S}^L$ and $\mathcal{S}^R$ are sets of short games. Define

$$J = \left\{G, H, \mathcal{G}^L \cup \mathcal{H}^L \mid S, \mathcal{S}^R\right\}.$$

Since $G, H, S \in \mathcal{L}$, then $J \in \mathcal{L}$. More, $J > G$: consider the game

$$J - G = \left\{G, H, \mathcal{G}^L \cup \mathcal{H}^L \mid S, \mathcal{S}^R\right\} + \left\{-\mathcal{G}^R \mid -\mathcal{G}^L\right\}.$$

Left, playing first, can move in the first component to $G - G = 0$ and wins.

Right, playing first, can move in either component. If he moves in the second component, Left has a corresponding one in the first component, so he loses. If he moves in the first component to $S - G$, since $S > G$, he loses. If he moves in the first component to some $S^R - G$, since, by the Identity Principle, 3.1.1, $S^R \mathrel{|\!\rhd} S$, and $S > G$, then $S^R \mathrel{|\!\rhd} G$, and he loses.

Analogously, $J > H$.

Consider now the game

$$S - J = \left\{\mathcal{S}^L \mid \mathcal{S}^R\right\} + \left\{-S, -\mathcal{S}^R \mid -G, -H, -(\mathcal{G}^L \cup \mathcal{H}^L)\right\}.$$

Left, playing first, can move in the second component to $S - S = 0$ and wins.

Right, playing first, can move in either component. If he moves in the first component, Left has a corresponding one in the second component, so he loses. If he moves in the second component to $S - G$ or to $S - H$, since $S > G, H$, he loses. If he moves in the second component to $S - X$ with $X \in \mathcal{G}^L \cup \mathcal{H}^L$, then, since $X \mathrel{\lhd\!|} G < S$ or $X \mathrel{\lhd\!|} H < S$, $X \mathrel{\lhd\!|} S$, and he loses.

Therefore, $S - J > 0$, which contradicts the definition of join. $\square$

3.2 Case S $\neq \emptyset$

In the previous section, the lattices $\mathcal{L}^n$ of games born by day n were studied. In [1], 2012, Albert and Nowakowski study the ordered structure of the games of $\mathcal{L}(S)$ for an arbitrary set of games S.

Definition 3.2.1. Let S be a set of games. For $G \in \mathcal{L}(S)$, $\lfloor G \rfloor = \{H \in S : H \mathrel{\triangleleft\!\!|} G\}$ is the *floor* of G relative to S and $\lceil G \rceil = \{H \in S : H \mathrel{|\!\!\triangleright} G\}$ is the *ceiling* of G relative to G.

Using the Identity, Gift Horse, and Greediness principles, it is possible to define complete form.

Definition 3.2.2. Let S be a set of games. For $G \in \mathcal{L}(S)$, G is in *complete form* if $G = \{\lfloor G \rfloor \mid \lceil G \rceil\}$.

Example 3.2.3. Let $S = \{0, \{1 \mid *\}, 1*\}$. Then $1 = \{0 \mid 1*\} \in \mathcal{L}(S)$, and its complete form is $\{0, \{1 \mid *\}, 1* \mid \{1 \mid *\}, 1*\}$.

The idea of defining the meet and join of two games using complete forms, and unions and intersections of sets is still feasible.

Theorem 3.2.4 (Albert, Nowakowski, [1])**.** *Let S be a set of games. Then $\mathcal{L}(S)$ is a complete lattice.*

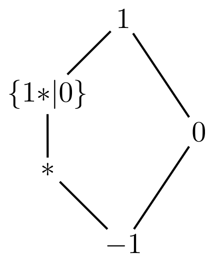
Proof. To prove that $\mathcal{L}(S)$ is complete, define

$$\bigvee G_i = \left\{ \bigcup \lfloor G_i \rfloor \,\middle|\, \bigcap \lceil G_i \rceil \right\} \text{ and } \bigwedge G_i = \left\{ \bigcap \lfloor G_i \rfloor \,\middle|\, \bigcup \lceil G_i \rceil \right\}$$

and the proof is analogous to the proof of Theorem 3.1.7. $\square$

Unlike the case $S = \emptyset$, for an arbitrary set of games S, $L(S)$ may not be distributive.

Example 3.2.5. Let $S = \{0, \{1 \mid *\}, 1*\}$. Then $\mathcal{L}(S) = \{-1, *, 0, \{1* \mid 0\}, 1\}$ is a N_5.



In the same paper [1], Albert and Nowakowski study conditions that ensure the distributivity of $\mathcal{L}(S)$. Observing the proof of Lemma 3.1.8, a fundamental fact was that $\mathcal{L}^{n-1} \subseteq \mathcal{L}^n$. In the previous example, this property isn't observed, because $\{1 \mid *\}$ and $1*$ are elements of S but are not elements of $\mathcal{L}(S)$. This motivates the following definition:

Definition 3.2.6. Let S be a set of games. S is *self-generated* if $S \subseteq \mathcal{L}(S)$.

Observation 3.2.7. Observe that self-generation is a property of a set of games in general (short and long).

Next, it will be proved that self-generation implies the distributivity of $\mathcal{L}(S)$. However, in order to check if a set of games S is self-generated, the computation of $\mathcal{L}(S)$ is needed. Because of this, Albert and Nowakowski defined the concept of hereditarily closure for short games.

Definition 3.2.8. Let S be a set of short games. S is *hereditarily closed* if for each $G \in S$, all the options in the canonical form of G belong to S.

Observation 3.2.9. Observe that hereditarily closure is a property of a set of short games because it needs the concept of canonical form. Also, it is a thinner condition that implies self-generation but, knowning the canonical forms of the games of S, avoids the computation of $\mathcal{L}(S)$.

Proposition 3.2.10. *Let S be a set of short games. If S is hereditarily closed then S is self-generated.*

Proof. Let $G \in S$, and suppose that $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$ is in canonical form. By Definition 3.2.8, $\mathcal{G}^L \subseteq S$ and $\mathcal{G}^R \subseteq S$. So, $G \in \mathcal{L}(S)$. □

It is now possible to obtain a lemma, analogous to Lemma 3.1.8:

Lemma 3.2.11. *Let S be a set of games and $G, H \in \mathcal{L}(S)$. If S is self-generated, then*

$$\begin{aligned} \lfloor G \vee H \rfloor &= \lfloor G \rfloor \cup \lfloor H \rfloor \text{ and} \\ \lceil G \wedge H \rceil &= \lceil G \rceil \cup \lceil H \rceil. \end{aligned}$$

Proof. Since S is self-generated, the proof is analogous to the proof of Lemma 3.1.8. □

The result correspondent to Lemma 3.1.9 is valid for any set of games.

Lemma 3.2.12. *Let S be a set of games, $G, H \in \mathcal{L}(S)$, and let $\left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$ and $\left\{\mathcal{H}^L \mid \mathcal{H}^R\right\}$ be any form of G and H. Then*

$$\begin{aligned} G \vee H &= \left\{\mathcal{G}^L \cup \mathcal{H}^L \mid \lceil G \rceil \cap \lceil H \rceil\right\} \text{ and} \\ G \wedge H &= \left\{\lfloor G \rfloor \cap \lfloor H \rfloor \mid \mathcal{G}^R \cup \mathcal{H}^R\right\}. \end{aligned}$$

Proof. Analogous to the proof of 3.1.9. □

Theorem 3.2.13. *Let S be a set of games. If S is self-generated, then $\mathcal{L}(S)$ is a distributive lattice.*

Proof. Analogous to the proof of Theorem 3.1.10 □

Although Corollary 3.2.14 appears in [1] as the main result, the authors refer that the key condition to its proof is the self-generation of S.

Corollary 3.2.14 (Albert, Nowakowski, [1])**.** *Let S be a set of short games. If S is hereditarily closed, then $\mathcal{L}(S)$ is a distributive lattice.*

In the remaining chapters, which constitute the main contribution of this work, a more detailed analysis of these results can be found. Also, the modularity of $\mathcal{L}(S)$ is studied, and the possibility of constructing all kind of lattices, finite and infinite, is discussed. Conway's transfinite construction is studied, and a condition guaranteing that the set of games born in days before a certain ordinal is is a lattice is obtained.

Chapter 4

Modularity of Lattices of Games

In face of the previous chapter, the following questions are natural:

Question 1: Characterize the sets S such that $\mathcal{L}(S)$ is a distributive lattice.

Question 2: What lattices are generated?

For example, Nowakowski conjectured that if $\mathcal{L}(S)$ is a modular lattice then it is also a distributive lattice. (More specifically, if the lattice M_3 is generated then so is N_5.)

Although Question 1 is not answered, in this chapter the condition presented in Theorem 3.2.13 is deepened, and non-distributive, modular lattices are generated. The method used to generate the "first" known example of a non-distributive, modular $\mathcal{L}(S)$, introduced in [7], uses the same techniques that are used in Section 4.2.

4.1 Strongly Self-Generation $\Leftrightarrow$ Hereditarily Closure

Theorem 3.2.13 states that the self-generation of S implies the distributivity of $\mathcal{L}(S)$. The question arises of if self-generation implies hereditarily closure in the universe of short games. The answer is negative as is shown in the next two examples.

Example 4.1.1. The set $S = \{1, 1*\}$ is self-generated: $1 = \{1* \mid 1*\}$ and $1* = \{1 \mid 1\}$. However, S is not hereditarily closed because, in canonical form, $1 = \{0 \mid \}$ and $0 \notin S$.

Considering infinite sets, it is possible to construct an example with only numbers.

Example 4.1.2. Consider the set

$$S = \{\text{all dyadic rationals } G \text{ such that } 0 < G < 1\}.$$

S is self-generated because $\frac{m}{2^j} = \left\{\frac{2m-1}{2^{j+1}} \,\middle|\, \frac{2m+1}{2^{j+1}}\right\}$ (Definition 2.4.2), but is not hereditarily closed because, in canonical form, $\frac{1}{2} = \{0\,|\,1\}$ and $0 \notin S$ (also, $1 \notin S$).

Next, hereditarily closure is characterized.

Definition 4.1.3. Let S be a set of games. S is *strongly self-generated* if, for every $G \in S$, there are games $A, B, \ldots, A', B', \ldots \in S$ with

$$\mathrm{b}(A), \mathrm{b}(B), \ldots, \mathrm{b}(A'), \mathrm{b}(B'), \ldots < \mathrm{b}(G)$$

and $G = \{A, B, \ldots \mid A', B' \ldots\}$.

Observation 4.1.4. Observe that strongly self-generation is a property of a set of games in general (short and long).

Theorem 4.1.5. *Let S be a set of short games. Then S is hereditarily closed if and only if S is strongly self-generated.*

Proof.

($\Rightarrow$) Exactly as in the proof of Proposition 3.2.10. (Note that, when a game is in canonical form, the options have smaller birthdays than the game itself.)

($\Leftarrow$) Because S is strongly self-generated, for every $G \in S$ a form

$$G = \{A_1, A_2, \ldots \mid B_1, B_2 \ldots\}$$

exists with $A_i, B_j \in S$ and $\mathrm{b}(A_i), \mathrm{b}(B_j) < \mathrm{b}(G)$ for all i and j.

Given a game $G \in S$ and some form of G, define a (infinite) birthday sequence $\langle c_i \rangle$, where c_i is the number of options in the form of G with birthday i. Order these sequences by $\langle c_i \rangle > \langle d_i \rangle$ if there is some k so that $c_i = d_i$ for $i > k$ and $c_k > d_k$. (A game may have more than one associated vector if it has more than one form.)

Suppose that S is not hereditarily closed. Then there is some game in S that has an option of the canonical form that is not in S. Choose

$$G = \{A_1, A_2, \ldots \mid B_1, B_2 \ldots\} \in S$$

to be such a game that also has the least associated birthday sequence.

Since dominated options can be eliminated, giving a smaller birthday sequence, then this form of G has no dominated options. Since this form cannot be the canonical form of G, it must have a reversible option, say the right option

$$B_1 = \{D_1, D_2, \ldots \mid E_1, E_2, \ldots\}$$

where $D_1, D_2, \ldots, E_1, E_2 \ldots \in S$, and with, say, $D_1 \geqslant G$ (necessarily, $\mathrm{b}(D_1) < \mathrm{b}(G)$).

Again, since S is strongly self-generated then there is a form

$$D_1 = \{F_1, F_2, \ldots \mid H_1, H_2, \ldots\},$$

$F_1, F_2, \ldots, H_1, H_2, \ldots \in S$ and $\mathrm{b}(F_i), \mathrm{b}(H_j) < \mathrm{b}(D_1)$.

Now, in the original $G = \{A_1, A_2, \ldots \mid B_1, B_2 \ldots\}$, it is possible to replace B_1 by $H_1, H_2, \ldots$, that is,

$$G = \{A_1, A_2, \ldots \mid H_1, H_2, \ldots, B_2, B_3, \ldots\}.$$

This is a contradiction since this form of G has a smaller birthday sequence than that chosen as the smallest (the coefficient of c_i where $\mathrm{b}(B_1) = i$ has been reduced by 1).

Thus S is hereditarily closed. □

Example 4.1.6.

1. Let

   $$S = \{-1, 0, 1, *, *2, *4, *5, *6, \{0, *, *3 \mid -1\}, \{1 \mid 0, *, *3\}, A, -A\},$$

   where $A = \{1 \mid \{0, *, *3 \mid -1\}\}$.

   S is not hereditarily closed, but it is self-generated.

   In this example, the canonical forms of 1, -1, 0, $*$, $*2$, A and $-A$ can be obtained with games in S.

   The game $*6 = \{A, *2, *4, *5 \mid -A, *2, *4, *5\}$ can be obtained with options with smaller birthdays, but it can not be obtained in the canonical form.

   The games $\{0, *, *3 \mid -1\} = \{A \mid -1\}$, $\{1 \mid 0, *, *3\} = \{1 \mid -A\}$, $*4 = \{A, *2 \mid -A, *2\}$ and $*5 = \{A, *2, *4 \mid -A, *2, *4\}$ can be obtained, but not with options with smaller birthdays.

2. Let $S = \{0, 1, \ldots \text{ (all nonnegative integers)}, \omega\}$.

   This set is strongly self-generated, but it cannot be said to be hereditarily closed because ω is not a short game (it doesn't have a canonical form).

Observation 4.1.7. Essentially, what the previous results established was that if S is strongly-self generated, all the options of canonical forms of the short games of S are in S and, as a natural corollary, in the universe of short games, strongly-self generation and hereditarily closure are equivalent.

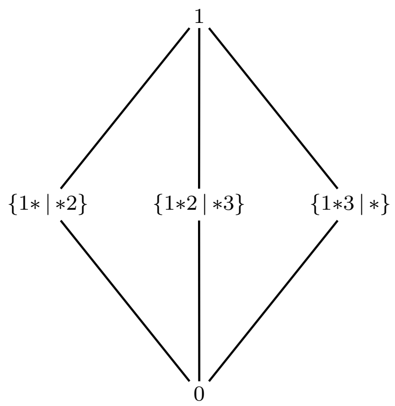
4.2 Modularity of Lattices of Games

In this section, the problem of determining when $\mathcal{L}(S)$ is modular is studied.

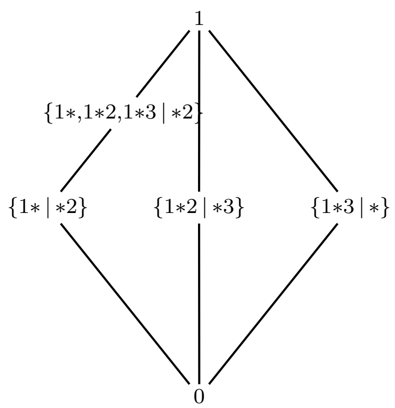
The next example illustrates some of the issues when trying to determine whether $\mathcal{L}(S)$ is modular.

Example 4.2.1. Consider the set of games $S = \{*, *2, *3, 1*, 1*2, 1*3\}$ and the associated $\mathcal{L}(S)$ (see Figure 4.1).

The following sublattice is a diamond (M_3).



However, there are elements in $\mathcal{L}(S)$ strictly between an atom and 1 (or 0) of the M_3, which construct a N_5 as well. The original conjecture was based on such evidence.



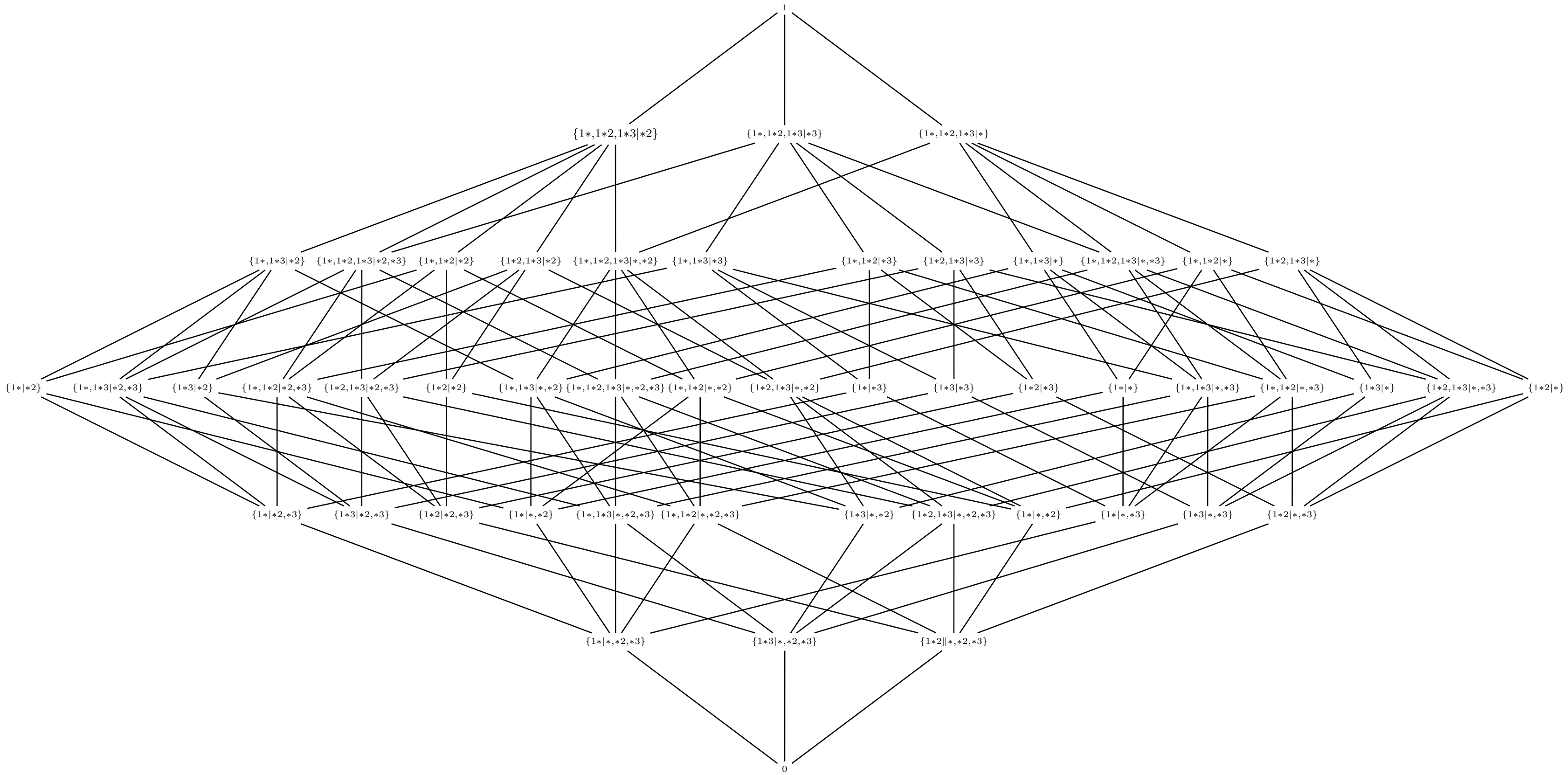
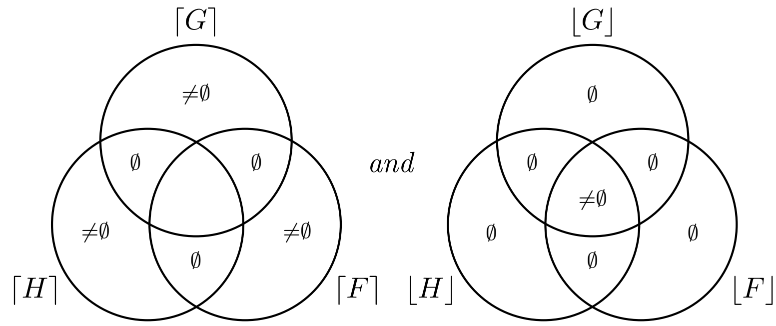


Figure 4.1: $\mathcal{L}(S)$

From 1.2.14, it is known that if a lattice $\mathcal{L}$ is finite, modular, and non-distributive, then it contains a cover-preserving diamond. So, to understand when $\mathcal{L}(S)$ is a non-distributive modular lattice, it is important to understand when cover-preserving diamonds exist.

The next result states a necessary condition for diamonds to exist.

Theorem 4.2.2. *Let S be a set of games where $\mathcal{L}(S)$ is a finite non-distributive modular lattice. Let $\{G \wedge H \wedge F, G, H, F, G \vee H \vee F\}$ be a cover preserving diamond in $\mathcal{L}(S)$. One of the following situations must happen:*

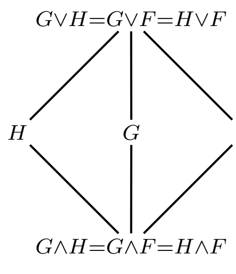


or the symmetric picture changing $\lfloor\,\rfloor$ with $\lceil\,\rceil$.

Proof. Consider the cover-preserving diamond

$$\mathcal{D} = \{G \wedge H \wedge F, G, H, F, G \vee H \vee F\}$$

of $\mathcal{L}(S)$:



Claim 1: $(\lceil G\rceil \cap \lceil H\rceil) \setminus \lceil F\rceil = \emptyset$,
$(\lceil G\rceil \cap \lceil F\rceil) \setminus \lceil H\rceil = \emptyset$,
$(\lceil H\rceil \cap \lceil F\rceil) \setminus \lceil G\rceil = \emptyset$ and
$(\lfloor G\rfloor \cap \lfloor H\rfloor) \setminus \lfloor F\rfloor = \emptyset$,
$(\lfloor G\rfloor \cap \lfloor F\rfloor) \setminus \lfloor H\rfloor = \emptyset$,
$(\lfloor H\rfloor \cap \lfloor F\rfloor) \setminus \lfloor G\rfloor = \emptyset$.

Proof. Suppose $X \in (\lceil G\rceil \cap \lceil H\rceil) \setminus \lceil F\rceil$. Then, $X \leqslant F$, $X \,|\!\!\!\rhd\, G$, and $X \,|\!\!\!\rhd\, H$. Consider

$$K = \{\lfloor G\rfloor \cup \lfloor H\rfloor \mid X\}.$$

Right, playing first, loses in the game

$$K - G = \{\lfloor G\rfloor \cup \lfloor H\rfloor \mid X\} + \{-\lceil G\rceil \mid -\lfloor G\rfloor\},$$

so $K \geqslant G$. Similarly $K \geqslant H$, so $K \geqslant G \vee H = G \vee F \geqslant F$. However, $K - F \lhd\!\!| \ 0$ because Right, playing first, wins moving to $X - F \leqslant 0$, a contradiction.

The other equalities are proved using a similar argument. $\square$

Claim 2: One of the sets $\lceil G\rceil \setminus (\lceil H\rceil \cup \lceil F\rceil)$,
$\lceil H\rceil \setminus (\lceil G\rceil \cup \lceil F\rceil)$,
$\lceil F\rceil \setminus (\lceil G\rceil \cup \lceil H\rceil)$,
$\lfloor G\rfloor \setminus (\lfloor H\rfloor \cup \lfloor F\rfloor)$,
$\lfloor H\rfloor \setminus (\lfloor G\rfloor \cup \lfloor F\rfloor)$,
$\lfloor F\rfloor \setminus (\lfloor G\rfloor \cup \lfloor H\rfloor)$ must be nonempty.

Proof. If these 6 sets are all empty then

$$\lceil G\rceil = \lceil H\rceil = \lceil F\rceil \text{ and } \lfloor G\rfloor = \lfloor H\rfloor = \lfloor F\rfloor.$$

So, $G = H = F$ which is a contradiction. $\square$

Claim 3: Suppose that, without loss of generality, $\lceil G\rceil \setminus (\lceil H\rceil \cup \lceil F\rceil) \neq \emptyset$. Then $\lfloor H\rfloor = \lfloor F\rfloor$.

Proof. Take $A \in \lceil G\rceil \setminus (\lceil H\rceil \cup \lceil F\rceil)$. Thus $A \leqslant H$, $A \leqslant F$ and $A \,|\!\!\!\rhd\, G$, and suppose that there is an

$$X \in \lfloor F\rfloor \setminus \lfloor H\rfloor,$$

that is, $X \lhd\!\!| \ F$ and $X \geqslant H$. Since $(\lfloor G\rfloor \cap \lfloor F\rfloor) \setminus \lfloor H\rfloor = \emptyset$, then $X \geqslant G$. Now, consider the game

$$K = \{\lfloor G\rfloor \cup \lfloor F\rfloor \mid \lceil G\rceil\}.$$

Then, $G \vee F > K$ because Right loses playing first in

$$\{\lfloor G\rfloor \cup \lfloor F\rfloor \mid \lceil G\rceil \cap \lceil F\rceil\} + \{-\lceil G\rceil \mid -(\lfloor G\rfloor \cup \lfloor F\rfloor)\}$$

and, if Left plays first, $G \vee F - A$ is a winning move ($A \leqslant F < G \vee F$). Also, $K > G$ because Right loses playing first in

$$\{\lfloor G\rfloor \cup \lfloor F\rfloor \mid \lceil G\rceil\} + \{-\lceil G\rceil \mid -\lfloor G\rfloor\}$$

since the left option $X - G$ is a winning move in the game $K - G$. But then $\mathcal{D}$ wouldn't be a cover-preserving diamond. $\square$

Given that $\lceil G \rceil \setminus (\lceil H \rceil \cup \lceil F \rceil) \neq \emptyset$, then either $\lceil H \rceil \setminus (\lceil G \rceil \cup \lceil F \rceil) \neq \emptyset$ or $\lceil F \rceil \setminus (\lceil H \rceil \cup \lceil G \rceil) \neq \emptyset$. If these two sets are both empty then $\lfloor H \rfloor = \lfloor F \rfloor$ and $\lceil H \rceil = \lceil F \rceil$ and so $H = F$, which is a contradiction. Therefore, say

$$\lceil H \rceil \setminus (\lceil G \rceil \cup \lceil F \rceil) \neq \emptyset$$

and it follows, as in Claim 3, that $\lfloor G \rfloor = \lfloor F \rfloor$.

Finally, it is also the case that

$$\lceil F \rceil \setminus (\lceil G \rceil \cup \lceil H \rceil) \neq \emptyset.$$

If not, $\lceil F \rceil \subseteq \lceil H \rceil$ and, by Lemma 3.1.3,

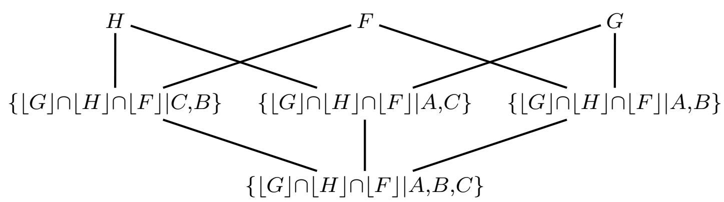
$$H = \{\lfloor H \rfloor \mid \lceil H \rceil\} = \{\lfloor F \rfloor \mid \lceil H \rceil\} \leqslant \{\lfloor F \rfloor \mid \lceil F \rceil\} = F,$$

which is a contradiction.

The assumption, in Claim 3, that $\lceil G \rceil \setminus (\lceil H \rceil \cup \lceil F \rceil) \neq \emptyset$, results in the first case presented in the theorem. Assuming $\lfloor G \rfloor \setminus (\lfloor H \rfloor \cup \lfloor F \rfloor)$ in Claim 3 would give the second case. $\square$

Even with this necessary condition, it is still not easy to get an example of a finite non-distributive modular $\mathcal{L}(S)$. To do so, start with the following scheme.

In many examples, the following "bad" (that is, almost distributive) picture occurs:



It is necessary to manipulate the sets so as to create the situation

$$\begin{aligned}
\{\lfloor G \rfloor \cap \lfloor H \rfloor \cap \lfloor F \rfloor \mid A, B, C\} &= \{\lfloor G \rfloor \cap \lfloor H \rfloor \cap \lfloor F \rfloor \mid C, B\} = \\
&= \{\lfloor G \rfloor \cap \lfloor H \rfloor \cap \lfloor F \rfloor \mid A, C\} = \\
&= \{\lfloor G \rfloor \cap \lfloor H \rfloor \cap \lfloor F \rfloor \mid A, B\}
\end{aligned}$$

so, in the game $\{\lfloor G \rfloor \cap \lfloor H \rfloor \cap \lfloor F \rfloor \mid A, B, C\}$, all the right options can be eliminated but, and very importantly, no more than one option at a time. This can be achieved by using reversible options. For instance, consider the game

$$\{1 \mid \{\{3 \mid *, *2\} \mid -3\}, \{\{3 \mid *, *3\} \mid -3\}, \{\{3 \mid *2, *3\} \mid -3\}\}.$$

All its right options can be eliminated, but no more than one at the same time. The game, in canonical form, is the game $\{1 \mid *, *2, *3\}$. With this in mind, it is possible to obtain a cover-preserving diamond.

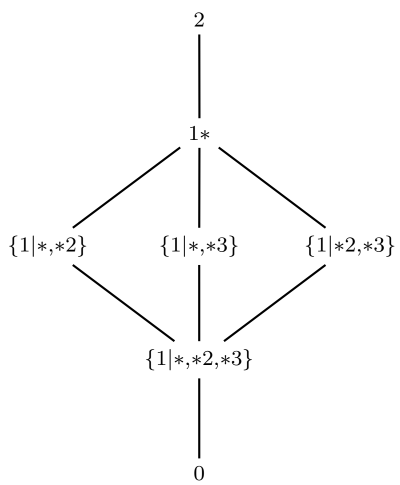
Let

$$\begin{aligned} A &= \{\{3 \mid *, *2\} \mid -3\}, \\ B &= \{\{3 \mid *, *3\} \mid -3\}, \\ C &= \{\{3 \mid *2, *3\} \mid -3\}, \text{ and} \\ S &= \{1, A, B, C\}. \end{aligned}$$

For this set of games, the antichain table is:

<table>
<tr><th></th><th>1</th><th>A</th><th>B</th><th>C</th><th>A,B</th><th>A,C</th><th>B,C</th><th>A,B,C</th><th>$\emptyset$</th></tr>
<tr><td>1</td><td>$1*$</td><td>$\{1 \mid *, *2\}$</td><td>$\{1 \mid *, *3\}$</td><td>$\{1 \mid *2, *3\}$</td><td colspan="4">$\{1 \mid *, *2, *3\}$</td><td>2</td></tr>
<tr><td>A</td><td colspan="9" rowspan="8">0</td></tr>
<tr><td>B</td></tr>
<tr><td>C</td></tr>
<tr><td>A,B</td></tr>
<tr><td>A,C</td></tr>
<tr><td>B,C</td></tr>
<tr><td>A,B,C</td></tr>
<tr><td>$\emptyset$</td></tr>
</table>

The Hasse diagram of $\mathcal{L}(S)$ is:



This example is also the first known non-distributive modular $\mathcal{L}(S)$.

The previous necessary condition can be useful to analyze some particular universes of games.

Corollary 4.2.3. *Let S be a set of games. If $\mathcal{L}(S)$ is a finite non-distributive modular lattice then there are $A, B, C \in S$ such that $A \parallel B$, $A \parallel C$ and $B \parallel C$.*

Proof. By Theorem 4.2.2, exists

$$\begin{array}{rcl} A & \in & \lceil G \rceil \setminus (\lceil H \rceil \cup \lceil F \rceil), \\ B & \in & \lceil H \rceil \setminus (\lceil G \rceil \cup \lceil F \rceil), \text{ and} \\ C & \in & \lceil F \rceil \setminus (\lceil H \rceil \cup \lceil G \rceil) \end{array}$$

or the symmetric situation (for which the proof is similar). So,

$$A \mathrel{|\!\!\rhd} G,\ A \leqslant H,\ A \leqslant F \text{ and } B \mathrel{|\!\!\rhd} H,\ B \leqslant G,\ B \leqslant F.$$

If $A \leqslant B$ then $A \leqslant B \leqslant G$ which contradicts $A \mathrel{|\!\!\rhd} G$. If $A \geqslant B$ then $H \geqslant A \geqslant B$ which contradicts $B \mathrel{|\!\!\rhd} H$. Therefore, $A \parallel B$.

The arguments for $A \parallel C$ and $B \parallel C$ are similar. □

Corollary 4.2.4. *Let S be a finite linear order of games. Then, modularity of $\mathcal{L}(S)$ implies distributivity of $\mathcal{L}(S)$.*

Proof. Because S is a finite linear order, it cannot contain A, B, C such that $A \parallel B$, $A \parallel C$ and $B \parallel C$. So, by Corollary 4.2.3, $\mathcal{L}(S)$ cannot be a non-distributive modular lattice. □

The question *What conditions on S force $\mathcal{L}(S)$ to be a modular lattice?* is still open. However, in the next chapter, Question 2 (*What lattices are generated?*) will be completely answered: all complete lattices can be generated. The finite case is shorter and more direct, the infinite case is more involved.

Chapter 5

Representation With Games

In this chapter, some results from [7] and [8] are presented. A *Representation Theorem with Games* allowing the generation of all finite lattices in game context is proved, and is extended to the infinite complete case, proving that, for a well chosen S, allowing the axiom of choice, every complete lattice is isomorphic to $\mathcal{L}(S) \setminus \{\bot, \top\}$. Also, when researching [1], sets S were found such that $\mathcal{L}^n(S) \neq \mathcal{L}^m(\emptyset)$ for any finite n and m. This motivated the general question of whether $\mathcal{L}^\gamma(S) = \mathcal{L}^\gamma(\emptyset)$ for some $\gamma \in \mathbb{O}rd$, that will be solved in the last section of this chapter. Also, a condition that ensures that the class of games born in days before a certain ordinal is a lattice.

Nimbers play a very important role in the choice of S. In particular, transfinite nimbers will be very useful so, when the word *nimber* occurs, it may be a transfinite nimber.

5.1 Representation Theorem: Finite Case

The construction of a non-distributive modular $\mathcal{L}(S)$ made in the previous chapter was based on the careful choice of a set of games S involving nimbers and reversible options. In this section, using the same ideas, a general representation theorem, which shows that, for every finite lattice $\mathcal{L}$, there is a set of games S such that $\mathcal{L}(S) \setminus \{\bot, \top\}$ is isomorphic to $\mathcal{L}$, is proven.

Consider the universe

$$\{J_A = \{A \mid -1\} : A \text{ a set of fuzzy nimbers}\}.$$

(Note, fuzzy nimbers include all nimbers except 0.) In this universe it is easy to see that:

U1: $J_A = J_B$ if and only if $A = B$;

U2: $J_A > J_B$ if and only if B is strictly contained in A;

U3: $J_A \parallel J_B$ if and only if $B \setminus A \neq \emptyset$ and $A \setminus B \neq \emptyset$.

These are consequences of the fact that, if $*k \in A \setminus B$, then, in the game $\{A \mid -1\} + \{1 \mid B\}$, Left, moving first, has the winning move $*k + \{1 \mid B\}$.

Using reversibility, it is possible to define a more thin and useful universe of games.

Definition 5.1.1. Let $\mathcal{F}$ be a family of sets of fuzzy nimbers. Let $\mathcal{F}^*$ be the set of games

$$\mathcal{F}^* = \{G_A : G_A = \{2 \mid \{A \mid -2\}\} \text{ and } A \in \mathcal{F}\};$$

and

$$\mathcal{U}_{\mathcal{F}} = \{J_A : J_A = \{G_A \mid -1\} \text{ and } G_A \in \mathcal{F}^*\}.$$

For any set $\mathcal{F}$, let

$$\overline{\mathcal{F}} = \bigcup_{\mathcal{T} \subseteq \mathcal{F}} \left(\bigcup_{A \in \mathcal{T}} A \right)$$

be the closure of $\mathcal{F}$ under unions, and define $\overline{\mathcal{F}}^*$ and $\mathcal{U}_{\overline{\mathcal{F}}}$ as in the previous sentence.

The following lemma highlights some important facts about the elements of $\mathcal{U}_{\mathcal{F}}$.

Lemma 5.1.2. *Let $\mathcal{F}$ be a family of sets of fuzzy nimbers. Then,*

$$J_A = \{\{2 \mid \{A \mid -2\}\} \mid -1\} = \{A \mid -1\}$$

and the elements of $\mathcal{U}_{\mathcal{F}}$ have the properties U1, U2 and U3 of the previous universe.

Proof. Since $\{A \mid -2\} < \{\{2 \mid \{A \mid -2\}\} \mid -1\}$ then, by reversibility,

$$\{\{2 \mid \{A \mid -2\}\} \mid -1\} = \{A \mid -1\}.$$

The proof of the second claim is the same as that already given. $\square$

For the intended purposes of this work, the set of games $\mathcal{F}^*$ needs to be extended to

$$S = \mathcal{F}^* \cup \{-1\}.$$

Following, some properties of $\mathcal{L}(S)$ for that specific S.

Proposition 5.1.3. *Let $\mathcal{F}$ be a family of sets of fuzzy nimbers and let $S = \mathcal{F}^* \cup \{-1\}$. Then, $\mathcal{L}(S) = \{0, -2\} \cup \{-1*\} \cup \mathcal{U}_{\overline{\mathcal{F}}}$.*

Proof. Consider $X \in \mathcal{L}(S)$.

Suppose that each right option of X is a G_A, for some A. Since each G_A is fuzzy, then, regardless of the left option, $X = 0$. Thus, to generate any other game, -1 must be a right option. In this case, again since each G_A is fuzzy, X is negative if there is a left option and $X = 0$ otherwise. Therefore, the maximum element of $\mathcal{L}(S)$ is 0.

Suppose

$$X = \left\{\mathcal{X}^L \mid G_{A_1}, G_{A_2}, \ldots, G_{A_k}, -1\right\}.$$

Since $X \leqslant 0$, a move by Right to some G_A gives Left a move to 2, which is clearly greater than X. Therefore, all of right's options, except -1, reverse out.

The game $\{\,\mid -1\} = -2$ is the mimimum element of $\mathcal{L}(S)$ (by Lemma 3.1.3, nothing can be worse for Left than having no options).

By definition of $\mathcal{L}(S)$, the other possibilities are the games $\{-1 \mid -1\} = -1*$ and $\{G_{A_1}, G_{A_2}, \ldots, G_{A_k} \mid -1\}$ such that $A_i \in \mathcal{F}$.

The fact

$$\begin{aligned}
\{G_{A_1}, G_{A_2}, \ldots, G_{A_k} \mid -1\} &= \{G_{A_1 \cup A_2 \cup \ldots \cup A_k} \mid -1\} \\
&= \{A_1 \cup A_2 \cup \ldots \cup A_k \mid -1\}
\end{aligned}$$

finishes the proof. $\square$

Proposition 5.1.4. *Let $\mathcal{F}$ be a family of sets of fuzzy nimbers and let $S = \mathcal{F}^* \cup \{-1\}$. Then $\{-1*\} \cup \mathcal{U}_{\overline{\mathcal{F}}}$ is a sublattice of $\mathcal{L}(S)$.*

Proof. This is a direct consequence of the inequalities

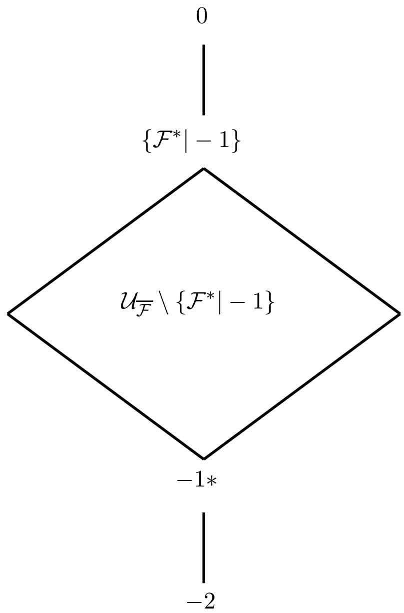
$$-1* < \{G_{A_1}, G_{A_2}, \ldots, G_{A_k} \mid -1\}$$

and

$$\{\mathcal{F}^* \mid -1\} \geqslant \{G_{A_1}, G_{A_2}, \ldots, G_{A_k} \mid -1\} \text{ (Lemma 3.1.3).}$$

$\square$

The following picture illustrates the structure of $\mathcal{L}(S)$:



Theorem 5.1.5. *Let $\mathcal{F}$ be a family of sets of fuzzy nimbers. Consider the set of games $S = \mathcal{F}^* \cup \{-1\}$ and the family of sets $\overline{\mathcal{F}} \cup \emptyset$. Consider the lattices $\big(\{-1*\} \cup \mathcal{U}_{\overline{\mathcal{F}}}, \leqslant\big)$, and $(\overline{\mathcal{F}} \cup \emptyset, \subseteq)$. Then, the map ψ defined by*

$$\psi : \{-1*\} \cup \mathcal{U}_{\overline{\mathcal{F}}} \longrightarrow \overline{\mathcal{F}} \cup \{\emptyset\},$$

where $\psi(-1*) = \emptyset$ and $\psi(\{G_{A_1 \cup A_2 \cup \ldots \cup A_k} \mid -1\}) = A_1 \cup \ldots \cup A_k$, is a order- and join-preserving isomorphism.

Proof. The order-preserving property $\psi(J_{S_1}) \leqslant \psi(J_{S_2})$ iff $S_1 \subseteqq S_2$ follows from applying Lemma 5.1.2:

$$\begin{aligned}
&\{G_{A_1 \cup A_2 \cup \ldots \cup A_k}| - 1\} \leqslant \big\{G_{B_1 \cup B_2 \cup \ldots \cup B_j}| - 1\big\} \\
&\quad \Leftrightarrow \quad \{A_1 \cup A_2 \cup \ldots \cup A_k \mid -1\} \leqslant \{B_1 \cup B_2 \cup \ldots \cup B_j \mid -1\} \\
&\quad \Leftrightarrow \quad A_1 \cup A_2 \cup \ldots \cup A_k \subseteqq B_1 \cup B_2 \cup \ldots \cup B_j.
\end{aligned}$$

Also, the one-to-one correspondence is consequence of

$$\begin{aligned}
&\{G_{A_1 \cup A_2 \cup \ldots \cup A_k}| - 1\} = \big\{G_{B_1 \cup B_2 \cup \ldots \cup B_j}| - 1\big\} \\
&\quad \Leftrightarrow A_1 \cup A_2 \cup \ldots \cup A_k = B_1 \cup B_2 \cup \ldots \cup B_j.
\end{aligned}$$

Finally, if, say, $J_{S_1} = -1*$, then

$$\psi(J_{S_1} \vee J_{S_2}) = \psi(J_{S_2}) = \emptyset \cup \psi(J_{S_2}) = \psi(J_{S_1}) \cup \psi(J_{S_2}).$$

Suppose that $J_{S_1} \neq -1*$ and $J_{S_2} \neq -1*$. Then,

$$\psi(J_{S_1} \vee J_{S_2}) = \psi(\{S_1 \cup S_2| - 1\}) = S_1 \cup S_2 = \psi(J_{S_1}) \cup \psi(J_{S_2}),$$

i.e. ψ is join-preserving. ∎

Propositions 5.1.3 and 5.1.4 and Theorem 5.1.5 establish a way to identify finite sets with finite lattices of games. Theorem 1.2.17 and Corollary 1.2.18 establish a way to identify finite lattices with finite sets. Therefore, it is possible to identify finite lattices with sets of games.

Theorem 5.1.6 (Representation Theorem with Games)**.** *Let $(\mathcal{L}, \leqslant)$ be a finite lattice. Consider M its set of meet irreducibles labelled with the nimbers, $*1, *2, \ldots, *k$. Then,*

$$\mathcal{L} \overset{\varphi}{\to} R(\varphi) \overset{\psi^{-1}}{\to} \{-1*\} \cup \mathcal{U}_{R(\varphi)\backslash\emptyset}$$

is a lattice isomorphism and $\psi^{-1} \circ \varphi(a \vee b) = \psi^{-1}(\varphi(a) \cup \varphi(b))$.

Proof. Note that $R(\varphi)\backslash\emptyset$ is a family of sets of fuzzy nimbers closed under unions. Thus this result is consequence of Theorem 5.1.5 and Corollary 1.2.18. $\square$

So, given a finite lattice $\mathcal{L}$, it is possible to choose a set S in such way that

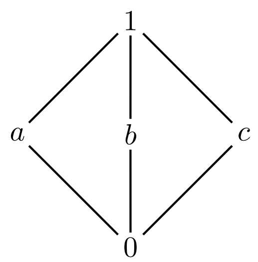
$$\mathcal{L}(S) \setminus \{0, -2\} \cong \mathcal{L}.$$

The set $S = (R(\varphi)\backslash\emptyset)^* \cup \{-1\}$ or, even better,

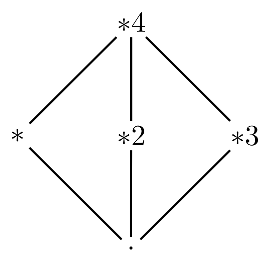
$$S = (\mathcal{J}(R(\varphi))\backslash\emptyset)^* \cup \{-1\}$$

produces the intended finite lattice, where $\mathcal{J}(R(\varphi))$ is the set of join-irreducibles of $R(\varphi)$.

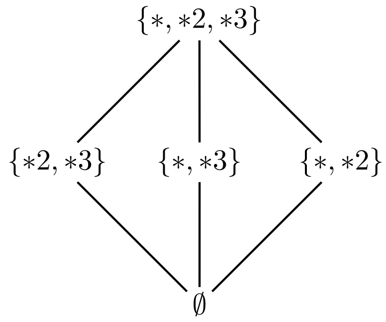
Example 5.1.7. Consider the lattice M_3 from Figure 1.1.



The set of meet-irreducibles of M_3 is $M = \{a, b, c, 1\}$. Consider it labeled with the fuzzy nimbers $*$, $*2$, $*3$ and $*4$, in the following way:



Then $R(\varphi)$ is given by the following diagram:

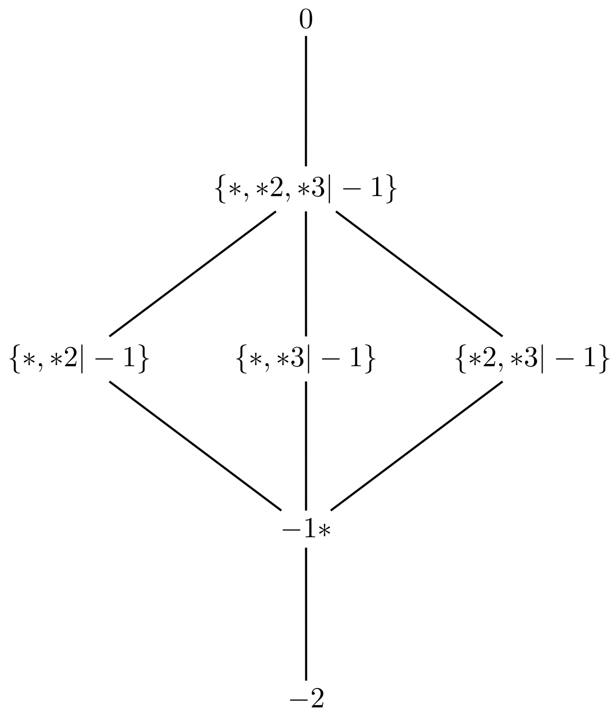


If S was considered to be the set $\left\{\{G \mid -1\} : G \subseteq 2^{\{*,*2,*3\}}\right\}$, then $\{* \mid -1\}$ would be an element of S, yielding that S would have more elements than the intended.

So, consider $\mathcal{F} = \left\{\{*, *3\}, \{*2, *3\}, \{*, *2\}, \{*, *2, *3\}\right\}$. Then

$$\mathcal{F}^* = \left\{\{2 \mid \{*, *3 \mid -2\}\}, \{2 \mid \{*2, *3 \mid -2\}\}, \{2 \mid \{*, *2 \mid -2\}\}, \{2 \mid \{*, *2, *3 \mid -2\}\}\right\}$$

and $\mathcal{U}_{\mathcal{F}} = \left\{\{*, *3 \mid -1\}, \{*2, *3 \mid -1\}, \{*, *2 \mid -1\}, \{*, *2, *3 \mid -1\}\right\}$. For $S = \mathcal{F}^* \cup \{-1\}$, the lattice $\mathcal{L}(S)$ is given in the following diagram.



Example 5.1.8. The lattice $M_3[D]$, the congruence preserving extension of D, where D is the three-element chain, is a non-distributive modular lattice with a beautiful tridimensional visualization (see [26]).

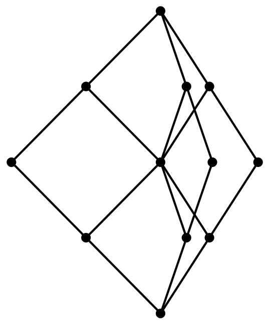
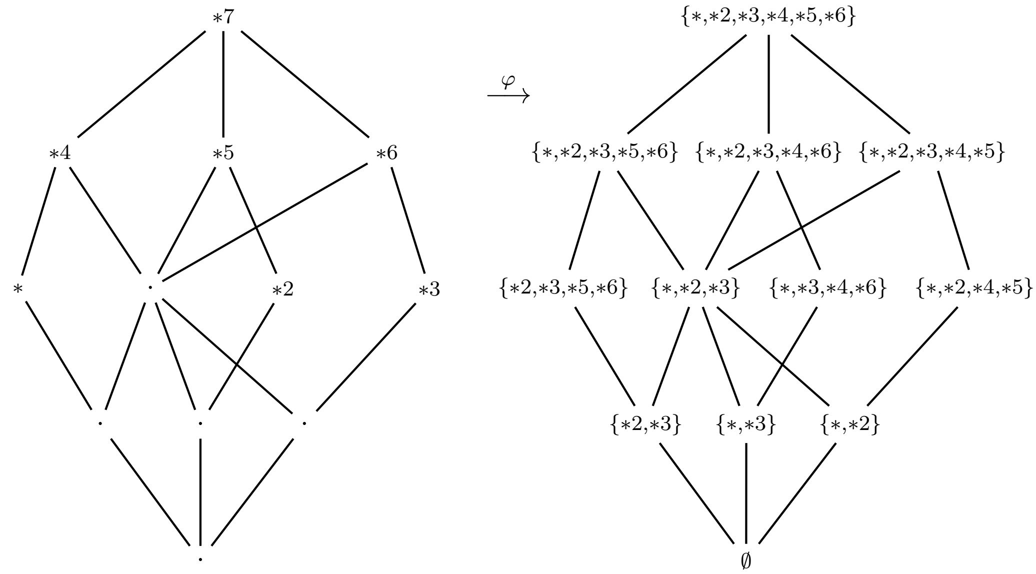


Figure 5.1: $\mathcal{L}(S) \cong M_3[D]$ (with triangular dipyramidal picture [25])

The next picture shows how to use the Representation Theorem 5.1.6 to generate $M_3[D]$ from games.



$M \subseteq M_3[D]$ labeled with fuzzy nimbers. $R(\varphi)$

Figure 5.2: Example of application of Theorem 5.1.6

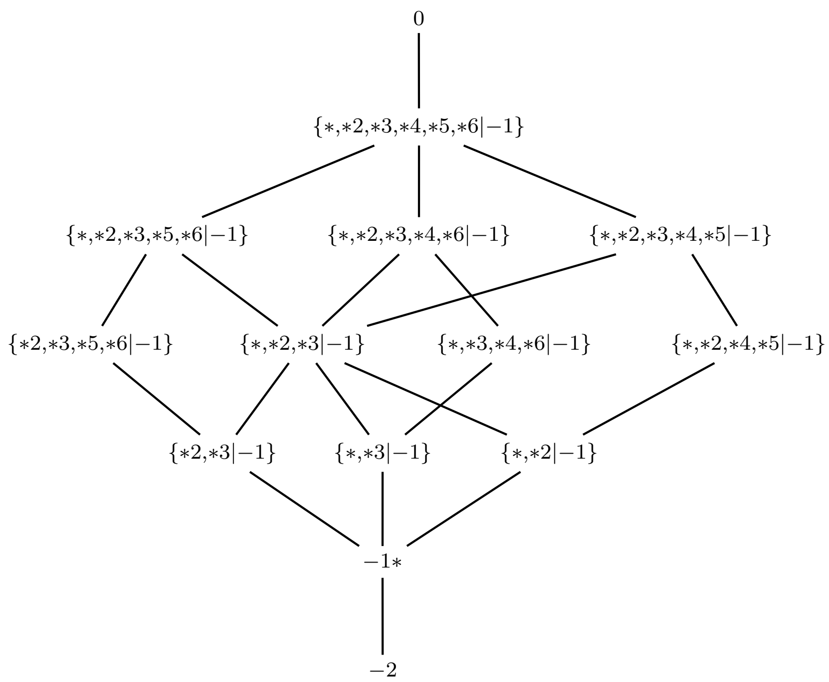
Considering

$$\begin{array}{rcl} A & = & \{2 \mid \{*2, *3 \mid -2\}\} \\ B & = & \{2 \mid \{*, *3 \mid -2\}\} \\ C & = & \{2 \mid \{*, *2 \mid -2\}\} \\ D & = & \{2 \mid \{*2, *3, *5, *6 \mid -2\}\} \\ E & = & \{2 \mid \{*, *3, *4, *6 \mid -2\}\} \\ F & = & \{2 \mid \{*, *2, *4, *5 \mid -2\}\} \end{array}$$

and

$$S = \{-1, A, B, C, D, E, F\},$$

then $\mathcal{L}(S)$ is



5.2 Representation Theorem for Complete Lattices

In the previous section, it was proved a *Representation Theorem with Games* allowing the generation of all finite lattices in game context. From Chapter 3, it is known that $\mathcal{L}(S)$ is a complete lattice. Therefore, a natural step is trying to generate all complete lattices (finite or infinite) in game context.

The idea for the finite case was to establish a composition

$$\begin{array}{ccccc} \mathcal{L} & \longrightarrow & \text{Set of sets} & \longrightarrow & \mathcal{L}(S) \setminus \{\bot, \top\} \\ \text{finite lattice} & & \text{(closed for unions)} & & \text{lattice of games.} \end{array}$$

The second part of the composition was based in some results that are *true allowing transfinite nimbers.* To summarize the previous results, it is possible to attach an isomorphism ψ to *any* family of sets of fuzzy nimbers $\mathcal{H}$ closed under unions, that is

$$\psi : \{-1*\} \cup \mathcal{U}_{\mathcal{H}} \longrightarrow \overline{\mathcal{H}} \cup \{\emptyset\} = \mathcal{H} \cup \{\emptyset\}.$$

From the set of games $S = \mathcal{H}^* \cup \{-1\}$, the lattice

$$\mathcal{L}(S) = \{0, -2\} \cup \{-1*\} \cup \mathcal{U}_{\mathcal{H}}$$

is obtained and, applying Theorem 5.1.5, the sublattice

$$(\mathcal{L}(S) \setminus \{\perp, \top\}, \subseteqq)$$

is isomorphic to $(\mathcal{H} \cup \{\emptyset\}, \subseteqq)$. The following results will establish the appropriate $(\mathcal{H} \cup \{\emptyset\}, \subseteqq)$, an object, in lattice theory, called *interior system*, in order to prove the more general *Representation Theorem with Games.* Observe that the top and bottom of $(\mathcal{H} \cup \{\emptyset\}, \subseteqq)$,

$$\top = \bigcup \mathcal{H} \text{ and } \perp = \emptyset$$

are identified with the games $\{\bigcup \mathcal{H} \mid -1\}$ and $\{-1 \mid -1\} = -1*$, respectively the top and bottom of $\mathcal{L}(S) \setminus \{\perp, \top\} = \{-1*\} \cup \mathcal{U}_{\mathcal{H}}$. All these previous results, with the same proofs, work for the transfinite case.

The crucial part to solve, when thinking about a generalization for complete lattices, is the first part of the composition. The results in [1] were based in Theorem 1.2.17 and Corollary 1.2.18 that are of *no use* for the general complete case:

Definition 5.2.1. Let $\mathcal{L}$ be a lattice. A family $\mathcal{I}$ of subsets of $\mathcal{L}$ is said to be a *interior system* if $\mathcal{I}$ is closed under unions, that is, for all $\mathcal{H} \subseteq \mathcal{I}$, $\bigcup \mathcal{H} \in \mathcal{I}$.

Observe that $\bigcup \emptyset = \emptyset \in \mathcal{I}$.

Interior systems, as families of subsets, have the following property.

Proposition 5.2.2. *Let $\mathcal{I}$ be an interior system. Then $(\mathcal{I}, \subseteq)$ is a complete lattice.*

Proof. If $\mathcal{A}$ is a set of elements of $\mathcal{I}$, it is easy to argue that

1. $\bigvee \mathcal{A} = \bigcup \mathcal{A}$, and
2. $\bigwedge \mathcal{A} = \bigcup (B \in \mathcal{I} : B \subseteq \bigcap \mathcal{A})$.

□

More, for any complete lattice, it is possible to construct an interior system in a rather natural way.

Proposition 5.2.3. *Let $\mathcal{L}$ be a complete lattice. Then, the family*

$$\mathcal{I} = \big\{\{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} : a \in \mathcal{L}\big\}$$

is an interior system.

Proof. To prove that $\displaystyle\bigcup_{a\in A\subseteq\mathcal{L}} \{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} \in \mathcal{I}$, it is enough to prove that

$$\bigcup_{a\in A\subseteq\mathcal{L}} \{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} = \left\{x \in \mathcal{L} : x \lhd\!\!\shortmid \bigvee_{a\in A} a\right\} \in \mathcal{I}.$$

This is a consequence of $x \lhd\!\!\shortmid \displaystyle\bigvee_{a\in A} a$ if and only if, for some $b \in A$, $x \lhd\!\!\shortmid b$ and the existence of $\displaystyle\bigvee_{a\in A} a \in \mathcal{L}$ is guaranteed because $\mathcal{L}$ is a complete lattice. $\square$

These results allow any complete lattice to be represented by an interior system.

Proposition 5.2.4. *Let $\mathcal{L}$ be a complete lattice. Then, $\mathcal{L}$ is isomorphic to some interior system.*

Proof. By Proposition 5.2.3, the family $\mathcal{I} = \{\{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} : a \in \mathcal{L}\}$ is an interior system and, by Proposition 5.2.2, this family, with $\subseteq$ is a complete lattice.

Consider the map

$$\varphi : a \in \mathcal{L} \mapsto \{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} \in \mathcal{I}.$$

The map φ is an isomorphism, and $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$:

1. Let $a, b \in \mathcal{L}$ and suppose that $a \leqslant b$. Let $x \in \mathcal{L}$ be such that $x \lhd\!\!\shortmid a$. It follows that $x \lhd\!\!\shortmid b$ as well, since otherwise $b \leqslant x$ and $a \leqslant b \leqslant x \Rightarrow a \leqslant x$. Therefore

   $$\varphi(a) = \{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} \subseteq \{x \in \mathcal{L} : x \lhd\!\!\shortmid b\} = \varphi(b),$$

   and φ is order-preserving.

2. For $a, b \in \mathcal{L}$,

   $$\begin{aligned}\varphi(a \vee b) &= \{x \in \mathcal{L} : x \lhd\!\!\shortmid a \vee b\}\\ &= \{x \in \mathcal{L} : x \lhd\!\!\shortmid a\} \cup \{x \in \mathcal{L} : x \lhd\!\!\shortmid b\}\\ &= \varphi(a) \cup \varphi(b).\end{aligned}$$

3. Let $a, b \in \mathcal{L}$, and suppose that $\varphi(a) = \varphi(b)$, that is,

$$\{x \in \mathcal{L} : x \lhd\!\mid a\} = \{x \in \mathcal{L} : x \lhd\!\mid b\}.$$

Then $a = b$, since, otherwise, $a \lhd\!\mid b$ or $b \lhd\!\mid a$. Without loss of generality, suppose that $a \lhd\!\mid b$. Then $a \in \{x \in \mathcal{L} : x \lhd\!\mid b\}$, which yields a contradiction with the fact that $a \notin \{x \in \mathcal{L} : x \lhd\!\mid a\}$.

□

It is now possible to establish the first part of the composition, and prove the rest of the infinite case.

Theorem 5.2.5 (Representation Theorem with Games for Complete Lattices)**.** *Let $(\mathcal{L}, \leqslant)$ be a complete lattice. Then, for some set of games S, $\mathcal{L}$ is isomorphic to $\mathcal{L}(S) \setminus \{\bot, \top\}$.*

Proof. Consider the isomorphism $\varphi : (\mathcal{L}, \leqslant) \mapsto (\mathcal{I}, \subseteq)$,

$$\varphi(a) = \{x \in \mathcal{L} : x \lhd\!\mid a\}$$

defined in Proposition 5.2.4).

By Theorem 1.3.11, for some $\alpha \in \mathbb{O}rd$, consider the bijection

$$\Psi : \mathcal{L} \mapsto \{\xi \in \mathbb{O}rd : \xi < \alpha\}.$$

Then, the map $*\Psi : (\mathcal{I}, \subseteq) \mapsto (*\mathcal{I}, \subseteq)$, where

$$*\Psi(\{x \in \mathcal{L} : x \lhd\!\mid a\}) = \{*(\Psi(x)) : x \in \mathcal{L}, x \lhd\!\mid a\}$$

is an isomorphism and $(*\mathcal{I}, \subseteq)$ is an interior system of sets of nimbers. Note that $*(\Psi(x))$ is a nimber, eventually transfinite.

Also, the map

$$\psi^{-1} : (*\mathcal{I}, \subseteq) \mapsto (\{-1*\} \cup \mathcal{U}_{*\mathcal{I}}, \leqslant)$$

given by Theorem 5.1.5 is an isomorphism.

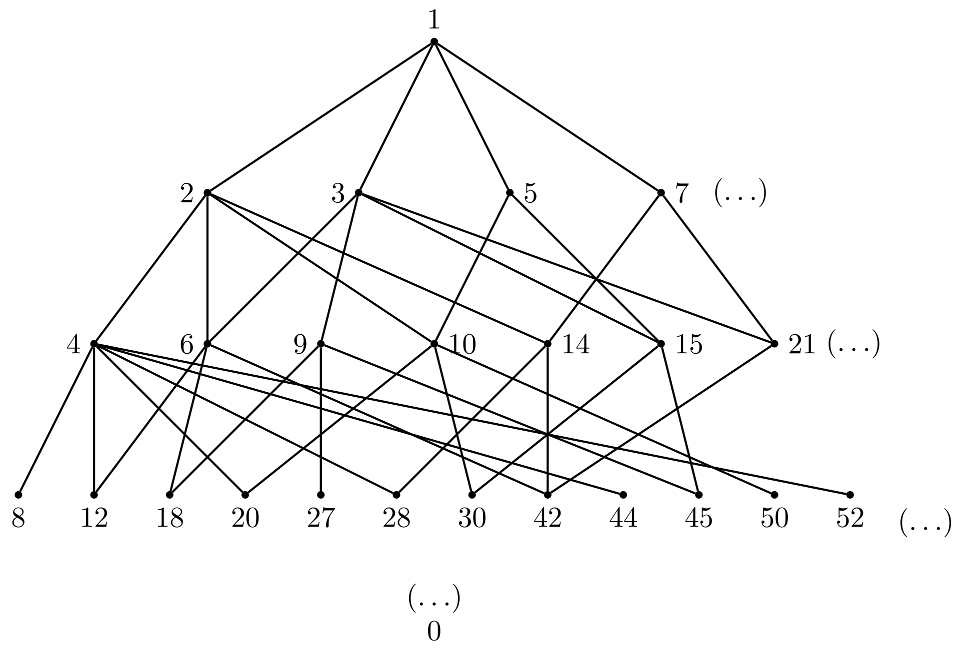
So, if $\mathcal{F}$ is the family of sets of fuzzy nimbers of $*\mathcal{I}$ and S is the set of games $\mathcal{F}^* \cup \{-1\}$ then, $\psi^{-1} \circ *\Psi \circ \varphi$ is an isomorphism mapping $\mathcal{L}$ to $\mathcal{L}(S) \setminus \{\bot, \top\}$. □

Some Applications

Next, two applications of the Representation Theorem with Games for Complete Lattices are shown.

Example 5.2.6. In an infinite lattice, there may be no meet-irreducible or no join-irreducible elements at all, not considering the trivial cases $\perp$ and $\top$. So, Theorem 1.2.17 is not applicable.

Consider lattice of nonnegative integers, ordered with the reverse of the usual divisibility ordering, $x \leqslant y$ when x divides y. The set of non-zero join-irreducible elements is empty.



In this example, $x \vee y = \gcd(x, y)$ and $x \wedge y = \operatorname{lcm}(x, y)$.

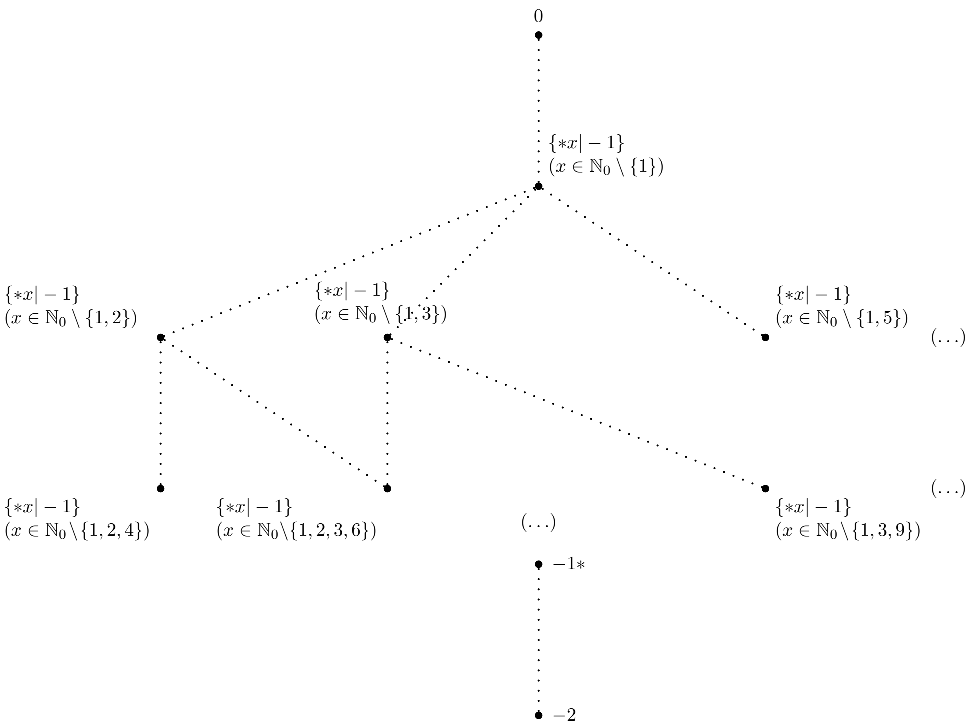
To find the representation with games, note that

$$\varphi(a) = \{x \in \mathbb{N}_0 : a \nmid x\}.$$

So, define the game $G_a = \{2 \mid \{*x \mid -2\}\}$ where $x \in \mathbb{N}_0$ and $a \nmid x$, and choose

$$S = \left(\bigcup_{a \in \mathbb{N}_0} \{G_a\} \right) \cup \{-1\}.$$

The following $\mathcal{L}(S)$ is obtained.



Example 5.2.7. The lattice $\mathbb{Q} \cup \{-\infty, +\infty\}$ is a not complete infinite lattice. In fact, sets such as $\{r \in \mathbb{Q} : r^2 < 2\}$ have no joins or meets. In this case, the family

$$\mathcal{I} = \{\{x \in \mathbb{Q} \cup \{-\infty, +\infty\} : x < a\} : a \in \mathbb{Q} \cup \{-\infty, +\infty\}\}$$

is not an interior system and the representation method exposed in the last section fails.

However, the lattice $\mathbb{R} \cup \{-\infty, +\infty\}$ is complete and, for this case, the family

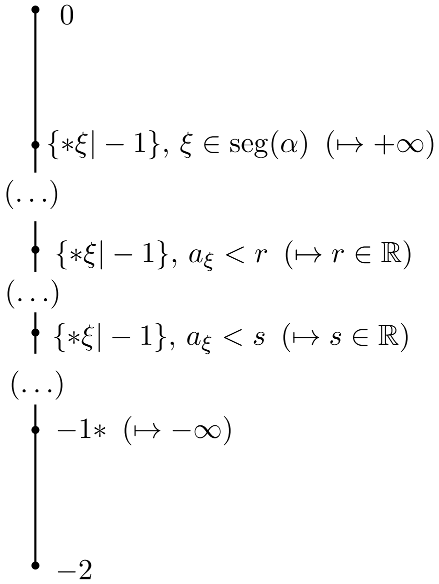
$$\mathcal{I} = \{\{x \in \mathbb{R} \cup \{-\infty, +\infty\} : x < a_\xi\} : \xi < \alpha\}$$

is an interior system. Consider a enumeration of $\mathbb{R}$, a_ξ, $\xi < \alpha$, $\xi, \alpha \in \mathbb{O}rd$.

Define, for $r \in \mathbb{R} \cup \{-\infty, +\infty\}$, $G_r = \{2 \mid \{\text{nimbers}{*\xi} \mid -2\}\}$ where $a_\xi < r$ and choose

$$S = \left(\bigcup_r \{G_r\}\right) \cup \{-1\}.$$

The following $\mathcal{L}(S)$ is obtained.



5.3 Transfinite Conway's Construction

Starting Conway's transfinite recursion with $S \neq \emptyset$ raises the question of whether $\mathcal{L}^\gamma(S) = \mathcal{L}^\gamma(\emptyset)$ for some $\gamma \in \mathbb{O}rd$. For instance, consider $S = \{0\}$. For every $n \in \mathbb{N}_0$, $\mathcal{L}^n(S) \neq \mathcal{L}^n(\emptyset)$ because, if $k \in \mathbb{N}$, $k \in \mathcal{L}^{k-1}(S)$ and $k \notin \mathcal{L}^{k-1}(\emptyset)$. However it is possible to prove a general convergence to Conway's construction, beginning with the empty set.

Theorem 5.3.1. *Let S be a set of games. Then, for some $\gamma \in \mathbb{O}rd$,*

$$\mathcal{L}^\gamma(S) = \mathcal{L}^\gamma(\emptyset).$$

Proof. There is a natural embedding of $B \subsetneq \mathbb{O}rd$, the ordinals that are birthdays of at least one game of S, into S. The images may be obtained applying the Axiom of Choice to the family

$$\{G \in S : \mathrm{b}(G) = \xi\}_{\xi \in B}.$$

Therefore B must be a set because S is a set. Also, B must be bounded (an unbounded class of ordinals is not a set).

Consider $\alpha \in \mathbb{O}rd$ such that $\mathrm{b}(G) \leqslant \alpha$ for all $G \in S$. Let γ be an additively indecomposable ordinal such that $\gamma > \alpha$ (it exists because, by Proposition 1.3.10, $\mathbb{H}$ is unbounded).

To prove that $\mathcal{L}^\gamma(\emptyset) \subseteq \mathcal{L}^\gamma(S)$, consider a game $J \in \mathcal{L}^\gamma(\emptyset)$. By definition, $J \in \mathcal{L}(U)$, where U is the set of games with birthday less than γ. Since

$0 \in \mathcal{L}^0(S)$, every game in U is also born before day n in the recursive process with initial set S. So, again by definition, $J \in \mathcal{L}^\gamma(S)$.

To prove that $\mathcal{L}^\gamma(S) \subseteq \mathcal{L}^\gamma(\emptyset)$, consider a game $J \in \mathcal{L}^\gamma(S)$. By definition, $J \in \mathcal{L}(U)$, where U is the set of games born on day $\beta < \omega$ in the recursive process with initial set S. But, since $\mathrm{b}(G) \leqslant \alpha$ for all $G \in S$, all games in S are also in $\mathcal{L}^\alpha(\emptyset)$. Therefore, $J \in \mathcal{L}(U')$, where U' is the set of games born on day $\alpha + \beta$ beginning with $\emptyset$. Observing that γ is additively indecomposable, $\alpha + \beta < \gamma$, and, so, U' is contained in the set of games that are born before day γ in the process beginning with $\emptyset$, and therefore $J \in \mathcal{L}^\gamma(\emptyset)$. $\square$

Example 5.3.2.

1. If $S = \{1, *7, \{20 \mid -30\}\}$, then $L^\omega(S) = L^\omega(\emptyset)$ and, indeed, if S is a set of short games, then $L^\omega(S) = L^\omega(\emptyset)$.
2. If $S = \{1, *7, *(\omega + 1)\}$, then $L^{\omega^2}(S) = L^{\omega^2}(\emptyset)$.

Theorem 3.1.12 states that the collection of short games is not a lattice. It is also interesting to understand what happens with the collection of games born in the days before γ, for $\gamma \in \mathbb{O}rd$. In fact, it can be observed that the key fact is not whether the games are short or long, but if $\gamma \in \mathbb{O}rd$ is a successor or a limit ordinal.

Theorem 5.3.3. *Let $\gamma \in \mathbb{O}rd$ be a limit ordinal. Then the collection of games born before γ is not a lattice.*

Proof. Let
$$\mathcal{C}_\gamma = \{G : \mathrm{b}(G) < \gamma\}$$
be the collection of games born before γ.

Let $G, H \in \mathcal{C}_\gamma$, $G \parallel H$, and suppose that the join of G and H in $\mathcal{C}_\gamma$, S, exists. Let $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$, $H = \left\{\mathcal{H}^L \mid \mathcal{H}^R\right\}$, and $S = \left\{\mathcal{S}^L \mid \mathcal{S}^R\right\}$ be forms of G, H, and S where the birthdays of the options are less than the birthday of the game itself, and define

$$J = \left\{G, H, \mathcal{G}^L \cup \mathcal{H}^L \mid S, \mathcal{S}^R\right\}.$$

Since $G, H, S \in \mathcal{C}_\gamma$, then $\mathrm{b}(G), \mathrm{b}(H), \mathrm{b}(S) < \gamma$, and

$$\mathrm{b}(J) \leqslant \max\{\mathrm{b}(G), \mathrm{b}(H), \mathrm{b}(S)\} + 1.$$

Since γ is a limit ordinal, then $\mathrm{b}(J) < \gamma$, and $J \in \mathcal{C}_\gamma$. More, $J > G$: consider the game

$$J - G = \left\{G, H, \mathcal{G}^L \cup \mathcal{H}^L \mid S, \mathcal{S}^R\right\} + \left\{-\mathcal{G}^R \mid -\mathcal{G}^L\right\}.$$

Left, playing first, can move in the first component to $G - G = 0$ and wins.

Right, playing first, can move in either component. If he moves in the second component, Left has a corresponding one in the first component, so he loses. If he moves in the first component to $S - G$, since $S > G$, he loses. If he moves in the first component to some $S^R - G$, since, by the Identity Principle, 3.1.1, $S^R \mathrel{|\!\!\triangleright} S$, and $S > G$, then $S^R \mathrel{|\!\!\triangleright} G$, and he loses.

Analogously, $J > H$.

Consider now the game

$$S - J = \{\mathcal{S}^L \mid \mathcal{S}^R\} + \{-S, -\mathcal{S}^R \mid -G, -H, -(\mathcal{G}^L \cup \mathcal{H}^L)\}.$$

Left, playing first, can move in the second component to $S - S = 0$ and wins.

Right, playing first, can move in either component. If he moves in the first component, Left has a corresponding one in the second component, so he loses. If he moves in the second component to $S - G$ or to $S - H$, since $S > G, H$, he loses. If he moves in the second component to $S - X$ with $X \in \mathcal{G}^L \cup \mathcal{H}^L$, then, since $X \mathrel{\triangleleft\!\!|} G < S$ or $X \mathrel{\triangleleft\!\!|} H < S$, $X \mathrel{\triangleleft\!\!|} S$, and he loses.

Therefore, $S - J > 0$, which contradicts the definition of join. $\square$

Observe that Theorem 3.1.12 is, in fact, a corollary of this result.

Final Remarks and Future Work

In this work, it is shown that all complete lattices can be generated in Combinatorial Game Theory context. This is an important result, for it cannot be expected to find any particular properties of lattices of combinatorial games. If all modular lattices of games were distributive, that should be because of game properties. However, statements like this one are false, since every complete lattice can be obtained. This conclusion is a main contribution of this work.

There are still many unanswered questions when the universe of games is restricted.

The obvious unanswered questions are:

Question 1: *Which families of games generate distributive lattices and which generate modular lattices?*

Other interesting lattices can be found. For example, in [22], it is shown that the set of option-closed games born on day n form a planar lattice. Preliminary examples suggest that S such that

$$\bigcap_{G\in S} I(G) = \emptyset$$

($I(G)$ is the confusion interval of G) is an interesting universe.

This text considered on normal games. Misère games also form an equivalence relation and a partial order. Unfortunately, most equivalence classes are small and most games are incomparable with each other. Moving to a restricted universe will increase the number of comparabilities and sizes of the equivalence classes. However, in general, few universes have been explored. The only lattices found in [4] are linear orders but most families considered there generated non-lattice partial orders.

Question 2: *Is there a universe of misère games that can be used to represent all lattices?*

OSLO (One-Sided LOopy) games are ones in which one player (say Left) is given an option to pass in all positions except in a terminal position. In [2] it is shown that the OSLO games born on day n form a distributive lattice. This lattice is not the same as that generated by all the short games born on day n. However, there is a lattice homomorphism from day n normal play games into the lattice of OSLO games born on day n.

Question 3: *Can sets of OSLO games be used to generate all lattices?*

In [12] and [29], universes of scoring games with elements (points) of some group ε that are endgames without options were proposed (the only Conway's form without options is $\{ \mid \}$).

Question 4: *Is there a lattice structure emerging of scoring recursive processes?*

Appendix A

Drawing Lattices of Games

In this appendix, a computational tool for drawing $\mathcal{L}(S)$ when S is finite is presented.

The software application LatticeMaker was developed to compute the lattice $\mathcal{L}(S)$ of a given set of games S. The application also produces a graphic, in PSTricks format, useful to visualize and include in mathematical texts (like most of the pictures from this work).

This was done by the implementation of the following tasks:

1. A CGsuite (see [27]) script that receives the initial set of games S, outputting two lists, the first list describing the set $\mathcal{L}(S)$, and the second with the pairs of the relation $R = \{(G_1, G_2) : G_1 < G_2, G_i \in \mathcal{L}(S)\}$;

2. These lists are then read by a Java program that translates them into the standard dot format for graphs;

3. This dot file is processed by a set of GraphViz tools such that a transitive reduction is found (i.e., the minimal relation over $\mathcal{L}(S)$ such that the transitive closure of the original relation R is preserved);

4. This reduced lattice is then translated into several PSTricks files, each with different focuses on the lattice structure, which can be easily integrated in LaTeX documents.

The current open source software version can be obtained at the following site `http://homepages.di.fc.ul.pt/~jpn/lattice`.

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Contactos:
Universidade de Évora
Instituto de Investigação e Formação Avançada - IIFA
Palácio do Vimioso | Largo Marquês de Marialva, Apart. 94
7002-554 Évora | Portugal
Tel: (+351) 266 706 581
Fax: (+351) 266 744 677
email: iifa@uevora.pt