Geometric conditions of regularity in some kind of minimal time problems

PhD Thesis by
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I am grateful to my Supervisor Prof. Vladimir V. Goncharov for the proposed problems, permanent support and the attention to the work.

I would like also to thank Prof. Giovanni Colombo and the scientific staff of the Mathematical Department of the Padua University, Italy, where some part of the Thesis was completed.

To my family

Geometric conditions of regularity in some kind of minimal time problems

Abstract

The work is devoted to the problem of reaching a closed subset of a Hilbert space in minimal time from a point situated near the target subject to a constant convex dynamics. Two types of geometric conditions guaranteeing existence and uniqueness of the end point of an optimal trajectory are given. We study the mapping, which associates to each initial state this end point, and under some supplementary assumptions prove its Hölder continuity outside the target. Then we estabilish the (Hölder) continuous differentiability of the value function in an open neighbourhood of the target set and give explicit formulas for its derivative. From the same point of view we treat the close problem with a nonlinear Lipschitzean perturbation and obtain some regularity results for viscosity solutions of a kind of Hamilton-Jacobi equations with non trivial boundary data.

Condições geométricas de regularidade numa classe de problemas de tempo mínimo

Resumo

O trabalho é dedicado ao problema de se atingir um subconjunto de um espaço de Hilbert em tempo mínimo a partir de um ponto situado próximo do conjunto-alvo com uma dinâmica convexa constante. São dados dois tipos de condições geométricas que garantem existência e unicidade do ponto final de uma trajectória óptima. Estudamos a aplicação que associa a cada estado inicial esse ponto final, e sob algumas condições suplementares provamos continuidade de Hölder da respectiva aplicação fora do conjunto-alvo. Depois mostramos a diferenciabilidade contínua (de Hölder) da função valor também numa vizinhança do alvo apresentando fórmulas explicitas para a sua derivada. Do mesmo ponto de vista tratamos o problema com uma perturbação não linear Lipschitzeana e obtemos alguns resultados de regularidade para soluções viscosas de um certo tipo de equações de Hamilton-Jacobi com dados na fronteira não triviais.

Extended abstract

Let H be a Hilbert space, $F \subset H$ be a closed convex bounded set containing the origin in its interior, and $C \subset H$ be nonempty and closed. The main problem considered in Thesis is to attain the (target) set C in minimal time from a point x close to C by trajectories of the control system $\dot{x} = u$, $u \in F$. Denoting by $\pi_C^F(x)$ the set of end points of all optimal trajectories in this problem (so called time-minimum projection), we study first the conditions, under which the mapping $x \mapsto \pi_C^F(x)$ is single-valued and continuous in some neighbourhood of C. These conditions have a geometric character and involve some concepts of Convex Analysis such as rotundity, uniform smoothness, curvature, duality mapping, which are introduced and studied in the first chapter. One of the hypotheses guaranteeing the well-posedness of the time-minimum projection is completely new, while the other is a sharp generalization of the assumptions known for the case $F = \overline{\mathbf{B}}$ ($\overline{\mathbf{B}}$ stands for the unit closed ball in H).

The next step in our investigation is the further regularity of the mapping $x \mapsto \pi_C^F(x)$. Assuming, in addition, smoothness either of the target set or of the dynamics F we establish the local (Hölder) regularity of $\pi_C^F(\cdot)$ first in a neighbourhood of a fixed boundary point $x_0 \in C$ and then in an open set around the target. It is proved that the well-posedness and the regularity of the time-minimum projection are strictly related to the regularity properties of the value function $x \mapsto \mathfrak{T}_C^F(x)$ (so called minimal time function). In particular, we obtain some results concerning the (Hölder) continuous differentiability of $\mathfrak{T}_C^F(\cdot)$ near (but outside) the target.

In the last part of the work we apply the same technique to the perturbed optimization problem $(\mathcal{P}_\theta)$ by adding a Lipschitz continuous function $\theta(\cdot)$ satisfying some controllability assumption. Denoting by $\pi_C^{F,\theta}(\cdot)$ and $u(\cdot)$ the set of minimizers in the problem above and the value function, respectively, we obtain results justifying the connection between the continuous differentiability of the mapping $u(\cdot)$, on one hand, and existence, uniqueness and stability of minimizers in $\pi_C^{F,\theta}(\cdot)$, on the other. Similarly to the case $\theta \equiv 0$ the (Hölder) continuity of both the (single-valued) mapping $\pi_C^{F,\theta}(\cdot)$ and the gradient $\nabla u(\cdot)$ near (but outside) the set C is proved under some natural assumptions involving the regularity of all three elements C, F, $\theta(\cdot)$ as well as their geometric compatibility. The last results are specially important because the value function $u(\cdot)$ is nothing else than the viscosity solution to a certain Hamilton-Jacobi equation with a non affine Lipschitzean boundary data, and it becomes the classical solution (at least near the boundary) whenever our conditions are fulfilled.

In Thesis we used the Geometry of Hilbert spaces as well as methods of Convex, Nonsmooth and Variational Analysis. The obtained results develop and generalize those known until now. In the particular case $F = \overline{\mathbf{B}}$ they reduce to the respective properties of the distance function and the metric projection. However, we consider much more general dynamics F (asymmetric, not necessarily strictly convex nor smooth) and very sharp sufficient conditions for the well-posedness, which in some situations are close to necessary ones. On the other hand, by our opinion there is a strong relation between the perturbed problem $(\mathcal{P}_\theta)$ and some minimal time problem with a regular but nonconstant dynamics (the respective research is out of our Thesis and we plan to occupy it in the nearest future).

The most of the results obtained in Thesis are new and were presented on various national and international scientific meetings and seminars. Among them let us indicate the following:

1. "Functional Analysis and Optimization", Bedlewo (Poland), 16 – 21 September, 2007;
2. "4^{th} Joint Meeting of CEOC/CIMA-UE on Optimization and Optimal Control", Aveiro (Portugal), 7 – 8 December, 2008;
3. "Workshop on Control, Nonsmooth Analysis and Optimization", Porto (Portugal), 4 – 8 May, 2009;
4. "51^{st} Workshop of the International School of Mathematics "Guido Stampacchia" on Variational Analysis and Applications", Erice (Italy), 9 – 17 May, 2009;
5. Seminar of the Mathematical Department, Università degli studi di Padova, Padua (Italy), 12 October, 2009, invited by Prof. G. Colombo;
6. Seminar of the Mathematical Department, Università di Milano Bicocca, Milan (Italy), 19 October, 2009, invited by Prof. A. Cellina.

The essential part of the work is contained in the paper "V. V. Goncharov and F. F. Pereira, Neighbourhood retractions of nonconvex sets in a Hilbert space via sublinear functionals" recently accepted for publication in the Journal of Convex Analysis. The other Thesis' results instead will be included into two papers, which are now in preparation (see [54, 68]).

The work is written on 133 pages. The bibliography consists of 84 items.

Key words: minimal time problem; metric projection; strict convexity; curvature; duality mapping; uniform smoothness; proximal, Fréchet, limiting and Clarke subdifferentials; normal cones; Hölder continuity; Hamilton-Jacobi equation; viscosity solutions.

Mathematical Subject Classification (2000): 49J52, 49N15.

Resumo alargado

Sejam H um espaço de Hilbert, $F \subset H$ um conjunto fechado convexo limitado que contem a origem no seu interior, e $C \subset H$ não vazio e fechado. O problema principal considerado na Tese é atingir o conjunto (alvo) C em tempo mínimo a partir de um ponto x próximo de C através de trajectórias do sistema de controlo $\dot{x} = u$, $u \in F$. Denotando por $\pi_C^F(x)$ o conjunto de todos os pontos finais de trajectórias deste problema (chamado projecção tempo-mínimo), estudamos primeiro as condições, sobre as quais a aplicação $x \mapsto \pi_C^F(x)$ é de valor singular e contínua nalguma vizinhança de C. Estas condições tem carácter geométrico e envolvem alguns conceitos da Análise Convexa, que foram introduzidos e estudados no primeiro capítulo, tais como rotundidade, suavidade uniforme, curvatura, aplicação dualizante. Uma das hipóteses que garante a boa posição da projecção tempo-mínimo é completamente nova, enquanto a outra é uma generalização abrangente das condições conhecidas para o caso $F = \overline{\mathbf{B}}$ ($\overline{\mathbf{B}}$ representa a bola unitária fechada em H).

O passo seguinte na nossa investigação é a regularidade mais forte da aplicação $x \mapsto \pi_C^F(x)$. Supondo ainda suavidade do conjunto-alvo ou da dinâmica F podemos estabelecer primeiro a regularidade local (de Hölder) para $\pi_C^F(\cdot)$ numa vizinhança de um ponto fixado na fronteira $x_0 \in C$, e depois num conjunto aberto em torno do alvo. É provado que a boa posição e a regularidade da projecção tempo-mínimo estão estritamente relacionadas com as propriedades de regularidade da função valor $x \mapsto \mathfrak{T}_C^F(x)$ (chamada função de tempo mínimo). Em particular, obtemos alguns resultados relativos à diferenciabilidade continua (de Hölder) para $\mathfrak{T}_C^F(\cdot)$ próximo (mas fora) do alvo.

Na última parte do trabalho aplicamos a mesma técnica para o problema de optimização perturbado $(\mathcal{P}_\theta)$ adicionando uma função Lipschitzeana $\theta(\cdot)$ que verifica alguma condição de controlabilidade. Denotando por $\pi_C^{F,\theta}(\cdot)$ e $u(\cdot)$ o conjunto dos minimizantes do problema acima e a função valor, respectivamente, obtemos resultados que justificam a ligação entre a diferenciabilidade contínua da aplicação $u(\cdot)$ com a existência, unicidade e estabilidade dos minimizantes em $\pi_C^{F,\theta}(\cdot)$. Analogamente ao caso $\theta \equiv 0$ supondo algumas hipóteses naturais é provada a continuidade (de Hölder) da aplicação unívoca $\pi_C^{F,\theta}(\cdot)$ e do gradiente $\nabla u(\cdot)$ próximo (mas fora) do conjunto C. Estas hipóteses envolvem regularidade dos três elementos C, F, $\theta(\cdot)$ assim como uma sua compatibilidade geométrica. Os últimos resultados são especialmente importantes porque a função valor $u(\cdot)$ é a solução viscosa de uma certa equação de Hamilton-Jacobi com dados na fronteira Lipschitzeanos não afins, e torna-se a solução clássica (pelo menos próximo da fronteira) sempre que as nossas condições se verificam.

Na Tese usamos a Geometria dos espaços de Hilbert assim como métodos de Análise Convexa, Não-suave e Variacional. Os resultados obtidos desenvolvem e generalizam os conhecidos até ao momento, e no caso particular $F = \overline{\mathbf{B}}$ reduzem-se às propriedades respectivas para a função distância e para a projecção métrica. Contudo, nós consideramos dinâmicas F muito mais gerais (assimétricas, não necessariamente convexas ou suaves) e condições suficientes para a boa posição muito abrangentes, que em certas situações são próximas das necessárias. Por outro lado, na nossa opinião, existe uma forte relação entre o problema perturbado $(\mathcal{P}_\theta)$ e algum problema de tempo mínimo com uma dinâmica regular mas não constante (a respectiva pesquisa não se encontra na Tese e pretendemos ocupar-nos disso no futuro próximo).

A maior parte dos resultados obtidos na Tese são novos e foram apresentados em vários encontros científicos nacionais e internacionais e em seminários. Entre eles indicamos os seguintes:

1. "Functional Analysis and Optimization", Bedlewo (Polónia), 16 – 21 Setembro, 2007;

2. "4^{th} Joint Meeting of CEOC/CIMA-UE on Optimization and Optimal Control", Aveiro (Portugal), 7 – 8 Dezembro, 2008;

3. "Workshop on Control, Nonsmooth Analysis and Optimization", Porto (Portugal), 4 – 8 Maio, 2009;

4. "51^{st} Workshop of the International School of Mathematics "Guido Stampacchia" on Variational Analysis and Applications", Erice (Itália), 9 – 17 Maio, 2009;

5. Seminário do Departamento de Matemática, Università degli studi di Padova, Pádua (Itália), 12 Outubro, 2009, convidada por Prof. G. Colombo;

6. Seminário do Departamento de Matemática, Università di Milano Bicocca, Milão (Itália), 19 Outubro, 2009, convidada por Prof. A. Cellina.

A parte essencial do trabalho está contida no artigo "V. V. Goncharov e F. F. Pereira, Neighbourhood retractions of nonconvex sets in a Hilbert space via sublinear functionals" que foi recentemente aceite para publicação no Journal of Convex Analysis. Os outros resultados da Tese serão incluidos em dois artigos que estão em preparação (ver [54, 68]).

O trabalho está escrito em 133 páginas. A bibliografia consiste de 84 itens.

Palavras chave: problema de tempo mínimo; convexidade estrita; aplicação dualizante; suavidade uniforme; subdiferenciais proximal, Fréchet, limite e Clarke; cones normais; continuidade de Hölder; equação de Hamilton-Jacobi; soluções viscosas.

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Contents

Introduction

The question is: for a point x in some space H there will be a unique $\overline{x}$ in a fixed $C \subset H$ which is more close to x? And how this $\overline{x}$ will depend on x?

The goal of the Thesis is to search the answer to this question specifying H and C, and giving a certain sense to the proximity. Let us first give a little of history.

We start with the case of a (real) normed space $(H, \|\cdot\|)$ and a linear subspace $C \subset H$. For any $x \in H$ the (possibly empty) set of *best approximations* (or *nearest points*) to x in C is defined by

$$\pi_C(x) := \{y \in C : \|x - y\| = \mathrm{d}_C(x)\},$$

where $\mathrm{d}_C(x) := \inf\{\|x - y\| : y \in C\}$, $x \in H$, is the *distance function* associated with C, and the set valued mapping $\pi_C : H \to C$ is called the *metric projection* onto C. The subspace C is said to be *proximal* (respectively, *Chebyshev*) if $\pi_C(x)$ is nonempty (respectively, is a singleton) for each $x \in H$. The last concept was introduced by S. B. Stechkin (see [48, 49]) in honour of the founder of the best approximation theory P. L. Chebyshev.

In 1859 P. L. Chebyshev showed that (in actual terminology) in the space $H = C([0, 1])$ of continuous functions on $[0, 1]$ the subspace $C = P_n$ of polynomials of degree non greater than n is a Chebyshev set. He also considered the set P_{nm} of rational functions $P_n(\cdot)/Q_m(\cdot)$, such that $Q_m(\cdot)$ does not have roots in $[0, 1]$, and proved that it is Chebyshev too (we refer to the papers [78, 77] for review of the classic P. L. Chebyshev's works).

Further on, the best approximation results were generalized to an arbitrary nonempty set $C \subset H$. The definitions of the distance and the projection remain the same.

A Chebyshev set is necessarily closed. On the other hand, it is a long-staying problem whether a Chebyshev set must be convex. In finite dimensions the answer is positive but in infinite-dimensional setting the problem is still open. However, as was conjectured first by V. Klee and proved then by E. Asplund [1], if a Hilbert space contains a nonconvex Chebyshev set then it contains also one whose complement is bounded and convex (so called *Klee cavern*).

Another problem, in some sense symmetric to the problem above, involves the farthest points. Namely, given a point x in a normed space H the elements of the (possible empty) set

$$\chi_C(x) := \left\{y \in C : \|x - y\| = \sup_{z \in C} \|z - x\|\right\}$$

are called *farthest points* from x in a bounded subset C. The set C is said to be *remotal* if $\chi_C(x) \neq \emptyset$ for every $x \in H$ and *uniquely remotal* if $\chi_C(x)$ is a singleton for every $x \in H$. An important question is: under what conditions on the space H does every uniquely remotal subset of H reduce to a single point? In other words, in a "good" space H, if C is not a singleton then it must have a "central" point which admits at least two farthest points in C.

This problem is strictly related with the problem of convexity of Chebyshev sets, and it is not also resolved till now. There are some opinions (see, e.g., [59, 25]) that the solution of one of them leads to the solution of the other.

Returning to the nearest points, we are interested to find necessary and suficient conditions for their existence and uniqueness. Many sufficient conditions for a closed set to be Chebyshev and also for a Chebyshev set to be convex have been obtained. We pay the main attention for the first question, while for the second we refer to [50, 59, 1, 3, 79, 24, 81, etc.], giving here some brief comments only.

In 1961 V. Klee [59] found the first condition in infinite dimensions guaranteeing convexity of a Chebyshev set. Namely, he showed that in a Banach space which is both uniformly smooth and uniformly strictly convex, every weakly closed Chebyshev set is convex. Thus, in such a space a set is closed and convex if and only if it is a weakly closed Chebyshev set.

In the same year N. V. Efimov and S. B. Stechkin (see [50]) proved that in an uniformly convex and smooth Banach space each Chebyshev *approximatively compact*[1] set is convex. Applying their criteria, they, in particular, established that the set P_{nm} of rational functions (see above) does not satisfy the Chebyshev property in the space $L^p[0,1]$, $p > 1$.

It was also V. Klee who proved in [59] that in a smooth reflexive Banach space a Chebyshev set is convex if the associated metric projection is both continuous and weakly continuous. Later E. Asplund in [1] essentially weakened the Klee theorem for Hilbert spaces showing that if a metric projection onto a Chebyshev set is (norm to norm) continuous at all points then the set is convex. Further, the last result was refined in [3, 79] where the authors proved that for the convexity of a Chebyshev set it is enough that the set of discontinuity points of the metric projection is at most countable. Finally, the papers [24, 81] contain another type of conditions for the convexity of a Chebyshev set involving differentiability of the distance function.

As we already said, in $\mathbb{R}^n$ the classes of Chebyshev and convex closed sets coincide (see, e.g., [78, 8]). Observe that in order to prove the existence of projection in this case it is not necessary to assume convexity (which is needed, however, for uniqueness). Furthermore, J. M. Borwein and S. Fitzpatrick proved in [11] that every nonempty closed subset of a Banach space H is proximal if and only if H is finite-dimensional. Thus, in infinite dimensions the convexity is important for the existence as well. In fact, it is sufficient for the proximality in each reflexive Banach space (see [11]). Later some generic results were obtained in the lack of convexity. For instance, M. Edelstein proved in [46] that in an arbitrary uniformly convex Banach space H for each nonempty closed subset $C \subset H$ the mapping $\pi_C : H \to C$ is well defined and single-valued

[1] A set C is said to be *approximatively compact* if any sequence $y_n \in C$ with $\|x - y_n\| \to d_C(x)$ admits a subsequence converging to an element of C.

the outside curvature of the set, while the others take a sense in more general setting. Speaking about the curvature we mean the following: for each $x \in \partial C$ and each normal vector v to C at x there exists a sphere with radius $1/(2\varphi(x))$ centred in the half-line $x + \lambda v$, $\lambda \geq 0$, which touches C at the point x only. Thus, this is a local property (some kind of an external sphere condition) but as we will see later it can be treated as a global one.

The class of φ-convex sets includes all convex sets and the sets with $C^{1,1}_{loc}$-boundary, i.e., those having the form $\{x \in H : g(x) \leq 0\}$, where the function $g(\cdot)$ is continuously differentiable with locally Lipschitzean gradient (we rigorously prove this fact in sequel). Moreover, if for each $x_0 \in \partial C$ there exists a neighbourhood $\mathcal{U}(x_0)$ such that either $C \cap \mathcal{U}(x_0)$ is convex or $\partial C \cap \mathcal{U}(x_0)$ is of class $C^{1,1}_{loc}$ then C is φ-convex as well. As a simple example of a φ-convex set, which does not satisfy any of these two properties (even locally) we can consider the set $\left\{x \in \mathbb{R}^n : \max|x_i| \leq 1,\ \Sigma x_i^2 \geq 1\right\}$. For a less trivial example of a φ-convex set (in infinite dimensions) we refer to [16, Theorem 1.8 and Proposition 1.9].

Let us recall some important results regarding φ-convexity. In [16] it was proved that the metric projection onto a φ-convex set C is a single-valued mapping defined and locally Lipschitzean near C. This result was obtained earlier by H. Federer (see [53]) but only for $C \subset \mathbb{R}^n$, while F. Clarke, R. Stern and P. Wolenski considered in [24] its infinite-dimensional uniform version. The last authors showed also that the distance function $\mathrm{d}_C(\cdot)$ is of class $C^{1,1}_{loc}$ in a neighbourhood of C (assuming C to be φ-convex with a constant $\varphi(\cdot)$). Later the complete characterization of φ-convexity was given (see [71, 27]). In particular, it was proved that C is φ-convex if and only if there exists an open set $\mathcal{U} \supset C$ such that each $x \in \mathcal{U}$ has a unique metric projection $\pi_C(x)$, and the mapping $x \mapsto \pi_C(x)$ is continuous in $\mathcal{U}$. Moreover, this is equivalent to the continuous differentiability of the distance function $\mathrm{d}_C(\cdot)$ on the set $\mathcal{U} \backslash C$. In this case the Fréchet derivative $\nabla \mathrm{d}_C(\cdot)$ is given by the formula

$$\nabla \mathrm{d}_C(x) = \frac{x - \pi_C(x)}{\mathrm{d}_C(x)}, \quad x \in \mathcal{U} \backslash C, \tag{1}$$

being Lipschitz continuous in a neighbourhood of each point $x \in \mathcal{U} \backslash C$ with the Lipschitz constant tending to infinity as x goes to the boundary $\partial \mathcal{U}$. This is the reason why φ-convex sets are said to be also proximaly smooth.

Let us observe that the distance function and the metric projection can be interpreted in another way. In fact, for $C \subset H$ and $x \in H$ the distance $\mathrm{d}_C(x)$ is nothing else than the minimum time necessary to reach the set C starting from the point x by trajectories of the control system

$$\dot{x}(t) = v(t), \quad \|v(t)\| \leq 1, \tag{2}$$

while the projection $\pi_C(x)$, for x not in C, is the set of all points in ∂C, attainable from x for the minimal time. Slightly extending this problem we can consider in the place of the closed unit ball in (2) (denoted further by $\overline{\mathbf{B}}$) an arbitrary closed convex bounded set $F \subset H$, containing the origin in its interior (we need the last condition in order to guarantee controllability). So that, given a point $x \in H$ we are led to study the following time optimal control problem with constant dynamics:

$$\min \left\{ T > 0 : \exists x(\cdot), \, x(T) \in C, \;\; x(0) = x, \text{ and } \dot{x}(t) \in F \;\; \text{a.e. in } [0, T] \right\}. \tag{3}$$

Taking into account this interpretation, we refer to the sets F and C as the *dynamics* and the *target set*, respectively. The value function in this problem (denoted further by $\mathfrak{T}_C^F(\cdot)$ and called the *minimal time function*) is the suitable substitute of $\mathrm{d}_C(\cdot)$. While the set of the terminal points $x(T)$ for all functions $x(\cdot)$, which are minimizers in (3), called further the *time-minimum projection* of x onto C (with respect to F) and denoted by $\pi_C^F(x)$, generalizes the metric projection $\pi_C(x)$. We keep the same name and notation for the unique element of $\pi_C^F(x)$ in the case when it is a singleton. Since each terminal point can be achieved by an affine trajectory (due to convexity of F), $\mathfrak{T}_C^F(\cdot)$ can be also given as

$$\mathfrak{T}_C^F(x) = \inf \{ t > 0 : C \cap (x + tF) \neq \emptyset \},$$

or, in other words,

$$\mathfrak{T}_C^F(x) = \inf_{y \in C} \rho_F(y - x),$$

where $\rho_F(\cdot)$ is the *Minkowski functional* (or *gauge function*) of the set F,

$$\rho_F(\xi) := \inf \{ \lambda > 0 : \xi \in \lambda F \}.$$

Clearly, if $F = \overline{\mathbf{B}}$ then $\rho_F(x)$, $\mathfrak{T}_C^F(x)$ and $\pi_C^F(x)$ are reduced to the usual norm $\|x\|$, to the distance $\mathrm{d}_C(x)$ and to the metric projection $\pi_C(x)$ of x onto C, respectively. Observe that in general we do not suppose the set F to be either symmetric or smooth or strictly convex unlike this particular case. The generic properties established by M. Edelstein [46] and I. Ekeland [51] for the metric projection were subsequently generalized in [43, 19, 20] to the function $\pi_C^F(\cdot)$ with an arbitrary dynamics F, even in Banach spaces.

Some conditions guaranteeing the well-posedness of the time-minimum projection (i.e., existence, uniqueness and continuity of the mapping $x \mapsto \pi_C^F(x)$) near the target were obtained in [31], which turned out to be appropriate for the regularity of the value function $\mathfrak{T}_C^F(\cdot)$ as well. These conditions combine φ-convexity of the target set C (with $\varphi = \text{const}$) and some type of uniform strict convexity of F controllable with a parameter $\gamma > 0$. Then a neighbourhood of C, where the well-posedness takes place, is given by some relation between φ and γ. However, these hypotheses are not so sharp as for the usual metric projection and can be essentially refined. The first part of the present work is devoted to this question.

On the other hand, in [20] a relationship between the existence of time-minimum projection and the directional derivatives of the minimal time function was proposed. Namely, under suitable suplementary conditions on the dynamics (including a kind of uniform convexity) it was proved that $x \in H \backslash C$ admits a unique time-minimum projection onto C if and only if $\mathbf{D}\mathfrak{T}_C^F(x)(v) = 1$ for some $v \in \partial F$. Here and further on we denote by $\mathbf{D}f(x)(v)$ the *directional* *derivative* of the function $f : H \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$, $\operatorname{dom} f := \{x \in H : f(x) < +\infty\}$, with respect to (w.r.t.) the vector $v \in H$, i.e.,

$$\mathbf{D}f(x)(v) := \lim_{\lambda \to 0+} \frac{f(x + \lambda v) - f(x)}{\lambda}. \tag{4}$$

It is interesting to observe that the "symmetric" property ($\mathbf{D}\mathfrak{T}_C^F(x)(v) = -1$ for some $v \in -\partial F$) is equivalent to the existence of at least one projection.

There are many papers devoted to the study of the subdifferentials of the distance function $\mathrm{d}_C(\cdot)$. For example, in [12, 15] the authors give explicit formulas for the Clarke subdifferential of $\mathrm{d}_C(\cdot)$ under various hypotheses on the closed (not necessarily convex) set $C \subset H$ and on the normed space H. These formulas link the Clarke subdifferential either with the Clarke normal cone to C at the point x (in the case $x \in \partial C$), or with the respective normal cone to the sublevel set $\{y \in H : \mathrm{d}_C(y) \leq \mathrm{d}_C(x)\}$ (if $x \in H \backslash C$), both generalizing the well-known relationships in the case of convex C. The similar relationship for the proximal subdifferential of $\mathrm{d}_C(x)$ in Hilbert spaces and x does not belonging to C was obtained in [24], while in [13] the proximal and the Fréchet subdifferentials (for both cases $x \in \partial C$ and $x \notin C$) in a normed space were considered. Further, the formulas for the various (Clarke, Fréchet and proximal) subdifferentials of $\mathrm{d}_C(\cdot)$ were generalized to an arbitrary dynamics F and to the respective minimal time function $\mathfrak{T}_C^F(\cdot)$ (see [80, 29] for $\mathbb{R}^n$ and [30, 31] for an infinite-dimensional Hilbert space). Recently, the case of a Banach space, or even of a normed space without completeness, was treated (see [83, 82]).

Let us pass now to once more interpretation of the distance function (and of the minimal time function as well), which comes from partial differential equations. We start by considering the general first order equation in finite dimensions

$$\Gamma(x, u(x), \nabla u(x)) = 0, \tag{5}$$

$x \in \Omega$, with the boundary data

$$u(x) = \theta(x), \quad x \in \partial\Omega, \tag{6}$$

where $\Gamma : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\theta : \overline{\Omega} \to \mathbb{R}$ are given functions, $\Omega \subset \mathbb{R}^n$ is an open bounded set with the closure $\overline{\Omega}$, and $\nabla u(x)$ means the gradient of $u(\cdot)$ at x. A function $u : \overline{\Omega} \to \mathbb{R}^n$ of class $C^1(\overline{\Omega})$ satisfying both (5) for all $x \in \Omega$ and (6) is said to be a *classical solution* of the boundary value problem (5)-(6). As simple examples show this problem may not admit any classical solution (for instance, the equation $|\nabla u(x)| = 1$ with $u(x) = 0$ on the boundary never has such solution). So that we are led to another weaker concept. Namely, we say that a Lipschitz continuous function $u : \overline{\Omega} \to \mathbb{R}^n$ is a *generalized solution* of the equation (5) if the relation (5) holds true for almost each (a.e.) $x \in \Omega$ (observe that the gradient $\nabla u(\cdot)$ exists almost everywhere by the *Rademacher's theorem* [52, p. 81]). But in this way we lose uniqueness of solution in most of the cases. So that one needs an intermediate definition.

Observe that (5) is the stationary version of the (time-dependent) Hamilton-Jacobi-Bellman equation

$$u_t + \widehat{\Gamma}(t, x, u, \nabla u) = 0, \tag{7}$$

appearing as a necessary condition of optimality in an optimal control problem. The first attempts to define a class of solutions to (7), where existence and uniqueness (and may be some regularity w.r.t. the initial data) take place, are due to O. A. Oleinik [68] and A. Douglis [45] in the scalar case. Their definitions were based on some kind of "semi-decreasing" property of solution, which is usefull in the study of nonlinear conservation laws. Later S. N. Kružkov studied the stationary equation (5) (see [61] and the bibliography therein) motivating his research by the problem arising from the geometrical optics. In particular, when $n = 3$ and $\Gamma(x, u, p) = |p| - a$, with a constant $a > 0$, one has the so-called *eikonal equation* describing the propagation of a light wave from a point source placed at the origin in a homogeneous medium with refraction index $1/a$. If, instead, this medium is anisotropic and has constant coefficients of refraction of light rays parallel to the coordinate axes (say c_i) then the propagation of light can be described by the (more general) eliptic equation

$$\sum_{i=1}^{3} c_i^2 u_{x_i}^2 - 1 = 0. \tag{8}$$

If, besides that, the medium moves with a constant velocity $w = (w_1, w_2, w_3)$ then the equation (8) contains a linear additive term and admits the form:

$$\sum_{i=1}^{3} c_i^2 u_{x_i}^2 + \frac{2}{c} \langle w, u_x \rangle - 1 = 0,$$

where c is the speed of light in a vacuum. Extending more, one gets the Hamilton-Jacobi equation (5) with the hamiltonian Γ in $\mathbb{R}^n$ do not depending of x and u and convex w.r.t. the third variable. For such type of equations (and with some dependence on x and u as well) S. N. Kružkov looked for solutions in the class $E(\Omega)$ of locally Lipschitzean functions with a supplementary property involving the uniform boundedness of its second order finite differences. In such a way S. N. Kružkov proved existence and uniqueness of solution in $E(\Omega)$ and its stability w.r.t. the so-called viscosity approximations. Observe that he was the first who related well--posedness of a solution with the "vanishing viscosity", proving that each solution $u(\cdot) \in E(\Omega)$ ($\Omega \subset \mathbb{R}^n$ is a bounded domain) is the uniform limit of the sequence of solutions $u^\varepsilon(\cdot)$ of the respective problems for the nonlinear elliptic equations[3]

$$\Gamma(x, u, \nabla u) - \varepsilon \Delta u = 0, \tag{9}$$

as $\varepsilon \to 0+$ (notice that the equation (9) has a unique classical solution for each $\varepsilon > 0$ small enough in accordance with Theorem 3.2 [61]). This construction itself can be admitted as the definition of solution to the boundary value problem and it was motivated by the method of "vanishing viscosity" in fluid mechanics.

Much later M. Crandall and P.-L. Lions (see [34]), considering the general problem (5)-(6) without convexity assumptions and basing on the same idea of "vanishing viscosity", introduced

[3] Δ is the Laplace operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

a new notion of solution, called *viscosity solution.* In [34, 32] also other equivalent definitions appeared. The exact definitions of viscosity solution will be given in Chapter 4, while now let us mention that two of them use the suitable test functions (similarly as the notion of the generalized solutions of linear PDE in the sense of distributions), and the other involves a generalization of the gradient of a continuous function at the points of nondifferentiability. Notice that so introduced solutions need not to be differentiable anywhere (they are supposed to be continuous only). Nevertheless, as follows directly from the definitions, a function $u(\cdot) \in \mathcal{C}^1(\overline{\Omega})$ is a classical solution of the problem (5)-(6) if and only if it is a solution in the viscosity sense.

At present, Theory of Viscosity Solutions is a very developed and powerful field of the modern mathematics having numerous applications in partial differential equations as well as in control theory, differential games and so on. The fundamental results in this theory besides its creators M. Crandall and P.-L. Lions were obtained by such mathematicians as L. Evans, H. Ishii, G. Barles, M. Bardi, I. Capuzzo-Dolcetta and many others (see, e.g., [32, 6, 38, 33, 63, 7, 5]). The large bibliography concerning this theory can be found in the last three books.

To summarize everything said above about viscosity solutions in finite-dimensional spaces we refer to the excellent tutorial lessons by A. Bressan [14] where the following properties were emphasized ($\mathfrak{W}(\Omega)$ denotes here the family of viscosity solutions):

(i) for each suitable boundary data $\theta(\cdot)$ a unique solution $u(\cdot) \in \mathfrak{W}(\Omega)$ of the problem (5)-(6) exists, and it is stable with respect to both $\theta(\cdot)$ and $\Gamma(\cdot)$;

(ii) the solution $u(\cdot) \in \mathfrak{W}(\Omega)$ is also stable with respect to the "vanishing viscosity" approximations. Namely, denoting by $u^{\varepsilon}(\cdot)$ the (unique) solution of the equation (9) one has $u^{\varepsilon}(x) \to u(x)$ as $\varepsilon \to 0+$ uniformly in $x \in \Omega$;

(iii) whenever (5) is the Hamilton-Jacobi equation for the value function in some optimization problem, the unique viscosity solution $u(\cdot) \in \mathfrak{W}(\Omega)$ should coincide exactly with that value function (see, e.g., [7]).

Afterwards, the concept and the main results concerning with the viscosity solutions were generalized to some classes of infinite-dimensional Banach spaces (see, e.g., [35, 36, 37, 7]). Notice that passing from finite to infinite dimensions one meets three main difficulties. First, in order to prove uniqueness of a viscosity solution in finite-dimensional setting, one essentially uses the fact that continuous functions attain their maximal and minimal values on a closed ball, which is false in infinite dimensions. However, in [35] the authors have proposed another way to do this basing on the *Radon-Nikodym property* (briefly, (RNP)), which is equivalent to attainability of maxima and minima for arbitrarily small linear perturbations of continuous functions (see [36] and bibliography therein). In what follows we have this property because H is always supposed to be Hilbert, and Hilbert spaces (and even reflexive ones) possess the (RNP) (see [44, p. 100]). Let us only mention here that the case of Banach spaces without (RNP) was treated in [37]. However, in this case an (alternative) coercivity condition for the mapping $\Gamma(\cdot)$ should be posed. Next, in infinite dimensions the property **(ii)** above has no sense by the simple reason that the Laplace operator is not defined. So that we can not use more the "vanishing viscosity" argument to motivate the necessity of introduction of such type of solutions. The motivation, however, comes now from the Theory of Differential Games. The third dificulty appearing in infinite dimensions is concerned with the Arzelá-Ascoli theorem which is no longer applicable in this case. So, one needs an alternative Convergence Theorem proved in [36].

Let us return now to the eikonal equation (with $\Gamma(x,u,p) = \|p\|$). It follows from (1) that the distance function $\mathrm{d}_C(\cdot)$ with $C = H \backslash \Omega$ is a generalized solution of the problem

$$\|\nabla u(x)\| - 1 = 0$$

with the boundary condition

$$u(x) = 0, \quad x \in \partial\Omega.$$

Moreover, it turns out that the distance is exactly the unique viscosity solution.

The existence of viscosity solution to (5)-(6), when $\Gamma(\cdot)$ is an arbitrary (continuous) function of the gradient, was investigated, e.g., in [18, 39], in the finite-dimensional case. Denoting by F the closed convex hull of the set of zeros $\{\xi \in H : \Gamma(\xi) = 0\}$ the authors of the first paper reduced (5)-(6) to the following specific boundary value problem

$$\begin{cases} \rho_{F^o}(-\nabla u(x)) - 1 = 0 & \text{if } x \in \Omega \\ u(x) = \theta(x) & \text{if } x \in \partial\Omega, \end{cases} \tag{10}$$

where F^o means the polar set. They proved that under appropriate conditions involving a kind of geometric compatibility of F, $\theta(\cdot)$ and the domain Ω the (unique) viscosity solution of (5)-(6) exists and coincides with the viscosity solution of (10). Moreover, this solution can be given by the formula

$$u(x) = \inf_{y \in \partial\Omega} \{\rho_F(y - x) + \theta(y)\}. \tag{11}$$

Notice that already S. N. Kružkov (see [61]) considered the function (11) as the candidate for solutions to eikonal equation or to its generalizations belonging to the class $E(\Omega)$ introduced by him. We postpone the direct proof of this fact and the study of the function (11) to the Chapter 4, while now let us observe that the viscosity solution of (10) (consequently, of (5)-(6)) is the further generalization of the minimal time function $\mathfrak{T}_C^F(\cdot)$, being reduced to this function whenever $\theta \equiv 0$. Probably, M. Bardi (see [4]) was the first who characterized the minimal time function as the unique solution to a Hamilton-Jacobi equation using viscosity methods.

In the second part of Thesis we are interested in attainability of the infimum in (11), in uniqueness of the minimum point as well as in the regularity of the function $u(\cdot)$. As in the case of the time-minimum projections we will see that these three questions are linked each with other. Furthermore, the answers for them permit to make the conclusions about resolvability of the problem in the classical sense. On the other hand, due to the dynamic programming principle the gradient equation in (10) is the Hamilton-Jacobi equation for the minimal time problem. Therefore, the study of such equations (with an arbitrary enough regular boundary function $\theta(\cdot)$) has also an independent interest due to possible applications to Optimal Control or to Differential Games Theory.

Besides this Introduction, Thesis consists of four chapters, conclusion and bibliography comments. Throughout the whole work we assume H to be a Hilbert space, $F \subset H$ (dynamics) to be a closed convex bounded set with the origin in its interior, and $C \subset H$ (the target set) to be nonempty and closed. Our main goal is to obtain local conditions on F and C, which would guarantee the existence of a neighbourhood of C, where the time-minimum projection is well-posed (i.e., continuous as a single-valued mapping), providing, furthermore, Lipschitz (or, in more general, Hölder) regularity of the time-minimum projection $\pi_C^F(\cdot)$ and the differentiability of the value function $\mathfrak{T}_C^F(\cdot)$. Moreover, we generalize these conditions for the respective problem with non linear perturbation $\theta(\cdot)$.

The Chapter 1 is an auxiliary one, having nevertheless an independent interest. Our purpose here is to bring together various concepts concerning the geometric structure of convex solids in a Hilbert space, to study quantitatively their dual properties such as rotundity and smoothness, and to put the introduced numerical characteristics into general settings of Convex Analysis.

In the Chapter 2 we present two types of geometric conditions on both F and C, which guarantee existence and uniqueness of the time-minimum projection locally (i.e., in a neighbourhood of the target), and the continuity of the mapping $\pi_C^F(\cdot)$ as well. One of the main tools used for proving of our theorems is the Ekeland's variational principle (see [51, Corollary 11]), which enables to establish some regularity property of minimizing sequences in the respective problems. Besides that, we strongly use the fuzzy calculus of the proximal subdifferentials of lower semicontinuous functions, which permits to prove the well-posedness theorems in the most general setting. Any way we give the exact formulations of these fundamental principles of Analysis in the preliminaries. Under these conditions we prove first a local retraction theorem (Theorem 2.2.1). In the case of φ-convex target this result leads then to the explicit formula for the neighbourhood of C where the retraction is defined (or, in other words, where the well--posedness holds). In the last section we concretize the obtained results for the case when either ∂C is smooth (theorems 2.3.1-2.3.3) or ∂F^o is of class $\mathcal{C}^2$ (Theorem 2.3.4).

Using the same geometric conditions as in Chapter 2 and the same technique (involving, in particular, the fuzzy sum rule) we show in Chapter 3 that under some natural extra hypotheses the regularity of the time-minimum projection $\pi_C^F(\cdot)$ can be essentially improved. Namely, we prove Lipschitz (or, in more general, Hölder) continuity of $\pi_C^F(\cdot)$ in a neighbourhood of C. Finally, we study differentiability of the value function $\mathfrak{T}_C^F(\cdot)$ near the target and give explicit formulas for its (Fréchet) gradient. Based on these formulas we conclude that $\nabla\mathfrak{T}_C^F(\cdot)$ is locally Lipschitzean (or Hölderian) as well.

In the last chapter we study the problem of minimization of the Minkowski functional with some additive nonlinear perturbation (see (11)), adjusting the existence and uniqueness results of the Chapter 2 for this case. Under appropriate assumptions we obtain also a local regularity result for the corresponding mapping $\pi_C^{F,\theta}(\cdot)$, which associates to each $x \in H$ the (unique) point where the minimum in (11) is attained. We prove, in fact, that $x \mapsto \pi_C^{F,\theta}(x)$ is Lipschitz continuous near C, deriving then the classic differentiability of the viscosity solution $u(\cdot)$.

In comments we give some remarks regarding to the place of our work among other investigations in this area. We clearify various intersections with the results known in the literature and compare our hypotheses with the known ones. We give also a more detailed and more concrete (related more to our particular problems) historical sketch than in Introduction.

Chapter 1

Rotundity and smoothness in a Hilbert space

This chapter is devoted to some notions of Convex Analysis that will be used in sequel. First of all we give in Section 1.1 the basic definitions and some notations concerning with the geometry of convex solids, which is the main technique throughout the whole Thesis. Further, in Section 1.2 we introduce some moduli of local rotundity for the convex set F that seem to be more suitable for our objectives. They are inspired essentially by the geometry of Banach spaces (see, e.g., [65]) and adapted here for the case of "asymmetric norms". By using of one of these moduli we define then the concept of strict convexity graduated by some parameter $\alpha > 0$ and associated with a dual pair of vectors (ξ, ξ^*). The main numerical characteristics resulting from these considerations is the curvature (and the respective curvature radius), which shows how rotund the set F is near a fixed boundary point watching along a given direction. The Section 1.3 is devoted to the dual notions. Namely, considering the polar set F^o we define the so-called modulus of smoothness of F^o and local smoothness (also associated with a dual pair, in this case (ξ^*, ξ)). In particular, we prove here the local asymmetric version of the Lindenstrauss duality theorem quantitatively establishing the duality between local smoothness and local rotundity. Thus, the curvature of F can be considered also as a numerical characteristics of the polar set F^o, showing how sleek F^o is in a neighbourhood of a boundary point ξ^* if one watchs along a direction ξ. Applying this theorem, we obtain a characterization of the curvature of F in terms of the second derivative of the dual Minkowski functional. In the last Section 1.4 we give some examples.

1.1 Basic notations and definitions

Let us consider a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, a closed convex bounded set $F \subset H$ such that $0 \in \operatorname{int} F$ ("int" stands for the *interior* of F), and denote by F^o its *polar set*, i.e.,

$$F^o := \{\xi^* \in H : \langle \xi, \xi^* \rangle \leq 1 \ \ \forall \xi \in F\}.$$

Together with the *Minkowski functional* $\rho_F(\cdot)$ defined by

$$\rho_F(\xi) := \inf\{\lambda > 0 : \xi \in \lambda F\}, \quad \xi \in H,$$

we consider the *support function* $\sigma_F : H \to \mathbb{R}^+$, $\sigma_F(\xi^*) := \sup\{\langle \xi, \xi^* \rangle : \xi \in F\}$, and observe that (see [31, Proposition 2.1])

$$\rho_F(\xi) = \sigma_{F^o}(\xi), \quad \xi \in H, \tag{1.1}$$

and, consequently,

$$\frac{1}{\|F\|}\|\xi\| \le \rho_F(\xi) \le \|F^o\|\,\|\xi\|, \quad \xi \in H, \tag{1.2}$$

where $\|F\| := \sup\{\|\xi\| : \xi \in F\}$. The inequalities (1.2) mean that $\rho_F(\cdot)$ is a sublinear functional "equivalent" to the norm $\|\cdot\|$. It is not a norm since $-F \neq F$ in general. As a consequence of (1.1) and (1.2) we have the Lipschitz property

$$|\rho_F(\xi_1) - \rho_F(\xi_2)| \le \|F^o\|\,\|\xi_1 - \xi_2\| \qquad \forall \xi_1, \xi_2 \in H. \tag{1.3}$$

In what follows we use the so-called *duality mapping* $\mathfrak{J}_F : \partial F^o \to \partial F$ that associates to each $\xi^* \in \partial F^o$ the set of all functionals that support F^o at ξ^* :

$$\mathfrak{J}_F(\xi^*) := \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\}.$$

We say also that (ξ, ξ^*) is a *dual pair* when $\xi^* \in \partial F^o$ and $\xi \in \mathfrak{J}_F(\xi^*)$.

Remark 1.1.1 *Notice that $\mathfrak{J}_F(\xi^*) \neq \emptyset$ for every $\xi^* \in \partial F^o$. Indeed, fixed $\xi^* \in \partial F^o$ let us consider a sequence $\{\xi_n\} \subset F$ such that*

$$1 = \rho_{F^o}(\xi^*) = \sup_{y \in F}\langle \xi^*, y \rangle < \langle \xi^*, \xi_n \rangle + \frac{1}{n}, \quad n \in \mathbb{N}. \tag{1.4}$$

Since F is obviously weakly compact, $\{\xi_n\}$ admits a subsequence converging weakly to some $\xi \in F$. Passing to limit in (1.4) we obtain

$$\langle \xi^*, \xi \rangle \ge 1.$$

On the other hand,

$$1 \le \langle \xi^*, \xi \rangle \le \sup_{y^* \in F^o}\langle y^*, \xi \rangle = \rho_F(\xi) \le 1,$$

which implies that $\rho_F(\xi) = 1$, i.e., $\xi \in \partial F$.

Let us denote by $\mathbf{N}_F(\xi)$ the *normal cone* to F at the point $\xi \in F$ and by $\partial\rho_F(\xi)$ the *subdifferential* of the function $\rho_F(\cdot)$ in the sense of Convex Analysis. Notice that for each $\xi^* \in \partial F^o$ the set $\mathfrak{J}_F(\xi^*)$ is nothing else than $\partial\rho_{F^o}(\xi^*)$, and $\mathbf{N}_F(\xi) \cap \partial F^o$ is the pre-image of the mapping $\mathfrak{J}_F(\cdot)$ calculated at the point $\xi \in \partial F$. Here we use the following result

Remark 1.2.1 *In the above formulas the infimum can be taken for* $\eta \in \partial F$. *Let us prove this for the modulus* $\widehat{\mathfrak{C}}_F(r,\xi,\xi^*)$, *the others are similar. We have always the inequality*

$$\widehat{\mathfrak{C}}_F(r,\xi,\xi^*) \le \inf\{\langle \xi-\eta,\xi^*\rangle : \eta \in \partial F,\ \|\xi-\eta\| \ge r\}. \tag{1.7}$$

Now we suppose that the equality does not hold. Then there exists $\overline{\eta} \in \operatorname{int} F$ *with* $\|\xi-\overline{\eta}\| \ge r$ *such that*

$$\langle \xi-\overline{\eta},\xi^*\rangle < \inf_{\substack{\eta\in\partial F\\ \|\xi-\eta\|\ge r}} \langle \xi-\eta,\xi^*\rangle. \tag{1.8}$$

Then by the separation theorem there exists a line passing through $\overline{\eta}$, *which does not intersect the open ball* $\xi + r\mathbf{B}$. *Since* $\overline{\eta} \in \operatorname{int} F$, *this line meets* ∂F *at exactly two points, let* η_1 *and* η_2. *Consequently, there exists* $\lambda \in [0,1]$ *with* $\overline{\eta} = \lambda\eta_1 + (1-\lambda)\eta_2$. *Hence*

$$\langle \xi-\overline{\eta},\xi^*\rangle = \lambda\langle \xi-\eta_1,\xi^*\rangle + (1-\lambda)\langle \xi-\eta_2,\xi^*\rangle > \langle \xi-\overline{\eta},\xi^*\rangle,$$

that contradicts (1.8).

Remark 1.2.2 *Observe that for all* $r>0$ *the inequality* $\mathfrak{C}_F^{+}(r,\xi,\xi^*) \ge \delta_F(r,\xi)$ *holds. Indeed, for each* $\eta \in F$ *with* $\rho_F(\eta-\xi) \ge r$ *by (1.1) we have*

$$\delta_F(r,\xi) \le 2-\rho_F(\xi+\eta) \le 2\langle \xi,\xi^*\rangle - \langle \xi+\eta,\xi^*\rangle = \langle \xi-\eta,\xi^*\rangle.$$

But the opposite inequality is violated even in the simplest cases. For example, if $F=\overline{\mathbf{B}}$, $\|\xi^*\|=1$ *and* $\xi=\xi^*$ *(*$\mathfrak{J}_F(\xi^*)=\{\xi\}$ *is singleton) then direct calculations give*

$$\delta_F(r,\xi) = \inf\{2-\|\xi+\eta\| :\ \eta\in S,\ \|\xi-\eta\|=r\} = \frac{r^2}{2+\sqrt{4-r^2}},$$

while $\mathfrak{C}_F^{\pm}(r,\xi,\xi^*) = \widehat{\mathfrak{C}}_F(r,\xi,\xi^*) = \frac{r^2}{2}$, $0<r\le 2$.

Due to (1.2) we also have the following inequalities:

$$\mathfrak{C}_F^{\pm}\left(\frac{r}{\|F\|},\xi,\xi^*\right) \le \widehat{\mathfrak{C}}_F(r,\xi,\xi^*) \le \mathfrak{C}_F^{\pm}(\|F^o\|\,r,\xi,\xi^*),\quad r>0. \tag{1.9}$$

The Definition 1.2.2 suggests another concept of strict convexity. Namely, the set F is said to be *strictly convex at the point* $\xi\in\partial F$ *w.r.t.* $\xi^*\in\mathfrak{J}_F^{-1}(\xi)$ if $\widehat{\mathfrak{C}}_F(r,\xi,\xi^*)>0$ for all $r>0$. The modulus $\widehat{\mathfrak{C}}_F(r,\xi,\xi^*)$ here can be, certainly, substituted by $\mathfrak{C}_F^{\pm}(r,\xi,\xi^*)$ (see (1.9)). This, obviously, implies that ξ is an *exposed point* of F, and the vector ξ^* *exposes* ξ in the sense that the hyperplane $\{\eta\in H : \langle\eta,\xi^*\rangle = \sigma_F(\xi^*)\}$ touches F only at the point ξ, or, in other words, that $\mathfrak{J}_F(\xi^*)=\{\xi\}$. Therefore, we can speak just about the *strict convexity w.r.t. the vector* ξ^* (do not refering to the unique $\xi\in\mathfrak{J}_F(\xi^*)$).

The following statement characterizes the local strict convexity in terms of the duality mapping (and of the dual Minkowski functional as well).

Proposition 1.2.1 *The set F is strictly convex w.r.t. $\xi^* \in \partial F^o$ if and only if one of the following assertions holds:*

(i) ξ is a strongly exposed point *of F w.r.t. ξ^*, i.e., $\mathfrak{J}_F(\xi^*) = \{\xi\}$ and each sequence $\{\xi_n\} \subset F$ such that $\langle \xi_n, \xi^* \rangle \to \langle \xi, \xi^* \rangle = 1$, $n \to \infty$, converges to ξ strongly ($\|\xi_n - \xi\| \to 0$ as $n \to \infty$);*

(ii) the duality mapping $\mathfrak{J}_F(\cdot)$ is Hausdorff continuous at ξ^* with $\mathfrak{J}_F(\xi^*) = \{\xi\}$, which in this case means

$$\sup_{\eta \in \mathfrak{J}_F(\eta^*)} \|\eta - \xi\| \to 0 \quad as\ \eta^* \to \xi^*,\ \eta^* \in \partial F^o; \tag{1.10}$$

(iii) the function $\rho_{F^o}(\cdot)$ is Fréchet differentiable at ξ^* and $\nabla \rho_{F^o}(\xi^*) = \xi$.

Proof.

Let us show first that the strict convexity of F w.r.t. ξ^* is equivalent to the property **(i)**. Assuming that the unique point $\xi \in \mathfrak{J}_F(\xi^*)$ (here and further on we write $\xi = \mathfrak{J}_F(\xi^*)$) is not strongly exposed for F (w.r.t. ξ^*) we can choose $\varepsilon > 0$ and a sequence $\{\xi_n\} \subset F$ with $\|\xi_n - \xi\| \geq \varepsilon$ such that $\langle \xi - \xi_n, \xi^* \rangle \to 0$ as $n \to \infty$. Hence, $0 \leq \widehat{\mathfrak{C}}_F(\varepsilon, \xi, \xi^*) \leq \langle \xi - \xi_n, \xi^* \rangle \to 0$, and the strict convexity is violated. On the other hand, if $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) = 0$ for some $r > 0$ then, by Definition 1.2.2, there exists a sequence $\{\xi_n\} \subset F$ such that $\|\xi_n - \xi\| \geq r$ and $\langle \xi - \xi_n, \xi^* \rangle \to 0$ as $n \to \infty$. But this is impossible if ξ^* strongly exposes $\xi \in \partial F$.

The equivalence **(i)**$\Leftrightarrow$**(iii)** was proved in [70, Proposition 5.11], while the equivalence between the conditions **(iii)** and **(ii)** follows from [2, p. 460] where general properties of the convex functions in topological vector spaces were studied. However, let us give here an alternative direct proof, which is useful from the methodological point of view (its idea is due to [65]).

(iii)$\Rightarrow$**(ii)** Assume that there exist $\varepsilon > 0$ and sequences $\{\eta_n^*\} \subset \partial F^o$, $\eta_n^* \to \xi^*$, $\eta_n \in \mathfrak{J}_F(\eta_n^*)$ such that $\rho_F(\eta_n - \xi) \geq \|\eta_n - \xi\| / \|F\| \geq \varepsilon$, $n = 1, 2, \ldots$. Let us fix $n \in \mathbb{N}$. Using (1.1) we can choose $v_n^* \in \partial F^o$ with

$$\langle \eta_n - \xi, v_n^* \rangle \geq \varepsilon. \tag{1.11}$$

By Fréchet differentiability of $\rho_{F^o}(\cdot)$ there exists $t > 0$ such that

$$\rho_{F^o}(\xi^* + t v_n^*) - \rho_{F^o}(\xi^*) - \langle \xi, t v_n^* \rangle \leq \frac{\varepsilon t}{2}. \tag{1.12}$$

On the other hand,

$$\begin{aligned} \rho_{F^o}(\xi^* + t v_n^*) - \rho_{F^o}(\xi^*) &\geq \langle \eta_n, \xi^* + t v_n^* \rangle - \langle \eta_n, \eta_n^* \rangle \\ &= \langle \eta_n, \xi^* - \eta_n^* \rangle + \langle \eta_n - \xi, t v_n^* \rangle + \langle \xi, t v_n^* \rangle, \end{aligned}$$

and combining with (1.11) and (1.12) we obtain

$$\varepsilon t + \langle \eta_n, \xi^* - \eta_n^* \rangle \leq \langle \eta_n - \xi, t v_n^* \rangle + \langle \eta_n, \xi^* - \eta_n^* \rangle \leq \frac{\varepsilon t}{2}.$$

Therefore

$$\frac{\varepsilon t}{2} \leq \langle \eta_n, \eta_n^* - \xi^* \rangle \leq \rho_{F^o}(\eta_n^* - \xi^*),$$

which contradicts the convergence $\rho_{F^o}(\eta_n^* - \xi^*) \to 0$, $n \to \infty$.

(ii)⇒(iii) By the Lipschitz continuity of $\rho_{F^o}(\cdot)$ we have

$$\rho_{F^o}(\xi^* + tv^*) \geq \rho_{F^o}(\xi^*) - \|F\| \|tv^*\| = 1 - \|F\| t > 0,$$

so that $\rho_{F^o}(\xi^* + tv^*) \neq 0$ for all $v^* \in \overline{\mathbf{B}}$ and for $t > 0$ small enough. Moreover,

$$\frac{\xi^* + tv^*}{\rho_{F^o}(\xi^* + tv^*)} \to \xi^* \quad \text{as } t \to 0+$$

uniformly in $v^* \in \overline{\mathbf{B}}$. Fix $\varepsilon > 0$ and by (1.10) choose $\delta > 0$ such that

$$|\langle \eta - \xi, v^* \rangle| \leq \|\eta - \xi\| \|v^*\| \leq \varepsilon$$

for all $\eta \in \mathfrak{J}_F(\eta^*)$ and $v^* \in \overline{\mathbf{B}}$, with $\eta^* \in \partial F^o$ such that $\|\eta^* - \xi^*\| \leq \delta$. In particular, we have $\langle \eta - \xi, v^* \rangle \leq \varepsilon$ for all $\eta \in \mathfrak{J}_F((\xi^* + tv^*)/\rho_{F^o}(\xi^* + tv^*))$, $v^* \in \overline{\mathbf{B}}$ and $0 < t \leq \delta$. For such t, v^* and η we have

$$\langle \xi, tv^* \rangle = \langle \xi, \xi^* + tv^* \rangle - \langle \xi, \xi^* \rangle \leq \rho_{F^o}(\xi^* + tv^*) - \rho_{F^o}(\xi^*) . \tag{1.13}$$

On the other hand, by the definition of the duality mapping we can represent

$$\rho_{F^o}(\xi^* + tv^*) = \langle \eta, \xi^* + tv^* \rangle ,$$

and, consequently,

$$\rho_{F^o}(\xi^* + tv^*) - \rho_{F^o}(\xi^*) = \langle \eta, \xi^* \rangle + \langle \eta, tv^* \rangle - \rho_{F^o}(\xi^*) \leq \langle \eta, tv^* \rangle . \tag{1.14}$$

Finally, from (1.13) and (1.14) we conclude that

$$0 \leq \rho_{F^o}(\xi^* + tv^*) - \rho_{F^o}(\xi^*) - \langle \xi, tv^* \rangle \leq \langle \eta - \xi, tv^* \rangle \leq \varepsilon t$$

whenever $0 < t \leq \delta$ and $v^* \in \overline{\mathbf{B}}$, and the Fréchet differentiability follows. ■

>From Definition 1.2.2 we get also a "strict monotonicity" inequality:

$$\langle \eta - \xi, \eta^* - \xi^* \rangle \geq \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) + \widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \tag{1.15}$$

whenever $\xi \in \mathfrak{J}_F(\xi^*)$ and $\eta \in \mathfrak{J}_F(\eta^*)$ with $\|\xi - \eta\| \geq r$, which permits to prove an uniform version of the previous statement. Namely, given $U \subset \partial F^o$ let us call the set F *uniformly strictly convex w.r.t. the set* U if

$$\beta_U(r) := \inf \left\{ \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) : \xi^* \in U \right\} > 0$$

for all $r > 0$. Here as usual ξ denotes the point $\mathfrak{J}_F(\xi^*)$ for respective $\xi^* \in U$. If, in the definition above, U is a neighbourhood of a point $\xi_0^* \in \partial F^o$ then we say that F is *uniformly strictly convex w.r.t.* ξ_0^*. This property makes sense mainly in infinite-dimensional spaces, where it is stronger than the strict convexity w.r.t. all the vectors near ξ_0^*.

Proposition 1.2.2 *If the set F is uniformly strictly convex w.r.t. $U \subset \partial F^o$ then the duality mapping $\mathfrak{J}_F(\cdot)$ is single-valued, uniformly continuous on U and coincides with the gradient $\nabla \rho_{F^o}(\cdot)$.*

Proof.

It is enough to show only the uniform continuity of the mapping $\mathfrak{J}_F(\cdot)$ on U. Let us assume the contrary, i.e., that there exist $\varepsilon > 0$ and two sequences $\{\xi_n^*\}$, $\{\eta_n^*\} \subset U$ such that $\|\xi_n^* - \eta_n^*\| \to 0$ as $n \to \infty$ but $\|\mathfrak{J}_F(\xi_n^*) - \mathfrak{J}_F(\eta_n^*)\| \geq \varepsilon$, $n = 1, 2, \ldots$. Denoting by $\xi_n := \mathfrak{J}_F(\xi_n^*)$ and $\eta_n := \mathfrak{J}_F(\eta_n^*)$, it follows from (1.15) that

$$\langle \eta_n - \xi_n, \eta_n^* - \xi_n^* \rangle \geq \widehat{\mathfrak{C}}_F(\varepsilon, \xi_n, \xi_n^*) + \widehat{\mathfrak{C}}_F(\varepsilon, \eta_n, \eta_n^*) \geq 2\beta_U(\varepsilon),$$

which implies

$$0 < 2\beta_U(\varepsilon) < \langle \eta_n - \xi_n, \eta_n^* - \xi_n^* \rangle \leq \|\eta_n - \xi_n\| \, \|\eta_n^* - \xi_n^*\| \leq 2\|F\| \, \|\eta_n^* - \xi_n^*\| \to 0,$$

but this is a contradiction. ∎

Let us give now a stronger (graduated) concept of (local) strict convexity.

Definition 1.2.3 *Fix $\xi^* \in \partial F^o$, and let ξ be the unique element of $\mathfrak{J}_F(\xi^*)$. The set F is said to be* strictly convex of order $\alpha > 0$ *(*at the point ξ*)* w.r.t. ξ^* *if*

$$\hat{\gamma}_{F,\alpha}(\xi, \xi^*) := \liminf_{\substack{(r,\eta,\eta^*) \to (0+,\xi,\xi^*) \\ \eta \in \mathfrak{J}_F(\eta^*),\, \eta^* \in \partial F^o}} \frac{\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^\alpha} > 0, \tag{1.16}$$

and α is the least number such that (1.16) holds.

Remark 1.2.3 *The condition (1.16) means that for some $\theta > 0$ and $\delta > 0$ the inequality*

$$\widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \geq \theta r^\alpha \tag{1.17}$$

takes place whenever $\|\eta^* - \xi^*\| \leq \delta$, $\|\eta - \xi\| \leq \delta$, $\eta \in \mathfrak{J}_F(\eta^*)$, $\eta^* \in \partial F^o$ and $0 < r \leq \delta$. By the monotonicity of the function $r \mapsto \widehat{\mathfrak{C}}_F(r, \eta, \eta^*)$, diminishing if necessary the constant $\theta > 0$, we may suppose that (1.17) is valid for all positive r. In fact, $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) = +\infty$ whenever $r > 2\|F\|$ and for $\delta \leq r \leq 2\|F\|$ we have

$$\widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \geq \widehat{\mathfrak{C}}_F(\delta, \eta, \eta^*) \geq \theta \delta^\alpha = \theta \left(\frac{\delta}{r}\right)^\alpha r^\alpha \geq \theta \left(\frac{\delta}{2\|F\|}\right)^\alpha r^\alpha.$$

Hence, F is uniformly strictly convex w.r.t. ξ^*, and by Proposition 1.2.2 the duality mapping is single-valued and uniformly continuous in a neighbourhood of ξ^*. In particular, the condition $\eta \to \xi$ in (1.16) is superfluous.

The numbers $\hat{\gamma}_{F,\alpha}(\xi, \xi^*)$ in (1.16) possess the following invariantness property (we do not assume here that $0 \in \operatorname{int} F$).

Proposition 1.2.3 *Let* $y_1, y_2 \in \operatorname{int} F$, $\xi \in \partial F$ *and* $\xi_1^* \in \mathfrak{J}_{F-y_1}^{-1}(\xi - y_1)$. *Then there exists a unique* $\xi_2^* \in \mathfrak{J}_{F-y_2}^{-1}(\xi - y_2)$ *colinear with* ξ_1^* *and such that*

$$\frac{1}{\|\xi_1^*\|}\hat{\gamma}_{F-y_1,\alpha}(\xi - y_1, \xi_1^*) = \frac{1}{\|\xi_2^*\|}\hat{\gamma}_{F-y_2,\alpha}(\xi - y_2, \xi_2^*), \tag{1.18}$$

for each $\alpha > 0$.

Proof.

First notice that $\xi_1^* \in (F - y_1)^o$ implies $\langle y - y_1, \xi_1^* \rangle < 1$ for each $y \in \operatorname{int} F$. Therefore $1 + \langle y_1 - y_2, \xi_1^* \rangle > 0$.

Now setting $\xi_2^* := \frac{\xi_1^*}{1+\langle y_1 - y_2, \xi_1^* \rangle}$ we see that ξ_2^* has the same direction as ξ_1^* and

$$\begin{aligned}\langle y - y_2, \xi_2^* \rangle &= \frac{1}{1 + \langle y_1 - y_2, \xi_1^* \rangle}\langle y - y_2, \xi_1^* \rangle \\ &= \frac{1}{1 + \langle y_1 - y_2, \xi_1^* \rangle}\left(\langle y - y_1, \xi_1^* \rangle + \langle y_1 - y_2, \xi_1^* \rangle\right) \leq 1\end{aligned}$$

for all $y \in F$, which implies that $\xi_2^* \in (F - y_2)^o$ and $\langle \xi - y_2, \xi_2^* \rangle = 1$, i.e., $\xi_2^* \in \mathfrak{J}_{F-y_2}^{-1}(\xi - y_2)$. Given $\eta \in \partial F$ close to ξ, $\eta_1^* \in \mathfrak{J}_{F-y_1}^{-1}(\eta - y_1)$ close to ξ_1^* and setting $\eta_2^* := \frac{\eta_1^*}{1+\langle y_1 - y_2, \eta_1^* \rangle}$ (which belongs to some neighbourhood of ξ_2^*) we obtain, directly from Definition 1.2.2,

$$\begin{aligned}\frac{1}{\|\eta_2^*\|}\widehat{\mathfrak{C}}_{F-y_2}(r, \eta - y_2, \eta_2^*) &= \frac{1 + \langle y_1 - y_2, \eta_1^* \rangle}{\|\eta_1^*\|}\inf\{\langle \eta - y, \eta_2^* \rangle : y \in F,\ \|\eta - y\| \geq r\} \\ &= \frac{1}{\|\eta_1^*\|}\inf\{\langle \eta - y, \eta_1^* \rangle : y \in F,\ \|\eta - y\| \geq r\} \\ &= \frac{1}{\|\eta_1^*\|}\widehat{\mathfrak{C}}_{F-y_1}(r, \eta - y_1, \eta_1^*),\end{aligned}$$

i.e.,

$$\frac{1}{\|\eta_2^*\|}\widehat{\mathfrak{C}}_{F-y_2}(r, \eta - y_2, \eta_2^*) = \frac{1}{\|\eta_1^*\|}\widehat{\mathfrak{C}}_{F-y_1}(r, \eta - y_1, \eta_1^*) \tag{1.19}$$

for all $r > 0$. Dividing both parts of (1.19) by r^α and passing to lim inf as $r \to 0+$, $\eta \to \xi$, $\eta_1^* \to \xi_1^*$ (and, consequently, $\eta_2^* \to \xi_2^*$) we easily come to (1.18) (see (1.16)). ■

Observing that the common direction of the vectors ξ_1^* and ξ_2^* from Proposition 1.2.3 is normal to F at the point ξ (since obviously $\mathbf{N}_{F-y_i}(\xi - y_i) = \mathbf{N}_F(\xi)$, $i = 1, 2$), we may extend the concept of strict convexity for the case of an arbitrary closed convex bounded solid (do not assuming that $0 \in \operatorname{int} F$). Indeed, given $\xi \in \partial F$ and $\nu \in \mathbf{N}_F(\xi)$, $\|\nu\| = 1$, we say that F is *strictly convex of order* $\alpha > 0$ *(at the point* ξ*) w.r.t. the vector* ν if the translated set $F - y$ is strictly convex of order α (at the point $\xi - y$) w.r.t. the same direction ν (or w.r.t. $\nu / \rho_{(F-y)^o}(\nu) \in \partial (F - y)^o$, see Definition 1.2.3), where y is an arbitrary element from $\operatorname{int} F$. We use such generalization in Section 2.3 (see Proposition 2.3.5 (i)). Furthermore, since this is a local property, it can be extended also for the case of an unbounded set.

In what follows we use the strict convexity of order $\alpha = 2$ only denoting $\hat{\gamma}_{F,2}(\xi, \xi^*)$ simply by $\hat{\gamma}_F(\xi, \xi^*)$.

Curvature

Finally we are able to define curvature (and the respective radius of curvature).

Definition 1.2.4 *Fix $\xi \in \partial F$ and $\xi^* \in \mathfrak{J}_F^{-1}(\xi)$. The numbers*

$$\hat{\varkappa}_F(\xi,\xi^*) := \frac{1}{\|\xi^*\|}\hat{\gamma}_F(\xi,\xi^*) \tag{1.20}$$

and

$$\widehat{\mathfrak{R}}_F(\xi,\xi^*) := \frac{1}{2\hat{\varkappa}_F(\xi,\xi^*)}. \tag{1.21}$$

*are said to be the (*square*)* curvature *and the* curvature radius *of the set F at the point $\xi \in \partial F$ w.r.t. ξ^*, respectively.*

Roughly speaking, the curvature shows how rotund the boundary ∂F is in a neighbourhood of ξ (watching from the end of the vector ξ^*). As follows from Proposition 1.2.3 it does not depend on the position of the origin in $\operatorname{int} F$ and can be defined also when $0 \notin \operatorname{int} F$. By using (1.21) we give the following geometric characterization of the curvature radius.

Proposition 1.2.4 *Given $\xi \in \partial F$ and $\xi^* \in \mathfrak{J}_F^{-1}(\xi)$ we have*

$$\frac{\widehat{\mathfrak{R}}_F(\xi,\xi^*)}{\|\xi^*\|} = \limsup_{\substack{(\varepsilon,\eta,\eta^*)\to(0+,\xi,\xi^*)\\ \eta\in\mathfrak{J}_F(\eta^*),\,\eta^*\in\partial F^o}} \inf\left\{r>0 : F\cap\left(\eta+\varepsilon\overline{\mathbf{B}}\right)\subset \eta - r\eta^* + r\|\eta^*\|\overline{\mathbf{B}}\right\}. \tag{1.22}$$

Proof.

Let us prove first the inequality "$\le$" in (1.22) assuming without loss of generality that the right-hand side (further denoted by R) is finite. Taking an arbitrary $\rho > R$ we can affirm that for each $\varepsilon > 0$ small enough and for each dual pair (η,η^*) from a neighbourhood of (ξ,ξ^*) the relation

$$\inf\left\{r>0 : F\cap\left(\eta+\varepsilon\overline{\mathbf{B}}\right)\subset \eta - r\eta^* + r\|\eta^*\|\overline{\mathbf{B}}\right\} < \rho$$

holds. In particular,

$$F\cap\left(\eta+\varepsilon\overline{\mathbf{B}}\right)\subset \eta - \rho\eta^* + \rho\|\eta^*\|\overline{\mathbf{B}},$$

implying that

$$\|\zeta-\eta+\rho\eta^*\|^2 \le \rho^2\|\eta^*\|^2$$

whenever $\zeta \in F$ with $\|\zeta-\eta\| = \varepsilon$, or, in another form,

$$\langle \eta-\zeta,\eta^*\rangle \ge \frac{\varepsilon^2}{2\rho}. \tag{1.23}$$

If $w \in F$ is an arbitrary point with $\|w-\eta\| \ge \varepsilon$ then setting $\zeta := \lambda w + (1-\lambda)\eta \in F$, where $\lambda := \varepsilon/\|w-\eta\| \le 1$, we have $\|\zeta-\eta\| = \varepsilon$ and $\langle\eta-\zeta,\eta^*\rangle = \lambda\langle\eta-w,\eta^*\rangle$. Using (1.23) we obtain

$$\frac{\varepsilon^2}{2\rho} \le \lambda\langle\eta-w,\eta^*\rangle \le \langle\eta-w,\eta^*\rangle,$$

and therefore (see Definition 1.2.2)

$$\frac{1}{2\rho} \leq \frac{\widehat{\mathfrak{C}}_F(\varepsilon, \eta, \eta^*)}{\varepsilon^2}.$$

Hence, passing to $\liminf$ as $\varepsilon \to 0+$, $(\eta, \eta^*) \to (\xi, \xi^*)$ and $\rho \to R+$ we conclude the fist part of the proof.

In order to show the opposite inequality let us assume that $R > 0$ (in the case $R = 0$ it is trivial). If now $0 < \rho < R$ then by the definition of $\limsup$ there exist an arbitrarily small $\varepsilon > 0$ and a dual pair (η, η^*) arbitrarily near (ξ, ξ^*) such that

$$\inf\left\{r > 0 : F \cap \left(\eta + \varepsilon\overline{\mathbf{B}}\right) \subset \eta - r\eta^* + r\|\eta^*\|\,\overline{\mathbf{B}}\right\} > \rho.$$

Therefore the set $F \cap \left(\eta + \varepsilon\overline{\mathbf{B}}\right)$ is not contained in $\eta - \rho\eta^* + \rho\|\eta^*\|\,\overline{\mathbf{B}}$, or, in other words, there exists $\zeta \in F$ with $\|\zeta - \eta\| \leq \varepsilon$ such that

$$\|\zeta - \eta + \rho\eta^*\|^2 > \rho^2 \|\eta^*\|^2.$$

Consequently, setting $r := \|\zeta - \eta\| \leq \varepsilon$ we have

$$2\langle \eta - \zeta, \rho\eta^*\rangle < \|\zeta - \eta\|^2 = r^2$$

that implies

$$\frac{\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^2} < \frac{1}{2\rho}. \tag{1.24}$$

Passing in (1.24) to $\liminf$ as $r \to 0+$, $(\eta, \eta^*) \to (\xi, \xi^*)$ and then to limit as $\rho \to R-$ we prove the inequality "$\geq$" in (1.22). ∎

Besides of $\hat{\gamma}_F(\xi, \xi^*)$ in what follows we also use the one-sided characteristics $\gamma_F^+(\xi, \xi^*)$ and $\gamma_F^-(\xi, \xi^*)$ defined by the same way as (1.16), $\alpha = 2$, but with the modulus $\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)$ substituted by $\mathfrak{C}_F^{\pm}(r, \eta, \eta^*)$, respectively. However, they do not satisfy the invariantness property given by Proposition 1.2.3 (see Example 1.4.3 from Section 1.4), being connected with the "true" curvature through the inequalities

$$\frac{1}{\|F^o\|^2}\,\hat{\varkappa}_F(\xi, \xi^*) \leq \frac{\gamma_F^{\pm}(\xi, \xi^*)}{\|\xi^*\|} \leq \|F\|^2\,\hat{\varkappa}_F(\xi, \xi^*)$$

(see (1.9) and (1.20)).

According to Remark 1.2.3 it makes sense to define

$$\gamma_F(\xi, \xi^*) := \sup\Big\{\theta > 0 : \exists\varepsilon > 0 \ \text{ such that } \ \widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \geq \theta r^2 \ \text{ whenever } \ \|\eta - \xi\| \leq \varepsilon,$$
$$\|\eta^* - \xi^*\| \leq \varepsilon,\ \eta \in \mathfrak{J}_F(\eta^*),\ \eta^* \in \partial F^o \ \text{ and } \ r > 0\Big\}, \tag{1.25}$$

or, in a compact form,

$$\gamma_F(\xi, \xi^*) = \liminf_{\substack{(\eta,\eta^*)\to(\xi,\xi^*) \\ \eta\in\mathfrak{J}_F(\eta^*),\, \eta^*\in\partial F^o}} \ \inf_{r>0} \frac{\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^2}. \tag{1.26}$$

Remark 1.2.4 *We see directly from the definition that the function* $(\xi,\xi^*) \mapsto \gamma_F(\xi,\xi^*)$ *is lower semicontinuous (and the functions* $(\xi,\xi^*) \mapsto \hat{\gamma}_F(\xi,\xi^*)$ *and* $(\xi,\xi^*) \mapsto \gamma_F^{\pm}(\xi,\xi^*)$ *as well). In fact, by (1.25), for each* $\nu > 0$ *small enough there exist* $\theta > 0$ *and* $\varepsilon > 0$ *such that*

$$\gamma_F(\xi,\xi^*) \le \theta + \nu \tag{1.27}$$

and

$$\widehat{\mathfrak{C}}_F(r,\eta,\eta^*) \ge \theta r^2 \tag{1.28}$$

whenever $\|\eta - \xi\| \le \varepsilon$, $\|\eta^* - \xi^*\| \le \varepsilon$, $\eta \in \mathfrak{J}_F(\eta^*)$, $\eta^* \in \partial F^o$ *and* $r > 0$. *Now, let us fix* $\zeta^* \in \partial F^o$ *and* $\zeta \in \mathfrak{J}_F(\zeta^*)$ *such that* $\|\xi - \zeta\| \le \varepsilon/2$, $\|\xi^* - \zeta^*\| \le \varepsilon/2$. *Thus, for every* $\eta \in \mathfrak{J}_F(\eta^*)$, $\eta^* \in \partial F^o$ *with* $\|\eta - \zeta\| \le \varepsilon/2$, $\|\eta^* - \zeta^*\| \le \varepsilon/2$ *and* $r > 0$ *the inequality (1.28) holds, since* $\|\eta - \xi\| \le \varepsilon$, $\|\eta^* - \xi^*\| \le \varepsilon$. *Consequently,*

$$\gamma_F(\zeta,\zeta^*) \ge \theta.$$

Combining this with (1.27) and passing to lim inf *as* $(\zeta,\zeta^*) \to (\xi,\xi^*)$ *and* $\nu \to 0+$ *we obtain*

$$\gamma_F(\xi,\xi^*) \le \liminf_{\substack{(\zeta,\zeta^*)\to(\xi,\xi^*) \\ \zeta\in\mathfrak{J}_F(\zeta^*),\, \zeta^*\in\partial F^o}} \gamma_F(\zeta,\zeta^*).$$

Furthermore, arguing as in Proposition 1.2.4 we have

$$\frac{1}{2\gamma_F(\xi,\xi^*)} = \limsup_{\substack{(\eta,\eta^*)\to(\xi,\xi^*) \\ \eta\in\mathfrak{J}_F(\eta^*),\, \eta^*\in\partial F^o}} \inf\left\{ r > 0 : F \subset \eta - r\eta^* + r\|\eta^*\|\,\overline{\mathbf{B}} \right\}. \tag{1.29}$$

It follows readily from (1.19) that

$$\varkappa_F(\xi,\xi^*) := \frac{\gamma_F(\xi,\xi^*)}{\|\xi^*\|} \tag{1.30}$$

is invariant with respect to translations similarly to the curvature $\hat{\varkappa}_F(\xi,\xi^*)$. On the other hand, $\varkappa_F(\xi,\xi^*)$ and

$$\mathfrak{R}_F(\xi,\xi^*) := \frac{1}{2\varkappa_F(\xi,\xi^*)} \tag{1.31}$$

are not only local characteristics of the boundary ∂F at the point ξ but depend also on the size of the set F. In particular, $\mathfrak{R}_F(\xi,\xi^*)$ can not be too small, namely (see (1.29)),

$$\mathfrak{R}_F(\xi,\xi^*) \ge \mathfrak{r}_F, \tag{1.32}$$

where $\mathfrak{r}_F > 0$ is the *Chebyshev radius*[1] of F (notice that, $1/\|F^o\| \le \mathfrak{r}_F \le \|F\|$). This distinguishes it from the "true" curvature radius $\widehat{\mathfrak{R}}_F(\xi,\xi^*)$. In what follows we sometimes call $\varkappa_F(\xi,\xi^*)$ and $\mathfrak{R}_F(\xi,\xi^*)$ *scaled curvature* and *scaled curvature radius*, respectively.

[1]The *Chebyshev radius* τ_K of a bounded set $K \subset H$ is defined as

$$\tau_K := \inf\left\{ \rho > 0 : \exists k \in H \text{ such that } K \subset k + \rho\overline{B} \right\}.$$

1.3 Local smoothness

As well-known (see [2, 56, 62, 64, 65, 66, 70, 84] and others) the strict convexity of a convex closed bounded set F with $0 \in \operatorname{int} F$ is strongly related to the smoothness of its polar set F^o. Here we are interested in quantitative aspect of such connection. In particular, we found some relationships between the functions $\gamma_F^{\pm}(\xi, \xi^*)$ introduced in the previous section and the local characteristics of F^o.

Definition 1.3.1 *Let us fix $\xi^* \in \partial F^o$ and $\xi \in \mathfrak{J}_F(\xi^*) \subset \partial F$. For $t \in \mathbb{R}$ we define a* modulus of smoothness of the set F^o at the point ξ^* w.r.t. ξ *by*

$$\mathfrak{S}_{F^o}(t, \xi^*, \xi) := \sup\{\rho_{F^o}(\xi^* + t\eta^*) - \rho_{F^o}(\xi^*) - t\langle \xi, \eta^* \rangle : \eta^* \in F^o\}. \tag{1.33}$$

Since $\xi \in \mathfrak{J}_F(\xi^*) = \partial\rho_{F^o}(\xi^*)$, we always have $\mathfrak{S}_{F^o}(t, \xi^*, \xi) \geq 0$. By Proposition 1.2.1 (iii), if F is strictly convex w.r.t. ξ^* then $\rho_{F^o}(\cdot)$ is Fréchet differentiable at ξ^* and consequently

$$\lim_{t \to 0} \frac{\mathfrak{S}_{F^o}(t, \xi^*, \xi)}{t} = 0, \tag{1.34}$$

where ξ is the unique element of $\mathfrak{J}_F(\xi^*)$. In the case of the condition (1.34) holds we say that F^o is *uniformly smooth at ξ^* w.r.t. ξ.*

Remark 1.3.1 *Notice that the uniform smoothness of F^o at ξ^* (w.r.t. ξ) is equivalent to the Fréchet differentiability of the functional $\rho_{F^o}(\cdot)$ at ξ^*. In turn it follows from the Proposition 1.2.1 (see the equivalence (ii) $\Leftrightarrow$ (iii)) that the gradient $\nabla\rho_{F^o}(\cdot)$ should be continuous at this point. Consequently, by the same result the uniform smoothness of F^o at ξ^* w.r.t. ξ is equivalent to the strict convexity of F at ξ w.r.t. ξ^*. In this case we have*

$$\mathfrak{J}_F(\xi^*) = \partial\rho_{F^o}(\xi^*) = \{\nabla\rho_{F^o}(\xi^*)\}$$

(see Section 1.1). Then

$$\partial F \cap \mathbf{N}_{F^o}(\xi^*) = \{\nabla\rho_{F^o}(\xi^*)\}$$

and consequently

$$\mathfrak{n}_{F^o}(\xi^*) := \frac{\nabla\rho_{F^o}(\xi^*)}{\|\nabla\rho_{F^o}(\xi^*)\|}$$

is the unique (unit) normal vector at the set F^o at the point ξ^*.

There is a relationship between the modulus of smoothness and the modulus of rotundity given by the following statement, which is nothing else than a one-sided local version of the *Lindenstrauss duality theorem* (see [62, Theorem 1]).

Proposition 1.3.1 *Let $\xi \in \partial F$ and $\xi^* \in \partial F^o$ be such that $\langle \xi, \xi^* \rangle = 1$. Then for each $t > 0$ the equalities*

$$\mathfrak{S}_{F^o}(\pm t, \xi^*, \xi) = \sup \left\{ tr - \mathfrak{C}_F^{\pm}(r, \xi, \xi^*) : \ r > 0 \right\} \tag{1.35}$$

hold.

Proof.

Let us prove the equality (1.35) for $\mathfrak{C}_F^{+}(r, \xi, \xi^*)$ only. The other one can be proved similarly. Given $\varepsilon > 0$, from (1.33) we choose $\eta^* \in F^o$ and $\eta \in F$ such that

$$\begin{aligned}
\mathfrak{S}_{F^o}(t, \xi^*, \xi) &\leq \langle \eta, \xi^* + t\eta^* \rangle - \langle \xi, \xi^* \rangle - t \langle \xi, \eta^* \rangle + \varepsilon \\
&= \langle \eta - \xi, \xi^* \rangle + t \langle \eta - \xi, \eta^* \rangle + \varepsilon \\
&\leq \langle \eta - \xi, \xi^* \rangle + t\rho_F(\eta - \xi) + \varepsilon \\
&\leq \sup_{\eta \in F} \{ t\rho_F(\eta - \xi) - \langle \xi - \eta, \xi^* \rangle \} + \varepsilon \\
&\leq \sup_{r>0} \sup \{ tr - \langle \xi - \eta, \xi^* \rangle : \eta \in F, \rho_F(\eta - \xi) = r \} + \varepsilon \\
&= \sup_{r>0} \left\{ tr - \mathfrak{C}_F^{+}(r, \xi, \xi^*) \right\} + \varepsilon,
\end{aligned}$$

and the inequality "$\leq$" in (1.35) follows.

In order to prove the opposite inequality let us fix $\varepsilon > 0$ and choose first $r > 0$, $\eta \in F$ with $\rho_F(\eta - \xi) \geq r$ and then $\eta^* \in F^o$ such that

$$\begin{aligned}
\sup_{r>0} \left\{ tr - \mathfrak{C}_F^{+}(r, \xi, \xi^*) \right\} &\leq tr - \langle \xi - \eta, \xi^* \rangle + \frac{\varepsilon}{2} \\
&\leq t\rho_F(\eta - \xi) - \langle \xi - \eta, \xi^* \rangle + \frac{\varepsilon}{2} \\
&\leq t \langle \eta - \xi, \eta^* \rangle - \langle \xi - \eta, \xi^* \rangle + \varepsilon \\
&= \langle \eta, \xi^* + t\eta^* \rangle - t \langle \xi, \eta^* \rangle - \langle \xi, \xi^* \rangle + \varepsilon \\
&\leq \rho_{F^o}(\xi^* + t\eta^*) - \rho_{F^o}(\xi^*) - t \langle \xi, \eta^* \rangle + \varepsilon \\
&\leq \mathfrak{S}_{F^o}(t, \xi^*, \xi) + \varepsilon,
\end{aligned}$$

and the proof is concluded. ■

If we put

$$\mathfrak{C}_F(r, \xi, \xi^*) := \begin{cases} \mathfrak{C}_F^{+}(r, \xi, \xi^*) & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ \mathfrak{C}_F^{-}(-r, \xi, \xi^*) & \text{if } r < 0 \end{cases}$$

then (1.35) can be written in a more symmetric form

$$\mathfrak{S}_{F^o}(\cdot, \xi^*, \xi) = \mathfrak{C}_F^{\bigstar}(\cdot, \xi, \xi^*), \tag{1.36}$$

where "★" means the *Legendre-Fenchel transform*.

Now, by using Proposition 1.3.1, we obtain a dual characterization of the second order strict convexity, which makes more precise the equality (1.34).

Proposition 1.3.2 *Let (ξ,ξ^*) be a dual pair of elements: $\xi \in \partial F$, $\xi^* \in \partial F^o$, $\xi \in \mathfrak{J}_F(\xi^*)$. Then*

$$\frac{1}{4\gamma_F^{\pm}(\xi,\xi^*)} = \limsup_{\substack{(t,\eta,\eta^*)\to(0\pm,\xi,\xi^*)\\ \eta\in\mathfrak{J}_F(\eta^*),\,\eta^*\in\partial F^o}} \frac{\mathfrak{S}_{F^o}(t,\eta^*,\eta)}{t^2}. \tag{1.37}$$

Proof.

We prove the formula (1.37) for $\gamma_F^+(\xi,\xi^*)$. The respective proof for $\gamma_F^-(\xi,\xi^*)$ is similar.

While proving the inequality "$\geq$" in (1.37) we can assume without loss of generality that $\gamma_F^+(\xi,\xi^*) > 0$ (i.e., F is strictly convex of second order w.r.t. ξ^*). Then the mapping $\mathfrak{J}_F(\cdot)$ is single-valued and continuous in a neighbourhood of ξ^* (see Remark 1.2.3), and taking an arbitrary $0 < \beta < \gamma_F^+(\xi,\xi^*)$ one can choose $\varepsilon > 0$ such that

$$\mathfrak{C}_F^+(r,\mathfrak{J}_F(\eta^*),\eta^*) > \beta r^2 \tag{1.38}$$

for all $0 < r \leq \varepsilon$ and $\eta^* \in \partial F^o$ with $\|\eta^* - \xi^*\| \leq \varepsilon$. As it is easy to see,

$$\sup\left\{tr - \mathfrak{C}_F^+(r,\mathfrak{J}_F(\eta^*),\eta^*) : 0 < r \leq \varepsilon\right\} \leq \sup\left\{tr - \beta r^2 : 0 < r \leq \varepsilon\right\} = \frac{t^2}{4\beta} \tag{1.39}$$

for all $0 < t \leq 2\varepsilon\beta$. On the other hand, as $\mathfrak{C}_F^+(r,\mathfrak{J}_F(\eta^*),\eta^*) = +\infty$ whenever $r > D := 2\|F^o\|\,\|F\|$ (see (1.2)) and the function $r \mapsto \mathfrak{C}_F^+(r,\mathfrak{J}_F(\eta^*),\eta^*)$ is increasing, using (1.38), we obtain

$$\begin{aligned}\sup\left\{tr - \mathfrak{C}_F^+(r,\mathfrak{J}_F(\eta^*),\eta^*) : r > \varepsilon\right\} &\leq \sup\left\{tr - \mathfrak{C}_F^+(\varepsilon,\mathfrak{J}_F(\eta^*),\eta^*) : \varepsilon < r < D\right\}\\ &\leq tD - \beta\varepsilon^2 \leq \frac{t^2}{4\beta}\end{aligned} \tag{1.40}$$

for all $0 < t \leq 2\beta\left(D - \sqrt{D^2 - \varepsilon^2}\right)$. Thus, applying the duality formula (1.35), we obtain from (1.39) and (1.40)

$$\frac{\mathfrak{S}_{F^o}(t,\eta^*,\mathfrak{J}_F(\eta^*))}{t^2} \leq \frac{1}{4\beta}.$$

Hence, passing to lim sup as $t \to 0+$, $\eta^* \to \xi^*$ and to limit as $\beta \to \gamma_F^+(\xi,\xi^*)-$ we conclude the first part of the proof.

In order to prove the converse inequality let us suppose that the right-hand side of (1.37) (further denoted by L) is finite. Then, taking any $\beta > L$ we can find $\varepsilon > 0$ such that

$$\mathfrak{S}_{F^o}(t,\eta^*,\eta) < \beta t^2 \tag{1.41}$$

for all $0 < t \leq \varepsilon$ and for each dual pair (η,η^*) such that $\|\eta - \xi\| \leq \varepsilon$, $\|\eta^* - \xi^*\| \leq \varepsilon$. Applying the Legendre-Fenchel transform to (1.41) we have

$$\begin{aligned}\mathfrak{S}_{F^o}^{\bigstar}(r,\eta^*,\eta) &\geq \sup\left\{tr - \mathfrak{S}_{F^o}(t,\eta^*,\eta) : 0 < t \leq \varepsilon\right\}\\ &\geq \sup\left\{tr - \beta t^2 : 0 < t \leq \varepsilon\right\} = \frac{r^2}{4\beta},\end{aligned} \tag{1.42}$$

$0 < r \leq 2\varepsilon\beta$. Since the double conjugate function is always below the original one, it follows from (1.42) and (1.36) that

$$\frac{r^2}{4\beta} \leq \mathfrak{S}^{\bigstar}_{F^o}(r, \eta^*, \eta) \leq \mathfrak{C}^{+}_{F}(r, \eta, \eta^*).$$

Dividing by r^2, passing to lim inf as $r \to 0+$, $\eta^* \to \xi^*$ $\eta \to \xi$ we obtain

$$\gamma^{\pm}_{F}(\xi, \xi^*) \geq \frac{1}{4\beta},$$

and now by passing to the limit as $\beta \to L+$ the desired inequality follows. ■

Let us concretize the formula (1.37) in the case when the boundary of F^o is second order smooth.

As we know (see Remark 1.2.3 and Proposition 1.2.1 (iii)) if $\gamma^{+}_{F}(\xi, \xi^*) > 0$ then $\rho_{F^o}(\cdot)$ is Fréchet differentiable on $\partial F^o \cap (\xi^* + \varepsilon\overline{\mathbf{B}})$ for some $\varepsilon > 0$, and, furthermore, the Fréchet derivative $\nabla\rho_{F^o}(\cdot)$ is (uniformly) continuous in a neighbourhood of ξ^*. Remind that the functional $\rho_{F^o}(\cdot)$ is said to be *twice (Fréchet) differentiable* at $\xi^* \in \partial F^o$ if there exists a (self-adjoint) linear bounded operator $\nabla^2\rho_{F^o}(\xi^*) : H \to H$ (called *second Fréchet derivative*) such that

$$\frac{\nabla\rho_{F^o}(\xi^* + tv^*) - \nabla\rho_{F^o}(\xi^*)}{t} \to \nabla^2\rho_{F^o}(\xi^*)\, v^* \ \text{ as } \ t \to 0+$$

uniformly in $v^* \in F^o$. Let us define the F^o-norm of the operator $\nabla^2\rho_{F^o}(\xi^*)$ by

$$\left\|\nabla^2\rho_{F^o}(\xi^*)\right\|_{F^o} := \sup_{v^* \in F^o} \left\langle \nabla^2\rho_{F^o}(\xi^*)\, v^*, v^* \right\rangle. \tag{1.43}$$

Finally, the boundary ∂F^o is said to be *of class* $\mathcal{C}^2$ (or *second order smooth*) *at the point* $\xi^* \in \partial F^o$ if $\rho_{F^o}(\cdot)$ is twice differentiable at each point of a neighbourhood of ξ^*, and the mapping $\eta^* \mapsto \nabla^2\rho_{F^o}(\eta^*)$ is continuous near ξ^* with respect to the operator topology. This is the same to require the continuous differentiability of the (unique) unit normal vector to F^o near the point ξ^* (see Proposition 1.2.1). Hence, in particular, the continuity of the functional $\eta^* \mapsto \left\|\nabla^2\rho_{F^o}(\eta^*)\right\|_{F^o}$ in a neighbourhood of ξ^* follows.

Proposition 1.3.3 *Assume that the boundary of the set F^o is of class $\mathcal{C}^2$ at the point $\xi^* \in \partial F^o$, and $\xi \in \partial F$ is the unique element of $\mathfrak{J}_F(\xi^*)$ (in other words $\xi = \nabla\rho_{F^o}(\xi^*)$). Then*

$$\gamma^{+}_{F}(\xi, \xi^*) = \gamma^{-}_{F}(\xi, \xi^*) = \frac{1}{2\left\|\nabla^2\rho_{F^o}(\xi^*)\right\|_{F^o}}. \tag{1.44}$$

Proof.

Given $\eta^* \in \partial F^o$ in a neighbourhood of the point ξ^* by the Taylor formula (see, e.g., [10, p. 75]) for each $v^* \in F^o$ and $t > 0$ small enough we have

$$\rho_{F^o}(\eta^* + tv^*) = \rho_{F^o}(\eta^*) + t\langle \eta, v^* \rangle + \int_0^t \left\langle \nabla^2\rho_{F^o}(\eta^* + \tau v^*)\, v^*, v^* \right\rangle (t - \tau)\, d\tau, \tag{1.45}$$

where $\eta := \nabla\rho_{F^o}(\eta^*) = \mathfrak{J}_F(\eta^*)$. Hence, by using the mean value theorem for integrals, given $t > 0$ and $v^* \in F^o$ we find $\tau^* = \tau(t, v^*)$, $0 < \tau^* < t$, such that

$$\int_0^t \langle \nabla^2\rho_{F^o}(\eta^* + \tau v^*) v^*, v^* \rangle (t - \tau)\, d\tau = \frac{t^2}{2} \langle \nabla^2\rho_{F^o}(\eta^* + \tau^* v^*) v^*, v^* \rangle .$$

Then (see (1.33))

$$\frac{\mathfrak{S}_{F^o}(t, \eta^*, \eta)}{t^2} = \frac{1}{2} \sup_{v^* \in F^o} \langle \nabla^2\rho_{F^o}(\eta^* + \tau^* v^*) v^*, v^* \rangle . \tag{1.46}$$

By continuity of the second derivative we have the convergence

$$\nabla^2\rho_{F^o}(\eta^* + \tau^* v^*) \to \nabla^2\rho_{F^o}(\xi^*)$$

as $\eta^* \to \xi^*$, $\eta^* \in \partial F^o$, and as $t \to 0+$ in the operator topology uniformly in $v^* \in F^o$. Therefore,

$$\begin{aligned} & \limsup_{\substack{(t,\eta,\eta^*)\to(0+,\xi,\xi^*) \\ \eta\in\mathfrak{J}_F(\eta^*),\, \eta^*\in\partial F^o}} \sup_{v^*\in F^o} \langle \nabla^2\rho_{F^o}(\eta^* + \tau^* v^*) v^*, v^* \rangle \\ \leq & \limsup_{\substack{(t,\eta,\eta^*)\to(0+,\xi,\xi^*) \\ \eta\in\mathfrak{J}_F(\eta^*),\, \eta^*\in\partial F^o}} \sup_{v^*\in F^o} \langle (\nabla^2\rho_{F^o}(\eta^* + \tau^* v^*) - \nabla^2\rho_{F^o}(\xi^*)) v^*, v^* \rangle + \left\| \nabla^2\rho_{F^o}(\xi^*) \right\|_{F^o} \\ = & \left\| \nabla^2\rho_{F^o}(\xi^*) \right\|_{F^o} . \end{aligned}$$

Since the reverse inequality is obvious, we conclude that

$$\frac{1}{4\gamma_F^+(\xi, \xi^*)} = \limsup_{\substack{(t,\eta,\eta^*)\to(0+,\xi,\xi^*) \\ \eta\in\mathfrak{J}_F(\eta^*),\, \eta^*\in\partial F^o}} \frac{\mathfrak{S}_{F^o}(t, \eta^*, \eta)}{t^2} = \frac{1}{2} \left\| \nabla^2\rho_{F^o}(\xi^*) \right\|_{F^o}$$

(see (1.37)). In order to find the same representation for $\gamma_F^-(\xi, \xi^*)$ it is enough to apply the Taylor formula (1.45) for $t < 0$ instead of $t > 0$. ■

1.4 Examples

Example 1.4.1 *In a Hilbert space H for a fixed $v \in H$, $\|v\| = 1$, $0 < \theta < 1$ and $\alpha > 1$ let us consider the set*

$$F := \{\xi \in H : \langle v, \xi + v \rangle \geq \theta \|\xi + v\|^\alpha\} .$$

Clearly, F is convex closed bounded with $0 \in \operatorname{int} F$. We prove that it is uniformly strictly convex (w.r.t. the whole ∂F^o) and even strictly convex of second order with curvature uniformly bounded from below whenever $1 < \alpha \leq 2$, while in the other case ($\alpha > 2$) the curvature of F is bounded from above, admiting the value 0 at some point. Indeed, for $\bar{\xi} := -v \in \partial F$ setting

$$\bar{\xi}^* := \frac{-v}{\rho_{F^o}(-v)} = -v$$

(the unique normal direction at $\bar{\xi}$) we directly have

$$\begin{aligned}\widehat{\mathfrak{C}}_F\left(r,\bar{\xi},\bar{\xi}^*\right) &= \inf\left\{\left\langle \bar{\xi}-\eta,\bar{\xi}^*\right\rangle : \eta\in F,\ \left\|\bar{\xi}-\eta\right\|\geq r\right\}\\ &= \inf\left\{\langle v+\eta, v\rangle : \eta\in\partial F,\ \|v+\eta\|\geq r\right\}\\ &= \theta r^{\alpha}\quad r>0,\end{aligned}$$

and, consequently, $\hat{\varkappa}_F\left(\bar{\xi},\bar{\xi}^*\right)>0$ if $\alpha\leq 2$ and $\hat{\varkappa}_F\left(\bar{\xi},\bar{\xi}^*\right)=0$ otherwise.

Taking instead $\xi\in\partial F$, $\xi\neq -v$, notice that the function $h(y):=\|y\|^{\alpha}$ is of classe $\mathcal{C}^2$ near ξ with

$$\nabla h(y)=\frac{\alpha}{\|y\|^{2-\alpha}}\,y\quad\text{and}\quad\left\langle\nabla^2 h(y)\,w,w\right\rangle=\alpha\left[\frac{\|w\|^2}{\|y\|^{2-\alpha}}-(2-\alpha)\frac{\langle y,w\rangle^2}{\|y\|^{4-\alpha}}\right],\quad w\in H.$$

Then for $\eta\in\partial F$ enough close to ξ we obtain by the second order Taylor formula (see, e.g., [10, p. 75])

$$\begin{aligned}&\|\eta+v\|^{\alpha}-\|\xi+v\|^{\alpha}-\frac{\alpha}{\|\xi+v\|^{2-\alpha}}\langle\eta-\xi,\xi+v\rangle\\ =\ &\alpha\int_0^1\left[\frac{\|\eta-\xi\|^2}{\|\eta_\tau+v\|^{2-\alpha}}-(2-\alpha)\frac{\langle\eta_\tau+v,\eta-\xi\rangle^2}{\|\eta_\tau+v\|^{4-\alpha}}\right](1-\tau)\,d\tau\\ \geq\ &\alpha(\alpha-1)\|\eta-\xi\|^2\int_0^1\frac{1-\tau}{\|\eta_\tau+v\|^{2-\alpha}}\,d\tau\geq\frac{\alpha(\alpha-1)\|\eta-\xi\|^2}{2\left(\|F\|+1\right)^{2-\alpha}},\end{aligned}\tag{1.47}$$

where $\eta_\tau:=\tau\eta+(1-\tau)\xi\in F$, $\tau\in[0,1]$, if $1<\alpha\leq 2$, while in the case $\alpha>2$ analogously we have

$$\begin{aligned}&\|\eta+v\|^{\alpha}-\|\xi+v\|^{\alpha}-\frac{\alpha}{\|\xi+v\|^{2-\alpha}}\langle\eta-\xi,\xi+v\rangle\\ \leq\ &\alpha(\alpha-1)\|\eta-\xi\|^2\int_0^1(1-\tau)\|\eta_\tau+v\|^{\alpha-2}\,d\tau\\ \leq\ &\frac{\alpha}{2}(\alpha-1)\|\eta-\xi\|^2\left(\|F\|+1\right)^{\alpha-2}.\end{aligned}\tag{1.48}$$

Observe that $\mathbf{N}_F(\xi)=\nabla g(\xi)\,\mathbb{R}^+$ where $g(\xi):=\theta\|\xi+v\|^{\alpha}-\langle v,\xi+v\rangle$. It follows from (1.47) and (1.48) that

$$\begin{aligned}\langle\xi-\eta,\nabla g(\xi)\rangle &= \left\langle\xi-\eta,\alpha\theta\frac{\xi+v}{\|\xi+v\|^{2-\alpha}}-v\right\rangle\\ &= \langle(\xi+v)-(\eta+v),-v\rangle+\frac{\alpha\theta}{\|\xi+v\|^{2-\alpha}}\langle\xi-\eta,\xi+v\rangle\\ &= \theta\left[\|\eta+v\|^{\alpha}-\|\xi+v\|^{\alpha}-\frac{\alpha}{\|\xi+v\|^{2-\alpha}}\langle\eta-\xi,\xi+v\rangle\right]\\ &\geq \frac{\theta\alpha(\alpha-1)}{2\left(\|F\|+1\right)^{2-\alpha}}\|\eta-\xi\|^2\end{aligned}\tag{1.49}$$

for $1 < \alpha \leq 2$ and

$$\langle \xi - \eta, \nabla g(\xi) \rangle \leq \frac{\theta\alpha}{2}(\alpha - 1)\|\eta - \xi\|^2 (\|F\| + 1)^{\alpha-2} \tag{1.50}$$

for $\alpha > 2$, respectively. Here we used the fact that $\xi, \eta \in \partial F$, i.e., $g(\xi) = g(\eta) = 0$. Moreover

$$\begin{aligned} \|\nabla g(\xi)\|^2 &= \frac{\alpha^2\theta^2}{\|\xi + v\|^{2(2-\alpha)}} \langle \xi + v, \xi + v \rangle - 2\frac{\alpha\theta}{\|\xi + v\|^{2-\alpha}} \langle \xi + v, v \rangle + \|v\|^2 \\ &= \alpha^2\theta^2 \|\xi + v\|^{2\alpha-2} - 2\frac{\alpha\theta}{\|\xi + v\|^{2-\alpha}}\theta \|\xi + v\|^\alpha + 1 \\ &= \alpha\theta^2 \|\xi + v\|^{2\alpha-2}(\alpha - 2) + 1 \leq 1 \end{aligned}$$

for $1 < \alpha \leq 2$, and $\|\nabla g(\xi)\| > 1$ if $\alpha > 2$. Therefore, denoting by $\xi^* := \nabla g(\xi) / \rho_{F^o}(\nabla g(\xi))$ from (1.6) and (1.49) (or (1.50)), we obtain for $r > 0$ small enough

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) \geq \frac{1}{\rho_{F^o}(\nabla g(\xi))} \frac{\theta\alpha(\alpha - 1)}{2(\|F\| + 1)^{2-\alpha}} r^2 \geq \frac{\theta\alpha(\alpha - 1)}{2\|F\|(\|F\| + 1)^{2-\alpha}} r^2 \tag{1.51}$$

if $1 < \alpha \leq 2$,

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) \leq \frac{\theta\alpha}{2}(\alpha - 1)\|F^o\|(\|F\| + 1)^{\alpha-2} r^2 \tag{1.52}$$

when $\alpha > 2$.

Finally, dividing both parts of the inequality (1.51) (respectively, of (1.52)) by r^2 and passing to lim inf as $r \to 0+$, we find the estimates for the curvatures

$$\hat{\varkappa}_F(\xi, \xi^*) \geq \frac{\theta\alpha(\alpha - 1)}{2\|F\|\|F^o\|(\|F\| + 1)^{2-\alpha}} > 0, \quad 1 < \alpha \leq 2,$$

and

$$\hat{\varkappa}_F(\xi, \xi^*) \leq \frac{\theta\alpha}{2}(\alpha - 1)\|F\|\|F^o\|(\|F\| + 1)^{\alpha-2} < +\infty, \quad \alpha > 2.$$

Example 1.4.2 *Let* $F := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq 1 - \xi_1^4,\ -1 \leq \xi_1 \leq 1\}$.

Observe that F is closed convex bounded with $0 \in \operatorname{int} F$. Let us estimate the curvatures $\varkappa_F(\xi, \xi^*)$ and $\hat{\varkappa}(\xi, \xi^*)$ for an arbitrary dual pair (ξ, ξ^*) (i.e., $\xi^* \in \partial F^o$ and $\xi \in \mathfrak{J}_F(\xi^*)$). Setting $\xi = (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}$, by symmetry we can consider, clearly, only the case when $\xi_2 \geq 0$ and $\xi_1 \leq 0$.

If $\xi_2 > 0$ then the (unique) normal vector ξ^* to F at ξ such that $\rho_{F^o}(\xi^*) = 1$ is given by

$$\xi^* = \frac{1}{1 + 3\xi_1^4}(4\xi_1^3, 1).$$

>From Remark 1.2.1 we have, after some simple transformations,

$$\begin{aligned}\widehat{\mathfrak{C}}_F(r,\xi,\xi^*) &= \inf\{\langle\xi-\eta,\xi^*\rangle : \eta\in\partial F,\ \|\xi-\eta\|\geq r\}\\ &= \frac{1}{1+3\xi_1^4}\ \inf\left\{4\xi_1^3(\xi_1-\eta_1)+\eta_1^4-\xi_1^4 : -1\leq\eta_1\leq 1,\ \|\xi-\eta\|\geq r\right\}\\ &= \frac{1}{1+3\xi_1^4}\ \inf\Big\{(\eta_1-\xi_1)^2\Big[(\eta_1-\xi_1)^2+\\ &\quad +4\xi_1(\eta_1-\xi_1)+6\xi_1^2\Big] : \|\xi-\eta\|\geq r,\ -1\leq\eta_1\leq 1\Big\},\quad r>0. \qquad (1.53)\end{aligned}$$

Since

$$\|\xi-\eta\|^2=(\eta_1-\xi_1)^2+\left(\eta_1^4-\xi_1^4\right)^2=(\eta_1-\xi_1)^2\left[1+\left(\eta_1^3+\eta_1^2\xi_1+\eta_1\xi_1^2+\xi_1^3\right)^2\right],$$

and $-1\leq\eta_1\leq 1$, we obtain the inequality

$$\|\xi-\eta\|\leq|\eta_1-\xi_1|\sqrt{1+\left(1+|\xi_1|+|\xi_1|^2+|\xi_1|^3\right)^2}, \qquad (1.54)$$

and the condition $\|\xi-\eta\|\geq r$ can be writen as

$$|\eta_1-\xi_1|\geq\frac{\|\xi-\eta\|}{\Sigma(\xi_1)}\geq\frac{r}{\Sigma(\xi_1)},$$

where $\Sigma(\xi_1):=\sqrt{1+\left(\sum\limits_{k=0}^{3}|\xi_1|^k\right)^2}$. Consequently (see (1.53))

$$\begin{aligned}\frac{\langle\xi-\eta,\xi^*\rangle}{r^2} &\geq \frac{\langle\xi-\eta,\xi^*\rangle}{\|\xi-\eta\|^2}\geq\frac{(\eta_1-\xi_1)^2+4\xi_1(\eta_1-\xi_1)+6\xi_1^2}{\Sigma^2(\xi_1)}\\ &\geq \frac{1}{\Sigma^2(\xi_1)}\left[\frac{r^2}{\Sigma^2(\xi_1)}+4r\frac{\xi_1}{\Sigma(\xi_1)}+6\xi_1^2\right].\end{aligned}$$

Then, by (1.53),

$$\frac{\widehat{\mathfrak{C}}_F(r,\xi,\xi^*)}{r^2}\geq\frac{1}{\left(1+3\xi_1^4\right)\Sigma^2(\xi_1)}\left[\frac{r^2}{\Sigma^2(\xi_1)}+4r\frac{\xi_1}{\Sigma(\xi_1)}+6\xi_1^2\right]. \qquad (1.55)$$

Notice that the right-hand side in this inequality is continuous in ξ. Therefore, in order to obtain an estimate of the scaled curvature from below it is enough only to pass to infimum in (1.55) for $r>0$ (see (1.26)), while for the "true" (local) curvature we let $r\to 0+$ (see (1.16)). Thus, since $\|\xi^*\|=\sqrt{1+16\xi_1^6}/\left(1+3\xi_1^4\right)$, we obtain

$$\varkappa_F(\xi,\xi^*)=\frac{\gamma_F(\xi,\xi^*)}{\|\xi^*\|}\geq K(\xi_1):=\frac{2\xi_1^2}{\sqrt{1+16\xi_1^6\Sigma^2(\xi_1)}} \qquad (1.56)$$

and

$$\hat{\varkappa}_F(\xi,\xi^*) = \frac{\hat{\gamma}_F(\xi,\xi^*)}{\|\xi^*\|} \geq 3K(\xi_1). \tag{1.57}$$

In the same way (employing the inequality $\|\xi-\eta\| \geq |\xi_1-\eta_1|$ instead of (1.54)) we obtain

$$\begin{aligned}
&\widehat{\mathfrak{C}}_F(r,\xi,\xi^*) \\
= \;& \frac{1}{1+3\xi_1^4} \inf\left\{(\eta_1-\xi_1)^2\left[(\eta_1-\xi_1)^2+4\xi_1(\eta_1-\xi_1)+6\xi_1^2\right] : -1\leq\eta\leq 1,\ \|\xi-\eta\|\geq r\right\} \\
\leq \;& \frac{1}{1+3\xi_1^4} \inf\left\{\|\xi-\eta\|^2\left[\|\xi-\eta\|^2+4\xi_1\|\xi-\eta\|+6\xi_1^2\right] : -1\leq\eta\leq 1,\ \|\xi-\eta\|= r\right\} \\
= \;& \frac{r^2}{1+3\xi_1^4}\left(r^2+4\xi_1 r+6\xi_1^2\right),
\end{aligned}$$

and hence

$$\varkappa_F(\xi,\xi^*) \leq \frac{2\xi_1^2}{\sqrt{1+16\xi_1^6}}, \quad \hat{\varkappa}_F(\xi,\xi^*) \leq \frac{6\xi_1^2}{\sqrt{1+16\xi_1^6}}. \tag{1.58}$$

Combining estimates (1.56), (1.57) and (1.58) we see that the curvatures $\varkappa_F(\xi,\xi^*)$ and $\hat{\varkappa}(\xi,\xi^*)$ are of order $O\left(\xi_1^2\right)$ (as $|\xi_1| \to 0$). In particular, both $\varkappa_F$ and $\hat{\varkappa}_F$ are equal to zero at the points $(0,\pm 1)$.

If $\bar{\xi} := (-1,0)$ then we have

$$\mathbf{N}_F(\bar{\xi}) = \left\{(v_1,v_2)\in\mathbb{R}^2 : v_1 \leq -4|v_2|\right\},$$

and for $\xi^* \in \partial\mathbf{N}_F(\bar{\xi})$ by the lower semicontinuity we can apply the same reasoning as above but not for $\xi^* \in \operatorname{int}\mathbf{N}_F(\bar{\xi})$. In this last case we have $\hat{\varkappa}_F(\bar{\xi},\xi^*) = +\infty$ (see (1.22)) while $\varkappa_F(\bar{\xi},\xi^*)$ is a finite positive number depending on the size of both sets F and F^o, and on the proximity of ξ^* to the boundary $\partial\mathbf{N}_F(\bar{\xi})$. To obtain a precise estimate we can proceed, e.g., as in the proof of Theorem 2.3.4 below (see Example 2.4.3 in the Section 2.4).

Now let us give a simple example illustrating the lack of the invariantness property for $\gamma_F^{\pm}(\xi,\xi^*)$ unlike the curvature (see Proposition 1.2.3).

Example 1.4.3 *Fix $a \in H$ with $\|a\| < 1$ and consider the set*

$$F := \{\xi \in H : \|\xi - a\| \leq 1\}.$$

It is easy to see that

$$\rho_{F^o}(\xi^*) = \sigma_F(\xi^*) = \sup_{\|\xi-a\|\leq 1}\langle\xi^*,\xi-a\rangle + \langle\xi^*,a\rangle = \|\xi^*\| + \langle\xi^*,a\rangle, \quad \xi^*\in H.$$

This function is twice continuously differentiable at each $\xi^* \neq 0$, and taking $\xi^* \in \partial F^o$ we have

$$\nabla\rho_{F^o}(\xi^*) = a + \frac{\xi^*}{\|\xi^*\|}$$

and

$$\nabla^2 \rho_{F^o}(\xi^*) v^* = \frac{\|\xi^*\|^2 v^* - \langle \xi^*, v^* \rangle \xi^*}{\|\xi^*\|^3}, \qquad v^* \in H.$$

The $\|F^o\|$-norm of this operator (see (1.43)) is

$$\begin{aligned}\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} &= \sup_{v^* \in F^o} \left\langle \nabla^2 \rho_{F^o}(\xi^*) v^*, v^* \right\rangle \\ &= \frac{1}{\|\xi^*\|^3} \sup \left\{ \|\xi^*\|^2 \|v^*\|^2 - \langle \xi^*, v^* \rangle^2 : \langle v^*, a \rangle + \|v^*\| \le 1 \right\}.\end{aligned}$$

Applying Lagrange multipliers after some calculations we find a vector $v^* \in F^o$ giving maximum to the quadratic form $v^* \mapsto \left\langle \nabla^2 \rho_{F^o}(\xi^*) v^*, v^* \right\rangle$, which satisfies the relations

$$\langle \xi^*, v^* \rangle = - \langle \xi^*, a \rangle \|v^*\|$$

and

$$\|v^*\|^2 = \frac{\|\xi^*\|^2 - 2\langle \xi^*, a \rangle^2 + \|\xi^*\|^2 \|a\|^2 + 2\sqrt{\left(\|\xi^*\|^2 - \langle \xi^*, a \rangle^2\right)\left(\|a\|^2 \|\xi^*\|^2 - \langle \xi^*, a \rangle^2\right)}}{\left(\|\xi^*\|^2 - \langle \xi^*, a \rangle^2\right)\left(1 - \|a\|^2\right)^2}.$$

Hence

$$\begin{aligned}\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} &= \frac{\|v^*\|^2}{\|\xi^*\|^3}\left(\|\xi^*\|^2 - \langle \xi^*, a \rangle^2\right) \\ &= \frac{1}{\left(1 - \|a\|^2\right)^2 \|\xi^*\|^3}\left(\|\xi^*\|^2\left(1 + \|a\|^2\right) - 2\langle \xi^*, a \rangle^2 + \right. \\ &\quad \left. +2\sqrt{\left(\|\xi^*\|^2 - \langle \xi^*, a \rangle^2\right)\left(\|a\|^2 \|\xi^*\|^2 - \langle \xi^*, a \rangle^2\right)}\right), \qquad (1.59)\end{aligned}$$

and the rotundity characteristics $\gamma_F^{\pm}(\xi, \xi^*)$ can be found from Proposition 1.3.3. Here as usual $\xi \in \partial F$ is the unique point with $\langle \xi, \xi^* \rangle = 1$. In particular cases when ξ^* is colinear to a the square root in (1.59) vanishes, and we obtain

$$\gamma_F^{\pm}(\xi, \xi^*) = \begin{cases} \frac{1-\|a\|}{2} & \text{if } \xi^* = \frac{a}{\|a\|(1+\|a\|)} \\ \frac{1+\|a\|}{2} & \text{if } \xi^* = -\frac{a}{\|a\|(1-\|a\|)}. \end{cases}$$

Thus, $\gamma_F^{\pm}(\xi, \xi^*)$ depend essentially on a (on position of the origin inside the ball). Namely, they tend either to 0 or to 1 as $\|a\| \to 1$ whenever the origin is either more distant from the point ξ or more close to ξ, respectively. This distinguishes $\gamma_F^{\pm}(\xi, \xi^*)$ from $\hat{\gamma}_F(\xi, \xi^*)$ (see Proposition 1.2.3). Observe that in the case $a = 0$ the formula (1.59) gives $\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} = 1$, and $\gamma_F^{\pm}(\xi, \xi^*) = 1/2$ for each $\xi^* \in H$ with $\|\xi^*\| = 1$ and $\xi = \xi^*$ (see Remark 1.2.2).

Chapter 2

Well-posedness of the time-minimum projection

We begin this chapter recalling some concepts regarding with the general (nonconvex) sets and functions in a Hilbert space, in particular, definitions of various kinds of subdifferentials and normal cones, which will be used throughout the work. We rigorously define φ-convex (proximal smooth) sets and construct the function $\varphi(\cdot)$ in the special case when the set has boundary of class $C^{1,1}_{loc}$. In the end of the first section we obtain a very useful property of minimizing sequences in the minimal time problem, whose proof is based on the Ekeland's variational principle. In the Section 2.2 we present two types of geometric conditions (on F and C) guaranteeing the well--posedness of the problem. One of them essentialy generalizes the konwn hypotheses, employing φ-convexity of the target set C and the second order rotundity of the dynamics F. The other does not use any of these properties. The boundary of C can even have "inward corner" points. We just require a certain Lipschitz condition on the duality mapping and (simple) strict convexity of F near given boundary points. Under these assumptions we prove a general (local) retraction theorem (Theorem 2.2.1) by using the property of minimizing sequences mentioned above. In this section we also give an explicit formula for the neighbourhood of C where the time-minimum projection $\pi^F_C(\cdot)$ is well-posed (although in the case of a φ-convex target only). The obtained results are then concretized for the case where either ∂C is smooth (see theorems 2.3.1-2.3.3 in Section 2.3) or ∂F^o is of class C^2 (Theorem 2.3.4). At the end of the chapter we present some examples (see Section 2.4).

2.1 Properties of nonconvex sets. Auxiliary results

Subdifferentials and normal cones

We start with definitions of some subdifferentials to a lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$.

For $x \in \operatorname{dom} \phi := \{x \in H : \phi(x) < +\infty\}$, the *proximal subdifferential* $\partial^p \phi(x)$ is the (possibly empty) convex not necessarily closed set (see [23, p. 29])

$$\partial^p \phi(x) = \left\{ \zeta : \exists \eta > 0,\ \sigma \geq 0 \ \text{ so that } \ \phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \forall y \in x + \eta \overline{\mathbf{B}} \right\}.$$

The *Fréchet subdifferential* $\partial^f \phi(x)$ is defined by

$$\partial^f \phi(x) = \left\{ \zeta : \liminf_{y \to x} \frac{\phi(y) - \phi(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

and it is convex and closed (see [13, Proposition 3.1]).

The *limiting (Mordukhovich) subdifferential* $\partial^l \phi(x)$, in the case of a Hilbert space, is given by (see [66, p. 240])

$$\partial^l \phi(x) = \left\{ w - \lim_{i \to \infty} \zeta_i, \quad \zeta_i \in \partial^p \phi(x_i), \quad x_i \to x, \quad \phi(x_i) \to \phi(x) \right\}, \tag{2.1}$$

where "$w - \lim\limits_{i \to \infty}$" denotes the weak limit. If $\phi(\cdot)$ is locally Lipschitzean around x then $\partial^l \phi(x) \neq \emptyset$ (see [66, Corollary 2.25]).

If $\phi(\cdot)$ is Lipschitzean near x then the *Clarke subdifferential* $\partial^c \phi(x)$ is the nonempty convex closed bounded set given by (see [22, Proposition 2.1.2])

$$\partial^c \phi(x) = \left\{ \zeta : \limsup_{\substack{y \to x \\ t \to 0+}} \frac{\phi(y + tv) - \phi(y)}{t} \geq \langle \zeta, v \rangle \quad \forall v \in H \right\}. \tag{2.2}$$

Given sequences $\{x_i\}$ and $\{\zeta_i\}$ in H such that $\zeta_i \in \partial^c \phi(x_i)$, if $x_i \to x$ and $\{\zeta_i\}$ converges to ξ weakly as $i \to \infty$, one has $\zeta \in \partial^c \phi(x)$ (see [22, Proposition 2.1.5]). We say that the multivalued mapping $\partial^c \phi(\cdot)$ has strongly$\times$weakly-closed (or briefly $s \times w$-closed) graph. Moreover, it was proved in [23, p. 88] that

$$\partial^c \phi(x) = \overline{\text{co}} \left\{ w - \lim_{n \to \infty} \zeta_n, \quad \zeta_n \in \partial^p \phi(x_n), \quad x_n \to x \right\} \tag{2.3}$$

and that

$$\partial^p \phi(x) \subset \partial^f \phi(x) \subset \partial^l \phi(x) \subset \partial^c \phi(x) \tag{2.4}$$

for each $x \in \text{dom}\, \phi$.

If ϕ is a convex function then all subdifferentials coincide with the subdifferential $\partial \phi(\cdot)$ in the sense of Convex Analysis.

Since the target set $C \subset H$ is assumed to be nonempty and closed we obtain the various concepts of normal cones to C at a point $x \in C$ through the respective subdifferential of its *indicator function* $\mathbf{I}_C(\cdot)$ at x (which is equal to zero on C and to $+\infty$ elsewhere). Namely, the *proximal normal cone*, the most used throughout the work, is the convex (not necessarily closed and possibly trivial, i.e., equal to $\{0\}$) cone defined by (see [23, p. 25])

$$\mathbf{N}_C^p(x) := \left\{ v \in H : \exists \sigma \geq 0 \textit{ such that } \langle v, y - x \rangle \leq \sigma \|y - x\|^2 \textit{ for all } y \in C \right\}.$$

The *Fréchet normal cone* is the closed convex (and hence weakly closed) cone in H defined by (see [13, p. 229])

$$\mathbf{N}_C^f(x) := \left\{ v \in H : \limsup_{\substack{y \to x \\ y \in C}} \left\langle v, \frac{y - x}{\|y - x\|} \right\rangle \leq 0 \right\}.$$

The *limiting (Mordukhovich) normal cone*, in the case of a Hilbert space, is given by (see [66, p. 240])

$$\mathbf{N}_C^l(x) := \left\{ w - \lim_{n\to\infty} \zeta_n, \ \zeta_n \in \mathbf{N}_C^p(x_n), \ x_n \to x, \ x_n \in C \right\} \tag{2.5}$$

can be trivial (see [66, Corollary 2.24]) and is not necessarily convex (see [66, p. 5]) neither closed, in general (see [66, p. 11]). However in $\mathbb{R}^n$ the mapping $\mathbf{N}_C^l(\cdot)$ has closed graph (see [66, p. 11]).

The *Clarke normal cone* defined as $\partial^c \mathbf{I}_C(x)$ satisfies the following useful property (see [23, p. 88])

$$\mathbf{N}_C^c(x) = \overline{\text{co}}\mathbf{N}_C^l(x).$$

It can be proved that

$$\mathbf{N}_C^p(x) \subset \mathbf{N}_C^f(x) \subset \mathbf{N}_C^l(x) \subset \mathbf{N}_C^c(x), \quad \forall x \in C.$$

If C is convex then all cones coincide with the normal cone $\mathbf{N}_C(\cdot)$ in the sense of Convex Analysis.

Proximal convexity

For each $v \in \mathbf{N}_C^p(x)$, $v \neq 0$, let us define the function

$$\psi_C(x,v) := \frac{1}{\|v\|} \sup_{y\in C\setminus\{x\}} \frac{\langle v, y-x\rangle}{\|y-x\|^2} < +\infty$$

that measures the degree of "prominence" (or "cavity") of the set C at the point x with respect to the direction v.

Recalling the distance function $\mathrm{d}_C(x) := \inf\{\|x-y\| : y \in C\}$ in the case $\psi_C(x,v) > 0$ we have another representation:

$$\frac{1}{2\|v\|\psi_C(x,v)} = \sup\{\lambda > 0 : \mathrm{d}_C(x+\lambda v) = \lambda\|v\|\},$$

i.e., each sphere centred on the half-line $\{x+\lambda v : \lambda > 0\}$ and touching the boundary ∂C at x has radius $r \leq \frac{1}{2\psi_C(x,v)}$. Otherwise (if $\psi_C(x,v) \leq 0$) such sphere can have a radius arbitrarily large. Setting

$$\hat{\psi}_C(x,v) := \frac{1}{\|v\|} \limsup_{\substack{y\to x\\ y\in C}} \frac{\langle v, y-x\rangle}{\|y-x\|^2}$$

we get a local characteristic of the set C. Observe that C is "concave" at x with respect to the direction v whenever $\hat{\psi}_C(x,v) > 0$, and $\frac{1}{2\hat{\psi}_C(x,v)}$ is the "concavity radius".

For some purposes (compare, for instance, with the definitions of the Section 1.2) the number $-\hat{\psi}_C(x,v)$ can be interpreted as exterior (negative) curvature of the (nonconvex) set C. It is convenient to set also $\psi_C(x,0) = \hat{\psi}_C(x,0) = 0$. Since $\hat{\psi}_C(x,v) < +\infty$ iff $\psi_C(x,v) < +\infty$ (see [23, p. 25]), we have

$$\mathbf{N}_C^p(x) = \left\{ v \in H : \hat{\psi}_C(x,v) < +\infty \right\}.$$

If $\psi_C(x,v)$ is majorized by some continuous nonnegative function (say $\varphi(\cdot)$) uniformly in $v \in \mathbf{N}_C^p(x)$ (i.e., $\psi_C(x,v) \leq \varphi(x)$ for all $x \in \partial C$ and $v \in \mathbf{N}_C^p(x)$) then the set C is said to be *φ-convex* (or *proximally smooth*). In this case we have

$$\langle v, y - x \rangle \leq \varphi(x) \|v\| \|y - x\|^2 \quad \forall x, y \in C \text{ and } \forall v \in \mathbf{N}_C^p(x).$$

Another definition in terms of "almost monotonicity" of the normal cone can be given. Namely, a closed set $C \subset H$ is φ-convex iff for some continuous function $\varphi : C \to \mathbb{R}^+$ the inequality

$$\langle v - w, x - y \rangle \geq -(\varphi(x) \|v\| + \varphi(y) \|w\|) \|x - y\|^2$$

holds whenever $x, y \in C$, $v \in \mathbf{N}_C^p(x)$ and $w \in \mathbf{N}_C^p(y)$. In [27] it was proved (see Theorem 6.3) that φ-convexity is equivalent to the following geometric property: for each $x \in C$ there exist $r > 0$ and $p \geq 0$ such that for all $x_1, x_2 \in C \cap (x + r\mathbf{B})$ one has

$$\mathrm{d}_C\left(\frac{x_1 + x_2}{2}\right) \leq p \|x_1 - x_2\|^2.$$

If C is convex then we clearly set $\varphi(x) \equiv 0$. Since all the normal cones defined above to a φ-convex set coincide (see, e.g., [17, 27]), there is no ambiguity to write $\mathbf{N}_C(x)$ in the place of $\mathbf{N}_C^p(x)$.

Finally, we say that a closed set $C \subset H$ *has smooth* (or $\mathcal{C}^1$) *boundary at the point* $x_0 \in \partial C$ if there exist $\varepsilon > 0$ and a continuous mapping $\mathfrak{n}$: $\partial C \cap \left(x_0 + \varepsilon\overline{\mathbf{B}}\right) \to \partial\overline{\mathbf{B}}$ such that $\mathfrak{n}(x)$ is the unique vector in $\mathbf{N}_C^l(x)$ with $\|\mathfrak{n}(x)\| = 1$. If this property is satisfied globally (i.e., $\mathbf{N}_C^l(x) \cap \partial\overline{\mathbf{B}}$ is a singleton continuously depending on $x \in \partial C$) then the boundary of C is said to be *smooth.*

Let us define the *"reduced" boundary*

$$\partial^* C := \left\{ x \in \partial C : \mathbf{N}_C^p(x) \neq \{0\} \right\},$$

which is dense in ∂C (see [23, p. 49]).

A special case

Let us consider now the case when the set C is represented via some regular function and prove its φ-convexity.

Let us suppose that $C := \{x \in H : f(x) \leq 0\}$ where $f : H \to \mathbb{R}$ is a differentiable function with local Lipschitz continuous derivative $\nabla f(\cdot)$ such that $\nabla f(x) \neq 0$ for all $x \in \partial C$ (following the tradition we write $f(\cdot) \in \mathcal{C}_{loc}^{1,1}$). This is the most simple and natural example of a non convex but φ-convex set (see, e.g., [16, 17]). Since it often appears in the further considerations and examples, we think that it will be opportune to set here the complete proof of the φ-convexity of C.

Let us fix $x_0 \in \partial C$ and $\delta = \delta(x_0) > 0$ such that $\nabla f(\cdot)$ is Lipschitzean in $x_0 + \delta \overline{\mathbf{B}}$ with the Lipschitz constant $L = L(x_0)$. We prove first that there exists $p = p(x_0) > 0$ such that

$$\langle v, x_2 - x_1 \rangle \leq p \|v\| \|x_1 - x_2\|^2 \tag{2.6}$$

for all $x_1, x_2 \in C \cap (x_0 + \delta \mathbf{B})$ and $v \in \mathbf{N}_C^p(x_1)$. Then we construct a continuous function $\varphi : C \to \mathbb{R}^+$ such that the inequality

$$\langle v, y - x \rangle \leq \varphi(x) \|v\| \|y - x\|^2 \tag{2.7}$$

takes place whenever x, $y \in C$ and $v \in \mathbf{N}_C^p(x)$.

To prove (2.6) we assume that $\mathbf{N}_C^p(x_1) \neq \{0\}$ (otherwise (2.6) holds trivially for any $p > 0$). Consequently, $x_1 \in \partial C$. Since $f(\cdot)$ is continuously differentiable at x_1 (it is obviously locally Lipschitzean), we have $\partial^c f(x_1) = \{\nabla f(x_1)\} \neq \{0\}$ (see [23, p. 78]). Hence (see [22, p. 56])

$$\{0\} \neq \mathbf{N}_C^p(x_1) \subset \mathbf{N}_C^c(x_1) = \bigcup_{\lambda \geq 0} \lambda \nabla f(x_1)$$

which implies

$$\mathbf{N}_C^p(x_1) = \bigcup_{\lambda \geq 0} \lambda \nabla f(x_1).$$

Observing that $f(x_1) = 0$ and $f(x_2) \leq 0$, we find by Lagrange theorem $y = \lambda x_1 + (1 - \lambda) x_2$, $\lambda \in (0, 1)$ such that $\langle \nabla f(y), x_2 - x_1 \rangle = f(x_2) - f(x_1) \leq 0$. Therefore, by the Lipschitz continuity of $\nabla f(\cdot)$,

$$\begin{aligned} \langle \nabla f(x_1), x_2 - x_1 \rangle &\leq \langle \nabla f(x_1) - \nabla f(y), x_2 - x_1 \rangle \\ &\leq \|\nabla f(x_1) - \nabla f(y)\| \|x_2 - x_1\| \\ &\leq L \|x_2 - x_1\|^2 . \end{aligned} \tag{2.8}$$

Now let us fix $v \in \mathbf{N}_C^p(x_1)$ different from zero (in the case $v = 0$ there is nothing to prove). Then $v = \lambda \nabla f(x_1)$ for some $\lambda > 0$, and by (2.8) we obtain

$$\begin{aligned} \left\langle \frac{v}{\|v\|}, x_2 - x_1 \right\rangle &= \frac{1}{\|\nabla f(x_1)\|} \langle \nabla f(x_1), x_2 - x_1 \rangle \\ &\leq \frac{L}{\|\nabla f(x_1)\|} \|x_2 - x_1\|^2 \\ &\leq \frac{L}{\|\nabla f(x_0)\| - L\delta} \|x_2 - x_1\|^2 . \end{aligned}$$

Certainly, we should choose $\delta > 0$ sufficiently small (remind that $\nabla f(x_0) \neq 0$).

Thus we choose $p := \frac{L}{\|\nabla f(x_0)\| - L\delta}$ and the inequality (2.6) is proved. The point x_2 in (2.6) can be any point from C in fact (it is not necessary that x_2 is close to x_0). Indeed, given $x \in C \cap (x_0 + \delta \mathbf{B})$, $v \in \mathbf{N}_C^p(x)$ and $y \in C \backslash (x_0 + \delta \mathbf{B})$ we obtain

$$\langle v, y - x \rangle \leq \|v\| \|y - x\| \leq \frac{1}{\delta} \|v\| \|y - x\|^2 \leq p(x_0) \|v\| \|y - x\|^2 ,$$

where $p(x_0) := p + \frac{1}{\delta}$.

Notice that in the case $x_0 \in \operatorname{int} C$ we can choose $\delta = \delta(x_0) > 0$ arbitrary small such that $x_0 + \delta\overline{\mathbf{B}} \subset C$. Then (2.6) in this point clearly holds as well.

The sets $(x_0 + \delta(x_0)\mathbf{B}) \cap C$, $x_0 \in C$, form an open covering of the paracompact set C (C is a closed subset of a metric space). Therefore it admits a locally finite refinement $\{\mathcal{U}_i\}_{i \in I}$. For each $i \in I$ let us choose a point $x_i \in C$ such that

$$\mathcal{U}_i \subset (x_i + \delta(x_i)\mathbf{B}) \cap C.$$

If $\{e_i(\cdot)\}_{i \in I}$ is a continuous partition of unity associated to $\{\mathcal{U}_i\}_{i \in I}$ (see, e.g., [47, p. 23]) then we set

$$\varphi(x) = \sum_{i \in I} e_i(x)\, p(x_i).$$

>From the local finiteness of the covering $\{\mathcal{U}_i\}_{i \in I}$ and the continuity of each function $e_i(\cdot)$ it follows that the function $\varphi : C \to \mathbb{R}^+$ is well defined, finite and continuous. For each $x \in C$ and $i \in I$ with $e_i(x) \neq 0$ we have $x \in \mathcal{U}_i$, and

$$\langle v, y - x \rangle \le p(x_i)\, \|v\|\, \|y - x\|^2 \qquad \forall y \in C.$$

Multiplying each inequality by $e_i(x)$ and summing them we obtain

$$\langle v, y - x \rangle = \sum_{i \in I} e_i(x)\langle v, y - x \rangle \le \sum_{i \in I} e_i(x)\, p(x_i)\, \|v\|\, \|y - x\|^2 = \varphi(x)\, \|v\|\, \|y - x\|^2$$

that proves (2.7).

A property of minimizing sequences

Let us formulate three fundamental results of nonsmooth analysis that will be used in sequel.

Theorem 2.1.1 (Fuzzy sum rule [23, p. 56]) *Let $\phi_1, \phi_2 : H \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions, not identically equal to $+\infty$, $x_0 \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$, and let $\zeta \in \partial^p(\phi_1 + \phi_2)(x_0)$. Suppose that either ϕ_1 and ϕ_2 are lower semicontinuous w.r.t. weak topology on H (automatically the case if H is finite dimensional) or one of the functions is Lipschitz continuous near x_0. Then, for any $\varepsilon > 0$, there exist points $x_i \in x_0 + \varepsilon\mathbf{B}$, $i = 1, 2$, with $|\phi_i(x_0) - \phi_i(x_i)| < \varepsilon$ such that*

$$\zeta \in \partial^p\phi_1(x_1) + \partial^p\phi_2(x_2) + \varepsilon\mathbf{B}.$$

While in the case of limiting subdifferentials we have the following exact sum rule:

Theorem 2.1.2 ([23, p. 62]) *Let $\phi_1, \phi_2 : H \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions, not identically equal to $+\infty$, $x_0 \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$. If one of ϕ_1, ϕ_2 is Lipschitz continuous near x_0, then*

$$\partial^l(\phi_1 + \phi_2)(x_0) \subset \partial^l\phi_1(x_0) + \partial^l\phi_2(x_0).$$

Theorem 2.1.3 (Ekeland's variational principle [51, Corollary 11]) *Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, not identically equal to $+\infty$. If $x_0 \in H$ is such that*

$$\phi(x_0) \leq \inf_{x \in H} \phi(x) + \varepsilon,$$

then there exists $\overline{x} \in H$ such that

$$\begin{aligned} &\phi(\overline{x}) \leq \phi(x_0); \\ &\|\overline{x} - x_0\| \leq \sqrt{\varepsilon}; \\ &\phi(\overline{x}) < \phi(y) + \sqrt{\varepsilon}\,\|\overline{x} - y\|, \qquad \forall y \neq \overline{x}. \end{aligned}$$

The following lemma is crucial for proving the main theorems contained in the next sections.

Lemma 2.1.1 *Let $C \subset H$ be a nonempty closed set, $z \in H \backslash C$, and $\{x_n\} \subset C$ be a* minimizing sequence *for the function $x \mapsto \rho_F(x - z)$ on C, i.e., such that $\rho_F(x_n - z) \to \mathfrak{T}_C^F(z) +$ as $n \to \infty$. Then there exist another minimizing sequence $\{x'_n\} \subset C$ and sequences $\{x''_n\}$, $\{v_n\}$, $\{\xi_n^*\}$ such that $v_n \in \mathbf{N}_C^p(x'_n)$, $\xi_n^* \in \partial\rho_F(x''_n - z)$ and*

$$\|x'_n - x_n\| + \|x''_n - x_n\| \to 0, \tag{2.9}$$

$$\|v_n + \rho_F(x''_n - z)\,\xi_n^*\| \to 0, \tag{2.10}$$

as $n \to \infty$.

Proof.
Given an arbitrary sequence $\varepsilon_n \to 0+$ with $\rho_F^2(x_n - z) \leq \left(\mathfrak{T}_C^F(z)\right)^2 + \varepsilon_n^2$, by Theorem 2.1.3 there exists $\{y_n\} \subset C$ satisfying the conditions

$$\rho_F^2(y_n - z) \leq \left(\mathfrak{T}_C^F(z)\right)^2 + \varepsilon_n^2; \tag{2.11}$$

$$\|x_n - y_n\| \leq \varepsilon_n;$$

$$\rho_F^2(y_n - z) \leq \rho_F^2(y - z) + \varepsilon_n\|y - y_n\| \qquad \forall y \in C, \tag{2.12}$$

$n = 1, 2, \ldots$. The inequality (2.12), in particular, means that y_n minimizes the functional

$$F(y) := \rho_F^2(y - z) + \varepsilon_n\|y - y_n\| + \mathbf{I}_C(y)$$

on H, where $\mathbf{I}_C(\cdot)$ is the indicator function of the set C. Then we obviously have $0 \in \partial^p F(y_n)$ (see [23, p. 37]). According to Theorem 2.1.1

$$\begin{aligned} 0 &\in \partial^p \mathbf{I}_C(x'_n) + \partial\left(\rho_F^2(x''_n - z) + \varepsilon_n\|x''_n - y_n\|\right) + \varepsilon_n\overline{\mathbf{B}} \\ &= \mathbf{N}_C^p(x'_n) + 2\rho_F(x''_n - z)\,\partial\rho_F(x''_n - z) + \varepsilon_n\frac{x''_n - y_n}{\|x''_n - y_n\|} + \varepsilon_n\overline{\mathbf{B}} \\ &\subset \mathbf{N}_C^p(x'_n) + 2\rho_F(x''_n - z)\,\partial\rho_F(x''_n - z) + 2\varepsilon_n\overline{\mathbf{B}} \end{aligned}$$

for some sequences $\{x'_n\} \subset C$ and $\{x''_n\} \subset H$, $\|x'_n - y_n\| \le \varepsilon_n$, $\|x''_n - y_n\| \le \varepsilon_n$, $n \in \mathbb{N}$. We are able to apply Theorem 2.1.1, since the function $x \mapsto \rho_F(x - z)$ and the norm are convex and, consequently, locally Lipschitzean. Therefore we can find vectors $v_n \in \mathbf{N}^p_C(x'_n)$ and $\xi^*_n \in \partial\rho_F(x''_n - z)$ such that

$$\left\|v_n + \rho_F\left(x''_n - z\right)\xi^*_n\right\| \le \varepsilon_n,$$

and the property (2.10) holds. It follows from (2.11) that $\{x'_n\}$ is a minimizing sequence of $x \mapsto \rho_F(x - z)$ on C:

$$\rho_F\left(x'_n - z\right) \le \rho_F\left(x'_n - y_n\right) + \rho_F\left(y_n - z\right) \le \mathfrak{T}^F_C(z) + \left(1 + \|F^o\|\right)\varepsilon_n,$$

and (2.9) is also valid. ■

Remark 2.1.1 *The relation (2.10), in particular, shows that x'_n belongs to $\partial^* C$ for all n large enough, since otherwise either $\xi^*_n \to 0$ or $x'' \to z$, but both are impossible since $\xi^*_n \in \partial F^o$ (see Proposition 1.1.1), while $x''_n \to z$ would imply $y_n \to z$ and $z \in \overline{C} = C$.*

Remark 2.1.2 *The vectors v_n in Lemma 2.1.1 can be chosen such that*

$$\rho_{F^o}(-v_n) = \rho_F\left(x''_n - z\right), \tag{2.13}$$

$n = 1, 2, \ldots$. Indeed, setting $v'_n := v_n \frac{\rho_F(x''_n - z)}{\rho_{F^o}(-v_n)} \in \mathbf{N}^p_C(x'_n)$ we have, by the Lipschitz continuity of $\rho_{F^o}(\cdot)$ (see (1.3)) and by (2.10),

$$\begin{aligned}\left\|v_n - v'_n\right\| &= \frac{\|v_n\|}{\rho_{F^o}(-v_n)}\left|\rho_F\left(x''_n - z\right)\rho_{F^o}(\xi^*_n) - \rho_{F^o}(-v_n)\right| \\ &\le \|F^o\|\,\|F\|\left\|\rho_F\left(x''_n - z\right)\xi^*_n + v_n\right\| \to 0,\end{aligned}$$

and, therefore, v'_n can substitute v_n.

Remark 2.1.3 *In the case when all the basic normal cones to the set C coincide (e.g., if C is φ-convex), in the proof of Lemma 2.1.1 we may use the limiting subdifferential (see (2.1)) in the place of $\partial^p F(y)$, and apply the precise sum rule (see 2.1.2) instead of the fuzzy one. In this way we obtain a stronger statement of Lemma 2.1.1, which gives $x'_n = x''_n$.*

2.2 Neighbourhood retractions to nonconvex sets

Basic hypotheses

Let us introduce two types of local hypotheses for the sets F and C.

We say that the pair of sets (F, C) satisfies *the condition* (**A**) at a point $x_0 \in \partial C$ if there exists $\delta > 0$ such that

($\mathbf{A}_1$) the mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}^p_C(x) \cap \partial F^o\right)$ is single-valued and Lipschitz continuous on

$$C_\delta(x_0) := \{x \in \partial^* C : \|x - x_0\| \le \delta\};$$

($\mathbf{A}_2$) F is uniformly strictly convex with respect to

$$U_{\delta,\delta'}(x_0) := \partial F^o \cap \bigcup_{x\in C_\delta(x_0)} \left[-\mathbf{N}_C^p(x) \cap \partial F^o + \delta'\overline{\mathbf{B}}\right] \tag{2.14}$$

for some $\delta' > 0$.

Alternatively, we say that (F, C) satisfies *the condition* (**B**) at $x_0 \in \partial C$ if for some $\delta > 0$

($\mathbf{B}_1$) the function $\psi_C(x, v)$ is upper bounded on the set

$$\left\{(x, v) : x \in C_\delta(x_0),\ v \in \mathbf{N}_C^p(x)\right\}$$

(or, in other words, C is proximally smooth in a neighbourhood of the point x_0);

($\mathbf{B}_2$) there exist $\delta' > 0$ and $K > 0$ such that

$$\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) \geq K \quad \text{for all } \xi^* \in \hat{U}_{\delta,\delta'}(x_0),$$

where

$$\hat{U}_{\delta,\delta'}(x_0) := \partial F^o \cap \bigcup_{x\in C_\delta(x_0)\setminus\{x_0\}} \left[-\mathbf{N}_C^p(x) \cap \partial F^o + \delta'\overline{\mathbf{B}}\right]. \tag{2.15}$$

We are ready now to formulate the main results of this chapter.

Main results

Theorem 2.2.1 *Assume that at each point $x_0 \in \partial C$ the pair of sets (F, C) satisfies either the condition* (**A**) *or* (**B**). *Then there exists an open set $\mathcal{U} \supset C$ such that for each $z \in \mathcal{U}$ the time-minimum projection $\pi_C^F(z)$ is a singleton, and the mapping $z \mapsto \pi_C^F(z)$ is continuous on $\mathcal{U}$.*

Proof.
We prove first that given $x_0 \in \partial C$ one can find an (open) neighbourhood $\mathcal{U}(x_0)$ such that for an arbitrary $z \in \mathcal{U}(x_0)$ each minimizing sequence $\{x_n\}$ of $x \mapsto \rho_F(x - z)$ on the set C is a Cauchy sequence.

Case 1. The condition (**A**) holds at the point x_0. Then we set

$$\mathcal{U}(x_0) := \left\{z \in H : \|z - x_0\| < \frac{\delta}{\|F\|\,\|F^o\| + 1},\ \ \mathfrak{T}_C^F(z) < \frac{1}{L}\right\}, \tag{2.16}$$

where $L > 0$ is the Lipschitz constant of $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^o\right)$ on $C_\delta(x_0)$ (see ($\mathbf{A}_1$)). Fix $z \in \mathcal{U}(x_0) \setminus C$ and a minimizing sequence $\{x_n\} \subset C$. Let us choose $\{x_n'\} \subset \partial^* C$, $\{x_n''\} \subset H$,

$v_n \in \mathbf{N}^p_C(x'_n)$ and $\xi^*_n \in \partial\rho_F(x''_n - z)$ as in Lemma 2.1.1 and such that $\rho_F(x''_n - z) = \rho_{F^o}(-v_n)$, $n = 1, 2, \ldots$ (see remarks 2.1.1 and 2.1.2). Since by (1.2)

$$\begin{aligned} \|x_n - x_0\| &\le \|F\|\,\rho_F(x_n - z) + \|z - x_0\| \\ &= \|F\|\left(\rho_F(x_n - z) - \mathfrak{T}^F_C(z)\right) + \|z - x_0\| + \|F\|\,\mathfrak{T}^F_C(z) \\ &\le \|F\|\left(\rho_F(x_n - z) - \mathfrak{T}^F_C(z)\right) + (\|F\|\,\|F^o\| + 1)\,\|z - x_0\|, \end{aligned} \tag{2.17}$$

and $\rho_F(x_n - z) - \mathfrak{T}^F_C(z) \to 0+$, $\|x_n - x'_n\| \to 0$ as $n \to \infty$, we can suppose, without loss of generality, that $x'_n \in C_\delta(x_0)$ for all $n = 1, 2, \ldots$. Consider a decreasing sequence $\nu_n \to 0+$ such that

$$\|x'_n - x_n\| + \|x''_n - x_n\| \le \nu_n; \tag{2.18}$$

$$\rho_F(x'_n - z) \le \mathfrak{T}^F_C(z) + \nu_n; \tag{2.19}$$

$$\|v_n + \rho_F(x''_n - z)\,\xi^*_n\| \le \frac{1}{2}\mathfrak{T}^F_C(z)\,\nu_n, \tag{2.20}$$

$n = 1, 2, \ldots$ (see Lemma 2.1.1). It follows, in particular, from (2.20), (2.18) and from

$$\mathfrak{T}^F_C(z) \le \rho_F(x_n - z) \le \|F^o\|\,\|x''_n - x_n\| + \rho_F(x''_n - z)$$

that

$$\left\|\frac{v_n}{\rho_{F^o}(-v_n)} + \xi^*_n\right\| \le \frac{\mathfrak{T}^F_C(z)}{2\rho_F(x''_n - z)}\nu_n \le \frac{\mathfrak{T}^F_C(z)}{2\left(\mathfrak{T}^F_C(z) - \|F^o\|\,\|x''_n - x_n\|\right)}\nu_n \le \nu_n, \tag{2.21}$$

for every n large enough. Furthermore (see Proposition 1.2.2), the hypothesis $(\mathbf{A}_2)$ implies that the (single-valued) mapping $\mathfrak{J}_F(\cdot)$ is uniformly continuous on $U_{\delta,\delta'}(x_0)$, and, therefore, the (decreasing) sequence

$$\beta_n := \sup_{\substack{\|\xi^* - \eta^*\| \le \nu_n \\ \xi^*, \eta^* \in U_{\delta,\delta'}(x_0)}} \|\mathfrak{J}_F(\xi^*) - \mathfrak{J}_F(\eta^*)\|$$

tends to zero as $n \to \infty$.

Observe that $\xi^*_n \in \partial\rho_F(x''_n - z) = \mathbf{N}_F\left(\frac{x''_n - z}{\rho_F(x''_n - z)}\right) \cap \partial F^o$ (see Proposition 1.1.1), and hence

$$\frac{x''_n - z}{\rho_F(x''_n - z)} = \mathfrak{J}_F(\xi^*_n) \tag{2.22}$$

(recall that $\mathfrak{J}^{-1}_F(\xi) = \mathbf{N}_F(\xi) \cap \partial F^o$ for $\xi \in \partial F$). By (2.21) we have ξ^*_n, $-\frac{v_n}{\rho_{F^o}(-v_n)} \in U_{\delta,\delta'}(x_0)$, and, consequently,

$$\left\|\mathfrak{J}_F(\xi^*_n) - \mathfrak{J}_F\left(-\frac{v_n}{\rho_{F^o}(-v_n)}\right)\right\| \le \beta_n, \quad n = 1, 2, \ldots . \tag{2.23}$$

Given $m \ge n$ we obtain from (2.18) and (2.19):

$$\begin{aligned} &\left|\rho_F(x''_m - z) - \rho_F(x''_n - z)\right| \\ \le\ &\|F^o\|\,\|x''_m - x'_m\| + \left|\rho_F(x'_m - z) - \rho_F(x'_n - z)\right| + \|F^o\|\,\|x''_n - x'_n\| \\ \le\ &2\,\|F^o\|\,\nu_n + 2\nu_n \end{aligned}$$

and

$$\rho_F\left(x''_m - z\right) \le \|F^o\| \left\|x''_m - x'_m\right\| + \rho_F\left(x'_m - z\right) \le \|F^o\| \nu_n + \mathfrak{T}_C^F(z) + \nu_n.$$

Notice that here we use the fact that (v_n) is decreasing. Now by using (2.22) and (2.23) we obtain

$$\begin{aligned}
\left\|x''_m - x''_n\right\| &= \left\|\rho_F\left(x''_m - z\right)\mathfrak{J}_F\left(\xi^*_m\right) - \rho_F\left(x''_n - z\right)\mathfrak{J}_F\left(\xi^*_n\right)\right\| \\
&\le \rho_F\left(x''_m - z\right)\left\|\mathfrak{J}_F\left(\xi^*_m\right) - \mathfrak{J}_F\left(\xi^*_n\right)\right\| + \left|\rho_F\left(x''_m - z\right) - \rho_F\left(x''_n - z\right)\right| \|F\| \\
&\le \left(\|F^o\|\nu_n + \nu_n + \mathfrak{T}_C^F(z)\right)\left\|\mathfrak{J}_F\left(\xi^*_m\right) - \mathfrak{J}_F\left(\xi^*_n\right)\right\| + 2\|F\|\left(\|F^o\| + 1\right)\nu_n \\
&\le \mathfrak{T}_C^F(z)\left\|\mathfrak{J}_F\left(\xi^*_m\right) - \mathfrak{J}_F\left(\xi^*_n\right)\right\| + 4\nu_n\|F\|\left(\|F^o\| + 1\right) \\
&\le \mathfrak{T}_C^F(z)\left\|\mathfrak{J}_F\left(-\frac{v_m}{\rho_{F^o}\left(-v_m\right)}\right) - \mathfrak{J}_F\left(-\frac{v_n}{\rho_{F^o}\left(-v_n\right)}\right)\right\| + \\
&\quad + 2\mathfrak{T}_C^F(z)\beta_n + 4\nu_n\|F\|\left(\|F^o\| + 1\right).
\end{aligned} \tag{2.24}$$

Since $-\frac{v_n}{\rho_{F^o}(-v_n)} \in -\mathbf{N}_C^p\left(x'_n\right) \cap \partial F^o$, applying the condition $(\mathbf{A}_1)$ we find from (2.18) and (2.24) that

$$\left(1 - L\mathfrak{T}_C^F(z)\right)\left\|x'_m - x'_n\right\| \le \nu'_n$$

for some sequence $\nu'_n \to 0+$, as $n \to \infty$. Hence, by the choice of z (see (2.16)) we conclude that $\{x'_n\}$ (and $\{x_n\}$ as well) is a Cauchy sequence.

Case 2. If at the point x_0 the condition $(\mathbf{B})$ holds then we set

$$\mathcal{U}(x_0) := \left\{z \in H : \|z - x_0\| < \frac{\delta}{\|F\|\|F^o\| + 1},\ \ \mathfrak{T}_C^F(z) < \frac{K}{M}\right\}, \tag{2.25}$$

where the constant $M > 0$ is such that $\psi_C(x, v) \le M$ for all $x \in C_\delta(x_0)$ and $v \in \mathbf{N}_C^p(x)$. Let $z \in \mathcal{U}(x_0)\backslash C$ and $\{x_n\} \subset C$ be a minimizing sequence of $x \mapsto \rho_F(x - z)$ on C. Everything is already proved if $x_n \to x_0$, $n \to \infty$. Otherwise, as we'll see in sequel, there is no loss of generality to suppose that x_0 is not a cluster point of $\{x_n\}$, and that the sequence $\{\rho_F(x_n - z)\}$ is nonincreasing. By using Lemma 2.1.1 similarly to the Case 1 we choose sequences $\{x'_n\} \subset \partial^* C$, $\{x''_n\} \subset H$, $v_n \in \mathbf{N}_C^p(x'_n)$ and $\xi^*_n \in \partial\rho_F(x''_n - z)$ satisfying (2.9), (2.10) and (2.13). Observe that, in virtue of the hypothesis $(\mathbf{B}_1)$, a simpler version of Lemma 2.1.1 holds that gives $x'_n = x''_n$ (see Remark 2.1.3). But, for the sake of uniformity, we prefer to keep all the notations. We can assume, certainly, that $0 < \|x'_n - x_0\| < \delta$, $n = 1, 2, \ldots$ (see (2.9) and (2.17)). Let us choose a decreasing sequence $\nu_n \to 0+$ satisfying the inequalities (2.18)-(2.21), and assume that $\nu_n \le \delta'$, $n \in \mathbb{N}$. Since $x'_n \in C_\delta(x_0) \setminus \{x_0\}$ and $-v_n/\rho_{F^o}(-v_n) \in -\mathbf{N}_C^p(x'_n) \cap \partial F^o$, we obtain from (2.21) that $\xi^*_n \in \hat{U}_{\delta,\delta'}(x_0)$ (see (2.15)).

For convenience we introduce the following notations:

$$\begin{aligned}
\rho_n &:= \rho_F\left(x''_n - z\right); \\
G_n &:= z + \rho_n F; \\
R_n &:= \frac{1}{\|\xi^*_n\|}\,\mathfrak{R}_F\left(\xi_n, \xi^*_n\right) = \frac{1}{2\gamma_F\left(\xi_n, \xi^*_n\right)} \quad \text{(see (1.30), (1.31))}; \\
\psi_n &:= \psi_C\left(x'_n, v_n\right).
\end{aligned}$$

Here $\xi_n := \mathfrak{J}_F(\xi_n^*)$ can be found as in the Case 1 (see (2.22)), because $(\mathbf{B}_2)$ implies that F is uniformly strictly convex w.r.t. the set $\hat{U}_{\delta,\delta'}(x_0)$ and, by Proposition 1.2.2, $\mathfrak{J}_F(\cdot)$ is single-valued on the same set. Combining the hypotheses $(\mathbf{B}_1)$ and $(\mathbf{B}_2)$ we have from the above arguments:

$$\begin{aligned}\frac{1}{2R_n} - \mathfrak{T}_C^F(z)\|\xi_n^*\|\psi_n &= \|\xi_n^*\|\left(\frac{\gamma_F(\xi_n,\xi_n^*)}{\|\xi_n^*\|} - \mathfrak{T}_C^F(z)\psi_C(x_n',v_n)\right)\\ &\geq \|\xi_n^*\|\left(K - \mathfrak{T}_C^F(z)M\right)\\ &\geq 2\nu := \frac{1}{\|F\|}\left(K - \mathfrak{T}_C^F(z)M\right) > 0.\end{aligned} \tag{2.26}$$

By (2.20), (2.18) and (2.19), respectively,

$$\begin{aligned}\|v_n\| &\leq \frac{1}{2}\mathfrak{T}_C^F(z)\nu_n + \rho_F(x_n'' - z)\|\xi_n^*\|\\ &\leq \frac{1}{2}\mathfrak{T}_C^F(z)\nu_n + \left(\|F^o\|\left\|x_n'' - x_n'\right\| + \rho_F(x_n' - z)\right)\|\xi_n^*\|\\ &\leq \frac{1}{2}\mathfrak{T}_C^F(z)\nu_n + \left(\|F^o\|\nu_n + \mathfrak{T}_C^F(z) + \nu_n\right)\|\xi_n^*\|\\ &\leq \mathfrak{T}_C^F(z)\|\xi_n^*\| + \mu_n,\end{aligned}$$

for some $\mu_n \to 0$ as $n \to \infty$. Using this, (2.26) and the boundedness of the sequence $\{\psi_n\}$ we can choose $\nu_n' > 0$ such that

$$\frac{1}{2(R_n + \nu_n')} - \|v_n\|\psi_n \geq \nu \tag{2.27}$$

for $n \in \mathbb{N}$ large enough (assume that for all n).

Let us consider the approximate *curvature centre* of the set G_n (at the point x_n'')

$$z_n := x_n'' - \rho_n(R_n + \nu_n')\xi_n^*. \tag{2.28}$$

We claim that for each $m \geq n$

$$\left\|z_n - x_m''\right\| \leq \left\|z_n - x_n''\right\| + 2\|F^o\|\|F\|\nu_n. \tag{2.29}$$

Indeed, the monotonicity of the sequence $\{\rho_F(x_n - z)\}$ implies

$$\begin{aligned}\rho_m &= \rho_F(x_m'' - z)\\ &\leq \|F^o\|\left\|x_m'' - x_m\right\| + \rho_F(x_m - z)\\ &\leq \|F^o\|\nu_n + \rho_F(x_n - z)\\ &\leq \|F^o\|\nu_m + \rho_F(x_n'' - z) + \|F^o\|\left\|x_n'' - x_n\right\|\\ &\leq \rho_n + 2\|F^o\|\nu_n.\end{aligned}$$

On the other hand, from the definition of G_n, from (1.29), (2.22) and (2.28) we obtain:

$$\begin{aligned}G_n &\subset z + \rho_n\left(\mathfrak{J}(\xi_n^*) - (R_n + \nu_n')\xi_n^* + (R_n + \nu_n')\|\xi_n^*\|\overline{\mathbf{B}}\right)\\ &= x_n'' - \rho_n(R_n + \nu_n')\xi_n^* + \rho_n(R_n + \nu_n')\|\xi_n^*\|\overline{\mathbf{B}}\\ &= z_n + \left\|x_n'' - z_n\right\|\overline{\mathbf{B}}.\end{aligned}$$

Since the sequence $\{\|z_n - x_n''\|\}$ is bounded (due to (2.28), (2.18) and (2.19)), by the claim above (see (2.29))

$$\left\|z_n - x_m''\right\|^2 \le \left\|z_n - x_n''\right\|^2 + \mu_n''$$

for some $\mu_n'' \to 0$ as $n \to \infty$. Now recalling (2.30), we can write the estimate (2.32) in the form:

$$\begin{aligned} \frac{1}{2}\left\|x_m' - x_n'\right\|^2 &\le \frac{1}{2}\left(2\nu_n + \left\|x_m'' - x_n''\right\|\right)^2 \\ &\le \frac{1}{2}\mu_n'' + \left(R_n + \nu_n'\right)\left(\psi_n \left\|v_n\right\| \left\|x_m' - x_n'\right\|^2 + \mu_n'\right) + 2\left(\nu_n \left\|x_m'' - x_n''\right\| + \nu_n^2\right). \end{aligned}$$

Hence, by the *a priori* estimate (1.32), we have

$$\left(\frac{1}{2\left(R_n + \nu_n'\right)} - \psi_n \left\|v_n\right\|\right)\left\|x_m' - x_n'\right\|^2 \le \overline{\mu}_n,$$

for some $\overline{\mu}_n \to 0$ as $n \to \infty$. The Cauchy property of the sequence $\{x_n'\}$ (and of $\{x_n\}$ as well) follows from this inequality together with (2.27).

Let us pass now to the second part of the proof. Denote by

$$\mathcal{U} = \bigcup_{x_0 \in C} \mathcal{U}(x_0) \supset C,$$

where we put $\mathcal{U}(x_0) := \operatorname{int} C$ for $x_0 \in \operatorname{int} C$. Given $x_0 \in \partial C$, $z \in \mathcal{U}(x_0) \setminus C$ and a minimizing sequence $\{x_n\} \subset C$ of $x \mapsto \rho_F(x - z)$ on C, in the Case 1 (i.e., when the condition (**A**) is valid at x_0) we immediately find the (unique) projection $\pi_C^F(z)$ as the limit of $\{x_n\}$, existing since it is a Cauchy sequence. Otherwise (when the condition (**B**) holds) we choose first a subsequence $\{x_{k_n}\}$ such that $\{\rho_F(x_{k_n} - z)\}$ is nonincreasing, and x_0 is not a cluster point of $\{x_{k_n}\}$. Being a Cauchy sequence it converges to an element $x \in \pi_C^F(z)$. Assuming that $x, y \in \pi_C^F(z)$ with $x \ne y$ we consider the sequence $\{x_n\}$ whose odd terms are equal to x and all even terms are equal to y. Since $\{\rho_F(x_n - z)\}$ is now stationary, we can again apply the first part of the proof and conclude the convergence of $\{x_n\}$ to $x = y$. Notice that the above arguments are applicable also if one of the points x or y coincides with x_0 (because for a pair of natural numbers n and m with $m \ge n$ we utilize the hypothesis ($\mathbf{B}_2$) at the point x_n' only). In order to show continuity at the point $z \in \mathcal{U}$ let us observe that for each $\{z_n\} \subset \mathcal{U}$ converging to z the sequence $\left\{\pi_C^F(z_n)\right\}$ minimizes $x \mapsto \rho_F(x - z)$ on C. Indeed,

$$\begin{aligned} \rho_F\left(\pi_C^F(z_n) - z\right) &\le \rho_F\left(\pi_C^F(z_n) - z_n\right) + \rho_F(z_n - z) \\ &\le \mathfrak{T}_C^F(z) + 2\left\|F^o\right\|\left\|z_n - z\right\| \to \mathfrak{T}_C^F(z) + . \end{aligned}$$

Thus, by the same reasons as above, each subsequence of $\left\{\pi_C^F(z_n)\right\}$ admits a subsequence converging to $\pi_C^F(z)$. So $\pi_C^F(z_n) \to \pi_C^F(z)$, and the theorem is completely proved. ■

Thus we have two types of local assumptions guaranteeing the well-posedness of the time--minimum projection in a neighbourhood of a fixed point $x_0 \in \partial C$. The first one (the condition (**A**)) provides regularity of the superposition operator involving both the proximal normal cone to C and the gradient $\nabla \rho_{F^o}(\cdot)$, while the other involves the curvatures of F and C being square characteristics of these sets. Therefore, we can refer to (**A**) and (**B**) as to the first and to the second order condition, respectively. Although there is a large class of problems, which satisfy both hypotheses (for instance, if $F = \overline{\mathbf{B}}$ and $C = \{x \in H : f(x) \leq 0\}$, where $f(\cdot)$ is a locally $C^{1,1}$ function with $\nabla f(x) \neq 0$ for $x \in \partial C$), simple examples show (see Section 2.4) that none of the two ((**A**) and (**B**)) implies the other. At the end of Section 2.3 we amplify a little bit this list of local conditions including some extreme cases.

If the set C is proximally smooth then we can give an explicit formula for a neighbourhood where the continuous retraction $\pi_C^F(\cdot)$ is defined, which has, however, mainly theoretic interest due to the fact that it involves approximations to the projection itself. To this end let us consider a slightly stronger hypothesis than ($\mathbf{B}_2$). Namely, we say that (F, C) satisfies the *condition* ($\mathbf{B}_2'$) at a point $x_0 \in \partial C$ if there exist $\delta, \delta' > 0$ and $K > 0$ such that

$$\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) \geq K \quad \text{for all } \xi^* \in U_{\delta,\delta'}(x_0),$$

where the set $U_{\delta,\delta'}(x_0)$ is defined by (2.14).

Theorem 2.2.2 *Assume that $C \subset H$ is φ-convex with a continuous function $\varphi : C \to \mathbb{R}^+$, and at each point $x_0 \in \partial C$ the pair (F, C) satisfies the condition ($\mathbf{B}_2'$). Then the mapping $z \mapsto \pi_C^F(z)$ is single-valued and continuous on the open set $\mathfrak{A}(C)$ of all points $z \in H$, which either belong to C or satisfy the inequality*

$$\liminf_{\mathfrak{F}(z)} \left\{\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\varphi(x)\right\} > 0, \tag{2.33}$$

where $\mathfrak{F}(z)$, $z \notin C$, is the filter in H^3 generated by the sets

$$\left\{(x, v, \xi^*) : \rho_F(x - z) < \mathfrak{T}_C^F(z) + \varepsilon,\ x \in \partial C,\ v \in \mathbf{N}_C(x),\ \|\xi^* + v\| < \varepsilon,\ \xi^*, -v \in \partial F^o\right\}, \quad \varepsilon > 0.$$

Proof.

In order to prove openness of $\mathfrak{A}(C)$ let us take first $z \in \mathfrak{A}(C) \setminus C$ and choose $\nu > 0$, $\varepsilon > 0$ such that

$$\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\varphi(x) \geq \nu \tag{2.34}$$

whenever $x \in \partial C$ with $\rho_F(x - z) \leq \mathfrak{T}_C^F(z) + \varepsilon$ and $v \in \mathbf{N}_C(x)$, $\xi^*, -v \in \partial F^o$ with $\|\xi^* + v\| \leq \varepsilon$. By the *a priori* estimate (1.32) the function $\varphi(\cdot)$ is bounded on the set of x satisfying (2.34), say $\varphi(x) \leq M$ with some $M > 0$. Set

$$\varepsilon' := \min\left\{\frac{\varepsilon}{4\|F^o\|}, \frac{\varepsilon}{2}, \frac{\nu}{2M\|F^o\|}\right\}.$$

Assuming, moreover, that $\left(z + \varepsilon'\overline{\mathbf{B}}\right) \cap C = \emptyset$, for each $z' \in z + \varepsilon'\overline{\mathbf{B}}$ let us define the set

$$P(z') := \left\{x \in \partial C : \rho_F(x - z') \leq \mathfrak{T}_C^F(z') + \varepsilon'\right\} \neq \emptyset.$$

For each $x \in P(z')$ by the choice of ε' we have

$$\begin{aligned}\rho_F(x-z) &\le \rho_F(x-z') + \|F^o\|\,\|z'-z\| \\ &\le \mathfrak{T}_C^F(z') + \varepsilon' + \|F^o\|\,\varepsilon' \\ &\le \mathfrak{T}_C^F(z) + \|F^o\|\,\|z'-z\| + \varepsilon'(1+\|F^o\|) \\ &\le \mathfrak{T}_C^F(z) + \varepsilon'(1+2\|F^o\|) \\ &\le \mathfrak{T}_C^F(z) + \varepsilon.\end{aligned}$$

Then, by (2.34) for arbitrary vectors $\xi^*, -v \in \partial F^o$ with $v \in \mathbf{N}_C(x)$, $x \in P(z')$ and $\|\xi^* + v\| \le \varepsilon' \le \varepsilon$, we have

$$\begin{aligned}\varkappa_F(\mathfrak{J}_F(\xi^*),\xi^*) - \mathfrak{T}_C^F(z')\varphi(x) &\ge \varkappa_F(\mathfrak{J}_F(\xi^*),\xi^*) - \left(\mathfrak{T}_C^F(z) + \|F^o\|\,\|z'-z\|\right)\varphi(x) \\ &\ge \varkappa_F(\mathfrak{J}_F(\xi^*),\xi^*) - \mathfrak{T}_C^F(z)\varphi(x) - \|F^o\|\,\varepsilon' M \\ &\ge \frac{\nu}{2},\end{aligned} \tag{2.35}$$

which implies that $z' \in \mathfrak{A}(C)$.

Let now $z := x_0 \in \partial C$. By the hypothesis $(\mathbf{B}_2')$ and continuity of the function $\varphi(\cdot)$ there exist $\delta, \delta' > 0$ and positive constants K, M such that $\varkappa_F(\mathfrak{J}_F(\xi^*),\xi^*) \ge K$ for all $\xi^* \in U_{\delta,\delta'}(x_0)$ and $\varphi(x) \le M$ for all $x \in C_\delta(x_0)$. Set

$$\varepsilon' := \frac{1}{2}\min\left\{\frac{K}{\|F^o\|\,M}, \frac{\delta}{\|F\|\,(\|F^o\|+1)}\right\}.$$

For $z' \in \left(x_0 + \varepsilon'\overline{\mathbf{B}}\right)\setminus C$ and $x \in P(z')$, taking into account that $\|F\|\,\|F^o\| \ge 1$, we have

$$\begin{aligned}\|x - x_0\| &\le \|F\|\,\rho_F(x-z') + \|z'-x_0\| \\ &\le \|F\|\left(\mathfrak{T}_C^F(z') + \varepsilon'\right) + \varepsilon' \\ &\le \|F\|\left(\|F^o\|\,\|x_0 - z'\| + \varepsilon'\right) + \varepsilon' \\ &\le \varepsilon'\left(\|F\|\,(\|F^o\|+1)+1\right) \le \delta,\end{aligned}$$

and

$$\mathfrak{T}_C^F(z')\,\varphi(x) \le \|F^o\|\,\|x_0 - z'\|\,M \le \|F^o\|\,\varepsilon' M \le \frac{K}{2}.$$

If, furthermore, $v \in \mathbf{N}_C(x)$, $\rho_{F^o}(-v) = 1$, and $\xi^* \in \partial F^o$ with $\|\xi^* + v\| \le \varepsilon' < \delta'$ then clearly $\xi^* \in U_{\delta,\delta'}(x_0)$, and we obtain the inequality (2.35) with $\nu = K$. Consequently, $z' \in \mathfrak{A}(C)$ and therefore $\mathfrak{A}(C)$ is an open neighbourhood of C.

Proving the well-posedness of the projection $\pi_C^F(\cdot)$ we can proceed as in the proof of Theorem 2.2.1 with some minor changements. Let us fix $z \in \mathfrak{A}(C)$, $z \notin C$, and take a minimizing sequence $\{x_n\}$ for $x \mapsto \rho_F(x-z)$ on the set C, assuming that $\{\rho_F(x_n - z)\}$ decreases (may be not strictly). Then we choose the sequences $\{x_n'\} \subset \partial C$, $v_n \in \mathbf{N}_C(x_n')$, $\xi_n^* \in \partial\rho_F(x_n' - z)$ from Lemma 2.1.1 (see remarks 2.1.2 and 2.1.3). Observe that (2.10) implies that

$$\mathfrak{T}_C^F(z)\,\|\xi_n^*\| - \|v_n\| \le \rho_F(x_n' - z)\,\|\xi_n^*\| - \|v_n\| \to 0$$

and

$$\begin{aligned}\|v_n\| - \mathfrak{T}_C^F(z)\|\xi_n^*\| &\le \|v_n\| - (\rho_F(x_n' - z) - \nu_n)\|\xi_n^*\| \\ &= (\|v_n\| - \rho_F(x_n' - z)\|\xi_n^*\|) + \nu_n\|\xi_n^*\| \to 0.\end{aligned}$$

Hence, we can choose a decreasing sequence $\nu_n \to 0+$ such that $\nu_n \le \delta'$, $\|x_n' - x_n\| \le \nu_n$, $\rho_F(x_n' - z) \le \mathfrak{T}_C^F(z) + \nu_n$ and

$$\left|\mathfrak{T}_C^F(z)\|\xi_n^*\| - \|v_n\|\right| \le \nu_n, \tag{2.36}$$

$n = 1, 2, \dots$.

Since $-v_n/\rho_{F^o}(-v_n) \in -\mathbf{N}_C(x_n') \cap \partial F^o$ and by (2.10)

$$\left\|\frac{v_n}{\rho_{F^o}(-v_n)} + \xi_n^*\right\| \to 0$$

as $n \to \infty$, we find (see (2.33)) a number $\nu > 0$ such that

$$\varkappa_F(\mathfrak{J}_F(\xi_n^*), \xi_n^*) - \mathfrak{T}_C^F(z)\varphi(x_n') \ge 2\nu\|F\| \tag{2.37}$$

for $n \ge 1$ large enough (assume that for all n).

Notice that the vectors ξ_n^* belong to the set $U_{\delta,\delta'}(x_n')$ where the mapping $\mathfrak{J}_F(\cdot)$ is single--valued. Let us set, as usual, $\xi_n = \mathfrak{J}_F(\xi_n^*)$. Denoting by

$$R_n := \frac{1}{2\|\xi_n^*\|\varkappa_F(\xi_n, \xi_n^*)}$$

and $\varphi_n := \varphi(x_n')$ we obtain from (2.37)

$$\frac{1}{2R_n} - \mathfrak{T}_C^F(z)\|\xi_n^*\|\varphi_n = \|\xi_n^*\|(\varkappa_F(\xi_n, \xi_n^*) - \mathfrak{T}_C^F(z)\varphi(x_n')) \ge 2\nu. \tag{2.38}$$

Due to the estimate (1.32) the sequence $\{1/R_n\}$ is bounded (and $\{\varphi_n\}$ is bounded too as follows from (2.38)). Taking into account the inequality (2.36), we come to (2.27), and the remainder of the proof is exactly the same as respective reasoning in Theorem 2.2.1. ■

Remark 2.2.1 *In a finite-dimensional space due to the compactness of the set $U_{\delta,\delta'}(x_0)$, the condition $(\mathbf{B}_2')$ can be substituted by the second order strict convexity of F w.r.t. each vector $\xi^* \in -\mathbf{N}_C(x_0) \cap \partial F^o$. However, in general, we have to require the local uniformity of this property through lack of the strong convergence of normals.*

Following the definition given in [31, Definition 5.2] we say that a convex closed set $F \subset H$ is *γ-strictly convex* (for some $\gamma > 0$) if

$$\langle \eta - \xi, \eta^* - \xi^* \rangle \ge \gamma\|\eta - \xi\|^2,$$

whenever $\xi, \eta \in F$, $\xi^* \in \mathbf{N}_F(\xi) \cap \overline{\mathbf{B}}$ and $\eta^* \in \mathbf{N}_F(\eta) \cap \overline{\mathbf{B}}$, or, equivalently, if

$$\langle \eta - \xi, \xi^* \rangle \leq -\frac{\gamma}{2} \|\eta - \xi\|^2 .$$

The last inequality implies obviously that

$$\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) \geq \frac{\gamma}{2} \tag{2.39}$$

for all $\xi^* \in \partial F^o$, which is the global version of second order rotundity. In this case as an immediate consequence of Theorem 2.2.2 we obtain the following well-posedness result.

Corollary 2.2.3 *Let $F \subset H$ be a closed bounded γ-strictly convex set with $0 \in \operatorname{int} F$, and let $C \subset H$ be a nonempty closed and φ-convex set with a continuous function $\varphi : C \to \mathbb{R}^+$. Then the projection $\pi_C^F(z)$ is a singleton continuously depending on $z \in \mathfrak{B}(C)$, where*

$$\mathfrak{B}(C) := \left\{ z \in H : \limsup_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ x \in C}} \rho_F(x-z)\varphi(x) < \frac{\gamma}{2} \right\} \tag{2.40}$$

is an open set containing C.

Proof.

Let prove that $\mathfrak{B}(C) \subset \mathfrak{A}(C)$. To this end fix $z \in \mathfrak{B}(C) \backslash C$. Then, by (2.39),

$$\begin{aligned} \liminf_{\mathfrak{F}(z)} \left\{ \varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\varphi(x) \right\} &\geq \liminf_{\mathfrak{F}(z)} \varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) + \liminf_{\mathfrak{F}(z)} \left(-\mathfrak{T}_C^F(z)\varphi(x) \right) \\ &\geq \frac{\gamma}{2} - \limsup_{\mathfrak{F}(z)} \mathfrak{T}_C^F(z)\varphi(x) \\ &\geq \frac{\gamma}{2} - \limsup_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ x \in \partial C}} \rho_F(x-z)\varphi(x) > 0 \end{aligned}$$

and consequently $z \in \mathfrak{A}(C)$.

The openness of the set $\mathfrak{B}(C)$ can be shown by the same reasons as the openness of $\mathfrak{A}(C)$ in Theorem 2.2.2, and the statement of the corollary follows from Theorem 2.2.2. ■

Remark 2.2.2 *If the target set C is* perfect, *i.e., it has no isolated points, then $\mathfrak{B}(C)$, which can be smaller than the neighbourhood $\mathfrak{A}(C)$, coincides with*

$$\mathfrak{B}(C) = \left\{ z \in H : \mathfrak{T}_C^F(z)\varphi\left(\pi_C^F(z)\right) < \frac{\gamma}{2} \right\}. \tag{2.41}$$

Let us take z from the set (2.41) and let $\{x_n\}$ be a minimizing sequence of $x \mapsto \rho_F(x-z)$ on C, i.e.,

$$\rho_F(x_n - z) \to \mathfrak{T}_C^F(z) = \rho_F(\bar{x} - z),$$

$\bar{x} := \pi_C^F(z)$ *(we assume that the projection exists and is unique). We should prove that* $x_n \to \bar{x}$. *To this end we fix an arbitrary subsequence (denoted also by* $\{x_n\}$*) and construct its subsequence converging to* $\bar{x}$. *Since* $\bar{x}$ *is not an isolated point of* C, *there exists a sequence* $\{x'_n\} \subset C$, $x'_n \to \bar{x}$, *and such that* $x'_n \neq \bar{x}$, *consequently, by the uniqueness of projection,*

$$\rho_F(x'_n - z) > \mathfrak{T}_C^F(z), \qquad n \in \mathbb{N}.$$

Since $\rho_F(x_n - z) \to \mathfrak{T}_C^F(z)$, *in particular, there exists a number* k_1 *such that* $\rho_F(x_{k_1} - z) < \rho_F(x'_1 - z)$. *But we have also* $\rho_F(x'_n - z) \to \mathfrak{T}_C^F(z)$. *Therefore, one can find* $k'_1 > k_1$ *with* $\rho_F\left(x'_{k'_1} - z\right) < \rho_F(x_{k_1} - z)$. *By the same reasons there exist* k_2 *and* k'_2, $k'_2 > k_2 > k'_1$ *such that*

$$\rho_F\left(x'_{k'_2} - z\right) < \rho_F(x_{k_2} - z) < \rho_F\left(x'_{k'_1} - z\right).$$

Continuing this process we construct two subsequences $\{x_{k_n}\}$ *of* $\{x_n\}$ *and* $\left\{x'_{k'_n}\right\}$ *of* $\{x'_n\}$, *respectively, such that*

$$\rho_F\left(x'_{k'_{n+1}} - z\right) < \rho_F(x_{k_{n+1}} - z) < \rho_F\left(x'_{k'_n} - z\right) < \rho_F(x_{k_n} - z)$$

for each n. *Let us define now another minimizing sequence* $\{y_n\}$ *of* $x \mapsto \rho_F(x - z)$ *on* C *by setting* $y_{2n-1} = x_{k_n}$ *and* $y_{2n} = x'_{k'_n}$, $n \in \mathbb{N}$. *Then by the previous inequalities we have* $\rho_{2n+1} < \rho_{2n} < \rho_{2n-1}$, $\rho_k := \rho_F(y_k - z)$ *and* $y_{2n} \to \bar{x}$. *By the continuity of* $\varphi(\cdot)$ *we can suppose that*

$$\mathfrak{T}_C^F(z)\,\varphi(x'_n) < \frac{\gamma}{2} \tag{2.42}$$

for all n. *Literally repeating the reasoning of Theorem 2.2.2 (see also the estimates in the proof of Theorem 2.2.1) we can show that* $\|y_m - y_{2n}\|$ *tends to zero as* $m \geq 2n \to \infty$. *Since* $y_{2n} \to \bar{x}$, *we have also* $x_{k_n} = y_{2n-1} \to \bar{x}$ *as we need. Thus, the minimizing sequence* $\{x_n\}$ *itself converges to* $\bar{x}$, *and for all* $n \in \mathbb{N}$ *large enough* $\rho_F(x_n - z)\,\varphi(x_n) < \gamma/2$. *In particular,*

$$\limsup_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ x \in C}} \rho_F(x - z)\,\varphi(x) < \frac{\gamma}{2}$$

and $z \in \mathfrak{B}(C)$. *The opposite inclusion is obvious.*

2.3 Some particular and special cases

Let us concretize the results obtained in the previous section. First, we consider the case of a φ-convex target set with smooth boundary, denoting by $\mathfrak{n}(x)$ the unit normal vector to C at the point $x \in \partial C$ and setting

$$\mathfrak{v}(x) := -\frac{\mathfrak{n}(x)}{\rho_{F^\circ}(-\mathfrak{n}(x))}. \tag{2.43}$$

Theorem 2.3.1 *Let C be a closed set with smooth boundary, which is φ-convex with a continuous function $\varphi : C \to \mathbb{R}^+$, and let F be a closed bounded set with $0 \in \operatorname{int} F$, which is strictly convex of second order w.r.t. each vector $\mathfrak{v}(x)$, $x \in \partial C$. Then the time-minimum projection $\pi_C^F(\cdot)$ is well-defined on the (open) set $\mathfrak{U}(C)$ (see (2.33)), which in this case admits the form*

$$\left\{ z \in H : \liminf_{\substack{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+ \\ \xi^*-\mathfrak{v}(x)\to 0 \\ x\in\partial C,\, \xi^*\in\partial F^o}} \left\{ \varkappa_F\left(\mathfrak{J}_F(\xi^*),\xi^*\right) - \mathfrak{T}_C^F(z)\,\varphi(x) \right\} > 0 \right\}. \tag{2.44}$$

We put naturally $\liminf$ *in (2.44) to be equal to $+\infty$ whenever $z \in \operatorname{int} C$.*

Proof.

Let us show the validity of the condition ($\mathbf{B}_2'$) under assumptions of the theorem. Indeed, for each $x_0 \in \partial C$ and fixed

$$\varepsilon := \frac{\varkappa_F\left(\mathfrak{J}_F\left(\mathfrak{v}(x_0)\right), \mathfrak{v}(x_0)\right)}{2} > 0$$

by the lower semicontinuity of the curvature at the point $\mathfrak{v}(x_0)$ there exists $\delta' = \delta'(x_0) > 0$ such that

$$\|\xi^* - \mathfrak{v}(x_0)\| \le 2\delta'$$

implies

$$\varkappa_F\left(\mathfrak{J}_F(\xi^*),\xi^*\right) > \varkappa_F\left(\mathfrak{J}_F\left(\mathfrak{v}(x_0)\right),\mathfrak{v}(x_0)\right) - \varepsilon. \tag{2.45}$$

On the other hand, by the continuity of $\mathfrak{v}(\cdot)$ we find $\delta = \delta(x_0)$ such that

$$\|\mathfrak{v}(x) - \mathfrak{v}(x_0)\| \le \delta'$$

for all $x \in \partial C$ with $\|x - x_0\| \le \delta$. Then given $\xi^* \in U_{\delta,\delta'}(x_0)$ let us choose $x \in \partial C$ satisfying both inequalities $\|x - x_0\| \le \delta$ and $\|\xi^* - \mathfrak{v}(x)\| \le \delta'$. Since

$$\|\xi^* - \mathfrak{v}(x_0)\| \le \|\xi^* - \mathfrak{v}(x)\| + \|\mathfrak{v}(x) - \mathfrak{v}(x_0)\| \le 2\delta',$$

we have (see (2.45))

$$\begin{aligned} \varkappa_F\left(\mathfrak{J}_F(\xi^*),\xi^*\right) &> \varkappa_F\left(\mathfrak{J}_F\left(\mathfrak{v}(x_0)\right),\mathfrak{v}(x_0)\right) - \varepsilon \\ &= K := \frac{\varkappa_F\left(\mathfrak{J}_F\left(\mathfrak{v}(x_0)\right),\mathfrak{v}(x_0)\right)}{2} > 0. \end{aligned}$$

Notice that, $\mathfrak{J}_F(\cdot)$ is single-valued and continuous near each $\mathfrak{v}(x)$, $x \in \partial C$, due to second order strict convexity.

Therefore the time-minimum projection $\pi_C^F(\cdot)$ is well-defined on the (open) set $\mathfrak{U}(C)$ given by (2.33) which can be represented obviously as (2.44). ■

Remark 2.3.1 *In terms of the time-minimum projection itself (already defined and single--valued on $\mathfrak{A}(C)$) we can represent this neighbourhood as*

$$\mathfrak{A}(C) = C \cup \{z \in H \backslash C : \mathfrak{T}_C^F(z)\,\varphi(\bar{x}) < \varkappa_F(\mathfrak{J}_F(\mathfrak{v}(\bar{x})), \mathfrak{v}(\bar{x}))\} \tag{2.46}$$

where $\bar{x} := \pi_C^F(z)$. Indeed, let us suppose that $z \notin C$ belongs to the right-hand side of (2.46) and fix a minimizing sequence $\{x_n\} \subset \partial C$ of the functional $x \mapsto \rho_F(x - z)$ on C and a sequence $\{\xi_n^*\} \subset \partial F^o$ such that $\|\mathfrak{v}(x_n) - \xi_n^*\| \to 0$ and

$$\liminf_{\substack{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+ \\ \xi^*-\mathfrak{v}(x)\to 0 \\ x\in\partial C,\, \xi^*\in\partial F^o}} \{\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\,\varphi(x)\} = \lim_{n\to\infty} \{\varkappa_F(\mathfrak{J}_F(\xi_n^*), \xi_n^*) - \mathfrak{T}_C^F(z)\,\varphi(x_n)\}. \tag{2.47}$$

We should prove that $x_n \to \bar{x}$. To this end we fix an arbitrary subsequence (denoted also by $\{x_n\}$) and construct its subsequence converging to $\bar{x}$. Proceeding as in Remark 2.2.2 we can find a sequence $\{x_n'\} \subset C$, $x_n' \to \bar{x}$ with $x_n' \neq \bar{x}$ and choose subsequences $\{x_{k_n}\}$ of $\{x_n\}$ and $\left\{x'_{k'_n}\right\}$ of $\{x_n'\}$ such that $\{\rho_F(y_n - z)\}$ is strictly decreasing (here $y_{2n-1} = x_{k_n}$ and $y_{2n} = x'_{k'_n}$, $n \geq 1$). By the continuity of $\varphi(\cdot)$ and $\mathfrak{b}(\cdot)$, and by the lower semicontinuity of the curvature we can suppose that

$$\mathfrak{T}_C^F(z)\,\varphi(x_n') - \varkappa_F(\mathfrak{J}_F(\mathfrak{v}(x_n')), \mathfrak{v}(x_n')) < 0, \quad \forall n \in \mathbb{N}.$$

Now using the estimates in the proof of Theorem 2.2.1 (see (2.31)-(2.32)) we show that $\|y_m - y_{2n}\| \to 0$ as $m \geq 2n \to \infty$. Hence, $x_{k_n} = y_{2n-1} \to \bar{x}$ (see Remark 2.2.2). Thus, the minimizing sequence $\{x_n\}$ itself converges to $\bar{x}$. Consequently, $\xi_n^* \to \mathfrak{v}(\bar{x})$ and by using both the continuity of $\varphi(\cdot)$ and the lower semicontinuity of the curvature we conclude that

$$\lim_{n\to\infty} \{\varkappa_F(\mathfrak{J}_F(\xi_n^*), \xi_n^*) - \mathfrak{T}_C^F(z)\,\varphi(x_n)\} \geq \varkappa_F(\mathfrak{J}_F(\mathfrak{v}(\bar{x})), \mathfrak{v}(\bar{x})) - \mathfrak{T}_C^F(z)\,\varphi(\bar{x}) > 0.$$

Taking into account now the equality (2.47) we have $z \in \mathfrak{A}(C)$. The opposite inclusion is obvious.

Remark 2.3.2 *If $\dim H < \infty$ then each minimizing sequence has a cluster point and consequently the neighbourhood (2.44) can be written in a simpler form:*

$$\mathfrak{A}(C) = \left\{ z \in H : \liminf_{\substack{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+ \\ x\in\partial C}} \{\varkappa_F(\mathfrak{J}_F(\mathfrak{v}(x)), \mathfrak{v}(x)) - \mathfrak{T}_C^F(z)\,\varphi(x)\} > 0 \right\}. \tag{2.48}$$

Indeed, let us suppose that z belongs to the right-hand side of (2.48) and fix a minimizing sequence $\{x_n\} \subset \partial C$ of the functional $x \mapsto \rho_F(x - z)$ on C and a sequence $\{\xi_n^*\} \subset \partial F^o$ such that $\|\mathfrak{v}(x_n) - \xi_n^*\| \to 0$ and the equality (2.47) holds. Taking a cluster point $x \in \partial C$ of $\{x_n\}$ we assume without loss of generality that $x_n \to x$, and by the hipothesis, in particular,

$$\tau := \varkappa_F(\mathfrak{J}_F(\mathfrak{v}(x)), \mathfrak{v}(x)) - \mathfrak{T}_C^F(z)\,\varphi(x) > 0. \tag{2.49}$$

On the other hand, by continuity $\mathfrak{v}(x_n) \to \mathfrak{v}(x)$, $n \to \infty$. *So that* $\xi_n^* \to \mathfrak{v}(x)$, *and, by Proposition 1.2.1 also* $\mathfrak{J}_F(\xi_n^*) \to \mathfrak{J}_F(\mathfrak{v}(x))$ *as* $n \to \infty$. *Hence, by the continuity of* $\varphi(\cdot)$ *and the lower semicontinuity of the curvature we have that*

$$\varkappa_F(\mathfrak{J}_F(\mathfrak{v}(x)), \mathfrak{v}(x)) - \mathfrak{T}_C^F(z)\varphi(x) \le \varkappa_F(\mathfrak{J}_F(\xi_n^*), \xi_n^*) - \mathfrak{T}_C^F(z)\varphi(x_n) + \frac{\tau}{2}$$

for $n \geq 1$ *sufficiently large. This together with (2.49) gives*

$$\varkappa_F(\mathfrak{J}_F(\xi_n^*), \xi_n^*) - \mathfrak{T}_C^F(z)\varphi(x_n) \ge \frac{\tau}{2}.$$

Finally, passing to the limit as $n \to \infty$ *we conclude*

$$\liminf_{\substack{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+\\ \xi^*-\mathfrak{v}(x)\to 0\\ x\in\partial C,\,\xi^*\in\partial F^o}} \left\{\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\varphi(x)\right\} \ge \frac{\tau}{2} > 0,$$

i.e., $z \in \mathfrak{A}(C)$. *The opposite inclusion is obvious.*

Concretizing now the local result given by Theorem 2.2.1 we have

Theorem 2.3.2 *Let* C *be a closed* φ*-convex set with smooth boundary and such that for each point* $x_0 \in \partial C$ *one of the following assumptions holds:*

(i) the set F *is uniformly strictly convex w.r.t. the vector* $\mathfrak{v}(x_0)$, *and the (single-valued) mapping* $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$ *is Lipschitz continuous near* x_0;

(ii) the set F *is strictly convex of second order w.r.t.* $\mathfrak{v}(x_0)$.

Then $z \mapsto \pi_C^F(z)$ *is a neighbourhood retraction of the set* C.

Proof.
The hypothesis (**i**) is nothing else than the condition (**A**) at the point x_0 specified for the case of smooth boundary, while (**ii**) implies the condition ($\mathbf{B}_2'$) at x_0, because $(\xi, \xi^*) \mapsto \varkappa_F(\xi, \xi^*)$ is lower semicontinuous (see Remark 1.2.4). Notice that, in this case, ($\mathbf{B}_2'$) is equivalent to ($\mathbf{B}_2$). Thus, we are able to apply directly Theorem 2.2.1. ■

Notice that if at each point $x_0 \in \partial C$ the dynamics satisfies the hypothesis (**i**) from the above theorem then we can entirely avoid the φ-convexity assumption for the target set.

Theorem 2.3.3 *Let* C *be a closed set with smooth boundary, and let* F *be uniformly strictly convex w.r.t. each vector* $\mathfrak{v}(x)$, $x \in \partial C$. *If, moreover, the (single-valued) mapping* $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$ *is locally Lipschitzean on* ∂C *then the statement of Theorem 2.3.2 holds.*

On the other hand, we can obtain the well-posedness of $\pi_C^F(\cdot)$ in a neighbourhood of a φ-convex set C even with lack of the strict convexity of F w.r.t. $\xi^* \in -\mathbf{N}_C^p(x) \cap \partial F^o$ for some isolated points $x \in \partial C$ where smoothness of the boundary is also violated (see Example 2.4.4).

Observe that the formulas (2.33), (2.44) as well as the neighbourhood $\mathcal{U}$ given by Theorem 2.2.1 (see (2.25)) involve the function $\varkappa_F(\xi, \xi^*)$, which can not be in general substituted by the "true" curvature $\hat{\varkappa}_F(\xi, \xi^*)$. Let us propose a method to estimate $\varkappa_F(\xi, \xi^*)$ from below basing on the differentiability properties of the duality mapping $\mathfrak{J}_F(\cdot)$ similarly as it was done for $\gamma_F^{\pm}(\xi, \xi^*)$ (see (1.44)). This permits us to find a smaller neighbourhood of C, expressed in other terms different from $\varkappa_F(\xi, \xi^*)$, where the well-posedness of the projection takes place. To this end we assume that the set ∂F^o is second order smooth (at $\xi^* \in \partial F^o$). Let us associate to each $\delta > 0$ some positive number $\beta(\delta, \xi^*)$ such that

$$\left\|\nabla^2 \rho_{F^o}(\eta^*) - \nabla^2 \rho_{F^o}(\xi^*)\right\| \le \delta$$

whenever $\eta^* \in \partial F^o$ with $\|\eta^* - \xi^*\| \le \beta(\delta, \xi^*)$. This number exists by the continuity of the mapping $\eta^* \mapsto \nabla^2 \rho_{F^o}(\eta^*)$ near ξ^*. In particular, given $\delta > 0$ and $0 < \lambda < 1$ the inequality

$$\left\|\nabla^2 \rho_{F^o}(\eta^* + tv^*) - \nabla^2 \rho_{F^o}(\xi^*)\right\| \le \delta \tag{2.50}$$

holds for all $0 < t \le (1-\lambda)\beta(\delta, \xi^*) / \|F^o\|$, $v^* \in \partial F^o$ and $\eta^* \in \partial F^o$ with $\|\eta^* - \xi^*\| < \lambda\beta(\delta, \xi^*)$. Recalling the proof of Proposition 1.3.3 we obtain from (1.46), (1.43) and (2.50) that

$$\begin{aligned}
\mathfrak{S}_{F^o}(t, \eta^*, \eta) &= \frac{t^2}{2} \sup_{v^* \in F^o} \left\langle \nabla^2 \rho_{F^o}(\eta^* + \tau v^*) v^*, v^* \right\rangle \\
\le\ & \frac{t^2}{2} \left(\sup_{v^* \in F^o} \left\langle \nabla^2 \rho_{F^o}(\xi^*) v^*, v^* \right\rangle + \sup_{v^* \in F^o} \left\langle \left(\nabla^2 \rho_{F^o}(\eta^* + \tau v^*) - \nabla^2 \rho_{F^o}(\xi^*)\right) v^*, v^* \right\rangle \right) \\
\le\ & \frac{1}{2} \left(\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} + \delta \|F^o\|^2 \right) t^2,
\end{aligned} \tag{2.51}$$

where as usual $\eta := \mathfrak{J}_F(\eta^*)$, and $\tau = \tau(t, v^*)$ is some number between 0 and t. Applying the Legendre-Fenchel transform to both parts of (2.51) we obtain

$$\begin{aligned}
\mathfrak{S}_{F^o}^{\star}(r, \eta^*, \eta) &\ge \sup\left\{ tr - \mathfrak{S}_{F^o}(t, \eta^*, \eta) : 0 < t \le (1-\lambda) \frac{\beta(\delta, \xi^*)}{\|F^o\|} \right\} \\
\ge\ & \sup\left\{ tr - \frac{1}{2} \left(\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} + \delta \|F^o\|^2 \right) t^2 : 0 < t \le (1-\lambda) \frac{\beta(\delta, \xi^*)}{\|F^o\|} \right\} \\
=\ & \frac{r^2}{2\left(\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} + \delta \|F^o\|^2 \right)},
\end{aligned} \tag{2.52}$$

which holds true for all $0 < r \le (1-\lambda)\, q(\delta, \xi^*)$, where

$$q(\delta, \xi^*) := \beta(\delta, \xi^*) \frac{\left\|\nabla^2 \rho_{F^o}(\xi^*)\right\|_{F^o} + \delta \|F^o\|^2}{\|F^o\|}.$$

By using the duality between the moduli of local smoothness and of local strict convexity (see (1.36)) we obtain from (1.9) and (2.52) that

$$\begin{aligned} \widehat{\mathfrak{C}}_F(r,\eta,\eta^*) &\geq \mathfrak{C}_F^+\left(\frac{r}{\|F\|},\eta,\eta^*\right) \geq \left(\mathfrak{C}_F^+\right)^{\star\star}\left(\frac{r}{\|F\|},\eta,\eta^*\right) = \mathfrak{S}_{F^o}^{\star}\left(\frac{r}{\|F\|},\eta^*,\eta\right) \\ &\geq \frac{r^2}{2\|F\|^2\left(\left\|\nabla^2\rho_{F^o}(\xi^*)\right\|_{F^o}+\delta\|F^o\|^2\right)} \end{aligned} \tag{2.53}$$

whenever $0<r\leq(1-\lambda)\,q(\delta,\xi^*)\,\|F\|$. Obviously, $\widehat{\mathfrak{C}}_F(r,\eta,\eta^*)=+\infty$ for $r>2\|F\|$, while in the case $(1-\lambda)\,q(\delta,\xi^*)\,\|F\|<r\leq 2\|F\|$, by the monotonicity of the function $\widehat{\mathfrak{C}}_F(\cdot,\eta,\eta^*)$, we have

$$\begin{aligned} \widehat{\mathfrak{C}}_F(r,\eta,\eta^*) &\geq \widehat{\mathfrak{C}}_F\left((1-\lambda)\,q(\delta,\xi^*)\,\|F\|,\eta,\eta^*\right) \\ &\geq \frac{\left((1-\lambda)\,q(\delta,\xi^*)\,\|F\|\right)^2}{2\|F\|^2\left(\left\|\nabla^2\rho_{F^o}(\xi^*)\right\|_{F^o}+\delta\|F^o\|^2\right)} \\ &\geq (1-\lambda)^2\frac{q^2(\delta,\xi^*)}{8\|F\|^2\left(\left\|\nabla^2\rho_{F^o}(\xi^*)\right\|_{F^o}+\delta\|F^o\|^2\right)}\,r^2. \end{aligned} \tag{2.54}$$

Finally, comparing the inequalities (2.53) and (2.54), which hold for all η^* near ξ^*, by arbitrarity of λ, $0<\lambda<1$, we obtain (see (1.26)):

$$\begin{aligned} \gamma_F(\xi,\xi^*) &= \liminf_{\substack{(\eta,\eta^*)\to(\xi,\xi^*)\\ \eta\in\mathfrak{J}_F(\eta^*),\,\eta^*\in\partial F^o}} \inf_{r>0}\frac{\widehat{\mathfrak{C}}_F(r,\eta,\eta^*)}{r^2} \\ &\geq \frac{1}{2\|F\|^2\left(\left\|\nabla^2\rho_{F^o}(\xi^*)\right\|_{F^o}+\delta\|F^o\|^2\right)}\min\left(\frac{q^2(\delta,\xi^*)}{4},1\right). \end{aligned} \tag{2.55}$$

This estimate together with Theorem 2.3.1 permits us to formulate the following result.

Theorem 2.3.4 *In addition to the hypotheses of Theorem 2.3.1 let us suppose that the polar set F^o has boundary of class C^2 near $\mathfrak{v}(x)$ for each $x\in\partial C$. Then, for a given $\delta>0$, the time-minimum projection $\pi_C^F(\cdot)$ is well-defined on the (open) set*

$$\mathfrak{A}_\delta(C)=\left\{z\in H:\liminf_{\substack{\rho_F(x-z)\to\mathfrak{T}_C^F(z)+\\ \xi^*-\mathfrak{v}(x)\to 0\\ x\in\partial C,\,\xi^*\in\partial F^o}}\left\{\Omega(\delta,\xi^*)-\mathfrak{T}_C^F(z)\,\varphi(x)\right\}>0\right\}, \tag{2.56}$$

where

$$\mathfrak{Q}(\delta,\xi^*) := \frac{1}{2\|F\|^2\|\xi^*\|}\min\left\{\frac{\beta^2(\delta,\xi^*)\left(\|\nabla^2\rho_{F^o}(\xi^*)\|_{F^o}+\delta\|F^o\|^2\right)}{4\|F^o\|^2},\right. \tag{2.57}$$
$$\left.\frac{1}{\|\nabla^2\rho_{F^o}(\xi^*)\|_{F^o}+\delta\|F^o\|^2}\right\}.$$

We put naturally lim inf *in (2.56) to be* $+\infty$ *whenever* $z \in \operatorname{int} C$.

Remark 2.3.3 *From (2.57) and the definition of* $\beta(\delta,\xi^*)$ *we see that the neighbourhood* $\mathfrak{U}_\delta(C)$ *is larger whenever the second derivative* $\nabla^2\rho_{F^o}(\cdot)$ *grows slower. Varying* $\delta > 0$ *we can control slightly the size of this neighbourhood.*

The Theorem 2.3.4 perfectly works, in particular, when $\nabla^2\rho_{F^o}(\cdot)$ *is Lipschitz continuous locally at each point* $\xi^* \in \partial F^o$ *(say in a* ε_{ξ^*}*-neighbourhood of* ξ^**) with Lipschitz constant* L_{ξ^*}*, in which case we can choose* $\beta(\delta,\xi^*)$ *equal to* $\min(\delta/L_{\xi^*},\varepsilon_{\xi^*})$ *(see Example 2.4.3).*

Concluding this section let us give two special hypotheses involving local convexity of the target set, which also guarantee the well-posedness of the projection.

Proposition 2.3.5 *Suppose that for a given* $x_0 \in \partial C$ *one of the following conditions holds:*

(i) *C has smooth boundary at* x_0*, and for some* $\varepsilon > 0$ *the set* $C \cap (x_0 + \varepsilon\overline{\mathbf{B}})$ *has nonempty interior, and it is strictly convex of second order at* x_0 *(w.r.t. the corresponding normal vector);*

(ii) *for some* $\varepsilon > 0$ *the set* $C \cap (x_0 + \varepsilon\overline{\mathbf{B}})$ *is convex, and F is strictly convex of second order w.r.t. each* $v \in -\mathbf{N}_C(x) \cap \partial F^o$*, where* $x \in \partial C$ *with* $\|x - x_0\| \le \varepsilon$.

Then the function $z \mapsto \pi_C^F(z)$ *is single-valued and continuous in a neighbourhood of* x_0.

Proof.

Let us consider each case separately.

(i) Without loss of generality (translating if necessary the set C) we can suppose that $0 \in \operatorname{int} G$, where $G := C \cap (x_0 + \varepsilon\overline{\mathbf{B}})$. Let us denote by

$$v_0 := \frac{\mathfrak{n}(x_0)}{\rho_{G^o}(\mathfrak{n}(x_0))},$$

where $\mathfrak{n}(x_0)$ is the unit normal vector to C (as well as to G, certainly) at the point x_0. Since G is convex, closed, bounded with $0 \in \operatorname{int} G$ and $v_0 \in \partial G^o \cap \mathbf{N}_G(x_0) = \mathfrak{J}_G^{-1}(x_0)$ (consequently, $\mathfrak{J}_G(v_0) = \{x_0\}$), the number $\nu := \gamma_G(x_0, v_0)$ is well defined and strictly positive by assumption. By (1.25) there exist $\theta \ge \nu/2$ and $\delta' > 0$ such that $\widehat{\mathfrak{C}}_G(r,x,v) \ge \theta r^2$ whenever $\|x - x_0\| \le \delta'$,

$\|v - v_0\| \le \delta'$, $x = \mathfrak{J}_G(v)$, $v \in \partial G^o$ and $r > 0$. Since the mapping $x \mapsto \mathfrak{n}(x)$ in continuous in a neighbourhood of x_0, setting $v := \mathfrak{n}(x) / \rho_{G^o}(\mathfrak{n}(x))$ let us find $0 < \delta < \min\{\delta', \varepsilon\}$ such that

$$\begin{aligned} \|v - v_0\| &\le \frac{1}{\rho_{G^o}(\mathfrak{n}(x))\,\rho_{G^o}(\mathfrak{n}(x_0))} [\rho_{G^o}(\mathfrak{n}(x))\,\|\mathfrak{n}(x_0) - \mathfrak{n}(x)\| + \\ &\quad + \|\mathfrak{n}(x_0)\|\,|\rho_{G^o}(\mathfrak{n}(x)) - \rho_{G^o}(\mathfrak{n}(x_0))|] \\ &\le \frac{\|G^o\|^2}{\|\mathfrak{n}(x_0)\|\,\|\mathfrak{n}(x)\|} \|G\|\,(\|\mathfrak{n}(x)\| + \|\mathfrak{n}(x_0)\|)\,\|\mathfrak{n}(x_0) - \mathfrak{n}(x)\| \\ &= 2\,\|G^o\|^2\,\|G\|\,\|\mathfrak{n}(x_0) - \mathfrak{n}(x)\| \le \delta' \end{aligned}$$

whenever $\|x - x_0\| \le \delta$. So that $\widehat{\mathfrak{C}}_G(r, x, v) \ge \theta r^2$ for all $x \in \partial C$, $\|x - x_0\| \le \delta$, and all $r > 0$.

Setting now

$$\mathcal{U}(x_0) := x_0 + \frac{\delta}{D}\overline{\mathbf{B}},$$

where $D := 2\,\|F^o\|\,\|F\|$, take $z \in \mathcal{U}(x_0)$ and a minimizing sequence $\{x_n\} \subset \partial C$ of the function $x \mapsto \rho_F(x - z)$ on C. Similarly as in the proof of Theorem 2.2.1 we see that $\|x_n - x_0\| \le \delta$. By Definition 1.2.2,

$$\frac{\nu}{2} r^2 \le \theta r^2 \le \widehat{\mathfrak{C}}_G(r, x, v) \le \langle x - y, v\rangle \quad \forall y \in G,\ \|y - x\| \ge r,$$

whenever $x \in \partial C$ with $\|x - x_0\| \le \delta$, $v = \mathfrak{n}(x) / \rho_{G^o}(\mathfrak{n}(x))$ and $r > 0$. In particular, substituing into the above inequality

$$x := x_n,\ \ v := \frac{\mathfrak{n}(x_n)}{\rho_{G^o}(\mathfrak{n}(x_n))},\ \ y := x_m,\ \text{ and }\ r := \|x_m - x_n\|,$$

for sufficiently large $m \ge n \ge 1$ we have

$$\frac{\nu}{2}\rho_{G^o}(\mathfrak{n}(x_n))\,\|x_m - x_n\|^2 \le \langle x_n - x_m, \mathfrak{n}(x_n)\rangle. \tag{2.58}$$

In accordance with Lemma 2.1.1 and remarks 2.1.2 and 2.1.3 we do not lose generality if suppose that for some vectors $\xi_n^* \in \partial\rho_F(x_n - z) = \mathbf{N}_F\left(\frac{x_n - z}{\rho_F(x_n - z)}\right) \cap \partial F^o$ and for some sequence $\nu_n \to 0+$ the inequality

$$\|\mathfrak{v}(x_n) - \xi_n^*\| \le \frac{\nu_n}{\mathfrak{T}_C^F(z)}, \tag{2.59}$$

$n = 1, 2, \ldots$, takes place, where $\mathfrak{v}(x_n)$ is given by (2.43). In fact, Lemma 2.1.1 affirms the existence of another minimizing sequence $\{x_n'\} \subset C$ close to $\{x_n\}$ (due to Remark 2.1.3 we can set $x_n'' = x_n'$) and of the vectors $\xi_n^* \in \mathbf{N}_F\left(\frac{x_n' - z}{\rho_F(x_n' - z)}\right) \cap \partial F^o$ and $v_n \in \mathbf{N}_C^p(x_n')$ such that $\|v_n + \rho_F(x_n' - z)\,\xi_n^*\| \to 0$, $n \to \infty$. By the smoothness we obviously have $v_n = -\mathfrak{v}(x_n)\,\rho_F(x_n - z)$ (see Remark 2.1.2). Therefore we can consider the sequence $\{x_n\}$ itself in the place of $\{x_n'\}$ and ξ_n^* satisfying the equality (2.59). Let us set

$$\lambda_n := \frac{\rho_{F^o}(-\mathfrak{n}(x_n))}{\rho_{G^o}(\mathfrak{n}(x_n))}$$

and $z_n := x_n + \lambda_n \xi_n^*$. Hence, by (2.58) and (2.59), we obtain that

$$\begin{aligned} \langle z_n - x_n, x_m - x_n \rangle &= \lambda_n \left[\langle \mathfrak{v}(x_n), x_m - x_n \rangle + \langle \xi_n^* - \mathfrak{v}(x_n), x_m - x_n \rangle \right] \\ &\geq \lambda_n \left[\left\langle -\frac{\mathfrak{n}(x_n)}{\rho_{F^o}(-\mathfrak{n}(x_n))}, x_m - x_n \right\rangle - \|\xi_n^* - \mathfrak{v}(x_n)\| \, \|x_m - x_n\| \right] \\ &\geq \frac{\lambda_n}{\rho_{F^o}(-\mathfrak{n}(x_n))} \frac{\nu}{2} \rho_{G^o}(\mathfrak{n}(x_n)) \|x_m - x_n\|^2 - \lambda_n \frac{\nu_n}{\mathfrak{T}_C^F(z)} \|x_m - x_n\| \\ &= \frac{\nu}{2} \|x_m - x_n\|^2 - \lambda_n \frac{\nu_n}{\mathfrak{T}_C^F(z)} \|x_m - x_n\| \qquad (2.60) \end{aligned}$$

for all $m \geq n \geq 1$. On the other hand, ξ_n^* is a normal vector to the set $z + \rho_F(x_n - z) F$ at the point x_n, and x_m belongs to this set by the eventual monotonicity of $\{\rho_F(x_n - z)\}$. Therefore, $\langle z_n - x_n, x_m - x_n \rangle = \lambda_n \langle \xi_n^*, x_m - x_n \rangle \leq 0$, and combining this with (2.60) we find

$$\frac{\nu}{2} \|x_m - x_n\| \leq \frac{\nu_n}{\mathfrak{T}_C^F(z)} \lambda_n,$$

for all $m \geq n \geq 1$. Hence $\{x_n\}$ is a Cauchy sequence because $\{\lambda_n\}$ is bounded, and the remainder follows by the same line as in the proof of Theorem 2.2.1.

(ii) In this case we set

$$\mathcal{U}(x_0) := x_0 + \frac{\varepsilon}{2(\|F^o\| \, \|F\| + 1)} \overline{\mathbf{B}}$$

and show directly that $\pi_C^F(z) \neq \emptyset$ for each $z \in \mathcal{U}(x_0)$. Indeed, if $\{x_n\} \subset C$ is a sequence with $\rho_F(x_n - z) \leq \mathfrak{T}_C^F(z) + 1/n$ then by the boundedness there exists an its subsequence converging weakly to some $x \in H$. Hence

$$x_n \in C \cap \left(z + \left(\mathfrak{T}_C^F(z) + \frac{1}{n} \right) F \right) \subset C \cap \left(x_0 + \varepsilon \overline{\mathbf{B}} \right) \qquad (2.61)$$

for $n \geq 1$ large enough. In fact, the relations $\rho_F \left(\frac{x_n - z}{\mathfrak{T}_C^F(z) + 1/n} \right) \leq 1$ and $\{x_n\} \subset C$ imply that

$$x_n \in C \cap \left(z + \left(\mathfrak{T}_C^F(z) + \frac{1}{n} \right) F \right),$$

and for every $y \in C \cap \left(z + \left(\mathfrak{T}_C^F(z) + \frac{1}{n} \right) F \right)$ we have

$$\begin{aligned} \|y - x_0\| &\leq \|F\| \rho_F(y - z) + \|z - x_0\| \leq \|F\| \left(\mathfrak{T}_C^F(z) + \frac{1}{n} \right) + \|z - x_0\| \\ &\leq \|F\| \left(\|F^o\| \, \|z - x_0\| + \frac{1}{n} \right) + \|z - x_0\| \leq \varepsilon, \qquad (2.62) \end{aligned}$$

for every n large enough. Since the last set in (2.61) is weakly closed, we have $x \in C$. On the other hand, choosing a sequence $y_n \in z + \mathfrak{T}_C^F(z) F$ such that $\rho_F(x_n - y_n) \leq 1/n$ we observe that the weak limit of some its subsequence is equal to x too. Notice that such subsequence exists because $\{y_n\}$ is bounded, and its weak limit belongs to the weakly closed set $z+\mathfrak{T}_C^F(z)\,F$. Hence $x \in \left(z+\mathfrak{T}_C^F(z)\,F\right)\cap C = \pi_C^F(z)$. Therefore we prove that $\pi_C^F(z) \neq \emptyset$ for each $z \in \mathcal{U}(x_0)$, as we want.

Let us assume now that the projection $\pi_C^F(z)$ consists at least of two different points, say x and y. As we said above $x, y \in \left(z+\mathfrak{T}_C^F(z)\,F\right)\cap C$. Also similarly as (2.62) we have $x, y \in x_0+\varepsilon\overline{\mathbf{B}}$ impling that $x, y \in G := C \cap \left(x_0+\varepsilon\overline{\mathbf{B}}\right)$. By the convexity of $z+\mathfrak{T}_C^F(z)\,F$ and of G the projection $\pi_C^F(z)$ contains the whole segment $\{\lambda x+(1-\lambda)\,y : \lambda \in [0,1]\}$. Fix some $\hat{x} := \hat{\lambda}x+\left(1-\hat{\lambda}\right)y$ with $0<\hat{\lambda}<1$. Then being $\hat{x} \in \pi_C^F(z)$ the point $\hat{x}$ minimizes the functional $y \mapsto \rho_F(y-z)+\mathbf{I}_G(y)$ on H. Consequently

$$0 \in \partial\left(\rho_F(\hat{x}-z)+\mathbf{I}_G(\hat{x})\right) = \partial\rho_F(\hat{x}-z)+\mathbf{N}_C(\hat{x}). \tag{2.63}$$

Hence, there exists a unit normal vector $\hat{\mathfrak{n}} \in \mathbf{N}_C(\hat{x})$ such that

$$\hat{\mathfrak{v}} := \frac{-\hat{\mathfrak{n}}}{\rho_{F^o}(-\hat{\mathfrak{n}})} \in \partial\rho_F(\hat{x}-z) = \mathbf{N}_F\left(\frac{\hat{x}-z}{\mathfrak{T}_C^F(z)}\right)\cap\partial F^o,$$

or, equivalently, $\hat{\mathfrak{v}} \in \mathfrak{J}_F^{-1}\left((\hat{x}-z)/\mathfrak{T}_C^F(z)\right)$. In fact $(\hat{x}-z)/\mathfrak{T}_C^F(z)$ is the unique element of $\mathfrak{J}_F(\hat{\mathfrak{v}})$ because F is strictly convex of second order w.r.t. $\hat{\mathfrak{v}} \in -\mathbf{N}_C(\hat{x})\cap\partial F^o$ and $\hat{x} \in \left(x_0+\varepsilon\overline{\mathbf{B}}\right)\cap\partial C$. It is easy to see that the vector $\hat{\mathfrak{v}}$ is orthogonal to the vector subspace

$$L := \left\{\lambda\frac{x-z}{\mathfrak{T}_C^F(z)}+(1-\lambda)\frac{y-z}{\mathfrak{T}_C^F(z)} : \lambda \in \mathbb{R}\right\}-\frac{\hat{x}-z}{\mathfrak{T}_C^F(z)}.$$

Indeed, for each $w \in L$, we have

$$\langle\hat{\mathfrak{v}}, w\rangle = \lambda\left\langle\hat{\mathfrak{v}}, \frac{x-z}{\mathfrak{T}_C^F(z)}-\frac{\hat{x}-z}{\mathfrak{T}_C^F(z)}\right\rangle+(1-\lambda)\left\langle\hat{\mathfrak{v}}, \frac{y-z}{\mathfrak{T}_C^F(z)}-\frac{\hat{x}-z}{\mathfrak{T}_C^F(z)}\right\rangle \leq 0,$$

since $\hat{\mathfrak{v}} \in \mathbf{N}_F\left((\hat{x}-z)/\mathfrak{T}_C^F(z)\right)$, and, on the other hand, $\hat{\mathfrak{v}} \in -\mathbf{N}_C(\hat{x})$ that implies

$$\langle\hat{\mathfrak{v}}, w\rangle = \frac{1}{\mathfrak{T}_C^F(z)}\left[\lambda\langle\hat{\mathfrak{v}}, x-\hat{x}\rangle+(1-\lambda)\langle\hat{\mathfrak{v}}, y-\hat{x}\rangle\right] \geq 0.$$

Thus $\mathfrak{R}_F(\mathfrak{J}_F(\hat{\mathfrak{v}}), \hat{\mathfrak{v}}) = +\infty$ contradicting the condition of theorem and showing that $\pi_C^F(z)$ is singleton for each $z \in \mathcal{U}(x_0)$.

Finally, let us consider a sequence $\{z_n\} \subset \mathcal{U}(x_0)$ converging to some $z \in \mathcal{U}(x_0)$. By the arguments above, without loss of generality we may suppose that $\left\{\pi_C^F(z_n)\right\}$, being a minimizing sequence for $x \mapsto \rho_F(x-z)$ on C (see the end of the proof of Theorem 2.2.1), converges weakly to the unique projection $\pi_C^F(z)$. So it remains to show that convergence is also strong. Setting $\hat{x} := \pi_C^F(z)$, from the relation (2.63) we find again a normal vector $\hat{\mathfrak{n}} \in \mathbf{N}_C(\hat{x})$ such that $(\hat{x}-z)/\mathfrak{T}_C^F(z)$ is the unique element of $\mathfrak{J}_F(\hat{\mathfrak{v}})$ where $\hat{\mathfrak{v}} := -\hat{\mathfrak{n}}/\rho_{F^o}(-\hat{\mathfrak{n}})$. Since by assumption F is strictly convex of second order w.r.t. $\hat{\mathfrak{v}}$ then (see Proposition 1.2.1(i)) $(\hat{x}-z)/\mathfrak{T}_C^F(z)$ is a strongly exposed point of F. So that the weak convergence of $\left\{\left(\pi_C^F(z_n)-z_n\right)/\mathfrak{T}_C^F(z_n)\right\} \subset F$ to $\left(\pi_C^F(z)-z\right)/\mathfrak{T}_C^F(z)$ implies its strong convergence, and the continuity of the mapping $z \mapsto \pi_C^F(z)$ follows. ■

2.4 Examples

Example 2.4.1 *In a Hilbert space H for a fixed $v \in H$, $\|v\| = 1$, and $0 < \theta < 1$ let us consider the convex closed cone*

$$K_{v,\theta} := \{x \in H : \langle v, x \rangle \geq \theta \|x\|\},$$

whose polar cone is

$$K^o_{v,\theta} = \left\{x \in H : \langle -v, x \rangle \geq \sqrt{1-\theta^2}\,\|x\|\right\}.$$

Taking now $0 < \theta_1 < \theta_2 < 1$, we define $C := \overline{H \backslash K_{v,\theta_1}}$ and $F := (K_{v,\theta_2} - v) \cap \overline{\mathbf{B}}$.

The set C neither has smooth boundary, nor is φ-convex, and, moreover, the origin is its "inward corner" point, $\mathbf{N}^p_C(0) = \{0\}$. On the other hand, F is not strictly convex, because the boundary ∂F contains a lot of linear segments.

However, the hypotheses of Theorem 2.2.1 are fulfilled, and $\pi^F_C(\cdot)$ is a (global) continuous retraction of C. Indeed, let us represent the target set in the form $C = \{x \in H : f(x) \leq 0\}$ where $f(x) := \langle v, x \rangle - \theta_1 \|x\|$. Then

$$\mathbf{N}^p_C(x) = \nabla f(x)\,\mathbb{R}^+ = \left(v - \theta_1 \frac{x}{\|x\|}\right)\mathbb{R}^+$$

for each $x \in \partial^* C = \partial C \backslash \{0\}$. In particular, taking $x \in \partial C \backslash \{0\}$ and $\xi^* \in -\mathbf{N}^p_C(x) \cap \partial F^o$ we have

$$\langle -v, \xi^* \rangle = \lambda \left\langle -v, -v + \theta_1 \frac{x}{\|x\|} \right\rangle = \lambda \left(1 + \frac{\theta_1}{\|x\|} \langle -v, x \rangle\right) = \lambda\left(1 - \theta_1^2\right),$$

for some $\lambda > 0$, and

$$\|\xi^*\|^2 = \lambda^2 \left\langle -v + \theta_1 \frac{x}{\|x\|}, -v + \theta_1 \frac{x}{\|x\|} \right\rangle = \lambda^2 \left(1 + 2\frac{\theta_1}{\|x\|} \langle -v, x \rangle + \theta_1^2\right) = \lambda^2\left(1 - \theta_1^2\right),$$

so that

$$\langle -v, \xi^* \rangle = \sqrt{1-\theta_1^2}\,\|\xi^*\| > \sqrt{1-\theta_2^2}\,\|\xi^*\|,$$

i.e., $\xi^* \in \operatorname{int} K^o_{v,\theta_2}$. Since

$$\xi^* \in K^o_{v,\theta_2} \cap \partial F^o = \mathbf{N}_K(0) \cap \partial F^o = \mathbf{N}_F(-v) \cap \partial F^o = \mathfrak{J}_F^{-1}(-v)$$

then $-v \in \mathfrak{J}_F(\xi^*)$ and consequently $\langle \xi^*, -v \rangle = 1$ and, on the other hand,

$$\begin{aligned} \xi^* \in \operatorname{int} K^o_{v,\theta_2} \quad &\Rightarrow \quad \langle \xi^*, y \rangle < 0 \quad \forall y \in K_{v,\theta_2} \backslash \{0\} \\ &\Rightarrow \quad \langle \xi^*, y - v \rangle < 1 \quad \forall y \in K_{v,\theta_2} \backslash \{0\} \\ &\Rightarrow \quad \langle \xi^*, \xi \rangle < 1 \quad \forall \xi \in F \backslash \{-v\}, \end{aligned}$$

so $\mathfrak{J}_F(\xi^*) = \{-v\}$. Therefore the mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}^p_C(x) \cap \partial F^o\right)$ is constant, and the condition $(\mathbf{A}_1)$ is satisfied trivially at the point $x_0 = 0$ (with arbitrary $\delta > 0$). In order to justify $(\mathbf{A}_2)$ let us choose σ, $\sqrt{1-\theta_2^2} < \sigma < \sqrt{1-\theta_1^2}$, and $\delta' > 0$ such that

$$\frac{\langle -v, \eta^* \rangle}{\|\eta^*\|} \geq \frac{\langle -v, \xi^* \rangle}{\|\xi^*\|} - \left(\sqrt{1-\theta_1^2} - \sigma\right)$$

for all $\xi^* \in \partial F^o$ and $\eta^* \in H$ with $\|\eta^* - \xi^*\| \le 2\delta'$. If, moreover, $\xi^* \in -\mathbf{N}_C^p(x) \cap \partial F^o$, $x \in \partial C$, $x \neq 0$, then

$$\langle -v, \eta^* \rangle \ge \sigma \|\eta^*\| . \tag{2.64}$$

Now for each $\eta^* \in U_{\delta,\delta'}(0)$ (see (2.14)) we have $\eta^* \in \partial F^o$, and there exists $\xi^* \in \partial F^o$ such that $\xi^* \in -\mathbf{N}_C^p(x)$ and $\|\xi^* - \eta^*\| \le \delta'$. Therefore $\|\eta^* + \delta' v - \xi^*\| \le 2\delta'$ and by (2.64) the inequality

$$\langle -v, \eta^* + \delta' v \rangle \ge \sigma \|\eta^* + \delta' v\| > \sqrt{1-\theta_2^2}\,\|\eta^* + \delta' v\|$$

follows, or $\eta^* + \delta' v \in K_{v,\theta_2}^o$. Hence, for each $\eta^* \in U_{\delta,\delta'}(0)$ and each $\eta \in F$ by duality of the cones we have $\langle \eta^* + \delta' v, v + \eta \rangle \le 0$, and recalling that $\mathfrak{J}_F(\eta^*) = \{-v\}$ and $v + \eta \in K_{v,\theta_2}$ we obtain

$$\begin{aligned}
\widehat{\mathfrak{C}}_F(r, -v, \eta^*) &= \inf\{\langle -v - \eta, \eta^* \rangle : \eta \in F,\ \|v + \eta\| \ge r\} \\
&\ge \delta' \inf\{\langle v + \eta, v \rangle : \eta \in F,\ \|v + \eta\| \ge r\} \\
&\ge \delta' \theta_2 r > 0,
\end{aligned}$$

which means the uniform strict convexity of F w.r.t. the set of directions $U_{\delta,\delta'}(0)$. In this example, certainly, it is easier to observe directly the uniform continuity of the mapping $\mathfrak{J}_F(\eta^*) \equiv -v$ on $U_{\delta,\delta'}(0)$ (this is what we really need for proving Theorem 2.2.1) than to construct an estimate of the modulus $\widehat{\mathfrak{C}}_F$.

Example 2.4.2 *Let us modify slightly the previous example, taking arbitrary $v \in H$ with $\|v\| = 1$; $0 < \theta_1, \theta_2 < 1$; $1 < \alpha < 2$ and setting*

$$C := \{x \in H : \langle v, x \rangle \le \theta_1 \|x\|^\alpha\};$$
$$F := \{\xi \in H : \langle v, \xi + v \rangle \ge \theta_2 \|\xi + v\|^\alpha\}.$$

Clearly, F is convex closed bounded with $0 \in \operatorname{int} F$, and C is closed admitting at each point $x \in \partial C \setminus \{0\}$ an unique unit normal vector directed as

$$\nabla f(x) = v - \alpha\theta_1 \frac{x}{\|x\|^{2-\alpha}}$$

(here $f(x) := \langle v, x \rangle - \theta_1 \|x\|^\alpha$), which is also continuously extendable up to the origin (we have $\nabla f(0) = v$). So that $\mathbf{N}_C^l(x) = \nabla f(x)\mathbb{R}^+$, $x \in \partial C$, and the boundary of C is smooth. However, $\mathbf{N}_C^p(0) = \{0\}$ (as it is easy to verify there is no point except the origin itself whose metric projection onto C is 0), while $\mathbf{N}_C^p(x) = \mathbf{N}_C^l(x)$ at other points $x \in \partial C$. Therefore, C is not φ-convex, and the condition (**B**) can not be applied (at least in a neighbourhood of the point 0). Notice that in Chapter 1 we already considered the set F and its rotundity properties (see Example 1.4.1). We showed there that F is uniformly strictly convex (w.r.t. the whole ∂F^o). It is even γ-strictly convex with some $\gamma > 0$. Therefore, the hypothesis ($\mathbf{A}_2$) is fulfiled.

In order to verify ($\mathbf{A}_1$) let us fix an arbitrary point $x \in \partial C$, $x \neq 0$, with the proximal normal vector $\nabla f(x)$ and determine a (unique) $\xi \in \partial F$ such that $-\nabla f(x)$ is normal to F at ξ. Since $\mathbf{N}_F(\xi) = \nabla g(\xi)\mathbb{R}^+$, where

$$g(\xi) := \theta_2 \|\xi + v\|^\alpha - \langle v, \xi + v \rangle,$$

solving the equation $-\nabla f(x) = \lambda \nabla g(\xi)$, $\lambda > 0$, we find immediately that $\lambda = 1$ and

$$\xi = \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\alpha-1}} x - v.$$

Thus, the (single-valued) mapping

$$x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^o\right) = \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\alpha-1}} x - v$$

is Lipschitz continuous on $C_\delta(0)$ with $\delta > 0$ arbitrarily large, and the Lipschitz constant is $L = \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\alpha-1}}$. Applying now Theorem 2.2.1 we can affirm that $\pi_C^F(\cdot)$ is a neighbourhood retraction defined on the open set (see (2.16))

$$\mathcal{U} = \left\{ z \in H : \mathfrak{T}_C^F(z) < \left(\frac{\theta_2}{\theta_1}\right)^{\frac{1}{\alpha-1}} \right\}.$$

The following example (in the space $H = \mathbb{R}^2$ for the sake of clarity) illustrates the second order condition (balance between the curvatures).

Example 2.4.3 *Let*

$$F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \le 1 - \xi_1^4,\ -1 \le \xi_1 \le 1\right\}$$

and

$$C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le x_2^2\right\}.$$

Observe that C is closed and has smooth boundary with the unit normal vector

$$\mathfrak{n}(x) = \frac{1}{\sqrt{1+4x_2^2}}(1, -2x_2), \quad x := (x_1, x_2) \in \partial C.$$

The target is also φ-convex with

$$\varphi(x) = \frac{1}{\sqrt{1+4x_2^2}}. \tag{2.65}$$

In fact, for any x, $y \in \partial C$ with x_1, $y_1 \ge 0$ (the other case is similar) and $v \in \mathbf{N}_C^p(x)$ we have

$$\begin{aligned}
\frac{1}{\|v\|}\langle v, y - x\rangle &= \frac{1}{\sqrt{1+4x_2^2}}\left\langle (1, -2x_2), \left(y_2^2 - x_2^2, y_2 - x_2\right)\right\rangle \\
&= \frac{1}{\sqrt{1+4x_2^2}}\left(y_2^2 + x_2^2 - 2x_2y_2\right) \\
&= \|y - x\|^2 \frac{(y_2 - x_2)^2}{\sqrt{1+4x_2^2}\,\|y - x\|^2} \le \|y - x\|^2 \frac{1}{\sqrt{1+4x_2^2}}.
\end{aligned}$$

The closed convex bounded set F with 0 in its interior already been considered in the Example 1.4.2. Let us recall some estimates obtained there.

Let us fix an arbitrary dual pair (ξ, ξ^*) (i.e., $\xi^* \in \partial F^o$ and $\xi \in \mathfrak{J}_F(\xi^*)$) with $\xi =: (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}$. By symmetry we can consider only the case when $\xi_2 \geq 0$ and $\xi_1 \leq 0$. If $\xi_2 > 0$ then we have (see Example 1.4.2)

$$K(\xi_1) := \frac{2\xi_1^2}{\sqrt{1 + 16\xi_1^6 \Sigma^2(\xi_1)}} \leq \varkappa_F(\xi, \xi^*) = \frac{\gamma_F(\xi, \xi^*)}{\|\xi^*\|} \leq \frac{2\xi_1^2}{\sqrt{1 + 16\xi_1^6}}, \tag{2.66}$$

where $\Sigma(\xi_1) := \sqrt{1 + \left(\sum_{k=0}^{3} |\xi_1|^k\right)^2}$, and

$$3K(\xi_1) \leq \hat{\varkappa}_F(\xi, \xi^*) = \frac{\hat{\gamma}_F(\xi, \xi^*)}{\|\xi^*\|} \leq \frac{6\xi_1^2}{\sqrt{1 + 16\xi_1^6}}. \tag{2.67}$$

In particular, both $\varkappa_F$ and $\hat{\varkappa}_F$ are equal to zero at the points $(0, \pm 1)$. Therefore, the set F is not γ-strictly convex, and the results of [31] can not be applied here. Notice that the estimates (2.66) and (2.67) are valid also at the point $\xi = (-1, 0)$ (i.e., when $\xi_2 = 0$) with respect to the normal vector $\xi^* = (-1, 1/4) \in \partial F^o$, which belongs to the boundary of the normal cone $\mathbf{N}_F(-1, 0)$.

However, there is a *local* uniform rotundity along the boundary of C that permits us to apply Theorem 2.3.2 (ii). To be more precise let us estimate the respective curvatures. Direct calculations give

$$\rho_{F^o}(\xi_1^*, \xi_2^*) = \sigma_F(\xi_1^*, \xi_2^*) = \begin{cases} \frac{3|\xi_1^*|^{4/3}}{4^{4/3}|\xi_2^*|^{1/3}} + |\xi_2^*| & \text{if } |\xi_2^*| \geq \frac{|\xi_1^*|}{4} \\ |\xi_1^*| & \text{if } |\xi_2^*| < \frac{|\xi_1^*|}{4}. \end{cases}$$

Considering now $x = (x_1, x_2) \in \partial C$ with $|x_2| \geq 1/8$ we see that for the vector

$$\mathfrak{v}(x) := -\frac{\mathfrak{n}(x)}{\rho_{F^o}(-\mathfrak{n}(x))} = \frac{8|x_2|^{1/3}}{3 + 16x_2^{4/3}}(-1, 2x_2) \in \partial F^o \tag{2.68}$$

there is a unique $\xi = (\xi_1, \xi_2) \in \mathfrak{J}_F(\mathfrak{v}(x))$, which can be easily found from the relation $\mathfrak{v}(x) \in \mathbf{N}_F(\xi)$ where $\mathbf{N}_F(\xi) = \{\lambda(4\xi_1^3, \operatorname{sgn}(\xi_2)) : \lambda \geq 0\}$. Namely,

$$\xi_1 = -\frac{1}{2|x_2|^{1/3}} \in [-1, 0[\quad \text{and} \quad \xi_2 = (1 - \xi_1^4)\operatorname{sgn}(x_2). \tag{2.69}$$

Setting for simplicity $\Sigma^2(\xi_1) \leq 17$, from (2.66) we have at this point:

$$\varkappa_F(\xi, \mathfrak{v}(x)) \geq \frac{1}{17}\frac{|x_2|^{1/3}}{\sqrt{1 + 4x_2^2}}. \tag{2.70}$$

Otherwise (if $|x_2| < 1/8$) the vector $\mathfrak{v}(x) = (-1, 2x_2)$ belongs to the interior of the normal cone

$$\mathbf{N}_F(-1,0) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 \leq -4|v_2|\}, \tag{2.71}$$

and the second order strict convexity also follows. In this case the curvature $\hat{\varkappa}_F$ at $\bar{\xi} = (-1,0)$ w.r.t. the vector $\mathfrak{v}(x)$ is equal to $+\infty$ (see (1.22)), while $\varkappa_F(\bar{\xi}, \mathfrak{v}(x))$ is a finite positive number depending on the size of both sets F and F^o, and on the proximity of $\mathfrak{v}(x)$ to the boundary $\partial\mathbf{N}_F(\bar{\xi})$. To obtain a precise estimate we can proceed, e.g., as in the proof of Theorem 2.3.4, since $\rho_{F^o}(\cdot)$ is of class C^2 at each $\xi^* \in \operatorname{int}\mathbf{N}_F(\bar{\xi})$. Namely, let us denote by $d(x)$ the minimal distance of $\mathfrak{v}(x)$ from $e^{\pm} := (-1, \pm 1/4)$ that are extreme vectors among those $\xi^* \in \partial F^o$ with $\nabla\rho_{F^o}(\xi^*) = \mathfrak{J}_F(\xi^*) = \bar{\xi}$. Therefore, the function $\xi^* \mapsto \nabla^2\rho_{F^o}(\xi^*)$ is Lipschitz continuous (it is to identical zero) on

$$\partial F^o \cap \left(\mathfrak{v}(x) + d(x)\overline{\mathbf{B}}\right) \subset \partial F^o \cap \mathbf{N}_F(-1,0).$$

Substituting $\left\|\nabla^2\rho_{F^o}(\mathfrak{v}(x))\right\|_{F^o} = 0$; $\beta(\delta, \mathfrak{v}(x)) = d(x)$ (see (1.43) and Remark 2.3.3)) and choosing a suitable $\delta > 0$ (e.g., such that (2.55) gives the better estimate of the curvature from below, namely $q(\delta, \mathfrak{v}(x))/2 = 1$) from the inequality (2.55) we obtain

$$\begin{aligned}\varkappa_F(\bar{\xi}, \mathfrak{v}(x)) &\geq \frac{1}{\|\mathfrak{v}(x)\|} \frac{1}{2\|F\|^2\left(\left\|\nabla^2\rho_{F^o}(\mathfrak{v}(x))\right\|_{F^o} + \delta\|F^o\|^2\right)} \frac{q(\delta, \mathfrak{v}(x))}{2} \\ &= \frac{d(x)}{4\|F\|^2\|F^o\|\,\|\mathfrak{v}(x)\|}\end{aligned} \tag{2.72}$$

where $\|F\|$ and $\|F^o\|$ can be found through the radii of two balls: one containing the set F and another contained in it. In our case, for instance, $\|F\| \leq 7/6$ and $\|F^o\| \leq 9/8$.

Summarizing everything said above, we affirm (by Theorem 2.3.2 (ii)) that the time-minimum projection $\pi_C^F(\cdot)$ is well-posed locally (near C), and, furthermore, the inequalities (2.70) and (2.72) together with (2.65) allow us to estimate the radius $r(x)$ of a ball centred at a given $x \in \partial C$ where such well-posedness takes place. In particular (see (2.25)), $r(x) = O\left(|x_2|^{1/3}\right)$ as $|x_2| \to \infty$.

Notice that in this example the mapping $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$ is locally Lipschitzean, and so we are able to apply the condition (**A**) as well (see Theorem 2.3.3), which gives even a larger radius $r(x) = O\left(|x_2|^{4/3}\right)$ as $|x_2| \to \infty$ (see (2.16) and (2.69)).

In the conclusion let us consider the mixed case (when there are points of both types: either satisfying the condition (**B**) only, or the condition (**A**)) emphasizing the situation when the boundedness of the curvature from below should be verified only in a neighbourhood of a given point $x_0 \in \partial C$ but not at x_0 itself.

Example 2.4.4 *Let us define two continuous real functions $f : [-1,1] \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ as follows:*

$$f(t) := \begin{cases} 1 - t^4 & \text{if } t \notin \left[-\frac{1}{\sqrt[3]{3}}, -\frac{1}{\sqrt[3]{5}}\right] \\ \text{affine} & \text{otherwise,} \end{cases}$$

$$g(t) := \begin{cases} \frac{1}{5}t - \frac{1}{100} & \text{if } 0 \le t \le \frac{1}{10} \\ t^2 & \text{if } \frac{1}{10} < t < \frac{3}{4} \\ \left(t - \frac{1}{2}\right)^2 + \frac{1}{2} & \text{if } t \ge \frac{3}{4}. \end{cases}$$

Set

$$F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \le f(\xi_1),\ -1 \le \xi_1 \le 1\right\};$$
$$C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le g(|x_2|)\right\}.$$

In this modification of the previous example the boundary ∂F has two affine pieces, and the target set is neither φ-convex (because it has an "inward corner" point $a = \left(-\frac{1}{100}, 0\right))$, nor smooth (besides of the point a, where the normal cone is trivial, it has multiple normals at $b^{\pm} = \left(\frac{9}{16}, \pm\frac{3}{4}\right)$).

For each $x_0 \in \partial C$, $x_0 \neq a, b^{\pm}$, we may proceed as in Example 2.4.3 since at these points both conditions (**A**) and (**B**) hold. If $x_0 = a$ then we can not apply (**B**) because the boundedness of $\psi_C(\cdot)$ near a fails. However, for each $x \in \partial C$ close to a the (nontrivial) cone $-\mathbf{N}_C^p(x)$ is contained in the interior of $\mathbf{N}_F(-1,0)$ (see (2.71)). In particular,

$$\mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^o\right) \equiv (-1,0),$$

therefore the mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^o\right)$ is (trivially) Lipschitzean on $C_\delta(a)$ and uniformly continuous on $U_{\delta,\delta'}(a)$, for some $\delta, \delta' > 0$, so the condition (**A**) follows (see the end of Example 2.4.1). The well-posedness of $\pi_C^F(\cdot)$ near a follows from Theorem 2.2.1.

Let now $x_0 = b^+$ (the symmetric point is considered similarly). Although at this point ∂C is not smooth (the normal cone is generated by two noncolinear vectors $e_1 = (1, -1/2)$ and $e_2 = (1, -3/2)$), the function $\psi_C(\cdot)$ is upper bounded in a neighbourhood of x_0, namely,

$$\psi_C(x,v) \le \max\left\{\frac{1}{\sqrt{1+4x_2^2}}, \frac{1}{\sqrt{1+(2x_2-1)^2}}\right\} \le 1, \tag{2.73}$$

$x = (x_1, x_2) \in C_\delta(x_0)$, $v \in \mathbf{N}_C^p(x)$, for some $\delta > 0$. Notice that $\mathfrak{J}_F\left(-\frac{e_1}{\rho_{F^o}(-e_1)}\right)$ and $\mathfrak{J}_F\left(-\frac{e_2}{\rho_{F^o}(-e_2)}\right)$ are different, hence the condition ($\mathbf{A}_1$) is violated. Also we have no strict convexity of the set F with respect to the vector $-e/\rho_{F^o}(-e)$, where

$$e := \left(\left(\frac{1}{3}\right)^{4/3} - \left(\frac{1}{5}\right)^{4/3}, \left(\frac{1}{5}\right)^{1/3} - \left(\frac{1}{3}\right)^{1/3}\right)$$

is normal to F at the affine part with $\xi_2 > 0$ and belongs to the interior of $-\mathbf{N}_C^p(b^+)$, impeding to apply the condition $(\mathbf{B}_2')$. Nevertheless, for each $x \in C_\delta(b^+) \setminus \{b^+\}$ the (unique) unit normal vector $\mathfrak{n}(x)$ to C (also belonging to $\mathbf{N}_C^p(b^+)$) is far enough from $e/\|e\|$, and F is strictly convex of second order w.r.t. $\mathfrak{v}(x) := -\mathfrak{n}(x)/\rho_{F^\circ}(-\mathfrak{n}(x))$. Moreover, the curvature is uniformly bounded from below, and the hypothesis $(\mathbf{B}_2)$ holds. In such a way constructing a neighbourhood of x_0, where $\pi_C^F(\cdot)$ is well-defined, we may take into account balance between (2.73) at the points $x \in \partial C$ near x_0 and the curvature of F only at $(\xi_1, \xi_2) \in \partial F$ with $\xi_1 \in \left[-\left(\frac{1}{2}\right)^{1/3}, -\left(\frac{1}{6}\right)^{1/3}\right]$, which are close to the end-points of the respective arc.

Chapter 3

Regularity results

We assume the same hypotheses as in the previous chapter, that is, H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, $F \subset H$ is a nonempty closed convex bounded set with $0 \in \operatorname{int} F$, and $C \subset H$ is nonempty and closed. Based on the geometric conditions (**A**), (**B**) used in Section 2.2 here we show that the time-minimum projection $\pi_C^F(\cdot)$ (well-defined and single-valued near the target set) satisfies in fact stronger regularity conditions than simple continuity (see Section 3.2). Namely, it is Lipschitz continuous under the hypothesis (**A**), while under (**B**) it is only Hölderian with exponent 1/2. However, as we will see in sequel the last result can be essentially improved if either the target set, or the dynamics is supposed to be enough regular (at least locally). Using these supplementary conditions in the next Section 3.3 we prove (Fréchet) differentiability of the minimal time function $\mathfrak{T}_C^F(\cdot)$ near the target C and give explicit formulas for its derivative (which vary according to the hypotheses involved). Observe that the results obtained in Section 3.2 also allow to prove the Hölder regularity of the derivatives. We finish the chapter with some examples (see Section 3.4).

3.1 Various concepts of regularity

Let us start this chapter by recalling some important properties of the subdiferentials of non convex functions in a Hilbert space introduced in Section 2.1, which will be used in sequel.

Definition 3.1.1 *Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and $x \in H$ with $\phi(x) < +\infty$. The function $\phi(\cdot)$ is said to be*

(i) proximally regular *at x if $\partial^p \phi(x) = \partial^l \phi(x)$;*

(ii) lower regular *at x if $\partial^f \phi(x) = \partial^l \phi(x)$;*

(iii) Fréchet regular *at x if $\partial^p \phi(x) = \partial^f \phi(x)$;*

(iv) Clarke regular *at x if $\partial^f \phi(x) = \partial^c \phi(x)$.*

Since the target set $C \subset H$ is assumed to be nonempty and closed that implies the lower semicontinuity of the indicator function $\mathbf{I}_C(\cdot)$, we are led to various notions of regularity for C at a point $x \in \partial C$ through the respective concept given for $\mathbf{I}_C(\cdot)$.

Definition 3.1.2 *A closed set C is said to be*

(i) proximally regular *at $x \in \partial C$ if* $\mathbf{N}_C^p(x) = \mathbf{N}_C^l(x)$;

(ii) normally regular *at x if* $\mathbf{N}_C^f(x) = \mathbf{N}_C^l(x)$;

(iii) Fréchet regular *at x if* $\mathbf{N}_C^p(x) = \mathbf{N}_C^f(x)$;

(iv) Clarke regular *at x if* $\mathbf{N}_C^f(x) = \mathbf{N}_C^c(x)$.

All the definitions of subdifferentials as well as of normal cones are given in Section 2.1. Notice that the proximal regularity is a very strong property. Even the continuous (Fréchet) differentiability of $\phi(\cdot)$ at a fixed point does not imply (in general) the proximal regularity at this point. A simple example of such situation just in $\mathbb{R}$ is given by the function $y = -|x|^{3/2}$ whose gradient at $x = 0$ is $-\frac{3}{2}\sqrt{|x|}$, but nevertheless $\partial^p\phi(0) = \emptyset$ (while $\partial^l\phi(0) = \{0\}$). However, if the gradient $\nabla\phi(\cdot)$ exists and is Lipschitz continuous near x then $\phi(\cdot)$ is proximally regular (and Clarke regular as well) at this point. Indeed it can be shown easily that $\partial^p\phi(y) = \{\nabla\phi(y)\}$ for y close to x (see [23, p. 36] for the respective proof in the case of $\phi(\cdot)$ of class $\mathcal{C}^2$) then by using the definition of the limiting subdifferential we obviously have $\partial^l\phi(x) = \{\nabla\phi(x)\}$, and $\partial^c\phi(x) = \{\nabla\phi(x)\}$ as well. Observe that the lower regularity (normal regularity in the case of sets) as well as Fréchet regularity are weaker properties than the proximal regularity.

Let us emphasize specially the class $\mathfrak{F}$ of functions $\phi(\cdot)$ (the class $\mathfrak{N}$ of sets C), which are simultaneously proximally and Clarke regular, i.e., such that for them all the subdifferentials (respectively, the normal cones) coincide at each point $x \in \operatorname{dom}\phi$ ($x \in \partial C$). In particular, all φ-convex sets (but not only) belong to $\mathfrak{N}$. In turn the class of φ-convex sets contains all convex sets as well as the sets C admitting at each point $x \in \partial C$ a (proximal) normal vector, which is locally Lipschitzean w.r.t. x (see Section 2.1). Similarly, the class $\mathfrak{F}$ contains all convex functions as well as the (Fréchet) differentiable functions with locally Lipschitzean gradient. The class of functions with the last property, which are defined on an open set $U \subset H$, is traditionally denoted by $\mathcal{C}_{loc}^{1,1}(U)$. Generalizing slightly this concept, we introduce the class of functions $\phi(\cdot) \in \mathcal{C}^1(U)$, whose gradient $\nabla\phi(\cdot)$ is Hölder continuous near each $x \in U$ with an exponent $0 < \alpha \leq 1$, i.e., given $x \in U$ there exist a constant $K = K(x) > 0$ and $\varepsilon > 0$ such that

$$\|\nabla\phi(x_1) - \nabla\phi(x_2)\| \leq K\|x_1 - x_2\|^\alpha$$

whenever $\|x_i - x\| \leq \varepsilon$, $i = 1, 2$. In this case we say that the function $\phi(\cdot)$ belongs to the class $\mathcal{C}_{loc}^{1,\alpha}(U)$.

One of the purposes of this chapter is to study the regularity introduced above for the minimal time function $\mathfrak{T}_C^F(\cdot)$, and, in particular, to emphasize the conditions guaranteeing that $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\alpha}$ for some $0 < \alpha \leq 1$ in an open neighbourhood of the target C.

First of all let us collect here well-known results concerning with the representation formulas for the various subdifferentials of the distance function (as well as for the minimal time function) and derived from them regularity properties.

As was said in Introduction one should consider separately two cases: $x \notin C$ and $x \in \partial C$. In the first case one kind of formulas links the subdiferential with the respective normal cone to the sublevel set $C(r) := \{y \in H : \mathrm{d}_C(y) \leq r\}$, $r := \mathrm{d}_C(x)$, for the distance (or to the enlargement set $C(r) := \{y \in H : \mathfrak{T}_C^F(y) \leq r\}$, $r := \mathfrak{T}_C^F(x)$, for the value function $\mathfrak{T}_C^F(\cdot)$). Namely, for a nonempty closed set $C \subset H$ and $x \notin C$ we always have the following relations (see [24, 13])

$$\partial^p \mathrm{d}_C(x) = \mathbf{N}_{C(r)}^p(x) \cap \partial \mathbf{B},$$

$$\partial^f \mathrm{d}_C(x) = \mathbf{N}_{C(r)}^f(x) \cap \partial \mathbf{B},$$

while in the case of an arbitrary dynamics

$$\partial^p \mathfrak{T}_C^F(x) = \mathbf{N}_{C(r)}^p(x) \cap (-\partial F^o), \tag{3.1}$$

$$\partial^f \mathfrak{T}_C^F(x) = \mathbf{N}_{C(r)}^f(x) \cap (-\partial F^o) \tag{3.2}$$

as was proved in [31] (see Theorem 3.1). Since inverting the formulas (3.1) and (3.2) we can equivalently write

$$\mathbf{N}_{C(r)}^p(x) = \bigcup_{\lambda \geq 0} \lambda \partial^p \mathfrak{T}_C^F(x)$$

and

$$\mathbf{N}_{C(r)}^f(x) = \bigcup_{\lambda \geq 0} \lambda \partial^f \mathfrak{T}_C^F(x),$$

it follows immediately that the function $\mathfrak{T}_C^F(\cdot)$ is Fréchet regular at $x \notin C$ if and only if the respective enlargement set $C(r)$, $r := \mathfrak{T}_C^F(x)$, is Fréchet regular at the same point. Observe that for the Clarke subdifferential the situation is more complicated: even in the case of the distance. For the validity of the formula

$$\partial^c \mathrm{d}_C(x) = \mathbf{N}_{C(r)}^c(x) \cap \partial \mathbf{B}$$

one should require some *a priori* regularity assumption (see [15, Theorem 14]). The respective regularity condition guaranteeng the validity of the relation

$$\partial^c \mathfrak{T}_C^F(x) = \mathbf{N}_{C(r)}^c(x) \cap (-\partial F^o) \tag{3.3}$$

was studied, e.g., in [83].

Assume now that $x \in \partial C$. Then for the distance we have (see [13]):

$$\partial^p \mathrm{d}_C(x) = \mathbf{N}_C^p(x) \cap \overline{\mathbf{B}},$$

$$\partial^f \mathrm{d}_C(x) = \mathbf{N}_C^f(x) \cap \overline{\mathbf{B}}.$$

Observe that in these formulas the whole ball $\overline{\mathbf{B}}$ stands in the place of the sphere, which, in particular, shows that the subdifferentials $\partial^p \mathrm{d}_C(x)$ and $\partial^f \mathrm{d}_C(x)$ can not be single-valued (equivalently, the distance is never differentiable at the boundary points). The respective formulas for the value function $\mathfrak{T}_C^F(\cdot)$ are (see [80, 83, 82]):

$$\partial^p \mathfrak{T}_C^F(x) = \mathbf{N}_C^p(x) \cap (-F^o), \tag{3.4}$$

$$\partial^f \mathfrak{T}_C^F(x) = \mathbf{N}_C^f(x) \cap (-F^o), \tag{3.5}$$

which could be complemented with the similar relation involving the limiting constructions as well (see [67, Theorem 3.6])

$$\partial^l \mathfrak{T}_C^F(x) \subset \mathbf{N}_C^l(x) \cap (-F^o). \tag{3.6}$$

Observe that in (3.6) we have only one sided inclusion, in general. In spite of this the following (inverse) representation takes place

$$\mathbf{N}_C^l(x) = \bigcup_{\lambda \geq 0} \lambda \partial^l \mathfrak{T}_C^F(x), \tag{3.7}$$

while it is obvious for other types of subdifferentials (as follows from (3.4) and (3.5)).

Hence, on one hand, we again see that $\mathfrak{T}_C^F(\cdot)$ can not be differentiable at $x \in \partial C$ while, on the other, it is Fréchet (proximally, lower) regular at some point $x \in \partial C$ if and only if the target set C is Fréchet (respectivelly, proximally or normally) regular at x.

As we see in the case of in-set points the subdifferentials of the minimal time function (in particular, of the distance) split into two parts: one depends only on the target set (the respective normal cone) while the other depends on the dynamics (minus polar set). For out-of-set points instead we have a different situation. Indeed, the enlargement set $C(r)$ involved into the right-hand of (3.1)-(3.3) is defined by stirring the properties of both sets C and F. In order to avoid this inconvenience we should, first of all, associate to each $x \notin C$ a point $\bar{x} \in \partial C$ and then try to represent the subdifferential of $\mathfrak{T}_C^F(\cdot)$ at x through the corresponding normal cone to C at $\bar{x}$. It turns out that this point $\bar{x}$ is nothing else than the (time-minimum) projection of x onto C, and the one-sided inclusions for $\partial^p \mathfrak{T}_C^F(x)$ and $\partial^f \mathfrak{T}_C^F(x)$ take the form (see [31, Theorem 3.3])

$$\partial^p \mathfrak{T}_C^F(x) \subset \mathbf{N}_C^p(\bar{x}) \cap (-\partial \rho_F(\bar{x} - x)), \tag{3.8}$$

$$\partial^f \mathfrak{T}_C^F(x) \subset \mathbf{N}_C^f(\bar{x}) \cap (-\partial \rho_F(\bar{x} - x)), \tag{3.9}$$

where $\partial \rho_F(\cdot)$ is the subdifferential in the sense of convex analysis of the (convex) guage function, which can be found by the formula

$$\partial \rho_F(\bar{x} - x) = \mathbf{N}_F\left(\frac{\bar{x} - x}{\rho_F(\bar{x} - x)}\right) \cap \partial F^o.$$

Unfortunately, it is not possible to obtain the exact equalities in (3.8) and (3.9) with no supplementary regularity conditions for C and/or F. But assuming such conditions we can prove the reverse inclusions. For instance, if the target set C is supposed to be proximally (normally) regular then by using a simple argument based on the necessary condition of optimality

$$0 \in \partial^l F(\bar{x}), \tag{3.10}$$

where $F(y) := \rho_F(y - x) + \mathbf{I}_C(y)$, $y \in H$, we deduce first that the right-hand side in (3.8) (respectively, in (3.9)) is nonempty (see [31, Proposition 5.5]) indeed, by the well-known formula for the limiting subdifferential (see Theorem 2.1.2)

$$0 \in \partial \rho_F(\bar{x} - x) + \partial^l \mathbf{I}_C(\bar{x}).$$

Then, taking into account that $\partial^l \mathbf{I}_C(\bar{x}) = \mathbf{N}_C^l(\bar{x})$, we see that the nonemptiness of the intersections above follows from the respective regularity properties. In Section 3.3 we exploit this fact to prove the equalities in (3.8), (3.9) and we even obtain there a stronger equality:

$$\partial^c \mathfrak{T}_C^F(x) = \mathbf{N}_C^p(\bar{x}) \cap (-\partial \rho_F(\bar{x} - x)),$$

which under some extra hypotheses would imply the (continuous) differentiability of the minimal time function.

3.2 Hölder continuity of the time-minimum projection

Let us consider separately the conditions (**A**) and (**B**) (see the beginning of Section 2.2).

The condition (A)

Theorem 3.2.1 *Let us fix a point $x_0 \in \partial C$. Suppose that the pair of sets (F, C) satisfies the condition* (**A**) *at x_0. Then the time-minimum projection $\pi_C^F(\cdot)$ is locally Lipschitz continuous in some neighbourhood of x_0 (outside of C).*

Proof.
We set

$$\mathcal{U}'(x_0) := \left\{ z \in H : \|z - x_0\| < \frac{\delta}{2(\|F\| \|F^o\| + 1)}, \ \mathfrak{T}_C^F(z) < \frac{1}{L} \right\},$$

where $L > 0$ is the Lipschitz constant of the mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^o\right)$ on $C_\delta(x_0)$ (see ($\mathbf{A}_1$)). Notice that $\mathcal{U}'(x_0)$ is contained in the neighbourhood $\mathcal{U}(x_0)$ defined by (2.16) where $\pi_C^F(\cdot)$ is well-posed, i.e., $\pi_C^F(x)$ is a singleton continuously depending on $x \in \mathcal{U}'(x_0)$. Let $\delta, \delta' > 0$ and the open set $U_{\delta,\delta'}(x_0)$ be as in ($\mathbf{A}_2$).

Let us fix $x \in \mathcal{U}'(x_0) \backslash C$ and let the numbers $\tau > 0$, $0 < \bar{\delta} < \frac{\tau}{2\|F^o\|}$ be such that

$$\mathfrak{T}_C^F(x) + \tau < \frac{1}{L};$$

$$x + \bar{\delta}\mathbf{B} \subset \mathcal{U}'(x_0) \backslash C.$$

Take arbitrary $z, w \in x + \bar{\delta}\mathbf{B}$, $z \neq w$, and set $\beta := \frac{\|z-w\|}{2} > 0$; $\bar{z} := \pi_C^F(z)$ and $\bar{w} =: \pi_C^F(w)$. By the condition ($\mathbf{A}_2$), there exists $0 < \nu \leq \delta'$ such that

$$\|\mathfrak{J}_F(u) - \mathfrak{J}_F(v)\| \leq \beta, \tag{3.11}$$

whenever $u, v \in U_{\delta,\delta'}(x_0)$, with $\|u - v\| \leq \nu$. We set also

$$\varepsilon := \min\left\{ \frac{\tau}{2\|F^o\|}, \nu, \beta \right\}.$$

Since $\overline{z}$ minimizes the functional $F(y) := \rho_F(y-z) + \mathbf{I}_C(y)$, by the necessary minimum condition (see [23, p. 37]) $0 \in \partial^p F(\overline{z})$ and using the fuzzy sum rule (Theorem 2.1.1) we find $z_1, z_2 \in \overline{z} + \varepsilon\mathbf{B}$ with $z_1 \in \partial^* C$ and vectors $v_z \in -\partial F^o \cap \mathbf{N}_C^p(z_1)$, $\xi_z^* \in \partial\rho_F(z_2 - z)$ such that

$$\|v_z + \xi_z^*\| \leq \varepsilon.$$

Notice that the Lipschitz continuity of $\mathfrak{T}_C^F(\cdot)$ implies

$$\mathfrak{T}_C^F(z) \leq \|F^o\| \, \|z - x_0\|, \tag{3.12}$$

and, hence,

$$\begin{aligned} \|z_1 - x_0\| &\leq \|z_1 - \overline{z}\| + \|F\| \, \rho_F(\overline{z} - z) + \|z - x_0\| \\ &\leq \varepsilon + \|F\| \, \mathfrak{T}_C^F(z) + \|z - x_0\| \\ &\leq \varepsilon + (\|F\| \, \|F^o\| + 1) \, \|z - x_0\| \leq \delta, \end{aligned}$$

i.e., $z_1 \in C_\delta(x_0)$. Since $\xi_z^*, -v_z \in \partial F^o \cap \left(-\mathbf{N}_C^p(z_1) \cap \partial F^o + \delta'\overline{\mathbf{B}}\right)$, by (3.11), we obtain

$$\|\mathfrak{J}_F(\xi_z^*) - \mathfrak{J}_F(-v_z)\| \leq \beta.$$

Similarly, considering the point w and its time-minimum projection $\overline{w}$ in the place of z and $\overline{z}$, we find $w_1, w_2 \in \overline{w} + \varepsilon\mathbf{B}$, $w_1 \in \partial^* C$, and $v_w \in -\partial F^o \cap \mathbf{N}_C^p(w_1)$, $\xi_w^* \in \partial\rho_F(w_2 - w)$, satisfying $\|v_w + \xi_w^*\| \leq \varepsilon$. By the same reasons as above we show that $\|\mathfrak{J}_F(\xi_w^*) - \mathfrak{J}_F(-v_w)\| \leq \beta$. Joining together these two inequalities and taking into account the hypothesis $(\mathbf{A}_1)$, we have

$$\begin{aligned} \|\mathfrak{J}_F(\xi_z^*) - \mathfrak{J}_F(\xi_w^*)\| &\leq 2\beta + \|\mathfrak{J}_F(-v_z) - \mathfrak{J}_F(-v_w)\| \\ &\leq 2\beta + L\,\|z_1 - w_1\| \\ &\leq 2\beta + L\,(\|z_1 - \overline{z}\| + \|\overline{z} - \overline{w}\| + \|\overline{w} - w_1\|) \\ &\leq 2\beta + L\,(2\varepsilon + \|\overline{z} - \overline{w}\|). \end{aligned} \tag{3.13}$$

On the other hand, from the inequalities

$$\left|\rho_F(z_2 - z) - \mathfrak{T}_C^F(z)\right| \leq \|F^o\|\,\varepsilon \quad \text{and} \quad \left|\rho_F(w_2 - w) - \mathfrak{T}_C^F(w)\right| \leq \|F^o\|\,\varepsilon$$

it follows that

$$\begin{aligned} |\rho_F(z_2 - z) - \rho_F(w_2 - w)| &\leq 2\,\|F^o\|\,\varepsilon + \left|\mathfrak{T}_C^F(z) - \mathfrak{T}_C^F(w)\right| \\ &\leq \|F^o\|\,(2\varepsilon + \|z - w\|). \end{aligned} \tag{3.14}$$

Also

$$\begin{aligned} \rho_F(z_2 - z) &\leq \rho_F(z_2 - \overline{z}) + \mathfrak{T}_C^F(z) \\ &\leq \|F^o\|\,\|z_2 - \overline{z}\| + \mathfrak{T}_C^F(x) + \|F^o\|\,\|x - z\| \\ &\leq \|F^o\|\,\varepsilon + \mathfrak{T}_C^F(x) + \|F^o\|\,\overline{\delta} \leq \mathfrak{T}_C^F(x) + \tau. \end{aligned} \tag{3.15}$$

By Proposition 1.1.1 $\mathfrak{J}_F(\xi_z^*) = \frac{z_2 - z}{\rho_F(z_2 - z)}$ and $\mathfrak{J}_F(\xi_w^*) = \frac{w_2 - w}{\rho_F(w_2 - w)}$. Hence, by using, respectively, (3.15), (3.13) and (3.14) we obtain

$$\begin{aligned}
\|\overline{z} - \overline{w}\| &\leq \|\overline{z} - z_2\| + \|z_2 - w_2\| + \|w_2 - \overline{w}\| \\
&\leq 2\varepsilon + \|\mathfrak{J}_F(\xi_z^*)\rho_F(z_2 - z) - \mathfrak{J}_F(\xi_w^*)\rho_F(w_2 - w)\| + \|z - w\| \\
&\leq 2\varepsilon + \rho_F(z_2 - z)\|\mathfrak{J}_F(\xi_z^*) - \mathfrak{J}_F(\xi_w^*)\| + \|\mathfrak{J}_F(\xi_w^*)\|\,|\rho_F(z_2 - z) - \rho_F(w_2 - w)| + \\
&\quad + \|z - w\| \\
&\leq 2\varepsilon + \left(\mathfrak{T}_C^F(x) + \tau\right)(2\beta + L(2\varepsilon + \|\overline{z} - \overline{w}\|)) + \|F\|\,\|F^o\|(2\varepsilon + \|z - w\|) + \\
&\quad + \|z - w\|.
\end{aligned}$$

Now taking into account the definitions of β and ε we obtain

$$\left(1 - \left(\mathfrak{T}_C^F(x) + \tau\right)L\right)\|\overline{z} - \overline{w}\| \leq \mathfrak{k}\,\|z - w\|,$$

where

$$\mathfrak{k} = 2(1 + \|F\|\,\|F^o\|) + \left(\mathfrak{T}_C^F(x) + \tau\right)(1 + L).$$

In the case $z = w$ the same inequality holds due to uniqueness of the projection. Thus, $\pi_C^F(\cdot)$ is Lipschitz continuous on $x + \overline{\delta}\mathbf{B}$ with the Lipschitz constant $\mathfrak{k}/\left(1 - \left(\mathfrak{T}_C^F(x) + \tau\right)L\right)$. ■

The condition (B)

Theorem 3.2.2 *Let us fix $x_0 \in \partial C$. If the pair of sets (F, C) satisfies the condition* (**B**) *at x_0 then $\pi_C^F(\cdot)$ is locally Hölder continuous with exponent $1/2$ in $\mathcal{U}'(x_0)\backslash C$ for some neighbourhood $\mathcal{U}'(x_0)$ of the point x_0. Furthermore, if the mapping*

$$x \mapsto -\mathbf{N}_C^p(\overline{x}) \cap \mathbf{N}_F\left(\frac{\overline{x} - x}{\rho_F(\overline{x} - x)}\right) \cap \partial F^o, \quad \textit{where } \overline{x} := \pi_C^F(x),$$

is single-valued, say $\{v(x)\}$, and $v(\cdot)$ is Hölder continuous near x_0 (outside of C) with an exponent $0 < \alpha \leq 1$, then $\pi_C^F(\cdot)$ is locally Hölder continuous with the exponent $\frac{1+\alpha}{2}$. In particular, the Lipschitz continuity of $v(\cdot)$ implies the same property for $\pi_C^F(\cdot)$.

Proof.

We set $\mathcal{U}'(x_0) := \mathcal{U}(x_0)$ the same neighbourhood as in (2.16) which is given by

$$\mathcal{U}(x_0) := \left\{z \in H : \|z - x_0\| < \frac{\delta}{\|F\|\,\|F^o\| + 1},\ \mathfrak{T}_C^F(z) < \frac{K}{M}\right\}, \tag{3.16}$$

where the constant $M > 0$ is such that $\psi_C(x, v) \leq M$ for all $x \in C_\delta(x_0)$ and all $v \in \mathbf{N}_C^p(x)$ (see ($\mathbf{B}_1$)). Let us fix $x \in \mathcal{U}(x_0)\backslash C$ and let $\tau > 0$ such that

$$\mathfrak{T}_C^F(x) + \tau < \frac{K}{M}.$$

Now we fix $0 < \overline{\delta} < \frac{\tau}{\|F^o\|}$ such that $x + \overline{\delta}\mathbf{B} \subset \mathcal{U}(x_0) \backslash C$, and take arbitrary $z, w \in x + \overline{\delta}\mathbf{B}$. Setting $\overline{z} := \pi_C^F(z)$, $\overline{w} =: \pi_C^F(w)$, by (3.12) we have

$$\|\overline{z} - x_0\| \leq \|F\| \rho_F(\overline{z} - z) + \|z - x_0\| \leq (\|F\| \|F^o\| + 1) \|z - x_0\| \leq \delta,$$

i.e., $\overline{z} \in C_\delta(x_0)$, and analogously $\overline{w} \in C_\delta(x_0)$.

Applying, similarly as in the proof of the previous theorem, the necessary condition of minimum (but in the limiting form, i.e., the condition (3.10)) and using the fact that the proximal normal cones to C at the points $\overline{z}$ and $\overline{w}$ coincide with the limiting ones (since C is φ-convex in a neighbourhood of x_0), we conclude that the sets

$$-\mathbf{N}_C^p(\overline{z}) \cap \mathbf{N}_F\left(\frac{\overline{z} - z}{\rho_F(\overline{z} - z)}\right) \cap \partial F^o \quad \text{and} \quad -\mathbf{N}_C^p(\overline{w}) \cap \mathbf{N}_F\left(\frac{\overline{w} - w}{\rho_F(\overline{w} - w)}\right) \cap \partial F^o$$

are nonempty (see the end of the previous section). Choosing an element from each of these sets (say n_z and n_w, respectively), we have $\psi_C(\overline{z}, -n_z) \leq M$ (see $(\mathbf{B}_1)$) and $\varkappa_F(\mathfrak{J}_F(n_z), n_z) \geq K$ (see $(\mathbf{B}_2)$). Then, denoting by

$$\begin{aligned} R_z &:= \frac{\mathfrak{R}_F(\mathfrak{J}_F(n_z), n_z)}{\|n_z\|} = \frac{1}{2\|n_z\| \varkappa_F(\mathfrak{J}_F(n_z), n_z)}, \\ \psi_z &:= \psi_C(\overline{z}, -n_z), \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{2R_z} - \mathfrak{T}_C^F(z) \psi_z \|n_z\| &= \|n_z\| \left(\varkappa_F(\mathfrak{J}_F(n_z), n_z) - \mathfrak{T}_C^F(z) \psi_z\right) \\ &\geq 2\nu := \frac{1}{\|F\|} \left(K - \left(\mathfrak{T}_C^F(x) + \tau\right) M\right) > 0. \end{aligned} \tag{3.17}$$

Furthermore, let $\nu' > 0$ be so small that

$$\frac{1}{2(R_z + \nu')} - \mathfrak{T}_C^F(z) \psi_z \|n_z\| > \nu. \tag{3.18}$$

By definition of R_z and taking into account that $\mathfrak{J}_F(n_z) = \frac{\overline{z} - z}{\rho_F(\overline{z} - z)}$ we have

$$F \subset \frac{\overline{z} - z}{\rho_F(\overline{z} - z)} - (R_z + \nu') n_z + (R_z + \nu') \|n_z\| \overline{\mathbf{B}}.$$

Since $\frac{\overline{w} - w}{\rho_F(\overline{w} - w)} \in F$, from the last inclusion follows, in particular, that

$$\left\| z - \overline{z} + \frac{\rho_F(\overline{z} - z)}{\rho_F(\overline{w} - w)} (\overline{w} - w) + \rho_F(\overline{z} - z)(R_z + \nu') n_z \right\| \leq \rho_F(\overline{z} - z)(R_z + \nu') \|n_z\|,$$

or, in another form,

$$\left\| z - \overline{z} + \frac{\rho_F(\overline{z} - z)}{\rho_F(\overline{w} - w)} (\overline{w} - w) \right\|^2 \leq -2\rho_F(\overline{z} - z)(R_z + \nu') \left\langle n_z, z - \overline{z} + \frac{\rho_F(\overline{z} - z)}{\rho_F(\overline{w} - w)} (\overline{w} - w) \right\rangle. \tag{3.19}$$

The rest of the proof is divided into two steps.

Step 1: Assume that $\rho_F(\overline{z}-z)=\rho_F(\overline{w}-w)$ and denote their common value by ρ. By (3.19) we have

$$\begin{aligned}\|(z-w)-(\overline{z}-\overline{w})\|^2 &\le 2\rho\left(R_z+\nu'\right)\langle -n_z,(z-w)-(\overline{z}-\overline{w})\rangle \\ &\le 2\rho\left(R_z+\nu'\right)\left(\langle n_z,w-z\rangle+\psi_z\|n_z\|\,\|\overline{z}-\overline{w}\|^2\right).\end{aligned}$$

Combining this inequality with

$$\begin{aligned}\|(z-w)-(\overline{z}-\overline{w})\|^2 &= \|z-w\|^2-2\langle z-w,\overline{z}-\overline{w}\rangle+\|\overline{z}-\overline{w}\|^2 \\ &\ge \|\overline{z}-\overline{w}\|^2-2\langle z-w,\overline{z}-\overline{w}\rangle\end{aligned}$$

we obtain

$$\left(1-2\rho\left(R_z+\nu'\right)\psi_z\|n_z\|\right)\|\overline{z}-\overline{w}\|^2\le 2\rho\left(R_z+\nu'\right)\langle n_z,w-z\rangle+2\|z-w\|\,\|\overline{z}-\overline{w}\|,$$

which implies

$$\left(\frac{1}{2\left(R_z+\nu'\right)}-\rho\psi_z\|n_z\|\right)\|\overline{z}-\overline{w}\|^2\le\rho\langle n_z,w-z\rangle+\frac{1}{R_z+\nu'}\|z-w\|\,\|\overline{z}-\overline{w}\|.$$

So that by (3.18)

$$\nu\|\overline{z}-\overline{w}\|^2\le\rho\langle n_z,w-z\rangle+\frac{1}{\mathfrak{r}_F}\|z-w\|\,\|\overline{z}-\overline{w}\|, \tag{3.20}$$

where $\mathfrak{r}_F>0$ is the Chebyshev radius of the convex set F (see (1.32)). Since $n_z\in F^o$, we can rewrite (3.20) in the form:

$$\|\overline{z}-\overline{w}\|\left(\nu\|\overline{z}-\overline{w}\|-\frac{1}{\mathfrak{r}_F}\|z-w\|\right)\le\rho\|F^o\|\,\|z-w\|.$$

Therefore, one of the inequalities

$$\|\overline{z}-\overline{w}\|\le\sqrt{\rho\|F^o\|}\,\|z-w\|^{1/2}$$

or

$$\nu\|\overline{z}-\overline{w}\|\le\sqrt{\rho\|F^o\|}\,\|z-w\|^{1/2}+\frac{1}{\mathfrak{r}_F}\|z-w\|, \tag{3.21}$$

clearly, takes place. Observe that $\rho=\mathfrak{T}_C^F(z)\le\|F^o\|\,\delta$ and $\|z-w\|\le\delta$. The last follows from the fact that $z,w\in\mathcal{U}(x_0)$ and from the inequality $\|F\|\,\|F^o\|\ge 1$. Finally, from (3.21) we obtain

$$\|\overline{z}-\overline{w}\|\le\mathfrak{k}\|z-w\|^{1/2} \tag{3.22}$$

where

$$\mathfrak{k}:=\sqrt{\delta}\max\left\{\|F^o\|,\frac{\|F^o\|}{\nu}+\frac{1}{\mathfrak{r}_F\nu}\right\}. \tag{3.23}$$

Step 2: Now let us fix $z, w \in x+\overline{\delta}_1\mathbf{B}$ where $\overline{\delta}_1 := \frac{\overline{\delta}}{2\|F\|\|F^o\|+1}$ and denote by $\rho_z := \rho_F(\overline{z}-z)$ and $\rho_w := \rho_F(\overline{w}-w)$ do not assuming more that $\rho_z = \rho_w$.

For the sake of determinancy we assume that $\rho_w > \rho_z$. Write $\overline{w} = w + \rho_w u$, with $\rho_F(u) = 1$, and let $w' = w + (\rho_w - \rho_z)u$. We have by [31, Proposition 2.6 (a)]

$$\mathfrak{T}_C^F(w') = \mathfrak{T}_C^F(w + (\rho_w - \rho_z)u) \leq \mathfrak{T}_C^F(w + \rho_w u) + \rho_z = \mathfrak{T}_C^F(\overline{w}) + \rho_z = \rho_z,$$

and, on the other hand,

$$\begin{aligned}
\mathfrak{T}_C^F(w') &= \inf_{y\in C} \rho_F(y - w') = \inf_{y\in C} \rho_F(y - w - (\rho_w - \rho_z)u) \\
&\geq \inf_{y\in C} (\rho_F(y-w) - \rho_F((\rho_w - \rho_z)u)) \\
&= \inf_{y\in C} \rho_F(y-w) - (\rho_w - \rho_z)\rho_F(u) \\
&= \mathfrak{T}_C^F(w) - (\rho_w - \rho_z) = \rho_z.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|w' - x\| &= \|w + (\rho_w - \rho_z)u - x\| \leq \|w - x\| + (\rho_w - \rho_z)\|u\| \\
&\leq \|w - x\| + \left|\mathfrak{T}_C^F(w) - \mathfrak{T}_C^F(z)\right| \|F\| < \overline{\delta}_1 + \|F^o\|\,\|F\|\,\|w - z\| \\
&< (1 + 2\|F^o\|\,\|F\|)\overline{\delta}_1 = \overline{\delta},
\end{aligned}$$

and also $\|z - x\| < \overline{\delta}_1 \leq \overline{\delta}$. Moreover, since

$$\rho_F(\overline{w} - w') = \rho_F(w + \rho_w u - w - (\rho_w - \rho_z)u) = \rho_F(\rho_z u) = \rho_z = \mathfrak{T}_C^F(w'),$$

and $w' \in \mathcal{U}(x_0)$, we have $\overline{w} = \pi_C^F(w')$. Thus, we can apply the Step 1 to the points z and w'. Namely, by (3.22) we deduce

$$\begin{aligned}
\|\overline{z} - \overline{w}\| &\leq \mathfrak{k}\|z - w'\|^{\frac{1}{2}} \leq \mathfrak{k}(\|z - w\| + \|w - w'\|)^{1/2} \\
&= \mathfrak{k}(\|z - w\| + (\rho_w - \rho_z)\|u\|)^{1/2} \\
&\leq \mathfrak{k}(1 + \|F^o\|\,\|F\|)^{1/2}\|z - w\|^{1/2}, \qquad (3.24)
\end{aligned}$$

where the constant $\mathfrak{k} > 0$ is given by (3.23).

Let us prove now the second part of the theorem assuming that the (single-valued) mapping $x \mapsto v(x) = -\mathbf{N}_C^p(\overline{x}) \cap \mathbf{N}_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right) \cap \partial F^o$ is Hölder continuous with an exponent $0 < \alpha \leq 1$ in the δ-neighbourhood of x_0.

We define $\mathcal{U}'(x_0) := \mathcal{U}(x_0)$ (see (3.16)), fix $x \in \mathcal{U}(x_0)\setminus C$ and choose constants $\tau > 0, \overline{\delta} > 0$ and $\nu > 0$ as above. Then for given $z, w \in x + \overline{\delta}\mathbf{B}$ and for the associated normal vectors n_z and n_w (see the beginning of the proof) we have $n_z = v(z)$ and $n_w = v(w)$.

Let us consider first the simpler case when $\rho_F(\overline{z} - z) = \rho_F(\overline{w} - w) =: \rho$. As already shown (see (3.20))

$$\nu\|\overline{z} - \overline{w}\|^2 \leq \rho\langle v(z), w - z\rangle + \frac{1}{\mathfrak{r}_F}\|z - w\|\,\|\overline{z} - \overline{w}\|.$$

By the same reasons changing the places of the points z and w we obtain

$$\nu \|\overline{z}-\overline{w}\|^2 \leq \rho \langle v(w), z-w\rangle + \frac{1}{\mathfrak{r}_F}\|z-w\| \|\overline{z}-\overline{w}\|.$$

Now adding the last two inequalities we get

$$\begin{aligned} 2\nu \|\overline{z}-\overline{w}\|^2 &\leq \rho \langle v(w)-v(z), z-w\rangle + \frac{2}{\mathfrak{r}_F}\|z-w\| \|\overline{z}-\overline{w}\| \\ &\leq \rho \|v(w)-v(z)\| \|z-w\| + \frac{2}{\mathfrak{r}_F}\|z-w\| \|\overline{z}-\overline{w}\|. \end{aligned} \tag{3.25}$$

Therefore

$$\begin{aligned} & 2\nu \|\overline{z}-\overline{w}\|^2 \leq \rho\mathfrak{h} \|w-z\|^{\alpha+1} + \frac{2}{\mathfrak{r}_F}\|z-w\| \|\overline{z}-\overline{w}\| \\ \Rightarrow \quad & \|\overline{z}-\overline{w}\| \left(2\nu \|\overline{z}-\overline{w}\| - \frac{2}{\mathfrak{r}_F}\|z-w\|\right) \leq \rho\mathfrak{h} \|w-z\|^{1+\alpha} \\ \Rightarrow \quad & \|\overline{z}-\overline{w}\| \leq \sqrt{\rho\mathfrak{h} \|w-z\|^{1+\alpha}} \quad \text{or} \quad 2\nu \|\overline{z}-\overline{w}\| \leq \sqrt{\rho\mathfrak{h} \|w-z\|^{1+\alpha}} + \frac{2}{\mathfrak{r}_F}\|z-w\| \\ \Rightarrow \quad & \|\overline{z}-\overline{w}\| \leq \max\left\{\sqrt{\rho\mathfrak{h}}, \frac{1}{2\nu}\left(\sqrt{\rho\mathfrak{h}} + \frac{2}{\mathfrak{r}_F}\|z-w\|^{\frac{1-\alpha}{2}}\right)\right\} \|z-w\|^{\frac{1+\alpha}{2}}, \end{aligned}$$

where $\mathfrak{h} > 0$ is the Hölder constant of $v(\cdot)$ (i.e., the inequality $\|v(z)-v(w)\| \leq \mathfrak{h}\|z-w\|^{\alpha}$ holds for all $z, w \in x_0 + \delta\overline{\mathbf{B}}$). So that

$$\|\overline{z}-\overline{w}\| \leq \mathfrak{k} \|z-w\|^{\frac{1+\alpha}{2}}, \tag{3.26}$$

where

$$\mathfrak{k} := \delta^{\frac{1-\alpha}{2}} \max\left\{\sqrt{\|F^o\|\,\mathfrak{h}}, \frac{\sqrt{\|F^o\|\,\mathfrak{h}}}{2\nu} + \frac{1}{\mathfrak{r}_F \nu}\right\}. \tag{3.27}$$

In the general case instead (i.e., $\rho_F(\overline{z}-z) \neq \rho_F(\overline{w}-w)$) we take arbitrary points z, w in a smaller neighbourhood of x, namely, $z, w \in x + \overline{\delta}_1\mathbf{B}$ where $\overline{\delta}_1 := \frac{\overline{\delta}}{2\|F\|\|F^o\|+1}$ and proceed as in the Step 2 above substituing the point with the larger minimum time value (say w) by another w' whose projection $\pi_C^F(w')$ coincides with $\overline{w} = \pi_C^F(w)$ and $\rho_F(\overline{w}-w') = \rho_F(\overline{z}-z)$. In this way employing the Hölderian inequality already proved for the last case

$$\|\overline{z}-\overline{w}\| \leq \mathfrak{k} \|z-w'\|^{\frac{1+\alpha}{2}},$$

similarly as in (3.24) we obtain

$$\|\overline{z}-\overline{w}\| \leq \mathfrak{k} (1 + \|F^o\| \|F\|)^{\frac{1+\alpha}{2}} \|z-w\|^{\frac{1+\alpha}{2}}.$$

That concludes the proof. In particular, for $\alpha = 1$ we have that the Lipschitz continuity of $v(\cdot)$ implies the (local) Lipschitz continuity of $\pi_C^F(\cdot)$. ■

Remark 3.2.1 *Observe that for proving of the previous theorem we need only that*

$$\varkappa_F\left(\mathfrak{J}_F\left(\xi^*\right),\xi^*\right)\geq K$$

for some $K>0$ *and all* $\xi^*\in-\mathbf{N}_C^p\left(\overline{x}\right)\cap\mathbf{N}_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right)\cap\partial F^o$, $x\in\mathcal{U}\left(x_0\right)\backslash C$.

Now we consider another type of regularity hypotheses.

Theorem 3.2.3 *Let us fix* $x_0\in\partial C$, *and assume that the pair of sets* (F,C) *satisfies the condition* $(\mathbf{B})$ *at* x_0. *Moreover, suppose that one of the following conditions holds:*

(i) C *has smooth boundary at the point* x_0, *and the function* $\mathfrak{n}_C\left(\cdot\right)$ *which associates to each* $x\in C_\delta\left(x_0\right)$ *the respective (unique) normal vector* $\mathfrak{n}_C\left(x\right)\in\mathbf{N}_C^l\left(x\right)\cap\partial\overline{\mathbf{B}}$ *is Hölder continuous with an exponent* $0<\alpha\leq1$;

(ii) F *is uniformly smooth (see Section 1.3) at* $\xi:=\mathfrak{J}_F\left(\xi^*\right)$ *(w.r.t.* ξ^**) for each* $\xi^*\in U_{\delta,\delta'}\left(x_0\right)$, *where the set* $U_{\delta,\delta'}\left(x_0\right)$ *is given by (2.14), and the gradient* $\nabla\rho_F\left(\cdot\right)$ *is Hölder continuous near* ξ *with an exponent* $0<\alpha\leq1$.

Then $\pi_C^F\left(\cdot\right)$ *is (locally) Hölder continuous near* x_0 *with the exponent* $\frac{1}{2-\alpha}$. *In particular,* $\pi_C^F\left(\cdot\right)$ *is (locally) Lipschitzean whenever either* $\mathfrak{n}_C\left(\cdot\right)$ *or* $\nabla\rho_F\left(\cdot\right)$ *is Lipschitzean.*

Proof.

Let us define the neighbouhood $\mathcal{U}'\left(x_0\right)$ of the target set C as in Theorem 3.2.2 (see (3.16)) and fix $x\in\mathcal{U}\left(x_0\right)\backslash C$. As usual take $\tau>0$ and $0<\overline{\delta}<\frac{\tau}{\|F^o\|}$ such that $\mathfrak{T}_C^F\left(x\right)+\tau<\frac{K}{M}$, and $x+\overline{\delta}\mathbf{B}\subset\mathcal{U}\left(x_0\right)\backslash C$.

Let us prove first that the mapping $z\mapsto\pi_C^F\left(z\right)$ is Hölder continuous with the exponent α. To this end given $z,w\in x+\overline{\delta}\mathbf{B}$ we set $\overline{z}:=\pi_C^F\left(z\right)$, $\overline{w}=:\pi_C^F\left(w\right)$ and denote by n_z (respectively, n_w) the unique vector belonging to $-\mathbf{N}_C^p\left(\overline{z}\right)\cap\mathbf{N}_F\left(\frac{\overline{z}-z}{\rho_F(\overline{z}-z)}\right)\cap\partial F^o$ (respectively, to $-\mathbf{N}_C^p\left(\overline{w}\right)\cap\mathbf{N}_F\left(\frac{\overline{w}-w}{\rho_F(\overline{w}-w)}\right)\cap\partial F^o$).

We consider the cases (i) and (ii) separately.

Case (i): Since $\overline{z}\in C_\delta\left(x_0\right)$ as shown in the proof of Theorem 3.2.2 and

$$\mathbf{N}_C^p\left(\overline{z}\right)\cap\partial\overline{\mathbf{B}}=\mathbf{N}_C^l\left(\overline{z}\right)\cap\partial\overline{\mathbf{B}}=\left\{\mathfrak{n}_C\left(\overline{z}\right)\right\},$$

we have clearly

$$n_z:=-\frac{\mathfrak{n}_C\left(\overline{z}\right)}{\rho_{F^o}\left(-\mathfrak{n}_C\left(\overline{z}\right)\right)}$$

and, similarly,

$$n_w:=-\frac{\mathfrak{n}_C\left(\overline{w}\right)}{\rho_{F^o}\left(-\mathfrak{n}_C\left(\overline{w}\right)\right)}.$$

Now, taking into account that $\|\mathfrak{n}_C(\overline{z})\| = \|\mathfrak{n}_C(\overline{w})\| = 1$ (consequently, $\frac{1}{\|F^o\|} \leq \rho_{F^o}(-\mathfrak{n}_C(\overline{z})) \leq \|F\|$ and similarly for $\overline{w}$), we obtain

$$\begin{aligned}\|n_w - n_z\| &\leq \|F^o\|^2 \|\rho_{F^o}(-\mathfrak{n}_C(\overline{w}))\mathfrak{n}_C(\overline{z}) - \rho_{F^o}(-\mathfrak{n}_C(\overline{z}))\mathfrak{n}_C(\overline{w})\| \\ &\leq 2\|F^o\|^2\|F\|\|\mathfrak{n}_C(\overline{z}) - \mathfrak{n}_C(\overline{w})\| \\ &\leq 2\mathfrak{h}\|F\|\|F^o\|^2\|\overline{z} - \overline{w}\|^\alpha, \end{aligned} \tag{3.28}$$

where $\mathfrak{h}$ is the Hölder constant of $\mathfrak{n}_C(\cdot)$. In the case $\rho_F(\overline{z} - z) = \rho_F(\overline{w} - w)$ (denoted this common value by ρ) we may proceed as in the second part of the proof of the previous theorem and obtain (see (3.25)):

$$\begin{aligned}2\nu\|\overline{z} - \overline{w}\|^2 &\leq \rho\|n_w - n_z\|\|z - w\| + \frac{2}{\mathfrak{r}_F}\|z - w\|\|\overline{z} - \overline{w}\| \\ &\leq 2\rho\mathfrak{h}\|F\|\|F^o\|^2\|\overline{z} - \overline{w}\|^\alpha\|z - w\| + \frac{2}{\mathfrak{r}_F}\|z - w\|\|\overline{z} - \overline{w}\|, \end{aligned} \tag{3.29}$$

where

$$2\nu := \frac{1}{\|F\|}\left(K - \left(\mathfrak{T}_C^F(x) + \tau\right)M\right) > 0$$

(see (3.17)). Hence, setting for the simplicity $\mu := 2\rho\mathfrak{h}\|F\|\|F^o\|^2$, we have the following chain of implications:

$$\begin{aligned}&\|\overline{z} - \overline{w}\|^{1-\alpha}\left(2\nu\|\overline{z} - \overline{w}\| - \frac{2}{\mathfrak{r}_F}\|z - w\|\right) \leq \mu\|z - w\| \\ \Rightarrow\ & \|\overline{z} - \overline{w}\|^{1-\alpha} \leq (\mu\|z - w\|)^{1-\alpha} \quad \text{or} \quad 2\nu\|\overline{z} - \overline{w}\| \leq \frac{2}{\mathfrak{r}_F}\|z - w\| + (\mu\|z - w\|)^\alpha \\ \Rightarrow\ & \|\overline{z} - \overline{w}\| \leq \max\left\{\mu\|z - w\|^{1-\alpha}, \frac{1}{2\nu}\left(\mu^\alpha + \frac{2}{\mathfrak{r}_F}\|z - w\|^{1-\alpha}\right)\right\}\|z - w\|^\alpha. \end{aligned} \tag{3.30}$$

Since $\rho = \mathfrak{T}_C^F(z) \leq \delta\|F^o\|$ (so that $\mu \leq 2\delta\mathfrak{h}\|F\|\|F^o\|^3$), the inequality in (3.30) can be written as

$$\|\overline{z} - \overline{w}\| \leq \overline{\mathfrak{h}}\|z - w\|^\alpha,$$

where

$$\overline{\mathfrak{h}} = \delta^{\min\{\alpha, 1-\alpha\}}\mathfrak{h}' \tag{3.31}$$

and the constant $\mathfrak{h}' > 0$ depends on the conditions of the problem and on the choice of the point $x \in \mathcal{U}(x_0)$.

In the general case (if $\rho_F(\overline{z} - z) \neq \rho_F(\overline{w} - w)$) we take arbitrary points z, w in the smaller neighbourhood $x + \overline{\delta}_1\mathbf{B}$ where $\overline{\delta}_1 := \frac{\overline{\delta}}{2\|F\|\|F^o\|+1}$ and proceed as in the Step 2 of Theorem 3.2.2. In this way using the inequality (3.24) we obtain

$$\|\overline{z} - \overline{w}\| \leq \overline{\mathfrak{h}}(1 + \|F^o\|\|F\|)^\alpha\|z - w\|^\alpha.$$

Case (ii): Since $\overline{z} \in C_\delta(x_0)$ and $n_z \in -\mathbf{N}_C^p(\overline{z}) \cap \partial F^o$ we have $n_z \in U_{\delta,\delta'}(x_0)$ and by assumption F is uniformly smooth at $\xi := \mathfrak{J}_F(n_z) = \frac{\overline{z} - z}{\rho_F(\overline{z} - z)}$. By Remark 1.3.1 we obtain

$$\partial F^o \cap \mathbf{N}_F(\xi) = \{\nabla\rho_F(\xi)\},$$

so we can represent n_z and n_w in terms of the gradient of $\rho_F(\cdot)$. Namely,

$$n_z := \nabla\rho_F\left(\frac{\overline{z}-z}{\rho_F(\overline{z}-z)}\right) \quad \text{and} \quad n_w := \nabla\rho_F\left(\frac{\overline{w}-w}{\rho_F(\overline{w}-w)}\right).$$

Then we have

$$\begin{aligned} \|n_z - n_w\| &= \left\|\nabla\rho_F\left(\frac{\overline{z}-z}{\rho_F(\overline{z}-z)}\right) - \nabla\rho_F\left(\frac{\overline{w}-w}{\rho_F(\overline{w}-w)}\right)\right\| \\ &\leq \mathfrak{h}\left\|\frac{\overline{z}-z}{\rho_F(\overline{z}-z)} - \frac{\overline{w}-w}{\rho_F(\overline{w}-w)}\right\|^{\alpha}, \end{aligned} \tag{3.32}$$

where $\mathfrak{h}$ is the Hölder constant of $\nabla\rho_F(\cdot)$. Now assuming that $\rho_F(\overline{z}-z) = \rho_F(\overline{w}-w)\ (= \rho)$ by the same arguments as above (see (3.25) and (3.29)) we find

$$\begin{aligned} & 2\nu\|\overline{z}-\overline{w}\|^2 \\ \leq\ & \rho\|n_z - n_w\|\,\|z-w\| + \frac{2}{\mathfrak{r}_F}\|z-w\|\,\|\overline{z}-\overline{w}\| \\ \leq\ & \rho\mathfrak{h}\left\|\frac{\overline{z}-z}{\rho} - \frac{\overline{w}-w}{\rho}\right\|^{\alpha}\|z-w\| + \frac{2}{\mathfrak{r}_F}\|z-w\|\,\|\overline{z}-\overline{w}\| \\ \leq\ & \rho^{1-\alpha}\mathfrak{h}\left(\|\overline{z}-\overline{w}\|^{\alpha} + \|z-w\|^{\alpha}\right)\|z-w\| + \frac{2}{\mathfrak{r}_F}\|z-w\|\,\|\overline{z}-\overline{w}\|. \end{aligned} \tag{3.33}$$

Here we use the numerical inequality

$$(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha} \tag{3.34}$$

which holds for every $a, b > 0$ and $0 \leq \alpha \leq 1$. In fact, for arbitrary $0 \leq \alpha \leq 1$ and $t > 0$ we have $(t+1)^{1-\alpha} \geq t^{1-\alpha}$. This implies $(t+1)^{\alpha-1} \leq t^{\alpha-1}$ and consequently

$$\int_0^t (\tau+1)^{\alpha-1}\,d\tau \leq \int_0^t \tau^{\alpha-1}d\tau,$$

which is equivalent to

$$\frac{(t+1)^{\alpha}}{\alpha} - \frac{1}{\alpha} \leq \frac{t^{\alpha}}{\alpha} \Leftrightarrow (t+1)^{\alpha} \leq t^{\alpha} + 1.$$

Now for given $a, b > 0$ setting $t = \frac{a}{b}$ we obtain (3.34).

It follows from (3.33) that

$$\begin{aligned} & \|\overline{z}-\overline{w}\|^{\alpha}\left(2\nu\|\overline{z}-\overline{w}\|^{2-\alpha} - \mu\|z-w\| - \frac{2}{\mathfrak{r}_F}\|z-w\|\,\|\overline{z}-\overline{w}\|^{1-\alpha}\right) \leq \mu\|z-w\|^{\alpha+1} \\ \Rightarrow\ & \|\overline{z}-\overline{w}\|^{\alpha} \leq \mu^{\alpha^2}\|z-w\|^{\alpha} \\ & \text{or}\ \ 2\nu\|\overline{z}-\overline{w}\|^{2-\alpha} - \mu\|z-w\| - \frac{2}{\mathfrak{r}_F}\|z-w\|\,\|\overline{z}-\overline{w}\|^{1-\alpha} \leq \mu^{1-\alpha^2}\|z-w\| \\ \Rightarrow\ & \|\overline{z}-\overline{w}\| \leq \mu^{\alpha}\|z-w\| \\ & \text{or}\ \ \|\overline{z}-\overline{w}\|^{1-\alpha}\left(2\nu\|\overline{z}-\overline{w}\| - \frac{2}{\mathfrak{r}_F}\|z-w\|\right) \leq \mu^{\alpha(1-\alpha)}\left(\mu^{1-\alpha} + \mu^{1-\alpha+\alpha^2}\right)\|z-w\|, \end{aligned} \tag{3.35}$$

where $\mu := \rho^{1-\alpha}\mathfrak{h}$. On the other hand,

$$\begin{aligned}
& \|\overline{z}-\overline{w}\|^{1-\alpha}\left(2\nu\|\overline{z}-\overline{w}\| - \frac{2}{\mathfrak{r}_F}\|z-w\|\right) \le \mu^{\alpha(1-\alpha)}\left(\mu^{1-\alpha}+\mu^{1-\alpha+\alpha^2}\right)\|z-w\| \\
\Rightarrow\quad & \|\overline{z}-\overline{w}\|^{1-\alpha} \le \mu^{\alpha(1-\alpha)}\|z-w\|^{1-\alpha} \\
\text{or}\quad & 2\nu\|\overline{z}-\overline{w}\| - \frac{2}{\mathfrak{r}_F}\|z-w\| \le \left(\mu^{1-\alpha}+\mu^{1-\alpha+\alpha^2}\right)\|z-w\|^{\alpha} \\
\Rightarrow\quad & \|\overline{z}-\overline{w}\| \le \mu^{\alpha}\|z-w\| \\
\text{or}\quad & \|\overline{z}-\overline{w}\| \le \left(\frac{\mu^{1-\alpha}\left(1+\mu^{\alpha^2}\right)}{2\nu} + \frac{1}{\nu\mathfrak{r}_F}\|z-w\|^{1-\alpha}\right)\|z-w\|^{\alpha}. \qquad (3.36)
\end{aligned}$$

Joining together the inequalities (3.35) and (3.36) we have

$$\|\overline{z}-\overline{w}\| \le \max\left\{\mu^{\alpha}\|z-w\|^{1-\alpha}, \frac{\mu^{1-\alpha}\left(1+\mu^{\alpha^2}\right)}{2\nu} + \frac{1}{\nu\mathfrak{r}_F}\|z-w\|^{1-\alpha}\right\}\|z-w\|^{\alpha}. \qquad (3.37)$$

So that we are led to the Hölder inequality

$$\|\overline{z}-\overline{w}\| \le \overline{\mathfrak{h}}\,\|z-w\|^{\alpha},$$

where the constant $\overline{\mathfrak{h}}$ for small $\delta > 0$ can be represented as

$$\overline{\mathfrak{h}} = \delta^{(1-\alpha)^2}\mathfrak{h}' \qquad (3.38)$$

and $\mathfrak{h}' > 0$ again depends only on the data of the problem and on $x \in \mathcal{U}(x_0)$ (compare with (3.31)).

In the general case (when $\mathfrak{T}_C^F(z)$ and $\mathfrak{T}_C^F(w)$ are different) as usual we conclude that

$$\|\overline{z}-\overline{w}\| \le \overline{\mathfrak{h}}\,(1+\|F^o\|\,\|F\|)^{\alpha}\|z-w\|^{\alpha}.$$

Let us show now that the Hölderianity above can be essentially improved. Observe that in both cases ((i) and (ii)) we have

$$-\mathbf{N}_C^p(\overline{x}) \cap \mathbf{N}_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right) \cap \partial F^o = \{v(x)\},$$

where $v(x)$ is either

$$-\frac{\mathfrak{n}_C(\overline{x})}{\rho_{F^o}(-\mathfrak{n}_C(\overline{x}))} \quad \text{or} \quad \nabla\rho_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right),$$

respectively, $\overline{x} := \pi_C^F(x)$. Then by the first part of the proof the mapping $v(\cdot)$ is Hölder continuous on $\mathcal{U}(x_0)$ with exponent α^2 as the composition of two Hölderian functions (either $\mathfrak{n}_C(\cdot)$ or $\nabla\rho_F(\cdot)$ and $\pi_C^F(\cdot)$) both with exponent α. By Theorem 3.2.2 $\pi_C^F(\cdot)$ is Hölder continuous with exponent $\frac{1+\alpha^2}{2}$. Applying again the first part we conclude that the Hölder exponent of $v(\cdot)$

is in fact $\frac{(1+\alpha^2)\alpha}{2}$. Continuing these iterations let us suppose that the time-minimum projection $\pi_C^F(\cdot)$ is Hölder continuous with an exponent $\beta_n > 0$. Therefore $v(\cdot)$ is Hölderian with the exponent $\alpha\beta_n$, Theorem 3.2.2 gives Hölderianity of $\pi_C^F(\cdot)$ with the exponent $\beta_{n+1} := \frac{\alpha\beta_n+1}{2}$. Setting naturally $\beta_1 := \alpha$, by induction we show that the sequence $\{\beta_n\}$ is upper bounded (namely, $\beta_n \le \frac{1}{2-\alpha}$) and increases. Consequently, $\{\beta_n\}$ converges to the limit (equal to $\frac{1}{2-\alpha}$ as well), which is also the Hölder exponent of $\pi_C^F(\cdot)$ on the neighbourhood $\mathcal{U}(x_0)$. The theorem is completely proved. ■

Remark 3.2.2 *Notice that in accordance with the representations (3.23), (3.27), (3.31) and (3.38) the Hölder constant of the time-minimum projection $\pi_C^F(\cdot)$ is proportional to some degree of $\delta > 0$ showing how distant the points are from the target set C. On the other hand, this constant depends essentially on the distance from the boundary of the neighbourhood $\mathcal{U}(x_0)$ controlled with the parameters τ and ν, tending to $+\infty$ whenever the strict inequality $\mathfrak{T}_C^F(z) < \frac{K}{M}$ tends to become an equality (see (3.23), (3.27), (3.30) and (3.37)).*

3.3 Smoothness of the value function

In this section we study the differential properties of the minimal time function through the regular projection $\pi_C^F(\cdot)$. One of the versions of such connections is given by the following theorem.

Theorem 3.3.1 *Let us fix $x \notin C$ and assume that the mapping $z \mapsto \pi_C^F(z)$ is well-defined and single-valued in some neighbourhood $U(x)$ of x where also the following "one-point" Hölder property holds: there are constants $K = K(x) > 0$ and $\frac{1}{2} < \beta \le 1$ such that*

$$\left\|\pi_C^F(y) - \pi_C^F(x)\right\| \le K \|y - x\|^\beta \quad \text{for all } y \in U(x). \tag{3.39}$$

If the target set C is proximally regular at $\overline{x} := \pi_C^F(x)$ then the function $\mathfrak{T}_C^F(\cdot)$ is Clarke (and lower) regular at x.

Proof.

Let us prove the following equalities

$$\partial^f \mathfrak{T}_C^F(x) = \partial^c \mathfrak{T}_C^F(x) = -\partial\rho_F(\overline{x} - x) \cap \mathbf{N}_C^p(\overline{x}). \tag{3.40}$$

First of all, since the (unique) projection $\overline{x} := \pi_C^F(x)$ exists, we have (see (3.9))

$$\partial^f \mathfrak{T}_C^F(x) \subset -\partial\rho_F(\overline{x} - x) \cap \mathbf{N}_C^f(\overline{x})$$

and using the proximally regularity of C at $\overline{x}$ (Definition 3.1.2) we conclude that

$$\partial^f \mathfrak{T}_C^F(x) \subset -\partial\rho_F(\overline{x} - x) \cap \mathbf{N}_C^p(\overline{x}). \tag{3.41}$$

Let us prove now the opposite inclusion in (3.41). To this end we fix ξ belonging to the right-hand side. Then by definition of proximal normals there exists $\sigma = \sigma(\overline{x}, \xi)$ such that

$$\langle \xi, z - \overline{x} \rangle \leq \sigma \|z - \overline{x}\|^2 \quad \forall z \in C. \tag{3.42}$$

On the other hand, by the definition of subdifferential of a convex function

$$\rho_F(u) \geq \rho_F(\overline{x} - x) + \langle -\xi, u - \overline{x} + x \rangle \quad \forall u \in H. \tag{3.43}$$

In particular, it follows from (3.43) that

$$\rho_F\left(\pi_C^F(z) - z\right) - \rho_F\left(\pi_C^F(x) - x\right) - \langle \xi, z - x \rangle \geq \left\langle -\xi, \pi_C^F(z) - \pi_C^F(x) \right\rangle \tag{3.44}$$

for all z enough close to x. Combining now (3.44) and (3.42) we obtain

$$\begin{aligned}
& \liminf_{z \to x} \frac{\mathfrak{T}_C^F(z) - \mathfrak{T}_C^F(x) - \langle \xi, z - x \rangle}{\|z - x\|} \\
= \ & \liminf_{z \to x} \frac{\rho_F\left(\pi_C^F(z) - z\right) - \rho_F\left(\pi_C^F(x) - x\right) - \langle \xi, z - x \rangle}{\|z - x\|} \\
\geq \ & \liminf_{z \to x} \frac{\left\langle -\xi, \pi_C^F(z) - \pi_C^F(x) \right\rangle}{\|z - x\|} \\
\geq \ & \liminf_{z \to x} \frac{-\sigma \left\|\pi_C^F(z) - \pi_C^F(x)\right\|^2}{\|z - x\|} \\
\geq \ & \liminf_{z \to x} -\sigma K \|z - x\|^{2\beta - 1} = 0,
\end{aligned}$$

where K is the constant from (3.39). Thus $\xi \in \partial^f \mathfrak{T}_C^F(x)$, and we have proved that

$$-\partial \rho_F(\overline{x} - x) \cap \mathbf{N}_C^p(\overline{x}) \subset \partial^f \mathfrak{T}_C^F(x). \tag{3.45}$$

Furthermore, the inclusion $\partial^f \mathfrak{T}_C^F(x) \subset \partial^c \mathfrak{T}_C^F(x)$ obviously holds (see (2.4)). In order to show the opposite inclusion we represent the subdifferential $\partial^c \mathfrak{T}_C^F(x)$ through the proximal subdifferentials at close points (see (2.3)). Then by using the inclusion (3.8), the definition of limiting normal cone in Hilbert spaces (see (2.5)) and the strong×weak closedness of the graph of the subdifferential $\partial \rho_F(\cdot)$ we successively obtain

$$\begin{aligned}
\partial^c \mathfrak{T}_C^F(x) & = \overline{\text{co}} \left\{ w - \lim_{i \to \infty} \xi_i, \ \xi_i \in \partial^p \mathfrak{T}_C^F(x_i), \ x_i \to x \right\} \\
& \subset \overline{\text{co}} \left\{ w - \lim_{i \to \infty} \xi_i, \ \xi_i \in -\partial \rho_F\left(\pi_C^F(x_i) - x_i\right) \cap \mathbf{N}_C^p\left(\pi_C^F(x_i)\right), \ x_i \to x \right\} \\
& \subset \overline{\text{co}} \left(-\partial \rho_F\left(\pi_C^F(x) - x\right) \cap \mathbf{N}_C^l\left(\pi_C^F(x)\right) \right).
\end{aligned}$$

Since $\mathbf{N}_C^l(\overline{x}) = \mathbf{N}_C^p(\overline{x}) = \mathbf{N}_C^f(\overline{x})$, from the closedness and convexity of the latter cone and from (3.45) we conclude that

$$\partial^c \mathfrak{T}_C^F(x) \subset -\partial \rho_F(\overline{x} - x) \cap \mathbf{N}_C^p(\overline{x}) \subset \partial^f \mathfrak{T}_C^F(x),$$

(3.40) follows. Since the inclusions $\partial^f \mathfrak{T}_C^F(x) \subset \partial^l \mathfrak{T}_C^F(x) \subset \partial^c \mathfrak{T}_C^F(x)$ are always true (see (2.4)) we have, in fact, the equalities

$$\partial^f \mathfrak{T}_C^F(x) = \partial^l \mathfrak{T}_C^F(x) = \partial^c \mathfrak{T}_C^F(x)$$

and the Clarke and the lower regularity of the function $\mathfrak{T}_C^F(\cdot)$ at x follows. ■

If, in the place of "one-point" Hölder condition we have the usual Hölder continuity of $\pi_C^F(\cdot)$ at x, i.e., if there exist constants $K = K(x) > 0$, $\frac{1}{2} < \beta \leq 1$ and a neighbourhood $U(x)$ of x such that

$$\left\|\pi_C^F(y) - \pi_C^F(z)\right\| \leq K \|y - z\|^\beta \tag{3.46}$$

for all $y, z \in U(x)$ then we obtain the following smoothness result.

Theorem 3.3.2 *Let us fix $x \notin C$ and suppose that the mapping $z \mapsto \pi_C^F(z)$ is well-defined, single-valued and Hölder continuous with an exponent $\frac{1}{2} < \beta \leq 1$ in some neighbourhood $U(x)$ of x. Assume, moreover, the target set C to be proximally regular at each point $y \in C_\delta(\overline{x}) := \partial C \cap (\overline{x} + \delta \mathbf{B})$ for some $\delta > 0$ where $\overline{x} := \pi_C^F(x)$. Then the function $\mathfrak{T}_C^F(\cdot)$ is (Fréchet) continuously differentiable at x if there exists a continuous mapping $\nu : U(x) \to \partial F^o$ such that*

$$-\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y} - y}{\rho_F(\overline{y} - y)}\right) \cap \partial F^o = \{\nu(y)\} \tag{3.47}$$

where $\overline{y} := \pi_C^F(y)$, $y \in U(x) \backslash C$. In this case $\nabla \mathfrak{T}_C^F(x) = -\nu(x)$.

Proof.
By assumption for each $y \in U(x)$ the intersection

$$-\mathbf{N}_C^p(\overline{y}) \cap \partial \rho_F(\overline{y} - y) = -\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y} - y}{\rho_F(\overline{y} - y)}\right) \cap \partial F^o$$

is reduced to the single vector $\nu(y)$ (see (3.47) and Proposition 1.1.1).

On the other hand, since the (unique) projection $\overline{y} = \pi_C^F(y)$ exists for each y enough close to x and it is close to $\overline{x}$, the set C is proximally regular at such $\overline{y}$ furthermore, the Hölder continuity of $\pi_C^F(\cdot)$ on $U(x)$ implies the "one-point" condition (3.39) centred at each y near x. So that by the previous theorem we obtain

$$\partial^f \mathfrak{T}_C^F(y) = -\partial \rho_F(\overline{y} - y) \cap \mathbf{N}_C^p(\overline{y}).$$

Hence

$$\partial^f \mathfrak{T}_C^F(y) = -\partial \rho_F(\overline{y} - y) \cap \mathbf{N}_C^p(\overline{y}) = \{-\nu(y)\}$$

and the statement of the theorem follows. ■

Remark 3.3.1 *Observe that in a finite dimensional space we could require only that the mapping $\nu(\cdot)$ in the Theorem 3.3.2 is single-valued because continuity follows immediately from the fact that $y \mapsto -\mathbf{N}_C^l(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o$ admits closed graph (see Section 2.1).*

For convenience of further references let us formulate the following obvious statement.

Corollary 3.3.3 *Under all the assumptions and notations of Theorem 3.3.2 suppose, moreover, that the single-valued mapping*

$$y \mapsto -\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o$$

is Hölder continuous with an exponent $0 < \alpha \leq 1$ *in a neighbourhood of* $x \notin C$. *Then the function* $\mathfrak{T}_C^F(\cdot)$ *is of class* $C^{1,\alpha}$ *in this neighbourhood.*

The condition (3.47) splits into the following two particular cases.

Corollary 3.3.4 *Assume as in Theorem 3.3.2 that the time-minimum projection* $\pi_C^F(\cdot)$ *is single-valued and Hölder continuous with an exponent* $1/2 < \beta \leq 1$ *in a neighbourhood* $U(x)$ *of a point* $x \notin C$, *and that the set* C *is proximally regular at each point* $y \in C_\delta(\overline{x})$, $\overline{x} := \pi_C^F(x)$, $\delta > 0$. *Then the value function* $\mathfrak{T}_C^F(\cdot)$ *is (Fréchet) continuously differentiable at* x *if at least one of the following conditions hold:*

(i) C *has smooth boundary at* $\overline{x}$;

(ii) F *is uniformly smooth at* $\xi := \frac{\overline{x}-x}{\rho_F(\overline{x}-x)}$.

Furthermore, in the first case we have

$$\nabla\mathfrak{T}_C^F(x) = \frac{\mathfrak{n}_C(\overline{x})}{\rho_{F^o}(-\mathfrak{n}_C(\overline{x}))}, \tag{3.48}$$

where $\mathfrak{n}_C(\overline{x})$ *is, as usually, the unit normal vector to* ∂C *at* $\overline{x}$, *while in the second*

$$\nabla\mathfrak{T}_C^F(x) = -\nabla\rho_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right). \tag{3.49}$$

Proof.

In order to prove this statement it is enough to observe that

$$-\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o \tag{3.50}$$

is contained either in

$$\left\{-\frac{\mathfrak{n}_C(\overline{y})}{\rho_{F^o}(-\mathfrak{n}_C(\overline{y}))}\right\} \quad \text{or in} \quad \left\{\nabla\rho_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right)\right\}$$

whenever the condition **(i)** or the condition **(ii)**, respectively, is fulfilled. Since the set (3.50) is nonempty, everything is proved. ■

Similarly to Corollary 3.3.3 we can formulate the following result on the Hölder regularity of the minimal time function.

Corollary 3.3.5 *Under all the assumptions and notations of Corollary 3.3.4 the following implications hold*

(i) if $\mathfrak{n}_C(\cdot)$ *is Hölderian near* $\overline{x} := \pi_C^F(x)$ *with an exponent* $0 < \alpha \leq 1$ *then* $\mathfrak{T}_C^F(\cdot)$ *is of class* $C^{1,\alpha\beta}$ *near the point* $x \notin C$*;*

(ii) if $\nabla\rho_F(\cdot)$ *is Hölderian near* $\xi := \frac{\overline{x}-x}{\rho_F(\overline{x}-x)}$ *with an exponent* $0 < \alpha \leq 1$ *then* $\mathfrak{T}_C^F(\cdot)$ *is of class* $C^{1,\alpha\beta}$ *near* x*.*

Proof.

(i) Take $\overline{\delta} > 0$ small enough such that all the conditions of Corollary 3.3.4 are fulfilled in the neighbourhood $x + \overline{\delta}\mathbf{B}$, and let us fix $x_1, x_2 \in x + \overline{\delta}\mathbf{B}\backslash C$. Using the formula (3.48) and proceeding, e.g., as in (3.28) we obtain

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \leq 2\mathfrak{h}_C \|F\| \|F^o\|^2 \|\overline{x}_1 - \overline{x}_2\|^\alpha$$

where $\mathfrak{h}_C > 0$ is the Hölder constant of $\mathfrak{n}_C(\cdot)$ near $\overline{x}$. Since $\overline{x}_i := \pi_C^F(x_i)$, $i = 1, 2$, we continue this estimate:

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \leq 2\mathfrak{h}_C \|F\| \|F^o\|^2 K^\alpha \|x_1 - x_2\|^{\alpha\beta},$$

where $K > 0$ is the Hölder constant of $\pi_C^F(\cdot)$ near x.

(ii) Taking $0 < \overline{\delta} \leq \frac{\mathfrak{T}_C^F(x)}{2\|F^o\|}$ again small enough such that all the conditions of Corollary 3.3.4 are fulfilled in $x + \overline{\delta}\mathbf{B}$, and using the formula (3.49) similarly as above for each $x_1, x_2 \in x + \overline{\delta}\mathbf{B}$ we have the inequality

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \leq \mathfrak{h}_F \|\xi_1 - \xi_2\|^\alpha, \tag{3.51}$$

where $\mathfrak{h}_F > 0$ is the Hölder constant of $\nabla\rho_F(\cdot)$ near ξ, and $\xi_i := \frac{\overline{x}_i - x_i}{\rho_F(\overline{x}_i - x_i)}$, $i = 1, 2$. Setting $\rho_i := \rho_F(\overline{x}_i - x_i)$, $i = 1, 2$, we have

$$\begin{aligned}
\|\xi_1 - \xi_2\| &= \frac{1}{\rho_1\rho_2} \|(\overline{x}_1 - x_1)\rho_2 - (\overline{x}_2 - x_2)\rho_1\| \\
&\leq \frac{1}{\rho_1\rho_2} \left(\|\overline{x}_1 - x_1\| |\rho_2 - \rho_1| + \rho_1 \|\overline{x}_1 - x_1 - (\overline{x}_2 - x_2)\|\right) \\
&\leq \frac{1}{\rho_1\rho_2} \left(\|F\| \|F^o\| \rho_1 \|\overline{x}_1 - x_1 - (\overline{x}_2 - x_2)\| + \rho_1 \|\overline{x}_1 - x_1 - (\overline{x}_2 - x_2)\|\right) \\
&\leq \frac{\|F\| \|F^o\| + 1}{\rho_2} \left(K \|x_2 - x_1\|^\beta + \|x_1 - x_2\|\right) \\
&= \frac{\|F\| \|F^o\| + 1}{\mathfrak{T}_C^F(x_2)} \left(K + \|x_2 - x_1\|^{1-\beta}\right) \|x_1 - x_2\|^\beta.
\end{aligned} \tag{3.52}$$

Since

$$\mathfrak{T}_C^F(x_2) \geq \mathfrak{T}_C^F(x) - \rho_F(x_2 - x) \geq \mathfrak{T}_C^F(x) - \|F^o\| \|x_2 - x\| \geq \mathfrak{T}_C^F(x) - \|F^o\| \overline{\delta} \geq \frac{\mathfrak{T}_C^F(x)}{2}$$

from (3.51) and (3.52) we obtain

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \leq \overline{\mathfrak{h}}\, \|x_1 - x_2\|^{\alpha\beta},$$

for some constant $\overline{\mathfrak{h}} > 0$. ■

Notice that the previous statements were proved under assumptions that the time-minimum projection $\pi_C^F(\cdot)$ is enough regular (Hölder continuous with an exponent $\beta > 1/2$) in a neighbourhood of a fixed point outside of the target set. Applying now the results of Section 3.2 we can emphasize the hypotheses guaranteeing such regularity. These hypotheses employ the first and second order conditions studied in the previous chapter and allow us to formulate global results.

Theorem 3.3.6 *Let $x_0 \in \partial C$. Suppose that the following assumptions hold:*

(i) the pair of sets (F, C) satisfies the condition $(\mathbf{A})$ *at x_0;*

(ii) the target set C is proximally regular near x_0;

(iii) for each $x \notin C$ enough close to x_0 one of the properties below takes place:

- *(a) C has smooth boundary at $\overline{x} := \pi_C^F(x)$, and the unit normal $\mathfrak{n}_C(\cdot)$ is Hölder continuous near $\overline{x}$ with an exponent $0 < \alpha \leq 1$;*
- *(b) F is uniformly smooth at $\xi := \frac{\overline{x}-x}{\rho_F(\overline{x}-x)}$, and $\nabla\rho_F(\cdot)$ is Hölder continuous near ξ with an exponent $0 < \alpha \leq 1$;*
- *(c) the mapping $v(y) := -\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o$ is single-valued and Hölder continuous near x with an exponent $0 < \alpha \leq 1$.*

Then the value function $\mathfrak{T}_C^F(\cdot)$ is of class $C_{loc}^{1,\alpha}$ in a neighbourhood of the point x_0.

Here and in what follows by the differentiability and the Hölder continuity of the gradient $\nabla\mathfrak{T}_C^F(\cdot)$ we mean that these properties hold true only outside the target set.

Proof.

It follows from theorems 2.2.1 and 3.2.1 that the projection $\pi_C^F(\cdot)$ is well-defined, single-valued and Lipschitz continuous (Hölderian with $\beta = 1$) in some neighbourhood $\mathcal{U}'(x_0)$ of x_0. Consequently, all the assumptions either of Theorem 3.3.2 or of Corollary 3.3.4 are fulfilled. Then applying, respectively, Corollary 3.3.3 or Corollary 3.3.5, we complete the proof. ■

Theorem 3.3.7 *Let $x_0 \in \partial C$. Suppose that the following assumptions hold:*

(i) the pair of sets (F, C) satisfies the condition $(\mathbf{B})$ *at x_0;*

(ii) for each $x \notin C$ enough close to x_0 one of the properties below takes place:

(a) C has smooth boundary at $\overline{x} := \pi_C^F(x)$, and the unit normal $\mathfrak{n}_C(\cdot)$ is Hölder continuous near $\overline{x}$ with an exponent $0 < \alpha \leq 1$;

(b) F is uniformly smooth at $\xi := \frac{\overline{x}-x}{\rho_F(\overline{x}-x)}$, and $\nabla\rho_F(\cdot)$ is Hölder continuous near ξ with an exponent $0 < \alpha \leq 1$;

*then the value function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\frac{\alpha}{2-\alpha}}$ in a neighbourhood of the point x_0. However, if in the place of the condition **(ii)** we assume that*

(ii') the single-valued mapping $y \mapsto v(y) := -\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o$ is Hölder continuous with an exponent $0 < \alpha \leq 1$ near x, for each $x \notin C$ close to x_0,

then $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\alpha}$ near x_0.

Proof.
Unlike the previous statement here we apply either Theorem 3.2.3 or Theorem 3.2.2 to be able to use the results of this section. Then it is enough to apply either Corollary 3.3.5 with $\beta = \frac{1}{2-\alpha}$ or Corollary 3.3.3. ■

Observe that although the assumption **(iii')** of the last theorem is really more general than the alternative hypotheses **(iii)**, we obtain a stronger regularity for the minimal time function ($\alpha > \frac{\alpha}{2-\alpha}$). This happens because the normal vector $v(\cdot)$ in **(iii')** depends on a moving point $x \notin C$ itself, while in **(iii)** it is a composed function involving the projection.

The following statement is the global version of the theorems 3.3.6 and 3.3.7. Notice that in order to formulate the regularity result in global setting we should reduce all the Hölder exponents of the gradient $\nabla\mathfrak{T}_C^F(\cdot)$ obtained earlier to the common one. So we have

Theorem 3.3.8 *Let us suppose that the pair of sets (F, C) satisfies either the condition* (**A**) *or* (**B**) *at each point $x_0 \in \partial C$, and that the target set C is proximally regular (in particular, φ-convex). Moreover, we assume that given $0 < \beta \leq 1$ in a neighbourhood of each $x_0 \in \partial C$ either the mapping $y \mapsto v(y) := -\mathbf{N}_C^p(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o$, $\overline{y} := \pi_C^F(y)$, is single-valued and Hölder continuous with the exponent β, or the hypothesis **(ii)** of Theorem 3.3.7 is verified, where $\alpha = \beta$ if at x_0 the condition* (**A**) *holds, and $\alpha = \frac{2\beta}{\beta+1}$ under the condition* (**B**). *Then there exists an open set $\mathfrak{U}(C) \supset C$ such that the minimal time function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\beta}$ on $\mathfrak{U}(C)\setminus C$.*

Proof.
This is immediate consequence of two previous theorems taking into account that the equality $\alpha = \frac{2\beta}{\beta+1}$ is equivalent to $\beta = \frac{\alpha}{2-\alpha}$. ■

Since

$$\alpha := \frac{2\beta}{\beta+1} \geq \beta,$$

we formulate another version of the global result, which does note make differences between Hölder exponents in the case of the hypothesis (**A**) or (**B**).

Theorem 3.3.9 *Let us suppose that the pair of sets* (F, C) *satisfies either the condition* (**A**) *or* (**B**) *at each point* $x_0 \in \partial C$*, and that the target set* C *is proximally regular. Assume, in addition, that either* C *has smooth boundary and the mapping* $x \mapsto \mathfrak{n}_C(x)$ *is locally Hölderian near* C *with the exponent* α*, or* F *is uniformly smooth (at each point) and* $\nabla\rho_F(\cdot)$ *is locally Hölderian near* F *with the same exponent* α*. Then the minimal time function* $\mathfrak{T}_C^F(\cdot)$ *is of class* $C_{loc}^{1,\beta}$ *on* $\mathfrak{U}(C)\setminus C$ *where* $\mathfrak{U}(C)$ *is an open neighbourhood of* C*.*

3.4 Examples

Here we illustrate the regularity results obtained in the previous sections by simple examples which have already been treated in this work but for other purposes. For the sake of clarity we consider only the finite dimensional case.

Example 3.4.1 *Let*

$$F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \le 1 - \xi_1^4,\ -1 \le \xi_1 \le 1\right\}$$

$$C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le \min\left(x_2^2, \left(x_2 - \frac{1}{2}\right)^2 + \frac{1}{2}, \left(x_2 + \frac{1}{2}\right)^2 + \frac{1}{2}\right)\right\}.$$

Notice that the same dynamics F was used also in the examples 1.4.2 and 2.4.3, while the target set C is similar to that considered in Example 2.4.4, but simpler because it does not contain an "inward corner" point. Then F is closed convex bounded with $0 \in \operatorname{int} F$, and C is closed, φ-convex with $\varphi(\cdot)$ given by

$$\varphi(x) = \max\left\{\frac{1}{\sqrt{1 + 4x_2^2}}, \frac{1}{\sqrt{1 + (2x_2 - 1)^2}}\right\}, \quad x = (x_1, x_2),$$

(see (2.73)). Furthermore, C has smooth boundary at each point except $b^{\pm} := \left(\frac{9}{16}, \pm\frac{3}{4}\right)$ with the unit normal vector

$$\mathfrak{n}_C(x) = \begin{cases} \frac{1}{\sqrt{1+4x_2^2}}(1, -2x_2) & \text{if } |x_2| < \frac{3}{4} \\ \frac{1}{\sqrt{1+4\left(x_2-\frac{1}{2}\right)^2}}\left(1, -2\left(x_2 - \frac{1}{2}\right)\right) & \text{if } x_2 > \frac{3}{4} \\ \frac{1}{\sqrt{1+4\left(x_2+\frac{1}{2}\right)^2}}\left(1, -2\left(x_2 + \frac{1}{2}\right)\right) & \text{if } x_2 < -\frac{3}{4}, \end{cases} \tag{3.53}$$

and the vector $\mathfrak{v}(x) := -\mathfrak{n}_C(x)/\rho_{F^o}(-\mathfrak{n}_C(x))$ is well-defined for $x \in \partial C$, $x \ne b^{\pm}$.

Proceeding as in Example 2.4.3, we conclude that the condition (**B**) holds for each $x_0 = (x_1^0, x_2^0) \in \partial C\setminus\{b^{\pm}\}$. Moreover, F is smooth at each point $\eta = (\eta_1, \eta_2) \in \partial F\setminus\{(\pm 1, 0)\}$ with the unit normal

$$\mathfrak{n}_F(\eta) := \frac{\nabla\rho_F(\eta)}{\|\nabla\rho_F(\eta)\|} = \frac{1}{\sqrt{16\eta_1^6 + 1}}\left(4\eta_1^3, \operatorname{sgn}(\eta_2)\right),$$

being the function $\eta \mapsto \mathfrak{n}_F(\eta)$ locally Lipschitzean.

On the other hand, it is easy to see that for each x_0, $x_0 \neq b^{\pm}$, with $|x_2^0| > 1/8$ and for some $\delta > 0$ small enough there is no $x \in C_\delta(x_0)$ with $\eta = \mathfrak{J}_F(\mathfrak{v}(x)) = (\pm 1, 0)$. Indeed, choosing an arbitrary $x \in \partial C$ close to x_0 and assuming $x_2 > 0$ (the other case is symmetric) we consider two cases. In the first one (when $1/8 < x_2 < 3/4$) the vector $\mathfrak{v}(x)$ is given by (2.68), and for $\xi = (\xi_1, \xi_2) = \mathfrak{J}_F(\mathfrak{v}(x))$ we have

$$\xi_1 = -\frac{1}{2|x_2|^{1/3}} \in [-1, 0[$$

(see (2.69)). Otherwise (i.e., if $x_2 > 3/4$) from (3.53) we find

$$\mathfrak{v}(x) = \frac{(-1, 2x_2 - 1)}{\rho_{F^o}(-1, 2x_2 - 1)},$$

where

$$\rho_{F^o}(-1, 2x_2 - 1) = \frac{3 + 4^{4/3}|1 - 2x_2|^{4/3}}{4^{4/3}|1 - 2x_2|^{1/3}},$$

and for $\xi = (\xi_1, \xi_2) \in \mathfrak{J}_F(\mathfrak{v}(x))$ simple calculations give

$$\xi_1 = \frac{1}{4^{1/3}(1 - 2x_2)^{1/3}} \in [-1, 0[\quad \text{and} \quad \xi_2 = \left(1 - \xi_1^4\right).$$

Thus, in these two cases $\mathfrak{n}_F(\cdot)$ is Lipschitz continuous near each $\mathfrak{J}_F(\mathfrak{v}(x_0))$ and the time-minimum projection $\pi_C^F(\cdot)$ is locally Lipschitzean near x_0 by Theorem 3.2.3 **(ii)**. Furthermore, according to Theorem 3.3.7 **(ii)b** the value function $\mathfrak{T}_C^F(\cdot)$ is of class $C_{loc}^{1,1}$. Notice that here we can also apply the condition **(i)** of Theorem 3.2.3 (as well as the hypothesis **(ii)a** of Theorem 3.3.7) because C has smooth boundary near x_0 with $\mathfrak{n}_C(\cdot)$ locally Lipschitzean (see (3.53)). However, if $|x_2^0| \leq 1/8$ then to guarantee Lipschitz continuity (which takes place as well) we can use only the last conditions, i.e., the local regularity of ∂C since in this case $\xi := \mathfrak{J}_F(\mathfrak{v}(x_0)) = (\pm 1, 0)$, and the smoothness of ∂F at ξ fails.

Finally, at the points $x_0 = b^{\pm}$ the boundary of C is not smooth. Nevertheless, calculating also the normal cone to C at these (singular) points we see that ∂F is smooth at each point $\eta = \mathfrak{J}_F(\eta^*)$, $\eta^* \in -\mathbf{N}_C^p(x) \cap \partial F^o$, $x \in C_\delta(x_0)$, where $\delta > 0$ is small enough, and the respective normal $\mathfrak{n}_F(\cdot)$ is Lipschitz continuous. Consequently, also in this case we conclude that $\pi_C^F(\cdot)$ is Lipschitzean near x_0, and $\mathfrak{T}_C^F(\cdot)$ is of class $C_{loc}^{1,1}$ in some neighbourhood $\mathcal{U}'(x_0)$.

Thus, joining together everything said above we see that the projection $\pi_C^F(\cdot)$ is locally Lipschitz continuous near C. Moreover, the value function $\mathfrak{T}_C^F(\cdot)$ is Fréchet continuously differentiable, and its gradient $\nabla\mathfrak{T}_C^F(\cdot)$ is also locally Lipschitzean on some neighbourhood $\mathfrak{U}(C)$ of the target set.

Example 3.4.2 *Let*

$$F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^2 + (\xi_2 - 1)^2 \leq 4,\ \xi_2 \leq 1\right\};$$
$$C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq f(x_2)\right\}$$

where $f:\mathbb{R}\to\mathbb{R}$ *is given by*

$$f(t):=\begin{cases} -\frac{t}{3}-\frac{1}{3} & \textit{if } t\leq -1\\ -\sqrt{1-t^2} & \textit{if } -1\leq t\leq \frac{\sqrt{3}}{2}\\ -\frac{1}{2} & \textit{if } t\geq \frac{\sqrt{3}}{2}.\end{cases}$$

Observe that F is closed convex bounded with $0\in \operatorname{int} F$, and C is closed, φ-convex with $\varphi(x)=\frac{1}{2}$. Furthermore, C has smooth boundary in all the points except in $(0,-1)$, $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ with the unit normal vector given by

$$\mathfrak{n}_C(x)=\begin{cases} (-x_1,-x_2) & \text{if } -1<x_2<\frac{\sqrt{3}}{2}\\ (1,0) & \text{if } x_2>\frac{\sqrt{3}}{2}\\ \frac{1}{\sqrt{10}}(3,1) & \text{if } x_2<-1,\end{cases}$$

$x=(x_1,x_2)\in\partial C$.

For an arbitrary dual pair (ξ,ξ^*), $\xi\in\mathfrak{J}_F(\xi^*)$, $\xi^*\in\partial F^o$, we have

$$\rho_{F^o}(\xi_1^*,\xi_2^*)=\begin{cases} 2|\xi_1^*|+\xi_2^* & \text{if } \xi_2^*\geq 0\\ 2\sqrt{(\xi_1^*)^2+(\xi_2^*)^2}+\xi_2^* & \text{if } \xi_2^*<0.\end{cases}$$

It is obvious that for each $\xi=(\xi_1,1)$, $\xi_1\in\,]-2,2[$, and the (unique) normal vector $\xi^*=(0,1)\in\partial F^o$ to F at ξ the curvature $\varkappa(\xi,\xi^*)$ is equal to zero, while $\varkappa(\xi,\xi^*)=\frac{1}{4\|\xi^*\|}$ for each $\xi^*=(\xi_1^*,\xi_2^*)\in\partial F^o$ with $\xi_2^*\leq 0$ and the unique $\xi\in\mathfrak{J}_F(\xi^*)$. Finally, if $\xi_2^*>0$ and $\xi_1^*\neq 0$ we have $\varkappa(\xi,\xi^*)\geq\frac{1}{4\|\xi^*\|}$, where again $\xi=\mathfrak{J}_F(\xi^*)$. In the last case, obviously, $\hat{\varkappa}(\xi,\xi^*)=+\infty$.

Given $x_0=\left(x_1^0,x_2^0\right)\in\partial C$ let us consider the various cases.

(i) If $-1<x_1^0<-\frac{1}{2}$, $x_2^0>0$ then for $x=(x_1,x_2)$ enough close to x_0 we have

$$\mathfrak{v}(x)=-\frac{\mathfrak{n}(x)}{\rho_{F^o}(-\mathfrak{n}(x))}=\frac{\left(-\sqrt{1-x_2^2},x_2\right)}{2\sqrt{1-x_2^2}+x_2}\in\partial F^o,$$

and $\xi=(-2,1)$ is the unique element of $\mathfrak{J}_F(\mathfrak{v}(x))$. Since $\|\mathfrak{v}(x)\|\leq\|F^o\|\,\rho_{F^o}(\mathfrak{v}(x))=1$ we have $\varkappa_F(\xi,\mathfrak{v}(x))\geq 1/4$.

(ii) If $-1<x_1^0<0$, $x_2<0$ then for each $x=(x_1,x_2)$ near x_0 we find

$$\mathfrak{v}(x)=\frac{\left(-\sqrt{1-x_2^2},x_2\right)}{2+x_2},$$

and there exists a unique $\xi\in\partial F$ such that the (unique) normal vector to F at ξ belonging to ∂F^o is exactly $\mathfrak{v}(x)$, i.e., $\xi=\mathfrak{J}_F(\mathfrak{v}(x))$. So that

$$\varkappa_F(\xi,\mathfrak{v}(x))=\frac{1}{4\|\mathfrak{v}(x)\|}\geq\frac{1}{4}.$$

(iii) The case $x_1^0 = -1$ (and $x_2^0 = 0$) is reduced to the previous cases because every neighbourhood of x_0 intersects both semiplanes $\{x_2 > 0\}$ and $\{x_2 < 0\}$, while for the point $x = (-1,0)$ itself $\mathfrak{v}(x) = (-1/2, 0)$, $\xi = (-2,1)$ and $\varkappa_F(\xi, \mathfrak{v}(x)) = 1/2$.

(iv) If $x_1^0 > 0$, $x_2^0 < -1$ then for each x from some neighbourhood of x_0 we have

$$\mathfrak{v}(x) = \frac{-1}{2\sqrt{10}-1}(3,1)$$

and for the unique $\xi \in \mathfrak{J}_F(\mathfrak{v}(x))$ we have

$$\varkappa_F(\xi, \mathfrak{v}(x)) = \frac{1}{4\|\mathfrak{v}(x)\|} = \frac{2\sqrt{10}-1}{4\sqrt{10}}.$$

(v) If $x_1^0 = -\frac{1}{2}$, $x_2^0 > \frac{\sqrt{3}}{2}$ then for each x from some neighbourhood of x_0 we have $\mathfrak{v}(x) = (-1/2, 0)$, and $\xi = (-2,1)$ is the unique element of $\mathfrak{J}_F(\mathfrak{v}(x))$. So that again $\varkappa_F(\xi, \mathfrak{v}(x)) = 1/2$.

(vi) If $x_0 = (0,-1)$ then $\mathbf{N}_C^p(x_0) = \{\lambda_1 e_1 + \lambda_2 e_2 : \lambda_1, \lambda_2 \geq 0\}$, where $e_1 := (0,1)$, $e_2 := (3,1)$. As it is easy to see for each $\xi^* \in -\mathbf{N}_C^p(x_0) \cap \partial F^o$ there exists a unique $\xi \in \mathfrak{J}_F(\xi^*)$ and we have $\varkappa_F(\xi, \xi^*) \geq 1/4$. But in each neighbourhood of x_0 there are points x either with $-1 < x_1 < 0$, $x_2 < 0$, or with $x_1 > 0$ and $x_2 < -1$. These cases we combine with **(ii)** and with **(iv)**, respectively.

(vii) If $x_0 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ then similarly to **(vi)** $\mathbf{N}_C^p(x_0) = \{\lambda_1 e_3 + \lambda_2 e_4 : \lambda_1, \lambda_2 \geq 0\}$, where $e_3 := (1,0)$, $e_4 := \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. For each $\xi^* \in -\mathbf{N}_C^p(x_0) \cap \partial F^o$ we have $\mathfrak{J}_F(\xi^*) = \{(-2,1)\}$ and $\varkappa_F(\xi, \xi^*) \geq 1/4$.

Therefore the condition **(B)** holds at every $x_0 \in \partial C$, and the projection $\pi_C^F(\cdot)$ is well--defined on an open set $\mathcal{U} \supset C$. Moreover, it is Hölder continuous with the exponent $1/2$. In order to improve the Hölder regularity of the projection as well as to show the smoothness of the value function (if any) we should study the smoothness either of the target set or of the dynamics. First, C has smooth boundary in all points $x_0 \in \partial C \setminus \left\{(0,-1), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}$ and the function $\mathfrak{n}_C(\cdot)$, which associates to each $x \in C_\delta(x_0)$ the respective (unique) normal vector $\mathfrak{n}_C(x) \in \mathbf{N}_C^l(x) \cap \partial\overline{\mathbf{B}}$, is Lipschitz continuous. Hence, by Theorem 3.2.3 $\pi_C^F(\cdot)$ is (locally) Lipschitzean near such x_0 and by Theorem 3.3.7 (ii)a $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,1}$ in some neighbourhood of x_0. Considering the point $x_0 = (0,-1)$ we see that F is uniformly smooth at $\xi := \mathfrak{J}_F(\xi^*)$ (w.r.t. ξ^*) for each $\xi^* \in U_{\delta,\delta'}(x_0)$ (see (2.14)) with some δ, $\delta' > 0$ and that the gradient $\nabla\rho_F(\xi)$ being normal to F at the point ξ is Lipschitz continuous in ξ. Thus we conclude by Theorem 3.2.3 that $\pi_C^F(\cdot)$ is (locally) Lipschitzean near x_0 and $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,1}$ in some neighbourhood of x_0. Let us consider now the point $\bar{x} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Since the normal cone to C at $\bar{x}$ being contained in $-\mathbf{N}_F(-2,1)$ does not touch the vertical semi-line $\mathbb{R}^+ e_1$, the condition **(A)** at $\bar{x}$ also holds. In turn this implies that the projection $\pi_C^F(\cdot)$ is not only Hölder continuous with the exponent $1/2$ but is (locally) Lipschitz continuous in fact in a neighbourhood of $\bar{x}$ (see

Theorem 3.2.1). On the other hand, denoting $e = (2, -1)$ observe that $\pi_C^F(\bar{x} - \varepsilon e) = \{\bar{x}\}$ for all $\varepsilon > 0$ small enough. Nevertheless, for $x := \bar{x} - \varepsilon e$ the intersection

$$-\mathbf{N}_C^p(\bar{x}) \cap \mathbf{N}_F\left(\frac{\bar{x} - x}{\rho_F(\bar{x} - x)}\right) \cap \partial F^o$$

is reduced to $-\{\lambda_1 e_3 + \lambda_2 e_4 : \lambda_1, \lambda_2 \geq 0\} \cap \partial F^o$. By the equality (3.40) obtained in Theorem 3.3.1 this intersection is exactly the Fréchet subdifferential $\partial^f \mathfrak{T}_C^F(x)$. So that we have no the Fréchet differentiability of the value function near the point $\bar{x} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Chapter 4

The problem with a nonlinear perturbation

In this chapter we exploit another view to the minimal time problem that allow us to somehow generalize a part of the obtained results. Namely, we are interested in studying of the regularity properties of the viscosity solutions to the equation

$$\rho_{F^{\circ}}(-\nabla u(x)) - 1 = 0 \quad \text{in} \quad H \backslash C, \tag{4.1}$$

such that

$$u(x) = \theta(x), \qquad x \in C. \tag{4.2}$$

Here as usual H is a Hilbert space, $F \subset H$ (dynamics) is convex closed bounded and with $0 \in \operatorname{int} F$ and $C \subset H$ is an arbitrary closed set. As we already said (see Introduction) in the case $\theta \equiv 0$ the unique viscosity solution of the problem (4.1)-(4.2) is exactly the minimal time function $\mathfrak{T}_C^F(\cdot)$. Generalizing this problem we suppose that $\theta : C \to \mathbb{R}$ is a non affine but enough regular function. In what follows we assume $\theta(\cdot)$ to be well-defined not only on C but on the convex closed hull $\overline{\text{co}}C$ or even on the whole space H. If $H = \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open domain with the boundary $C := \partial\Omega$ then (4.1)-(4.2) becomes the boundary value problem for the Hamilton-Jacobi equation associated with some optimal control problem. Consideration of this relationship is out of Thesis. Here we just occupy with the solution of (4.1)-(4.2) itself and with their properties. First of all, in Section 4.1 we give the main definitions regarding the theory of viscosity solutions. Furthermore, for the sake of completeness we place here the direct proof of the well-known fact (see, e.g., [61]) that the (unique) viscosity solution of (4.1) is given by

$$u(x) := \inf_{y \in C} \{\rho_F(y - x) + \theta(y)\}, \quad x \in H. \tag{4.3}$$

As we will see later the regularity of the function (4.3), similarly to the case $\theta \equiv 0$, strongly depends on the attainability of the minimum in (4.3) and on the uniqueness of the point $\bar{x} \in C$, where this minimum is attained. So that we are led to study the (multivalued in general) mapping $x \mapsto \pi_C^{F,\theta}(x)$, where

$$\pi_C^{F,\theta}(x) := \{x \in H : u(x) = \rho_F(y - x) + \theta(y)\},$$

and give in Section 4.2 some effective (local) conditions guaranteeing that the above mapping is well-defined, single-valued and continuous in a neighbourhood of the set C. These conditions extend the hypotheses (**A**) considered in the chapters 2 and 3 to the case $\theta \neq 0$ and ensures the Lipschitz continuity of $\pi_C^{F,\theta}(\cdot)$ as well. The last Section 4.3 is devoted just to the regularity of the function (4.3), and we prove here that $u(\cdot)$ is of class $C_{loc}^{1,1}$ near the set C and give an explicit formula for its derivative. This is similar to the respective results for the function $\mathfrak{T}_C^F(\cdot)$ (see Section 3.3).

4.1 A short introduction to viscosity solutions and to Hamilton-Jacobi equations

We start from the definition of viscosity solution given by M. G. Crandall and P.-L. Lions in [34] for the general first order (nonlinear) partial differential equation

$$\Gamma(x, u(x), \nabla u(x)) = 0, \quad x \in \Omega, \tag{HJ}$$

in a finite dimensional space, where Ω is an open domain in $\mathbb{R}^n$ and $\Gamma(\cdot,\cdot,\cdot)$ is a continuous function.

Notice that such solution may be nowhere differentiable in classical sense, and its gradient is understood by another way similar to the distributional derivative in the case of linear partial differential equations. However, due to essential nonlinearity of the considered problems the integration by parts here is replaced by "differentiation by parts". Surprisingly, another definition (see below) can be given in terms of the generalized (multivalued) differentials, which always exist (but often can be empty).

Denoting as usual by $\mathcal{C}(\overline{\Omega})$ the space of all real continuous functions defined on $\overline{\Omega}$, for each $\psi \in \mathcal{C}(\overline{\Omega})$ we set

$$E_+(\psi) := \left\{ y \in \Omega : \psi(y) = \sup_{\Omega} \psi > 0 \right\} \quad \text{and} \quad E_-(\psi) := \left\{ y \in \Omega : \psi(y) = \inf_{\Omega} \psi < 0 \right\}.$$

Let us consider also the space $\mathcal{C}_c^\infty(\Omega)$ of all (test) functions $\varphi : \Omega \to \mathbb{R}$ of the class $\mathcal{C}^\infty$ (infinitely differentiable) with compact *support* $\operatorname{supp}\varphi := \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$, and denote by $\mathcal{C}_c^\infty(\Omega)^+$ the set of nonnegative functions belonging to $\mathcal{C}_c^\infty(\Omega)$.

Definition 4.1.1 *A function $u \in \mathcal{C}(\overline{\Omega})$ is said to be* viscosity subsolution *(respectively,* supersolution*) of the equation (HJ) if for every $\varphi \in \mathcal{C}_c^\infty(\Omega)^+$ and $k \in \mathbb{R}$*

$$\exists x \in E_+(\varphi(u(\cdot) - k)) \quad \textit{such that} \quad \Gamma\left(x, u(x), -\frac{u(x) - k}{\varphi(x)} \nabla\varphi(x)\right) \leq 0, \tag{4.4}$$

(respectively,

$$\exists x \in E_-(\varphi(u(\cdot) - k)) \quad \textit{such that} \quad \Gamma\left(x, u(x), -\frac{u(x) - k}{\varphi(x)} \nabla\varphi(x)\right) \geq 0) \tag{4.5}$$

whenever the set $E_+(\varphi(u(\cdot)-k))$ *(respectively,* $E_-(\varphi(u(\cdot)-k))$*) is nonempty. We say that* $u \in C(\overline{\Omega})$ *is a* viscosity solution *of (HJ) if both implications (4.4) and (4.5) hold, i.e.,* $u(\cdot)$ *is both a viscosity subsolution and a viscosity supersolution.*

Due to the following result an equivalent definitions of viscosity solution can be given (see [32, Theorem 1.1]).

Proposition 4.1.1 *Let* $u \in C(\overline{\Omega})$. *Then* $u(\cdot)$ *is a viscosity subsolution (supersolution) of (HJ) if and only if for each* $\varphi \in C^1(\overline{\Omega})$ *such that the function* $u(\cdot)-\varphi(\cdot)$ *attains a local maximum (respectively, a local minimum) at* $x_0 \in \Omega$ *the inequality* $\Gamma(x_0, u(x_0), \nabla\varphi(x_0)) \le 0$ *(respectively,* $\Gamma(x_0, u(x_0), \nabla\varphi(x_0)) \ge 0$*) holds.*

In this simpler form subsequently the notion of viscosity solution was defined and studied in an arbitrary Banach space with the Radon-Nikodym property (see [35, 36]). Further on, naturally, we will consider only the case of a Hilbert space H (which possesses this property, see [44, p. 100]), setting $\Omega := H\backslash C$ where C is the target set.

Once more equivalent definition in each (Hilbert) space can be given basing on the concept of the Fréchet subdifferential $\partial^f u(\cdot)$ (see Section 2.1) and the symmetrical *Fréchet superdifferential* denoted by $\partial_f u(\cdot)$. Let us recall the respective definitions

$$\partial^f u(x) := \left\{ p \in H : \ \liminf_{y\to x} \frac{u(y)-u(x)-\langle p, y-x\rangle}{\|y-x\|} \ge 0 \right\};$$

$$\partial_f u(x) := \left\{ p \in H : \ \limsup_{y\to x} \frac{u(y)-u(x)-\langle p, y-x\rangle}{\|y-x\|} \le 0 \right\}.$$

The following result was obtained already by M. G. Crandall and P.-L. Lions in finite dimensional spaces (see [34, Proposition I.18]) and then was generalized to Banach spaces in [35, Proposition 1].

Proposition 4.1.2 *Let* $u \in C(\overline{\Omega})$. *Then*

(i) $u(\cdot)$ *is a viscosity subsolution of (HJ) if and only if*

$$\Gamma(x, u(x), p) \le 0 \qquad \forall x \in \Omega, \ \forall p \in \partial_f u(x);$$

(ii) $u(\cdot)$ *is a viscosity supersolution of (HJ) if and only if*

$$\Gamma(x, u(x), p) \ge 0 \qquad \forall x \in \Omega, \ \forall p \in \partial^f u(x).$$

Remark 4.1.1 *Notice that if both* $\partial_f u(x)$ *and* $\partial^f u(x)$ *are nonempty at some* x *then the function* $u(\cdot)$ *is differentiable at this point and* $\partial_f u(x) = \partial^f u(x) = \{\nabla u(x)\}$.

In what follows we consider only a particular case of (HJ), namely, the equation (4.1), and use only the definition of viscosity solutions given by the latter proposition. According to general existence and uniqueness theorems (see [36, Theorem 1.1] and [35, Theorem 1], respectively) we can affirm that the equation (4.1) admits a unique viscosity solution satisfying the boundary condition $u\,|_{\partial\Omega} = u\,|_C = \theta$ where $\theta(\cdot)$ is a fixed function. Furthermore, it is known that this viscosity solution is given by the exact formula (4.3) (see [61, 18] in the case of $\mathbb{R}^n$). However, for the sake of completeness let us prove the respective assertion.

Proposition 4.1.3 *Let $\theta : \overline{\mathrm{co}}C \to \mathbb{R}$ be a Lipschitz continuous function such that $\partial^c\theta(x) \subset -F^o$ for all $x \in \overline{\mathrm{co}}C$. Then the function $u(\cdot)$ defined by (4.3) coincides with $\theta(\cdot)$ on C and is the (unique) viscosity solution of (4.1) with this boundary condition.*

Proof.

Let us prove first that $u(x) = \theta(x)$ for all $x \in C$. The inequality $u(x) \le \theta(x)$ is obvious. In order to prove the opposite inequality let us fix $y \in C$. Then by the Lebourg Theorem (see [22, Theorem 2.3.7]), there exist $\eta = \lambda x + (1-\lambda)y \in \overline{\mathrm{co}}C$, $\lambda \in (0,1)$ and $\eta^* \in \partial^c\theta(\eta) \subset -F^o$ such that

$$\theta(x) - \theta(y) = \langle \eta^*, x - y\rangle \le \sup_{\xi^* \in F^o} \langle \xi^*, y - x\rangle = \rho_F(y - x). \tag{4.6}$$

Hence, passing to infimum in $y \in C$, we have

$$u(x) = \inf_{y\in C}\{\rho_F(y-x) + \theta(y)\} \ge \theta(x).$$

Thus

$$u(x) = \theta(x) \qquad \forall x \in C. \tag{4.7}$$

The function $u(\cdot)$ is continuous. It is even Lipschitzean on whole H with the Lipschitz constant $\|F^o\|$. Indeed, taking arbitrary x, $y \in H$ and $\varepsilon > 0$ we find $w \in C$ such that

$$u(y) \ge \rho_F(w-y) + \theta(w) - \varepsilon,$$

and hence

$$\begin{aligned} u(x) - u(y) &\le \rho_F(w-x) + \theta(w) - \rho_F(w-y) - \theta(w) + \varepsilon \\ &= \rho_F(w-x) - \rho_F(w-y) + \varepsilon \\ &\le \|F^o\|\,\|x-y\| + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0+$ we obtain the Lipschitz inequality.

In order to prove that $u(\cdot)$ given by (4.3) is the viscosity solution of (4.1), let us verify the conditions (i) and (ii) of Proposition 4.1.2.

(i) We have

$$\begin{aligned} u(y) - u(x) &= \inf_{z\in C}\{\rho_F(z-y) + \theta(z)\} - \inf_{z\in C}\{\rho_F(z-x) + \theta(z)\} \\ &\ge \inf_{z\in C}\{\rho_F(z-y) - \rho_F(z-x)\} \\ &\ge \inf_{z\in C}\{\rho_F(z-y) - \rho_F(z-y) - \rho_F(y-x)\} \\ &= -\rho_F(y-x) \end{aligned} \tag{4.8}$$

for every $x, y \notin C$.

Let us fix $x \notin C$ and $p \in \partial_f u(x)$. Then for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{u(y) - u(x) - \langle p, y - x\rangle}{\rho_F(y - x)} \leq \varepsilon \frac{\|y - x\|}{\rho_F(y - x)},$$

for all $y \neq x$ with $\|y - x\| < \delta$. By (4.8)

$$-1 - \left\langle p, \frac{y - x}{\rho_F(y - x)}\right\rangle \leq \varepsilon \frac{\|y - x\|}{\rho_F(y - x)},$$

and passing to supremum we have

$$-1 + \sup_{\|y-x\|<\delta} \left\langle p, \frac{x - y}{\rho_F(y - x)}\right\rangle \leq \varepsilon \sup_{\|y-x\|<\delta} \frac{\|y - x\|}{\rho_F(y - x)}. \tag{4.9}$$

Since

$$\sup_{\|y-x\|<\delta} \frac{\|y - x\|}{\rho_F(y - x)} = \sup_{z\neq 0} \frac{\|z\|}{\rho_F(z)} = \|F\|$$

and

$$\rho_{F^\circ}(-p) = \sup_{z\in F} \langle z, -p\rangle = \sup_{\|y-x\|<\delta} \left\langle -p, \frac{y - x}{\rho_F(y - x)}\right\rangle,$$

(4.9) implies that

$$-1 + \rho_{F^\circ}(-p) \leq \varepsilon \|F\|$$

and letting $\varepsilon \to 0+$ we obtain $\rho_{F^\circ}(-p) \leq 1$, proving that $u(\cdot)$ is a viscosity subsolution of (4.1).

(ii) Let us fix $x \notin C$ and $p \in \partial^f u(x)$. We should prove that $\rho_{F^\circ}(-p) \geq 1$. Let us suppose the contrary, i.e., $\rho_{F^\circ}(-p) < 1$, and let $\varepsilon > 0$ be so small that $\rho_{F^\circ}(-p) < 1 - \varepsilon$. Then by using the definition of Fréchet subdifferential we find $\delta > 0$ such that

$$u(y) - u(x) - \langle p, y - x\rangle \geq -\frac{\varepsilon}{4}\rho_F(y - x) \Leftrightarrow u(x) - u(y) - \langle -p, y - x\rangle \leq \frac{\varepsilon}{4}\rho_F(y - x) \tag{4.10}$$

for all $y \neq x$, $\|y - x\| < \delta$. Assume, moreover, that $(x + \delta\overline{\mathbf{B}}) \cap C = \emptyset$. On the other hand, let $z \in C$ be such that

$$u(x) \geq \rho_F(z - x) + \theta(z) - \frac{\varepsilon\delta}{8\|F\|},$$

and, consequently,

$$\begin{aligned} u(x) - u(y) &\geq \rho_F(z - x) + \theta(z) - \frac{\varepsilon\delta}{8\|F\|} - \rho_F(z - y) - \theta(z) \\ &= \rho_F(z - x) - \rho_F(z - y) - \frac{\varepsilon\delta}{8\|F\|} \quad \forall y \notin C. \end{aligned} \tag{4.11}$$

Since $\|z - x\| > \delta$, there exists $\lambda \in (0, 1)$ with $\|\overline{y} - x\| = \frac{\delta}{2}$ where $\overline{y} := \lambda x + (1 - \lambda) z$.

Notice that

$$\rho_F(\overline{y}-x) = (1-\lambda)\,\rho_F(z-x) \quad \text{and} \quad \rho_F(z-\overline{y}) = \lambda\rho_F(z-x)$$

and consequenlty

$$\rho_F(z-x) - \rho_F(z-\overline{y}) = \rho_F(\overline{y}-x) \geq \frac{\|\overline{y}-x\|}{\|F\|} = \frac{\delta}{2\|F\|}. \tag{4.12}$$

Therefore, applying successively the inequalities (4.12), (4.11) and (4.10), we obtain

$$\begin{aligned} \rho_F(\overline{y}-x) - \frac{\varepsilon\delta}{8\|F\|} - \langle -p, \overline{y}-x\rangle &= \rho_F(z-x) - \rho_F(z-\overline{y}) - \frac{\varepsilon\delta}{8\|F\|} - \langle -p, \overline{y}-x\rangle \\ &\leq u(x) - u(\overline{y}) - \langle -p, \overline{y}-x\rangle \leq \frac{\varepsilon}{4}\rho_F(\overline{y}-x). \end{aligned} \tag{4.13}$$

On the other hand, since $\rho_{F^o}(-p) \geq \left\langle -p, \frac{\overline{y}-x}{\rho_F(\overline{y}-x)}\right\rangle$ by the choice of ε we have

$$\langle -p, \overline{y}-x\rangle < \rho_F(\overline{y}-x)(1-\varepsilon).$$

Combining this with (4.13) and (4.12) we obtain

$$\begin{aligned} \rho_F(\overline{y}-x) &\leq \frac{\varepsilon\delta}{8\|F\|} + \langle -p, \overline{y}-x\rangle + \frac{\varepsilon}{4}\rho_F(\overline{y}-x) \\ &< \frac{\varepsilon}{4}\rho_F(\overline{y}-x) + \frac{\varepsilon}{4}\rho_F(\overline{y}-x) + \rho_F(\overline{y}-x)(1-\varepsilon) \\ &= \rho_F(\overline{y}-x)\left(1-\frac{\varepsilon}{2}\right), \end{aligned}$$

which is a contradiction. So the proposition is proved. ■

4.2 Existence, uniqueness and Lipschitz continuity of minimizers

Our goal in this section is to prove existence and uniqueness of minimizers in the problem

$$\min\{\rho_F(y-x) + \theta(y) : y \in C\}$$

for x from some neighbourhood of the target set, and then to establish the Lipschitz regularity of these minimizers w.r.t. x. The main condition used here is the relation

$$\partial^c\theta(x) \subset -\gamma F^o, \tag{4.14}$$

which holds for all $x \in \overline{\text{co}}C$, where $0 < \gamma < 1$ is some fixed constant. In particular, this implies that

$$-\partial^c\theta(x) + \eta\overline{\mathbf{B}} \subset \gamma F^o + (1-\gamma)F^o = F^o, \tag{4.15}$$

where $\eta := \frac{1-\gamma}{\|F\|}$ (in sequel we frequently use this constant). Oberve that if $\theta(\cdot)$ is defined in a neighbourhood of C and is such that $\partial^p\theta(x) \subset -\gamma F^o$ then (4.14) automatically holds. Indeed, any $\xi \in \partial^l\theta(x)$ can be represented as the weak limit of a sequence $\xi_n \in \partial^p\theta(x_n) \subset -\gamma F^o$, $x_n \to x$ (see (2.1)). Thus $\partial^l\theta(x) \subset -\gamma F^o$, and (4.14) follows because $\partial^c\theta(x) = \overline{\text{co}}\partial^l\theta(x)$ and F^o is convex. The condition (4.14) implies obviously that $\theta(\cdot)$ is Lipschitzean with the Lipschitz constant $\gamma\|F^o\|$.

Notice that for $x \in C$ we always have $\pi_C^{F,\theta}(x) = \{x\}$. Indeed, as we have already shown (see Proposition 4.1.3) $u(x) = \theta(x)$ and, consequenlty, $x \in \pi_C^{F,\theta}(x)$. On the other hand if there exists $y \in C$, $y \neq x$, with $u(x) = \theta(x) = \rho_F(y-x) + \theta(y)$ then by the Lebourg Theorem we find $\eta = \lambda x + (1-\lambda)y \in \overline{\text{co}}C$, $\lambda \in (0,1)$ and $\eta^* \in \partial^c\theta(\eta) \subset -\gamma F^o$ such that

$$\rho_F(y-x) = \theta(x) - \theta(y) = \langle -\eta^*, y-x\rangle \leq \sup_{\xi^*\in\gamma F^o}\langle \xi^*, y-x\rangle = \gamma\rho_F(y-x). \tag{4.16}$$

But (4.16) is possible only if $y = x$.

Now we emphasize the hypotheses guarateeing that the infinum

$$u(x) = \inf_{y\in C}\{\rho_F(y-x) + \theta(y)\}$$

is also achieved at a unique point if $x \notin C$ but enough close to C. First we prove the following auxiliar lemma (similar to Lemma 2.1.1).

Lemma 4.2.1 *Let $C \subset H$ be a nonempty closed set and $\theta : H \to \mathbb{R}$ be a function of class $C^{1,1}$. Let $z \in H\backslash C$, and $\{x_n\} \subset C$ be a minimizing sequence for the function $x \mapsto \rho_F(x-z) + \theta(x)$ on C. Then there exist another minimizing sequence $\{x_n'\} \subset C$ and sequences $\{x_n''\}$, $\{v_n\}$, $\{\xi_n^*\}$ such that $v_n \in \nabla\theta(x_n') + \mathbf{N}_C^p(x_n')$, $\xi_n^* \in \partial\rho_F(x_n''-z)$ and*

$$\|x_n' - x_n\| + \|x_n'' - x_n\| \to 0, \tag{4.17}$$

$$\|v_n + \xi_n^*\| \to 0, \tag{4.18}$$

as $n \to \infty$.

Proof.
Given an arbitrary sequence $\varepsilon_n \to 0+$ with $\rho_F(x_n - z) + \theta(x_n) \leq u(z) + \varepsilon_n$, by Theorem 2.1.3 there exists $\{y_n\} \subset C$ satisfying the conditions

$$\rho_F(y_n - z) + \theta(y_n) \leq u(z) + \varepsilon_n; \tag{4.19}$$

$$\|x_n - y_n\| \leq \sqrt{\varepsilon_n};$$

$$\rho_F(y_n - z) + \theta(y_n) \leq \rho_F(y-z) + \theta(y) + \sqrt{\varepsilon_n}\|y - y_n\| \qquad \forall y \in C, \tag{4.20}$$

$n = 1, 2, \ldots$. The inequality (4.20), in particular, means that y_n minimizes the functional

$$F(y) := \rho_F(y-z) + \theta(y) + \sqrt{\varepsilon_n}\|y - y_n\| + \mathbf{I}_C(y)$$

on H. Then we obviously have $0 \in \partial^p F(y_n)$ (see [23, p. 37]). According to Theorem 2.1.1

$$0 \in \partial\rho_F(x_n'' - z) + \sqrt{\varepsilon_n}\frac{x_n'' - y_n}{\|x_n'' - y_n\|} + \partial^p(\theta + \mathbf{I}_C)(x_n') + \sqrt{\varepsilon_n}\overline{\mathbf{B}}$$

for some sequences $\{x_n'\} \subset C$ and $\{x_n''\} \subset H$, $\|x_n' - y_n\| \leq \sqrt{\varepsilon_n}$, $\|x_n'' - y_n\| \leq \sqrt{\varepsilon_n}$, $n \in \mathbb{N}$. In the same way as in the proof of [23, Proposition 2.11, p. 38] but using the Lipschitz continuity of $\nabla\theta(\cdot)$ instead of the differentiability we obtain

$$\partial^p (\theta + \mathbf{I}_C)(x_n') = \nabla\theta(x_n') + \mathbf{N}_C^p(x_n'). \tag{4.21}$$

Therefore

$$0 \in \partial\rho_F(x_n'' - z) + \nabla\theta(x_n') + \mathbf{N}_C^p(x_n') + 2\sqrt{\varepsilon_n}\overline{\mathbf{B}}.$$

Consequently there exist vectors $v_n \in \nabla\theta(x_n') + \mathbf{N}_C^p(x_n')$ and $\xi_n^* \in \partial\rho_F(x_n'' - z)$ such that

$$\|v_n + \xi_n^*\| \leq 2\sqrt{\varepsilon_n},$$

and the property (4.18) holds. It follows from (4.19) that $\{x_n'\}$ is a minimizing sequence of $x \mapsto \rho_F(x-z) + \theta(x)$ on C:

$$\begin{aligned}
\rho_F(x_n' - z) + \theta(x_n') &\leq \rho_F(x_n' - y_n) + \rho_F(y_n - z) + \theta(y_n) + \mathfrak{h}\|x_n' - y_n\| \\
&\leq u(z) + (\|F^o\| + \mathfrak{h})\|x_n' - y_n\| + \varepsilon_n \\
&\leq u(z) + (\|F^o\| + \mathfrak{h})\sqrt{\varepsilon_n} + \varepsilon_n,
\end{aligned}$$

where $\mathfrak{h} > 0$ is the Lipschitz constant of $\theta(\cdot)$, and (4.17) is also valid. ■

Remark 4.2.1 *The vectors v_n in Lemma 4.2.1 can be chosen such that $-v_n \in \partial F^o$, $n = 1, 2, \ldots$. Indeed, setting*

$$v_n' := \frac{v_n - \nabla\theta(x_n')}{\rho_{F^o + \nabla\theta(x_n')}(-(v_n - \nabla\theta(x_n')))} + \nabla\theta(x_n')$$

we have

$$\rho_{F^o + \nabla\theta(x_n')}(-(v_n' - \nabla\theta(x_n'))) = 1,$$

i.e.,

$$-(v_n' - \nabla\theta(x_n')) \in \partial(F^o + \nabla\theta(x_n'))$$

and consequently $-v_n' \in \partial F^o$. On the other hand, taking into account that

$$\rho_{F^o + \nabla\theta(x_n')}(\xi_n^* + \nabla\theta(x_n')) = 1$$

we obtain

$$\begin{aligned}
\|v_n - v_n'\| &= \left\| v_n - \frac{v_n - \nabla\theta(x_n')}{\rho_{F^o + \nabla\theta(x_n')}(-(v_n - \nabla\theta(x_n')))} - \nabla\theta(x_n') \right\| \\
&= \frac{\|v_n - \nabla\theta(x_n')\|}{\rho_{F^o + \nabla\theta(x_n')}(-(v_n - \nabla\theta(x_n')))} \left| \rho_{F^o + \nabla\theta(x_n')}(-(v_n - \nabla\theta(x_n'))) - 1 \right| \\
&\leq \|F^o + \nabla\theta(x_n')\| \, \|(F^o + \nabla\theta(x_n'))^o\| \, \|(v_n - \nabla\theta(x_n')) + (\xi_n^* + \nabla\theta(x_n'))\| \\
&= \|F^o + \nabla\theta(x_n')\| \, \|(F^o + \nabla\theta(x_n'))^o\| \, \|v_n + \xi_n^*\| \to 0,
\end{aligned}$$

and, therefore, v_n' can substitute v_n.

Theorem 4.2.1 *Assume that* $\theta : H \to \mathbb{R}$ *is a function of class* $C^{1,1}$ *such that* $\nabla\theta(x) \in -\gamma F^o$ *for all* $x \in \overline{\text{co}}C$, *with the constant* $0 < \gamma < \frac{1}{\|F\|\|F^o\|}$. *Let us fix* $x_0 \in \partial C$ *and suppose that for some* $\delta = \delta(x_0) > 0$ *the following conditions hold*

(i) the mapping $x \mapsto \mathfrak{J}_F\left(-\left(\nabla\theta(x) + \mathbf{N}_C^p(x)\right) \cap \partial F^o\right)$ *is single-valued and Lipschitz continuous with the Lipschitz constant* $L := L(x_0) > 0$ *on*

$$C_\delta(x_0) := \left\{x \in \partial C : \|x - x_0\| \le \delta,\ \mathbf{N}_C^p(x) \ne \{0\}\right\},$$

and

(ii) F *is uniformly strictly convex with respect to*

$$U_{\delta,\delta'}(x_0) := \partial F^o \cap \bigcup_{x \in C_\delta(x_0)} \left[-\left(\nabla\theta(x) + \mathbf{N}_C^p(x)\right) \cap \partial F^o + \delta'\mathbf{B}\right]$$

for some $\delta' > 0$.

Then there exists a neighbourhood $\mathcal{U}(x_0)$ *such that for each* $z \in \mathcal{U}(x_0)$ *the set* $\pi_C^{F,\theta}(z)$ *is singleton, and the mapping* $z \mapsto \pi_C^{F,\theta}(z)$ *is continuous on* $\mathcal{U}(x_0)$.

Proof.

Whithout loss of generality we can assume that $\delta > 0$ is such that

$$\delta\gamma\|F^o\| < \frac{1 - \gamma\|F\|\|F^o\|}{L}.$$

We set

$$\mathcal{U}(x_0) := \left\{z \in H : \|z - x_0\| < \frac{(1 - \gamma\|F\|\|F^o\|)\delta}{2\|F\|\|F^o\|},\ u(z) < \frac{1 - \gamma\|F\|\|F^o\|}{L} + \theta(x_0) - \delta\gamma\|F^o\|\right\}, \tag{4.22}$$

where $L > 0$ is the Lipschitz constant of $x \mapsto \mathfrak{J}_F\left(-\left(\nabla\theta(x) + \mathbf{N}_C^p(x)\right) \cap \partial F^o\right)$ on $C_\delta(x_0)$.

Fix $z \in \mathcal{U}(x_0)\setminus C$ and a minimizing sequence $\{x_n\} \subset C$ for $x \mapsto \rho_F(x - z) + \theta(x)$ on C. Let us choose $\{x_n'\} \subset \partial C$, $\{x_n''\} \subset H$, $v_n \in \nabla\theta(x_n') + \mathbf{N}_C^p(x_n')$ and $\xi_n^* \in \partial\rho_F(x_n'' - z)$ as in Lemma 4.2.1. Moreover, by Remark 4.2.1 the vectors v_n can be chosen so that $-v_n \in \partial F^o$, $n = 1, 2, \ldots$. Similarly as in Remark 2.1.1 observe that x_n' in fact belongs to $\partial^* C$ for all n large enough, i.e., $\mathbf{N}_C^p(x_n') \ne \{0\}$. Indeed, otherwise $v_n = \nabla\theta(x_n') \in -\gamma F^o \subset -\operatorname{int} F^o$.

By the definition of $u(\cdot)$ (see (4.3)) we have

$$\begin{aligned}
\frac{\|x_n - x_0\|}{\|F\|} &\le \rho_F(x_n - x_0) \le \rho_F(x_n - z) + \rho_F(z - x_0) \\
&= (\rho_F(x_n - z) + \theta(x_n) - u(z)) + (\rho_F(z - x_0) + u(z) - \theta(x_n)) \\
&\le (\rho_F(x_n - z) + \theta(x_n) - u(z)) + \rho_F(z - x_0) + \rho_F(x_0 - z) + \theta(x_0) - \theta(x_n) \\
&\le (\rho_F(x_n - z) + \theta(x_n) - u(z)) + 2\|F^o\|\|z - x_0\| + \gamma\|F^o\|\|x_0 - x_n\|,
\end{aligned}$$

which implies

$$\left(\frac{1}{\|F\|} - \gamma \|F^o\|\right) \|x_n - x_0\| \leq (\rho_F (x_n - z) + \theta (x_n) - u (z)) + 2 \|F^o\| \|z - x_0\|.$$

Since $\rho_F (x_n - z) + \theta (x_n) - u (z) \to 0+$, $\|x_n - x'_n\| \to 0$ as $n \to \infty$, we can suppose, without loss of generality, that

$$\|x'_n - x_0\| \leq \frac{2 \|F^o\| \|F\|}{1 - \gamma \|F^o\| \|F\|} \|z - x_0\| < \delta, \tag{4.23}$$

for all $n = 1, 2, ..$, which implies that $x'_n \in C_\delta (x_0)$. Consider a decreasing sequence $\nu_n \to 0+$ such that

$$\|x'_n - x_n\| + \|x''_n - x_n\| \leq \nu_n; \tag{4.24}$$
$$\rho_F (x'_n - z) + \theta (x'_n) \leq u (z) + \nu_n; \tag{4.25}$$
$$\|v_n + \xi^*_n\| \leq \nu_n, \tag{4.26}$$

$n = 1, 2, ...$ (see Lemma 4.2.1). By hypothesis **(ii)** the (single-valued) mapping $\mathfrak{J}_F (\cdot)$ is uniformly continuous on $U_{\delta,\delta'} (x_0)$ (see Proposition 1.2.2), and, therefore, the (decreasing) sequence

$$\beta_n := \sup_{\substack{\|\xi^* - \eta^*\| \leq \nu_n \\ \xi^*, \eta^* \in U_{\delta,\delta'}(x_0)}} \|\mathfrak{J}_F (\xi^*) - \mathfrak{J}_F (\eta^*)\|$$

tends to zero as $n \to \infty$.

Observe that $\xi^*_n \in \partial\rho_F (x''_n - z) = \mathbf{N}_F \left(\frac{x''_n - z}{\rho_F (x''_n - z)}\right) \cap \partial F^o$ and hence

$$\frac{x''_n - z}{\rho_F (x''_n - z)} = \mathfrak{J}_F (\xi^*_n) \tag{4.27}$$

(recall that $\mathfrak{J}_F^{-1} (\xi) = \mathbf{N}_F (\xi) \cap \partial F^o$ for $\xi \in \partial F$). By (4.26) we have ξ^*_n, $-v_n \in U_{\delta,\delta'} (x_0)$, and, consequently,

$$\|\mathfrak{J}_F (\xi^*_n) - \mathfrak{J}_F (-v_n)\| \leq \beta_n, \quad n = 1, 2, \dots . \tag{4.28}$$

Given $m \geq n$ we obtain from (4.24) and (4.25)

$$\begin{aligned}
& \left|\rho_F (x''_m - z) - \rho_F (x''_n - z)\right| \\
\leq\ & \left|\rho_F (x''_m - z) - \rho_F (x'_m - z)\right| + \left|\rho_F (x'_m - z) - \rho_F (x'_n - z)\right| + \\
& + \left|\rho_F (x'_n - z) - \rho_F (x''_n - z)\right| \\
\leq\ & 2 \|F^o\| \nu_n + \left|\rho_F (x'_m - z) + \theta (x'_m) - u (z)\right| + \left|\theta (x'_m) - \theta (x'_n)\right| + \\
& + \left|\rho_F (x'_n - z) + \theta (x'_n) - u (z)\right| \\
\leq\ & 2 \|F^o\| \nu_n + 2\nu_n + \gamma \|F^o\| \|x'_m - x'_n\|
\end{aligned} \tag{4.29}$$

and by (4.23) we have

$$\begin{aligned}
\rho_F (x''_m - z) &\leq \|F^o\| \|x''_m - x'_m\| + \rho_F (x'_m - z) \\
&\leq \|F^o\| \nu_n + u (z) - \theta (x'_m) + \nu_n \\
&\leq \|F^o\| \nu_n + u (z) - \theta (x_0) + \gamma \|F^o\| \|x'_m - x_0\| + \nu_n \\
&\leq \|F^o\| \nu_n + u (z) - \theta (x_0) + \delta\gamma \|F^o\| + \nu_n.
\end{aligned}$$

Then using (4.27) we obtain

$$\begin{aligned}\|x''_m - x''_n\| &= \|\rho_F(x''_m - z)\mathfrak{J}_F(\xi^*_m) - \rho_F(x''_n - z)\mathfrak{J}_F(\xi^*_n)\| \\ &\le \rho_F(x''_m - z)\|\mathfrak{J}_F(\xi^*_m) - \mathfrak{J}_F(\xi^*_n)\| + |\rho_F(x''_m - z) - \rho_F(x''_n - z)|\,\|F\| \\ &\le ((\|F^o\|+1)\nu_n + u(z) - \theta(x_0) + \delta\gamma\|F^o\|)\|\mathfrak{J}_F(\xi^*_m) - \mathfrak{J}_F(\xi^*_n)\| + \\ &+ (2(\|F^o\|+1)\nu_n + \gamma\|F^o\|\,\|x'_m - x'_n\|)\,\|F\|.\end{aligned}$$

Now by using (4.24), (4.28) and the hypothesis **(i)** we obtain

$$\begin{aligned}\|x'_m - x'_n\| &\le 2\nu_n + \|x''_m - x''_n\| \\ &\le 2\nu_n + ((\|F^o\|+1)\nu_n + u(z) - \theta(x_0) + \delta\gamma\|F^o\|)(2\beta_n + L\|x'_m - x'_n\|) + \\ &+ (2(\|F^o\|+1)\nu_n + \gamma\|F^o\|\,\|x'_m - x'_n\|)\,\|F\|,\end{aligned}$$

which implies

$$(1 - ((\|F^o\|+1)\nu_n + u(z) - \theta(x_0) + \delta\gamma\|F^o\|)L - \gamma\|F^o\|\,\|F\|)\,\|x'_m - x'_n\| \le \mu_n$$

for some $\mu_n \to 0$ as $n \to \infty$. Hence, by the choice of z we conclude that $\{x_n\}$ is a Cauchy sequence.

Let us pass now to the second part of the proof. Denote by

$$\mathcal{U} = \bigcup_{x_0 \in C} \mathcal{U}(x_0) \supset C,$$

where we put $\mathcal{U}(x_0) := \operatorname{int} C$ for $x_0 \in \operatorname{int} C$. Given $x_0 \in \partial C$, $z \in \mathcal{U}(x_0)\backslash C$ and a minimizing sequence $\{x_n\} \subset C$ of $x \mapsto \rho_F(x - z) + \theta(x)$ on C we find a minimizer of this function as the limit of $\{x_n\}$, which exists because $\{x_n\}$ is a Cauchy sequence. Assuming that there are two elements $x, y \in \pi_C^{F,\theta}(z)$ with $x \ne y$ we consider the sequence $\{x_n\}$ whose odd terms are equal to x and all even terms are equal to y. Since $\{x_n\}$ is a minimizing sequence for $x \mapsto \rho_F(x - z) + \theta(x)$ on C by the same reasons we also conclude that $\{x_n\}$ converges to $x = y$. In order to show the continuity at $z \in \mathcal{U}$ it is enough to observe that for each $z_n \to z$ the sequence $\left\{\pi_C^{F,\theta}(z_n)\right\}$ minimizes $x \mapsto \rho_F(x - z) + \theta(x)$ on C. Indeed,

$$\begin{aligned}u(z) &\le \rho_F\left(\pi_C^{F,\theta}(z_n) - z\right) + \theta\left(\pi_C^{F,\theta}(z_n)\right) \\ &\le \rho_F\left(\pi_C^{F,\theta}(z_n) - z_n\right) + \rho_F(z_n - z) + \theta\left(\pi_C^{F,\theta}(z_n)\right) \\ &\le u(z) + 2\|F^o\|\,\|z_n - z\| \to u(z) + .\end{aligned}$$

Thus $\pi_C^{F,\theta}(z_n) \to \pi_C^{F,\theta}(z)$, and the theorem is completely proved. ■

Notice that similarly as in Section 3.2 the regularity of the minimizer $\pi_C^{F,\theta}(x)$ with respect to x can be essentially improved.

Theorem 4.2.2 *For fixed $x_0 \in \partial C$ under the same hypotheses as in Theorem 4.2.1, the mapping $\pi_C^{F,\theta}(\cdot)$ is locally Lipschitzean in some neighbourhood of x_0 (outside of C).*

Proof.

We set

$$\mathcal{U}'(x_0) \; : \; = \left\{ z \in H : \|z - x_0\| < \frac{(1 - \gamma \|F\| \|F^o\|)\,\delta}{4\|F\| \|F^o\|}, \right.$$
$$\left. u(z) < \frac{1 - \gamma \|F\| \|F^o\|}{L} + \theta(x_0) - \delta\gamma \|F^o\| \right\} \subset \mathcal{U}(x_0).$$

Let us fix $x \in \mathcal{U}'(x_0) \backslash C$ and let $\tau > 0$, $0 < \overline{\delta} < \frac{\tau}{2\|F^o\|}$ be such that

$$u(x) - \theta(x_0) + \delta\gamma \|F^o\| + \tau < \frac{1 - \gamma \|F\| \|F^o\|}{L};$$

$$x + \overline{\delta}\mathbf{B} \subset \mathcal{U}'(x_0) \backslash C.$$

Take arbitrary $z, w \in x + \overline{\delta}\mathbf{B}$, $z \neq w$, and set $\beta := \frac{\|z-w\|}{2} > 0$; $\overline{z} := \pi_C^{F,\theta}(z)$ and $\overline{w} =: \pi_C^{F,\theta}(w)$. By the condition (**ii**), there exists $0 < \nu \leq \delta'$ such that

$$\|\mathfrak{J}_F(u) - \mathfrak{J}_F(v)\| \leq \beta, \tag{4.30}$$

whenever $u, v \in U_{\delta,\delta'}(x_0)$, with $\|u - v\| \leq \nu$. We set also

$$\varepsilon := \min\left\{ \frac{\tau}{2\|F^o\|}, \nu, \beta \right\}.$$

Since $\overline{z}$ minimizes the functional

$$F(y) := \rho_F(y - z) + \theta(y) + \mathbf{I}_C(y)$$

on H, by the necessary minimum condition (see [23, p. 37]) $0 \in \partial^p F(\overline{z})$. Then by using the fuzzy sum rule (Theorem 2.1.1) we find $z_1, z_2 \in \overline{z} + \varepsilon\mathbf{B}$ with $z_1 \in \partial^* C$ and vectors $v_z \in -\partial F^o \cap \partial^p(\theta + \mathbf{I}_C)(z_1)$, $\xi_z^* \in \partial\rho_F(z_2 - z)$ such that

$$\|v_z + \xi_z^*\| \leq \varepsilon.$$

Notice that the definition of $u(\cdot)$ implies

$$u(z) \leq \rho_F(x_0 - z) + \theta(x_0)$$

and, hence,

$$\begin{aligned}
\frac{\|\overline{z} - x_0\|}{\|F\|} &\leq \rho_F(\overline{z} - x_0) \leq \rho_F(\overline{z} - z) + \rho_F(z - x_0) \\
&= u(z) - \theta(\overline{z}) + \rho_F(z - x_0) \\
&\leq \rho_F(x_0 - z) + \theta(x_0) - \theta(\overline{z}) + \rho_F(z - x_0) \\
&\leq 2\|F^o\| \|z - x_0\| + \gamma \|F^o\| \|x_0 - \overline{z}\|,
\end{aligned}$$

which implies

$$\left(\frac{1}{\|F\|} - \gamma \|F^o\|\right) \|\overline{z} - x_0\| \leq 2 \|F^o\| \|z - x_0\| . \tag{4.31}$$

Consequently

$$\|z_1 - x_0\| \leq \|z_1 - \overline{z}\| + \|\overline{z} - x_0\| \leq \varepsilon + \frac{2\|F^o\| \|F\|}{1 - \gamma \|F^o\| \|F\|} \|z - x_0\| \leq \delta,$$

i.e., $z_1 \in C_\delta(x_0)$. Since $\xi_z^*, -v_z \in \partial F^o \cap \left(-\partial^p (\theta + \mathbf{I}_C)(z_1) \cap \partial F^o + \delta' \overline{\mathbf{B}}\right)$, by (4.30), we obtain

$$\|\mathfrak{J}_F(\xi_z^*) - \mathfrak{J}_F(-v_z)\| \leq \beta$$

(see (4.21)). Similarly, considering w and $\overline{w}$ in the place of z and $\overline{z}$, we find $w_1, w_2 \in \overline{w} + \varepsilon \mathbf{B}$, $w_1 \in \partial^* C$, and

$$v_w \in -\partial F^o \cap \partial^p (\theta + \mathbf{I}_C)(w_1), \quad \xi_w^* \in \partial \rho_F (w_2 - w),$$

satisfying $\|v_w + \xi_w^*\| \leq \varepsilon$. By the same reasons as above we show that $\|\mathfrak{J}_F(\xi_w^*) - \mathfrak{J}_F(-v_w)\| \leq \beta$. Joining together the last two inequalities and taking into account the hypothesis (**i**), we have (see (3.13))

$$\|\mathfrak{J}_F(\xi_z^*) - \mathfrak{J}_F(\xi_w^*)\| \leq 2\beta + L(2\varepsilon + \|\overline{z} - \overline{w}\|). \tag{4.32}$$

On the other hand,

$$\begin{aligned} |\rho_F(\overline{w} - w) - \rho_F(\overline{z} - z)| &\leq |\rho_F(\overline{w} - w) + \theta(\overline{w}) - (\rho_F(\overline{z} - z) + \theta(\overline{z}))| + |\theta(\overline{w}) - \theta(\overline{z})| \\ &\leq |u(w) - u(z)| + \gamma \|F^o\| \|\overline{w} - \overline{z}\| \\ &\leq \|F^o\| \|z - w\| + \gamma \|F^o\| \|\overline{w} - \overline{z}\|, \end{aligned}$$

which implies

$$\begin{aligned} |\rho_F(w_2 - w) - \rho_F(z_2 - z)| &\leq |\rho_F(w_2 - w) - \rho_F(\overline{w} - w)| + |\rho_F(\overline{z} - z) - \rho_F(z_2 - z)| \\ &\quad + |\rho_F(\overline{w} - w) - \rho_F(\overline{z} - z)| \\ &\leq \|F^o\| (\|w_2 - \overline{w}\| + \|\overline{z} - z_2\|) + \|F^o\| \|z - w\| + \gamma \|F^o\| \|\overline{w} - \overline{z}\| \\ &\leq \|F^o\| (2\varepsilon + \|z - w\|) + \gamma \|F^o\| \|\overline{w} - \overline{z}\| . \end{aligned} \tag{4.33}$$

By using the Lipschitz continuity of both $u(\cdot)$ and $\theta(\cdot)$ we have

$$\begin{aligned} \rho_F(z_2 - z) &\leq \rho_F(z_2 - \overline{z}) + \rho_F(\overline{z} - z) \\ &\leq \|F^o\| \|z_2 - \overline{z}\| + u(z) - \theta(\overline{z}) \\ &\leq \|F^o\| \varepsilon + u(x) + \|F^o\| \|x - z\| - \theta(x_0) + \gamma \|F^o\| \|\overline{z} - x_0\| . \end{aligned}$$

Since (4.31) implies, in particular, that $\|\overline{z} - x_0\| \leq \delta$, by the choice $\overline{\delta}$ and ε of we obtain

$$\begin{aligned} \rho_F(z_2 - z) &\leq \|F^o\| \varepsilon + u(x) + \|F^o\| \overline{\delta} - \theta(x_0) + \gamma \|F^o\| \delta \\ &\leq u(x) - \theta(x_0) + \gamma \|F^o\| \delta + \tau. \end{aligned} \tag{4.34}$$

By Proposition 1.1.1 $\mathfrak{J}_F(\xi_z^*) = \frac{z_2-z}{\rho_F(z_2-z)}$ and $\mathfrak{J}_F(\xi_w^*) = \frac{w_2-w}{\rho_F(w_2-w)}$. Hence, by using, respectively, (4.34), (4.32) and (4.33) we obtain

$$\begin{aligned}
\|\overline{z}-\overline{w}\| &\leq \|\overline{z}-z_2\| + \|z_2-w_2\| + \|w_2-\overline{w}\| \\
&\leq 2\varepsilon + \|\mathfrak{J}_F(\xi_z^*)\,\rho_F(z_2-z) - \mathfrak{J}_F(\xi_w^*)\,\rho_F(w_2-w)\| + \|z-w\| \\
&\leq 2\varepsilon + \rho_F(z_2-z)\,\|\mathfrak{J}_F(\xi_z^*) - \mathfrak{J}_F(\xi_w^*)\| + \|\mathfrak{J}_F(\xi_w^*)\|\,|\rho_F(z_2-z) - \rho_F(w_2-w)| + \\
&\quad + \|z-w\| \\
&\leq 2\varepsilon + (u(x) - \theta(x_0) + \gamma\|F^o\|\,\delta + \tau)\,(2\beta + L(2\varepsilon + \|\overline{z}-\overline{w}\|)) \\
&\quad + \|F\|\,(\|F^o\|\,(2\varepsilon + \|z-w\|) + \gamma\|F^o\|\,\|\overline{w}-\overline{z}\|) + \|z-w\|\,.
\end{aligned}$$

Now taking into account the definitions of β and ε we obtain

$$(1 - (u(x) - \theta(x_0) + \gamma\|F^o\|\,\delta + \tau)\,L - \gamma\|F^o\|\,\|F\|)\,\|\overline{z}-\overline{w}\| \leq \mathfrak{k}\,\|z-w\|\,,$$

where

$$\mathfrak{k} = 2(1 + \|F\|\,\|F^o\|) + (u(x) - \theta(x_0) + \gamma\|F^o\|\,\delta + \tau)\,(1+L)\,.$$

In the case $z = w$ the same inequality holds due to uniqueness of the minimizer. Thus, $\pi_C^{F,\theta}(\cdot)$ is Lipschitz continuous on $x + \overline{\delta}\mathbf{B}$ with the Lipschitz constant $\mathfrak{k}/\nu$, where

$$\nu := 1 - (u(x) - \theta(x_0) + \gamma\|F^o\|\,\delta + \tau)\,L - \gamma\|F^o\|\,\|F\| > 0.$$

■

Remark 4.2.2 *Notice that the Lipschitz constant of the mapping $\pi_C^{F,\theta}(\cdot)$ depends essentially on the distance from the boundary of the neighbourhood $\mathcal{U}(x_0)$ controlled by the parameters τ and ν, tending to $+\infty$ whenever the strict inequality $u(z) < \frac{1-\gamma\|F\|\|F^o\|}{L} + \theta(x_0) - \delta\gamma\|F^o\|$ tends to become an equality.*

4.3 Regularity of the viscosity solution near the boundary

In this section we study the differentiability of the function $u(\cdot)$ through the regularity of the mapping $\pi_C^{F,\theta}(\cdot)$. Here we assume that $\theta(\cdot)$ is defined and continuous on the whole space H and satisfies the relation $\partial^p\theta(x) \subset -\gamma F^o$ for all $x \in \overline{\text{co}}C$, where $0 < \gamma < 1$ is some suitable constant.

We start with a technical result, adapting Proposition 2.6 and Corollary 2.7 from [31].

Lemma 4.3.1 *Suppose $x \notin C$.*

(i) For all $\xi \in F$ and $t \geq 0$

$$u(x - t\xi) \leq u(x) + t.$$

(ii) Assume that $\pi_C^{F,\theta}(x) = \{\overline{x}\}$. For $\overline{\xi} := \frac{\overline{x}-x}{\rho_F(\overline{x}-x)} \in \partial F$ and $0 < t < \rho_F(\overline{x}-x)$

$$u(x + t\overline{\xi}) \leq u(x) - t.$$

Proof.

(i) Let $\xi \in F$, $t \geq 0$ and $\varepsilon > 0$. There exists $y \in C$ such that $\rho_F(y-x) + \theta(y) < u(x) + \varepsilon$. By subadditivity and positive homogeneity of the Minkowski functional, we have

$$u(x - t\xi) \leq \rho_F(y - x + t\xi) + \theta(y) \leq \rho_F(y-x) + t\rho_F(\xi) + \theta(y) < u(x) + \varepsilon + t.$$

Letting $\varepsilon \to 0+$ proves (**i**).

(**ii**) Let us define $y_t := x + t\overline{\xi}$, $t > 0$. We have

$$\begin{aligned} u(y_t) &\leq \rho_F(\overline{x} - y_t) + \theta(\overline{x}) = \rho_F\left(\overline{x} - x - t\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right) + \theta(\overline{x}) \\ &= \rho_F\left((\overline{x}-x)\left(1 - \frac{t}{\rho_F(\overline{x}-x)}\right)\right) + \theta(\overline{x}) \\ &= \rho_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}(\rho_F(\overline{x}-x) - t)\right) + \theta(\overline{x}) \\ &= \rho_F(\overline{x}-x) - t + \theta(\overline{x}) = u(x) - t, \end{aligned}$$

and the lemma is proved. ■

Theorem 4.3.1 *Let us fix $x \notin C$ and assume that the mapping $z \mapsto \pi_C^{F,\theta}(z)$ is well-defined and single-valued in some neighbourhood $U(x)$ of x where also the following "one-point" Hölder property holds: there are constants $K = K(x) > 0$ and $\frac{1}{2} < \beta \leq 1$ such that*

$$\left\|\pi_C^{F,\theta}(y) - \pi_C^{F,\theta}(x)\right\| \leq K\|y-x\|^\beta \quad \textit{for all } y \in U(x).$$

If the function $(\theta + \mathbf{I}_C)(\cdot)$ is proximally regular at each point $y \in C_\delta(\overline{x}) := \partial C \cap (\overline{x} + \delta\mathbf{B})$ for some $\delta > 0$ where $\overline{x} := \pi_C^{F,\theta}(x)$ then the function $u(\cdot)$ is Clarke (and lower) regular at x.

Proof.

Let us prove the following equalities

$$\partial^c u(x) = \partial^f u(x) = -\partial\rho_F(\overline{x}-x) \cap \partial^f(\theta + \mathbf{I}_C)(\overline{x}). \tag{4.35}$$

We divide the proof into several parts.

(**i**) We start proving that $\partial^f u(x) \subset -\partial\rho_F(\overline{x}-x)$. Notice that, by Proposition 1.1.1,

$$\partial\rho_F(\overline{x}-x) = \partial F^o \cap \mathbf{N}_F\left(\frac{\overline{x}-x}{\rho_F(\overline{x}-x)}\right).$$

Let us fix $v \in \partial^f u(x)$ and $\varepsilon, \delta > 0$ such that

$$\frac{u(y) - u(x) - \langle v, y-x\rangle}{\|x-y\|} \geq -\varepsilon \tag{4.36}$$

for all $y \in x + \delta\overline{\mathbf{B}}$. Now fix $\xi \in F$ and denote by $y_t := x - t\xi$, $t > 0$. By Lemma 4.3.1 (i) we have $u(x - t\xi) - u(x) \leq t$. So, by (4.36), we get

$$\begin{aligned} \frac{1 + \langle v, \xi \rangle}{\|\xi\|} &= \frac{t + \langle v, t\xi \rangle}{\|t\xi\|} \\ &\geq \frac{u(x - t\xi) - u(x) + \langle v, t\xi \rangle}{\|t\xi\|} \\ &= \frac{u(y_t) - u(x) - \langle v, y_t - x \rangle}{\|y_t - x\|} \geq -\varepsilon \end{aligned}$$

for every $t > 0$ small enough. By letting $\varepsilon \to 0+$ we obtain

$$1 \geq \langle -v, \xi \rangle \tag{4.37}$$

and consequently

$$\rho_{F^o}(-v) = \sup_{\xi \in F} \langle -v, \xi \rangle \leq 1.$$

For the opposite inequality remind that by Proposition 4.1.3 $u(\cdot)$ is the (unique) viscosity solution of (4.1). This implies, in particular, that $\rho_{F^o}(-v) \geq 1$ for all $v \in \partial^f u(x)$. Hence $\partial^f u(x) \subset -\partial F^o$. Now let us fix $v \in \partial^f u(x)$ and prove that $v \in \mathbf{N}_F(\overline{\xi})$ where $\overline{\xi} := \frac{\overline{x} - x}{\rho_F(\overline{x} - x)}$. Choose again $\varepsilon, \delta > 0$ such that the inequality (4.36) holds for all $y \in x + \delta\overline{\mathbf{B}}$. Consider $y_t := x + t\overline{\xi}$, $0 < t < \rho_F(\overline{x} - x)$. By using Lemma 4.3.1 (ii) and (4.36) we obtain

$$\begin{aligned} u(x) - t &\geq u(y_t) \\ &\geq -\varepsilon \|x - y_t\| + \langle v, y_t - x \rangle + u(x) \\ &= -\varepsilon t \left\|\overline{\xi}\right\| + t \left\langle v, \overline{\xi} \right\rangle + u(x), \end{aligned}$$

implying that

$$-1 \geq -\varepsilon \left\|\overline{\xi}\right\| + \left\langle v, \overline{\xi} \right\rangle.$$

Now passing to the limit as $\varepsilon \to 0+$ we find $\left\langle v, \overline{\xi} \right\rangle \leq -1$ and, consequently (see (4.37)),

$$\left\langle v, \overline{\xi} \right\rangle = -1.$$

On the other hand, as we have already shown $\langle -v, \xi \rangle \leq 1$ for every $\xi \in F$. So that

$$\left\langle -v, \xi - \overline{\xi} \right\rangle = \langle -v, \xi \rangle - \left\langle -v, \overline{\xi} \right\rangle \leq 0$$

for all $\xi \in F$, which means $-v \in \mathbf{N}_F\left(\overline{\xi}\right)$.

(ii) Now we fix $v \in \partial^f u(x)$. Let us denote by $y_s := s - \overline{x} + x$, for $s \in C$. Since $\overline{x} \in C$, $y_s \to x$ is equivalent to $s \to \overline{x}$ and

$$u(y_s) \leq \rho_F(s - y_s) + \theta(s) = \rho_F(\overline{x} - x) + \theta(s)$$

we successively obtain

$$\begin{aligned}
0 &\leq \liminf_{x'\to x}\frac{u(x')-u(x)-\langle v,x'-x\rangle}{\|x-x'\|}\\
&\leq \liminf_{\substack{s\to\overline{x}\\ s\in C}}\frac{u(y_s)-u(x)-\langle v,y_s-x\rangle}{\|x-y_s\|}\\
&\leq \liminf_{\substack{s\to\overline{x}\\ s\in C}}\frac{\rho_F(\overline{x}-x)+\theta(s)-\rho_F(\overline{x}-x)-\theta(\overline{x})-\langle v,s-\overline{x}\rangle}{\|s-\overline{x}\|}\\
&= \liminf_{\substack{s\to\overline{x}\\ s\in C}}\frac{\theta(s)-\theta(\overline{x})-\langle v,s-\overline{x}\rangle}{\|s-\overline{x}\|}\\
&= \liminf_{s\to\overline{x}}\frac{(\theta+\mathbf{I}_C)(s)-(\theta+\mathbf{I}_C)(\overline{x})-\langle v,s-\overline{x}\rangle}{\|s-\overline{x}\|},
\end{aligned}$$

concluding that $v\in\partial^f(\theta+\mathbf{I}_C)(\overline{x})$. Thus, we have already proved that

$$\partial^f u(x)\subset-\partial\rho_F(\overline{x}-x)\cap\partial^f(\theta+\mathbf{I}_C)(\overline{x}). \tag{4.38}$$

Furthermore, since the mapping $z\mapsto\pi_C^{F,\theta}(z)$ is well-defined and single-valued in some neighbourhood $U(x)$ of x we have

$$\partial^f u(y)\subset-\partial\rho_F(\overline{y}-y)\cap\partial^f(\theta+\mathbf{I}_C)(\overline{y}) \tag{4.39}$$

for every y enough close to x.

(iii) Let us prove the inclusion $-\partial\rho_F(\overline{x}-x)\cap\partial^p(\theta+\mathbf{I}_C)(\overline{x})\subset\partial^f u(x)$. To this end we fix v from its left-hand side. Then there exist constants $\eta>0$ and $\sigma>0$ such that both relations

$$(\theta+\mathbf{I}_C)(w)\geq(\theta+\mathbf{I}_C)(\overline{x})+\langle v,w-\overline{x}\rangle-\sigma\|w-\overline{x}\|^2\qquad\forall w\in\overline{x}+\eta\overline{\mathbf{B}} \tag{4.40}$$

and

$$\rho_F(z)\geq\rho_F(\overline{x}-x)+\langle -v,z-\overline{x}+x\rangle\qquad\forall z\in H. \tag{4.41}$$

hold. Setting in (4.41) $z=\overline{y}-y$ for y sufficiently close to x we have, in particular,

$$\rho_F(\overline{y}-y)-\rho_F(\overline{x}-x)-\langle v,y-x\rangle\geq\langle -v,\overline{y}-\overline{x}\rangle.$$

Hence, by (4.40) and the Hölderianity of $\pi_C^{F,\theta}(\cdot)$ we obtain

$$\begin{aligned}
&\liminf_{y\to x}\frac{u(y)-u(x)-\langle v,y-x\rangle}{\|y-x\|}\\
&= \liminf_{y\to x}\frac{\rho_F(\overline{y}-y)+\theta(\overline{y})-\rho_F(\overline{x}-x)-\theta(\overline{x})-\langle v,y-x\rangle}{\|y-x\|}\\
&\geq \liminf_{y\to x}\frac{\langle -v,\overline{y}-\overline{x}\rangle+\theta(\overline{y})-\theta(\overline{x})}{\|y-x\|}\\
&\geq \liminf_{y\to x}\frac{-\sigma\|\overline{y}-\overline{x}\|^2}{\|y-x\|}\geq\liminf_{y\to x}-\sigma\mathfrak{h}\|y-x\|^{2\beta-1}=0,
\end{aligned}$$

where $\mathfrak{h}$ is the Hölder constant of $\pi_C^{F,\theta}(\cdot)$.

(iv) Finally, the inclusion $\partial^f u(x) \subset \partial^c u(x)$ obviously holds (see (2.4)). In order to show the opposite inclusion, similarly as in Theorem 3.3.1, we represent the subdifferential $\partial^c u(x)$ through the proximal subdifferentials at close points (see (2.3)) and, using the relation (4.39), we arrive at

$$\begin{aligned} \partial^c u(x) &= \overline{\text{co}}\left\{ w - \lim_{i\to\infty} \xi_i,\ \xi_i \in \partial^p u(x_i),\ x_i \to x \right\} \\ &\subset \overline{\text{co}}\left\{ w - \lim_{i\to\infty} \xi_i,\ \xi_i \in \partial^f u(x_i),\ x_i \to x \right\} \\ &\subset \overline{\text{co}}\left\{ w - \lim_{i\to\infty} \xi_i,\ \xi_i \in -\partial\rho_F(\overline{x}_i - x_i) \cap \partial^f(\theta + \mathbf{I}_C)(\overline{x}_i),\ x_i \to x \right\}. \end{aligned} \tag{4.42}$$

Since the subdifferential of the convex function $\rho_F(\cdot)$ has strongly$\times$weakly closed graph, by using the proximal regularity of $(\theta + \mathbf{I}_C)(\cdot)$ near the point $\overline{x}$ and the definition of the limiting subdifferential we deduce from (4.42) that

$$\begin{aligned} \partial^c u(x) &\subset \overline{\text{co}}\left(-\partial\rho_F(\overline{x} - x) \cap \partial^l(\theta + \mathbf{I}_C)(\overline{x})\right) \\ &= -\partial\rho_F(\overline{x} - x) \cap \partial^f(\theta + \mathbf{I}_C)(\overline{x}). \end{aligned}$$

The last equality follows from the convexity and the closedness of both subdifferentials $\partial\rho_F(\overline{x} - x)$ and $\partial^f(\theta + \mathbf{I}_C)(\overline{x}) = \partial^l(\theta + \mathbf{I}_C)(\overline{x})$. Finally, from the proximal regularity and from the item (iii) above we conclude

$$\partial^c u(x) \subset -\partial\rho_F(\overline{x} - x) \cap \partial^p(\theta + \mathbf{I}_C)(\overline{x}) \subset \partial^f u(x) \subset \partial^c u(x).$$

Since the inclusions $\partial^f u(x) \subset \partial^l u(x) \subset \partial^c u(x)$ are always true (see (2.4)) we have, in fact, the equalities

$$\partial^f u(x) = \partial^l u(x) = \partial^c u(x)$$

and the Clarke and the lower regularity of the function $u(\cdot)$ at x follow. ■

Theorem 4.3.2 *Let us fix $x \notin C$ and assume that the mapping $z \mapsto \pi_C^{F,\theta}(z)$ is well-defined, single-valued and Hölder continuous with an exponent $\frac{1}{2} < \beta \leq 1$ in some neighbourhood $U(x)$ of x. Assume, moreover, that the function $(\theta + \mathbf{I}_C)(\cdot)$ is proximally regular at each point $y \in C_\delta(\overline{x})$, where $\overline{x} := \pi_C^{F,\theta}(x)$. Then the function $u(\cdot)$ is (Fréchet) continuously differentiable at x if there exists a continuous mapping $\nu : U(x) \to \partial F^o$ such that*

$$-\partial^p(\theta + \mathbf{I}_C)(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y} - y}{\rho_F(\overline{y} - y)}\right) \cap \partial F^o = \{\nu(y)\} \tag{4.43}$$

where $\overline{y} := \pi_C^F(y)$, $y \in U(x)$. In this case $\nabla u(x) = -\nu(x)$.

Proof.

Similarly to the proof of Theorem 3.3.2, the statement follows from Theorem 4.3.1 taking into account that (see Proposition 1.1.1)

$$-\partial^p (\theta + \mathbf{I}_C)(\overline{y}) \cap \partial\rho_F(\overline{y} - y) = -\partial^p (\theta + \mathbf{I}_C)(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y} - y}{\rho_F(\overline{y} - y)}\right) \cap \partial F^o$$

for every $y \in U(x)$. ■

For convenience of further references let us formulate the following obvious statement.

Corollary 4.3.3 *Under all the assumptions and notations of Theorem 4.3.2 suppose, moreover, that the single-valued mapping*

$$y \mapsto -\partial^p (\theta + \mathbf{I}_C)(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y} - y}{\rho_F(\overline{y} - y)}\right) \cap \partial F^o$$

is Hölder continuous with an exponent $0 < \alpha \leq 1$ in a neighbourhood of $x \notin C$. Then the function $u(\cdot)$ is of class $C^{1,\alpha}$ in this neighbourhood.

Remark 4.3.1 *Notice that if the function $\theta(\cdot)$ and the set C are proximally regular at $x \in \partial C$ then the function $(\theta + \mathbf{I}_C)(\cdot)$ is also proximally regular at x.*

The condition (4.43) splits into the following two particular cases.

Corollary 4.3.4 *Assume as in Theorem 4.3.2 that the mapping $\pi_C^{F,\theta}(\cdot)$ is single-valued and Hölder continuous with an exponent $1/2 < \beta \leq 1$ in a neighbourhood $U(x)$ of a point $x \notin C$, and that the function $(\theta + \mathbf{I}_C)(\cdot)$ is proximally regular at each $y \in C_\delta(\overline{x})$, $\overline{x} := \pi_C^{F,\theta}(x)$, $\delta > 0$. Then the function $u(\cdot)$ is (Fréchet) continuously differentiable at x if at least one of the following conditions holds:*

(i) C has smooth boundary at $\overline{x}$, and $\theta(\cdot)$ is of class C^1 at $\overline{x}$;

(ii) F is uniformly smooth at $\xi := \frac{\overline{x} - x}{\rho_F(\overline{x} - x)}$.

Furthermore, in the first case we have

$$\nabla u(x) = \nabla\theta(\overline{x}) + \lambda \mathrm{n}_C(\overline{x}), \tag{4.44}$$

where $\lambda = \lambda(\overline{x}) > 0$ is the unique positive root of the equation $\rho_{F^o}(-\nabla\theta(\overline{x}) - \lambda \mathrm{n}_C(\overline{x})) = 1$ and $\mathrm{n}_C(\overline{x})$ is, as usual, the (unique) unit normal vector to ∂C at $\overline{x}$, while in the second

$$\nabla u(x) = -\nabla\rho_F\left(\frac{\overline{x} - x}{\rho_F(\overline{x} - x)}\right). \tag{4.45}$$

Proof.

In the case (i) we have

$$\partial^l (\theta + \mathbf{I}_C)(\overline{y}) = \nabla\theta(\overline{y}) + \mathbf{N}^l_C(\overline{y})$$

(see [66, p. 112]) for every y enough close to x. Since C has smooth boundary at $\overline{x}$ then the nonempty (see (4.35))

$$-\partial^l (\theta + \mathbf{I}_C)(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o \tag{4.46}$$

can be reduced to $\{-\nabla\theta(\overline{y}) - \lambda \mathrm{n}_C(\overline{y})\}$, where $\lambda > 0$ is the such that$\rho_{F^o}(-\nabla\theta(\overline{x}) - \lambda \mathrm{n}_C(\overline{x})) = 1$. While in the case (ii), by Remark 1.3.1, we have

$$-\partial^l (\theta + \mathbf{I}_C)(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right) \cap \partial F^o = \left\{\nabla\rho_F\left(\frac{\overline{y}-y}{\rho_F(\overline{y}-y)}\right)\right\}$$

>From (4.35) also the formulas (4.44) and (4.45) follow. Let us prove now the continuity of the gradient $\nabla u(\cdot)$. We consider only the case (i). The case (ii) can be treated similarly. Given a sequence $\{x_n\}$ converging to x we represent

$$\nabla u(x_n) = \nabla\theta(\overline{x}_n) + \lambda_n \mathrm{n}_C(\overline{x}_n) \tag{4.47}$$

for some $\lambda_n > 0$, $n \geq 1$. Since $-\nabla u(x_n) \in \partial F^o$, we have

$$\begin{aligned} 1 &= \rho_{F^o}(-\nabla\theta(\overline{x}_n) - \lambda_n \mathrm{n}_C(\overline{x}_n)) \\ &\geq \lambda_n \rho_{F^o}(-\mathrm{n}_C(\overline{x}_n)) - \rho_{F^o}(\nabla\theta(\overline{x}_n)) \\ &\geq \frac{\lambda_n}{\|F^o\|} - \gamma \|F\| \, \|F^o\|, \end{aligned}$$

and $\{\lambda_n\}$ is bounded. Let $\lambda \geq 0$ be its cluster point. Without loss of generality we assume that $\lambda_n \to \lambda$. Hence, by the smoothness properties of $\theta(\cdot)$ and C we conclude from (4.47) that $\nabla u(x_n) = \nabla\theta(\overline{x}_n) + \lambda_n \mathrm{n}_C(\overline{x}_n)$ converges to $\nabla\theta(\overline{x}) + \lambda \mathrm{n}_C(\overline{x}) \in -\partial F^o$. On the other hand, by the definition of the normal cone to the convex set F we have

$$\left\langle y - \frac{\overline{x}_n - x_n}{\rho_F(\overline{x}_n - x_n)}, \nabla u(x_n)\right\rangle \leq 0 \quad \forall y \in F,$$

where $\overline{x}_n = \pi^{F,\theta}_C(x_n) \to \pi^{F,\theta}_C(x) = \overline{x}$ by a hypothesis of theorem. Consequently,

$$\left\langle y - \frac{\overline{x} - x}{\rho_F(\overline{x} - x)}, \nabla\theta(\overline{x}) + \lambda \mathrm{n}_C(\overline{x})\right\rangle \leq 0 \quad \forall y \in F.$$

Since the intersection (4.46) is reduced to the singleton $\{\nabla u(x)\}$, we finally conclude $\nabla\theta(\overline{x}) + \lambda \mathrm{n}_C(\overline{x}) = \nabla u(x)$, and the continuity of the derivative follows. ■

Proving the next statement we need the properties of the Hausdorff distance between convex closed sets. Let us remind some definitions. For nonempty sets $A, B \subset H$ the *Hausdorff distance* is defined by

$$\mathcal{D}(A,B) := \max\left\{\sup_{x\in A} \mathrm{d}_B(x), \sup_{y\in B} \mathrm{d}_A(y)\right\},$$

where, as usual, $\mathrm{d}_A(x) := \inf\{\|x-y\| : y \in A\}$, or, equivalently,

$$\mathcal{D}(A,B) := \inf\{r > 0 : A \subset B + r\overline{\mathbf{B}} \text{ and } B \subset A + r\overline{\mathbf{B}}\}.$$

>From the last definition in the case of convex closed sets we can prove the useful inequality

$$\mathcal{D}(A,B) \geq \sup_{\|v\|=1} |\sigma_A(v) - \sigma_B(v)|. \tag{4.48}$$

Indeed, given any $v \in H$ with $\|v\| = 1$ and any $r > 0$ such that $A \subset B + r\overline{\mathbf{B}}$ and $B \subset A + r\overline{\mathbf{B}}$ we have

$$\sigma_A(v) \leq \sigma_{B+r\overline{\mathbf{B}}}(v) \leq \sigma_B(v) + r\|v\| = \sigma_B(v) + r$$

and similarly $\sigma_B(v) \leq \sigma_A(v) + r$. Hence

$$|\sigma_A(v) - \sigma_B(v)| \leq r.$$

Taking the supremum in v and the infimum in r we obtain (4.48).

Corollary 4.3.5 *Under the same hypotheses as in Corollary 4.3.4 the following statements hold*

(i) if C has smooth boundary at $\overline{x} := \pi_C^{F,\theta}(x)$ and both $\mathfrak{n}_C(\cdot)$ and $\nabla\theta(\cdot)$ are Hölderian near $\overline{x}$ with exponents $0 < \alpha_C \leq 1$ and $0 < \alpha_{\nabla\theta} \leq 1$, respectively, then $u(\cdot)$ is of class $C^{1,\alpha\beta}$ near the point x, where $\alpha := \min\{\alpha_C, \alpha_{\nabla\theta}\}$;

(ii) if F is uniformly smooth at $\xi := \frac{\overline{x}-x}{\rho_F(\overline{x}-x)}$ and $\nabla\rho_F(\cdot)$ is Hölderian near ξ with an exponent $0 < \alpha \leq 1$ then $u(\cdot)$ is of class $C^{1,\alpha\beta}$ near x.

Proof.
Let us denote by $\mathfrak{h}_\pi = \mathfrak{h}_\pi(x)$, $\mathfrak{h}_C = \mathfrak{h}_C(\overline{x})$, $\mathfrak{h}_F = \mathfrak{h}_F(\xi)$ and $\mathfrak{h}_{\nabla\theta} = \mathfrak{h}_{\nabla\theta}(\overline{x})$ the Hölder constants of the functions $\pi_C^{F,\theta}(\cdot)$, $\mathfrak{n}_C(\cdot)$, $\mathfrak{n}_F(\cdot)$ and $\nabla\theta(\cdot)$ respectively.

(i) Let $\overline{\delta} > 0$ be such that $\mathfrak{n}_C(\cdot)$ and $\nabla\theta(\cdot)$ are well defined and Hölder continuous on $x + \overline{\delta}\mathbf{B}$. Given $x_1, x_2 \in x + \overline{\delta}\mathbf{B}$ by using the formula (4.44) we obtain

$$\|\nabla u(x_1) - \nabla u(x_2)\| \leq \|\nabla\theta(\overline{x}_1) - \nabla\theta(\overline{x}_2)\| + \|\lambda_1\mathfrak{n}_C(\overline{x}_1) - \lambda_2\mathfrak{n}_C(\overline{x}_2)\|,$$

where $\lambda_i > 0$ is the unique positive root of the equation

$$\rho_{F^\circ}(-\nabla\theta(\overline{x}_i) - \lambda_i\mathfrak{n}_C(\overline{x}_i)) = 1, \quad i = 1,2.$$

But

$$\|\nabla\theta(\overline{x}_1) - \nabla\theta(\overline{x}_2)\| \leq \mathfrak{h}_{\nabla\theta}\|\overline{x}_1 - \overline{x}_2\|^{\alpha_{\nabla\theta}} \leq \mathfrak{h}_{\nabla\theta}\mathfrak{h}_\pi\|x_1 - x_2\|^{\alpha_{\nabla\theta}\beta} \tag{4.49}$$

and

$$\|\lambda_1 \mathfrak{n}_C(\overline{x}_1) - \lambda_2 \mathfrak{n}_C(\overline{x}_2)\| \le \lambda_1 \|\mathfrak{n}_C(\overline{x}_1) - \mathfrak{n}_C(\overline{x}_2)\| + |\lambda_1 - \lambda_2| \|\mathfrak{n}_C(\overline{x}_2)\|, \tag{4.50}$$

with

$$\|\mathfrak{n}_C(\overline{x}_1) - \mathfrak{n}_C(\overline{x}_2)\| \le \mathfrak{h}_C \|\overline{x}_1 - \overline{x}_2\|^{\alpha_C} \le \mathfrak{h}_C \mathfrak{h}_\pi \|x_1 - x_2\|^{\alpha_C \beta}. \tag{4.51}$$

By the definition of λ_i, $i = 1, 2$, we have

$$\begin{aligned} \rho_{F^o}(-\nabla\theta(\overline{x}_i) - \lambda_i \mathfrak{n}_C(\overline{x}_i)) = 1 \ &\Leftrightarrow\ -\lambda_i \mathfrak{n}_C(\overline{x}_i) \in \partial F^o + \nabla\theta(\overline{x}_i) \\ &\Leftrightarrow\ \rho_{F^o + \nabla\theta(\overline{x}_i)}(-\lambda_i \mathfrak{n}_C(\overline{x}_i)) = 1 \\ &\Leftrightarrow\ \rho_{F^o + \nabla\theta(\overline{x}_i)}(-\mathfrak{n}_C(\overline{x}_i)) = \frac{1}{\lambda_i}, \end{aligned} \tag{4.52}$$

and

$$\rho_i := \rho_{F^o + \nabla\theta(\overline{x}_i)}(-\mathfrak{n}_C(\overline{x}_i)) \ge \frac{1}{\|F^o + \nabla\theta(\overline{x}_i)\|} \|\mathfrak{n}_C(\overline{x}_i)\| \ge \frac{1}{(1+\gamma)\|F^o\|}. \tag{4.53}$$

Hence

$$\begin{aligned} |\lambda_1 - \lambda_2| &= \frac{1}{\rho_1 \rho_2} |\rho_1 - \rho_2| \\ &\le (1+\gamma)^2 \|F^o\|^2 \Big(\left| \rho_{F^o + \nabla\theta(\overline{x}_1)}(-\mathfrak{n}_C(\overline{x}_1)) - \rho_{F^o + \nabla\theta(\overline{x}_1)}(-\mathfrak{n}_C(\overline{x}_2)) \right| + \\ &\quad + \left| \rho_{F^o + \nabla\theta(\overline{x}_1)}(-\mathfrak{n}_C(\overline{x}_2)) - \rho_{F^o + \nabla\theta(\overline{x}_2)}(-\mathfrak{n}_C(\overline{x}_2)) \right| \Big). \end{aligned} \tag{4.54}$$

Now by (4.51) and taking into account that

$$(F^o + \nabla\theta(\overline{x}_1))^o \subset \frac{\|F\|}{1-\gamma} \overline{\mathbf{B}}$$

(see (4.15)) we obtain

$$\begin{aligned} &\left| \rho_{F^o + \nabla\theta(\overline{x}_1)}(-\mathfrak{n}_C(\overline{x}_1)) - \rho_{F^o + \nabla\theta(\overline{x}_1)}(-\mathfrak{n}_C(\overline{x}_2)) \right| \\ \le\ & \|(F^o + \nabla\theta(\overline{x}_1))^o\| \|\mathfrak{n}_C(\overline{x}_1) - \mathfrak{n}_C(\overline{x}_2)\| \\ \le\ & \frac{\|F\|}{1-\gamma} \mathfrak{h}_C \mathfrak{h}_\pi \|x_1 - x_2\|^{\alpha_C \beta}. \end{aligned} \tag{4.55}$$

On the other hand, using the relation (1.1) and (4.48) we have

$$\begin{aligned} &\left| \rho_{F^o + \nabla\theta(\overline{x}_1)}(-\mathfrak{n}_C(\overline{x}_2)) - \rho_{F^o + \nabla\theta(\overline{x}_2)}(-\mathfrak{n}_C(\overline{x}_2)) \right| \\ =\ & \left| \sigma_{(F^o + \nabla\theta(\overline{x}_1))^o}(-\mathfrak{n}_C(\overline{x}_2)) - \sigma_{(F^o + \nabla\theta(\overline{x}_2))^o}(-\mathfrak{n}_C(\overline{x}_2)) \right| \\ \le\ & \sup_{\|v\|=1} \left| \sigma_{(F^o + \nabla\theta(\overline{x}_1))^o}(v) - \sigma_{(F^o + \nabla\theta(\overline{x}_2))^o}(v) \right| \\ \le\ & \mathcal{D}\left((F^o + \nabla\theta(\overline{x}_1))^o, (F^o + \nabla\theta(\overline{x}_2))^o \right). \end{aligned}$$

By Lemma 2 in [40] (which can be easily adapted to a Hilbert space) we have

$$\begin{aligned} &\left|\rho_{F^o+\nabla\theta(\overline{x}_1)}\left(-\mathfrak{n}_C\left(\overline{x}_2\right)\right)-\rho_{F^o+\nabla\theta(\overline{x}_2)}\left(-\mathfrak{n}_C\left(\overline{x}_2\right)\right)\right| \\ \leq\ & \left(\frac{\|F\|}{1-\gamma}\right)^2\left\|\nabla\theta\left(\overline{x}_1\right)-\nabla\theta\left(\overline{x}_2\right)\right\| \\ \leq\ & \left(\frac{\|F\|}{1-\gamma}\right)^2\mathfrak{h}_{\nabla\theta}\mathfrak{h}_\pi\left\|x_1-x_2\right\|^{\alpha_{\nabla\theta}\beta}. \end{aligned} \tag{4.56}$$

Therefore, by (4.54), (4.55), (4.56), respectively, we obtain

$$|\lambda_1-\lambda_2|\leq(1+\gamma)^2\left\|F^o\right\|^2\left(\frac{\|F\|}{1-\gamma}\mathfrak{h}_C\mathfrak{h}_\pi\left\|x_1-x_2\right\|^{\alpha_C\beta}+\left(\frac{\|F\|}{1-\gamma}\right)^2\mathfrak{h}_{\nabla\theta}\mathfrak{h}_\pi\left\|x_1-x_2\right\|^{\alpha_{\nabla\theta}\beta}\right),$$

and by (4.50), (4.53), (4.52), (4.51) we have

$$\begin{aligned} &\left\|\lambda_1\mathfrak{n}_C\left(\overline{x}_1\right)-\lambda_2\mathfrak{n}_C\left(\overline{x}_2\right)\right\| \\ \leq\ & 2\left\|F^o\right\|\mathfrak{h}_C\mathfrak{h}_\pi\left\|x_1-x_2\right\|^{\alpha_C\beta}+ \\ &+(1+\gamma)^2\left\|F^o\right\|^2\frac{\|F\|}{1-\gamma}\mathfrak{h}_\pi\left(\mathfrak{h}_C\left(2\delta\right)^{(\alpha_C-\alpha)\beta}+\frac{\|F\|}{1-\gamma}\mathfrak{h}_{\nabla\theta}^{(\alpha_{\nabla\theta}-\alpha)\beta}\left(2\delta\right)\right)\left\|x_1-x_2\right\|^{\alpha\beta}, \end{aligned}$$

where

$$\alpha:=\min\left\{\alpha_C,\alpha_{\nabla\theta}\right\}.$$

Consequently

$$\begin{aligned} \left\|\nabla u\left(x_1\right)-\nabla u\left(x_2\right)\right\| &\leq \left\|\nabla\theta\left(\overline{x}_1\right)-\nabla\theta\left(\overline{x}_2\right)\right\|+\left\|\lambda_1\mathfrak{n}_C\left(\overline{x}_1\right)-\lambda_2\mathfrak{n}_C\left(\overline{x}_2\right)\right\| \\ &\leq \overline{\mathfrak{h}}\left\|x_1-x_2\right\|^{\alpha\beta}, \end{aligned}$$

for some $\mathfrak{h}>0$.

(ii) In this case let us choose $\overline{\delta}>0$ such that

$$\overline{\delta}+\gamma\mathfrak{h}_\pi\overline{\delta}^\beta<\frac{u\left(x\right)-\theta\left(\overline{x}\right)}{\left\|F^o\right\|}$$

and $\nabla\rho_F\left(\cdot\right)$ is Hölder continuous on

$$\left\{\frac{\overline{y}-y}{\rho_F\left(\overline{y}-y\right)}:y\in x+\overline{\delta}\mathbf{B}\right\}.$$

Using the formula (4.45) for each $x_1,x_2\in x+\overline{\delta}\mathbf{B}$ and proceeding in the same way as in the proof of Corollary 3.3.5 (ii) we obtain

$$\left\|\nabla u\left(x_1\right)-\nabla u\left(x_2\right)\right\|\leq\mathfrak{h}_F\left\|\xi_1-\xi_2\right\|^\alpha,$$

where $\mathfrak{h}_F>0$ is the Hölder constant of $\nabla\rho_F\left(\cdot\right)$ near ξ, and $\xi_i:=\frac{\overline{x}_i-x_i}{\rho_F\left(\overline{x}_i-x_i\right)}$, $i=1,2$. Moreover,

$$\left\|\xi_1-\xi_2\right\|\leq\frac{\|F\|\left\|F^o\right\|+1}{\rho_2}\left(\mathfrak{h}_\pi+\left\|x_1-x_2\right\|^{1-\beta}\right)\left\|x_1-x_2\right\|^\beta,$$

where $\rho_2 := \rho_F(\overline{x}_2 - x_2) = u(x_2) - \theta(\overline{x}_2)$. Since

$$\theta(\overline{x}_2) \leq \gamma \|F^o\| \|\overline{x}_2 - \overline{x}\| + \theta(\overline{x}) \leq \gamma \mathfrak{h}_\pi \|F^o\| \|x_2 - x\|^\beta + \theta(\overline{x})$$

we have

$$\begin{aligned} \rho_2 &= u(x_2) - \theta(\overline{x}_2) \geq u(x) - \|F^o\| \|x - x_2\| - \gamma \mathfrak{h}_\pi \|F^o\| \|x - x_2\|^\beta - \theta(\overline{x}) \\ &\geq \nu := u(x) - \left(\|F^o\| \overline{\delta} + \gamma \mathfrak{h}_\pi \|F^o\| \overline{\delta}^\beta\right) - \theta(\overline{x}) > 0, \end{aligned}$$

by the choice of $\overline{\delta}$. Therefore

$$\|\nabla u(x_1) - \nabla u(x_2)\| \leq \overline{\mathfrak{h}}^\alpha \|x_1 - x_2\|^{\beta\alpha},$$

for some constant $\overline{\mathfrak{h}} > 0$. ■

Notice that the previous statements were proved under assumptions that the mapping $\pi_C^{F,\theta}(\cdot)$ is enough regular (Hölder continuous with an exponent $\beta > 1/2$) in a neighbourhood of a fixed point. Applying now the results of Section 4.2 we can emphasize the hypotheses guaranteeing such regularity and formulate global results.

Theorem 4.3.6 *Assume that $\theta : H \to \mathbb{R}$ is a function of class $C^{1,1}$ such that $\nabla\theta(x) \in -\gamma F^o$ for all $x \in \overline{\text{co}}C$, with the constant $0 < \gamma < \frac{1}{\|F\|\|F^o\|}$. Let $x_0 \in \partial C$. Suppose that the sets F, C and the function $\theta(\cdot)$ satisfy the same conditions as in Theorem 4.2.1. Assume, moreover, that for each $x \notin C$ close enough to x_0 the function $(\theta + \mathbf{I}_C)(\cdot)$ is proximally regular at each $y \in (\overline{x} + \delta\mathbf{B}) \cap \partial C$, $\overline{x} := \pi_C^{F,\theta}(x)$, for some $\delta > 0$, and one of the properties below holds:*

(i) C has smooth boundary at $\overline{x}$, and $\mathfrak{n}_C(\cdot)$ is Hölder continuous near $\overline{x}$ with an exponent $0 < \alpha \leq 1$;

(ii) F is uniformly smooth at $\xi := \frac{\overline{x} - x}{\rho_F(\overline{x} - x)}$, and $\nabla\rho_F(\cdot)$ is Hölder continuous near ξ with an exponent $0 < \alpha \leq 1$;

(ii) the mapping $y \mapsto v(y) := -\partial^p(\theta + \mathbf{I}_C)(\overline{y}) \cap \mathbf{N}_F\left(\frac{\overline{y} - y}{\rho_F(\overline{y} - y)}\right) \cap \partial F^o$ is single-valued and Hölder continuous near x with an exponent $0 < \alpha \leq 1$.

Then the value function $u(\cdot)$ is of class $C^{1,\alpha}_{loc}$ in a neighbourhood of the point x_0 (outside of C).

Proof.

It follows from theorems 4.2.1 and 4.2.2 that the mapping $\pi_C^{F,\theta}(\cdot)$ is well-defined, single-valued and (locally) Lipschitz continuous (i.e., Hölderian with $\beta = 1$) in some neighbourhood $\mathcal{U}'(x_0)$ of x_0. Consequently, all the assumptions either of Theorem 4.3.2 or of Corollary 4.3.4 are fulfilled. Then applying, respectively, Corollary 4.3.5 or Corollary 4.3.3, we complete the proof. ■

Finally let us formulate the global version of the previous theorem.

Theorem 4.3.7 *Let $\theta : H \to \mathbb{R}$ be a function of class $C^{1,1}$ such that $\nabla\theta(x) \in -\gamma F^o$ for all $x \in \overline{\mathrm{co}}C$, with the constant $0 < \gamma < \frac{1}{\|F\|\|F^o\|}$. Suppose that the sets F, C and the mapping $\theta(\cdot)$ satisfy the conditions of Theorem 4.2.1 at each point $x_0 \in \partial C$, and that the function $(\theta + \mathbf{I}_C)(\cdot)$ is proximally regular (this is the case, for instance, when C is φ-convex (see Remark 4.3.1)). Assume, furthermore, that either C has smooth boundary and $\mathfrak{n}_C(\cdot)$ is locally Hölderian with an exponent $0 < \alpha \leq 1$ near the target or F is uniformly smooth (at each point of ∂F) and $\nabla\rho_F(\cdot)$ is locally Hölderian near ∂F with the same exponent α. Then the value function $u(\cdot)$ is of class $C^{1,\alpha}_{loc}$ on $\mathfrak{U}(C)\backslash C$, for some open neighbourhood $\mathfrak{U}(C)$ of C.*

Proof.

The statement follows easily from the previous results. ■

Comments

To Chapter 1

It was J. A. Clarkson who introduced in 1936 the notion of *strict* and *uniform convexity* of a normed space intending the respective property of its closed unit ball. Roughly speaking, the space is strictly convex if any sphere in it does not contain nontrivial line segments, and it is uniformly convex if whenever the midpoint of a variable chord of the ball approaches the boundary, the length of the chord tends to zero (see, e.g., [64, 62, 65]). If a normed space is uniformly convex then it is strictly convex. Sometimes one says also that the norm is *strictly convex* (or *rotund*) and *uniformly convex* (*uniformly rotund*), respectively. Notice that the term "rotund" appeared only in 1958 in the M. M. Day's book [42]. Further, A. R. Lovaglia introduced (see [64]) a local version of the uniform convexity, requiring that one of the end points of the variable chord remains fixed.

The notions above can be easily described by means of the so called moduli of rotundity (or strict convexity). For instance, the modulus used by A. Clarkson is the function $\delta : [0, 2] \to \mathbb{R}^+$,

$$\delta(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S, \|x-y\| = \varepsilon \right\}, \tag{4.57}$$

and the uniform rotundity means that $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$ (see, e.g., [64, 62]).

In 1965 V. I. Gurariĭ (see [56]) defined a modulus of rotundity $\beta(\cdot)$ slightly different from the Clarkson's one substituting the midpoint in (4.57) by the point of that segment closer to the origin. Namely, for $\varepsilon \in [0, 2]$ he put

$$\beta(\varepsilon) := \inf \left\{ 1 - \inf_{t \in [0,1]} \|tx + (1-t)y\| : x, y \in S, \|x-y\| = \varepsilon \right\}.$$

Much later, in 1998, L. Sánches and A. Ullán [73] showed that

$$\delta(\varepsilon) \le \beta(\varepsilon) \le 2\delta(\varepsilon) \qquad \forall \varepsilon \in [0, 2]$$

and found examples of spaces for which these moduli are different.

On the other hand, M. M. Day in 1944 introduced the notions of *uniform smoothness* and the respective *modulus of smoothness* as its numerical characteristic (see [41]). He proved that a Banach space X is uniformly smooth if and only if its topological dual space X^* is uniformly convex. While the first results on such duality involving the various types of differentiability of the norm are due to the works by V. L. Šmulian (see [75, 76]). It was then J. Lindenstrauss who found in 1963 a nice quantitative version of that duality (see [62, Theorem 1]): the modulus of smoothness of X is nothing else than the conjugate of the modulus of rotundity of X^*.

All the moduli considered above as well as the geometric characteristics of the closed unit ball in a Banach space can be applied to an arbitrary subset of this space with suitable properties (permitting to associate to this set something like an equivalent "asymmetric norm"). Namely, restricting ourselves to the case of a Hilbert space H we consider a convex closed bounded set $F \subset H$ containing the origin in its interior in the place of the closed unit ball $\overline{\mathbf{B}}$. Notice that F is not necessarily symmetric (i.e., $F \neq -F$) in contrast to $\overline{\mathbf{B}}$. In spite of this one can extend all the concepts concerning the norms to the set F through its Minkowski functional (see, e.g., [60, 43]). It should be mentioned that by the properties of Hilbert spaces the ball $\overline{\mathbf{B}}$ is both uniformly rotund and uniformly smooth while F is not so, in general. Furthermore, the notions above can be specified taking into account various normal directions in a fixed point $\xi \in F$ where the modulus is determined (this does not appear in the case of the ball $\overline{\mathbf{B}}$ since at each point $\xi \in \overline{\mathbf{B}}$ there is exactly one unit normal vector $\xi^* = \xi$). Thereby we obtain the so called (local) modulus of rotundity (of smoothness) associated to the pair (ξ, ξ^*) (where $\xi \in F$ and ξ^* is a normal direction to F at ξ).

The notions of the curvatures and the respective curvature radius similar to those introduced in Section 1.2 appeared in the past even in spaces more general than Hilbert (see [72, 21]). However, the authors applied these concepts to (not necessarily convex) finite-dimensional manifolds imbedded in a Banach space. Besides that unlike the definitions given by E. R. Rozema and P. W. Smith in [72] we take into account also the structure of the boundary of F at points near ξ, which is watched along the normal vectors to F close to ξ^* (see (1.16) and (1.22)). This gives the lower semicontinuity of the curvature w.r.t. (ξ, ξ^*) that is essentially used throughout the work.

Another approach to definition of moduli of rotundity and smoothness for convex sets was proposed in [84] based on the Gurarii's modulus. The authors considered also the duality between these notions, proving some inequalities of the Lindenstrauss' type (see Proposition 1.3.1 in Section 1.3).

To Chapter 2

The idea to use the Ekeland's variational principle for proving that each minimizing sequence of the function $y \mapsto \rho_F(y - x)$ is in fact a Cauchy sequence was taken from [26] where the target set C was supposed to be φ-convex and $F = \overline{\mathbf{B}}$, i.e., the case of metric projection was treated. It should be said that by using another technique the existence and uniqueness of the metric projection onto a φ-convex closed subset of a Hilbert space was proved first in [16]. Moreover, an open neighbourhood of the target C where such well-posedness takes place was explicitly given. Notice that the open set $\mathfrak{A}(C)$ from Section 2.2 (see (2.40)) reduces to the neighbourhood constructed in [16] whenever $F = \overline{\mathbf{B}}$. Subsequently, G. Colombo and P.

Wolenski in [31] considering the case of γ-strictly convex dynamics F and φ-convex target C (with a constant function $\varphi(x) \equiv \varphi$) proved the well-posedness of the time-minimum projection in an uniform open set around C, which is given by a balance between γ and φ. Our well--posedness condition (see Corollary 2.2.3) involves also the case of the continuous function $\varphi(\cdot)$, while for $\varphi(x) \equiv \varphi$ it admits the form $2\varphi\mathfrak{T}_C^F(z) < \gamma$, which is slightly weaker than the hypotheses of Theorem 5.6 in [31]. The minimal time problem with a constant dynamics and a convex target set was detaily studied also in [31]. In Theorem 4.2 the authors used a simple argument based on weak convergence, which permitted us to prove existence of a time-minimum projection in the convex case (see Proposition 2.3.5 (ii)). However, [31] does not contain any result on uniqueness for a convex target, while in Proposition 2.3.5 we treated two alternative hypotheses when the uniqueness (and stability as well) takes place.

To Chapter 3

Various results concerning the regularity of the time-minimum projection in the case of γ-strictly convex dynamics appeared in [31]. Namely, the authors have already proved that $\pi_C^F(\cdot)$ is always Hölder continuous with the exponent 1/2 in the neighbourhood of C defined by a balance between the curvatures of C and F, where $\pi_C^F(x)$ is a singleton (see [31, Theorem 5.7]). In the same theorem they consider also some special cases when the mapping $\pi_C^F(\cdot)$ satisfies even the local Lipschitz condition. Notice that in Section 3.2 we study Hölder continuity with various exponents $0 < \alpha \leq 1$ and under much more general assumptions. Proving the Hölder regularity in theorems of Section 3.2 we reduce the general situation (with arbitrary points $z, w \in x + \bar{\delta}\mathbf{B}$) to the particular case when $\mathfrak{T}_C^F(z) = \mathfrak{T}_C^F(w)$. This argument, which simplifies essencially the respective estimates, was used earlier in [31, Theorem 5.7].

The regularity result given by Theorem 3.3.1 in Section 3.3 is close to [31, Theorem 5.12]. However, the authors assume there much stronger "one-point" Lipschitz condition for the time--minimum projection around a fixed point. It should be noticed that the proof of Theorem 3.3.1 was partially inspired by the reasoning in [58, Theorem 22].

The continuous differentiability of the value function (and the Lipschitz continuity of its gradient) appeared also in [31] (see Theorem 5.14 and Remark after it) but again under stronger hypotheses than those formulated in Section 3.3. Besides that, our results are more graduated (in the sense of arbitrarity of a Hölder exponent).

Let us emphasize once more that all the results obtained in chapters 2 and 3 are regarded to the well-posedness of $\pi_C^F(\cdot)$ and to the regularity of the value function $\mathfrak{T}_C^F(\cdot)$ in a neighbourhood of the target set. We refer to [28], where these properties were studied near an arbitrary point $x \notin C$, and the complete characterization of the well-posedness of the time-minimum projection by means of differentiability properties of $\mathfrak{T}_C^F(\cdot)$ was obtained. In the case of the distance of the metric projections (even in Banach spaces) these questions were treated, for instance, in [12, 58].

Conclusion

Let us emphasize the main results obtained in Thesis.

1. Properties of convex closed solids in a Hilbert space such as rotundity and smoothness are quantitatively studied. A new concept of curvature is introduced and placed into the general setting of Convex Analysis.

   (a) A local asymmetric version of the Lindenstrauss duality theorem is proved.

   (b) A relation between the curvature of a convex closed solid and the second derivative of its dual Minkowski functional is found.

2. The minimal time problem with a constant convex dynamics and a closed target set in a Hilbert space is studied. The conditions guaranteeing the well-posedness of this problem near the target are obtained by using the concepts of curvature and duality mapping.

   (a) It is proved that under those conditions for each point from a neighbourhood of the target there exists a unique point on the boundary of this set (called time-minimum projection), which is attained for a minimal time. Stability of this point with respect to small perturbations also follows.

   (b) A theorem on connection between the regularity of the time-minimum projection and the regularity of the value function is proved.

   (c) A series of results on the Hölder regularity of the time-minimum projection as well as on the (Hölder) continuous differentiability of the value function is obtained.

3. Regularity of viscosity solutions to some kind of Hamilton-Jacobi equations in a Hilbert space near the boundary is studied. The approach developed for proving of the local regularity of the time-minimum projection is adapted to this problem, which is treated as a minimal time problem with an additive Lipschitzean perturbation.

   (a) Under certain compatibility condition involving the duality mapping of the dynamics, on one hand, and both the normal cone to the target set and the gradient of the boundary function, on the other, the local existence, uniqueness and stability of minimizers in the perturbed problem are proved.

   (b) A connection between the regularity of the (unique) minimizer in the perturbed problem above and the regularity of the respective value function is clearified.

(c) Some results on the (Hölder) continuous differentiability of the viscosity solution to a kind of Hamilton-Jacobi equation close to boundary are obtained.

The obtained results are illustrated by series of examples in finite-dimensional as well as in infinite-dimensional Hilbert spaces.

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