Essays on Option Pricing under Alternative One-Dimensional Diffusions

Maria Manuela Coelho Larguinho

Tese apresentada à Universidade de Évora
para obtenção do Grau de Doutor em Matemática
Especialidade: Estatística

ORIENTADORES: *Carlos Alberto dos Santos Braumann*
José Carlos Gonçalves Dias

ÉVORA, Outubro, 2013

INSTITUTO DE INVESTIGAÇÃO E FORMAÇÃO AVANÇADA

To my son Manuel

Summary

Essays on Option Pricing under Alternative One-Dimensional Diffusions

Given its analytical attractiveness, the process most commonly used in the financial and real options literature is the geometric Brownian motion. However, this assumption embodies some unrealistic implications for the dynamical behavior of the underlying asset price. To overcome this issue, alternative stochastic processes have been considered in the valuation of financial and real options. This thesis examines financial and real options using alternative one-dimensional diffusions, namely the constant elasticity of variance (CEV) and mean-reverting CEV diffusions.

This thesis has two main purposes. First, it derives closed-form solutions for computing Greeks of European-style options under both the CEV and CIR (Cox, Ingersoll and Ross) models. Second, it analyzes the optimal entry and exit policy of a firm in the presence of output price uncertainty and costly reversibility of investment under a generalized class of one-dimensional diffusions and shows how the hysteretic band is affected.

Resumo

Ensaios sobre a Avaliação de Opções sob Difusões Unidimensionais Alternativas

Na avaliação de opções financeiras e reais, o processo mais utilizado na literatura é o movimento Browniano geométrico. Contudo, esta suposição incorpora algumas implicações irrealistas para o comportamento dinâmico do preço do activo subjacente. Para ultrapassar estas limitações, têm sido considerados processos estocásticos alternativos para a avaliação de opções financeiras e reais. Esta tese analisa opções, financeiras e reais, utilizando difusões unidimensionais alternativas, nomeadamente as difusões elasticidade constante da variância (CEV - *constant elasticity of variance*) e CEV com reversão à média.

Esta tese tem dois objectivos principais. Primeiro, derivar soluções analíticas para calcular as letras gregas de opções de tipo Europeu para os modelos CEV e CIR (Cox, Ingersoll e Ross). Segundo, analisar a política óptima de entrada e de saída de uma empresa na presença de incerteza no preço do *output* e de reversibilidade dos custos de investimento, para uma classe generalizada de difusões unidimensionais, e mostrar a influência sobre a banda de histerese económica.

Acknowledgements

I take this opportunity to warmly thank both Professor José Carlos Dias and Professor Carlos Alberto dos Santos Braumann for their total availability, support and hard working throughout these last five years. It has been a privilege working with them.

I would also like to thank my family for their constant encouragement and unconditional support in my quest for knowledge. A special thank to my son Manuel for his love, smile, care and understanding when I was not around, and to Miguel for his patience and help along these years. I could not forget to thank my mother and father for always believing me, for their continuous love, and for understanding my absence.

I would like to thank my friends and colleagues for providing me all the support and friendship that I needed.

Finally, I would like to thank the PROTEC program and ISCAC's Board of Directors for the financial support and partial reduction of service.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

This thesis examines financial options and real options using alternative one-dimensional diffusions in six separate and self-contained papers.

The *Theorie de la Speculation*, Louis Bachelier's PhD, published in 1900, marks the beginning of options theory, nowdays an integrant part of the modern finance theory, and applies the concept of the Brownian motion to model the stochastic behavior of stock prices. Louis Bachelier has proposed a modeling framework for valuing stock options assuming that the underlying asset follows an arithmetic Brownian motion. However, since this process allows the possibility of negative values of the state variable, its application for pricing financial options has been questioned.

The geometric Brownian motion (GBM hereafter), i.e. the multiplicative version of the arithmetic Brownian motion, has been used in the embryonal work of Black and Scholes (1973) and Merton (1973). Black and Scholes (1973) present a general methodology for evaluating stock options without dividends during the option's life and derive a simple formula to evaluate a European-style call option. Merton (1973) generalizes the Black and Scholes model by including dividend payments during the life of the option (i.e with a constant dividend yield) and stochastic interest rates. The subsequent studies on financial options have focused on the development of more realistic model for valuing financial option contracts, incorporating less restrictive assumptions in order to accommodate the empirical evidence in the literature (see, for example, Bakshi et al. (1997)).

A financial option gives the holder the right to buy or sell the underlying asset at a certain price, which is called strike price or exercise price, by a certain date. The date in the contract is known as the expiration date or maturity. We can have call and put options. A call option gives the holder the right to buy the underlying asset for a strike price within or at a maturity date. A put option gives the holder the right to sell the underlying asset for a strike price within or at a maturity date. In general terms, the underlying asset of the financial options can be thought of as a stock, a stock index, an exchange rate, or a financial futures contracts. Options can be classified in European-style options, where the option can be exercised only at the expiration date itself, or American-style options, where the option can be exercised at any time up to the maturity date of the contract. Options also can be classified by their moneyness, which results from the comparison between the spot price of the underlying asset and the strike price of the option contract. Options are referred to as in-the-money, at-the-money, or out-of-the-money. Let S be the spot price of the underlying asset and K the strike price. Then, a call (put) option is in-the-money if $S > K$ $(S < K)$, at-the-money if $S = K$, and out-of-the-money if $S < K$ $(S > K)$.

Consider a European-style call option that pays $c_T = max(S_T - K, 0) = (S_T - K)^+$ at maturity T. Taking the expectation of c_T, and discounting to time t, yields the option valuation formula for a call option with striking price K:

$$c_t = \mathbb{E}_t{}^{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+], \tag{1.1}$$

where r is the risk-free interest rate, and $\mathbb{E}_t{}^{\mathbb{Q}}[.] \equiv \mathbb{E}^{\mathbb{Q}}[.|\mathscr{F}_t]$ is the expected value under the risk-neutral probability measure $\mathbb{Q}$ conditional on the $\mathscr{F}_t$ σ-field generate by the standard Brownian motion $W_t{}^{\mathbb{Q}}$, $\mathscr{F}_t = \sigma(W_s{}^{\mathbb{Q}}; 0 \leqslant s \leqslant t)$. Similarly, consider now a European-style put option that pays $p_T = max(K - S_T, 0) = (K - S_T)^+$ at maturity T. Taking the expectation of p_T, and discounting to time t, yields the option valuation formula for a put option with striking price K:

$$p_t = \mathbb{E}_t{}^{\mathbb{Q}}[e^{-r(T-t)}(K - S_T)^+]. \tag{1.2}$$

Throughout this thesis we will consider one-dimensional stochastic differential equation (sde) of the form

$$dS_t = a(t, S_t)dt + b(t, S_t)dW_t \tag{1.3}$$

where the S_t is the underlying asset price, $a(t, S_t)$ and $b^2(t, S_t)$ are given functions, called drift and diffusion coefficients, and W_t is the Wiener process, with initial condition $S_0 = s_0$.

Real options can be viewed as an extension of financial options to value capital investment opportunities in real asset such as land, buildings, plant, or investment projects. A real option is the right, but not the obligation to take an action (for example, deferring, expanding, contracting or abandoning) at a predetermined investment cost called the exercise price for a predetermined period of time - the life of the option. The investment opportunity in a project can be seen as a call option. Similarly, an option to abandon a project is analogous to a put option on the project's value. The exercise price is the residual value of the equipment.

In the literature of the real options, most of the developed work is based on the papers of McKean (1965), Black and Scholes (1973), and Merton (1973) by assuming that the underlying state variable follows a diffusion process with lognormal distribution. Dixit and Pindyck (1994) made an extensive review for the case in which the state variable follows a GBM.

The GBM assumption can be a reasonable approximation of the stochastic behavior of some financial and real asset prices. However, the empirical evidence indicates that this assumption is not sufficient to capture the implied volatility smile effect or skew founded in the equity options market (see, for example, Jackwerth and Rubinstein (2001)).

To overcome these limitations, several alternative diffusion models have emerged in the literature. Along this thesis we will look with particular emphasis to the constant elasticity of variance (CEV henceforth) model of Cox (1975) and to the mean-reverting CEV model, namely the the mean-reverting square-root process of Cox et al. (1985).

The CEV model of Cox (1975) allows the instantaneous conditional variance of the asset returns to depend on the asset price level, thus exhibiting an implied volatility smile (or skew) similar to the volatility curves observed in practice (see, for example, Dennis and Mayhew (2002)). The CEV framework is also consistent with the so-called leverage effect, i.e. the existence of a negative correlation between stock returns and realized stock volatility, as documented, for instance, in Bekaert and Wu (2000). Thus, it is necessary to use a more general assumption for the state variable process that better captures the empirical regularities found in the finance literature.

The mean-reverting square-root model proposed by Cox et al. (1985), known as the CIR model, is an equilibrium asset pricing model for the term structure of interest rates and has been a benchmark for many years because of its analytical tractability. The interest rate behavior implied by this model has empirically relevant properties, namely, does not allow negative interest rates; if the interest rate reaches zero, it can subsequently become positive; the absolute variance of the interest rate increases when the interest rate itself increases; and there is a steady distribution for the interest rate. Moreover, this model produces relatively simple closed-form solutions for valuing zero-coupon bonds, coupon-bearing bonds, and various interest rate derivative securities.

This thesis is essentially divided in three parts. The first part is dedicated to the CEV model, while the second part is dedicated to the CIR model, both applied to financial options. Finally, the third part is dedicated to an application of the CEV and mean-reverting CEV models to real options.

The first part comprises two papers:

1. Absolute Diffusion Process: Sensitive Measures
2. On the Computation of Options Prices and Greeks under the CEV Model.

In the first of these papers, we analyze a particular case of the general CEV model, namely, the absolute diffusion model, for which the European-style call and put option pricing formulas are expressed in terms of the cumulative univariate standard normal distribution. In this paper, we present the analytical expressions of sensitivity measures, commonly referred in the financial literature as “Greeks”. These measures are considered vital tools for risk management and they all represent sensitivity measures of the option to a small change of a given parameter. The most common Greeks are the first order derivatives, namely delta, theta, vega, and rho, as well as gamma, a second order derivative of the value function.

The second paper analyzes the general CEV model whose option pricing formulae typically involve the so-called complementary non-central chi-square distribution function. This paper essentially offers two contributions for the existent option pricing literature. First, it provides comparative results in terms of accuracy and computation time for existing alternative algorithms for computing the non-central chi-square distribution function to be used for option pricing and hedging under the CEV model for a large set of parameter values. Since option valuation under the CEV model is computationally expensive, we think that it is important to know which algorithm offers the best speed-accuracy tradeoff for pricing options under the CEV model. We believe that the computational results obtained in this article are an important contribution not only for pricing and hedging European-style options under the CEV model, but also for a wide spectrum of options contracts, as for example the valuation of plain-vanilla American-style options or European and American-style (double) barrier options using the CEV diffusion. The second contribution of this paper is the derivation of closed-form solutions for computing Greeks of European-style options under the CEV model. These closed-form solutions are important for practitioners since they are generally preferable to finite difference schemes because of their computational speed advantage. The knowledge of closed-form solutions for deltas was fundamental to implement the static hedge approach proposed by Chung and Shih (2009) to price American-style options under the CEV model for other values of the elasticity parameter besides the $4/3$ benchmark used by the authors, as can be viewed in the paper of Ruas et al. (2013).

The second part of this thesis is dedicated to the CIR model and comprises two articles:

1. Valuation of Bond Options under the CIR Model: Some Computational Remarks
2. Bond Options, Sensitivity Measures, and Sinking-Fund Bonds under CIR Framework.

In the first paper, we compare the performance in terms of accuracy and computational time of alternative methods for computing the noncentral chi-square distribution used for pricing bond options under the CIR model against an externally tested benchmark. We have examined the methods proposed by Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) against a method based on series of incomplete gamma functions.

In the second article, we offer two contributions to the existent literature on CIR diffusions. First, we obtain closed-form solutions to efficiently and accurately calculate the sensitivity measures of both zero-coupon and coupon-bearing bonds options under the CIR model. These analytical solutions are relevant for practitioners of the derivatives industry since they reduce substantially the computational burden when dealing with large portfolios of bonds that have to be re-evaluated frequently. The second contribution is the extension of the results of Bacinello et al. (1996) by providing analytical tractable formulae required for valuing and analyzing comparative statics of sinking-fund bonds in the CIR diffusion. Based on our analytic solutions, we prove a result that compares the stochastic duration of the sinking-fund bond to those of corresponding serial and coupon bonds.

Finally, the third part, as we have referred, is dedicated to the real options literature and also comprises two articles:

1. A Note on (Dis)Investment Options and Perpetuities under CIR Interest Rates
2. Entry and Exit Decisions under Uncertainty for a Generalized Class of One-Dimensional Diffusions.

In the first article, we analyze alternative ways of computing the options to invest in and divest from an investment project in a CIR economy. In particular, we consider the situation of a firm that can invest $\bar{I}$ at any time and receive a perpetuity (project) with constant cash flow, but the perpetuity value is stochastic, since the interest rate used to discount the perpetual flows is assumed to follow a CIR model. We examine different methods of determining CIR perpetuities, namely, two methods proposed by Delbaen (1993), and one proposed by Geman and Yor (1993).

In the second article, we consider the problem of a firm's entry and exit decisions originally addressed by Dixit (1989a) and Tsekrekos (2010), and analyze how output price uncertainty and costly reversibility affects the optimal entry and exit policy of a competitive price-taking firm. We extend these previous studies in two ways. First, we assume that the underlying output price dynamics follows a generalized one-dimensional diffusion, namely the CEV process and the mean-reverting CEV process, which takes the modeling assumptions of Dixit (1989a) and Tsekrekos (2010) as two special cases. Second, we analyze the impact of costly reversibility on the dynamic entry and exit problem, this is, we examine how the choice of the stochastic process for the output price affects the hysteretic band. We believe that our results should be important for academics and practitioners, since our modeling framework admits the analysis of the general properties of entry and exit decisions under alternative underlying driving stochastic factor dynamics and characterizes the circumstances under which the obtained results are significantly different or remain qualitatively valid, depending on the assumption made for the underlying output price dynamics.

This thesis proceeds as follows. Chapter 2 presents the first paper. Chapter 3 presents the second paper. Chapter 4 presents the third paper. Chapter 5 presents the fourth paper. Chapter 6 presents the fifth paper. Chapter 7 presents the sixth paper. Finally, Chapter 8 concludes.

Chapter 2

Absolute Diffusion Process: Sensitivity Measures*

Abstract: The constant elasticity of variance (CEV) model of Cox (1975) captures the implied volatility smile that is similar to the volatility curves observed in practice. This diffusion process has been used for pricing several financial options contracts.

In this paper we present the analytical expressions of sensitivity measures for the absolute diffusion process, commonly known as Greeks, and we analyze numerically the behavior of the measures for European options under the CEV model.

JEL Classification: G13.

Keywords: Sensitivity Measures; Greeks; CEV model; Absolute Diffusion.

*This paper is a joint work with José Carlos Dias and Carlos A. Braumann and is published in Advances in Regression, Survival Analysis, Extreme Values, Markov Processes and Other Statistical Applications. Studies in Theoretical and Applied Statistics, Springer, 2013, 249 - 257. ISBN 978-3-642-34903-4

1 Introduction

Under the risk-neutral probability measure $\mathbb{Q}$, the constant elasticity of variance (CEV) process of Cox (1975) assumes that the asset price $\{S_t; t \geq 0\}$ is described by the following stochastic differential equation

$$dS_t = (r-q)S_t\,dt + \delta\,S_t^{\beta/2}\,dW_t^{\mathbb{Q}}, \tag{2.1}$$

where $W_t^{\mathbb{Q}}$ is a Wiener process under $\mathbb{Q}$, $r \geq 0$ represents the instantaneous riskless interest rate, which is assumed to be constant, $q \geq 0$ denotes the dividend yield for the underlying asset price, with a local volatility function given by

$$\sigma(S_t) = \delta\,S_t^{\beta/2-1}, \tag{2.2}$$

where β is a real number, and δ is a positive constant.

The elasticity of return variance with respect to price is equal to $\beta - 2$ given that $dv(S_t)/v(S_t) = (\beta - 2)\,dS_t/S_t$ where $v(S_t) = \delta^2\,S_t^{\beta-2}$ is the instantaneous variance of asset returns. Since volatility is proportional to a power of the underlying asset price, the elasticity of variance is independent of the asset price. The model parameter δ can be interpreted as the scale parameter fixing the initial instantaneous volatility at time $t = t_0$, $\sigma_0 = \sigma(S_{t_0}) = \delta\,S_{t_0}^{\beta/2-1}$.

The CEV specification given by equation (2.1) nests the log-normal assumption of Black and Scholes (1973) and Merton (1973) ($\beta = 2$), as well as the square-root diffusion ($\beta = 1$) and the absolute diffusion ($\beta = 0$) models of Cox and Ross (1976), as special cases. For $\beta < 2$ ($\beta > 2$) the local volatility given by equation (2.2) is a decreasing (increasing) function of the asset price. If $\beta = 2$, the stock price has no influence on the volatility, since the volatility will be a constant over time, $\sigma(S_t) = \delta$, regardless of the underlying asset price.

2 European Options under the CEV Diffusion

The CEV call option pricing formula for valuing European options has been initially expressed in terms of the standard complementary gamma distribution by Cox (1975) for $\beta < 2$, and by Emanuel and MacBeth (1982) for $\beta > 2$. Schroder (1989) has subsequently extended the CEV model by expressing the corresponding formulae in terms of the non-central chi-square distribution as

$$c_t := \begin{cases} S_t\, e^{-q\tau}\, Q(2y; 2+\frac{2}{2-\beta}, 2x) - X\, e^{-r\tau}\, [1 - Q(2x; \frac{2}{2-\beta}, 2y)] & \text{if } \beta < 2 \\ \\ S_t\, e^{-q\tau}\, Q(2x; \frac{2}{\beta-2}, 2y) - X\, e^{-r\tau}\, [1 - Q(2y; 2+\frac{2}{\beta-2}, 2x))] & \text{if } \beta > 2 \end{cases}, \qquad (2.3)$$

with X being the strike price of option, $Q(w; v, \lambda)$ being the complementary distribution function of a non-central chi-square law with v degrees of freedom and noncentrality parameter λ, and where

$$k = \frac{2(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)\tau} - 1]}, \qquad (2.4)$$

$$x = kS_t^{2-\beta}\, e^{(r-q)(2-\beta)\tau}, \qquad (2.5)$$

$$y = kX^{2-\beta}, \qquad (2.6)$$

$$\delta = \sigma_0 S_0^{1-\beta/2}, \qquad (2.7)$$

$$\tau = T - t. \qquad (2.8)$$

By put-call parity, the CEV put option pricing formulae are

$$p_t := \begin{cases} X\, e^{-r\tau}\, Q(2x; \frac{2}{2-\beta}, 2y) - S_t\, e^{-q\tau}\, [1 - Q(2y; 2 + \frac{2}{2-\beta}, 2x)] & \text{if } \beta < 2 \\ \\ X\, e^{-r\tau}\, Q(2y; 2 + \frac{2}{\beta-2}, 2x) - S_t\, e^{-q\tau}\, [1 - Q(2x; \frac{2}{\beta-2}, 2y))] & \text{if } \beta > 2 \end{cases} . \tag{2.9}$$

In general terms, the underlying asset of the CEV diffusion can be thought of as a stock, a stock index, an exchange rate, or a financial futures contracts, so long as the parameter q is understood as, respectively, a dividend yield, an average dividend yield, the foreign default-free interest rate, or the domestic risk-free interest rate.

There are several alternative methods for computing the cumulative distribution function of the non-central chi-square in the literature (see, for instance, Schroder (1989); Ding (1992); Benton and Krishnamoorthy (2003)).

In this work, we use a method based on series of incomplete gamma functions to compute the complementary non-central chi-square distribution function given by

$$Q(w; v, \lambda) = \sum_{i=0}^{\infty} \frac{(\lambda/2)^i e^{-\lambda/2}}{i!} \frac{\Gamma(v/2 + i, w/2)}{\Gamma(v/2 + i)}, \tag{2.10}$$

with $\Gamma(m, n)$ and $\Gamma(m)$ being, respectively, the complementary incomplete gamma function and the Euler gamma function as defined by Abramowitz and Stegun (1972, Equations 6.5.3 and 6.1.1).

The next figures show the behavior of European call and put options prices. We consider the following parameters for our analysis: the initial asset price is $S_0 = 100$, the strike price is $X = 100$, the instantaneous volatility at this price level is 25% per annum ($\sigma_0 = 0.25$), the risk-free interest rate is 10% per annum $(r = 0.1)$, the asset pays no dividends ($q = 0$), and all options have six months to expiration ($\tau = 0.5$). We employ seven different values to β $(-6, -4, -2, 0, 1, 2, 4)$ to show its effects on options prices. The constant volatility case ($\beta = 2$) corresponds to the Black and Scholes model. Let σ_0 be the instantaneous volatility for Black and Scholes model, then the value of δ to be used for models with different β values is adjusted to be $\delta = \sigma_0 S_0^{1-\beta/2}$.

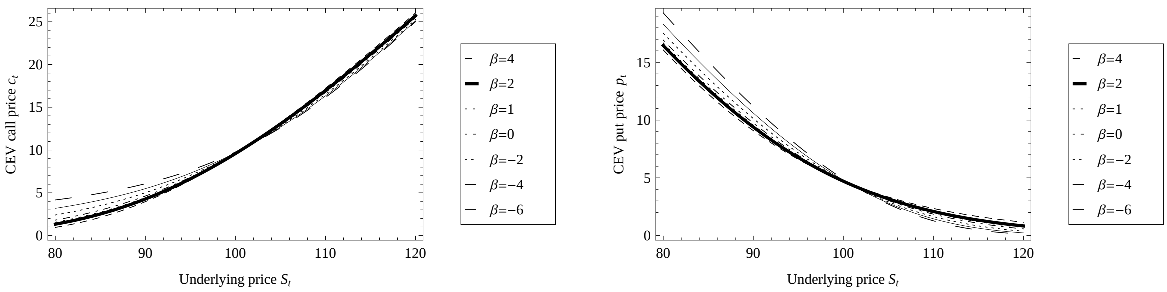

Figure 2.1: European call and put option prices under CEV processes and Black-Scholes model as functions of underlying asset price S_t. Parameters: $S_0 = 100, X = 100, \sigma_0 = 0.25, \tau = 0.5, r = 0.1$, and $q = 0$.

3 Sensitivity Measures for the Absolute Diffusion

The absolute diffusion process proposed by Cox and Ross (1976) is a particular case of the CEV diffusion process in equation (2.1) with $\beta = 0$.

Proposition 2.1 *Under the CEV diffusion* (2.1) *with* $\beta = 0$*, i.e., with a local volatility function given by* $\sigma(S_t) = \delta S_t^{-1}$*, the European call and put option prices are equal to:*[2.1]

$$c_t = (S_t e^{-q\tau} - X e^{-r\tau}) N(y_1) + (S_t e^{-q\tau} + X e^{-r\tau}) N(y_2) + u\,[n(y_1) - n(y_2)], \qquad (2.11)$$

$$p_t = (X e^{-r\tau} - S_t e^{-q\tau}) N(-y_1) + (S_t e^{-q\tau} + X e^{-r\tau}) N(y_2) + u\,[n(y_1) - n(y_2)], \qquad (2.12)$$

[2.1] Equation (2.11) of Proposition 1 is standard in the literature and can be found, for example, in Cox and Ross (1976). Equation (2.12) is then easily obtained through the put-call parity relation.

where $N(x)$ is the cumulative univariate standard normal distribution function, $n(x)$ is the standard normal density function, and

$$u = \delta\left(\frac{e^{-2q\tau} - e^{-2r\tau}}{2(r-q)}\right)^{1/2}, \tag{2.13}$$

$$y_1 = \frac{S_t e^{-q\tau} - X e^{-r\tau}}{u}, \tag{2.14}$$

$$y_2 = \frac{-S_t e^{-q\tau} - X e^{-r\tau}}{u}. \tag{2.15}$$

The sensitivity measures, commonly referred in financial literature as "Greek letters" or simply "Greeks", are vital tools for risk management and they all represent sensitivity measures of the option price to a small change of a given parameter. The most common Greeks are the first order derivatives: delta, theta, vega and rho as well as gamma, a second-order derivative of the value function.

In the following we give the analytical expressions for the greek letters under the absolute diffusion process.[2.2]

Delta

The delta, Δ, of an option is defined as the rate of change of the option price, V, with respect to the price of the underlying asset, S_t, that is, $\Delta = \partial V / \partial S_t$. It is the slope of the curve that relates the option price to the underlying asset price. The delta plays a crucial role for hedging portfolios. For European call and put options under the absolute diffusion process on an asset paying a dividend yield q we have

[2.2]Due to constraints of space, we have not included proofs of the analytical expressions of sensitivity measures but they are available upon request.

$$\Delta_{call} = e^{-q\tau}\Big(N(y_1) + N(y_2)\Big), \tag{2.16}$$

$$\Delta_{put} = e^{-q\tau}\Big(-N(-y_1) + N(y_2)\Big), \tag{2.17}$$

where y_1 and y_2 are defined as in equations (2.14) and (2.15).

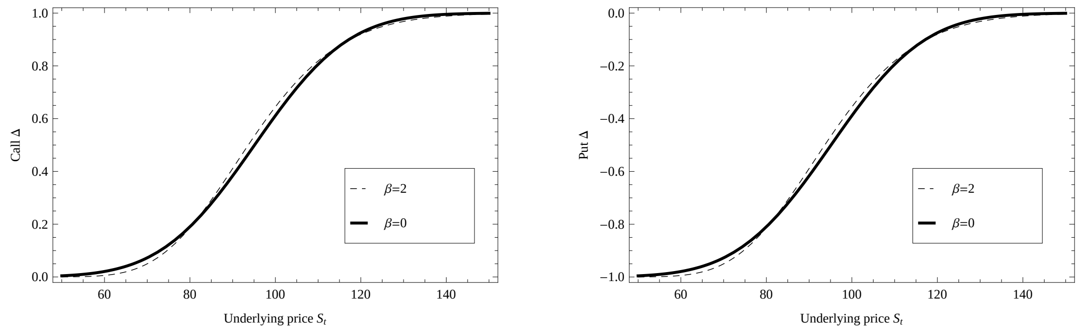

Figure 2.2: Variation of delta, Δ, with respect to the underlying asset price S_t. Parameters: $S_0 = 100$, $X = 100, \sigma_0 = 0.25, \tau = 0.5$, $r = 0.1$, and $q = 0$.

Theta

The theta, Θ, of an option is the rate of change of the option price, V, with respect to the passage of time, t, with all else remaining the same, that is, $\Theta = \partial V/\partial t$. Theta is sometimes referred to a time decay effect of the option. The theta of European call and put options under the absolute diffusion process, are found, respectively, to be

$$\Theta_{call} = S_t q e^{-q\tau}\Big(N(y_1) + N(y_2)\Big) - X r e^{-r\tau}\Big(N(y_1) - N(y_2)\Big) + A, \tag{2.18}$$

$$\Theta_{put} = -S_t q e^{-q\tau}\Big(N(-y_1) - N(y_2)\Big) + X r e^{-r\tau}\Big(N(-y_1) + N(y_2)\Big) + A, \tag{2.19}$$

where

$$A = u\Big(n(y_1) - n(y_2)\Big)\frac{qe^{-2q\tau} - re^{-2r\tau}}{e^{-2q\tau} - e^{-2r\tau}}, \tag{2.20}$$

with u, y_1 and y_2 being defined as in equations (2.13), (2.14) and (2.15).

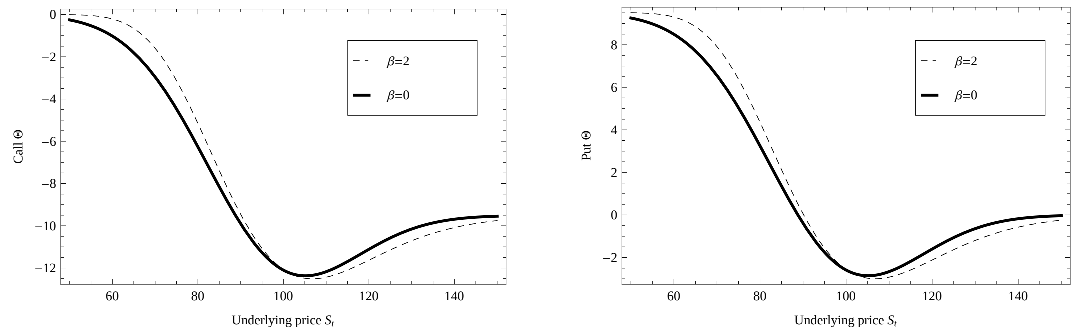

Figure 2.3: Variation of theta, Θ, with respect to the underlying asset price S_t. Parameters: $S_0 = 100$, $X = 100, \sigma_0 = 0.25, \tau = 0.5$, $r = 0.1$, and $q = 0$.

Vega

The vega, $\mathscr{V}$, of an option is defined to be the rate of change of the value of option, V, with respect to asset price volatility, σ, that is, $\mathscr{V} = \partial V/\partial\sigma$. For European call and put options under the absolute diffusion process, their vegas are found to be

$$\mathscr{V}_{call} = \mathscr{V}_{put} = \frac{u}{\sigma}\Big(n(y_1) - n(y_2)\Big), \tag{2.21}$$

where u, y_1 and y_2 are defined as in equations (2.13), (2.14) and (2.15).

Rho

The rho, ρ, of an option is defined to be the rate of change of the value of an option, V, with respect to the interest rate, r, that is, $\rho = \partial V/\partial r$. The rhos of the European call

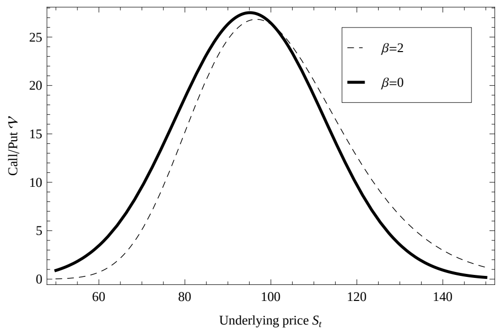

Figure 2.4: Variation of vega, $\mathcal{V}$, with respect to the underlying asset price S_t. Parameters: $S_0 = 100$, $X = 100, \sigma_0 = 0.25, \tau = 0.5$, $r = 0.1$, and $q = 0$.

and put options prices under absolute diffusion process are found to be

$$\rho_{call} = X\tau e^{-r\tau}\Big(N(y_1) - N(y_2)\Big) + B, \tag{2.22}$$

$$\rho_{put} = -X\tau e^{-r\tau}\Big(N(-y_1) + N(y_2)\Big) + B, \tag{2.23}$$

where

$$B = u\Big(n(y_1) - n(y_2)\Big)\left(\frac{\tau e^{-2r\tau}}{e^{-2q\tau} - e^{-2r\tau}} - \frac{1}{2(r-q)}\right), \tag{2.24}$$

with u, y_1 and y_2 being defined as in equations (2.13), (2.14) and (2.15).

Gamma

The gamma, Γ, of an option is defined as the rate of change of delta, Δ, with respect to the asset price, S_t, that is, $\Gamma = \partial^2 V/\partial S_t^2 = \partial\Delta/\partial S_t$. For European call and put options under the absolute diffusion process, their gammas are found to be

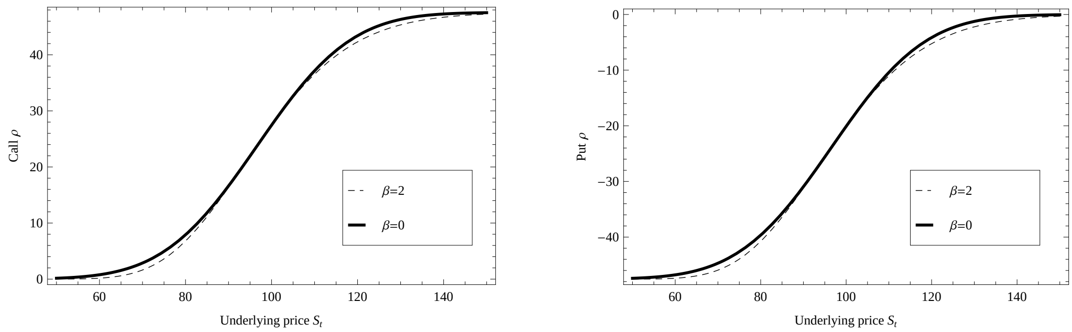

Figure 2.5: Variation of rho, ρ, with respect to the underlying asset price S_t. Parameters: $S_0 = 100$, $X = 100, \sigma_0 = 0.25, \tau = 0.5$, $r = 0.1$, and $q = 0$.

$$\Gamma_{call} = \Gamma_{put} = \frac{e^{-2q\tau}}{u}\Big(n(y_1) - n(y_2)\Big), \tag{2.25}$$

where u, y_1 and y_2 are defined as in equations (2.13), (2.14) and (2.15).

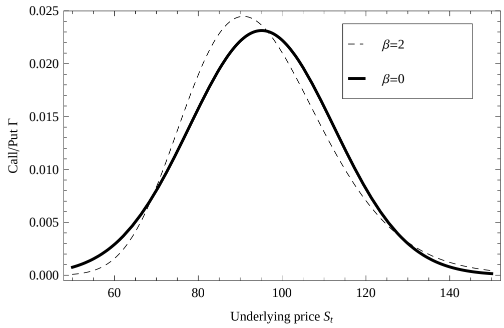

Figure 2.6: Variation of gamma, Γ, with respect to underlying asset price S_t. Parameters: $S_0 = 100$, $X = 100, \sigma_0 = 0.25, \tau = 0.5$, $r = 0.1$, and $q = 0$.

The following tables report values of call and put European options under the absolute diffusion process and the Black and Scholes model, as well as their corresponding greeks. Overall, our results show that the misspecification of β may result in significant errors. Thus, similarly to other numerical analysis available in the literature (e.g., Dias and Nunes (2011)) we conclude that care must be taken when choosing the appropriate diffusion process for pricing and hedging options.

Table 2.1: Values for call options and greeks under absolute and GBM diffusion processes

X		Call price	Delta	Theta	Vega	Rho	Gamma
	$\beta = 0$	12.7426	0.7118	-12.0286	23.5433	30.7141	0.0198
95	$\beta = 2$	12.5880	0.7458	-11.8663	22.6677	30.9969	0.0181
	% Diff	1.23	4.56	1.37	3.86	0.91	9.14
	$\beta = 0$	9.5915	0.6113	-12.1002	26.4399	27.4513	0.0222
100	$\beta = 2$	9.5822	0.6448	-12.0722	26.3311	27.4472	0.0211
	% Diff	0.10	5.18	0.23	0.41	0.01	5.52
	$\beta = 0$	6.9403	0.5028	-11.5632	27.5180	23.4182	0.0231
105	$\beta = 2$	7.0996	0.5379	-11.6899	28.0819	23.3470	0.0225
	% Diff	2.24	6.53	1.08	2.01	0.31	2.97

Parameters used in calculations: $S_0 = 100, \sigma_0 = 0.25, \tau = 0.5, r = 0.1$, and $q = 0$

Table 2.2: Values for put options and greeks under absolute and GBM diffusion processes

X		Put price	Delta	Theta	Vega	Rho	Gamma
	$\beta = 0$	3.1094	-0.2882	-2.9920	23.5433	-14.4693	0.0198
95	$\beta = 2$	2.9548	-0.2542	-2.8296	22.6677	-14.1865	0.0181
	% Diff	5.23	13.39	5.74	3.86	1.99	9.14
	$\beta = 0$	4.7145	-0.3887	-2.5879	26.4399	-20.1102	0.0222
100	$\beta = 2$	4.7052	-0.3552	-2.5599	26.3311	-20.1142	0.0211
	% Diff	0.20	9.41	1.09	0.41	0.02	5.52
	$\beta = 0$	6.8194	-0.4972	-1.5752	27.5180	-26.5213	0.0231
105	$\beta = 2$	6.9786	-0.4621	-1.7019	28.0819	-26.5926	0.0225
	% Diff	2.28	7.60	7.44	2.01	0.27	2.97

Parameters used in calculations: $S_0 = 100, \sigma_0 = 0.25, \tau = 0.5, r = 0.1$, and $q = 0$.

4 Conclusion

The results of this paper clearly highlight the importance of the model choice for option pricing and hedging purposes. In fact, we have obtained quite different results when using the Black and Scholes model, the absolute diffusion model, or some other more generalized CEV model that is able to capture both direct $(\beta < 2)$ and inverse $(\beta > 2)$ leverage effects frequently observed in options markets.

Acknowledgements

Dias is member of the Finance Research Center (UNIDE/ISCTE), Braumann and Larguinho are members of the Research Center Centro de Investigação em Matemática e Aplicações (CIMA), both centers financed by the Fundação para a Ciência e Tecnologia (FCT). Dias gratefully acknowledges the financial support from the FCTs grant number PTDC/EGE-ECO/099255/2008.

References

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Benton, Denise and K. Krishnamoorthy, 2003, Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral t and the Distribution of the Square of the Sample Multiple Correlation Coefficient, *Computational Statistics and Data Analysis* 43, 249–267.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Cox, John C., 1975, Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working Paper, Stanford University. Reprinted in *Journal of Portfolio Management*, 23 (1996), 15-17.

Cox, John C. and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics* 3, 145–166.

Dias, José Carlos and João Pedro Nunes, 2011, Pricing Real Options under the Constant Elasticity of Variance Diffusion, *Journal of Futures Markets* 31, 230–250.

Ding, Cherng G., 1992, Algorithm AS 275: Computing the Non-Central χ^2 Distribution Function, *Applied Statistics* 41, 478–482.

Emanuel, David C. and James D. MacBeth, 1982, Further Results on the Constant Elasticity of Variance Call Option Pricing Model, *Journal of Financial and Quantitative Analysis* 17, 533–554.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Schroder, Mark, 1989, Computing the Constant Elasticity of Variance Option Pricing Formula, *Journal of Finance* 44, 211–219.

Chapter 3

On the Computation of Option Prices and Greeks under the CEV Model*

Abstract: Pricing options and evaluating Greeks under the constant elasticity of variance (CEV) model requires the computation of the non-central chi-square distribution function. In this article, we compare the performance, in terms of accuracy and computational time, of alternative methods for computing such probability distributions against an externally tested benchmark. In addition, we present closed-form solutions for computing Greek measures under the unrestricted CEV option pricing model, thus being able to accommodate direct leverage effects as well as inverse leverage effects that are frequently observed in the options markets.

JEL Classification: G1, G13.

Keywords: Option pricing; Derivatives hedging; Computational finance; Statistical methods.

*This paper is a joint work with José Carlos Dias ad Carlos A. Braumann and is published in Quantitative Finance, 2013, Vol. 13, No. 6, 907-917. ISSN 1469-7688 (print) 1469-7696 (online)

1 Introduction

Every option pricing model has to make a key assumption regarding the “right” distribution to be used when discounting the option’s expected payoff. This expectation is typically computed by integrating the payoff function over a risk-neutral density function. Under the log-normal models of Black and Scholes (1973) and Merton (1973) (BSM model) it is assumed that the underlying asset price follows a geometric Brownian motion. Yet, this prediction has been convincingly rejected in the finance literature. For instance, it is well documented—see, for example, Jackwerth and Rubinstein (1996)—that the log-normal assumption is unable to accommodate the negative skewness and the high kurtosis that are usually implicit in empirical asset return distributions.

The constant elasticity of variance (CEV) model of Cox (1975) is consistent with two well known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized volatility (leverage effect), as observed, for instance, in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (implied volatility skew)—see, for example, Dennis and Mayhew (2002). More importantly, being a “local volatility” model, the CEV diffusion is consistent with a “complete market” setup and, therefore, allows the hedging of short option positions only through the underlying asset.

Computing option prices under the CEV model typically involves the use of the so-called complementary non-central chi-square distribution function. There exists an extensive literature devoted to the efficient computation of this distribution function, with several alternative representations available (see, for instance, Farebrother (1987), Posten (1989), Schroder (1989), Ding (1992), Knüsel and Bablok (1996), Benton and Krishnamoorthy (2003), and Dyrting (2004)). The complementary non-central chi-square distribution function can also be computed using a method based on series of incomplete gamma functions. For certain ranges of parameter values, some of the alternative representations available are more computationally efficient than the series of incomplete gamma functions. Moreover, for some parameter configurations the use of analytic approximations (e.g. Sankaran (1963), Fraser et al. (1998), and Penev and Raykov (2000)) may be preferable.

The main purpose of this article is to provide comparative results in terms of accuracy and computation time of existing alternative algorithms for computing the non-central chi-square distribution function to be used for option pricing and hedging under the CEV model for a large set of parameter values. A similar study has been conducted by Ağca and Chance (2003) to price compound options and min-max options whose computation requires approximations of the bivariate normal probability.

All tested methods are generally accurate over a wide range of parameters that are frequently needed for pricing options, though they all present relevant differences in terms of running times. As expected, the analytic approximations run quickly but have an accuracy that varies significantly over the considered parameter space. Option pricing under the CEV assumption is computationally expensive especially when the elasticity parameter β of the local volatility function is close to two, volatility is low, or the time to maturity is small in the CEV formulae. Overall, we conclude that the iterative procedure of Benton and Krishnamoorthy (2003) clearly offers the best speed-accuracy tradeoff for pricing options under the CEV model.

Even though our numerical analysis focus on CEV European-style options, our findings are important also for valuing others European options contracts using the non-central chi-square distribution, including options on bonds (e.g. Cox et al. (1985)), volatility options (e.g. Grünbichler and Longstaff (1996)), and warrants (e.g. Bajo and Barbi (2010)).

Furthermore, our results are also of interest for some options contracts with early exercise features and/or exotic payoffs. For instance, the valuation of plain-vanilla American options under the optimal stopping approach as proposed by Nunes (2009) requires an explicit solution of its European counterpart option contract and knowledge of the transition density function of the underlying price process. Thus, an efficient method in terms of accuracy and computation time for pricing European-style options should be similarly efficient for valuing plain-vanilla American options within this framework and under the CEV diffusion. The same line of reasoning applies when valuing both European- and American-style (double) barrier options using the CEV assumption within the general multi-factor pricing model offered by Nunes and Dias (2012).

In other strand of the literature, Chung and Shih (2009) use the static hedge portfolio (SHP) approach to price and hedge standard American options under both the BSM and CEV models. The main idea of this approach is to create a static portfolio of standard European options whose values match the payoff of the option been hedged at expiration and along the boundary. The static hedge portfolio of an American option is formulated by applying the value-matching and smooth-pasting conditions on the early exercise boundary. Under the CEV diffusion, the numerical efficiency of the resulting valuation formulae of the n-point static hedge portfolio on the early exercise boundary requires also the use of an efficient algorithm for computing non-central chi-square distribution functions.

In summary, the results of this paper are important not only for pricing and hedging European-style options under the CEV model, but also for a wide spectrum of options contracts with early exercise features and exotic payoffs.

The theoretical contribution of this paper is the derivation of closed-form solutions for computing Greeks of European-style options under the CEV model that to our knowledge are not known in the finance literature. These new formulae are important for practitioners since closed-form solutions, when available, are generally preferable to finite difference schemes because of their computational speed advantage. Furthermore, the use of these solutions becomes a viable alternative for many practical applications arising within the context of the financial engineering industry. In addition, as we will discuss later, the existence of closed-form solutions for deltas is also relevant for pricing standard American options under the CEV model using the SHP approach proposed by Chung and Shih (2009).

The structure of the paper is organized as follows. Section 2 outlines the non-central chi-square distribution and presents different methods for computing it. Section 3 briefly reviews the CEV option pricing formulae expressed in terms of the non-central chi-square distribution for valuing European-style options. Section 4 compares the alternative methods in terms of speed and accuracy. Section 5 gives some concluding remarks. Proofs are presented in the Appendix.

2 Alternative Methods for Computing the Non-central Chi-Square Distribution

2.1 The Non-central Chi-Square Distribution

If $Z_1, Z_2, ..., Z_v$ are independent unit normal random variables, and $\delta_1, \delta_2, ..., \delta_v$ are constants, then

$$Y = \sum_{j=1}^{v} (Z_j + \delta_j)^2 \tag{3.1}$$

possesses a non-central chi-square distribution with v degrees of freedom and non-centrality parameter $\lambda = \sum_{j=1}^{v} \delta_j^2$, which is denoted as $\chi_v^2(\lambda)$. When $\delta_j = 0$ for all j, then Y is distributed as the central chi-square distribution with v degrees of freedom, and is denoted as χ_v^2.

Hereafter, $p_{\chi_v^2(\lambda)}(w) = p(w; v, \lambda)$ is the probability density function of a non-central chi-square distribution $\chi_v^2(\lambda)$, and $p_{\chi_v^2}(w) = p(w; v, 0)$ is the probability density function of a central chi-square distribution χ_v^2. Likewise, $P[\chi_v^2(\lambda) \leq w] = F(w; v, \lambda)$ is the cumulative distribution function of $\chi_v^2(\lambda)$, and $P[\chi_v^2 \leq w] = F(w; v, 0)$ is the cumulative distribution function of χ_v^2. The complementary distribution functions of $\chi_v^2(\lambda)$ and χ_v^2 are denoted as $Q(w; v, \lambda)$ and $Q(w; v, 0)$, respectively.

The cumulative distribution function of $\chi_v^2(\lambda)$ is given by (see, for instance, Johnson et al. (1995, equation 29.2)):

$$\begin{aligned} P[\chi_v^2(\lambda) \leq w] &= F(w; v, \lambda) \\ &= e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!\, 2^{v/2+j}\, \Gamma(v/2+j)} \int_0^w y^{v/2+j-1} e^{-y/2}\, dy, \quad w > 0, \end{aligned} \tag{3.2}$$

with $\Gamma(m)$ being the Euler gamma function as defined by Abramowitz and Stegun (1972, equation 6.1.1), while $F(w; v, \lambda) = 0$ for $w < 0$. Alternatively, it is possible to express $F(w; v, \lambda)$, for $w > 0$, as a weighted sum of central chi-square probabilities with weights equal to the probabilities of a Poisson distribution with expected value $\lambda/2$. This is (see, for instance, Johnson et al. (1995, equation 29.3), or Abramowitz and Stegun (1972, equation 26.4.25)),

$$
\begin{aligned}
F(w;v,\lambda) &= \sum_{j=0}^{\infty}\left(\frac{(\lambda/2)^j}{j!}e^{-\lambda/2}\right)P[\chi^2_{v+2j}\leq w] \\
&= \sum_{j=0}^{\infty}\left(\frac{(\lambda/2)^j}{j!}e^{-\lambda/2}\right)F(w;v+2j,0),
\end{aligned}
\tag{3.3}
$$

where the central chi-square probability function $F(w;v+2j,0)$ is given by Abramowitz and Stegun (1972, equation 26.4.1).

The complementary distribution function of $\chi^2_v(\lambda)$ is

$$
\begin{aligned}
Q(w;v,\lambda) &= 1-F(w;v,\lambda) \\
&= \sum_{j=0}^{\infty}\left(\frac{(\lambda/2)^j}{j!}e^{-\lambda/2}\right)Q(w;v+2j,0),
\end{aligned}
\tag{3.4}
$$

where the complementary central chi-square probability function $Q(w;v+2j,0)$ is given by Abramowitz and Stegun (1972, equation 26.4.2).

The probability density function of $\chi^2_v(\lambda)$ can, similarly, be expressed as a mixture of central chi-square probability density functions (see, for instance, Johnson et al. (1995, equation 29.4)):

$$
\begin{aligned}
p_{\chi^2_v(\lambda)}(w) &= p(w;v,\lambda) \\
&= \sum_{j=0}^{\infty}\left(\frac{(\lambda/2)^j}{j!}e^{-\lambda/2}\right)p(w;v+2j,0) \\
&= \frac{e^{-(\lambda+w)/2}}{2^{v/2}}\sum_{j=0}^{\infty}\left(\frac{\lambda}{4}\right)^j\frac{w^{v/2+j-1}}{j!\,\Gamma(v/2+j)} \\
&= \frac{1}{2}e^{-(\lambda+w)/2}\left(\frac{w}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda w}), \quad w>0, \qquad (3.5)
\end{aligned}
$$

where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q, as defined by Abramowitz and Stegun (1972, equation 9.6.10):

$$
I_q(z) = \left(\frac{z}{2}\right)^q \sum_{j=0}^{\infty}\frac{(z^2/4)^j}{j!\,\Gamma(q+j+1)}. \qquad (3.6)
$$

Using equation (3.5) we may also express the functions $F(w;v,\lambda)$ and $Q(w;v,\lambda)$ as integral representations:

$$
F(w;v,\lambda) = \int_0^w \frac{1}{2}e^{-(\lambda+u)/2}\left(\frac{u}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda u})\,du, \qquad (3.7)
$$

$$
Q(w;v,\lambda) = \int_w^\infty \frac{1}{2}e^{-(\lambda+u)/2}\left(\frac{u}{\lambda}\right)^{(v-2)/4} I_{(v-2)/2}(\sqrt{\lambda u})\,du. \qquad (3.8)
$$

2.2 The Gamma Series Method

It is well-known that the functions $F(w;v+2n,0)$ and $Q(w;v+2n,0)$ are related to the so-called incomplete gamma functions (see, for instance, Abramowitz and Stegun (1972, equation 26.4.19)). Hence, we may express non-central chi-square distribution functions (3.3) and (3.4) using series of incomplete gamma functions as follows:

$$F(w;v,\lambda) = \sum_{i=0}^{\infty} \frac{(\lambda/2)^i e^{-\lambda/2}}{i!} \frac{\gamma(v/2+i, w/2)}{\Gamma(v/2+i)}, \tag{3.9}$$

$$Q(w;v,\lambda) = \sum_{i=0}^{\infty} \frac{(\lambda/2)^i e^{-\lambda/2}}{i!} \frac{\Gamma(v/2+i, w/2)}{\Gamma(v/2+i)}, \tag{3.10}$$

with $\gamma(m,t)$ and $\Gamma(m,t)$ being, respectively, the incomplete gamma function and the complementary incomplete gamma function as defined by Abramowitz and Stegun (1972, equations 6.5.2 and 6.5.3).

The gamma series method has been applied by Fraser et al. (1998) as a benchmark for computing exact probabilities to be compared with several alternative methods for approximating the non-central chi-square distribution function, and by Dyrting (2004) for computing the non-central chi-square distribution function to be used under Cox et al. (1985) diffusion processes. Carr and Linetsky (2006) also use the gamma series approach but for computing option prices under a jump-to-default CEV framework.

While this method is accurate over a wide range of parameters, the number of terms that must be summed increases with the non-centrality parameter λ. To avoid the infinite sum of the series we use the stopping rule proposed by Knüsel and Bablok (1996), which allows the specification of a given error tolerance by the user.

There have been several alternative proposals for evaluating expressions (3.9) and (3.10)—see, for instance, Farebrother (1987), Posten (1989), Schroder (1989), Ding (1992), Knüsel and Bablok (1996), Benton and Krishnamoorthy (2003), and Dyrting (2004)—all of which involve a partial summation of the series. For certain ranges of parameter values, some of the alternative representations available are more computationally efficient than the series of incomplete gamma functions. Hence, it is important to evaluate the speed and accuracy of each method for computing the non-central chi-square distribution as well as for option pricing and hedging purposes.

For the numerical analysis of this article we will concentrate the discussion on Schroder (1989) and Ding (1992) methods since both are commonly used in the finance literature. The algorithm provided by Schroder (1989) has been subsequently used by Davydov and Linetsky (2001). The popular book on derivatives of Hull (2008) suggests the use of the Ding (1992) procedure. We will also use the suggested approach of Benton and Krishnamoorthy (2003), since it is argued by the authors that their algorithm is more computationally efficient than the one suggested by Ding (1992).

2.3 Analytic Approximations

The cumulative distribution function of the non-central chi-square distribution with degrees of freedom $v > 0$ and a non-centrality parameter $\lambda \geq 0$ is usually expressed as an infinite weighted sum of central chi-square cumulative distribution functions. For numerical evaluation purposes this infinite sum is being approximated by a finite sum. For large values of the non-centrality parameter, the sum converges slowly. To overcome this issue, a number of approximations have been proposed in the literature. A comparison of early approximation methods is given in Johnson et al. (1995, chapter 29).

In this article, we will consider the approximation method of Sankaran (1963) which is well-known in the finance literature due to Schroder (1989) who recommends its use for large values of w and λ. In addition, two more recent approximations, namely Fraser et al. (1998) and Penev and Raykov (2000), will be considered also since both of them are commonly referenced by the statistic literature as accurate methods for approximating the non-central chi-square distribution.

3 The CEV Option Pricing Model

Under the risk-neutral probability measure $\mathbb{Q}$, the CEV process of Cox (1975) assumes that the asset price $\{S_t, t \geq 0\}$ is described by the following stochastic differential equation:

$$dS_t = (r-q)\,S_t\,dt + \sigma(S_t)\,S_t\,dW_t^{\mathbb{Q}}, \tag{3.11}$$

with a local volatility function given by

$$\sigma(S_t) = \delta\,S_t^{\frac{\beta}{2}-1}, \tag{3.12}$$

for $\delta \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, and where $r \geq 0$ denotes the instantaneous riskless interest rate, which is assumed to be constant, $q \geq 0$ represents the dividend yield for the underlying asset price, $\sigma(S_t)$ corresponds to the instantaneous volatility per unit of time of asset returns, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Brownian motion under $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathscr{F}_t : t \geq t_0\}$.

The CEV specification given by equation (3.11) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) ($\beta = 2$), as well as the absolute diffusion ($\beta = 0$) and the square-root diffusion ($\beta = 1$) models of Cox and Ross (1976), as special cases. For $\beta < 2$ ($\beta > 2$) the local volatility given by equation (3.12) is a decreasing (increasing) function of the asset price. If $\beta = 2$, the stock price has no influence on the volatility, since the volatility will be a constant over time, $\sigma(S_t) = \delta$, regardless of the underlying asset price.

The elasticity of return variance with respect to price is equal to $\beta - 2$ given that

$dv(S_t)/v(S_t) = (\beta - 2)\, dS_t/S_t$, where $v(S_t) = \delta^2 S_t^{\beta-2}$ is the instantaneous variance of asset returns. Since volatility is proportional to a power of the underlying asset price, the elasticity of the variance is independent of the asset price. If $\beta = 2$, the elasticity is zero thus implying that asset prices are lognormally distributed and the variance of returns is constant, as is assumed in the BSM framework. The model parameter δ is a positive constant that can be interpreted as the scale parameter fixing the initial instantaneous volatility at time $t = 0$, $\sigma_0 = \sigma(S_0) = \delta\, S_0^{\beta/2-1}$.

While Cox (1975) has restricted the β parameter to the range $0 \leq \beta \leq 2$, Jackwerth and Rubinstein (2001) document that typical values of β implicit in the S&P 500 stock index option prices are as low as $\beta = -6$ in the post-crash of 1987. Elasticity values of $\beta < 2$ (i.e., with a direct leverage effect) are expected for stock index options and crude oil prices, whereas values of $\beta > 2$ (i.e., with an inverse leverage effect) are characteristic of some commodity spot prices and futures options with upward sloping implied volatility smiles (see, for instance, Davydov and Linetsky (2001), Geman and Shih (2009), and Dias and Nunes (2011)).

The CEV call option pricing formula for valuing European options has been initially expressed in terms of the standard complementary gamma distribution function by Cox (1975) for $\beta < 2$, and by Emanuel and MacBeth (1982) for $\beta > 2$. Schroder (1989) has subsequently extended the CEV model by expressing the time-t value of a European-style call option on the asset price S at time t, with strike X, and maturity at time T ($\geq t$) in terms of the non-central chi-square distribution as

$$c_t(S_t, X, T) = \begin{cases} S_t\, e^{-q\tau}\, Q\left(2y; 2 + \frac{2}{2-\beta}, 2x\right) - X\, e^{-r\tau}\left[1 - Q\left(2x; \frac{2}{2-\beta}, 2y\right)\right] \Leftarrow \beta < 2 \\ \\ S_t\, e^{-q\tau}\, Q\left(2x; \frac{2}{\beta-2}, 2y\right) - X\, e^{-r\tau}\left[1 - Q\left(2y; 2 + \frac{2}{\beta-2}, 2x\right)\right] \Leftarrow \beta > 2 \end{cases}, \quad (3.13)$$

with $Q(w;v,\lambda)$ being the complementary distribution function of a non-central chi-square law with v degrees of freedom and non-centrality parameter λ, and where

$$k = \frac{2(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)\tau}-1]}, \tag{3.14a}$$

$$x = k S_t^{2-\beta} e^{(r-q)(2-\beta)\tau}, \tag{3.14b}$$

$$y = kX^{2-\beta}, \tag{3.14c}$$

$$\delta^2 = \sigma_0^2 S_0^{2-\beta}, \tag{3.14d}$$

$$\tau = T - t. \tag{3.14e}$$

Even though we will concentrate our analysis on call options, the corresponding CEV put option formulae arise immediately if one applies the put-call parity, whose analytical solutions will be also used for deriving the Greeks of put options. Thus, the time-t value of a European-style put is given by

$$p_t(S_t,X,T) = \begin{cases} X e^{-r\tau} Q\left(2x; \frac{2}{2-\beta}, 2y\right) - S_t e^{-q\tau}\left[1 - Q\left(2y; 2+\frac{2}{2-\beta}, 2x\right)\right] \Leftarrow \beta < 2 \\ \\ X e^{-r\tau} Q\left(2y; 2+\frac{2}{\beta-2}, 2x\right) - S_t e^{-q\tau}\left[1 - Q\left(2x; \frac{2}{\beta-2}, 2y\right)\right] \Leftarrow \beta > 2 \end{cases}. \tag{3.15}$$

4 Computational Results

This section aims to present computational comparisons of the alternative methods of computing the non-central chi-square distribution function for pricing and hedging European options under the CEV diffusion. Hence, to better assess the speed-accuracy tradeoff between these competing methods we follow the guidelines of Broadie and Detemple (1996) by conducting a careful large sample evaluation of 2,500 randomly generated European-style call option prices under the CEV model, and therefore 5,000 randomly generated complementary non-central chi-square distribution functions. The probability distribution of option parameters and the benchmark selection are described next.

We fix the initial asset price at $S_{t_0} = 100$ and take the strike price X to be uniform between 70 and 130. The β parameter is distributed uniformly between -6 and 5. The volatility σ is distributed uniformly between 0.10 and 0.60, and the scale parameter δ is then computed. Time to maturity is, with probability 0.75, uniform between 0.1 and 1.0 years and, with probability 0.25, uniform between 1.0 and 5.0 years. The dividend yield q is uniform between 0.0 and 0.1. The riskless rate r is uniform between 0.0 and 0.1.

To compare methods, in terms of speed and accuracy, for computing non-central chi-square probabilities for pricing and hedging under the CEV model we need to choose a benchmark. An obvious candidate for a benchmark is to use the non-central chi-square distribution $F(w;v,\lambda)$ and its complementary function $Q(w;v,\lambda)$ expressed as gamma series as given by equations (3.9) and (3.10), respectively. For instance, Fraser et al. (1998) use the gamma series method as a benchmark for computing exact probabilities to be compared with several alternative methods for approximating the non-central chi-square distribution function. Alternatively, we can employ a standard numerical integration method for computing equations (3.7) and (3.8) or use a routine from an external source, such as *Matlab* or *R*, for computing non-central chi-square probabilities. The non-central chi-square distribution function $F(w;v,\lambda)$ as well as its complementary function $Q(w;v,\lambda)$ require values for w, v, and λ. For option pricing and hedging under the CEV model both w and λ can assume values of $2x$ or $2y$. We have performed extensive computational experiments (which are available upon request) with more than 200,000 parameters combinations and we have concluded that the gamma series method is an appropriate choice for our benchmark.

All the calculations in this article were made using *Mathematica* 7.0 running on a Pentium IV (2.53 GhZ) personal computer. Option prices and Greeks are computed using each of the alternative algorithms for approximating the complementary non-central chi-square distribution. We have truncated all the series with an error tolerance of 1E$-$10. In order to understand the computational speed of the alternative algorithms, we have computed the CPU times for all the algorithms using the function Timing[.] available in *Mathematica*. Since the CPU time for a single evaluation is very small, we have computed the CPU time for multiple computations.

4.1 Non-central Chi-Square Distribution Using Alternative Methods

Now we want to evaluate the differences in approximations of non-central chi-square probabilities $F(w;v,\lambda)$ for the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03), and the analytic approximations of Sankaran (1963) (S63), Fraser et al. (1998) (FWW98) and Penev and Raykov (2000)[3.1] compared against the benchmark based on the gamma series approach. We have excluded option parameter configurations with both $2x \geq 5,000$ and $2y \geq 5,000$, since it is well-known—see, for instance, Schroder (1989)—that the algorithms for computing the non-central chi-square distribution may converge slowly when $2x$ and $2y$ are large, and overflow and underflow errors may also be encountered. Out of the 2,500 options, 2,477 (and thus 4,954 non-central chi-square distribution functions) satisfied

[3.1] For the analytic method of Penev and Raykov (2000) we have considered both the second order Wiener germ approximation (PR00a) and the improved first order Wiener germ approximation (PR00b).

Table 3.1: Differences in approximations of non-central chi-square probabilities $F(w;v,\lambda)$ for each method compared against a benchmark based on the gamma series approach.

Methods	MaxAE	MaxRE	RMSE	MeanAE	CPU time	k_1
S89	7.74E−10	3.56E−03	7.77E−11	2.65E−11	7,088.97	0
D92	9.73E−11	3.73E−05	5.07E−11	4.37E−11	6,333.75	0
BK03	2.31E−11	4.85E−10	2.52E−12	1.27E−12	3,560.02	0
S63	1.10E−02	7.69E−01	3.22E−03	2.14E−03	532.23	4,910
FWW98	2.03E−01	6.20E−01	2.21E−02	7.04E−03	425.59	4,912
PR00a	3.36E−01	4.99E−01	3.96E−02	1.15E−02	1,610.15	4,675
PR00b	3.36E−01	4.99E−01	3.95E−02	1.15E−02	1,543.30	4,835

This table values the differences in approximations of 4,954 non-central chi-square probabilities $F(w;v,\lambda)$ for the iterative procedures of Schroder (1989), Ding (1992) and Benton and Krishnamoorthy (2003), and the analytic approximations of Sankaran (1963), Fraser et al. (1998) and Penev and Raykov (2000) compared against a benchmark based on the gamma series approach, which took a CPU time of 33,689.60 seconds to compute 1,000 times the whole set of 4,954 non-central chi-square distribution functions. The second rightmost column of the table reports the CPU time for computing 1,000 times the 4,954 non-central chi-square distribution functions under each alternative method. The MaxAE, MaxRE, RMSE, MeanAE, and k_1 denote, respectively, the maximum absolute error, the maximum relative error, the root mean absolute error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds E−07.

the criterion $2x < 5,000$ and $2y < 5,000$.

Table 3.1 reports the differences in approximations of non-central chi-square probabilities $F(w;v,\lambda)$ for each method compared against a benchmark based on the gamma series approach, which took a CPU time of 33,689.60 seconds to compute 1,000 times the whole set of 4,954 non-central chi-square distribution functions. The second rightmost column of the table reports the CPU time for computing 1,000 times the 4,954 non-central chi-square distribution functions under each alternative method. The MaxAE, MaxRE, RMSE, MeanAE, and k_1 denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds 1E−07.

The iterative procedures based on S89, D92, and BK03 methods are accurate for determining non-central chi-square probabilities that are needed for computing option prices under the CEV model. The S89, D92, and BK03 methods are, respectively, 4.75, 5.32, and 9.46 times faster than the benchmark GS method. The computational results show that the iterative procedure of BK03 is the best choice for computing the non-central chi-square distribution function since it is the most accurate one and the fastest.

As expected, the analytic approximations run quickly but have an accuracy that varies significantly over the considered parameter space. Thus, for small to moderate values of $2y$ and $2x$ none of the approximation methods should be used and the preference is to use the BK03 method.

It is well-known that the running time needed for computing the non-central chi-square distribution $F(w;v,\lambda)$, and its complementary distribution function $Q(w;v,\lambda)$, increases when w and λ are large. Option pricing under the CEV assumption is computationally expensive especially when β is close to two, volatility is low, or the time to maturity is small in the CEV formulae. For this reason, Schroder (1989) has suggested a two-part strategy for computing the non-central chi-square distribution where for small to moderate values of w and λ the iterative procedure is used, otherwise the distribution is evaluated using the analytic approximation of Sankaran (1963).

Computational experiments of the remaining 46 parameter configurations reveal, however, that while GS, S89, and D92 methods may converge slowly or produce overflow and underflow errors, the BK03 method always gives the correct value without additional commensurate computational burden. Thus, even though S63, FWW98, PR00a, and PR00b methods all produce accurate results when w and λ are very large, our computational results have shown that it is not necessary to use a combined two-part strategy for computing the non-central chi-square distribution, since the Benton and Krishnamoorthy (2003) algorithm clearly offers the best speed-accuracy tradeoff.

4.2 Option Pricing under the CEV Model

Even thought we have already analyzed the speed and accuracy of alternative methods for computing the non-central chi-square distribution at the statistic level, it is also relevant to understand how quickly and accurate are those competing methods for pricing and hedging purposes under the CEV model. We will concentrate our analysis on call options, but the same line of reasoning applies also for put options. Following Broadie and Detemple (1996), and to make the relative error meaningful, we have excluded call options whose price was lower than 50 cents. Out of the 2,477 options, 2,393 satisfied this criterion. However, all error measures yielded qualitatively similar results when we have included option values less than 50 cents.

Table 3.2 values the differences in call option prices under the CEV assumption using the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03), and the analytic approximations of Sankaran (1963) (S63), Fraser et al. (1998) (FWW98) and Penev and Raykov (2000) (PR00a and PR00b) compared against the benchmark based on the gamma series approach, which took a CPU time of 22,460.00 seconds to compute 1,000 times the whole set of 2,393 call option prices. The second rightmost column of the table reports the CPU time for computing 1,000 times the 2,393 call option prices under each alternative method. The MaxAE, MaxRE, RMSE, MeanAE, and k_2 denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds \$0.01.

The results of Table 3.2 highlight that the iterative procedures of S89, D92, and BK03 are all accurate for computing options prices under the CEV assumption, though the iterative procedure of BK03 is still the most efficient for determining option prices. The

Table 3.2: Differences in call option prices using each alternative method for computing the noncentral chi-square distribution compared against a benchmark based on the gamma series approach.

Methods	MaxAE	MaxRE	RMSE	MeanAE	CPU time	k_2
S89	1.31E−07	2.48E−07	1.32E−08	4.48E−09	5,740.57	0
D92	2.09E−08	3.92E−08	9.05E−09	8.09E−09	5,082.62	0
BK03	1.92E−09	3.21E−09	3.65E−10	2.20E−10	3,155.60	0
S63	1.46E+00	8.26E−02	2.60E−01	1.58E−01	475.10	1,887
FWW98	1.87E+01	4.28E−01	2.82E+00	1.10E+00	378.54	2,129
PR00a	2.51E+01	4.69E+00	4.79E+00	2.01E+00	1,498.78	1,849
PR00b	2.50E+01	4.67E+00	4.78E+00	2.00E+00	1,441.67	1,910

This table values the differences in call option prices under the CEV model using the iterative procedures of Schroder (1989), Ding (1992) and Benton and Krishnamoorthy (2003), and the analytic approximations of Sankaran (1963), Fraser et al. (1998) and Penev and Raykov (2000) compared against a benchmark based on the gamma series approach, which took a CPU time of 22,460.00 seconds to compute 1,000 times the whole set of 2,393 call option prices. The second rightmost column of the table reports the CPU time for computing 1,000 times the 2,393 call option prices under each alternative method. The MaxAE, MaxRE, RMSE, MeanAE, and k_2 denote, respectively, the maximum absolute error, the maximum relative error, the root mean absolute error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds \$0.01.

S89, D92, and BK03 methods are, respectively, 3.91, 4.42, and 7.12 times faster than the benchmark GS method.

The analytic approximations run quickly but all have an unsatisfactory accuracy given the high k_2 value they generate, though all approximation methods returns a value $k_2 = 0$ when w and λ are very large.

To sum up, we can conclude that the GS benchmark method is not worth considering due to speed and that the Benton and Krishnamoorthy (2003) algorithm offers the best speed-accuracy tradeoff for pricing and hedging options under the CEV model. This findings are clearly relevant for future work aiming to value options contracts with early exercise features and/or exotic payoffs under the one-dimensional CEV model.

4.3 Greeks under the CEV Model

Information about derivatives of options (commonly known as Greeks) is of practical and theoretical importance. In addition to pricing an option, a dealer of the financial services industry must also be able to hedge it. Thus, a practitioner needs to have knowledge regarding the sensitivity measures of derivative securities for designing hedging strategies to reduce the risk of a given security or a portfolio of securities, when closing the position is not viable or desirable. Greeks also enjoy many other multiple applications such as for market risk measurement, profit and loss attribution, model risk assessment and optimal contract design, and to imply out parameters from market prices.

For European-type options on dividend paying assets under the log-normal assumption, closed-form expressions for *delta* ($\partial/\partial S$), *gamma* ($\partial^2/\partial S^2$), *vega* ($\partial/\partial\sigma$), *theta* ($\partial/\partial t$), *rho* ($\partial/\partial r$), and *phi* (or *rho-q*, ($\partial/\partial q$)) are well documented in the literature (e.g., Hull (2008, Chapter 17)). Shadwick and Shadwick (2002) consider the implications of the use of the general BSM model for pricing and sensitivities calculations. Pelsser and Vorst (1994) discuss the computation of these Greeks under the binomial option pricing model of Cox et al. (1979). Garman (1992) introduces three more partial derivatives for derivative instruments, namely the *speed* ($\partial^3/\partial S^3$), the *charm* ($\partial^2/\partial S\partial t$), and the *colour* ($\partial^3/\partial S^2\partial t$). Many other Greeks of options are discussed in Haug (2006).

Derivative information of option prices are also important at a theoretical level. For instance, Breeden and Litzenberger (1978) show that the second derivative with respect to the strike price ($\partial^2/\partial X^2$) can be interpreted as a state price density. Carr (2001) shows how delta, gamma, speed and other higher-order derivatives of an option's price with respect to the initial price of the underlying asset can be viewed as an expectation, through an appropriate change of measure, of the corresponding derivative at the terminal date. Bergman et al. (1996) derive a general theoretical expression for delta when volatility is a function of stock price and time.

In the Appendix we present closed-form solutions for delta, gamma, vega, theta[3.2], and rho under the CEV option pricing model for both $\beta < 2$ and $\beta > 2$ that, to our knowledge, have not been published in the literature.[3.3] Based on these new closed-form solutions, we should also consider how different methods for computing the complementary non-central chi-square distribution affect the computation of Greeks.

Table 3.3 shows results for deltas (Δ), gammas (Γ), vegas ($\mathscr{V}$), thetas (Θ), and rhos (ρ) for European-style standard call and put options under the CEV assumption for different specifications of the option parameters. The last five lines of the table report the CPU times for computing 1,000 times the Greeks of the twenty one option contracts using the closed-form solutions presented in the Appendix based on the gamma series method (CPU time 1), on the iterative procedures of Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) (CPU time 2-4, respectively), and via elementary differentiation of the gamma series method through *Mathematica* with $nmax = 200$ (CPU time 5).

Several points are noteworthy from Table 3.3. While symbolic algebra programs such as *Mathematica* or *Maple* can derive such sensitivity measures[3.4], these new closed-form solutions for determining Greeks under the CEV model are important at least for three reasons. Firstly, as stated by Carr (2001), the derivation of Greeks through

[3.2]The sign of the theta of an option is generally negative, but there are some exceptions: for example, an in-the-money European put option on a non-dividend-paying stock or an in-the-money European call option on a currency with a very high interest rate. For this reason, we have omitted the sign behavior for thetas. However, it is well-known that if the underlying asset pays no dividends over the life of the contingent claim, then the corresponding call option is always a wasting asset in any one-dimensional diffusion setting (see, for instance, Bergman et al. (1996, Proposition 3)).

[3.3]In addition, we have determined also other partial derivatives for options under the CEV model. Even though their closed-form solutions are omitted here due to constraints of space, they are available upon request.

[3.4]For instance, Shaw (1998) shows how to derive Greeks under the geometric Brownian motion assumption via elementary differentiation using *Mathematica*. A similar symbolic algebra procedure can be used to derive any other arbitrary Greek under alternative stochastic processes.

Table 3.3: Greeks for European-style standard call and put options under the CEV assumption.

X	β	European standard call options					European standard put options				
		Delta	Gamma	Vega	Theta	Rho	Delta	Gamma	Vega	Theta	Rho
95	5	0.7912	0.0161	21.7198	-11.7066	31.3830	-0.2088	0.0161	21.7198	-2.6699	-13.8004
95	3	0.7616	0.0174	22.3075	-11.8030	31.1304	-0.2384	0.0174	22.3075	-2.7663	-14.0530
95	1	0.7293	0.0189	23.0771	-11.9410	30.8583	-0.2707	0.0189	23.0771	-2.9043	-14.3251
95	0	0.7118	0.0198	23.5433	-12.0286	30.7141	-0.2882	0.0198	23.5433	-2.9920	-14.4693
95	-2	0.6735	0.0218	24.6998	-12.2559	30.4045	-0.3265	0.0218	24.6998	-3.2192	-14.7789
95	-4	0.6286	0.0244	26.3712	-12.6035	30.0534	-0.3714	0.0244	26.3712	-3.5668	-15.1300
95	-6	0.5743	0.0277	28.5840	-13.0736	29.6381	-0.4257	0.0277	28.5840	-4.0369	-15.5453
100	5	0.6938	0.0197	26.5816	-12.1366	27.4562	-0.3062	0.0197	26.5816	-2.6243	-20.1053
100	3	0.6611	0.0206	26.3580	-12.0791	27.4483	-0.3389	0.0206	26.3580	-2.5669	-20.1132
100	1	0.6282	0.0216	26.3580	-12.0791	27.4483	-0.3718	0.0216	26.3580	-2.5669	-20.1132
100	0	0.6113	0.0222	26.4399	-12.1002	27.4513	-0.3887	0.0222	26.4399	-2.5879	-20.1102
100	-2	0.5763	0.0236	26.7924	-12.1907	27.4628	-0.4237	0.0236	26.7924	-2.6784	-20.0986
100	-4	0.5380	0.0255	27.5108	-12.3736	27.4796	-0.4620	0.0255	27.5108	-2.8613	-20.0818
100	-6	0.4946	0.0278	28.6855	-12.6709	27.4975	-0.5054	0.0278	28.6855	-3.1586	-20.0640
105	5	0.5944	0.0218	29.4169	-12.0217	23.3376	-0.4056	0.0218	29.4169	-2.0338	-26.6020
105	3	0.5561	0.0222	28.4533	-11.7794	23.3304	-0.4439	0.0222	28.4533	-1.7915	-26.6092
105	1	0.5202	0.0228	27.7716	-11.6182	23.3764	-0.4798	0.0228	27.7716	-1.6303	-26.5632
105	0	0.5028	0.0231	27.5180	-11.5632	23.4182	-0.4972	0.0231	27.5180	-1.5752	-26.5213
105	-2	0.4686	0.0240	27.1780	-11.5023	23.5387	-0.5314	0.0240	27.1780	-1.5144	-26.4008
105	-4	0.4344	0.0251	27.0996	-11.5166	23.7086	-0.5656	0.0251	27.0996	-1.5287	-26.2309
105	-6	0.3988	0.0265	27.3464	-11.6229	23.9313	-0.6012	0.0265	27.3464	-1.6350	-26.0082
CPU time 1		61.43	5.15	3.04	115.33	57.46	61.64	5.15	3.04	115.04	57.80
CPU time 2		28.97	5.15	3.04	55.88	30.56	29.28	5.15	3.04	55.99	30.76
CPU time 3		26.44	5.15	3.04	51.64	27.78	26.52	5.15	3.04	51.08	28.38
CPU time 4		23.62	5.15	3.04	42.74	24,13	23.57	5.15	3.04	43.23	23.42
CPU time 5		1,170.49	2,860.75	2,386.21	3,193.73	3,085.76	1,168.17	2,860.75	2,386.21	3,192.33	3,086.65

This table values deltas (Δ), gammas (Γ), vegas ($\mathscr{V}$), thetas (Θ), and rhos (ρ) for European-style standard call and put options under the CEV assumption for different specifications of the option parameters. The last five lines of the table report the CPU times for computing 1,000 times the Greeks of the twenty one option contracts using the closed-form solutions presented in the appendix based on the gamma series method (CPU time 1), on the iterative procedures of Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) (CPU time 2-4, respectively), and via elementary differentiation of the gamma series method through *Mathematica* with $nmax = 200$ (CPU time 5). Parameters used in the calculations: $S_0 = 100$, $\sigma_0 = \sigma(S_0) = 0.25$, $r = 0.10$, $q = 0$, and $\tau = 0.50$.

symbolic algebra programs cannot replace an intuitive understanding of the role, genesis, and relationships between all the various Greeks. Secondly, the computation time needed for computing analytic Greeks will diminish substantially, which is extremely relevant when one needs to design hedging strategies through time. For example, while options under the CEV model have non-zero gammas and vegas, these two Greek measures are not affected by the complementary non-central chi-square distribution. Thus, the small computational expense needed for computing gammas and vegas is especially notable. For the other Greeks (i.e., delta, theta, and rho) the BK03 is, as expected, the most efficient method. Lastly, the existence of analytical solutions allows that they can be coded in any desired computer language such as *Matlab*, *Fortran*, *R*, or *C*.

The knowledge of these new analytical solutions for deltas is also of interest for pricing standard American options under the CEV model using the SHP approach offered by Chung and Shih (2009). Even though they argue that their SHP approach works well for pricing American options under the CEV model, they only consider the case where $\beta = 4/3$ to simplify their numerical analysis and given the absence of closed-form solutions of deltas for a general β parameter. As observed by Schroder (1989), the prices of standard European calls and puts under the CEV assumption with $\beta = 4/3$ are easy to compute since the corresponding complementary non-central chi-square distribution functions $Q(w;1,\lambda)$, $Q(w;3,\lambda)$, and $Q(w;5,\lambda)$ can be determined using only the standard normal density function $N'(.)$ and the standard normal distribution function $N(.)$. This also means that when $\beta = 4/3$ we may determine the analytical expression for the corresponding delta using the same type of algebra as in a BSM world. With our closed-form solutions for deltas we may price (and hedge) plain-vanilla American options under the SHP approach as proposed by Chung and Shih (2009), but now for any β parameter able to accommodate direct leverage effects or inverse leverage effects. This will be left for future research.

5 Conclusions

In this article, we compare the performance of alternative algorithms for computing the non-central chi-square distribution function in terms of accuracy and computation time for evaluating option prices and Greeks under the CEV model. We find that the gamma series method and the iterative procedures of Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003) are all accurate over a wide range of parameters, though presenting significative speed computation differences. The analytic approximations of Sankaran (1963), Fraser et al. (1998), and Penev and Raykov (2000) run quickly but produce significant errors when w and λ are not too large. Our computational experiments have shown that the Benton and Krishnamoorthy (2003) algorithm clearly offers the best speed-accuracy tradeoff. Finally, we present closed-form solutions for computing Greek measures under the unrestricted CEV option pricing model for both $\beta < 2$ and $\beta > 2$, thus being able to accommodate direct leverage effects as well as inverse leverage effects that are frequently observed in the options markets.

Acknowledgements

The authors thank the helpful comments and suggestions addressed by two anonymous referees. The authors thank also the participants of the 6th International Conference of the Portuguese Finance Network, the Mathematical Finance Days 2011 Conference, and the 18th International Conference on Forecasting Financial Markets for helpful comments on a previous draft of the paper. Dias gratefully acknowledges the financial support from the Fundação para a Ciência e Tecnologia (FCT) grant number PTDC/EGE-ECO/099255/2008 and Larguinho and Braumann acknowledge the financial support from the Centro de Investigação em Matemática e Aplicações (CIMA) financed by the FCT.

Appendix A: Greeks for the CEV Option Pricing Model

This appendix shows in detail how to compute Greeks for European options under the CEV diffusion. To simplify notation all subscripts t are omitted throughout this appendix. To determine all Greeks we need to compute the auxiliary functions $2x$ and $2y$. After straightforward calculations it follows that

$$2x = \frac{4(r-q)\,e^{(r-q)(2-\beta)\tau}}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)\tau}-1]}\,S^{2-\beta}, \tag{A.1a}$$

$$2y = \frac{4(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)\tau}-1]}\,X^{2-\beta}. \tag{A.1b}$$

There are a few relations that are useful for deriving Greeks under the CEV diffusion. Let us start with the well-known relation $Q(w;v,\lambda) = 1 - F(w;v,\lambda)$. Now, based on Johnson et al. (1995, pp. 442-443) we conclude that

$$\frac{\partial Q(w;v,\lambda)}{\partial w} = \frac{\partial[1-F(w;v,\lambda)]}{\partial w} = -\frac{\partial F(w;v,\lambda)}{\partial w} = -p(w;v,\lambda), \tag{A.2a}$$

$$\frac{\partial Q(w;v,\lambda)}{\partial \lambda} = \frac{\partial[1-F(w;v,\lambda)]}{\partial \lambda} = -\frac{\partial F(w;v,\lambda)}{\partial \lambda} = p(w;v+2,\lambda), \tag{A.2b}$$

where $p(w;v,\lambda)$ is the probability density function of a non-central chi-square distribution as given by equation (3.5). Armed with these formulae, we are now ready to derive all Greeks under the CEV diffusion process. For each Greek letter, four cases will be considered.

A.1 Delta

To compute deltas (and gammas) we also need the following auxiliary functions:

$$\frac{\partial 2x}{\partial S} = \frac{2x(2-\beta)}{S}, \tag{A.3a}$$

$$\frac{\partial 2y}{\partial S} = 0. \tag{A.3b}$$

1. Call delta for $\beta < 2$.

$$\Delta_{\text{call}} = \frac{\partial c}{\partial S} = e^{-q\tau} Q\Big(2y; 2+\frac{2}{2-\beta}, 2x\Big) + Se^{-q\tau} \frac{\partial Q\Big(2y; 2+\frac{2}{2-\beta}, 2x\Big)}{\partial S}$$
$$+ Xe^{-r\tau} \frac{\partial Q\Big(2x; \frac{2}{2-\beta}, 2y\Big)}{\partial S}. \tag{A.4}$$

Using equations (A.2a), (A.2b), (A.3a), and (A.3b) we are able to compute the following partial derivatives:

$$\frac{\partial Q(2y;v,2x)}{\partial S} = \frac{\partial Q(2y;v,2x)}{\partial 2x}\frac{\partial 2x}{\partial S} = \frac{2x(2-\beta)}{S} p(2y;v+2,2x), \tag{A.5}$$

$$\frac{\partial Q(2x;v,2y)}{\partial S} = \frac{\partial Q(2x;v,2y)}{\partial 2x}\frac{\partial 2x}{\partial S} = -\frac{2x(2-\beta)}{S} p(2x;v,2y). \tag{A.6}$$

Substituting equations (A.5) and (A.6) in equation (A.4) yields

$$\Delta_{\text{call}} = e^{-q\tau} Q\Big(2y; 2+2/(2-\beta), 2x\Big) + \frac{2x(2-\beta)}{S} \Big[Se^{-q\tau} p(2y; 4+2/(2-\beta), 2x) - Xe^{-r\tau} p(2x; 2/(2-\beta), 2y)\Big] > 0. \tag{A.7}$$

2. Put delta for $\beta < 2$.

$$\Delta_{\text{put}} = \frac{\partial p}{\partial S} = -e^{-q\tau}\Big[1 - Q\Big(2y; 2+2/(2-\beta), 2x\Big)\Big] + \frac{2x(2-\beta)}{S} \Big[Se^{-q\tau} p(2y; 4+2/(2-\beta), 2x) - Xe^{-r\tau} p(2x; 2/(2-\beta), 2y)\Big] < 0. \tag{A.8}$$

3. Call delta for $\beta > 2$.

$$\Delta_{\text{call}} = \frac{\partial c}{\partial S} = e^{-q\tau} Q\Big(2x; 2/(\beta-2), 2y\Big) - \frac{2x(2-\beta)}{S} \Big[Se^{-q\tau} p(2x; 2/(\beta-2), 2y) - Xe^{-r\tau} p(2y; 4+2/(\beta-2), 2x)\Big] > 0. \tag{A.9}$$

4. Put delta for $\beta > 2$.

$$\Delta_{\text{put}} = \frac{\partial p}{\partial S} = -e^{-q\tau}\Big[1 - Q\Big(2x; 2/(\beta-2), 2y\Big)\Big] - \frac{2x(2-\beta)}{S} \Big[Se^{-q\tau} p(2x; 2/(\beta-2), 2y) - Xe^{-r\tau} p(2y; 4+2/(\beta-2), 2x)\Big] < 0. \tag{A.10}$$

A.2 Gamma

To compute gammas we also need to use the following recurrence relations obtained by Cohen (1988):

$$\begin{aligned}
\frac{\partial p(w;v,\lambda)}{\partial \lambda} &= \frac{1}{2}[-p(w;v,\lambda)+p(w;v+2,\lambda)], && \text{(A.11a)}\\
\frac{\partial p(w;v,\lambda)}{\partial w} &= \frac{1}{2}[-p(w;v,\lambda)+p(w;v-2,\lambda)], && \text{(A.11b)}\\
p(w;v-2,\lambda) &= \frac{\lambda}{w}p(w;v+2,\lambda)+\frac{v-2}{w}p(w;v,\lambda). && \text{(A.11c)}
\end{aligned}$$

1. Call and put gamma for $\beta < 2$.

$$\begin{aligned}
\Gamma_{\text{call}} = \frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta_{\text{call}}}{\partial S} = \Gamma_{\text{put}} = \frac{\partial^2 p}{\partial S^2} = \frac{\partial \Delta_{\text{put}}}{\partial S} &= \frac{2x(2-\beta)^2}{S}e^{-q\tau} \qquad \text{(A.12)}\\
&\Big[((3-\beta)/(2-\beta)-x)\,p(2y;4+2/(2-\beta),2x)+x\,p(2y;6+2/(2-\beta),2x)\Big]\\
&+\frac{2x(2-\beta)^2}{S}e^{-r\tau}\frac{X}{S}\Big[x\,p(2x;2/(2-\beta),2y)-y\,p(2x;2+2/(2-\beta),2y)\Big] > 0.
\end{aligned}$$

2. Call and put gamma for $\beta > 2$.

$$\begin{aligned}
\Gamma_{\text{call}} = \frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta_{\text{call}}}{\partial S} = \Gamma_{\text{put}} = \frac{\partial^2 p}{\partial S^2} = \frac{\partial \Delta_{\text{put}}}{\partial S} &= \frac{2x(2-\beta)^2}{S}e^{-r\tau}\frac{X}{S} \qquad \text{(A.13)}\\
&\Big[((1-\beta)/(2-\beta)-x)\,p(2y;4+2/(\beta-2),2x)+x\,p(2y;6+2/(\beta-2),2x)\Big]\\
&+\frac{2x(2-\beta)^2}{S}e^{-q\tau}\Big[x\,p(2x;2/(\beta-2),2y)-y\,p(2x;2+2/(\beta-2),2y)\Big] > 0.
\end{aligned}$$

A.3 Vega

1. Call and put vega for $\beta < 2$.

$$\begin{aligned}\mathscr{V}_{\text{call}} &= \frac{\partial c}{\partial \sigma} = \mathscr{V}_{\text{put}} = \frac{\partial p}{\partial \sigma} \\ &= -\frac{4x}{\delta}\Big[S e^{-q\tau}\, p(2y;4+2/(2-\beta),2x) - X e^{-r\tau}\, p(2x;2/(2-\beta),2y)\Big]\, S_0^{(2-\beta)/2} \\ &= -\frac{4x}{\sigma_0}\Big[S e^{-q\tau}\, p(2y;4+2/(2-\beta),2x) - X e^{-r\tau}\, p(2x;2/(2-\beta),2y)\Big] > 0.\end{aligned} \tag{A.14}$$

2. Call and put vega for $\beta > 2$.

$$\begin{aligned}\mathscr{V}_{\text{call}} &= \frac{\partial c}{\partial \sigma} = \mathscr{V}_{\text{put}} = \frac{\partial p}{\partial \sigma} \\ &= \frac{4x}{\delta}\Big[S e^{-q\tau}\, p(2x;2/(\beta-2),2y) - X e^{-r\tau}\, p(2y;4+2/(\beta-2),2x)\Big]\, S_0^{(2-\beta)/2} \\ &= \frac{4x}{\sigma_0}\Big[S e^{-q\tau}\, p(2x;2/(\beta-2),2y) - X e^{-r\tau}\, p(2y;4+2/(\beta-2),2x)\Big] > 0.\end{aligned} \tag{A.15}$$

A.4 Theta

1. Call theta for $\beta < 2$.

$$\begin{aligned}\Theta_{\text{call}} &= \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} = S q e^{-q\tau}\, Q\Big(2y;2+2/(2-\beta),2x\Big) \\ &\quad - X r e^{-r\tau}\Big[1 - Q\Big(2x;2/(2-\beta),2y\Big)\Big] + \frac{2x(r-q)(2-\beta)}{e^{(r-q)(2-\beta)\tau}-1} \\ &\quad \Big[S e^{-q\tau}\, p(2y;4+2/(2-\beta),2x) - X e^{-r\tau}\, p(2x;2/(2-\beta),2y)\Big].\end{aligned} \tag{A.16}$$

2. Put theta for $\beta < 2$.

$$
\begin{aligned}
\Theta_{\text{put}} = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} &= X r e^{-r\tau} Q\Big(2x; 2/(2-\beta), 2y\Big) \\
&-S q e^{-q\tau}\Big[1 - Q\Big(2y; 2+2/(2-\beta), 2x\Big)\Big] + \frac{2x(r-q)(2-\beta)}{e^{(r-q)(2-\beta)\tau}-1} \\
&\Big[S e^{-q\tau} p(2y; 4+2/(2-\beta), 2x) - X e^{-r\tau} p(2x; 2/(2-\beta), 2y)\Big].
\end{aligned}
\tag{A.17}
$$

3. Call theta for $\beta > 2$.

$$
\begin{aligned}
\Theta_{\text{call}} = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} &= S q e^{-q\tau} Q\Big(2x; 2/(\beta-2), 2y\Big) \\
&-X r e^{-r\tau}\Big[1 - Q\Big(2y; 2+2/(\beta-2), 2x\Big)\Big] - \frac{2x(r-q)(2-\beta)}{e^{(r-q)(2-\beta)\tau}-1} \\
&\Big[S e^{-q\tau} p(2x; 2/(\beta-2), 2y) - X e^{-r\tau} p(2y; 4+2/(\beta-2), 2x)\Big].
\end{aligned}
\tag{A.18}
$$

4. Put theta for $\beta > 2$.

$$
\begin{aligned}
\Theta_{\text{put}} = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} &= X r e^{-r\tau} Q\Big(2y; 2+2/(\beta-2), 2x\Big) \\
&-S q e^{-q\tau}\Big[1 - Q\Big(2x; 2/(\beta-2), 2y\Big)\Big] - \frac{2x(r-q)(2-\beta)}{e^{(r-q)(2-\beta)\tau}-1} \\
&\Big[S e^{-q\tau} p(2x; 2/(\beta-2), 2y) - X e^{-r\tau} p(2y; 4+2/(\beta-2), 2x)\Big].
\end{aligned}
\tag{A.19}
$$

A.5 Rho

1. Call rho for $\beta < 2$.

$$\rho_{\text{call}} = \frac{\partial c}{\partial r} = X\,\tau\,e^{-r\tau}\left[1 - Q\Big(2x; 2/(2-\beta), 2y\Big)\right] + 2x\left(\frac{1}{r-q} - \frac{(2-\beta)\tau}{e^{(r-q)(2-\beta)\tau} - 1}\right)$$
$$\left[S\,e^{-q\tau}\,p(2y; 4+2/(2-\beta), 2x) - X\,e^{-r\tau}\,p(2x; 2/(2-\beta), 2y)\right] > 0. \qquad \text{(A.20)}$$

2. Put rho for $\beta < 2$.

$$\rho_{\text{put}} = \frac{\partial p}{\partial r} = -X\,\tau\,e^{-r\tau}\,Q\Big(2x; 2/(2-\beta), 2y\Big) + 2x\left(\frac{1}{r-q} - \frac{(2-\beta)\tau}{e^{(r-q)(2-\beta)\tau} - 1}\right)$$
$$\left[S\,e^{-q\tau}\,p(2y; 4+2/(2-\beta), 2x) - X\,e^{-r\tau}\,p(2x; 2/(2-\beta), 2y)\right] < 0. \qquad \text{(A.21)}$$

3. Call rho for $\beta > 2$.

$$\rho_{\text{call}} = \frac{\partial c}{\partial r} = X\,\tau\,e^{-r\tau}\left[1 - Q\Big(2y; 2+2/(\beta-2), 2x\Big)\right] - 2x\left(\frac{1}{r-q} - \frac{(2-\beta)\tau}{e^{(r-q)(2-\beta)\tau} - 1}\right)$$
$$\left[S\,e^{-q\tau}\,p(2x; 2/(\beta-2), 2y) - X\,e^{-r\tau}\,p(2y; 4+2/(\beta-2), 2x)\right] > 0. \qquad \text{(A.22)}$$

4. Put rho for $\beta > 2$.

$$\rho_{\text{put}} = \frac{\partial p}{\partial r} = -X\,\tau\,e^{-r\tau}\,Q\Big(2y; 2+2/(\beta-2), 2x\Big) - 2x\left(\frac{1}{r-q} - \frac{(2-\beta)\tau}{e^{(r-q)(2-\beta)\tau} - 1}\right)$$
$$\left[S\,e^{-q\tau}\,p(2x; 2/(\beta-2), 2y) - X\,e^{-r\tau}\,p(2y; 4+2/(\beta-2), 2x)\right] < 0. \qquad \text{(A.23)}$$

References

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Ağca, Senay and Don M. Chance, 2003, Speed and Accuracy Comparison of Bivariate Normal Distribution Approximations for Option Pricing, *Journal of Computational Finance* 6, 61–96.

Bajo, Emanuele and Massimiliano Barbi, 2010, The Risk-Shifting Effect and the Value of a Warrant, *Quantitative Finance* 10, 1203–1213.

Bekaert, Geert and Guojun Wu, 2000, Asymetric Volatility and Risk in Equity Markets, *Review of Financial Studies* 13, 1–42.

Benton, Denise and K. Krishnamoorthy, 2003, Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral *t* and the Distribution of the Square of the Sample Multiple Correlation Coefficient, *Computational Statistics and Data Analysis* 43, 249–267.

Bergman, Yaacov Z., Bruce D. Grundy, and Zvi Wiener, 1996, General Properties of Option Prices, *Journal of Finance* 51, 1573–1610.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Breeden, Douglas T. and Robert H. Litzenberger, 1978, Prices of State-Contingent Claims Implicit in Option Prices, *Journal of Business* 51, 621–651.

Broadie, Mark and Jérôme Detemple, 1996, American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods, *Review of Financial Studies* 9, 1211–1250.

Carr, Peter, 2001, Deriving Derivatives of Derivative Securities, *Journal of Computational Finance* 4, 5–29.

Carr, Peter and Vadim Linetsky, 2006, A Jump to Default Extended CEV Model: An Application of Bessel Processes, *Finance and Stochastics* 10, 303–330.

Chung, San-Lin and Pai-Ta Shih, 2009, Static Hedging and Pricing American Options, *Journal of Banking and Finance* 33, 2140–2149.

Cohen, Jonathan D., 1988, Noncentral Chi-Square: Some Observations on Recurrence, *American Statistician* 42, 120–122.

Cox, John C., 1975, Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working Paper, Stanford University. Reprinted in *Journal of Portfolio Management*, 23 (1996), 15-17.

Cox, John C. and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics* 3, 145–166.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229–263.

Davydov, Dmitry and Vadim Linetsky, 2001, Pricing and Hedging Path-Dependent Options under the CEV Process, *Management Science* 47, 949–965.

Dennis, Patrick and Stewart Mayhew, 2002, Risk-Neutral Skewness: Evidence from Stock Options, *Journal of Financial and Quantitative Analysis* 37, 471–493.

Dias, José Carlos and João Pedro Nunes, 2011, Pricing Real Options under the Constant Elasticity of Variance Diffusion, *Journal of Futures Markets* 31, 230–250.

Ding, Cherng G., 1992, Algorithm AS 275: Computing the Non-Central χ^2 Distribution Function, *Applied Statistics* 41, 478–482.

Dyrting, S., 2004, Evaluating the Noncentral Chi-Square Distribution for the Cox-Ingersoll-Ross Process, *Computational Economics* 24, 35–50.

Emanuel, David C. and James D. MacBeth, 1982, Further Results on the Constant Elasticity of Variance Call Option Pricing Model, *Journal of Financial and Quantitative Analysis* 17, 533–554.

Farebrother, R. W., 1987, Algorithm AS 231: The Distribution of a Noncentral χ^2 Variable with Nonnegative Degrees of Freedom, *Applied Statistics* 36, 402–405.

Fraser, D. A. S., J. Wu, and A. C. M. Wong, 1998, An Approximation for the Noncentral Chi-Squared Distribution, *Communications in Statistics - Simulation and Computation* 27, 275–287.

Garman, Mark, 1992, Charm School, *Risk* 5, 53–56.

Geman, Hélyette and Yih Fong Shih, 2009, Modeling Commodity Prices Under the CEV Model, *Journal of Alternative Investments* 11, 65–84.

Grünbichler, Andreas and Francis A. Longstaff, 1996, Valuing Futures and Options on Volatility, *Journal of Banking and Finance* 20, 985–1001.

Haug, Espen Gaarder, 2006, *The Complete Guide to Option Pricing Formulas*, 2nd ed. (McGraw-Hill, New York, NY).

Hull, John C., 2008, *Options, Futures, and Other Derivatives*, 7th ed. (Prentice Hall, Upper Saddle River, NJ).

Jackwerth, Jens Carsten and Mark Rubinstein, 1996, Recovering Probability Distributions from Option Prices, *Journal of Finance* 51, 1611–1631.

Jackwerth, Jens Carsten and Mark Rubinstein, 2001, Recovering Stochastic Processes from Option Prices. Working Paper, London Business School.

Johnson, Norman L., Samuel Kotz, and N. Balakrishnan, 1995, *Continuous Univariate Distributions*, Vol. 2, 2nd ed. (John Wiley & Sons, New York, NY).

Knüsel, L. and B. Bablok, 1996, Computation of the Noncentral Gamma Distribution, *SIAM Journal of Scientific Computing* 17, 1224–1231.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Nunes, João Pedro, 2009, Pricing American Options under the Constant Elasticity of Variance Model and Subject to Bankruptcy, *Journal of Financial and Quantitative Analysis* 44, 1231–1263. Correction: Nunes, J. P., 2012, American-style Puts under the JDCEV Model: A Correction. JFQA web site at http://depts.washington.edu/jfqa/ in Unpublished Appendixes, Notes, Comments, and Corrections.

Nunes, João Pedro and José Carlos Dias, 2012, The Valuation of Double Barrier Options under Multifactor Pricing Models. Working Paper, BRU-UNIDE and ISCTE-IUL Business School.

Pelsser, Antoon and Ton Vorst, 1994, The Binomial Model and the Greeks, *Journal of Derivatives* 1, 45–49.

Penev, Spiridon and Tenko Raykov, 2000, A Wiener Germ Approximation of the Noncentral Chi Square Distribution and of its Quantiles, *Computational Statistics* 15, 219–228.

Posten, Harry O., 1989, An Effective Algorithm for the Noncentral Chi-Squared Distribution Function, *American Statistician* 43, 261–263.

Sankaran, Munuswamy, 1963, Approximations to the Non-Central Chi-Square Distribution, *Biometrika* 50, 199–204.

Schroder, Mark, 1989, Computing the Constant Elasticity of Variance Option Pricing Formula, *Journal of Finance* 44, 211–219.

Shadwick, Brad A. and William F. Shadwick, 2002, On the Computation of Option Prices and Sensitivities in the Black-Scholes-Merton Model, *Quantitative Finance* 2, 158–166.

Shaw, William, 1998, *Modelling Financial Derivatives with Mathematica* (Cambridge University Press, Cambridge, UK).

Chapter 4

Valuation of Bond Options under the CIR Model: Some Computational Remarks*

Abstract: Pricing bond options under the Cox, Ingersoll and Ross (CIR) model of the term structure of interest rates requires the computation of the noncentral chi-square distribution function. In this article, we compare the performance in terms of accuracy and computational time of alternative methods for computing such probability distributions against an externally tested benchmark. All methods are generally accurate over a wide range of parameters that are frequently needed for pricing bond options, though they all present relevant differences in terms of running times. The iterative procedure of Benton and Krishnamoorthy (2003) is the most efficient in terms of accuracy and computational burden for determining bond option prices under the CIR assumption.

JEL Classification: G13.

Keywords: CIR model; Bond options; Computational finance.

*This paper is a joint work with José Carlos Dias and Carlos A. Braumann and is accepted for publication in selected papers of SPE 2011.

1 Introduction

The CIR model is a general single-factor equilibrium model developed by Cox et al. (1985), and has been used throughout the years because of its analytical tractability and the fact that the short rate is always positive, contrary to the well-known Vasicek model of Vasicek (1977).

The CIR model is used to price zero-coupon bonds, coupon bonds and to price options on these bonds. To compute option prices under this process we need to use the non-central chi-square distribution function. There exists an extensive literature devoted to the efficient computation of this distribution function. In this article, we will examine the methods proposed by Schroder (1989), Ding (1992), and Benton and Krishnamoorthy (2003). The non-central chi-square distribution function can also be computed using methods based on series of incomplete gamma series, which will be used as our benchmark.

2 Non-central χ^2 Distribution and Alternative Methods

If $Z_1, Z_2, ..., Z_v$ are independent unit normal random variables, and $\delta_1, \delta_2, ..., \delta_v$ are constants, then $Y = \sum_{j=1}^{v} (Z_j + \delta_j)^2$ has a non-central chi-square distribution with v degrees of freedom and non-centrality parameter $\lambda = \sum_{j=1}^{v} \delta_j^2$, which is denoted as $\chi_v'^2(\lambda)$. When $\delta_j = 0$ for all j, then Y is distributed as the central chi-square distribution with v degrees of freedom, which is denoted as χ_v^2. Hereafter, $P[\chi_v'^2(\lambda) \leq w] = F(w; v, \lambda)$ is the cumulative distribution function of $\chi_v'^2(\lambda)$and $P[\chi_v^2 \leq w] = F(w; v, 0)$ is the cumulative distribution function of χ_v^2.

The cumulative distribution function of $\chi_v'^2(\lambda)$ is given by:

$$F(w;v,\lambda) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!\,2^{v/2+j}\,\Gamma(v/2+j)} \int_0^w y^{v/2+j-1} e^{-y/2}\,dy, \quad w > 0, \tag{4.1}$$

while $F(w;v,\lambda) = 0$ for $w < 0$. Alternatively, it is possible to express $F(w;v,\lambda)$, for $w > 0$, as a weighted sum of central chi-square probabilities with weights equal to the probabilities of a Poisson distribution with expected value $\lambda/2$, that is,

$$F(w;v,\lambda) = \sum_{j=0}^{\infty} \left(\frac{(\lambda/2)^j}{j!} e^{-\lambda/2} \right) F(w;v+2j,0), \tag{4.2}$$

where the central chi-square probability function $F(w;v+2j,0)$ is given by Abramowitz and Stegun (1972, Equation 26.4.1).

The Gamma Series Method

It is well-known that the function $F(w;v+2n,0)$ is related to the so-called incomplete gamma function (see, for instance, Abramowitz and Stegun (1972, Equation 26.4.19)). Hence, we may express the function (4.2) using series of incomplete gamma functions as follows:

$$F(w;v,\lambda) = \sum_{i=0}^{\infty} \frac{(\lambda/2)^i e^{-\lambda/2}}{i!} \frac{\gamma(v/2+i, w/2)}{\Gamma(v/2+i)}, \tag{4.3}$$

with $\gamma(m,t)$ and $\Gamma(m)$ being, respectively, the incomplete gamma function and the Euler gamma function as defined by Abramowitz and Stegun (1972, Equations 6.5.2 and 6.1.1).

While this method is accurate over a wide range of parameters, the number of terms that must be summed increases with the non-centrality parameter λ. To avoid the infinite sum of the series we use the stopping rule as proposed by Knüsel and Bablok (1996) which allows the specification of a given error tolerance by the user.

For the numerical analysis of this article we will concentrate the discussion on Schroder (1989) and Ding (1992) methods since both are commonly used in the finance literature. We will also use the suggested approach of Benton and Krishnamoorthy (2003), since it is argued by the authors that their algorithm is computationally more efficient than the one suggested by Ding (1992). A detailed explanation of how to compute the non-central chi-square distribution function using these three algorithms is presented below.

The Schroder Method

In the method proposed by Schroder (1989), the non-central chi-square distribution is expressed as an infinite double sum of gamma densities which does not require the computation of incomplete gamma functions, that is

$$F(w;v,\lambda)=\sum_{n=1}^{\infty}g(n+v/2,w/2)\sum_{i=1}^{n}g(i,\lambda/2), \tag{4.4}$$

where $g(m,u)=e^{-u}u^{m-1}/\Gamma(m)$ is the standard form of the gamma density function. As noted by Schroder (1989), equation (4.4) allows the following simple iterative algorithm to be used for computing the infinite sum when w and λ are not too large. First, initialize the following four variables (with $n=1$): $gA=\frac{e^{-w/2}(w/2)^{v/2}}{\Gamma(1+v/2)}$, $gB=e^{-\lambda/2}$, $Sg=gB$, and $R=gA\times Sg$. Then repeat the following loop beginning with $n=2$ and incrementing n by one after each iteration: $gA\leftarrow gA\times\frac{w/2}{n+v/2-1}$, $gB\leftarrow gB\times\frac{\lambda/2}{n-1}$, $Sg\leftarrow Sg+gB$, and $R\leftarrow R+gA\times Sg$. The loop is terminated when the contributions to the sum, R, are declining and very small.

The Ding Method

A similar simple recursive algorithm for evaluating the non-central chi-square distribution is provided also by Ding (1992). Let us define $t_0 = \frac{1}{\Gamma(v/2+1)} \left(\frac{w}{2}\right)^{v/2} e^{-w/2}$, $t_i = t_{i-1}\frac{w}{v+2i}$, $y_0 = u_0 = e^{-\lambda/2}$, $u_i = \frac{u_{i-1}\lambda}{2i}$, and $y_i = y_{i-1} + u_i$. Then the required probability that the variable with the non-central chi-square distribution will take values smaller than w is

$$F(w; v, \lambda) = \sum_{i=0}^{\infty} y_i t_i. \tag{4.5}$$

By taking a sufficient number of terms in the series or using the bound as defined by Ding (1992) for the error tolerance incurred by truncating the series, the required accuracy can be obtained.

The Benton and Krishnamoorthy Method

The function $F(w; v, \lambda)$ is also expressed by Benton and Krishnamoorthy (2003) using series of incomplete gamma functions as given by equation (4.3), where $P(m,t) = \frac{\gamma(m,t)}{\Gamma(m)}$ is the standard gamma distribution function, with $\gamma(m,t)$ and $\Gamma(m)$ as defined in (4.3). To compute $F(w; v, \lambda)$, Benton and Krishnamoorthy (2003) makes use of the following recurrence relations obtained from Abramowitz and Stegun (1972, Equation 6.5.21):

$$P(a+1,x) = P(a,x) - \frac{x^a e^{-x}}{\Gamma(a+1)}, \tag{4.6}$$

$$P(a-1,x) = P(a,x) + \frac{x^{a-1} e^{-x}}{\Gamma(a)}. \tag{4.7}$$

From equation (4.6) it follows that

$$P(a,x) = \frac{x^a e^{-x}}{\Gamma(a+1)} \left(1 + \frac{x}{(a+1)} + \frac{x^2}{(a+1)(a+2)} + \cdots \right), \tag{4.8}$$

which can be used to evaluate $P(a,x)$. The computational algorithm also differs from the others essentially because, in order to compute the noncentral chi-square distribution function $F(w;v,\lambda)$, it starts by evaluating the kth term, where k is the integer part of $\lambda/2$, and then the other terms $k \pm i$ are computed recursively. The proposed method runs in the following steps. First, evaluate $P_k = P(Y=k) = e^{-\lambda/2}(\lambda/2)^k/k!$ and $P(v/2 + k, w/2)$ using equation (4.8). Then, compute $P(Y=k+i)$ and $P(Y=k-i)$, for $i = 1,2,\ldots$, using the initial value $P(Y=k)$, and the recursion relations for Poisson probabilities $P_{i+1} = \frac{\lambda/2}{i+1} P_i$, $P_{i-1} = \frac{i}{\lambda/2} P_i$. Finally, using recursion relations (4.6) and (4.7) compute $P(v/2+k+i, w/2)$ and $P(v/2+k-i, w/2)$. By taking a sufficient number of terms in the series or by specifying a given error tolerance the required accuracy is then obtained.

3 Bond Options under the CIR Model

Under the risk-neutral measure $\mathbb{Q}$, Cox et al. (1985) modeled the evolution of the interest rate, r_t, by the stochastic differential equation (sde):

$$dr_t = \left[\kappa\theta - (\lambda + \kappa) r_t\right] dt + \sigma \sqrt{r_t} dW_t^{\mathbb{Q}}, \tag{4.9}$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$, κ, θ and σ are positive constants representing reversion rate, asymptotic rate and volatility parameters, respectively, and λ is the market risk. The condition $2\kappa\theta > \sigma^2$ has to be imposed to ensure that the interest rate remains positive. Following Cox et al. (1985), the price of a general interest rate claim $F(r,t)$ with cash flow rate $C(r,t)$ satisfies the following partial differential equation (pde)

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 F(r,t)}{\partial r^2}+\kappa(\theta-r)\frac{\partial F(r,t)}{\partial r}+\frac{\partial F(r,t)}{\partial t}-\lambda r\frac{\partial F(r,t)}{\partial r}-rF(r,t)+C(r,t)=0. \quad (4.10)$$

Zero-Coupon and Coupon Bonds

A bond is a contract that pays its holder a known amount, the principal, at a known future date, called maturity. The bond may also pay periodically to its holder fixed cash dividends, called coupons. When it gives no dividends, it is known as a zero-coupon bond, sometimes referred to as pure discount bond. The price of a zero-coupon bond with maturity at time s, $Z(r,t,s)$, satisfies the pde (4.10), with $C(r,t)=0$, subject to the boundary condition $Z(r,s,s)=1$, and is given by

$$Z(r,t,s)=A(t,s)e^{-B(t,s)r} \quad (4.11)$$

where $A(t,s)=\left(\frac{2\gamma e^{\left((\kappa+\lambda+\gamma)(s-t)\right)/2}}{(\kappa+\lambda+\gamma)\left(e^{\gamma(s-t)}-1\right)+2\gamma}\right)^{\frac{2\kappa\theta}{\sigma^2}}$, $B(t,s)=\frac{2\left(e^{\gamma(s-t)}-1\right)}{(\kappa+\lambda+\gamma)\left(e^{\gamma(s-t)}-1\right)+2\gamma}$, and $\gamma=\left((\kappa+\lambda)^2+2\sigma^2\right)^{1/2}$.

Since a coupon bond is just a portfolio of zero-coupon bonds of different maturities, the value of any riskless coupon bond can be expressed as a weighted sum of zero-coupon bond prices

$$P(r,t,s)=\sum_{i=1}^{N}a_i Z(r,t,s_i), \quad (4.12)$$

where $s_1,s_2,\cdots,s_N$ represent the N dates on which payments are made, and the $a_i>0$

terms denote the amount of the payments made. [4.1]

Bond Options

A bond option provides the investor with the right, but not the obligation, to buy or sell a given bond at a fixed price either or before a specific date. In this article, we analyze European-style plain-vanilla options on bonds, which confer the right to buy or sell at a known future date for a predetermined price, i.e. the exercise price. Denote by $c^{zc}(r,t,T,s,K)$ the price at time t of a European call option with maturity $T > t$, strike price K, written on a zero-coupon bond with maturity at time $s > T$ and with the instantaneous rate at time t given by r_t. K is restricted to be less than $A(T,s)$, the maximum possible bond price at time T, since otherwise the option would never be exercised and would be worthless. The option price will follow the basic valuation equation with terminal condition $c^{zc}(r,t,T,s,K) = max[Z(r,T,s) - K, 0]$ to the pde (4.10), with $C(r,t) = 0$, and is given by

$$c^{zc}(r,t,T,s,K) = Z(r,t,s)F(x_1;a,b_1) - KZ(r,t,T)F(x_2;a,b_2), \tag{4.13}$$

where $x_1 = 2r^*[\phi+\psi+B(T,s)]$, $x_2 = 2r^*[\phi+\psi]$, $a = \frac{2\kappa\theta}{\sigma^2}$, $b_1 = \frac{2\phi^2 re^{\gamma(T-t)}}{\phi+\psi+B(T,s)}$, $b_2 = \frac{2\phi^2 re^{\gamma(T-t)}}{\phi+\psi}$, $\psi = \frac{\kappa+\lambda+\gamma}{\sigma^2}$, $\phi = \frac{2\gamma}{\sigma^2\left(e^{\gamma(T-t)}-1\right)}$, $r^* = \left[\ln\left(\frac{A(T,s)}{K}\right)\right]/B(T,s)$, $F(.;\nu,\lambda)$ is the non-central chi-square distribution function with ν degrees of freedom and non-centrality parameter λ and, r^* is the critical rate below which exercise will occur, this is, $K = Z(r^*,T,s)$.

To compute options on coupon bonds we will use the Jamshidian's approach, Jamshidian (1989), which states that an option on a portfolio of zero-coupon bonds is equivalent to a portfolio of options with appropriate strike prices. The individual options all have

[4.1] As an example, consider a 10-year 6% bond with a face amount of 100. In this case, $N = 20$ since the bond makes 19 semiannual coupon payments of 3% as well as a final payment of 103%. That is, $a_i = 3\%, i = 1,2,\cdots,19$, $a_{20} = 3+100 = 103\%$, and $s_1 = 0.5, s_2 = 1, \cdots, s_{19} = 9.5, s_{20} = 10$.

the same maturity and are written on the individual zero-coupon bonds in the bond portfolio. Based on this result, a European call option with exercise price K and maturity T on a bond portfolio consisting of N zero-coupon bonds with distinct maturities s_i $(i = 1, 2, \cdots, N$ and $T < s_1 < s_2 < \cdots < s_N)$ and $a_i(a_i > 0, i = 1, 2, \cdots, N)$ issues of each can be priced as

$$c^{cb}(r, t, T, s, K) = \sum_{i=1}^{N} a_i c^{zc}(r, t, T, s_i, K_i), \tag{4.14}$$

where $K_i = Z(r^{**}, T, s_i)$ and r^{**} is the solution to $\sum_{i=1}^{N} a_i Z(r^{**}, T, s_i) = K$.

4 Numerical Analysis

This section aims to present computational comparisons of the alternative methods of computing the non-central chi-square distribution function for pricing European options on bonds under the CIR diffusion. We examine this CIR option pricing model using alternative combinations of input values over a wide range parameter space. All the calculations in this article were made using *Mathematica* 7.0 running on a Pentium IV (2.53 GhZ) personal computer. We have truncated all the series with an error tolerance of 1E−10. All values are rounded to four decimal places. In order to understand the computational speed of the alternative algorithms, we have computed the CPU times for all the algorithms using the function Timing[.] available in *Mathematica*. Since the CPU time for a single evaluation is very small, we have computed the CPU time for multiple computations. Note that the difference in computation time among the alternative tested methods is clearly due to the specific definition of each algorithm and the corresponding stopping rule, and not on the particular software implementation.

Benchmark Selection

The non-central chi-square distribution function $F(w;\nu,\lambda)$ requires values for w, ν, and λ. Our benchmark is the non-central chi-square distribution $F(w;\nu,\lambda)$ expressed as a gamma series (GS) as given by Equation (4.3), with a pre-defined error tolerance of 1E$-$10, which is tested against three external benchmarks based on the *Mathematica*, *Matlab*, and *R* built-in-functions that are available for computing the cumulative distribution function (CDF) of the non-central chi-square distribution. The set of parameters used in the benchmark selection: is $\kappa \in \{0.15, 0.25, \cdots, 0.85\}$, $\theta \in \{0.03, 0.06, \cdots, 0.15\}$, $r \in \{0.01, 0.02, \cdots, 0.15\}$, $\sigma \in \{0.03, 0.05, \cdots, 0.15\}$, and $\lambda \in \{-0.1, 0\}$. We also consider the next two set of parameters: for the bond maturity $s = 2$, we have $T \in \{1, 1.5, 1.75\}$, and in this case the strike price set is $K \in \{0.90, 0.95\}$; for the bond maturity of $s = 10$, we consider $T \in \{3, 5, 7\}$, and in this situation the strike prices are $K \in \{0.25, 0.35\}$. These combinations of parameters produce $98,280$ probabilities.[4.2] Table 4.1 reports the results obtained.

Table 4.1: Benchmark selection.

Methods	MaxAE	RMSE	k_1	k_2
GS vs CDF of *Mathematica*	1.29E$-$04	4.13E$-$07	79	1,769
GS vs CDF of *Matlab*	6.46E$-$11	1.16E$-$11	0	0
GS vs CDF of *R*	6.45E$-$11	1.16E$-$11	0	0

The results show that the maximum absolute error (MaxAE) and root mean absolute error (RMSE) are higher for the comparison between the GS vs CDF of *Mathematica*, though the number of times the absolute difference between the two methods exceeds 1E$-$07 (k_1) is small in relative terms (it represents about 0.08% of the $98,280$ computed probabilities). However, the number of times a computed probability is greater than 1 (k_2) is slightly higher for the CDF of *Mathematica*[4.3] (about 1.80% of computed proba-

[4.2]We obtained these probabilities by computing the values of $F(x_1;\nu,b_1)$ for this set of parameters.

[4.3]This means that care must be taken if one wants to use the CDF built-in-function of *Mathematica* for computing the non-central chi-square distribution function.

bilities computed). The results comparing the GS vs CDF of *Matlab* and GS vs CDF of *R* show that the corresponding differences are smaller and very similar (never exceeds 1E−07). Under the selected wide parameter space we have not obtained any probability value greater than 1 either in the gamma series method, *Matlab* or *R*. In summary, the results show that the gamma series method is an appropriate choice for the benchmark.

Bond options with alternative methods

Now we want to evaluate the differences in approximations of non-central chi-square probabilities $F(w;\nu,\lambda)$ and in zero-coupon and coupon bond option prices using the iterative procedures of Schroder (1989) (S89), Ding (1992) (D92) and Benton and Krishnamoorthy (2003) (BK03) compared against the benchmark based on the gamma series approach. We will concentrate our analysis on call options, but the same line of reasoning applies also for put options. Panels A and B of Table 4.2 report such comparison results using the following set of parameters: $\kappa \in \{0.35, 0.65\}$, $\theta = 0.08$, $\sigma \in \{0.04, 0.10, 0.16\}$, $r \in \{0.01, 0.02, \cdots, 0.15\}$, $\lambda \in \{-0.1, 0.0\}$, $K \in \{0.25, 0.30\}$, $T \in \{2, 5\}$, and $s \in \{10, 15\}$. Panel C of Table 4.2 analyzes the impact of these competing methods for pricing call options on coupon bonds under the CIR diffusion. In this analysis we used the following set of parameters:$\kappa \in \{0.35, 0.65\}$, $\theta = 0.08$, $\sigma \in \{0.04, 0.10, 0.16\}$, $r \in \{0.01, 0.02, \cdots, 0.15\}$, $\lambda \in \{-0.1, 0.0\}$, $K \in \{95, 100, 105\}$, face value $= 100$, $T \in \{2, 5\}$, $s \in \{10, 15\}$, and a coupon rate $\in \{0.10, 0.12\}$. The third rightmost column of the table reports the CPU time for computing $1,000$ times the $2,880$ probabilities and $1,440$ unique contracts of zero-coupon bond options[4.4] and the CPU time for determining 100 times the $4,320$ unique contracts of coupon bond options. The MaxRE, MeanAE and k_3 denote, respectively, the maximum relative error, the mean absolute error, and the number of times the absolute difference between the two methods exceeds $\$0.01$.

[4.4] The CPU time for the gamma series method is 3,303.23 seconds for probabilities, 3,340.48 seconds for zero-coupon bond options and 10,714.40 for coupon bond options.

Table 4.2: Differences in approximations for each method compared against a benchmark.

Methods	MaxAE	MaxRE	RMSE	MeanAE	CPU time	k_1	k_3
Panel A: Differences in probabilities							
S89	3.79E−10	4.19E−01	1.21E−10	9.03E−11	9,773.37	0	–
D92	9.60E−11	1.83E−02	6.20E−11	5.94E−11	9,085.95	0	–
BK03	4.23E−11	1.71E−07	4.29E−12	1.29E−12	1,946.11	0	–
Panel B: Differences in call option prices on zero-coupon bonds							
S89	1.22E−10	5.24E+00	2.60E−11	1.63E−11	9,796.68	–	0
D92	3.96E−11	3.43E−02	1.25E−11	1.00E−11	9,013.49	–	0
BK03	6.97E−12	4.76E−05	6.61E−13	1.59E−13	1,967.84	–	0
Panel C: Differences in call option prices on coupon bonds							
S89	1.24E−08	1.52E+00	1.98E−09	1.35E−09	14,101.40	–	0
D92	7.00E−09	2.58E−01	2.06E−09	1.66E−09	13,309.00	–	0
BK03	2.03E−09	1.97E−06	1.58E−10	3.03E−11	6,274.70	–	0

5 Conclusion

In this article, we compare the performance of alternative algorithms for computing the non-central chi-square distribution function in terms of accuracy and computation time for evaluating option prices under the CIR model. We find that all algorithms are accurate over a wide range of parameters, though presenting significative differences on computational expenses. Overall, we find that the Benton and Krishnamoorthy (2003) algorithm is clearly the most accurate and efficient in terms of computation time needed for determining option prices under the CIR assumption. Moreover, it has a running time that does not vary significantly with the parameters w, v and λ.

Acknowledgements

Dias is member of the BRU-UNIDE and Larguinho and Braumann are members of the Research Center Centro de Investigação em Matemática e Aplicações (CIMA), both centers financed by the Fundação para a Ciência e Tecnologia (FCT). Dias gratefully acknowledges the financial support from the FCTs grant number PTDC/EGE-ECO/099255/2008.

References

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Benton, Denise and K. Krishnamoorthy, 2003, Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral *t* and the Distribution of the Square of the Sample Multiple Correlation Coefficient, *Computational Statistics and Data Analysis* 43, 249–267.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Ding, Cherng G., 1992, Algorithm AS 275: Computing the Non-Central χ^2 Distribution Function, *Applied Statistics* 41, 478–482.

Jamshidian, Farshid, 1989, An Exact Bond Option Formula, *Journal of Finance* 44, 205–209.

Knüsel, L. and B. Bablok, 1996, Computation of the Noncentral Gamma Distribution, *SIAM Journal of Scientific Computing* 17, 1224–1231.

Schroder, Mark, 1989, Computing the Constant Elasticity of Variance Option Pricing Formula, *Journal of Finance* 44, 211–219.

Vasicek, Oldrich, 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.

Chapter 5

Bond Options, Sensitivity Measures, and Sinking-Fund Bonds under the CIR Framework*

Abstract: In this article, we derive simple closed-form solutions for computing sensitivity measures, commonly referred as Greeks, for both zero-coupon and coupon-paying bond options under the Cox et al. (1985) (CIR) model of the term structure of interest rates, which are shown to be accurate, easy to implement, and computationally highly efficient. We then apply these new formulae to analytically derive the comparative static properties of sinking-fund bonds in a CIR economy.

JEL Classification: G1, G13.

Keywords: CIR model; Zero-coupon bonds; Coupon-paying bonds; Bond options; Greeks; Sinking-fund bonds

*This paper is a joint work with José Carlos Dias and Carlos A. Braumann and is submitted to an international journal.

1 Introduction

A bond is a contract which pays its holder a known amount, the principal, at a known future date, called the maturity of the contract. The bond may also pay periodically to its holder fixed cash dividends, called the coupons. This type of bonds are known as coupon bonds (sometimes also called coupon-paying or coupon-bearing bonds). If the bond pays no dividends, it is known as a zero-coupon bond (or pure discount bond). Several bonds may contain special clauses or some embedded options. There are also some derivative contracts whose underlying asset is a bond.

In this paper, we want to discuss how to compute, efficiently and accurately, sensitivity measures of bond options under the Cox et al. (1985) mean-reverting square-root model (hereafter, the CIR model). Moreover, the option-like features of a sinking-fund bond will be also analyzed under the same interest rate dynamics setting.

The CIR model is an equilibrium asset pricing model for the term structure of interest rates. The CIR framework is important for several reasons: it provides a link between intertemporal asset pricing theory and the term structure of interest rates; it is extensible to several factors; it allows a complete characterization of the term structure of interest rates, which incorporates risk premiums and expectations for future interest rates; it ensures that interest rates remain non-negative (thus avoiding the potentially undesirable feature of the Gaussian interest rate model, as in Vasicek (1977), which allows negative interest rates); and it produces relatively simple closed-form solutions for valuing zero-coupon bonds, coupon-bearing bonds, and various interest rate derivative securities.

Cox et al. (1985) derive analytic solutions for the prices of both discount bonds and European call options on discount bonds. Given its analytical tractability and its ability to accommodate several desirable properties expected from an attractive term structure model, the CIR framework has been extensively studied in the finance literature and has lead to different generalizations in several directions.

For instance, Jamshidian (1989) shows that in all one-factor term structure models an option on a portfolio is equivalent to a portfolio of options with appropriate strike prices. Thus, using the decomposition technique offered by Jamshidian (1989) it is possible to price coupon-bearing bond options under the CIR model using an analytically tractable formula. Longstaff (1993) derives also equivalent closed-form expressions for valuing European-style calls and puts on coupon-paying bonds under the CIR framework, but by directly solving a partial differential equation. Wei (1997) shows that in the one-dimensional CIR model the price of a European coupon bond option can be accurately approximated by a multiple of the price of a European zero-coupon bond option having a time to maturity equal to the stochastic duration of the coupon bond.

The form and risk-sensitivity of zero-coupon bonds and other interest rate contingent claims have been analyzed by Alvarez (2001) for a broad class of single-factor diffusion models of interest rates. More recently, Mallier and Alobaidi (2004) price interest rate swaps under the CIR one-factor model, while Allegretto et al. (2003), ShuJin and ShengHong (2006), and Zhou et al. (2011) discuss the valuation of American-style put options on zero-coupon bonds under single-factor models of the short-term rate, including the CIR model.

An extended CIR model of the term structure of interest rates which allows for time-dependent parameters has been first proposed by Hull and White (1990), though requiring numerical procedures for computing European zero-coupon bond options. Analytically tractable solutions for zero-coupon bond options under the CIR model with time-varying coefficients have been proposed by Jamshidian (1995), Maghsoodi (1996), and Brigo and Mercurio (2001). Using the method of eigenfunction expansions, Gorovoi and Linetsky (2004) derive analytical solutions for zero-coupon bonds and bond options under a shifted CIR process which is particularly useful to model low interest rate regimes.

Extensions from a single-factor CIR model to a two-factor CIR-type interest rate setting have been proposed by Beaglehole and Tenney (1991), Chen and Scott (1992), and Longstaff and Schwartz (1992), while Chen and Scott (1995) provide solutions for the multi-factor version of the CIR model.[5.1]

The CIR model has also been used for valuing other assets with option-like features. For instance, Bacinello et al. (1996) propose a framework for valuing the delivery option associated to a sinking-fund bond in a CIR economy with only one sinking-fund date, while Bacinello and Ortu (1999) extend this valuation problem for the case where multiple sinking-fund dates are allowed, though requiring a backward induction pricing procedure. More recently, Ben-Ameur et al. (2007) propose a dynamic programming approach for pricing call and put options embedded in bonds in a CIR framework, while Dias and Shackleton (2011) provide closed-form solutions for valuing real investment and divestment options on a project, and thus model hysteresis effects under the same interest rate dynamics setting.

This paper offers two contributions to the existent literature on the CIR diffusion. First, we provide closed-form solutions to efficiently and accurately calculate the so-called Greeks (that is, sensitivities of the price of financial derivatives with respect to perturbations of the parameters in the underlying model) of both pure discount and coupon-bearing bonds under the CIR model, which, to the best of our knowledge, are new in the option pricing literature. We restrict ourselves to the derivation of Greeks under the

[5.1] Several empirical tests to estimate the parameters of the single- and multi-factor versions of the CIR model of the term structure of interest rates have been conducted in the literature (see, for instance, Brown and Dybvig (1986), Chan et al. (1992), Chen and Scott (1993), Brown and Schaefer (1994), Pearson and Sun (1994), and Chen and Scott (2003)).

time-independent parameters version of the single-factor CIR model, but the same line of reasoning can be applied to derive some Greeks under the time-varying coefficients version of the one-factor CIR models of Jamshidian (1995) and Maghsoodi (1996), or under the CIR++ model of Brigo and Mercurio (2001).

The analytical solutions proposed for the Greeks under the CIR model are relevant for practitioners of the derivatives industry since they reduce substantially the computational burden when dealing with large portfolios of bonds that have to be re-evaluated frequently, e.g. in a risk or portfolio management context, and allow that they can be easily coded in any desired computer language. Given these aforementioned reasons, many attempts have been made to offer Greek formulae in closed-form for asset price dynamics described by many well-known stochastic processes of practical interest, besides the usual log-normal model of Black and Scholes (1973) and Merton (1973), e.g. Kawai and Takeuchi (2011) for gamma processes, or Larguinho et al. (2013) under the constant elasticity of variance (CEV) diffusion.

Second, we extend Bacinello et al. (1996) results by providing analytical tractable formulae required for valuing and analyzing comparative statics of sinking-fund bonds in the CIR framework. While Bacinello et al. (1996) have been able to study such issues in closed-form under the Vasicek (1977) model, they analyze numerically the comparative static properties of the sinking-fund bond in the CIR framework. Based on our analytic solutions, we prove a result that compares the stochastic duration of the sinking-fund bond to those of the corresponding serial and coupon ones.

The structure of the paper is organized as follows. Section 2 outlines a brief summary of the CIR interest rate dynamics and the analytical formulae for computing discount bonds, coupon-bearing bonds, and European-style options on discount bonds and coupon-paying bonds in a CIR economy. Section 3 derives analytical tractable solutions of some sensitivity measures of bond options under the same interest rate dynamics setting. Section 4 presents some numerical examples to enhance the computational efficiency of our closed-form solutions. Section 5 provides analytically tractable formulae to analyze the comparative-statics properties of a sinking-fund bond in the CIR framework. Section 6 concludes.

2 Model Setup and Bond Option Valuation

In this section, we will present a brief remainder of the CIR interest rate dynamics and the analytical formulae for computing discount bonds, coupon-bearing bonds, and European-style call and put options on zero-coupon bonds and coupon-paying bonds in a CIR economy.

2.1 CIR Interest Rate Dynamics

Consider a CIR economy in which $\mathbb{E}_t^{\mathbb{Q}}$ denotes expectations, at time t, under the martingale (or risk-neutral) probability measure $\mathbb{Q}$, with respect to the risk-adjusted process for the instantaneous interest rate r_t

$$dr_t = (\kappa+\lambda)\left[\frac{\kappa\theta}{(\kappa+\lambda)} - r_t\right]dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}, \tag{5.1}$$

where $\kappa^* = \kappa+\lambda$ is the parameter that the speed of adjustment (reversion rate or reverting rate), $\theta^* = \kappa\theta/k^*$ is the long-run mean of the instantaneous interest rate (asymptotic interest rate or reverting level), σ is the volatility of the process, λ is the market price of risk parameter, and $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. It is well known that the $\kappa\theta$ term plays a key role under this diffusion and has important implications for capture of the interest rate process r at a value of zero. The condition $2\kappa\theta \geq \sigma^2$ ensures that the interest rate remains positive.[5.2]

2.2 Zero-Coupon Bonds under the CIR Model

Following Cox et al. (1985), the price of a general interest rate claim $F(r,t)$ with cash flow rate $C(r,t)$ satisfies the following partial differential equation

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 F(r,t)}{\partial r^2} + \kappa(\theta - r)\frac{\partial F(r,t)}{\partial r} + \frac{\partial F(r,t)}{\partial t} - \lambda r \frac{\partial F(r,t)}{\partial r} - rF(r,t) + C(r,t) = 0. \tag{5.2}$$

In a CIR economy, the price of a zero-coupon bond, at the valuation date t, and maturity date at time s (with $s > t$), $Z(r,t,s)$, satisfies this equation with $C(r,t) = 0$ subject to the boundary condition $Z(r,s,s) = 1$, and is given by

$$Z(r,t,s) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\right] = A(t,s)\, e^{-B(t,s)\, r}, \tag{5.3}$$

where constants $A(t,s)$, $B(t,s)$, and $\gamma > 0$ are given by

$$A(t,s) := \left[\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(s-t)]/2}}{(\kappa+\lambda+\gamma)(e^{\gamma(s-t)}-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2}, \tag{5.4a}$$

$$B(t,s) := \frac{2(e^{\gamma(s-t)}-1)}{(\kappa+\lambda+\gamma)(e^{\gamma(s-t)}-1)+2\gamma}, \tag{5.4b}$$

$$\gamma := \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{1/2}. \tag{5.4c}$$

[5.2] See Feller (1951) for a complete description of the boundary conditions.

2.3 Coupon-Paying Bonds under the CIR Model

Since a coupon bond is just a portfolio of zero-coupon bonds of different maturities, the value of a riskless coupon bond, at the valuation date t, and maturity date at time s (with $s > t$), $P(r,t,s)$, can be expressed as a weighted sum of zero-coupon bond prices, that is

$$P(r,t,s) = \sum_{i=1}^{N} a_i Z(r,t,s_i), \tag{5.5}$$

where $s_1, s_2, \cdots, s_N$ represent the N dates on which payments are made, and each $a_i > 0$ term denote the amount of the payments made.[5.3]

2.4 Zero-Coupon Bond Options under the CIR Model

Analytic solutions for pricing call options on discount bonds have been proposed by Cox et al. (1985). Denote as $c^{\text{zc}}(r,t,T,s,K)$ the price, at the valuation date t, of a European call option with expiration date T, strike price K, written on a zero-coupon bond with maturity date s (with $s > T > t$), and with the instantaneous interest rate at time t given by r.[5.4]

Applying the terminal condition $c^{\text{zc}}(r,t,T,s,K) = \max[Z(r,T,s) - K, 0]$ to the partial differential equation (5.2) with $C(r,t) = 0$, then the zero-coupon bond call option price is given by

[5.3] For example, consider a 10-year 6% coupon bond with a par value of 100 and semiannual coupon payments. In this case, $N = 20$ since the bond makes 19 semiannual coupon payments of 3% as well as a final payment of 103%. Thus, $a_i = 100 \times 6\%/2 = 3\%$ for $i = 1, 2, \cdots, 19$, $a_{20} = 3 + 100 = 103\%$, and $s_1 = 0.5, s_2 = 1, \cdots, s_{19} = 9.5$, and $s_{20} = 10$.

[5.4] It is well-known that K is restricted to be less than $A(T,s)$, the maximum possible bond price at time T, since otherwise the option would never be exercised and would be worthless.

$$c^{zc}(r,t,T,s,K) = Z(r,t,s)\,F(x_1;a,b_1) - K\,Z(r,t,T)\,F(x_2;a,b_2), \tag{5.6}$$

where $F(x;a,b)$ is the non-central chi-square distribution function with a degrees of freedom and non-centrality parameter b,[5.5]

$$x_1 := 2r^*\left[\phi + \psi + B(T,s)\right], \tag{5.7a}$$

$$x_2 := 2r^*\left[\phi + \psi\right], \tag{5.7b}$$

$$a := \frac{4\kappa\theta}{\sigma^2}, \tag{5.7c}$$

$$b_1 := \frac{2\phi^2 r e^{\gamma(T-t)}}{\phi + \psi + B(T,s)}, \tag{5.7d}$$

$$b_2 := \frac{2\phi^2 r e^{\gamma(T-t)}}{\phi + \psi}, \tag{5.7e}$$

where

$$\phi := \frac{2\gamma}{\sigma^2\left(e^{\gamma(T-t)} - 1\right)}, \tag{5.8a}$$

$$\psi := \frac{\kappa + \lambda + \gamma}{\sigma^2}, \tag{5.8b}$$

$$r^* := \left[\ln\left(\frac{A(T,s)}{K}\right)\right] / B(T,s), \tag{5.8c}$$

with r^* being the critical interest rate below which exercise will occur, i.e. $K = Z(r^*,T,s)$.

The price of the corresponding European put option, $p^{zc}(r,t,T,s,K)$, can be obtained

[5.5] An exhaustive numerical analysis comparing different alternative approximation schemes for efficiently compute the non-central chi-square distribution function in the context of the CEV model is provided by Larguinho et al. (2013).

through the put-call parity relation for bond options, that is

$$\begin{aligned} p^{\mathrm{zc}}(r,t,T,s,K) &= c^{\mathrm{zc}}(r,t,T,s,K) + KZ(r,t,T) - Z(r,t,s) \\ &= KZ(r,t,T)\,Q(x_2;a,b_2) - Z(r,t,s)\,Q(x_1;a,b_1), \end{aligned} \tag{5.9}$$

where $Q(.;a,b)$ is the complementary non-central chi-square distribution function with a degrees of freedom and non-centrality parameter b.

2.5 Coupon-Paying Bond Options under the CIR Model

Jamshidian (1989) shows that in all one-factor term structure models an option on a portfolio of zero-coupon bonds decomposes into a portfolio of options on the individual discount bonds in the portfolio. Based on this result, a European call option with strike price K and maturity date T on a portfolio consisting of N zero-coupon bonds with different expiry dates s_i is given by

$$c^{\mathrm{cb}}(r,t,T,s,K) = \sum_{i=1}^{N} a_i c^{\mathrm{zc}}(r,t,T,s_i,K_i), \tag{5.10}$$

with $T < s_1 < s_2 < \cdots < s_N$, $a_i > 0$, $K_i = Z(r^{**},T,s_i)$, and where r^{**} is the solution to $\sum_{i=1}^{N} a_i Z(r^{**},T,s_i) = K$. Alternatively, we could use the equivalent closed-form expression offered by Longstaff (1993, Equation 7). The corresponding coupon bond put prices can be obtained through put-call parity, that is

$$
\begin{aligned}
p^{\text{cb}}(r,t,T,s,K) &= c^{\text{cb}}(r,t,T,s,K) + KZ(r,t,T) - \sum_{i=1}^{N} a_i Z(r,t,s_i) \\
&= \sum_{i=1}^{N} a_i p^{\text{zc}}(r,t,T,s_i,K_i). \qquad (5.11)
\end{aligned}
$$

Note that the underlying asset for coupon bond options is actually the portfolio of discount bonds expiring after the option's maturity date. However, the value of this portfolio is strictly less than the current price of the coupon bond if the bond pays coupons before the expiry date of the option. As shown in Longstaff (1993), the value of the underlying asset for a 5-year option on a 10-year bond is not the current price of a 15-year bond, but the price of a 15-year bond minus the present value of coupon payments to be made during the next 5 years. In other words, the option's payoff, and hence the coupon bond option price, does not depend on the payments of the coupon bond to be made before the expiry date of the option.

3 Greeks of Bond Options under the CIR Model

3.1 Preliminaries

Let us begin with two general relations which will be used for deriving Greeks under the CIR diffusion process. From Johnson et al. (1995, pp. 442-443) we know that

$$
\begin{aligned}
\frac{\partial F(x;a,b)}{\partial x} &= p(x;a,b), && (5.12a) \\
\frac{\partial F(x;a,b)}{\partial b} &= -p(x;a+2,b), && (5.12b)
\end{aligned}
$$

where $p(x;a,b)$ is the probability density function of a non-central chi-square distribution as given by Johnson et al. (1995, Equation 29.4), that is

$$p(x;a,b) = \frac{1}{2} e^{-(b+x)/2} \left(\frac{x}{b}\right)^{(a-2)/4} I_{(a-2)/2}(\sqrt{bx}), \quad x > 0, \tag{5.13}$$

with $I_q(\cdot)$ being the modified Bessel function of the first kind of order q, as defined by Abramowitz and Stegun (1972, Equation 9.6.10). We will need to use the first derivative of the probability density function (5.13) with respect to the non-centrality parameter b, which can be computed through the following recurrence relation offered by Cohen (1988):

$$\frac{\partial p(x;a,b)}{\partial b} = \frac{1}{2}[-p(x;a,b) + p(x;a+2,b)]. \tag{5.14}$$

3.2 Greeks Formulas

The next propositions and remarks present the closed-form solutions for computing the *rho* (or *interest rate delta*), *delta*, *interest rate gamma*, *theta*, and *eta* (or *strike delta*) sensitivity measures of options on bonds under a CIR economy.[5.6]

Proposition 5.1 *Consider the formulae for pricing zero-coupon bond options under the CIR model as defined in equations* (5.6) *and* (5.9).

[5.6]Note that the so-called *vega*—which is the sensitivity of the bond option price with respect to the volatility parameter σ—depends on the degrees of freedom parameter a of the noncentral chi-square distribution function, for which (to the authors knowledge) there is no simple relationship as those given in equations (5.12a) and (5.12b). See Alvarez (2001) who discusses the conditions which determine the sign of the effect of increased volatility on the price of a general interest rate claim under a broad class of interest rate models.

i. Call rho (or interest rate call delta) for zero-coupon bond options

$$\rho_c^{zc} := \frac{\partial c^{zc}(.)}{\partial r} = Z(r,t,s)\left[-B(t,s)\,F(x_1;a,b_1) - \frac{b_1}{r}\,p(x_1;a+2,b_1)\right]$$
$$-KZ(r,t,T)\left[-B(t,T)\,F(x_2;a,b_2) - \frac{b_2}{r}\,p(x_2;a+2,b_2)\right]. \quad (5.15)$$

ii. Put rho (or interest rate put delta) for zero-coupon bond options

$$\rho_p^{zc} := \frac{\partial p^{zc}(.)}{\partial r} = KZ(r,t,T)\left[-B(t,T)\,Q(x_2;a,b_2) + \frac{b_2}{r}\,p(x_2;a+2,b_2)\right]$$
$$-Z(r,t,s)\left[-B(t,s)\,Q(x_1;a,b_1) + \frac{b_1}{r}\,p(x_1;a+2,b_1)\right]. \quad (5.16)$$

Proof. Let us first consider the following auxiliary functions:

$$\frac{\partial Z(r,t,j)}{\partial r} = -B(t,j)Z(r,t,j),\ \ j \in \{T,s\}, \quad (5.17a)$$

$$\frac{\partial x_1}{\partial r} = \frac{\partial x_2}{\partial r} = 0, \quad (5.17b)$$

$$\frac{\partial b_1}{\partial r} = \frac{b_1}{r}, \quad (5.17c)$$

$$\frac{\partial b_2}{\partial r} = \frac{b_2}{r}. \quad (5.17d)$$

The call rho for a zero-coupon bond option is computed as

$$\rho_c^{zc} := \frac{\partial c^{zc}(.)}{\partial r} = \frac{\partial Z(r,t,s)}{\partial r}\,F(x_1;a,b_1) + Z(r,t,s)\,\frac{\partial F(x_1;a,b_1)}{\partial r}$$
$$-K\left[\frac{\partial Z(r,t,T)}{\partial r}\,F(x_2;a,b_2) + Z(r,t,T)\,\frac{\partial F(x_2;a,b_2)}{\partial r}\right]. \quad (5.18)$$

Using equations (5.12b), (5.17a), (5.17c), and (5.17d) we are able to compute the following partial derivatives:

$$\frac{\partial F(x_1;a,b_1)}{\partial r} = \frac{\partial F(x_1;a,b_1)}{\partial b_1}\frac{\partial b_1}{\partial r} = -\frac{b_1}{r}\, p(x_1;a+2,b_1), \tag{5.19}$$

$$\frac{\partial F(x_2;a,b_2)}{\partial r} = \frac{\partial F(x_2;a,b_2)}{\partial b_2}\frac{\partial b_2}{\partial r} = -\frac{b_2}{r}\, p(x_2;a+2,b_2). \tag{5.20}$$

Finally, substituting equations (5.19) and (5.20) in equation (5.18) yields equation (5.15). The put rho for a zero-coupon bond option is treated similarly or obtained via put-call parity. ■

The next remark shows that it is straightforward to compute the call (and put) interest rate deltas in closed-form for coupon-paying bond options under the CIR framework. Then, we are able to compare the obtained solutions with the values shown in Wei (1997, Table II).

Remark 5.1 *The call and put rho (or interest rate delta) for coupon bond options arise immediately if one applies the decomposition technique of Jamshidian (1989), that is*

$$\rho_c^{cb} := \frac{\partial c^{cb}(.)}{\partial r} = \sum_{i=1}^{N} a_i \frac{\partial c^{zc}(r,t,T,s_i,K_i)}{\partial r} = \sum_{i=1}^{N} a_i \rho_c^{zc}(r,t,T,s_i,K_i), \tag{5.21}$$

$$\rho_p^{cb} := \frac{\partial p^{cb}(.)}{\partial r} = \sum_{i=1}^{N} a_i \frac{\partial p^{zc}(r,t,T,s_i,K_i)}{\partial r} = \sum_{i=1}^{N} a_i \rho_p^{zc}(r,t,T,s_i,K_i). \tag{5.22}$$

Longstaff (1993, Tables 1 and 2) report the first and the second derivatives of the option prices with respect to the underlying bond price, that is the so-called delta and gamma of the options, that have been computed using finite difference schemes. The next remark shows that we are also able to compute call and put deltas in closed-form for both zero-coupon and coupon-paying bond options under the CIR framework (gammas can be treated similarly).

Remark 5.2 *The call and put delta (with respect to the underlying bond price) for both zero-coupon and coupon-paying bond options arise immediately if one uses the results obtained in Proposition 5.1 and Remark 5.1, that is*

$$\Delta_c^{zc} := \frac{\partial c^{zc}(r,t,T,s,K)}{\partial Z(r,t,s)} = \frac{\partial c^{zc}(r,t,T,s,K)/\partial r}{\partial Z(r,t,s)/\partial r} = -\frac{\rho_c^{zc}(r,t,T,s,K)}{B(t,s)Z(r,t,s)}, \tag{5.23}$$

$$\Delta_p^{zc} := \frac{\partial p^{zc}(r,t,T,s,K)}{\partial Z(r,t,s)} = \frac{\partial p^{zc}(r,t,T,s,K)/\partial r}{\partial Z(r,t,s)/\partial r} = -\frac{\rho_p^{zc}(r,t,T,s,K)}{B(t,s)Z(r,t,s)}, \tag{5.24}$$

$$\Delta_c^{cb} := \frac{\partial c^{cb}(r,t,T,s,K)}{\partial P(r,t,s)} = \frac{\partial c^{cb}(r,t,T,s,K)/\partial r}{\partial P(r,t,s)/\partial r} = -\frac{\rho_c^{cb}(r,t,T,s,K)}{\sum_{i=1}^N a_i B(t,s_i)Z(r,t,s_i)}, \tag{5.25}$$

$$\Delta_p^{cb} := \frac{\partial p^{cb}(r,t,T,s,K)}{\partial P(r,t,s)} = \frac{\partial p^{cb}(r,t,T,s,K)/\partial r}{\partial P(r,t,s)/\partial r} = -\frac{\rho_p^{cb}(r,t,T,s,K)}{\sum_{i=1}^N a_i B(t,s_i)Z(r,t,s_i)}. \tag{5.26}$$

Proposition 5.2 *Consider the formulae for pricing zero-coupon bond options under the CIR model as defined in equations* (5.6) *and* (5.9)*.*

i. Interest rate call gamma for zero-coupon bond options

$$\begin{aligned}\Gamma_{r,c}^{zc} := \frac{\partial^2 c^{zc}(.)}{\partial r^2} &= Z(r,t,s)\Bigg[B^2(t,s)F(x_1;a,b_1) + 2B(t,s)\frac{b_1}{r}p(x_1;a+2,b_1) \\ &\quad -\frac{1}{2}\left(\frac{b_1}{r}\right)^2\big(-p(x_1;a+2,b_1)+p(x_1;a+4,b_1)\big)\Bigg] \\ &\quad -KZ(r,t,T)\Bigg[B^2(t,T)F(x_2;a,b_2) + 2B(t,T)\frac{b_2}{r}p(x_2;a+2,b_2) \\ &\quad -\frac{1}{2}\left(\frac{b_2}{r}\right)^2\big(-p(x_2;a+2,b_2)+p(x_2;a+4,b_2)\big)\Bigg]. \end{aligned} \tag{5.27}$$

ii. Interest rate put gamma for zero-coupon bond options

$$\begin{aligned}\Gamma^{zc}_{r,p} := \frac{\partial^2 p^{zc}(.)}{\partial r^2} &= Z(r,t,s)\Bigg[-B^2(t,s)Q(x_1;a,b_1) + 2B(t,s)\frac{b_1}{r}p(x_1;a+2,b_1) \\ &\quad -\frac{1}{2}\left(\frac{b_1}{r}\right)^2\big(-p(x_1;a+2,b_1)+p(x_1;a+4,b_1)\big)\Bigg] \\ &\quad -KZ(r,t,T)\Bigg[-B^2(t,T)Q(x_2;a,b_2)+2B(t,T)\frac{b_2}{r}p(x_2;a+2,b_2) \\ &\quad -\frac{1}{2}\left(\frac{b_2}{r}\right)^2\big(-p(x_2;a+2,b_2)+p(x_2;a+4,b_2)\big)\Bigg]. \qquad (5.28)\end{aligned}$$

Proof. Use equation (5.14) and follow the same line of reasoning as in Proposition 5.1. ■

Remark 5.3 *The call and put interest rate gamma for coupon bond options arise immediately if one applies the decomposition technique of Jamshidian (1989), that is*

$$\Gamma^{cb}_{r,c} := \frac{\partial^2 c^{cb}(.)}{\partial r^2} = \sum_{i=1}^{N} a_i \frac{\partial \rho^{zc}_c(r,t,T,s_i,K_i)}{\partial r} = \sum_{i=1}^{N} a_i \Gamma^{zc}_{r,c}(r,t,T,s_i,K_i), \qquad (5.29)$$

$$\Gamma^{cb}_{r,p} := \frac{\partial^2 p^{cb}(.)}{\partial r^2} = \sum_{i=1}^{N} a_i \frac{\partial \rho^{zc}_p(r,t,T,s_i,K_i)}{\partial r} = \sum_{i=1}^{N} a_i \Gamma^{zc}_{r,p}(r,t,T,s_i,K_i). \qquad (5.30)$$

Proposition 5.3 *Consider the formulae for pricing zero-coupon bond options under the CIR model as defined in equations (5.6) and (5.9).*

i. Call theta for zero-coupon bond options

$$\begin{aligned}\theta^{zc}_c := \frac{\partial c^{zc}(.)}{\partial t} &= Z(r,t,s)\Big[\zeta_s F(x_1;a,b_1) + \xi\, p(x_1;a,b_1) - \rho_1\, p(x_1;a+2,b_1)\Big] \qquad (5.31) \\ &\quad -KZ(r,t,T)\Big[\zeta_T F(x_2;a,b_2) + \xi\, p(x_2,a,b_2) - \rho_2 p(x_2;a+2,b_2)\Big].\end{aligned}$$

ii. Put theta for zero-coupon bond options

$$\begin{aligned}\theta_p^{zc} &:= \frac{\partial p^{zc}(.)}{\partial t}\\ &= KZ(r,t,T)\Big[\zeta_T\,Q(x_2;a,b_2)-\xi\,p(x_2,a,b_2)+\rho_2 p(x_2;a+2,b_2)\Big]\\ &\quad -Z(r,t,s)\Big[\zeta_s\,Q(x_1;a,b_1)-\xi\,p(x_1;a,b_1)+\rho_1\,p(x_1;a+2,b_1)\Big]. \qquad (5.32)\end{aligned}$$

Proof. Let us first consider the following auxiliary functions:

$$\frac{\partial A(t,T)}{\partial t}=\frac{\kappa\theta}{\sigma^2}\frac{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)(2\gamma-(\kappa+\lambda+\gamma)}{(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}A(t,T), \qquad (5.33a)$$

$$\frac{\partial B(t,T)}{\partial t}=-\frac{4\gamma^2e^{\gamma(T-t)}}{\Big[(\kappa+\lambda+\gamma)(e^{\gamma(T-t)}-1)+2\gamma\Big]^2}, \qquad (5.33b)$$

$$\frac{\partial Z(r,t,j)}{\partial t}=Z(r,t,j)\left[\frac{1}{A(t,j)}\frac{\partial A(t,j)}{\partial t}-r\frac{\partial B(t,j)}{\partial t}\right]=Z(r,t,j)\,\zeta_j,\ \ j\in\{T,s\}, \qquad (5.33c)$$

$$\xi=\frac{\partial x_1}{\partial t}=\frac{\partial x_2}{\partial t}=\frac{4r^*\gamma^2e^{\gamma(T-t)}}{\sigma^2(e^{\gamma(T-t)}-1)^2}, \qquad (5.33d)$$

$$\rho_1=\frac{\partial b_1}{\partial t}=b_1\gamma\frac{(\phi+\psi+B(T,s))+(\psi+B(T,s))e^{\gamma(T-t)}}{(e^{\gamma(T-t)}-1)(\phi+\psi+B(T,s))}, \qquad (5.33e)$$

$$\rho_2=\frac{\partial b_2}{\partial t}=b_2\gamma\frac{(\phi+\psi)+\psi e^{\gamma(T-t)}}{(e^{\gamma(T-t)}-1)(\phi+\psi)}. \qquad (5.33f)$$

The call theta for a zero-coupon bond option is computed as

$$\begin{aligned}\theta_c^{zc} := \frac{\partial c^{zc}(.)}{\partial t} &= \frac{\partial Z(r,t,s)}{\partial t}F(x_1;a,b_1)+Z(r,t,s)\frac{\partial F(x_1;a,b_1)}{\partial t}\\ &\quad -K\left[\frac{\partial Z(r,t,T)}{\partial t}F(x_2;a,b_2)+Z(r,t,T)\frac{\partial F(x_2;a,b_2)}{\partial t}\right]. \qquad (5.34)\end{aligned}$$

Using equations (5.12a), (5.12b), (5.33d), (5.33e), and (5.33f) we are able to compute the following partial derivatives:

$$\begin{aligned}
\frac{\partial F(x_1;a,b_1)}{\partial t} &= \frac{\partial F(x_1;a,b_1)}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial F(x_1;a,b_1)}{\partial b_1}\frac{\partial b_1}{\partial t} \\
&= p(x_1;a,b_1)\xi - p(x_1;a+2,b_1)\rho_1, \qquad (5.35) \\
\frac{\partial F(x_2;a,b_2)}{\partial t} &= \frac{\partial F(x_2;a,b_2)}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial F(x_2;a,b_2)}{\partial b_2}\frac{\partial b_2}{\partial t} \\
&= p(x_2;a,b_2)\xi - p(x_2;a+2,b_2)\rho_2. \qquad (5.36)
\end{aligned}$$

Finally, substituting equations (5.33c), (5.35) and (5.36) in equation (5.34) yields equation (5.31). The put theta for a zero-coupon bond option is treated similarly or obtained via put-call parity. ■

Remark 5.4 *The call and put theta for coupon bond options arise immediately if one applies the decomposition technique of Jamshidian (1989), that is*

$$\theta_c^{cb} := \frac{\partial c^{cb}(.)}{\partial t} = \sum_{i=1}^{N} a_i \frac{\partial c^{zc}(r,t,T,s_i,K_i)}{\partial t} = \sum_{i=1}^{N} a_i \theta_c^{zc}(r,t,T,s_i,K_i), \qquad (5.37)$$

$$\theta_p^{cb} := \frac{\partial p^{cb}(.)}{\partial t} = \sum_{i=1}^{N} a_i \frac{\partial p^{zc}(r,t,T,s_i,K_i)}{\partial t} = \sum_{i=1}^{N} a_i \theta_p^{zc}(r,t,T,s_i,K_i). \qquad (5.38)$$

Remark 5.5 *Clearly, equations* (5.15), (5.27), *and* (5.31) *satisfy the required partial differential equation* (5.2), *with* $C(r,t) = 0$, *assuming the interest rate claim is a zero-coupon bond call option contract. The same line of reasoning applies also for the other three interest rate claims under analysis, that is call options on coupon-paying bonds and put options on both zero-coupon and coupon-bearing bonds.*

Proposition 5.4 *Consider the formulae for pricing zero-coupon bond options under the CIR model as defined in equations* (5.6) *and* (5.9)*.*

i. Call eta for zero-coupon bond options

$$\eta_c^{zc} := \frac{\partial c^{zc}(.)}{\partial K} = Z(r,t,T)\left[-F(x_2;a,b_2) + 2\,p(x_2;a,b_2)\frac{(\phi+\psi)}{B(T,s)}\right] - 2Z(r,t,s)\,p(x_1;a,b_1)\frac{(\phi+\psi+B(T,s))}{B(T,s)\,K}. \tag{5.39}$$

ii. Put eta for zero-coupon bond options

$$\eta_p^{zc} := \frac{\partial p^{zc}(.)}{\partial K} = Z(r,t,T)\left[Q(x_2;a,b_2) + 2\,p(x_2;a,b_2)\frac{(\phi+\psi)}{B(T,s)}\right] - 2Z(r,t,s)\,p(x_1;a,b_1)\frac{(\phi+\psi+B(T,s))}{B(T,s)\,K}. \tag{5.40}$$

Proof. Let us first consider the following auxiliary functions:

$$\frac{\partial x_1}{\partial K} = -\frac{2(\phi+\psi+B(T,s))}{B(T,s)K}, \tag{5.41a}$$

$$\frac{\partial x_2}{\partial K} = -\frac{2(\phi+\psi)}{B(T,s)K}, \tag{5.41b}$$

$$\frac{\partial b_1}{\partial K} = \frac{\partial b_2}{\partial K} = 0, \tag{5.41c}$$

The call eta for a zero-coupon bond option is computed as

$$\eta_c^{zc} := \frac{\partial c^{zc}(.)}{\partial K} = Z(r,t,s)\,\frac{\partial F(x_1;a,b_1)}{\partial K} - \left[Z(r,t,T)\,F(x_2;a,b_2) + K\,Z(r,t,T)\,\frac{\partial F(x_2;a,b_2)}{\partial K}\right]. \tag{5.42}$$

Using equations (5.12a), (5.41a), and (5.41b) we are able to compute the following partial derivatives:

$$\frac{\partial F(x_1;a,b_1)}{\partial K} = \frac{\partial F(x_1;a,b_1)}{\partial x_1}\frac{\partial x_1}{\partial K} = -2\,p(x_1;a,b_1)\frac{(\phi+\psi+B(T,s))}{B(T,s)\,K}, \tag{5.43}$$

$$\frac{\partial F(x_2;a,b_2)}{\partial K} = \frac{\partial F(x_2;a,b_2)}{\partial x_2}\frac{\partial x_2}{\partial K} = -2\,p(x_2;a,b_2)\frac{(\phi+\psi)}{B(T,s)\,K}. \tag{5.44}$$

Finally, substituting equations (5.43) and (5.44) in equation (5.42) yields equation (5.39). The put eta for a zero-coupon bond option is treated similarly or obtained via put-call parity. ■

4 Numerical Experiments

To access the robustness of our formulae we have performed some numerical experiments to compare the results of our closed-form solutions with other results available in the literature, but obtained through finite difference schemes or numerical integration.

Tables 5.1 and 5.2 show the values of 5-year call and put option prices and the corresponding deltas on a 15-year 8% (in Table 5.1) and 14% (in Table 5.2) coupon bond—with the 10 coupons being paid annually—with par value 1,000 for different levels of the riskless interest rate (r) and strike price (K) assuming a CIR framework, with parameter values borrowed from Longstaff (1993, Tables 1 and 2), and using the Benton and Krishnamoorthy (2003) algorithm for computing the noncentral chi-square distribution function.

Table 5.1: Call and put coupon bond option prices and deltas under the CIR model.

	Call prices			Put prices		
r	$K = 960$	$K = 980$	$K = 1,000$	$K = 960$	$K = 980$	$K = 1,000$
0.01	22.9307	12.3091	4.9562	3.0980	7.1924	14.5555
0.02	22.4987	12.0551	4.8427	3.1123	7.1962	14.5115
0.03	22.0746	11.8061	4.7318	3.1262	7.1994	14.4667
0.04	21.6583	11.5622	4.6234	3.1398	7.2018	14.4210
0.05	21.2495	11.3231	4.5174	3.1531	7.2034	14.3745
0.06	20.8481	11.0888	4.4138	3.1659	7.2044	14.3272
0.07	20.4541	10.8593	4.3125	3.1784	7.2047	14.2791
0.08	20.0673	10.6343	4.2135	3.1905	7.2044	14.2302
0.09	19.6875	10.4139	4.1167	3.2023	7.2033	14.1807
0.10	19.3147	10.1980	4.0221	3.2137	7.2016	14.1304
0.11	18.9487	9.9863	3.9296	3.2247	7.1992	14.0794
0.12	18.5894	9.7790	3.8392	3.2354	7.1962	14.0277
0.13	18.2367	9.5758	3.7508	3.2458	7.1926	13.9753
0.14	17.8905	9.3768	3.6645	3.2557	7.1883	13.9223
0.15	17.5507	9.1817	3.5801	3.2654	7.1834	13.8687
	Call deltas			Put deltas		
r	$K = 960$	$K = 980$	$K = 1000$	$K = 960$	$K = 980$	K = 1,000
0.01	0.0456	0.0269	0.0120	−0.0015	−0.0004	0.0046
0.02	0.0454	0.0267	0.0119	−0.0015	−0.0004	0.0047
0.03	0.0452	0.0265	0.0118	−0.0015	−0.0003	0.0049
0.04	0.0449	0.0263	0.0117	−0.0015	−0.0002	0.0050
0.05	0.0447	0.0261	0.0116	−0.0014	−0.0001	0.0052
0.06	0.0445	0.0259	0.0115	−0.0014	−0.0001	0.0053
0.07	0.0442	0.0257	0.0113	−0.0014	0.0000	0.0055
0.08	0.0440	0.0256	0.0112	−0.0014	0.0001	0.0056
0.09	0.0438	0.0254	0.0111	−0.0013	0.0002	0.0058
0.10	0.0435	0.0252	0.0110	−0.0013	0.0002	0.0060
0.11	0.0433	0.0250	0.0109	−0.0013	0.0003	0.0061
0.12	0.0431	0.0248	0.0108	−0.0013	0.0004	0.0063
0.13	0.0428	0.0247	0.0107	−0.0012	0.0004	0.0065
0.14	0.0426	0.0245	0.0106	−0.0012	0.0006	0.0066
0.15	0.0424	0.0243	0.0105	−0.0012	0.0007	0.0068

This table values 5-year call and put option prices and the corresponding deltas on a 15-year 8% coupon bond—with the ten coupons being paid annually—with par value 1,000 for different levels of the riskless interest rate (r) and strike price (K) assuming a CIR framework, and using the Benton and Krishnamoorthy (2003) algorithm for computing the non-central chi-square distribution function. Parameter values borrowed from Longstaff (1993, Table 1): $\kappa = 0.75$, $\theta = 0.08$, $\sigma^2 = 0.014$, and $\lambda = 0$.

Table 5.2: Call and put coupon bond option prices and deltas under the CIR model.

	Call prices			Put prices		
r	$K = 1,340$	$K = 1,360$	$K = 1,380$	$K = 1,340$	$K = 1,360$	$K = 1,380$
0.01	37.1791	25.3832	15.4984	3.1703	6.0904	10.9216
0.02	36.4970	24.8882	15.1747	3.1895	6.1083	10.9224
0.03	35.8269	24.4025	14.8575	3.2083	6.1255	10.9221
0.04	35.1687	23.9260	14.5467	3.2267	6.1420	10.9207
0.05	34.5222	23.4585	14.2423	3.2447	6.1578	10.9183
0.06	33.8871	22.9998	13.9441	3.2624	6.1729	10.9149
0.07	33.2633	22.5498	13.6519	3.2797	6.1873	10.9105
0.08	32.6506	22.1083	13.3657	3.2966	6.2010	10.9052
0.09	32.0488	21.6752	13.0853	3.3131	6.2141	10.8988
0.10	31.4578	21.2503	12.8107	3.3293	6.2265	10.8915
0.11	30.8772	20.8335	12.5416	3.3451	6.2383	10.8833
0.12	30.3070	20.4246	12.2781	3.3605	6.2494	10.8741
0.13	29.7470	20.0235	12.0199	3.3756	6.2598	10.8640
0.14	29.1969	19.6300	11.7670	3.3902	6.2696	10.8530
0.15	28.6567	19.2440	11.5193	3.4045	6.2788	10.8411
	Call deltas			Put deltas		
r	$K = 1,340$	$K = 1,360$	$K = 1,380$	$K = 1,340$	$K = 1,360$	$K = 1,380$
0.01	0.0513	0.0373	0.0244	−0.0014	−0.0014	−0.0001
0.02	0.0511	0.0370	0.0242	−0.0014	−0.0013	0.0000
0.03	0.0508	0.0368	0.0240	−0.0014	−0.0013	0.0001
0.04	0.0506	0.0366	0.0239	−0.0014	−0.0013	0.0001
0.05	0.0504	0.0364	0.0237	−0.0014	−0.0012	0.0002
0.06	0.0501	0.0362	0.0235	−0.0014	−0.0012	0.0003
0.07	0.0499	0.0360	0.0233	−0.0014	−0.0011	0.0004
0.08	0.0496	0.0357	0.0232	−0.0014	−0.0011	0.0005
0.09	0.0494	0.0355	0.0230	−0.0014	−0.0011	0.0006
0.10	0.0492	0.0353	0.0228	−0.0013	−0.0010	0.0007
0.11	0.0489	0.0351	0.0226	−0.0013	−0.0010	0.0007
0.12	0.0487	0.0349	0.0225	−0.0013	−0.0009	0.0008
0.13	0.0485	0.0347	0.0223	−0.0013	−0.0009	0.0009
0.14	0.0482	0.0345	0.0221	−0.0013	−0.0008	0.0010
0.15	0.0480	0.0343	0.0220	−0.0013	−0.0008	0.0011

This table values 5-year call and put option prices and the corresponding deltas on a 15-year 14% coupon bond—with the ten coupons being paid annually—with par value 1,000 for different levels of the riskless interest rate (r) and strike price (K) assuming a CIR framework, and using the Benton and Krishnamoorthy (2003) algorithm for computing the non-central chi-square distribution function. Parameter values borrowed from Longstaff (1993, Table 2): $\kappa = 0.75$, $\theta = 0.08$, $\sigma^2 = 0.014$, and $\lambda = 0$.

Both tables report call and put bond option prices that are very close to the results shown in Longstaff (1993, Tables 1 and 2). While Longstaff (1993) uses the approximation method offered by Sankaran (1963) for computing the noncentral chi-square distribution function, we have used the Benton and Krishnamoorthy (2003) algorithm which is clearly much more accurate, thus explaining the small differences found when rounding our option prices to two decimal places. The tables also highlight that we can efficiently (and more accurately) value call and put deltas of bond options using our new proposed closed-form solutions (5.25) and (5.26). Even though the results shown in Longstaff (1993, Tables 1 and 2) are similar, they have been obtained through finite difference schemes, which are clearly, by its nature, less accurate and computationally more demanding than our analytical formulae.

Table 5.3 shows the values of 5-year call and put option prices, rhos (i.e. interest rate deltas), deltas, and thetas on a 15-year 10% coupon bond—with the 10 coupons being paid annually—with par value 100 for different levels of the riskless interest rate (r) and strike price ($K = 100$) assuming a CIR framework, with parameter values borrowed from Wei (1997, Table II), and using the Benton and Krishnamoorthy (2003) algorithm for computing the non-central chi-square distribution function.

One of the salient features of this table is that while our call option prices are similar (as expected) to the ones presented in the third and fourth columns of Wei (1997, Table II, Panel A), who also uses the Sankaran (1963) approximation for computing the non-central chi-square distribution function, our interest rate deltas, obtained via equation (5.21), do not completely agree with the ones presented in the third and fourth columns of Wei (1997, Table II, Panel B), especially when $r \geq 24\%$.

Table 5.3: Coupon bond option prices, rhos, deltas, and thetas under the CIR model.

	Call options				Put options			
r	Price	Rho	Delta	Theta	Price	Rho	Delta	Theta
0.04	9.1833	−92.5420	0.3029	1.3791	0.0382	1.7847	−0.0058	1.7847
0.06	7.4484	−81.0065	0.2853	0.9106	0.0885	3.3390	−0.0118	3.3390
0.08	5.9407	−69.8268	0.2647	0.5076	0.1754	5.4324	−0.0206	5.4324
0.10	4.6525	−59.0753	0.2410	0.1782	0.3084	7.9183	−0.0323	7.9183
0.12	3.5737	−48.9233	0.2148	−0.0726	0.4932	10.5569	−0.0463	10.5569
0.14	2.6902	−39.5845	0.1870	−0.2452	0.7299	13.0718	−0.0618	13.0718
0.16	1.9836	−31.2550	0.1589	−0.3464	1.0135	15.2090	−0.0773	15.2090
0.18	1.4323	−24.0685	0.1317	−0.3880	1.3345	16.7814	−0.0918	16.7814
0.20	1.0129	−18.0749	0.1064	−0.3846	1.6803	17.6903	−0.1041	17.6903
0.22	0.7016	−13.2408	0.0839	−0.3514	2.0375	17.9239	−0.1135	17.9239
0.24	0.4762	−9.4665	0.0645	−0.3019	2.3931	17.5407	−0.1195	17.5407
0.26	0.3168	−6.6099	0.0485	−0.2466	2.7357	16.6445	−0.1220	16.6445
0.28	0.2067	−4.5109	0.0356	−0.1931	3.0563	15.3605	−0.1212	15.3605
0.30	0.1324	−3.0114	0.0256	−0.1456	3.3484	13.8147	−0.1173	13.8147

This table values 5-year call option and put prices, and the corresponding rhos, deltas, and thetas on a 15-year 10% coupon bond—with the ten coupons being paid annually—with par value 100 for different levels of the riskless interest rate (r) and strike price $K = 100$ assuming a CIR framework, and using the Benton and Krishnamoorthy (2003) algorithm for computing the non-central chi-square distribution function. Parameter values borrowed from Wei (1997, Table II): $\kappa = 0.25$, $\theta = 0.085$, $\sigma = 0.05$, and $\lambda = 0$.

Apparently, there are some typos in Wei (1997, Table II) for these cases—the same line of reasoning is observed in Wei (1997, Tables I and IV), but for interest rate deltas under the Vasicek (1977) model—because the corresponding absolute values are approximately equal. Cox et al. (1985) and Longstaff (1993) show that zero-coupon and coupon-paying bond call options are strictly decreasing functions of the riskless interest rate. Thus, the first derivative of bond call options with respect to interest rates (i.e. interest rate deltas) illustrated in the third and fourth columns of Wei (1997, Table II, Panel B) should always be negative, as shown in the third column of our Table 5.3. An increase in the riskless interest rate, however, has an indeterminate effect on the value of a bond put as highlighted in Tables 5.1 and 5.2. The implications of this issue for the hedging behavior of interest rate puts has been discussed in Longstaff (1993).

5 Valuation and Comparative Statics of Sinking-Fund Bonds in the CIR Framework

Bonds are said to have embedded sinking-fund provisions when the issuer is required to retire portions of the bond issue before maturity, according to a pre-specified amortization schedule. The *delivery option* associated to this clause allows the issuer to retire the portions of the issue either by (i) calling the bonds by lottery at a pre-determinate value, usually at par, or (ii) buying back the bonds at the prevailing market value.

Bacinello et al. (1996) provide an elegant framework for analyzing the delivery option embedded in the sinking-fund bond provision (with only one sinking-fund date before maturity) under the one-dimensional stochastic term structure interest rate models of Vasicek (1977) and Cox et al. (1985). Bacinello et al. (1996) were able to analyze the comparative statics properties of the sinking-fund bond in the Vasicek (1977) framework analytically, but they use a numerical approach for the Cox et al. (1985) model. Thus, the main purpose of this section is to extend the Bacinello et al. (1996) approach by analyzing, in closed-form, the comparative statics properties of a default-free sinking-fund bond in the CIR framework.

Following Bacinello et al. (1996), a sinking-fund bond is characterized by a coupon rate i_c and an amortization schedule $\{(t_j, C_j)\}$, where $C_j > 0$ is the principal that the issuer is required to retire at time t_j. We also assume that $j = 1,2$ and, without loss of generality, $C_1 + C_2 = 1$, i.e. the sinking-fund bond is issued with a normalized principal, retired in two dates only. Letting t_0 denote the time of issuance of the bond, its coupon payments, I_j, are then assumed to be given by $I_1 = (1+i_c)^{(t_1-t_0)} - 1$, and $I_2 = C_2\left[(1+i_c)^{(t_2-t_1)} - 1\right]$. At time t_1 the issuer has the (delivery) option to retire the fraction C_1 of the principal either by calling it by lottery at par value, or by buying it back at the market value.

Bacinello et al. (1996, Proposition 2.1) show that the time-t price of the sinking-fund bond, $B^{sf}(r,t)$, can be expressed either in terms of the corresponding serial bond and a bond put option, or in terms of the corresponding coupon bond and a bond call option, that is

$$B^{sf}(r,t) = B^{s}(r,t) - C_1(1+i_c)^{(t_2-t_1)}p^{zc}\Big(r,t,t_1,t_2,(1+i_c)^{-(t_2-t_1)}\Big), \quad (5.45)$$

$$B^{sf}(r,t) = B^{cb}(r,t) - C_1(1+i_c)^{(t_2-t_1)}c^{zc}\Big(r,t,t_1,t_2,(1+i_c)^{-(t_2-t_1)}\Big), \quad (5.46)$$

where $B^{s}(r,t)$ and $B^{cb}(r,t)$ represent, respectively, the time-t price of the corresponding serial and coupon bonds as given by Bacinello et al. (1996, Equations 2.2 and 2.3).

Let us now assume that $t_2 - t_1 = t_1 - t_0 = 1$. Following the same line of reasoning applied by Bacinello et al. (1996) for the Vasicek (1977) framework, we substitute the relations given by Bacinello et al. (1996, Equations 2.1 and 2.3) and the bond call option pricing formula (5.6) in equation (5.46). We then obtain, for $t < t_1$,

$$B^{sf}(r,t) = Z(r,t,t_1)\Big[i_c + C_1F(x_2;a,b_2)\Big] + (1+i_c)Z(r,t,t_2)\Big[1 - C_1F(x_1;a,b_1)\Big], \quad (5.47)$$

with x_1 and x_2 defined as in equations (5.7a) and (5.7b), but with $K = (1+i_c)^{-1}$ in equation (5.8c). Thus, the sinking-fund bond is shown to depend explicitly on the fraction C_1 of outstanding capital to be retired at t_1, the coupon rate i_c, the spot rate r prevailing on the market, and the CIR parameters κ, θ, σ, and λ. We are now able to extend the analytical results provided by Bacinello et al. (1996) under the Vasicek (1977) framework for the CIR model case.

The sinking-fund bond under the CIR model is an increasing function of the coupon rate. To establish this fact, take the derivative of (5.47) with respect to i_c, and observe that the relation

$$Z(r,t,t_2)p(x_1;a,b_1)(\phi+\psi+B(t_1,t_2)) = Z(r,t,t_1)p(x_2;a,b_2)(\phi+\psi)(1+i_c)^{-1} \tag{5.48}$$

holds as an identity, so that, after some algebraic manipulations, we have

$$\frac{\partial B^{sf}(.)}{\partial i_c} = Z(r,t,t_1) + Z(r,t,t_2)\Big(1 - C_1 F(x_1;a,b_1)\Big) > 0, \tag{5.49}$$

where the strict positivity follows from the fact that, by assumption, $0 < C_1 < 1$. Considering now the premiums $B^{cb}(.) - B^{sf}(.)$ and $B^{s}(.) - B^{sf}(.)$ of the corresponding coupon and serial bonds over the sinking-fund bond, and using respectively equations (5.46) and (5.45), coupled with $t_2 - t_1 = t_1 - t_0 = 1$, we obtain

$$\frac{\partial(B^{cb}(.) - B^{sf}(.))}{\partial i_c} = -\left(\frac{K}{c^{\mathrm{zc}}(.)}\frac{\partial c^{\mathrm{zc}}(.)}{\partial K} - 1\right)C_1 c^{\mathrm{zc}}(.) = C_1 Z(r,t,t_2)F(x_1;a,b_1) > 0, \tag{5.50}$$

$$\frac{\partial(B^{s}(.) - B^{sf}(.))}{\partial i_c} = -\left(\frac{K}{p^{\mathrm{zc}}(.)}\frac{\partial p^{\mathrm{zc}}(.)}{\partial K} - 1\right)C_1 p^{\mathrm{zc}}(.) = -C_1 Z(r,t,t_2)Q(x_1;a,b_1) < 0, \tag{5.51}$$

so that the higher the coupon rate, the larger is the premium demanded by the corresponding coupon bond over the sinking-fund bond, and the smaller is the premium commanded by the corresponding serial bond over the sinking-fund bond. Note that the sign of the above derivatives depends entirely on the elasticity of the option prices to the strike price, in particular on the fact that such elasticity is negative for the call and exceeds 1 for the put.

We can analyze also explicitly the comparative statics properties of the sinking-fund bond with respect to the spot rate r (rho) and time t (theta). The first sensitivity measure is given by

$$\begin{aligned}\rho_B^{sf} := \frac{\partial B^{sf}(.)}{\partial r} &= \frac{\partial Z(r,t,t_1)}{\partial r}\Big(i_c + C_1 F(x_2;a,b_2)\Big) + \frac{\partial Z(r,t,t_2)}{\partial r}(1+i_c)\Big(1 - C_1 F(x_1;a,b_1)\Big) \\ &\quad + C_1(1+i_c)Z(r,t,t_2)p(x_1;a+2,b_1)\frac{2\phi^2 e^{\gamma(t-t_1)}}{\phi+\psi+B(t_1,t_2)} \\ &\quad - C_1 Z(r,t,t_1)p(x_2;a+2,b_2)\frac{2\phi^2 e^{\gamma(t-t_1)}}{\phi+\psi}, \end{aligned} \tag{5.52}$$

with $\partial Z(r,t,t_i)/\partial r$, for $i = t_1, t_2$, given by equation (5.17a). The effect on the premiums $B^{cb}(.) - B^{sf}(.)$ and $B^{s}(.) - B^{sf}(.)$ of an infinitesimal change in the spot interest rate r can be stated as

$$\frac{\partial(B^{cb}(.) - B^{sf}(.))}{\partial r} = C_1(1+i_c)\frac{\partial c^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})}{\partial r}, \tag{5.53}$$

$$\frac{\partial(B^{s}(.) - B^{sf}(.))}{\partial r} = C_1(1+i_c)\frac{\partial p^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})}{\partial r}, \tag{5.54}$$

where $\partial c^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})/\partial r$ and $\partial p^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})/\partial r$ are given, respectively, by equations (5.15) and (5.16), but with $K = (1+i_c)^{-1}$.

The effect on $B^{sf}(.)$ of an infinitesimal change in t can be computed explicitly as

$$\theta_B^{sf} := \frac{\partial B^{sf}(.)}{\partial t} = \frac{\partial Z(r,t,t_1)}{\partial t}\Big(i_c + C_1 F(x_2;a,b_2)\Big) + \frac{\partial Z(r,t,t_2)}{\partial t}(1+i_c)\Big(1 - C_1 F(x_1;a,b_1)\Big)$$
$$-C_1(1+i_c)Z(r,t,t_2)\Big(p(x_1;a,b_1)\xi - p(x_1;a+2,b_1)\rho_1\Big)$$
$$+C_1 Z(r,t,t_1)\Big(p(x_2;a,b_2)\xi - p(x_2;a+2,b_2)\rho_2\Big), \tag{5.55}$$

where $\partial Z(r,t,t_i)/\partial t$, with $i = t_1, t_2$, is given by equation (5.33c). As for the influence of the parameter t on the premiums $B^{cb}(.) - B^{sf}(.)$ and $B^s(.) - B^{sf}(.)$, we have

$$\frac{\partial(B^{cb}(.) - B^{sf}(.))}{\partial t} = C_1(1+i_c)\frac{\partial c^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})}{\partial t}, \tag{5.56}$$
$$\frac{\partial(B^{s}(.) - B^{sf}(.))}{\partial t} = C_1(1+i_c)\frac{\partial p^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})}{\partial t}, \tag{5.57}$$

where $\partial c^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})/\partial t$ and $\partial p^{\text{zc}}(r,t,t_1,t_2,(1+i_c)^{-1})/\partial t$ are given, respectively, by equations (5.31) and (5.32), but with $K = (1+i_c)^{-1}$.

Now we want to prove a result that compares the stochastic durations of the sinking-fund bond with those of the corresponding serial and coupon bonds in the CIR model. Following Cox et al. (1979), the relative basis risk of a zero-coupon bond (under the CIR model), with maturity $\tau := s-t$, is given by $g(\tau) = 2(e^{\gamma\tau}-1)/\big((\kappa+\lambda+\gamma)(e^{\gamma\tau}-1)+2\gamma\big) = B(t,s) = B(\tau)$, a function that is strictly increasing ($\partial B(\tau)/\partial\tau > 0$) and continuous on all positive reals, with the inverse function given by $g^{-1}(\tau) = (1/\gamma)\ln(1-2\gamma\tau/((\kappa+\lambda+\gamma)\tau - 2))$, and defined on the interval $]0, 2/(\kappa+\lambda+\tau)[$. Moreover, the stochastic duration of any interest rate sensitive instrument with price $f(r,t)$ is given by

$$D^f = g^{-1}(x), \tag{5.58}$$

where $x = -(\partial f(r,t)/\partial r)/f(r,t)$ is the basis risk of f. Next proposition explicitly relates the stochastic durations of the sinking-fund, corresponding coupon and corresponding serial bonds under the CIR framework, thus extending the analytical results provided by Bacinello et al. (1996, Proposition 4.1), but for the Vasicek (1977) model.

Proposition 5.5 *For any set of parameters, the stochastic durations* $D^{sf}(r,t)$, $D^{cb}(r,t)$, *and* $D^{s}(r,t)$ *of the sinking-fund, corresponding coupon and corresponding serial bonds under the CIR model satisfy the relation*

$$D^{s}(r,t) < D^{sf}(r,t) < D^{cb}(r,t). \tag{5.59}$$

Proof. To verify the first inequality, use equations (5.45) and (5.52), along with the fact that $g^{-1}(x)$ is (positive and) increasing, to observe that this inequality becomes

$$\frac{1}{\gamma}\ln\left(1-\frac{2\gamma\rho_B^{s}}{(\kappa+\lambda+\gamma)\rho_B^{s}+2B^{s}(.)}\right) < \frac{1}{\gamma}\ln\left(1-\frac{2\gamma\rho_B^{sf}}{(\kappa+\lambda+\gamma)\rho_B^{sf}+2B^{sf}(.)}\right),$$

which is equivalent to $p^{zc}(r,t,t_1,t_2,(1+i_c)^{-1})\rho_B^{sf} - B^{s}(r,t)\rho_p^{zc} < 0$. To check the second inequality, use equation (5.46) and follow the same reasoning to obtain $B^{cb}(r,t)\rho_c^{zc} - c^{zc}(r,t,t_1,t_2,(1+i_c)^{-1})\rho_B^{sf} < 0$, which concludes the proof. ■

Using the same set of parameters as in Bacinello et al. (1996), Figure 5.1 highlights that the stochastic duration of the sinking-fund bond is between the stochastic duration of the corresponding serial and coupon bonds. While this issue has been shown already by Bacinello et al. (1996, Figure 13) through numerical differentiation, we have now established this property analytically via Proposition 5.5 and using our closed-form solutions.

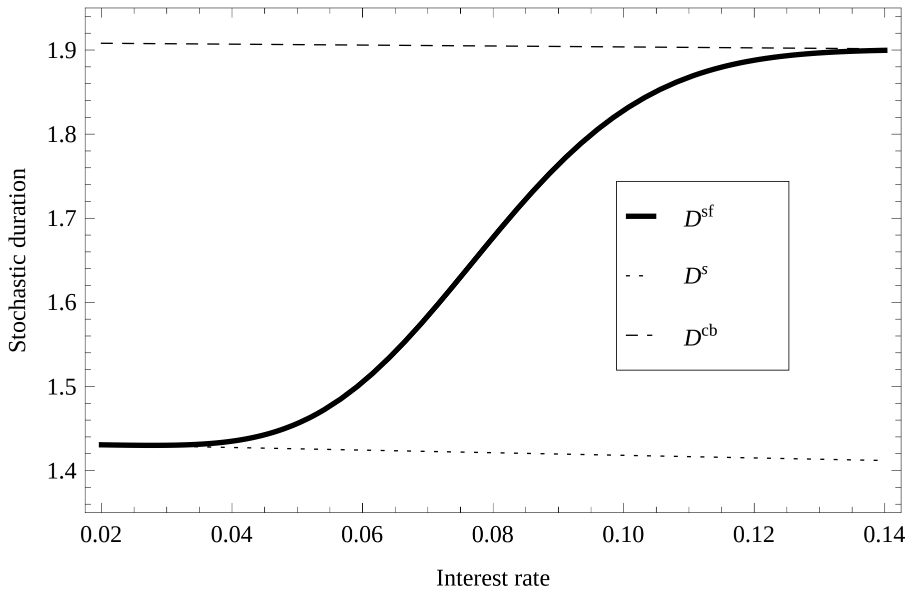

Figure 5.1: Stochastic duration of a sinking-fund bond under the CIR framework.

6 Conclusions

In this paper, we have derived closed-form solutions for determining sensitivity measures of both pure discount and coupon-paying bond options under the CIR framework, which are shown to be accurate, easy to implement, and computationally very efficient. Finally, we offer analytically tractable formulae to analyze the comparative-statics properties of a sinking-fund bond under the same one-dimensional interest rate dynamics setting.

Acknowledgements

Dias gratefully acknowledges the financial support from the Fundação para a Ciência e Tecnologia (FCT) grant number PTDC/EGE-ECO/099255/2008, while Larguinho and Braumann acknowledge the financial support from the Centro de Investigação em Matemática e Aplicações (CIMA) financed by the FCT.

References

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Allegretto, Walter, Yanping Lin, and Hongtao Yang, 2003, Numerical Pricing of American Put Options on Zero-Coupon Bonds, *Applied Numerical Mathematics* 46, 113–134.

Alvarez, Luis H. R., 2001, On the Form and Risk-Sensitivity of Zero Coupon Bonds for a Class of Interest Rate Models, *Insurance: Mathematics and Economics* 28, 83–90.

Bacinello, Anna Rita and Fulvio Ortu, 1999, Arbitrage Valuation and Bounds for Sinking-Fund Bonds with Multiple Sinking-Fund Dates, *Applied Mathematical Finance* 6, 293–312.

Bacinello, Anna Rita, Fulvio Ortu, and Patrizia Stucchi, 1996, Valuation of Sinking-Fund Bonds in the Vasicek and CIR Frameworks, *Applied Mathematical Finance* 3, 269–294.

Beaglehole, David R. and Mark S. Tenney, 1991, General Solutions of Some Interest Rate-Contingent Claim Pricing Equations, *Journal of Fixed Income* 1, 69–83.

Ben-Ameur, Hatem, Michèle Breton, Lotfi Karoui, and Pierre L'Ecuyer, 2007, A Dynamic Programming Approch for Pricing Options Embedded in Bonds, *Journal of Economic Dynamics and Control* 31, 2212–2233.

Benton, Denise and K. Krishnamoorthy, 2003, Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral t and the Distribution of the Square of the Sample Multiple Correlation Coefficient, *Computational Statistics and Data Analysis* 43, 249–267.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Brigo, Damiano and Fabio Mercurio, 2001, A Deterministic-Shift Extension of Analytically-Tractable and Time-Homogeneous Short-Rate Models, *Finance and Stochastics* 5, 369–387.

Brown, Roger H. and Stephen M. Schaefer, 1994, The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model, *Journal of Financial Economics* 35, 3–42.

Brown, Stephen J. and Philip H. Dybvig, 1986, The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates, *Journal of Finance* 41, 617–630.

Chan, K. C., G. Andrew Karolyi, Francis A. Longstaff, and Anthony B. Sanders, 1992, An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *Journal of Finance* 47, 1209–1227.

Chen, Ren-Raw and Louis Scott, 1992, Pricing Interest Rate Options in a Two-Factor Cox-Ingersoll-Ross Model of the Term Structure, *Review of Financial Studies* 5, 613–636.

Chen, Ren-Raw and Louis Scott, 1993, Maximum Likelihood Estimators for a Multifactor Equilibrium Model of the Term Structure of Interest Rates, *Journal of Fixed Income* 3, 14–31.

Chen, Ren-Raw and Louis Scott, 1995, Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure, *Journal of Derivatives* 3, 53–72.

Chen, Ren-Raw and Louis Scott, 2003, Multi-Factor Cox-Ingersoll-Ross Models of the

Term Structure: Estimates and Tests from a Kalman Filter Model, *Journal of Real Estate Finance and Economics* 27, 143–172.

Cohen, Jonathan D., 1988, Noncentral Chi-Square: Some Observations on Recurrence, *American Statistician* 42, 120–122.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1979, Duration and the Measurement of Basis Risk, *Journal of Business* 52, 51–61.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Dias, José Carlos and Mark B. Shackleton, 2011, Hysteresis Effects under CIR Interest Rates, *European Journal of Operational Research* 211, 594–600.

Feller, William, 1951, Two Singular Diffusion Problems, *Annals of Mathematics* 54, 173–182.

Gorovoi, Viatcheslav and Vadim Linetsky, 2004, Black's Model of Interest Rates as Options, Eigenfunction Expansions and Japanese Interest Rates, *Mathematical Finance* 14, 49–78.

Hull, John and Alan White, 1990, Pricing Interest-Rate-Derivative Securities, *Review of Financial Studies* 3, 573–592.

Jamshidian, Farshid, 1989, An Exact Bond Option Formula, *Journal of Finance* 44, 205–209.

Jamshidian, Farshid, 1995, A Simple Class of Square-Root Interest-Rate Models, *Applied Mathematical Finance* 2, 61–72.

Johnson, Norman L., Samuel Kotz, and N. Balakrishnan, 1995, *Continuous Univariate Distributions*, Vol. 2, 2nd ed. (John Wiley & Sons, New York, NY).

Kawai, Reiichiro and Atsushi Takeuchi, 2011, Greeks Formulas for an Asset Price Model with Gamma Processes, *Mathematical Finance* 21, 723–742.

Larguinho, Manuela, José Carlos Dias, and Carlos A. Braumann, 2013, On the Computation of Option Prices and Greeks under the CEV Model, *Quantitative Finance* 13, 907–917.

Longstaff, Francis A., 1993, The Valuation of Options on Coupon Bonds, *Journal of Banking and Finance* 17, 27–42.

Longstaff, Francis A. and Eduardo S. Schwartz, 1992, Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model, *Journal of Finance* 47, 1259–1282.

Maghsoodi, Yoosef, 1996, Solution of the Extended CIR Term Structure and Bond Option Valuation, *Mathematical Finance* 6, 89–109.

Mallier, R. and G. Alobaidi, 2004, Interest Rate Swaps under CIR, *Journal of Computational and Applied Mathematics* 164-165, 543–554.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Pearson, Neil D. and Tong-Sheng Sun, 1994, Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model, *Journal of Finance* 49, 1279–1304.

Sankaran, Munuswamy, 1963, Approximations to the Non-Central Chi-Square Distribution, *Biometrika* 50, 199–204.

ShuJin, Li and Li ShengHong, 2006, Pricing American Interest Rate Option on Zero-Coupon Bond Numerically, *Applied Mathematics and Computation* 175, 834–850.

Vasicek, Oldrich, 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.

Wei, Jason Z., 1997, A Simple Approach to Bond Option Pricing, *Journal of Futures Markets* 17, 131–160.

Zhou, Hong Jun, Ka Fai Cedric Yiu, and Leong Kwan Li, 2011, Evaluating American Put Options on Zero-Coupon Bonds by a Penalty Method, *Journal of Computational and Applied Mathematics* 235, 3921–3931.

Chapter 6

A Note on (Dis)Investment Options and Perpetuities under CIR Interest Rates*

Abstract: In this chapter, we discuss alternative ways of computing the options to invest in and divest from an investment project in a CIR economy Cox et al. (1985). Moreover, different methods of determining CIR perpetuities will also be analyzed.

JEL Classification: G1, G13.

Keywords: CIR economy; Perpetuities; Option to invest

*This paper is a joint work with José Carlos Dias and Carlos A. Braumann and is published in Recent Developments in Modeling and applications in Statistics. Studies in Theoretical and Applied Statistics, Springer, 2013, 203 - 211. ISBN 978-3-642-32419-2

1 Introduction

The most realistic case of the capital theory of investment is characterized by investments with costly reversibility in which a firm can purchase capital at a given price (by paying an investment cost $\overline{I}$) and sell capital at a lower price (by receiving the divestment proceeds $\underline{I}$), i.e. there is a fraction α of the invested capital, $\alpha := \underline{I}/\overline{I}$ $(0 < \alpha < 1)$, that a firm can recoup when divesting.

Decisions made under an uncertain environment where it is costly to reverse economic actions lead to an intermediate range, called the hysteretic band, where the continuation is the optimal policy until some threshold is reached.

In this chapter, such as in Dias (2006), we consider the situation of a firm that can invest $\overline{I}$ at any time and receive a perpetuity (a project) with constant cash flow rate. Even though the project's cash flows are fixed, its perpetuity value (negatively related to rates) is stochastic because the interest rate r used to discount the perpetual flows is assumed to follow a CIR (Cox et al. (1985)) diffusion process and thus r can change. Investment in this perpetual project will be triggered when interest rates are low (high perpetuity value) and in particular at a critical level $\underline{r}$.

Let us denote by $F_1(r) + P(r)$ and $F_0(r)$ the value functions for the active project (with perpetuity) which has the option to shut, and for the inactive project (without flow) but with the option to open, respectively. Investment will be triggered as the interest rate r falls to the lower threshold $\underline{r}$ whilst divestment will be triggered as rates rise to the upper trigger $\overline{r}$ $(\overline{r} > \underline{r})$. Thus, the decision to switch from the idle state to the operating state, and vice-versa, can be described by the following value matching conditions

$$\text{idle state} \rightarrow \text{operating state, that is } F_0(\underline{r}) + \bar{I} \rightarrow F_1(\underline{r}) + P(\underline{r})$$

$$\text{idle state} \leftarrow \text{operating state, that is } F_0(\bar{r}) + \underline{I} \leftarrow F_1(\bar{r}) + P(\bar{r}).$$

Thus, an idle firm invests when rates fall to $\underline{r}$ and an operating firm will disinvest once the interest rate rise to $\bar{r}$. Denote by V the value premium of active to idle firms, this depend on current rates r:

$$V(r) = F_1(r) + P(r) - F_0(r), \tag{6.1}$$

and the value matching conditions can be coupled with two smooth pasting (first order) conditions

$$V(\underline{r}) = \bar{I},\ V'(\underline{r}) = 0,\ V(\bar{r}) = \underline{I},\ V'(\bar{r}) = 0. \tag{6.2}$$

The range $(\underline{r}, \bar{r})$ is the hysteretic band of the problem since idle firms do not invest and operating firms do not suspend within this intermediate level of interest rate.

To find the entry and exit interest rate thresholds, and thus the optimal policy of the firm, we need to numerically evaluate a system of four highly non-linear equations (i.e., with two value matching and two smooth pasting conditions).

In this chapter, we discuss alternative ways of computing the options to invest in and divest from a project in a CIR economy. Furthermore, different methods of determining CIR perpetuities will also be analyzed.

The remainder of this chapter is as follows. Section 2 discusses in detail the necessary components for analyzing the optimal entry and exit decision of a firm with CIR interest rates. Section 3 presents computational results comparing the alternative methods of computing the options components and perpetuities under CIR diffusions. Section 4 presents the concluding remarks.

2 Optimal entry and exit decisions under CIR interest rates

Under the risk-neutral measure $\mathbb{Q}$, Cox et al. (1985) modeled the evolution of the interest rate, r_t, by the stochastic differential equation (sde):

$$dr_t = \left[\kappa\theta - (\lambda + \kappa)r_t\right]dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}, \tag{6.3}$$

where $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$, κ, θ and σ are positive constants representing reversion rate, asymptotic rate and volatility parameters, respectively, and λ is the market risk. The condition $2\kappa\theta > \sigma^2$ has to be imposed to ensure that the interest rate remains positive.[6.1]

2.1 CIR General Claims

Following Cox et al. (1985), the price of a general interest rate claim $F(r,t)$ with cash flow rate $C(r,t)$ satisfies the following partial differential equation

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 F(r,t)}{\partial r^2} + \kappa(\theta - r)\frac{\partial F(r,t)}{\partial r} + \frac{\partial F(r,t)}{\partial t} - \lambda r\frac{\partial F(r,t)}{\partial r} - rF(r,t) + C(r,t) = 0. \tag{6.4}$$

[6.1] See Feller (1951) for a complete description of the boundary conditions.

The price of a zero coupon bond with maturity at T, $Z(r,t,T)$, satisfies the equation (6.4) with $C(r,t) = 0$ subject to the boundary condition $Z(r,T,T) = 1$ and is given by

$$Z(r,t,T) = A(t,T)e^{-B(t,T)r} \tag{6.5}$$

where $A(t,T) = \left(\frac{2\gamma e^{\left((\kappa+\lambda+\gamma)(T-t)\right)/2}}{(\kappa+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma} \right)^{\frac{2\kappa\theta}{\sigma^2}}$, $B(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\kappa+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma}$, and $\gamma = \left((\kappa+\lambda)^2 + 2\sigma^2\right)^{1/2}$.

2.2 CIR Perpetuity

In a CIR diffusion, the value of a perpetuity, denoted by $P(r)$, that pays coupons at a constant unit rate $C(r,t) = 1$, should satisfy the following ordinary differential equation (ode):[6.2]

$$\frac{1}{2}\sigma^2 r \frac{d^2P(r)}{dr^2} + \kappa(\theta - r)\frac{dP(r)}{dr} - \lambda r \frac{dP(r)}{dr} - rP(r) + 1 = 0. \tag{6.6}$$

The solution of ode (6.6) is the value of a perpetuity given by

$$P(r) = E^{\mathbb{Q}}_{t_0}\left[\int_{t_0}^{\infty} e^{-\int_{t_0}^{t} r_s ds} dt\right] = \int_{t_0}^{\infty} Z(r,t_0,t)dt, \tag{6.7}$$

where $Z(r,t,T)$ is the price of a zero coupon bond. There are several ways to evaluate the value of this perpetuity. In this chapter, we analyze two methods proposed by Delbaen (1993) and one method proposed by Geman and Yor (1993). In the first

[6.2] We must note that since $\lim_{t\to\infty} \frac{\partial P(r,t)}{\partial t} = 0$, the value of a perpetuity is not time dependent.

method of Delbaen (1993) (method D_1), the value of a perpetuity is computed in the following manner:

$$P(r) = \int_0^1 \frac{2}{\eta} e^{-z(2r/\eta)} (1+\beta z)^{(\eta/2\gamma)(2\kappa\theta/\sigma^2)-1} (1-z)^{[(\gamma-\kappa-\lambda)/2\gamma](2\kappa\theta/\sigma^2)-1} dz, \qquad (6.8)$$

where $\eta = \kappa + \lambda + \gamma$ and $\beta = (\gamma - \kappa - \lambda)/\eta$. In the second method of Delbaen (1993) (method D_2), the value of a perpetuity is given by

$$P(r) = \frac{\gamma}{\kappa\theta} \Phi_1(a,b,c,x,y), \qquad (6.9)$$

where Φ_1 is the degenerate hypergeometric function defined as

$$\Phi_1(a,b,c,x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{m!} \frac{(a)_{m+n}}{(c)_{m+n}} (b)_m x^m y^n, \qquad (6.10)$$

where $(\alpha)_j$ is the Pochhammer symbol, and where $a = 1$, $b = 1 - \frac{\kappa+\lambda+\gamma}{2\gamma}\frac{2\kappa\theta}{\sigma^2}$, $c = 1 + \frac{\gamma-\kappa-\lambda}{2\gamma}\frac{2\kappa\theta}{\sigma^2}$, $x = -\frac{\gamma-\kappa-\lambda}{\kappa+\lambda+\gamma}$, and $y = -\frac{2r}{\kappa+\lambda+\gamma}$.

The alternative formulation proposed by Geman and Yor (1993) (method GY) is given by

$$P(r) = \frac{\psi}{\omega} e^{\phi\psi r/2} \int_0^1 \frac{(1+z)^p (1-z)^q e^{\frac{r\omega(z+\varphi)}{2(1+\varphi z)}}}{(1+\varphi z)^{\delta\psi/2}} dz, \qquad (6.11)$$

where $\delta = \kappa\theta$, $\phi = \frac{\kappa+\lambda}{2}$, $\psi = \frac{4}{\sigma^2}$, $\omega = (2\psi + \phi^2\psi^2)^{1/2}$, $\varphi = \frac{\phi\psi}{\omega}$, $p = \frac{\phi\delta\psi^2}{4\omega} + \frac{\delta\psi}{4} - 1$, and $q = \frac{\delta\psi}{4} - \frac{\phi\delta\psi^2}{4\omega} - 1$.

2.3 Complementary Functions

Particularizing the time homogeneous situation gives a simple ode that determines the perpetual option to invest in or divest from a project. Furthermore for the options themselves no cash flows are present, so $C(r) = 0$

$$\frac{1}{2}\sigma^2 r \frac{d^2F(r)}{dr^2} + \kappa(\theta - r)\frac{dF(r)}{dr} - \lambda r \frac{dF(r)}{dr} - rF(r) = 0. \tag{6.12}$$

Dias and Shackleton (2011) have shown that the ode (6.12) can be transformed into Kummer's equation given by:

$$z_0 g''(z_0) + (b - z_0)g'(z_0) - a_0 g(z_0) = 0, \tag{6.13}$$

$$z_1 g''(z_1) + (b - z_1)g'(z_1) - a_1 g(z_1) = 0, \tag{6.14}$$

with $a_{0,1} = \frac{\kappa\theta}{\sigma^2}\left(1 \mp \frac{\kappa+\lambda}{\gamma}\right)$, $b = \frac{2\kappa\theta}{\sigma^2}$, and $z_{0,1} = \pm\frac{2\gamma r}{\sigma^2}$. The solutions to (6.13) and (6.14) corresponds to the complementary functions to be used for an idle firm and an operating firm, respectively. Since there are many alternative solutions to Kummer's equation (see, Abramowitz and Stegun (1972, Equations 13.1.12-19) care must be taken to choose the ones that give simpler solutions and are easier to apply boundary conditions. In Dias and Shackleton (2011), the complete solutions of equations (6.13) and (6.14) are given by (see, Abramowitz and Stegun (1972, Equation 13.1.11))

$$y = A_0 M(a_0, b, z_0) + B_0 U(a_0, b, z_0), \tag{6.15}$$

$$y = A_1 M(a_1, b, z_1) + B_1 U(a_1, b, z_1), \tag{6.16}$$

where $A_{0,1}$ and $B_{0,1}$ are arbitrary constants, $M(a,b,z)$ is the Kummer's confluent hypergeometric function (e.g., Abramowitz and Stegun (1972, Equation 13.1.2)), and finally $U(a,b,z)$ is the Tricomi confluent hypergeometric function (e.g., Abramowitz and Stegun (1972, Equation 13.1.3)).

If we multiply the complete solutions (6.15) and (6.16) by $e^{v_0 r}$ and $e^{v_1 r}$, respectively, with $v_{0,1} = \frac{\kappa+\lambda\mp\gamma}{\sigma^2}$, and reverse the change of variables, we obtain

$$F_0(r) = C_1 e^{v_0 r} M(a_0,b,z_0) + C_2 e^{v_0 r} U(a_0,b,z_0), \tag{6.17}$$

$$F_1(r) = C_3 e^{v_1 r} M(a_1,b,z_1) + C_4 e^{v_1 r} U(a_1,b,z_1), \tag{6.18}$$

which are also solutions of the ode (6.12), and where C_{1-4} are constants to be determined from boundary conditions.

The solution to (6.12) can also be obtained via *Mathematica* or *Maple*. For instance, Dias (2006) provides the following solutions for an idle firm and an operating firm obtained through *Mathematica*:

$$F_0(r) = C_{1M} e^{v_0 r+\mu\log(r)} U(a,1+\mu,z_0) + C_{2M} e^{v_0 r+\mu\log(r)} L_n^{\beta}(z_0), \tag{6.19}$$

$$F_1(r) = C_{3M} e^{v_0 r+\mu\log(r)} U(a,1+\mu,z_0) + C_{4M} e^{v_0 r+\mu\log(r)} L_n^{\beta}(z_0), \tag{6.20}$$

with $\mu = 1 - \frac{2\kappa\theta}{\sigma^2}$, $a = -\frac{\kappa\theta(\kappa+\lambda+\gamma)-\sigma^2\gamma}{\sigma^2\gamma}$, $\beta = \mu$, $n = -a$, and where $L_n^{\beta}(z)$ is the Laguerre polynomial as defined in Abramowitz and Stegun (1972, Equation 22.5.54).

Carmona and León (2007) provide also an alternative solution of equation (6.12) for the valuation of an investment project with the option to wait, $F_0(r)$, given by[6.3]

$$F_0(r) = Ce^{v_0 r}\left(M(a_0,b,z_0) - \frac{\Gamma(b)}{\Gamma(a_0)}\frac{\Gamma(1+a_0+b)}{\Gamma(2-b)}z_0^{1-b}M(a_0-b+1,2-b,z_0)\right). \qquad (6.21)$$

Even though equations (6.17)-(6.21) are all solutions to equation (6.12) we should choose the ones that are easier to use. For instance, to solve the entry and exit problems we need to numerically evaluate a system of four highly non-linear equations. Thus, the appropriate choice of the options components solutions is relevant for simplifying numerical computations. The analytical study of the Kummer's confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$ allow us to conclude that the solution of Dias and Shackleton (2011) is the easiest to apply and turns the economic hysteresis problem much more simple to analyze and understand. Based on these observations, we describe below the necessary boundary conditions that should be applied to equations (6.17) and (6.18).

2.4 Boundary and First Order Conditions

Given the near-zero and asymptotic behavior of the functions $M(a,b,z)$ and $U(a,b,z)$ and the necessary boundary conditions, Dias and Shackleton (2011) found that the expected net present value in the idle state with the option to open and the option to switch out of the perpetuity are respectively given by:

$$F_0(r) = C_2 e^{v_0 r} U(a_0,b,z_0), \qquad (6.22)$$

[6.3] Even though Carmona and León (2007) does not present solutions for the option to divest, $F_1(r)$ could also be easily obtained.

$$F_1(r) = C_3 e^{v_1 r} M(a_1, b, z_1), \tag{6.23}$$

which, using (6.1) and (6.2), lead to two value matching conditions

$$F_1(\underline{r}) + P(\underline{r}) - F_0(\underline{r}) = \overline{I} \text{ and } F_1(\overline{r}) + P(\overline{r}) - F_0(\overline{r}) = \underline{I}. \tag{6.24}$$

The first order (smooth pasting) conditions are given by

$$\frac{dF_0(\underline{r})}{d\underline{r}} = \frac{dF_1(\underline{r})}{d\underline{r}} + \frac{dP(\underline{r})}{d\underline{r}} \text{ and } \frac{dF_0(\overline{r})}{d\overline{r}} = \frac{dF_1(\overline{r})}{d\overline{r}} + \frac{dP(\overline{r})}{d\overline{r}}. \tag{6.25}$$

Thus, the solution of the two sided control problem rests on the determination of the two embedded constants C_2, C_3, and two thresholds $\underline{r}$, $\overline{r}$.

3 Numerical Analysis

This section aims to present computational comparisons of the alternative methods of computing the value of a perpetuity, and then using these results to analyze the investment hysteresis problem.

3.1 Perpetuities

To compare, in terms of speed and accuracy, the computation of a perpetuity under the CIR diffusion we need to choose a benchmark. An obvious candidate for a benchmark is to use numerical integration using, for instance, Gauss-Kronrod's method. In order to simplify the numerical computations, we start to analyze if the use of a fixed number T_{max} in the upper limit of the integral of the price of a CIR zero coupon bond $\int_{t_0}^{\infty} Z(r,t_0,t)dt$, instead of using infinity, will not generate any problem.[6.4] The results are shown in Table 6.1. Considering $T_{max} = 5,000$ seems to be quite reasonable for the analysis and it will simplify the numerical computations if we use this approach.

Table 6.1: Benchmark selection.

		κ			T_{max}		θ		
T_{max}	θ	0.15	0.50	0.90	T_{max}	κ	0.03	0.09	0.15
100	0.09	15.983797	12.295701	11.795340	100	0.50	31.816054	12.295701	8.052518
500	0.09	15.986306	12.296527	11.706291	500	0.50	33.154585	12.296527	8.052518
5,000	0.09	15.986306	12.296527	11.706291	5,000	0.50	33.154593	12.296527	8.052518
∞	0.09	15.986306	12.296527	11.706291	∞	0.50	33.154593	12.296527	8.052518

Parameters used in calculations: $\sigma = 0.125, \lambda = -0.05$, and $r = 0$.

Now we want to evaluate the differences in methods proposed by Delbaen (1993) and Geman and Yor (1993) compared against the selected benchmark. Table 6.2 reports such comparison results using the following set of parameters: $\sigma \in \{0.05, 0.07, \cdots, 0.17, 0.19\}$, $r \in \{0, 0.01, \cdots, 0.14, 0.15\}$, $\lambda \in \{-0.10, -0.05, 0\}$, $\kappa \in \{0.15, 0.20, \cdots, 0.85, 0.90\}$, and $\theta \in \{0.030, 0.045, \cdots, 0.885, 0.900\}$. This combination of parameters produce $55,296$ perpetuity values.

Table 6.2: Differences in perpetuity values for each method compared against the benchmark.

Methods	MaxAE	MaxRE	RMSE	MeanAE	n	CPU time
D_1	$2.97E-01$	$8.45E-03$	$3.43E-04$	$2.97E-03$	$2,263$	$275s$
D_2	$4.93E-03$	$1.52E-04$	$4.44E-10$	$9.41E-08$	0	$9,147s$
GY	$2.90E-01$	$8.27E-03$	$3.27E-04$	$2.89E-03$	$2,238$	$377s$

The $MaxAE$, $MaxRE$, $RMSE$, $MeanAE$, and n denote, respectively, the maximum absolute error, the maximum relative error, the root mean squared error, the mean absolute

[6.4] We have tried other combinations and we have reached the same conclusions.

error, and the number of times the absolute difference between the two methods exceeds one penny. The last column of the Table 2 reports the CPU time, in seconds, for computing $55,296$ values of perpetuities, using the function $Timing[.]$ available in *Mathematica 7.0*, running on a Pentium IV $(2.53GhZ)$ personal computer. The results from Table 2 indicate that the Method D_2 proposed by Delbaen (1993) performs best in terms of accuracy, and the method D_1 proposed by Delbaen (1993) is the most efficient in terms of computation time.

3.2 Entry and Exit Problem

Table 6.3 presents the results for investment options on a CIR economy using the three alternative formulations described above (and after applying the appropriate boundary conditions) and under the same set of parameters as in Carmona and León (2007), i.e., $\kappa = 0.45$, $\theta = 0.03$, $\sigma = 0.15$, and $\lambda = 0$. In addition, we use an investment cost of $\bar{I} = 5$. The perpetuity value is computed using the method D_2 proposed by Delbaen (1993). As expected, the option value is the same under the three alternative solutions. However, given that the solution proposed by Dias and Shackleton (2011) is much more simple the preference is to use this one. Table 4 illustrates the upper and lower interest rate thresholds for the entry and exit problem in a CIR economy for the same parameters set defined in Table 3. When $\alpha = 0$, an operating firm never shuts its project. Therefore, the corresponding threshold represents the interest rate level that will induce an idle firm to enter in a project and continue its operations forever since the option to shut down is worthless. Table 4 reveals also that the increasing levels of flexibility reduce, as expected, the hysteric band. These results provide useful insights for practitioners since they clearly highlight that managerial consideration of abandonment options at the time of project initiation can add value.

Table 6.3: Option to invest in a CIR economy.

$Equation$	$Constant$	$\underline{r}$	$F_0(\underline{r})$
(6.17)	34.4582	0.1759	21.1574
(6.19)	16.1497	0.1759	21.1574
(6.21)	25.8925	0.1759	21.1574

CIR parameters used in calculations: $\kappa = 0.45$, $\theta = 0.03$, $\sigma = 0.15$, and $\lambda = 0$. The investment cost is set at $\bar{I} = 5$.

Table 6.4: Upper and lower interest rate thresholds for the entry and exit problem in a CIR economy.

α	$\underline{r}$	$\bar{r}$
0.00	0.1759	$+\infty$
0.25	0.1759	0.9439
0.50	0.1759	0.8897
0.75	0.1759	0.5496
1.00	0.2000	0.2000

CIR parameters used in calculations: $\kappa = 0.45$, $\theta = 0.03$, $\sigma = 0.15$, and $\lambda = 0$. The investment cost and the disinvestment proceeds are set at $\bar{I} = 5$ and $\underline{I} := \alpha \bar{I}$, respectively.

4 Conclusion

In this paper, we analyzed the alternative methods to calculate the value of a perpetuity under a CIR diffusion, and we conclude that the method D_2 proposed by Delbaen (1993) performs best in terms of accuracy. We also discussed the alternative ways to calculate the options to invest in or disinvest from a project under a CIR economy, and we conclude that the solution given in Dias and Shackleton (2011) is more simple for applying the smooth pasting conditions, particularly for entry and exit problems.

Acknowledgements

Dias is member of the BRU-UNIDE and Larguinho and Braumann are members of the Research Center Centro de Investigação em Matemática e Aplicações (CIMA), both centers financed by the Fundação para a Ciência e Tecnologia (FCT). Dias gratefully acknowledges the financial support from the FCTs grant number PTDC/EGE-ECO/099255/2008.

References

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Carmona, Julio and Angel León, 2007, Investment Option under CIR Interest Rates, *Finance Research Letters* 4, 242–253.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Delbaen, Freddy, 1993, Consols in the CIR Model, *Mathematical Finance* 3, 125–134.

Dias, J. C., "Essays in real options models under interest rate uncertainty." PhD dissertation, ISCTE Business School 2006.

Dias, José Carlos and Mark B. Shackleton, 2011, Hysteresis Effects under CIR Interest Rates, *European Journal of Operational Research* 211, 594–600.

Feller, William, 1951, Two Singular Diffusion Problems, *Annals of Mathematics* 54, 173–182.

Geman, Hélyette and Marc Yor, 1993, Bessel Processes, Asian Options, and Perpetuities, *Mathematical Finance* 3, 349–375.

Chapter 7

Entry and Exit Decisions under Uncertainty for a Generalized Class of One-Dimensional Diffusions*

Abstract: We consider the optimal entry and exit policy of a firm in the presence of output price uncertainty and costly reversibility of investment under a generalized class of one-dimensional diffusions, namely, the CEV process and the mean-reverting CEV process. For these processes, we analyze how output price uncertainty and costly reversibility affects the optimal entry and exit policy of a competitive price-taking firm, and how the hysteretic band is affected by the choice of the stochastic process.

JEL Classification: G1, G13.

Keywords: Real options; Price uncertainty; Investment opportunity; Divestment opportunity; Costly reversibility; Hysteresis

---

*This paper is a joint work with José Carlos Dias and Carlos A. Braumann.

1 Introduction

It is broadly accepted by academics and corporate managers that traditional valuation techniques based on discounted cash flows (e.g. the standard net present value method) are not the most appropriate tool in decision making, especially in the presence of uncertainty, complete irreversibility or costly reversibility of investment, and when there is some leeway for conducting a flexible management. The main reason for this observation is that, in the presence of irreversibility, the firm is unable to instantaneously adjust its operations to a desired optimal level if market conditions unexpectedly deteriorate and change in an unfavorable direction after an investment decision has been made. As expected, the presence of uncertainty augments this effect and raises the required investment premium associated with the irreversible decision by increasing the option value of waiting.

Given its analytical attractiveness, the process most used in the literature of real options is the geometric Brownian motion (GBM henceforth). However, it is well documented in the literature that the GBM assumption embodies some unrealistic implications for the dynamic behavior of real asset prices. Namely, there is empirical evidence indicating that this assumption is not rich enough to capture the volatility smiles or skews found in the equity options market - see, for example Jackwerth and Rubinstein (2001). To overcome this issue, alternative stochastic processes have been considered in the real options literature. Such alternatives beyond the classic GBM assumption and the vast range of possible applications for a practitioner, bring up a number of important issues deserving a detailed examination.

One of the first attempts to investigate the biases in value provoked by the use of traditional methods of valuation was done using a mean-reverting process, which can be more suitable under equilibrium conditions. Bhattacharya (1978) studies the accuracy of traditional valuation methods when cash flows follow a mean-reverting process, as opposed to the standard GBM, and the ensuing biases in value. There are several works in the real options literature where the stochastic processes used are the mean-reverting type. For example, Sarkar (2003) assumes that the stochastic costs (not stochastic revenues) follow a mean-reverting process, concluding that the mean reversion, in general, have a significant impact on investment. Thus, it is generally inappropriate to use a GBM process to approximate a mean-reverting process. This work does not consider reversibility nor disinvestment and is only concentrated on irreversible entry. To overcome this situation, an extension has been proposed by Tsekrekos (2010). Dias and Shackleton (2011) also examine the investment and divestment decisions problem assuming that the stochastic interest rate follows a mean-reverting process.

The constant elasticity of variance (CEV hereafter) model of Cox (1975) is another stochastic process used to overcome the drawbacks pointed out to the GBM process. This model is consistent with two well known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized volatility (leverage effect), as observed, for instance, in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (implied volatility skew)—see, for example, Dennis and Mayhew (2002).

To our knowledge, there are only a few empirical studies on real options where the CEV model has been used. Nevertheless, there is evidence supporting the use of this stochastic process. Choi and Longstaff (1985) have examined the stochastic behavior of soybean future quasi-returns. Their empirical study suggests that the CEV process is theoretical superior to the GBM process for pricing options on soybean futures. The dynamic of crude oil prices by region, time period, and observation frequency using the Chan et al. (1992) general diffusion formulation has been examined by Lee and Heo

(2008), where they have conclude that the CEV model is the most suitable process to explain the dynamics of crude oil prices. An interesting study was performed by Geman and Shih (2009), where they analyze the performance of the CEV process (the mean-reverting CEV process is also considered) to model the crude oil, coal, copper, and gold prices. They conclude that the CEV exponent plays an important role in metal and energy commodities after the year 2000. Recently, Dias and Nunes (2011) derive analytical solutions for perpetual American-style call and put options under the CEV model. Their results strongly highlight the case for moving beyond the simplistic real models based on the GBM assumption to more realistic models incorporating volatility smile effects.

In this paper, we reconsider the problem originally addressed by Dixit (1989a) and Tsekrekos (2010), and analyze how output price uncertainty and costly reversibility affects the optimal entry and exit policy of a competitive price-taking firm. We extend these previous studies in two ways. First, we assume that the underlying output price dynamics follows a generalized one-dimensional diffusion which takes the modeling assumptions of Dixit (1989a) and Tsekrekos (2010) as two special cases. Second, we analyze the impact of costly reversibility on the dynamic entry and exit problem. This latter issue as also been considered by Dias and Shackleton (2011), but in a real options model where uncertainty stems from the interest rate uncertainty.

Hence, our analysis covers a broad class of descriptions both for the reversibility degree and for the underlying stochastic price dynamics which, within our generalized class of one-dimensional diffusions, includes most typically applied mean-reverting models as well as different volatility specifications. These issues should be important for academics and practitioners, since our modeling framework admits the analysis of the general properties of entry and exit decisions under alternative underlying driving stochastic factor dynamics and characterizes the circumstances under which the obtained results are significantly different or remain qualitatively valid, depending on the assumption made for the underlying output price dynamics.

The structure of the paper is organized as follows. Section 2 presents the firm's policy, the general output price dynamics, the value-matching condition, the smooth-pasting condition, and define the hysteretic band. Section 3 specializes the architecture modeling framework for the GBM, CEV, and the mean-reverting CEV processes. Section 4 compares the optimal entry-exit policy under the several processes. In Section 5, we compute the *ex ante* probabilities of entry and exit and compare the results, and finally Section 6 concludes. The Monte Carlo methods used to compute the probabilities are shown in Appendix.

2 Modeling architecture

For the analysis to remain self-contained, the next four subsections provide the necessary building blocks for modeling entry and exit decisions under alternative output price dynamics.

2.1 The firm's policy

Following Dixit (1989a) and Tsekrekos (2010), we shall consider a price-taking firm that has the possibility to invest (at any time) a lump-sum entry cost $\overline{K}$ to enter in a market (i.e. it needs to pay $\overline{K}$ to switch from the idle or inactive state to the operating state). As usual, the entry mode is in the form of a discrete unit of investment, namely a single project of a given size. While active in the market, the firm can produce a unit flow of output at a variable cost C. Moreover, the firm can decide to suspend operations (at any time) if market conditions deteriorate.

Similarly to Abel et al. (1996), Abel and Eberly (1996), Alvarez (2011), and Dias and Shackleton (2011), and to accommodate the generalization for different costly reversibility levels, we assume that by divesting the firm receives the disinvestment proceeds $\underline{K}$, i.e. there is a fraction α of the invested capital, with $\alpha := \underline{K}/\overline{K}$ (the ratio of the *direct* switching costs), that a firm can recoup when divesting.[7.1] Such prescription for the α parameter encompasses different reversibility degrees contemplated in the literature, namely:

- $\alpha = 1$ represents the traditional costlessly reversible investment case in which the wedge between the investment cost and the divestment proceeds is zero, and the optimal investment policy of a firm maintains the marginal revenue product of capital equal to the Jorgenson (1963) marginal user cost of capital. As expected, such standard myopic investment rule is unrealistic since, in the presence of irreversibility and uncertainty, it is not expected that a firm can divest at no cost due to the so-called *lemons* problem of Akerlof (1970).
- $\alpha = 0$ stands for the completely irreversible investment case in which the sale price of capital is zero (so that the wedge is 100% of the purchase price of capital) initiated by Arrow (1968), and then employed in much of the subsequent work on optimal investment under uncertainty.
- There are also more realistic investment cases characterized by costly reversibility in which a firm can purchase capital at a given price and sell capital at a lower price, i.e. with $\alpha \in (0,1)$. In other words, even though capital has resale value, it is below its acquisition cost, thus making part of the initial entry costs sunk. For example, this modeling specification has been considered by Abel et al. (1996),

[7.1] As in Dixit (1989a), Tsekrekos (2010), or Dias and Shackleton (2011), we assume that entry and exit takes place immediately after the decision to invest or divest has been made, thus ignoring the so-called time to build (or investment lags) effects discussed in Majd and Pindyck (1987), Bar-Ilan and Strange (1996), and Milne and Whalley (2000).

Abel and Eberly (1996), Alvarez (2011), and Dias and Shackleton (2011). Such partial reversibility case is of paramount importance because, as it was shown by Keswani and Shackleton (2006), a project's option value increases with incremental levels of investment and disinvestment flexibility.

- In the previous case capital can be abandoned at a cost since only a fraction of the entry cost can be recovered on exit. There may be, however, situations where it is necessary to pay a lump-sum cost to close a project, such as the cases of a copper mine or a nuclear power station where environmental clean costs may have to be supported. In our modeling framework, this is equivalent to assume $\alpha < 0$. For instance, such assumption was taken by Dixit (1989a) and Tsekrekos (2010).

In order to simplify the exposition and keep our generalized modeling framework similar (and thus comparable) to the work of Dixit (1989a) and Tsekrekos (2010), we assume that the parameter values $\overline{K}$, α, and C are constant and non-stochastic. Moreover, uncertainty stems from the output equilibrium price P which is assumed to be exogenous to the firm (i.e. the firm is a price-taker as already stated).

Let $V_0(P)$ be the expected net present value of the firm (with an initial output price P in the idle state) and following dynamic optimal entry-exit policies. The optimal entry and exit policy is determined through two time independent values of the state variable P, one upper threshold price $\overline{P}$ (reached from below) and one lower trigger $\underline{P}$ (reached from above), with $\overline{P} > \underline{P}$, at which a firm optimally switches from the idle to the operating state and vice versa. At the optimal entry threshold $\overline{P}$, the idle firm exercises its entry option by paying $\overline{K}$ in order to receive an "underlying asset" of value $V_1(\overline{P})$, which includes both an option to exit and a flow reward component. Similarly, at the optimal exit threshold $\underline{P}$, the active firm exercises its exit option in favour of regaining an "underlying asset" worth $V_0(\underline{P})$ (i.e. an option to enter the market again) and a cash amount $\underline{K} := \alpha\overline{K}$ (positive, if $\alpha \in (0,1)$, or negative, if $\alpha < 0$).[7.2]

2.2 Output price dynamics

Hereafter, we assume the equilibrium output time-*t* price P_t, evolving on the complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, is characterized by the following one-dimensional Itô diffusion:

$$dP_t = \mu(P_t)dt + \sigma(P_t)dW_t^{\mathbb{P}},\ P_0 = P \in \mathbb{R}_+, \tag{7.1}$$

where $W_t^{\mathbb{P}}$ is a standard Brownian motion under the physical probability measure $\mathbb{P}$.[7.3] Furthermore, we assume that the drift coefficient $\mu : \mathbb{R}_+ \to \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy the conditions $\sigma(P) > 0$, $\forall P \in (0,\infty)$, and

$$\int_{P-\varepsilon}^{P+\varepsilon} \frac{1+|\mu(y)|}{\sigma^2(y)} dy < \infty,$$

for some $\varepsilon > 0$, $\forall P \in (0,\infty)$. As shown by Karatzas and Shreve (1991, pp. 342-351), these conditions guarantee the existence of a weak solution for the stochastic differential equation (7.1).

Using Itô's lemma for our *infinite-horizon stochastic problem*, it follows that

[7.2]In other words, we are assuming a two-sided discrete regulator problem with lump-sum costs (or discrete adjustments) $\overline{K}$ and $\underline{K}$, in which controls are applied only when the state variable P hits the threshold levels $\overline{P}$ and $\underline{P}$, thus making our economic application resembling a *stochastic optimal impulse control problem* in the Constantinides and Richard (1978), Harrison et al. (1983), Dixit (1991), and Dumas (1991) sense.

[7.3]To lighten notation, the subscript 't' is dropped in the remainder of the paper.

$$
\begin{aligned}
dV_0(P) &= V_0^{'}(P)dP + \frac{1}{2}V_0^{''}(P)(dP)^2 \\
&= \left[\frac{1}{2}\sigma^2(P)V_0^{''}(P) + \mu(P)V_0^{'}(P)\right]dt + \sigma(P)V_0^{'}(P)dW^{\mathbb{P}}.
\end{aligned} \tag{7.2}
$$

Note that the time partial derivative usually appearing in (7.2) is zero due to the perpetual nature of the problem. The expected return and the standard deviation of the return of the firm are respectively given by:

$$
\mathbb{E}[R] = \frac{\frac{1}{2}\sigma^2(P)V_0^{''}(P) + \mu(P)V_0^{'}(P)}{V_0(P)}, \tag{7.3}
$$

and

$$
\mathbb{D}[R] = \frac{\sigma(P)V_0^{'}(P)}{V_0(P)}. \tag{7.4}
$$

As usual, the firm value must satisfy the following risk-return relationship

$$
\mathbb{E}[R] = r + \lambda^*(P)\,\mathbb{D}[R] = r + \lambda^*(P)\,\frac{\sigma(P)V_0^{'}(P)}{V_0(P)}, \tag{7.5}
$$

where $\lambda^*(P)$ is the compensation per unit risk above the (constant) riskless rate r. Note that the functional form of the market price of risk $\lambda^*(P)$ will depend on the respective stochastic process that is chosen for modeling the output price dynamics.

Substituting equation (7.3) into equation (7.5), multiplying both sides by $V_0(P)$, and then rearranging terms yields the following ordinary differential equation (ode):

$$
\frac{1}{2}\sigma^2(P)V_0^{''}(P) + \left[\mu(P) - \lambda^*(P)\sigma(P)\right]V_0^{'}(P) - rV_0(P) = 0. \tag{7.6}
$$

This is the ordinary differential equation that the value of the firm must satisfy over the range of output prices that is optimal for an idle firm to remain in the inactive state, i.e. for $P \in (0, \overline{P})$.

Similarly, over the range of prices where it is optimal for an active firm to continue in the operating state, i.e. for $P \in (\underline{P}, \infty)$, the total return of the expected net present value of the firm, $V_1(P)$, comprises the expected capital gain $\mathbb{E}\left[dV_1(P)\right]/dt$, plus a cash inflow $(P - C)$ per unit of time. Following the same line of reasoning, the value $V_1(P)$ must satisfy the following ordinary differential equation (ode):

$$\frac{1}{2}\sigma^2(P)V_1''(P) + \left[\mu(P) - \lambda^*(P)\sigma(P)\right]V_1'(P) - rV_1(P) + P - C = 0. \tag{7.7}$$

Solving equations (7.6) and (7.7) subject to appropriate boundary conditions yields the value functions for an idle and an active firm.

2.3 Solutions of the value functions $V_0(P)$ and $V_1(P)$

Omitting the term $f(P) := P - C$ in the ode (7.7), one notes that both (7.6) and (7.7) are linear differential equations possessing the same general solution for the homogeneous equation, which can be expressed as a linear combination of any two independent solutions. In other words, the corresponding linearly independent complementary functions are similar, and can thus be solved together. What determines the difference between the two contingent solutions and the options they represent is their boundary conditions.

As usual, the firm's option value to enter the market should be nearly worthless as the output price P becomes very small. To ensure such economic rationale, the ode (7.6) must be solved subject to the following boundary condition:

$$\lim_{P \to 0^+} V_0(P) = 0. \tag{7.8}$$

However, the general solution of the non-homogeneous ode (7.7) should be expressed as the sum of two parts: The general solution of the homogeneous equation neglecting the flow reward function $f(P)$ and an arbitrarily chosen particular solution of the full equation (7.7). As shown by Dixit (1991), a very convenient particular solution of (7.7) is the expected discounted flow payoff

$$F(P) := \mathbb{E}\left[\int_0^{+\infty} e^{-rs} f(P_s) ds | P_0 = P\right], \tag{7.9}$$

that is calculated ignoring both (upper and lower) barriers on the one-dimensional diffusion process P.[7.4]

As expected, the firm's option value to exit the market (while in the active state) should be nearly worthless as the output price P becomes very high. Thus, to rule out any explosive growth of firm value with high output price, we must impose the so-called *no-bubbles condition* which implies that, for high equilibrium output prices, the exit option becomes worthless and the value function $V_1(P)$ converges to the expected present value of operating in the market perpetually given in equation (7.9), that is

$$\lim_{P \to +\infty} V_1(P) = \mathbb{E}\left[\int_0^{+\infty} (P_s - C) e^{-rs} ds | P_0 = P\right]. \tag{7.10}$$

[7.4]The particular solution $F(P)$ can be interpreted as the expected present value payoff when the (uncontrolled) state variable P is allowed to fluctuate without regulation, while the corresponding full solutions are interpreted similarly, but when the stochastic process is assumed to be regulated using the impulse form of control.

2.4 Boundary and first order conditions

The optimal switching policy (i.e. the displacement strategies idle state $\rightarrow$ active state and active state $\rightarrow$ idle state) is determined through two time independent trigger prices $\overline{P}$ and $\underline{P}$. Each threshold level is similar in spirit to the critical asset price (or early exercise boundary) that separates the *continuation* and *stopping* (or exercise) regions of an American-style option contract, thus turning the dynamic entry and exit decision of a firm an *optimal stopping* problem with two barriers.

Such entry and exit thresholds are determined numerically through a set of *value-matching* and *smooth-pasting* conditions. The former are stated as

$$V_0(\overline{P}) + \overline{K} = V_1(\overline{P}) \tag{7.11}$$

$$V_0(\underline{P}) + \underline{K} = V_1(\underline{P}), \tag{7.12}$$

which ensure that the gain in value from exercising the option is exactly equal to the cost of doing so. These value-matching conditions reflect an intuitive requirement for continuity at the optimal thresholds.

The optimality condition of such (optimal impulse control) two-sided discrete regulator problems arise, however, from the so-called smooth-pasting (also known as high-contact or first-order) conditions[7.5]

[7.5] Samuelson (1965), McKean (1965), and Merton (1973) established conditions of optimality for such *optimal stopping* problems. A rigorous exposition of these conditions is provided by Dixit (1991) and Dumas (1991).

$$V_0'(\overline{P}) = V_1'(\overline{P}) \tag{7.13}$$

$$V_0'(\underline{P}) = V_1'(\underline{P}), \tag{7.14}$$

requiring that the first derivative of the firm value function must take the same value before and after the option (to enter or exit) has been exercised. In other words, these conditions require that *marginal utility* should take the same value before and after the action has been taken. This is equivalent to say that, at the optimum thresholds $\overline{P}$ and $\underline{P}$, the marginal cost of discounting the payoff function that is obtained by exercising the switching option equals the marginal net benefit from further waiting. As shown by Shackleton and Sødal (2005), such conditions guarantee the equalization of the rate of return of the firm both prior and after the decision to invest or divest has been taken.

To sum up, equations (7.11)-(7.14) constitute a set of four highly non-linear equations with four unknowns, represented in matrix form and denoted by $F(X)$, whose optimal solution $X = [\overline{P}, \underline{P}, A, B]'$ is uniquely determined by (numerically) solving the system $F(X) = 0$. Such solution highlights the optimal entry and exit policy of a firm acting dynamically in the aforementioned generalized stochastic environment by simultaneously determining both the thresholds $\overline{P}$ and $\underline{P}$ and the pair of constants A and B (to be determined from the boundary conditions) associated, respectively, to the idle and operating states of the firm.

2.5 Hysteresis

Whenever the underlying output price is between the two critical boundaries $\overline{P}$ and $\underline{P}$, the firm remains in its current state (idle or active). Thus, the firm takes no action at all over the region of the state space $(\underline{P},\overline{P})$. In other words, the firm's actions (to enter or exit) are only triggered when the state variable P reaches the boundary of the *region of no action* (or *zone of no intervention*). This range of inaction results in hysteresis (i.e. permanent effects of temporary shifts). The duration of these hysteretic periods depends on the expected growth rate of the underlying price and on its volatility. As shown by Dixit (1989a), the presence of fixed entry and exit costs under uncertainty widens the hysteretic band, since

$$\overline{W} := C + r\overline{K} < \overline{P} \tag{7.15}$$

$$\underline{W} := C + r\underline{K} > \underline{P}, \tag{7.16}$$

with $\overline{W}$ and $\underline{W}$ being, respectively, the Marshallian investment and divestment trigger prices based on the standard myopic investment rule. One of the purposes of this article is to show if this zone of no intervention changes substantially under alternative assumption for modeling the output price dynamics.

3 Applications

In this section we specialize the architecture modeling framework for some special cases, namely: The classic GBM process, the CEV process, and the mean-reverting CEV process.

3.1 The classic geometric Brownian motion process

Definition 7.1 *The classic GBM process underlying most of the real options literature can be nested into the general framework described by equations* (7.1) *to* (7.7) *through the following restrictions:* $\mu(P) = \mu P$, $\sigma(P) = \sigma P$, *and* $\lambda^*(P) = \lambda\rho$, *where* μ *and* σ *denote, respectively, the (constant) growth rate and the (constant) volatility of the market price* P, *and* $\lambda = (\mathbb{E}[R_m] - r)/\mathbb{D}[R_m]$ *is the market price of risk (with* $\mathbb{E}[R_m]$ *and* $\mathbb{D}[R_m]$ *being, respectively, the expected return and standard deviation of the market portfolio), and* ρ *is the correlation between the output price* P *and the market portfolio, i.e.* $dW^{\mathbb{P}}dW_m^{\mathbb{P}} = \rho dt$. *Both* λ *and* ρ *are assumed constant.*

Proposition 7.1 *Under the restrictions stated in Definition 7.1, the optimal solution* $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ *is uniquely determined by solving the system* ***F(X) = 0****, where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0\overline{P}^{\xi_1} + B_1\overline{P}^{\xi_2} + \varphi\overline{P} - \overline{X} \\ -A_0\underline{P}^{\xi_1} + B_1\underline{P}^{\xi_2} + \varphi\underline{P} - \underline{X} \\ -A_0\xi_1\overline{P}^{\xi_1} + B_1\xi_2\overline{P}^{\xi_2} + \varphi\overline{P} \\ -A_0\xi_1\underline{P}^{\xi_1} + B_1\xi_2\underline{P}^{\xi_2} + \varphi\underline{P} \end{bmatrix}, \tag{7.17}$$

with

$$\xi_1 = \frac{1}{2} - \frac{(\mu - \lambda\rho\sigma)}{\sigma^2} + \sqrt{\left(\frac{(\mu - \lambda\rho\sigma)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{7.18}$$

$$\xi_2 = \frac{1}{2} - \frac{(\mu - \lambda\rho\sigma)}{\sigma^2} - \sqrt{\left(\frac{(\mu - \lambda\rho\sigma)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0, \tag{7.19}$$

$$\varphi = (r + \lambda\rho\sigma - \mu)^{-1}, \tag{7.20}$$

$$\overline{X} = \frac{C}{r} + \overline{K}, \tag{7.21}$$

and

$$\underline{X} = \frac{C}{r} + \underline{K}. \tag{7.22}$$

Proof. The proof of this proposition is standard in the literature and can be found, for example, in Dixit (1989a) and Tsekrekos (2010, Appendix A). For completeness and easier comparison with our new results, we derive the optimal entry and exit policy that will determine the entry and exit trigger prices.

Under the restrictions stated in Definition 7.1, equations (7.6) and (7.7) become, respectively, equal to:

$$\frac{1}{2}\sigma^2 P^2 V_0''(P) + \big(\mu - \lambda\rho\sigma\big) P V_0'(P) - rV_0(P) = 0, \tag{7.23}$$

and

$$\frac{1}{2}\sigma^2 P^2 V_1''(P) + \big(\mu - \lambda\rho\sigma\big) P V_1'(P) - rV_1(P) + P - C = 0. \tag{7.24}$$

Note that the use of contingent claims analysis requires the critical assumption that stochastic changes in P must be *spanned* by an asset (or a dynamic portfolio of assets) available in the underlying economy. In other words, it is assumed that spanning holds so that the uncertainty over future values of P can be replicated by existing assets.[7.6] Under the assumptions of the intertemporal capital asset pricing model of Merton (1973b), the expected rate of return from holding this asset or portfolio of assets, μ^*, should reflect only the asset's systematic (or nondiversifiable) risk, that is $\mu^* = r + \lambda\rho\sigma$. This implies that $\lambda\rho\sigma = \mu^* - r$. As usual, it is necessary to assume that the expected percentage rate of change of P, μ, is less that the risk-adjusted return μ^*. This requirement is accomplished by introducing the so-called *rate of return shortfall* $q := \mu^* - \mu$ considered in McDonald and Siegel (1984).

[7.6]Nevertheless, if spanning does not hold, dynamic programming can still be used to solve the entry and exit problem, though subject to an arbitrary discount rate.

Since both equations are linear and have the same homogeneous part, we can find the complementary functions together. Trying a solution of the form $V_\phi(P) = P^\xi$ (with $\phi \in \{0,1\}$), it follows that $V'_\phi(P) = \xi P^{\xi-1}$ and $V''_\phi(P) = \xi(\xi-1)p^{\xi-2}$. Substituting these functions into the above homogeneous equations and rearranging terms yields the following quadratic equation:

$$\frac{1}{2}\sigma^2\xi(\xi-1) + (\mu - \lambda\rho\sigma)\xi - r = 0, \tag{7.25}$$

whose corresponding solutions are real and equal to equations (7.18) and (7.19). Therefore, the general solution of the ode (7.23) and of the homogeneous part of the ode (7.24) can be expressed as a linear combination of two independent solutions as

$$V_\phi(P) = A_\phi P^{\xi_1} + B_\phi P^{\xi_2}, \tag{7.26}$$

where A_ϕ and B_ϕ are two constants to be determined from boundary conditions.

The solution of ode (7.23) subject to the boundary condition (7.8) is given by

$$V_0(P) = A_0 P^{\xi_1}, \tag{7.27}$$

which corresponds to the value function of an idle firm over the range of prices $(0, \overline{P})$. In other words, it is the option value to become active in the market at the trigger price $\overline{P}$. Similarly, the solution of the ode (7.24) subject to the boundary condition (7.10) is given by

$$V_1(P) = B_1 P^{\xi_2} + \frac{P}{r + \lambda\rho\sigma - \mu} - \frac{C}{r}, \tag{7.28}$$

which corresponds to the value function of an active firm over the range of prices $(\underline{P}, +\infty)$. Note that the first component on the right-hand side of equation (7.28) is the value of the option to exit at the trigger price $\underline{P}$ while the corresponding second and third components represent the expected value of operating in the market forever given by the particular solution (7.9).

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_\phi(P)$. Using straightforward calculus, it follows that:

$$V_0'(P) = A_0 \xi_1 P^{\xi_1 - 1}, \tag{7.29}$$

and

$$V_1^{''}(P) = B_1 \xi_2 P^{\xi_2 - 1} + \frac{1}{r + \lambda\rho\sigma - \mu}. \tag{7.30}$$

Finally, substituting (7.27)-(7.30) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.17). ■

3.2 The constant elasticity of variance process

The CEV model of Cox (1975) was originally studied to the case where the elasticity parameter, β, is less than two ($\beta < 2$), and then extended to the case where $\beta > 2$ by Emanuel and MacBeth (1982). While Cox (1975) has restricted the β parameter to the range $0 \leq \beta \leq 2$, Jackwerth and Rubinstein (2001) document that typical values of β implicit in the S&P 500 stock index option prices are as low as $\beta = -6$ in the post-crash of 1987. Elasticity values of $\beta < 2$ (i.e. with a direct leverage effect) are expected for stock index options and crude oil prices, whereas values of $\beta > 2$ (i.e. with an inverse leverage effect) are characteristic of some commodity spot prices and futures options with upward sloping implied volatility smiles (see, for instance, Davydov and Linetsky (2001), Geman and Shih (2009), and Dias and Nunes (2011)).

The CEV process assumption has been used in many different contexts, e.g. by Dias and Nunes (2011) to compute the analytical solutions for perpetual American-style call and put options, and Lee and Heo (2008) where these authors have concluded that the CEV process is the most suitable to explain the dynamics of crude oil prices.

Definition 7.2 *The class of CEV processes can be nested into the general framework described by equations* (7.1) *to* (7.7) *through the following restrictions:* $\mu(P) = \mu P$*,* $\sigma(P) = \delta P^{\beta/2}$*, and* $\lambda^*(P) = \lambda\rho P^{1-\beta/2}$*, where* β *represents the elasticity parameter,* μ *and* δ *denote, respectively, the (constant) growth rate and the (constant) scale parameter fixing the initial instantaneous volatility at time* $t = 0$*,* $\sigma_0 = \sigma(P_0) = \delta P_0^{\beta/2}$*, and* $\lambda = (\mathbb{E}[R_m] - r)/\mathbb{D}[R_m]$ *is the market price of risk (with* $\mathbb{E}[R_m]$ *and* $\mathbb{D}[R_m]$ *being, respectively, the expected return and standard deviation of the market portfolio), and* ρ *is the correlation between the output price* P *and the market portfolio, i.e.* $dW^{\mathbb{P}} dW_m^{\mathbb{P}} = \rho dt$*. Both* λ *and* ρ *are assumed constant.*

To obtain the optimal solution for the CEV process, we need to distinguish six situations: for the cases $\beta < 2$ and $\beta > 2$, we have to consider three situations: $(\mu - \lambda\rho\delta) > 0$, $(\mu - \lambda\rho\delta) < 0$ and $(\mu - \lambda\rho\delta) = 0$. The next proposition gives the optimal solution when $\beta < 2$ for the cases: A) $(\mu - \lambda\rho\delta) > 0$ and B) $(\mu - \lambda\rho\delta) < 0$.

Proposition 7.2 *Under the restrictions stated in Definition 7.2, and for* $\beta < 2$*, the optimal solution* $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ *is uniquely determined by solving the system* ***F(X) = 0****:*

A) In case $(\mu - \lambda\rho\delta) > 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M_1(\overline{P}) + B_1 U_1(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -A_0 M_1(\underline{P}) + B_1 U_1(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -A_0 W_1(\overline{P}) + B_1 V_1(\overline{P}) + \varphi\overline{P} \\ -A_0 W_1(\underline{P}) + B_1 V_1(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{7.31}$$

with

$$M_1(y) = \zeta_1 y e^{-x(y)} M(a_1, b_1, x(y)), \tag{7.32}$$

$$U_1(y) = \zeta_1 y e^{-x(y)} U(a_1, b_1, x(y)), \tag{7.33}$$

$$W_1(y) = M_1(y) - \vartheta y^{2-\beta} \left(M_1(y) - \zeta_1 y e^{-x(y)} \frac{a_1}{b_1} M(a_1 + 1, b_1 + 1, x(y)) \right), \tag{7.34}$$

$$V_1(y) = U_1(y) - \vartheta y^{2-\beta} \left(U_1(y) + \zeta_1 y e^{-x(y)} a_1 U(a_1 + 1, b_1 + 1, x(y)) \right), \tag{7.35}$$

$$x(y) = \frac{2|\mu - \lambda\rho\delta|}{\delta^2 |\beta - 2|} y^{2-\beta}, \tag{7.36}$$

$$a_1 = 1 - \frac{r}{(\mu - \lambda\rho\delta)(\beta - 2)}, \tag{7.37}$$

$$b_1 = 1 - \frac{1}{\beta - 2}, \tag{7.38}$$

$$\vartheta = \frac{2(\mu - \lambda\rho\delta)}{\delta^2}, \tag{7.39}$$

$$\zeta_1 = \left[-\frac{\vartheta}{(\beta - 2)} \right]^{\frac{1}{2} - \frac{1}{2(\beta - 2)}}, \tag{7.40}$$

and φ, $\overline{X}$, *and* $\underline{X}$ *as defined as in* (7.20), (7.21), *and* (7.22), *respectively, and where* $M(a,b,x)$ *and* $U(a,b,x)$ *are the Kummer functions, as defined by Abramowitz and Stegun (1972, expressions 13.1.2 and 13.1.3).*

B) In case $(\mu - \lambda\rho\delta) < 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M_2(\overline{P}) + B_1 U_2(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -A_0 M_2(\underline{P}) + B_1 U_2(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -A_0 W_2(\overline{P}) + B_1 V_2(\overline{P}) + \varphi\overline{P} \\ -A_0 W_2(\underline{P}) + B_1 V_2(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{7.41}$$

with

$$M_2(y) = \zeta_2 y M(a_2, b_2, x(y)), \tag{7.42}$$

$$U_2(y) = \zeta_2 y U(a_2, b_2, x(y)), \tag{7.43}$$

$$W_2(y) = M_2(y) - \zeta_2 \vartheta y^{3-\beta} \frac{a_2}{b_2} M\big(a_2 + 1, b_2 + 1, x(y)\big), \tag{7.44}$$

$$V_2(y) = U_2(y) + \zeta_2 \vartheta y^{3-\beta} a_2 U\big(a_2 + 1, b_2 + 1, x(y)\big), \tag{7.45}$$

$$a_2 = \frac{r}{(\mu - \lambda\rho\delta)(\beta - 2)} - \frac{1}{\beta - 2}, \tag{7.46}$$

$$b_2 = 1 - \frac{1}{\beta - 2}, \tag{7.47}$$

$$\zeta_2 = \left[\frac{\vartheta}{(\beta - 2)}\right]^{\frac{1}{2} - \frac{1}{2(\beta-2)}}, \tag{7.48}$$

and φ, $\overline{X}$, $\underline{X}$, $x(y)$, *and,* ϑ *as defined as in* (7.20), (7.21), (7.22), (7.36), *and,* (7.39), *respectively.*

Proof. To prove this proposition we will apply the results of Davydov and Linetsky (2001) and the same methodology as used in the GBM process.

Under the restrictions stated in Definition 7.2, equations (7.6) and (7.7) become, respectively:

$$\frac{1}{2}\delta^2 P^\beta V_0^{''}(P) + \big(\mu - \lambda\rho\delta\big)PV_0^{'}(P) - rV_0(P) = 0 \tag{7.49}$$

and

$$\frac{1}{2}\delta^2 P^\beta V_1^{''}(P) + \big(\mu - \lambda\rho\delta\big)PV_1^{'}(P) - rV_1(P) + P - C = 0. \tag{7.50}$$

Since both equations have the same homogeneous part, we start by solving the homogeneous ode to find the complementary functions together. Trying a solution of the form $V_\phi(P) = P^{\frac{\beta-1}{2}} e^{\varepsilon\frac{x(P)}{2}} w(x(P))$ (with $\phi \in \{0,1\}$), with $x(P)$ as defined in expression (7.36) and

$$\varepsilon = \begin{cases} +1\ if\ (\mu - \lambda\rho\delta)(\beta-2) > 0 \\ -1\ if\ (\mu - \lambda\rho\delta)(\beta-2) < 0. \end{cases} \tag{7.51}$$

It follows that

$$V_\phi'(P) = e^{\varepsilon\frac{x(P)}{2}}\left(\Big[\Big(\frac{\beta-1}{2}\Big)P^{\frac{\beta-3}{2}} - \frac{(\mu-\lambda\rho\sigma)}{\delta^2}\Big]w(x(P)) - \varepsilon\frac{2(\mu-\lambda\rho\sigma)}{\delta^2}P^{\frac{1-\beta}{2}}w'(x(P))\right)$$

and

$$\begin{aligned} V_\phi''(P) = e^{\varepsilon\frac{x(P)}{2}}\Bigg(&\Big[\Big(\frac{\beta-1}{2}\Big)\Big(\frac{\beta-3}{2}\Big)P^{\frac{\beta-5}{2}} + \Big(\frac{(\mu-\lambda\rho\sigma)}{\delta^2}\Big)^2 P^{\frac{3-3\beta}{2}}\Big]w(x(P)) \\ &+\varepsilon\Big(\frac{2(\mu-\lambda\rho\sigma)}{\delta^2}\Big)^2 P^{\frac{3-3\beta}{2}}w'(x(P)) + \Big(\frac{2(\mu-\lambda\rho\sigma)}{\delta^2}\Big)^2 P^{\frac{3-3\beta}{2}}w^{''}(x(P))\Bigg). \end{aligned}$$

Substituting these functions into the homogeneous equation and rearranging terms yields the following equation:

$$w''(x(P)) + \left[-\frac{1}{4} + \frac{k}{x(P)} + \frac{(\frac{1}{4} - m)^2}{x^2(P)} \right] w(x(P)) = 0, \tag{7.52}$$

with

$$k = \varepsilon \left(\frac{1}{2} + \frac{1}{2(\beta - 2)} \right) - \frac{r}{|(\mu - \lambda \rho \delta)(\beta - 2)|}, \tag{7.53}$$

$$m = \frac{1}{2|\beta - 2|}. \tag{7.54}$$

The equation (7.52) is the Whittaker's form of the confluent hypergeometric equation (see, for instance, Abramowitz and Stegun (1972, expression 13.1.31)), which has the general solution given by

$$w(x(P)) = A_\phi M_{k,m}(x(P)) + B_\phi W_{k,m}(x(P)), \tag{7.55}$$

where $M_{k,m}(x)$ and $W_{k,m}(x)$ are the Whittaker functions, as defined by Abramowitz and Stegun (1972, expressions 13.1.32 and 13.1.33) and, A_ϕ and B_ϕ are constants to be determined from boundary conditions. Multiplying this general solution by $P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}}$ we obtain the general solution of the ode (7.49) and of the homogeneous part of the ode (7.50), and is given by:

$$V_\phi(P) = A_\phi P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}} M_{k,m}(x(P)) + B_\phi P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}} W_{k,m}(x(P)). \tag{7.56}$$

The solution of ode (7.49), for $\beta < 2$, subject to the boundary condition (7.8) is given by

$$V_0(P) = A_0 P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}} M_{k,m}(x(P)), \tag{7.57}$$

which corresponds to the value function of an idle firm over the range of prices $(0, \overline{P})$.

In other words, it is the option value to become active in the market at the trigger price $\overline{P}$. Similarly, the solution of the ode (7.50) subject to the boundary condition (7.10) is given by

$$V_1(P) = B_1 P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}} W_{k,m}(x(P)) + \frac{P}{r+\lambda\rho\sigma-\mu} - \frac{C}{r}, \tag{7.58}$$

which corresponds to the value function of an active firm over the range of prices $(\underline{P}, +\infty)$. Note that the first component on the right-hand side of equation (7.58) is the value of the option to exit at the trigger price $\underline{P}$ while the corresponding second and third components represent the expected value of operating in the market forever given by the particular solution (7.9). Using the following relations between Whittaker's function with the Kummer's function

$$M_{k,m}(x) = e^{-\frac{x}{2}} x^{\frac{1}{2}+m} M\left(\frac{1}{2}+m-k, 1+2m, x\right) \tag{7.59}$$

and

$$W_{k,m}(x) = e^{-\frac{x}{2}} x^{\frac{1}{2}+m} U\left(\frac{1}{2}+m-k, 1+2m, x\right), \tag{7.60}$$

we can express (7.57) and (7.58) in terms of the Kummer's functions. Now, we must distinguish between the cases A) and B).

A) In terms of the Kummer's functions the solutions to $V_0(P)$ and $V_1(P)$ are given by:

$$V_0(P) = A_0 \left[-\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}} e^{-x(P)} P M(a_1, b_1, x(P)), \tag{7.61}$$

$$V_1(P) = B_1 \left[-\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}} e^{-x(P)} P U(a_1, b_1, x(P)) + \frac{P}{r+\lambda\rho\delta-\mu} - \frac{C}{r}. \tag{7.62}$$

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_\phi(P)$. Using straightforward calculus, the relations for derivatives of the Kummer's functions as in Abramowitz and Stegun (1972, expressions 13.4.8 and 13.4.21), it follows that:

$$V_0'(P) = A_0 \left[-\frac{2(\mu - \lambda\rho\delta)}{\delta^2(\beta-2)} \right]^{\frac{1}{2} - \frac{1}{2(\beta-2)}} e^{-x(P)} \tag{7.63}$$
$$\times \left(M(a_1, b_1, x(P)) - \frac{2(\mu - \lambda\rho\delta)}{\delta^2} P^{2-\beta} \left(M(a_1, b_1, x(P)) - \frac{a_1}{b_1} M(a_1+1, b_1+1, x(P)) \right) \right)$$

and

$$V_1'(P) = B_1 \left[-\frac{2(\mu - \lambda\rho\delta)}{\delta^2(\beta-2)} \right]^{\frac{1}{2} - \frac{1}{2(\beta-2)}} e^{-x(P)} \tag{7.64}$$
$$\times \left(U(a_1, b_1, x(P)) - \frac{2(\mu - \lambda\rho\delta)}{\delta^2} P^{2-\beta} \left(U(a_1, b_1, x(P)) + a_1 U(a_1+1, b_1+1, x(P)) \right) \right)$$
$$+ \frac{1}{r + \lambda\rho\delta - \mu}.$$

Finally, substituting (7.61)-(7.64) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.31) for $\beta < 2$ and $(\mu - \lambda\rho\delta) > 0$.

B) In terms of the Kummer's functions the solutions to the $V_0(P)$ and $V_1(P)$ are given by:

$$V_0(P) = A_0 \left[\frac{2(\mu - \lambda\rho\delta)}{\delta^2(\beta-2)} \right]^{\frac{1}{2} - \frac{1}{2(\beta-2)}} P M(a_1, b_1, x(P)), \tag{7.65}$$

$$V_1(P) = B_1 \left[\frac{2(\mu - \lambda\rho\delta)}{\delta^2(\beta-2)} \right]^{\frac{1}{2} - \frac{1}{2(\beta-2)}} P U(a_1, b_1, x(P)) + \frac{P}{r + \lambda\rho\delta - \mu} - \frac{C}{r}. \tag{7.66}$$

Using straightforward calculus and the derivatives relations to the Kummer functions, the first derivatives of the value functions $V_\phi(P)$ are given by:

$$V_0'(P) = A_0\left[\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}} \times\left(M(a_2,b_2,x(P)) - \frac{2(\mu-\lambda\rho\delta)}{\delta^2}P^{2-\beta}\frac{a_2}{b_2}M(a_2+1,b_2+1,x(P))\right), \tag{7.67}$$

and

$$V_1'(P) = B_1\left[\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}} \times\left(U(a_2,b_2,x(P)) + \frac{2(\mu-\lambda\rho\delta)}{\delta^2}P^{2-\beta}a_2U(a_2+1,b_2+1,x(P))\right) + \frac{1}{r+\lambda\rho\delta-\mu}. \tag{7.68}$$

Finally, substituting (7.65)-(7.68) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.41) for $\beta < 2$ and $(\mu-\lambda\rho\delta) < 0$. ■

The next proposition gives the optimal solution to the $\beta > 2$ for the cases: A) $(\mu - \lambda\rho\delta) > 0$ and B) $(\mu-\lambda\rho\delta) < 0$.

Proposition 7.3 *Under the restrictions stated in Definition 7.2, and for $\beta > 2$, the optimal solution **X** $= [\overline{P}, \underline{P}, A_1, B_0]'$ is uniquely determined by solving the system **F(X) = 0**:*

A) In case $(\mu - \lambda\rho\delta) > 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0U_3(\overline{P}) + A_1M_3(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -B_0U_3(\underline{P}) + A_1M_3(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -B_0V_3(\overline{P}) + A_1W_3(\overline{P}) + \varphi\overline{P} \\ -B_0V_3(\underline{P}) + A_1W_3(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{7.69}$$

with

$$M_3(y) = \zeta_3 M(a_3, b_3, x(y)), \tag{7.70}$$

$$U_3(y) = \zeta_3 U(a_3, b_3, x(y)), \tag{7.71}$$

$$W_3(y) = -\zeta_3\,\vartheta y^{2-\beta}\frac{a_3}{b_3}M\big(a_3+1, b_3+1, x(y)\big), \tag{7.72}$$

$$V_3(y) = \zeta_3\,\vartheta y^{2-\beta}a_3 U\big(a_3+1, b_3+1, x(y)\big), \tag{7.73}$$

$$a_3 = \frac{r}{(\mu - \lambda\rho\delta)(\beta-2)}, \tag{7.74}$$

$$b_3 = 1 + \frac{1}{\beta-2}, \tag{7.75}$$

$$\zeta_3 = \left[\frac{\vartheta}{(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}}, \tag{7.76}$$

and φ, $\overline{X}$, $\underline{X}$, $x(y)$, *and,* ϑ *as defined as in* (7.20), (7.21), (7.22), (7.36), *and,* (7.39), *respectively.*

B) In case $(\mu - \lambda\rho\delta) < 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0U_4(\overline{P}) + A_1M_4(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -B_0U_4(\underline{P}) + A_1M_4(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -B_0V_4(\overline{P}) + A_1W_4(\overline{P}) + \varphi\overline{P} \\ -B_0V_4(\underline{P}) + A_1W_4(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{7.77}$$

with

$$M_4(y) = \zeta_4 e^{-x(y)} M(a_4, b_4, x(y)), \tag{7.78}$$

$$U_4(y) = \zeta_4 e^{-x(y)} U(a_4, b_4, x(y)), \tag{7.79}$$

$$W_4(y) = -\vartheta y^{2-\beta} \left(M_4(y) - \zeta_4 e^{-x(y)} \frac{a_4}{b_4} M(a_4+1, b_4+1, x(y)) \right), \tag{7.80}$$

$$V_4(y) = -\vartheta y^{2-\beta} \left(U_4(y) + \zeta_4 e^{-x(y)} a_4 U(a_4+1, b_4+1, x(y)) \right), \tag{7.81}$$

$$a_4 = 1 + \frac{1}{\beta-2} - \frac{r}{(\mu-\lambda\rho\delta)(\beta-2)}, \tag{7.82}$$

$$b_4 = 1 + \frac{1}{\beta-2}, \tag{7.83}$$

$$\zeta_4 = \left[-\frac{\vartheta}{(\beta-2)} \right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}}, \tag{7.84}$$

and φ*,* $\overline{X}$*,* $\underline{X}$*,* $x(y)$*, and,* ϑ *as defined as in* (7.20)*,* (7.21)*,* (7.22)*,* (7.36)*, and* (7.39)*, respectively.*

Proof. The proof of this proposition is similar to the proof of the Proposition 7.2, but here the solution of ode (7.49), for $\beta > 2$, subject to the boundary condition (7.8) is given by

$$V_0(P) = B_0 P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}} W_{k,m}(x(P)), \tag{7.85}$$

which corresponds to the value function of an idle firm over the range of prices $(0, \overline{P})$. Similarly, the solution of the ode (7.50) subject to the boundary condition (7.10) is given by

$$V_1(P) = A_1 P^{\frac{\beta-1}{2}} e^{\varepsilon \frac{x(P)}{2}} W_{k,m}(x(P)) + \frac{P}{r+\lambda\rho\delta-\mu} - \frac{C}{r}, \tag{7.86}$$

which corresponds to the value function of an active firm over the range of prices $(\underline{P}, +\infty)$. Again, we must distinguish between the two cases: A) $(\mu - \lambda\rho\delta) > 0$ and B) $(\mu - \lambda\rho\delta) < 0$, and we will give the solutions in terms of the Kummer functions.

A) The value functions $V_\phi(P)$ and the respective derivatives are given by:

$$V_0(P) = B_0 \left[\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} U\left(a_3, b_3, x(P)\right), \tag{7.87}$$

$$V_1(P) = A_1 \left[\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} M\left(a_3, b_3, x(P)\right) + \frac{P}{r+\lambda\rho\delta-\mu} - \frac{C}{r}, \tag{7.88}$$

$$V_0'(P) = B_0 \left[\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} \frac{2(\mu-\lambda\rho\delta)}{\delta^2} P^{1-\beta} a_3 U(a_3+1, b_3+1, x(P)), \tag{7.89}$$

and,

$$\begin{aligned} V_1'(P) \;=\; & -A_1 \left[\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} \frac{2(\mu-\lambda\rho\delta)}{\delta^2} P^{1-\beta} \frac{a_3}{b_3} M(a_3+1, b_3+1, x(P)) \\ & + \frac{1}{r+\lambda\rho\delta-\mu}. \end{aligned} \tag{7.90}$$

Finally, substituting (7.87)-(7.90) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.69) for $\beta > 2$ and $(\mu - \lambda\rho\delta) > 0$.

B) The value functions $V_\phi(P)$ and the respective derivatives are given by:

$$V_0(P) = B_0 \left[-\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} e^{-x(P)} U\left(a_4, b_4, x(P)\right), \tag{7.91}$$

$$V_1(P) = A_1 \left[-\frac{2(\mu-\lambda\rho\delta)}{\delta^2(\beta-2)}\right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} e^{-x(P)} M\left(a_4, b_4, x(P)\right) + \frac{P}{r+\lambda\rho\delta-\mu} - \frac{C}{r}, \tag{7.92}$$

$$V_0'(P) = -B_0 \left[-\frac{2(\mu - \lambda\rho\delta)}{\delta^2(\beta-2)} \right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} \frac{2(\mu-\lambda\rho\delta)}{\delta^2} e^{-x(P)} P^{1-\beta} \times \Big(U(a_4, b_4, x(P)) + a_4 U(a_4+1, b_4+1, x(P)) \Big), \tag{7.93}$$

$$V_1'(P) = -A_1 \left[-\frac{2(\mu - \lambda\rho\delta)}{\delta^2(\beta-2)} \right]^{\frac{1}{2}+\frac{1}{2(\beta-2)}} \frac{2(\mu-\lambda\rho\delta)}{\delta^2} e^{-x(P)} P^{1-\beta} \times \Big(M(a_4, b_4, x(P)) - \frac{a_4}{b_4} M(a_4+1, b_4+1, x(P)) \Big) + \frac{1}{r+\lambda\rho\delta-\mu}. \tag{7.94}$$

Finally, substituting (7.91)-(7.94) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.77) for $\beta > 2$ and $(\mu - \lambda\rho\delta) < 0$. ■

We analyze now the particular case of the CEV model where $(\mu - \lambda\rho\delta) = 0$. In the next proposition we will give the optimal solution $\mathbf{X} = [\overline{P}, \underline{P}, A_\phi, B_\phi]'$ to the cases: A) $\beta < 2$ and B) $\beta) > 2$.

Proposition 7.4 *Under the restrictions stated in Definition 7.2, and for $(\mu - \lambda\rho\delta) = 0$, the optimal solution $\mathbf{X} = [\overline{P}, \underline{P}, A_\phi, B_\phi]'$ is uniquely determined by solving the system **F(X) = 0**:*

A) In case $\beta < 2$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 I(\overline{P}) + B_1 K(\overline{P}) + r^{-1}\overline{P} - \overline{X} \\ -A_0 I(\underline{P}) + B_1 K(\underline{P}) + r^{-1}\underline{P} - \underline{X} \\ -A_0 S_1(\overline{P}) + B_1 T_1(\overline{P}) + r^{-1}\overline{P} \\ -A_0 S_1(\underline{P}) + B_1 T_1(\underline{P}) + r^{-1}\underline{P} \end{bmatrix}, \tag{7.95}$$

with

$$I(y) = y^{\frac{1}{2}} I_\nu(z(y)), \tag{7.96}$$

$$K(y) = y^{\frac{1}{2}} K_\nu(z(y)), \tag{7.97}$$

$$S_1(y) = y^{\frac{1}{2}} I_\nu(z(y)) + \frac{\sqrt{2r}}{\delta} y^{\frac{3}{2}-\frac{\beta}{2}} I_{\nu+1}(z(y)), \tag{7.98}$$

$$T_1(y) = y^{\frac{1}{2}} K_\nu(z(y)) - \frac{\sqrt{2r}}{\delta} y^{\frac{3}{2}-\frac{\beta}{2}} K_{\nu+1}(z(y)), \tag{7.99}$$

$$z(y) = \frac{2\sqrt{2r}}{\delta|\beta-2|} y^{1-\beta/2}. \tag{7.100}$$

$$\overline{X} = \frac{C}{r} + \overline{K}, \tag{7.101}$$

$$\underline{X} = \frac{C}{r} + \underline{K}, \tag{7.102}$$

and, where $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of the first and second kind of order ν as defined by Abramowitz and Stegun (1972, expressions 9.6.3 and 9.6.4).

B) In case $\beta > 2$, where

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0 K(\overline{P}) + A_1 I(\overline{P}) + r^{-1}\overline{P} - \overline{X} \\ -B_0 K(\underline{P}) + A_1 I(\underline{P}) + r^{-1}\underline{P} - \underline{X} \\ -B_0 T_2(\overline{P}) + A_1 S_2(\overline{P}) + r^{-1}\overline{P} \\ -B_0 T_2(\underline{P}) + A_1 S_2(\underline{P}) + r^{-1}\underline{P} \end{bmatrix}, \tag{7.103}$$

with

$$S_2(y) = -\frac{\sqrt{2r}}{\delta} y^{\frac{3}{2}-\frac{\beta}{2}} I_{\nu+1}(z(y)), \tag{7.104}$$

$$T_2(y) = \frac{\sqrt{2r}}{\delta} y^{\frac{3}{2}-\frac{\beta}{2}} K_{\nu+1}(z(y)), \tag{7.105}$$

and $I(y)$, $K(y)$, $z(y)$, $\overline{X}$, and $\underline{X}$ as defined as in (7.96), (7.97), (7.100), (7.101), *and* (7.102), *respectively.*

Proof. The proof of this proposition is similar to the above proofs.

Under the restrictions stated in Definition 7.2 and $(\mu - \lambda\rho\delta) = 0$, equations (7.6) and (7.7) become, respectively:

$$\frac{1}{2}\delta^2 P^\beta V_0^{''}(P) - rV_0(P) = 0, \tag{7.106}$$

and

$$\frac{1}{2}\delta^2 P^\beta V_1^{''}(P) - rV_1(P) + P - C = 0. \tag{7.107}$$

Since both equations have the same homogeneous part, we start by solving the homogeneous ode to find the complementary functions together. Following Davydov and Linetsky (2001) we will try a solution of the form $V_\phi(P) = P^{\frac{1}{2}}h(z(P))$ (with $\phi \in \{0,1\}$), where $z(P)$ as defined as in equation (7.100). It follows that

$$V_\phi'(P) = \frac{1}{2}P^{\frac{\beta-3}{2}}h(z(P)) + P^{\frac{1-\beta}{2}}\frac{\sqrt{2r}(2-\beta)}{\delta|\beta-2|}h'(z(P))$$

and

$$V_\phi''(P) = -\frac{1}{4}P^{-\frac{3}{2}}h(z(P)) + (1-\frac{\beta}{2})P^{\frac{-1-\beta}{2}}\frac{\sqrt{2r}(2-\beta)}{\delta|\beta-2|}h'(z(P)) + \frac{2r}{\delta^2}P^{\frac{1}{2}-\beta}h^{''}(z(P)).$$

Substituting $V_\phi(P)$ and $V_\phi^{''}(P)$ functions into the above homogeneous equations and rearranging terms yields the following equation:

$$z^2(P)h''(z(P)) + z(P)h'(z(P)) - \left(\frac{1}{(2-\beta)^2} + z^2(P)\right)h(z(P)) = 0 \tag{7.108}$$

The equation (7.108) is the Bessel's modified equation (see, for instance, Abramowitz and Stegun (1972, expression 9.6.1)), which, for all values of $v\,(v \geq 0)$, has the general solution given by

$$h(z) = A_\phi I_v(z) + B_\phi K_v(z), \tag{7.109}$$

with $v = \frac{1}{|2-\beta|}$, and A_ϕ and B_ϕ are constants to be determined. Multiplying this general solution by $P^{\frac{1}{2}}$ we obtain the general solution of the ode (7.106) and of the homogeneous part of the ode (7.107), and is given by

$$V(P) = A_\phi P^{\frac{1}{2}} I_v(z(P)) + B_\phi P^{\frac{1}{2}} K_v(z(P)). \tag{7.110}$$

A) The solution of ode (7.106), for $\beta < 2$, subject to the boundary condition (7.8) is given by

$$V_0(P) = A_0 P^{\frac{1}{2}} I_v(z(P)), \tag{7.111}$$

which corresponds to the value function of an idle firm over the range of prices $(0, \overline{P})$. Similarly, the solution of the ode (7.107) subject to the boundary condition (7.10) is given by

$$V_1(P) = B_1 P^{\frac{1}{2}} K_v(z(P)) + \frac{P}{r} - \frac{C}{r}, \tag{7.112}$$

which corresponds to the value function of an active firm over the range of prices $(\underline{P}, +\infty)$.

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_\phi(P)$. Using straightforward calculus, the relations for derivatives of the Bessel modified functions as in Abramowitz and Stegun (1972, expression 9.6.26), it follows that:

$$V_0'(P) = A_0 \left[P^{-\frac{1}{2}} I_v(z(P)) + \frac{\sqrt{2r}}{\delta} P^{\frac{1}{2}-\frac{\beta}{2}} I_{v+1}(z(P)) \right] \tag{7.113}$$

and

$$V_1'(P) = B_1 \left[P^{-\frac{1}{2}} K_v(z(P)) - \frac{\sqrt{2r}}{\delta} P^{\frac{1}{2}-\frac{\beta}{2}} K_{v+1}(z(P)) \right] + \frac{1}{r} \tag{7.114}$$

Finally, substituting (7.111)-(7.114) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.95) for $\beta < 2$ and $(\mu - \lambda\rho\delta) = 0$.

B) The solution of ode (7.106), for $\beta > 2$, subject to the boundary condition (7.8) is given by

$$V_0(P) = B_0 P^{\frac{1}{2}} K_v(z(P)), \tag{7.115}$$

which corresponds to the value function of an idle firm over the range of prices $(0, \overline{P})$. Similarly, the solution of the ode (7.107) subject to the boundary condition (7.10) is given by

$$V_1(P) = A_1 P^{\frac{1}{2}} I_v(z(P)) + \frac{P}{r} - \frac{C}{r}, \tag{7.116}$$

which corresponds to the value function of an active firm over the range of prices $(\underline{P}, +\infty)$.

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_\phi(P)$, which are given by:

$$V_0'(P) = B_0 \frac{\sqrt{2r}}{\delta} P^{\frac{1}{2}-\frac{\beta}{2}} K_{v+1}(z(P)) \tag{7.117}$$

and

$$V_1'(P) = A_1 \frac{\sqrt{2r}}{\delta} P^{\frac{1}{2}-\frac{\beta}{2}} I_{v+1}(z(P)) + \frac{1}{r} \tag{7.118}$$

Finally, substituting (7.115)-(7.118) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.103) for $\beta > 2$ and $(\mu - \lambda\rho\delta) = 0$. ■

3.3 The mean-reverting CEV process

By a class of mean-reverting CEV processes we mean the volatility modeling specification considered in much of the literature on stochastic volatility models, e.g. Kahl and Jäckel (2006), Andersen and Piterbarg (2007), and Lord et al. (2010), that is:

Definition 7.3 *The generalized class of mean-reverting CEV processes can be nested into the general framework described by equations* (7.1) *to* (7.7) *through the following restrictions:* $\mu(P) = \kappa(\theta - P)$, $\sigma(P) = \delta P^\gamma$, *and* $\lambda^*(P) = \lambda\rho P^{1-\gamma}$, *where* γ *represents the elasticity parameter,* κ, θ, *and* δ *denote, respectively, the (constant) speed of reversion, the (constant) long-run mean price level, and the (constant) scale parameter fixing the initial instantaneous volatility at time* $t = 0$, $\sigma_0 = \sigma(P_0) = \delta P_0^\gamma$, *and* $\lambda = (\mathbb{E}[R_m] - r)/\mathbb{D}[R_m]$ *is the market price of risk (with* $\mathbb{E}[R_m]$ *and* $\mathbb{D}[R_m]$ *being, respectively, the expected return and standard deviation of the market portfolio), and* ρ *is the correlation between the* *output price P and the market portfolio, i.e. $dW^{\mathbb{P}}dW_m^{\mathbb{P}} = \rho dt$. Both λ and ρ are assumed constant.*

The inhomogeneous geometric Brownian motion process

The inhomogeneous geometric Brownian motion process (hereafter IGBM) (also known as the geometric mean reversion process or the geometric Ornstein-Uhlenbeck process)[7.7] is obtained with $\gamma = 1$.

This modeling assumption has been been used in many different contexts, e.g. by Brennan and Schwartz (1980) for analyzing convertible bonds, Insley (2002) to model the optimal tree harvesting decision, Sarkar (2003) to study the effect of mean reversion on investment under cost uncertainty, Abadie and Chamorro (2008) to analyze the choice between an inflexible and a flexible technology for producing electricity, and Tsekrekos (2010) to study the effect of mean reversion on entry and exit decisions under output price uncertainty.

Proposition 7.5 *Under the restrictions stated in Definition 7.3, with $\gamma = 1$, the optimal solution **X** $= [\overline{P}, \underline{P}, A_1, B_0]'$ is uniquely determined by solving the system **F(X) = 0**, where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0 U(\overline{P}) + A_1 M(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -B_0 U(\underline{P}) + A_1 M(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -B_0 V(\overline{P}) + A_1 W(\overline{P}) + \varphi\overline{P} \\ -B_0 V(\underline{P}) + A_1 W(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{7.119}$$

[7.7] Note that, with $\gamma = 1$, the variance rate grows with P, so that the variance is zero if P is zero. This is clearly a more appealing feature than the one associated to the simple Ornstein-Uhlenbeck process in which the variance rate is σdz. In this latter case, as the output price becomes small, the constant volatility could cause prices to became negative, which is not economically reasonable for a practitioner.

with

$$M(y) = y^{-a_1} M(a_1, b_1, z(y)), \tag{7.120}$$

$$U(y) = y^{-a_2} U(a_2, b_2, z(y)), \tag{7.121}$$

$$W(y) = a_1 \left[M(y) + \frac{z(y)}{b_1} y^{-a_1} M(a_1 + 1, b_1 + 1, z(y)) \right], \tag{7.122}$$

$$V(y) = a_2 \left[U(y) - z(y) y^{-a_2} U(a_2 + 1, b_2 + 1, z(y)) \right], \tag{7.123}$$

$$z(y) = \frac{2k\theta}{\delta^2 y}, \tag{7.124}$$

$$a_1 = -\frac{2(k + \lambda\rho\delta) + \delta^2 - \sqrt{8r\delta^2 + (-2k - 2\lambda\rho\delta - \delta^2)^2}}{2\delta^2}, \tag{7.125}$$

$$a_2 = -\frac{2(k + \lambda\rho\delta) + \delta^2 + \sqrt{8r\delta^2 + (-2k - 2\lambda\rho\delta - \delta^2)^2}}{2\delta^2}, \tag{7.126}$$

$$b_1 = 2 + 2a_1 + \frac{2(k + \lambda\rho\delta)}{\delta^2}, \tag{7.127}$$

$$b_2 = 2 + 2a_2 + \frac{2(k + \lambda\rho\delta)}{\delta^2}, \tag{7.128}$$

$$\varphi = (r + k + \lambda\rho\delta)^{-1}, \tag{7.129}$$

$$\overline{X} = -\frac{k\theta}{r(k + \lambda\rho\delta)} + \frac{k\theta}{(r + k + \lambda\rho\delta)(k + \lambda\rho\delta)} + \frac{C}{r} + \overline{K}, \tag{7.130}$$

$$\underline{X} = -\frac{k\theta}{r(k + \lambda\rho\delta)} + \frac{k\theta}{(r + k + \lambda\rho\delta)(k + \lambda\rho\delta)} + \frac{C}{r} + \underline{K}. \tag{7.131}$$

Proof. To proof of this proposition is similar to the proof of the above propositions.

Under the restrictions stated in Definition 7.3 and with $\gamma = 1$, equations (7.6) and (7.7) become, respectively:

$$\frac{1}{2}\sigma^2 P^2 V_0''(P) + \Big(k\theta - (k + \lambda\rho\delta)P\Big) V_0'(P) - rV_0(P) = 0, \tag{7.132}$$

and

$$\frac{1}{2}\sigma^2 P^2 V_1''(P) + \Big(k\theta - (k+\lambda\rho\delta)P\Big)V_1'(P) - rV_1(P) + P - C = 0. \tag{7.133}$$

Since both equations have the same homogeneous part, we start by solving the homogeneous ode to find the complementary functions together. Trying a solution of the form $V_\phi(P) = P^\xi h(P)$ (with $\phi \in \{0,1\}$), where ξ is a constant that will be chosen in order to make $h(P)$ satisfy a differential equation with a known solution. It follows that $V_\phi'(P) = \gamma P^{\gamma-1}h(P) + P^\gamma h'(P)$ and $V_\phi''(P) = \gamma(\gamma-1)P^{\gamma-2}h(P) + 2\gamma P^{\gamma-1}h(P) + P^\gamma h''(P)$. Substituting $V_\phi(P)$, $V_\phi'(P)$, and $V_\phi''(P)$ into the homogeneous equations and rearranging terms yields gives the following equation:

$$\begin{aligned} &P^{\xi-1}h(P)\left[\xi(\xi-1) - \tfrac{2(k+\lambda\rho\delta)\xi}{\delta^2} - \tfrac{2r}{\delta^2}\right] + \\ &+P^\xi\left[Ph''(P) + \left[\left(2\xi - \tfrac{2(k+\lambda\rho\delta)}{\delta^2}\right) + \tfrac{2k\theta}{\delta^2 P}\right]h'(P) + \tfrac{2k\theta\xi}{\delta^2 P^2}h(P)\right] = 0. \end{aligned} \tag{7.134}$$

The equation (7.134) must hold for any value of P, so the bracketed terms in the both the first and second lines of equation must equal zero. Let us first choose ξ to set the bracketed terms of the first line of equation (7.134) equal to zero:

$$\xi(\xi-1) - \frac{2(k+\lambda\rho\delta)\xi}{\delta^2} - \frac{2r}{\delta^2} = 0. \tag{7.135}$$

This quadratic equation has two solutions for ξ, which are equal to $-a_1$ and $-a_2$, where a_1 and a_2 are defined by equations (7.125) and (7.126), respectively.

To find a general solution to bracketed terms in the second line of (7.134) we make the change of variable $z(P) = 2k\theta/(\delta^2 P)$ and we consider a new function $h(P) = g(z(P))$, so that $h'(P) = -\frac{z^2\delta^2 g'(z(P))}{2k\theta}$ and $h''(P) = \frac{2\delta^4 z^3 g'(z(P))}{(2k\theta)^2} + \frac{\delta^4 z^4 g''(z(P))}{(2k\theta)^2}$. By making this substitution we transform this equation into

$$zg''(z(P)) + (b - z(P))g'(z) - ag(z(P)) = 0, \tag{7.136}$$

with $a = -\xi$ and $b = 2 - 2\xi + 2(k + \lambda\rho\delta)/\delta^2$. The equation (7.136) is the Kummer's equation, as defined in Abramowitz and Stegun (1972, expression 13.1.1), which have a general solution given by

$$g(z(P)) = A_1 M(a, b, z(P)) + A_2 U(a, b, z(P)). \tag{7.137}$$

Multiplying this general solution by P^{ξ} we will get the general solution of the ode (7.132) and of the homogeneous part of the equation (7.133), which can be expressed as a linear solution of two independent solutions as

$$V_{\phi}(P) = A_{\phi} P^{\xi} M\left(a, b, z(P)\right) + B_{\phi} P^{\xi} U\left(a, b, z(P)\right). \tag{7.138}$$

The solution of the ode (7.132), subject to the boundary condition (7.8) is given by

$$V_0(P) = B_0 P^{-a_2} U\left(a_2, b_2, z(P)\right), \tag{7.139}$$

which is valid over the range os prices $(0, \overline{P})$. Analogously, the solution of the ode (7.133) subject to the boundary condition (7.10) is given by

$$\begin{aligned} V_1(P) = {} & A_1 P^{-a_1} M(a_1, b_1, z(P)) \\ & + \frac{P}{r + k + \lambda\rho\delta} + \frac{k\theta}{r(k + \lambda\rho\delta)} - \frac{k\theta}{(r + k + \lambda\rho\delta)(k + \lambda\rho\delta)} - \frac{C}{r}, \end{aligned} \tag{7.140}$$

which is valid for P over the range of prices $(\underline{P}, \infty)$.

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_\phi(P)$. Using the results for the derivatives of the Kummer functions and straightforward calculus, we obtain

$$V_0'(P) = B_0 a_2 P^{-a_2-1}\left[U(a_2, b_2, z(P)) - z(P)U(a_2+1, b_2+1, z(P))\right] \tag{7.141}$$

and

$$V_1'(P) = A_1 a_1 P^{-a_1-1}\left[M(a_1, b_1, z(P)) + \frac{z(P)}{b_1} M(a_1+1, b_1+1, z(P))\right] + \frac{1}{r+k+\lambda\rho\delta}. \tag{7.142}$$

Finally, substituting (7.139)-(7.142) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.119). ■

The mean-reverting square-root process

The mean-reverting square-root process, also known as the Cox-Ingersoll-Ross process (hereafter CIR process), due to Cox et al. (1985) is obtained with $\gamma = 1/2$. This process has been widely used to model volatility, interest rates, and other financial instruments.

In the context of real options, this model assumption has been used by Dias and Shackleton (2011) to study the investment hysteresis problem under stochastic interest rates, while Alvarez (2011) use it to model optimal capital accumulation under price uncertainty and cost reversibility of investment.

Proposition 7.6 *Under the restrictions stated in Definition 7.3, with $\gamma = 1/2$, the optimal solution $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ is uniquely determined by solving the system* ***F(X) = 0****, where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M(\overline{P}) + B_1 U(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -A_0 M(\underline{P}) + B_1 U(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -A_0 W(\overline{P}) + B_1 V(\overline{P}) + \varphi\overline{P} \\ -A_0 W(\underline{P}) + B_1 V(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{7.143}$$

with

$$M(y) = M(a_1, b_1, z(y)), \tag{7.144}$$

$$U(y) = y^{\xi_2} U(a_2, b_2, z(y)), \tag{7.145}$$

$$W(y) = y\frac{r}{k\theta} M(a_1+1, b_1+1, z(y)), \tag{7.146}$$

$$V(y) = \left(\xi_2 U(y) - a_2 z(y) y^{\xi_2} U(a_2+1, b_2+1, z(y))\right), \tag{7.147}$$

$$z(y) = \frac{2(k+\lambda\rho\delta)}{\delta^2} y, \tag{7.148}$$

$$a_1 = \frac{r}{k+\lambda\rho\delta}, \tag{7.149}$$

$$a_2 = \xi_2 + \frac{r}{k+\lambda\rho\delta}, \tag{7.150}$$

$$b_1 = \frac{2k\theta}{\delta^2}, \tag{7.151}$$

$$b_2 = 2\xi_2 + \frac{2k\theta}{\delta^2}, \tag{7.152}$$

$$\xi_2 = 1 - \frac{2k\theta}{\delta^2}, \tag{7.153}$$

and φ, $\overline{X}$, and $\underline{X}$ as defined as in (7.129), (7.130), *and* (7.131), *respectively.*

Proof. To proof of this proposition is similar to the proof of the above propositions.

Under the restrictions stated in Definition 7.3 and with $\gamma = 1/2$, equations (7.6) and (7.7) become, respectively:

$$\frac{1}{2}\sigma^2 P V_0''(P) + \Big(k\theta - (k + \lambda\rho\delta)P\Big)V_0'(P) - rV_0(P) = 0 \tag{7.154}$$

and

$$\frac{1}{2}\sigma^2 P V_1''(P) + \Big(k\theta - (k + \lambda\rho\delta)P\Big)V_1'(P) - rV_1(P) + P - C = 0. \tag{7.155}$$

Since both equations have the same homogeneous part, we start by solving the homogeneous ode to find the complementary functions together. Trying again a solution of the form $V_\phi(P) = P^\xi h(P)$ (with $\phi \in \{0,1\}$), where ξ is a constant that will be chosen in order to make $h(P)$ satisfy a differential equation with a known solution. It follows that $V_\phi'(P) = \gamma P^{\gamma-1}h(P) + P^\gamma h'(P)$ and $V_\phi''(P) = \gamma(\gamma-1)P^{\gamma-2}h(P) + 2\gamma P^{\gamma-1}h(P) + P^\gamma h''(P)$. Substituting these functions into homogeneous equations and rearranging terms yields gives the following equation:

$$\begin{aligned} &P^{\xi-1}h(P)\left[\xi(\xi-1) + \tfrac{2k\theta}{\delta^2}\xi\right] + \\ &+P^\xi\left[Ph''(P) + \left(2\xi + \tfrac{2k\theta}{\delta^2} - \tfrac{2(k+\lambda\rho\delta)P}{\delta^2}\right)h'(P) - \left(\tfrac{2(k+\lambda\rho\delta)\xi}{\delta^2} + \tfrac{2r}{\delta^2}\right)h(P)\right] = 0. \end{aligned} \tag{7.156}$$

The equation (7.156) must hold for any value of P, so the bracketed terms in the both the first and second lines of equation must equal zero. Let us first choose ξ to set the bracketed terms of the first line of equation (7.156) equal to zero:

$$\xi(\xi-1) + \frac{2k\theta}{\delta^2}\xi = 0. \tag{7.157}$$

This quadratic equation has two solutions for ξ, which are equal to $\xi = \xi_1 = 0$ and

$\xi = \xi_2$, as defined by equation (7.153).

To find a general solution of the function in the bracketed terms in the second line of (7.156) we make the change of variable $z(P) = 2(k+\lambda\rho\delta)P/\delta^2$ and we consider a new function $h(P) = g(z(P))$, so that $h'(P) = \frac{2(k+\lambda\rho\delta)}{\delta^2}g'(z(P))$ and $h''(P) = (\frac{2(k+\lambda\rho\delta)}{\delta^2})^2 g''(z)$. By making this substitution we transform this equation, again, into the Kummer's equation (7.136), but now with $a = \xi + r/(k+\lambda\rho\delta)$ and $b = 2\xi + 2k\theta/\delta^2$. The general solution of this equation is given by equation (7.137).

Multiplying this general solution by P^{ξ} we will get the general solution of the ode (7.154) and of the homogeneous part of the equation (7.155), which can be expressed as a linear solution of two independent solutions as

$$V_{\phi}(P) = A_{\phi}P^{\xi}M\left(a,b,z(P)\right) + B_{\phi}P^{\xi}U\left(a,b,z(P)\right). \tag{7.158}$$

The solution of the ode (7.154), subject to the boundary condition (7.8) is given by

$$V_0(P) = A_0 M\left(a_1,b_1,z(P)\right), \tag{7.159}$$

which is valid over the range os prices $(0,\overline{P})$. Analogously, the solution of the ode (7.155) subject to the boundary condition (7.10) is given by

$$\begin{aligned} V_1(P) &= B_1 P^{\xi_2} U(a_2,b_2,z(P)) \\ &+ \frac{P}{r+k+\lambda\rho\delta} + \frac{k\theta}{r(k+\lambda\rho\delta)} - \frac{k\theta}{(r+k+\lambda\rho\delta)(k+\lambda\rho\delta)} - \frac{C}{r}, \end{aligned} \tag{7.160}$$

which is valid for P over the range of prices $(\underline{P}, \infty)$.

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_{\phi}(P)$. Using the results for the derivatives of the Kummer functions, we obtain after straightforward calculus

$$V_0'(P) = A_0 \frac{r}{k\theta} M\left(a_1+1, b_1+1, z(P)\right) \tag{7.161}$$

and

$$V_1'(P) = B_1 P^{\xi_2 - 1}\left[\xi_2 U\left(a_2, b_2, z(P)\right) - a_2 z(P) U\left(a_2+1, b_2+1, z(P)\right)\right] + \frac{1}{r+k+\lambda\rho\delta}. \tag{7.162}$$

Finally, substituting (7.159)-(7.162) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.143). ■

The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process (hereafter OU process) is obtained with $\gamma = 0$.

Despite its apparently less desirable feature of allowing paths with negative prices, the OU process is often used in many capital budgeting decisions given its analytic tractability and its ability to fit historical and futures price data (see, for instance, Smith and McCardle (1999) for a specific application in evaluating investments in the oil and gas industry).

Proposition 7.7 *Under the restrictions stated in Definition 7.3, with $\gamma = 0$, the optimal* *solution* $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ *is uniquely determined by solving the system* ***F(X) = 0****, where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M(\overline{P}) + B_1 U(\overline{P}) + \varphi \overline{P} - \overline{X} \\ -A_0 M(\underline{P}) + B_1 U(\underline{P}) + \varphi \underline{P} - \underline{X} \\ -A_0 W(\overline{P}) + B_1 V(\overline{P}) + \varphi \overline{P} \\ -A_0 W(\underline{P}) + B_1 V(\underline{P}) + \varphi \underline{P} \end{bmatrix}, \tag{7.163}$$

with

$$M(y) = M(a, b, z(y)), \tag{7.164}$$

$$U(y) = U(a, b, z(y)), \tag{7.165}$$

$$W(y) = -\frac{2r(k\theta - (k + \lambda\rho\delta)y)}{\delta^2(k + \lambda\rho\delta)} y M(a+1, b+1, z(y)), \tag{7.166}$$

$$V(y) = \frac{r(k\theta - (k + \lambda\rho\delta)y)}{\delta^2(k + \lambda\rho\delta)} y U(a+1, b+1, z(y)), \tag{7.167}$$

$$z(y) = \frac{\big(k\theta - (k + \lambda\rho\delta)y\big)^2}{\delta^2(\kappa + \lambda\rho\delta)}, \tag{7.168}$$

$$a = \frac{r}{2(k + \lambda\rho\delta)}, \tag{7.169}$$

$$b = \frac{1}{2}, \tag{7.170}$$

and φ, $\overline{X}$, *and* $\underline{X}$ *as defined as in* (7.129), (7.130), *and* (7.131), *respectively.*

Proof. To proof of this proposition is similar to the proof of the above propositions.

Under the restrictions stated in Definition 7.3 and with $\gamma = 0$, equations (7.6) and (7.7) become, respectively:

$$\frac{1}{2}\sigma^2 V_0''(P) + \Big(k\theta - (k + \lambda\rho\delta)P\Big) V_0'(P) - rV_0(P) = 0, \tag{7.171}$$

and

$$\frac{1}{2}\sigma^2 V_1''(P) + \left(k\theta - (k+\lambda\rho\delta)P\right)V_1'(P) - rV_1(P) + P - C = 0. \tag{7.172}$$

Since both equations have the same homogeneous part, we start by solving the homogeneous ode to find the complementary functions together. We make the change of variable $z(P) = \frac{\left(k\theta-(k+\lambda\rho\delta)y\right)^2}{\delta^2(\kappa+\lambda\rho\delta)}$, which converts the homogeneous equation in the so-called Kummer equation (7.136), but now with a and b as defined as in (7.169) and (7.170), respectively. The general solution of this equation is given by (7.137). So, we will get the general solution of the ode (7.171) and of the homogeneous part of the equation (7.172), which can be expressed as a linear solution of two independent solutions as

$$V_\phi(P) = A_\phi M\left(a,b,z(P)\right) + B_\phi U\left(a,b,z(P)\right). \tag{7.173}$$

The solution of the ode (7.154), subject to the boundary condition (7.8) is given by

$$V_0(P) = A_0 M\left(a,b,z(P)\right), \tag{7.174}$$

which is valid over the range of prices $(0,\overline{P})$. Analogously, the solution of the ode (7.172) subject to the boundary condition (7.10) is given by

$$\begin{aligned} V_1(P) = {} & B_1 U\left(a,b,z(P)\right) \\ & + \frac{P}{r+k+\lambda\rho\delta} + \frac{k\theta}{r(k+\lambda\rho\delta)} - \frac{k\theta}{(r+k+\lambda\rho\delta)(k+\lambda\rho\delta)} - \frac{C}{r}, \end{aligned} \tag{7.175}$$

which is valid for P over the range of prices $(\underline{P},\infty)$.

To apply the smooth-pasting conditions (7.13) and (7.14), one needs to compute the first derivatives of the value functions $V_\phi(P)$. Using the results for the derivatives of the Kummer functions, it is straightforward to obtain

$$V_0'(P) = -A_0 \frac{2r(k\theta-(k+\lambda\rho\delta)P)}{\delta^2(k+\lambda\rho\delta)} M(a+1,b+1,z(P)) \tag{7.176}$$

and

$$V_1'(P) = B_1 \frac{r(k\theta-(k+\lambda\rho\delta)P)}{\delta^2(k+\lambda\rho\delta)} U(a+1,b+1,z(P)) + \frac{1}{r+k+\lambda\rho\delta}. \tag{7.177}$$

Finally, substituting (7.174)-(7.177) into the value-matching and smooth-pasting conditions (7.11)-(7.14) and rearranging terms we obtain the non-linear system in matrix form given in (7.163). ■

4 Analysis of optimal entry-exit policy

In this section we analyze the optimal entry-exit policy assuming that the underlying output price dynamics follows the aforementioned generalized one-dimensional diffusion subject to the restrictions stated in definitions 7.1 to 7.3. Panels (a) and (b) of Figure 7.1 plot entry and exit thresholds prices, $\overline{P}$ and $\underline{P}$, respectively, as a function of lump-sum entry and exit costs $\overline{K} = -\underline{K}$ for different parameter values. In our modeling framework, this means that $\alpha = -1$. The range $(\overline{P},\underline{P})$ is the hysteretic band of the problem since idle firms do not invest and operating firms do not abandon the activity within this intermediate level of output prices. Panel (a) is for different β values of the CEV process, namely, $\beta \in \{-4,-2,0,1,2,3\}$ ($\beta = 2$ corresponds to the GBM assumption), where we have used the following parameter values: $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$,

$\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$, and panel (b) is for different γ values of the mean-reverting CEV processes, namely, $\gamma = 0$ (OU process), $\gamma = 1/2$ (CIR process) and $\gamma = 1$ (IGBM process), where we have used the following parameter values: $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$. As we can see from this figure, the hysteresis emerges only when entry and exit costs are present, otherwise the entry and exit thresholds $\overline{P}, \underline{P}$ drop to the level of the variable cost C. From panels (a) and (b) of Figure 7.1, we also can see that the hysteresis increases with the diffusion coefficient (i.e, with the parameter β in the CEV and with the parameter γ in the mean-reverting CEV processes). This results are consistent with the findings of Dias and Nunes (2011).

Panels (c) and (d) of the Figure 7.1 plot the optimal decisions thresholds scaled by the corresponding Marshallian triggers as functions of volatility. Panel (c) is for different β values of the CEV process, namely, $\beta \in \{-4, -2, 0, 1, 2, 3\}$ ($\beta = 2$ corresponds to the GBM assumption), where we have used the following parameter values:$\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$ and $P_0 = 1$, and panel (b) is for different γ values of the mean-reverting CEV processes, namely, $\gamma \in \{0, 1/2, 1\}$ (as well as the case of GBM with no drift), where we have used the following parameter values: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$. Both panels show that the entry and exit thresholds under the CEV and the mean-reverting CEV processes converge to the Marshallian triggers, $\overline{W}$ and $\underline{W}$, respectively, when $\delta \to 0$, so $\overline{P}/\overline{W} \to 1$ and $\underline{P}/\underline{W} \to 1$. From these panels, we can say that there is a clear trend for a wider range of inaction as the volatility coefficient rises.

An idle firm will enter in the market if the output price rises to high values, but it owns an option to exit later if the output prices fall to a sufficient low level and return to the idle state. Once the project is abandoned, the firms owns an option to reinvest again if the output prices reverse to high levels again. Thus, it is important to evaluate the

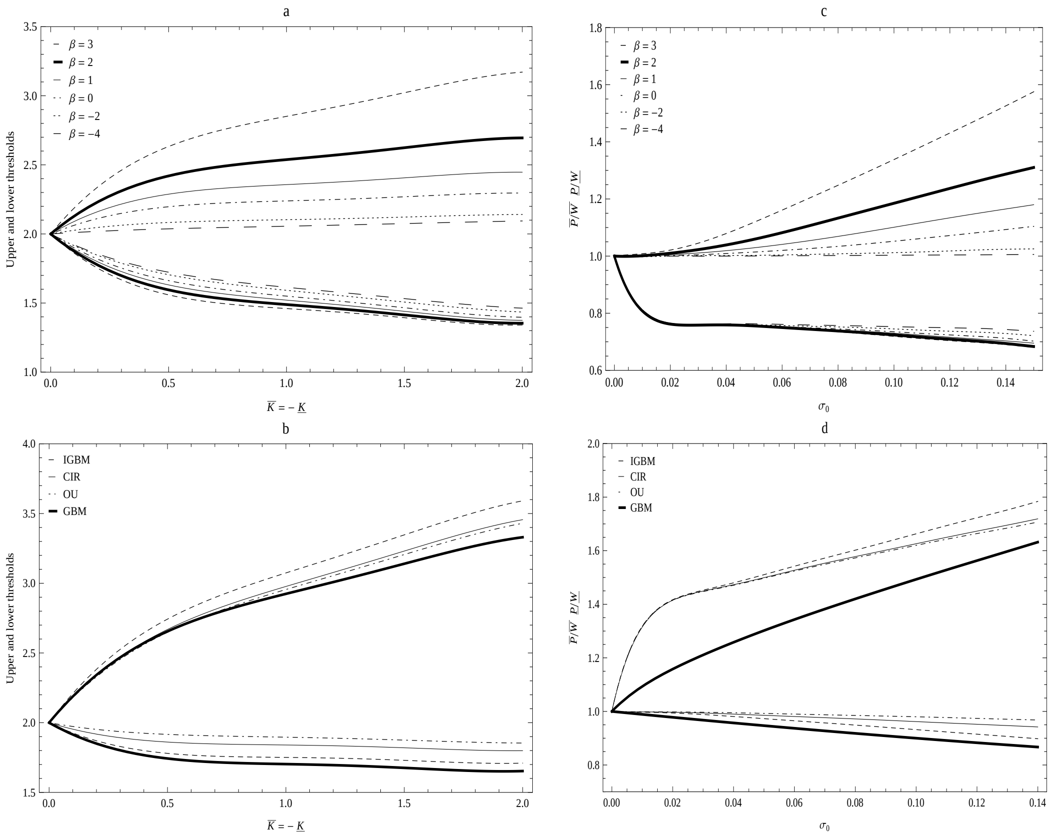

Figure 7.1: Panels (a) and (b): the optimal entry and exit triggers, $\overline{P}$ and $\underline{P}$, under the CEV and the mean reverting CEV processes, as function of entry and exit cost, in the particular case $\overline{K} = -\underline{K}$. Panels (c) and (d): the ratios of the optimal entry and exit triggers $\overline{P}$, $\underline{P}$ under the CEV and the mean-reverting processes, respectively, over the corresponding thresholds, $\overline{W}$, $\underline{W}$, as function of volatility, σ_0. The rest parameters used are $C = 2$, $r = 0.04$, $\lambda = 0.4$, $\rho = 1$, $P_0 = 1$, and in panel (a) $\beta \in \{-4, -2, 0, 1, 2, 3\}$, $\sigma_0 = 0.15$, $\mu = 0.08$; panel (b) $\gamma \in \{0, 1/2, 1\}$, $\sigma_0 = 0.15$, $k = 0.05$, $\theta = 1$, and the GBM with no drift; panel (c) $\overline{K} = 3$, $\underline{K} = -2$, $\beta \in \{-4, -2, 0, 1, 2, 3\}$, $\mu = 0.08$; and in panel (d) $\overline{K} = 3$, $\underline{K} = -2$, $\gamma \in \{0, 1/2, 1\}$, $k = 0.15$, $\theta = 1$, and the GBM with no drift.

no-action region, this is, the hysteretic band. Figures 7.2 and 7.3 highlight the value of an idle firm, $V_0(P)$, and the value of an active firm, $V_1(P)$, both as functions of the output price P, for the CEV process and mean-reverting CEV process, respectively. Also shown are the entry and exit thresholds prices, $\overline{P}$ and $\underline{P}$. Since the option to invest is exercised as soon as P reaches $\overline{P}$, the option value does not exist for values of P above $\overline{P}$. Similarly, since the abandonment is exercised as soon as P falls to $\underline{P}$, the option value does not exist for values of P below $\underline{P}$. Note that, for all β values, at $P = \underline{P}$, $V_0(P)$ exceeds $V_1(P)$ by the abandonment cost $-\underline{K}$, since at that price is optimal to exercise the abandonment option, giving up $-\underline{K}+V_1$ and receiving V_0. Likewise, at $P=\overline{P}$ it is optimal to invest, so $V_1=V_0+\overline{K}$.

In Figure 7.2 we have used the following parameter values: $\overline{K}=3$, $\underline{K}=-2$, $C=2$, $r=0.04$, $\sigma_0=0.15$, $\mu=0.08$, $\lambda=0.4$, $\rho=1$, and $P_0=1$ for $\beta\in\{-4,-2,0,1,2,3\}$. Considering these parameter values, and for $\beta=-4$, the entry and exit thresholds prices are, respectively, $\overline{P}=2.133$ and $\underline{P}=1.422$, which originates a range of inaction with width 0.711. Under the GBM process, i.e. $\beta=2$, the entry and exit trigger points are respectively $\overline{P}=2.782$ and $\underline{P}=1.319$, which gives a larger range of inaction of 1.463. With $\beta=3$, the range of inaction increases to 2.023, where entry and exit thresholds are given by $\overline{P}=3.34$ and $\underline{P}=1.312$, respectively. It is clear that the hysteretic band increases with the parameter β in the CEV process, keeping all else equal.

In Figure 7.3 we have used the following parameter values borrowed from Tsekrekos (2010): $\overline{K}=3$, $\underline{K}=-2$, $C=2$, $k=0.05$, $\theta=1$, $\lambda=0.4$, $\sigma_0=0.15$, $\rho=1$, $P_0=1$, and $r=0.04$ for $\gamma=0,1/2,1$. We also plot the GBM process with no drift for reference. We conclude that hysteresis increases with the volatility parameter of the mean-reverting processes, from de 1.809 in the OU process, to 1.892 in the CIR process, and to 2.136 in the IGBM process.

Figures 7.2 and 7.3 allow us to conclude that the value of an idle firm and the value of an active firm increase when the β and γ parameters rise.

In summary, we can draw the following conclusions from our analysis, for both CEV and mean-reverting CEV processes:

i) The entry threshold price rise and the exit threshold price falls as the parameters β and γ rises, keeping all remainder parameters equal. Thus, the hysteretic band will be wider;

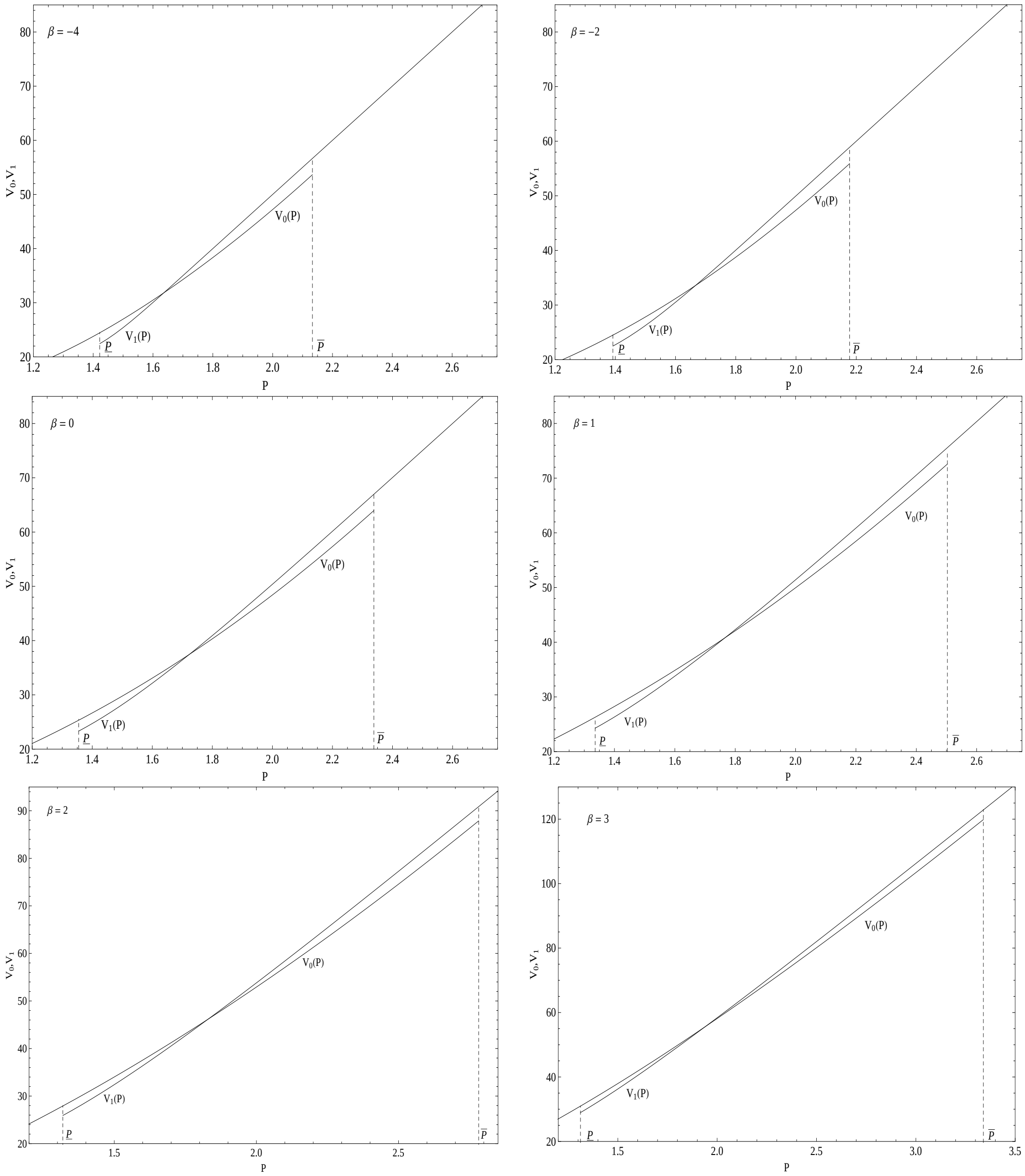

Figure 7.2: The value of the firm when idle, $V_0(P)$, and active, $V_1(P)$, as a function of the output price P, when the output price follows a CEV process. At the entry and exit thresholds, $\overline{P}$, $\underline{P}$, the firm optimally switches between the idle and active states by sinking the entry and exit costs, $\overline{K}$, $\underline{K}$. The rest of parameters used are: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$, $\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$.

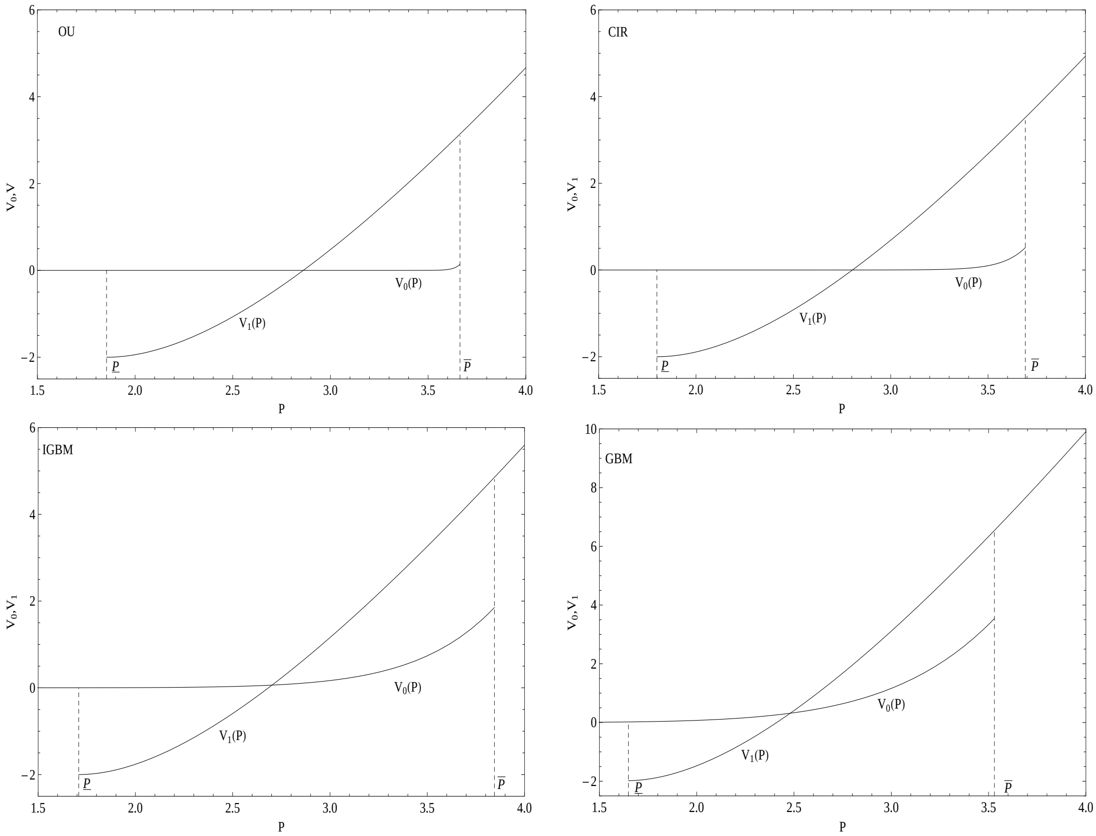

Figure 7.3: The value of the firm when idle, $V_0(P)$, and active, $V_1(P)$, as a function of the output price P, when the output price follows a CEV mean-reverting process. At the entry and exit thresholds, $\overline{P}$, $\underline{P}$, the firm optimally switches between the idle and active states by sinking the entry and exit costs, $\overline{K}$, $\underline{K}$. The rest of parameters used are: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\rho =$, $\sigma_0 = 0.15$, and $P_0 = 1$.

ii) When equal entry and abandonment cost, $\overline{K} = -\underline{K}$, increase, the entry trigger increases while the exit trigger decreases, leading to a higher inactive region. This holds for all β and γ values.

iii) Keeping all remainder parameter values constants, the hysteric band increases with the volatility parameter.

5 First passage time distributions for entry and exit thresholds

In this section we will compute and analyze the *ex ante* probability of entry for an inactive firm, and the *ex ante* probability of exit for an active firm within an specified horizon. Following Tsekrekos (2010), we will consider that the *ex ante* probability that a single idle firm will enter in an industry/market during the time horizon T will be a measure of the fraction of idle firms, under competitive equilibrium, that will enter the market during this time horizon. The same line of reasoning applies for the *ex ante* probability that a single active firm will exit during time T.

For any optimal policy pair $(\overline{P}, \underline{P})$, the *ex ante* probability that an inactive firm will enter during time T is equal to $P(\tau_U \leq T)$, where $\tau_U := inf\{t \geq t_0 : P_t = \overline{P}\}$ is the first hitting time of the underlying process to the investment threshold $\overline{P}$. Conversely, the *ex ante* probability that an active firm will exit the market during the time horizon T is equal to $P(\tau_L \leq T)$, with $\tau_L := inf\{t \geq t_0 : P_t = \underline{P}\}$ is the first hitting time of the underlying process to the divestment threshold $\underline{P}$.

Considering the particular case of the CEV model, $\beta = 2$ (GBM model), the *ex ante* probabilities of entry and exit can be computed in closed-form, using the following expressions as given by Jeanblanc et al. (2009, expressions 3.3.2 and 3.3.3), respectively,

$$P(\tau_U \leq T) = N\left(\frac{\ln(\frac{P_0^*}{\overline{P}}) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) + \left(\frac{\overline{P}}{P_0^*}\right)^{\frac{2\mu}{\sigma^2}-1} N\left(\frac{\ln(\frac{P_0^*}{\overline{P}}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \quad (7.178)$$

$$P(\tau_L \leq T) = N\left(\frac{\ln(\frac{\underline{P}}{P_0^*}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) + \left(\frac{\underline{P}}{P_0^*}\right)^{\frac{2\mu}{\sigma^2}-1} N\left(\frac{\ln(\frac{\underline{P}}{P_0^*}) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \quad (7.179)$$

where $N(.)$ is the normal standard cumulative distribution function, and $P_0^* = \frac{\overline{P}+\underline{P}}{2}$ is the current level of the price process. Expression (7.179) corrects the typo in Tsekrekos (2010, expression 27).

For the CEV process, with $\beta \neq 2$, and for the mean-reverting CEV process there are no simple closed-form solutions as for the GBM process. Thus, in order to simulate these probabilities, it will be necessary to compute first an approximate solution of the CEV and mean-reverting CEV processes prices.

Table 7.1 shows the *ex ante* entry and *ex ante* exit probabilities for the CEV diffusion model using the parameter values $\beta \in \{-4,-2,0,1,2,3\}$, $\sigma_0 \in \{0.10, 0.15, 0.20\}$, $r = 0.04$, and $T = 10$ years. The parameter μ will be adjusted so that the difference $(\mu - \lambda\rho\delta)$ remains constant and equal to 2 (so, for $\sigma_0 = 0.10, 0.15,$ and 0.20 we will have $\mu = 0.06, 0.08,$ and $0.10,$ respectively). To compute these probabilities we have used the Euler scheme, described in Glasserman (2004). Details on how these probabilities were computed are provided in the Appendix.

Table 7.1 show that, for $\beta < 2$, there is a direct relation between the volatility parameter and the *ex ante* probability of entry, while for $\beta > 2$, there is an inverse relation. For the case of the exit probability, and for all β values, the probability increases with volatility.

We can also conclude that there appears to be evidence that the value of entry and exit probabilities decreases when β increases, keeping the remainder parameter values constants.

Table 7.1: Probability of entry for an idle firm and the probability of exit for an active firm within a specified horizon, T, for the CEV process, computed via Monte Carlo simulation.

Volatility	$\beta=3$	$\beta=2$	$\beta=1$	$\beta=0$	$\beta=-2$	$\beta=-4$
Panel A: Probability of entry						
$\sigma_0=0.10$	0.5025	0.5888	0.6340	0.6816	0.7446	0.7976
$\sigma_0=0.15$	0.4896	0.5818	0.6354	0.6918	0.8063	0.8862
$\sigma_0=0.20$	0.4880	0.5820	0.6372	0.7056	0.8505	0.9147
Panel B: Probability of exit						
$\sigma_0=0.10$	0.0597	0.1602	0.2300	0.2760	0.3134	0.3174
$\sigma_0=0.15$	0.1263	0.2942	0.3758	0.4298	0.4626	0.4637
$\sigma_0=0.20$	0.1765	0.3945	0.4720	0.5082	0.5407	0.5343

This table values the probabilities of entry and exit in the market during the time horizon T under the CEV diffusion model. In each simulation, $70,000$ paths and $1,000$ time steps are used. σ_0 is the volatility of the market price, and μ is the growth rate of the market price, which is adjusted for each σ_0 value ($\mu \in \{0.06, 0.08, 0.10\}$, respectively). In all cases $P_0^* = (\overline{P}+\underline{P})/2$, the time horizon is $T=10$ years, the variable flow cost is $C=2$, $r=0.04$ is the riskless interest rate, $\lambda=0.4$ is the market price of the risk, $\rho=1$ is the correlation between output price and the market portfolio, and the entry and exit sunk costs are $\overline{K}=3$ and $\underline{K}=-2$, respectively.

To compute an approximate solution to the IGBM and CIR processes we will use the *Pathwise Adapted Linearization* and the *Pathwise Adapted Linearization Quadratic* schemes, both proposed by Kahl and Jäckel (2006), and explained in the Appendix. To compute an approximate solution to the OU process, we will simulate the output prices using the following exact discretization:

$$P_{t_{+1}} = e^{-k\Delta_t}P_{t_i} + \theta(1-e^{-k\Delta_t}) + \delta\sqrt{\frac{(1-e^{-2k\Delta_t})}{2k}}Z_{t-i+1}, \tag{7.180}$$

where Z_i are independent $N(0,1)$ variables, and $\Delta_t = t_{i+1}-t_i$.

Tables 7.2 and 7.3 give us the *ex ante* probabilities of entry and exit, respectively, for the mean-reverting CEV processes, namely, for the OU, CIR, and IGBM processes.

Our first remark is concerned with the effect of the volatility parameter, σ_0, on the *ex ante* probabilities. As reported in Table 7.2, the probability of entry increases with σ_0 in all cases. Similarly, we reach the same conclusion for the exit probability, except when the price process reverts to low levels $(\theta = 0.4)$ with higher speed $(k = 0.10)$. These conclusions are similar to the ones reaches by Tsekrekos (2010) for the IGMB process, and by Sarkar (2000) for the GBM process. So, we can also conclude that the non-monotonic relationship, as reported by the authors referenced above, is present in the all mean-reverting CEV processes analyzed.

We analyze also the effect of the log-run output price level, θ. By the results reported, we can argue that the increase of the θ parameter has a positive effect on the entry decisions and a negative effect on the exit decisions.

Now, we analyze the effect of the mean reverting speed parameter, k, on the investment and disinvestment decisions. There is some evidence of a negative effect on the entry probability, except when the price of the output process revert to hight level (when $\rho = 0$, and for $\rho = 1$ with low volatility).

Finally, we report our conclusions of the effect of the γ parameter. From Table 7.2 we can conclude that when γ increases, the probability of entry also increases, except when the correlation between the equilibrium output price and the market portfolio is null, $\rho = 0$, the mean reversion speed is hight, $k = 0.10$, and the volatility is hight, $\sigma_0 = 0.15, 0.25$. From Table 7.3, we may conclude that the probability of exit, for $\rho = 1$, increases when γ increases, except when volatility is low, $\sigma_0 = 0.10$, and the price process reverts to hight level, $\theta = 1.4$. However, there is some evidence of decrease of the exit probability, except when the price process reverts to low levels, $\theta = 0.4$.

Table 7.2: Probability of entry for an inactive firm within an specified horizon, T, for the mean-reverting CEV processes, computed via Monte Carlo simulation.

	$\theta = 0.40$			$\theta = 1.00$			$\theta = 1.40$		
	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$
Panel A: Probability of entry with $\rho = 1$									
OU process									
$k = 0.05$	0.0001	0.0045	0.0478	0.0100	0.0424	0.1207	0.1021	0.1344	0.2019
$k = 0.10$	0.0000	0.0000	0.0014	0.0000	0.0020	0.0249	0.1568	0.0949	0.1212
CIR process									
$k = 0.05$	0.0045	0.0395	0.1607	0.0448	0.1179	0.2408	0.1614	0.2192	0.3125
$k = 0.10$	0.0000	0.0012	0.0404	0.0026	0.0273	0.1241	0.2002	0.1749	0.2449
IGBM process									
$k = 0.05$	0.0295	0.1194	0.2745	0.1012	0.2015	0.3333	0.2195	0.2905	0.3816
$k = 0.10$	0.0010	0.0274	0.1544	0.0233	0.1002	0.2425	0.2359	0.2482	0.3338
Panel B: Probability of entry with $\rho = 0$									
OU process									
$k = 0.05$	0.0050	0.0550	0.2007	0.1233	0.2153	0.3384	0.3284	0.3771	0.4417
$k = 0.10$	0.0000	0.0005	0.0443	0.0371	0.1240	0.2657	0.4886	0.5005	0.5111
CIR process									
$k = 0.05$	0.0194	0.0982	0.2506	0.1487	0.2433	0.3575	0.3395	0.3888	0.4455
$k = 0.10$	0.0000	0.0079	0.1059	0.0578	0.1534	0.2911	0.4764	0.4902	0.5004
IGBM process									
$k = 0.05$	0.0488	0.1531	0.2966	0.1729	0.2724	0.3750	0.3383	0.3897	0.4437
$k = 0.10$	0.0022	0.0401	0.1792	0.0826	0.1871	0.3148	0.4524	0.4616	0.4741

This table values the probabilities of entry in the market during a time horizon T for the mean-reverting CEV. In each simulation, $100,000$ paths and $1,000$ time steps are used. θ is the long-run output price level, k is the speed of mean reversion, σ is the volatility of the output price process, and ρ is the correlation between the equilibrium output price and the market portfolio. In all cases, $P_0^* = (\overline{P} + \underline{P})/2$, the time horizon is $T = 10$ years, the market price of risk is $\lambda = 0.4$, the variable flow cost is $C = 1$, and the entry and exit sunk costs are $\overline{K} = 3$ and $\underline{K} = -2$, respectively.

Table 7.3: Probability of exit for an active firm within an specified horizon, T, for the mean-reverting CEV processes, computed via Monte Carlo simulation.

	$\theta = 0.40$			$\theta = 1.00$			$\theta = 1.40$		
	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$
Panel A: Probability of exit with $\rho = 1$									
OU process									
$k = 0.05$	0.3074	0.3562	0.4140	0.1098	0.1916	0.3012	0.0414	0.1133	0.2333
$k = 0.10$	0.6487	0.6080	0.5898	0.1018	0.1952	0.3207	0.0019	0.0431	0.1731
CIR process									
$k = 0.05$	0.3391	0.3983	0.4751	0.1264	0.2259	0.3623	0.0457	0.1340	0.2876
$k = 0.10$	0.6568	0.6299	0.6355	0.1225	0.2341	0.3887	0.0007	0.0480	0.2202
IGBM process									
$k = 0.05$	0.3621	0.4306	0.5340	0.1455	0.2585	0.4228	0.0491	0.1535	0.3442
$k = 0.10$	0.6604	0.6423	0.6711	0.1431	0.2723	0.4549	0.0002	0.0532	0.2709
Panel B: Probability of exit with $\rho = 0$									
OU process									
$k = 0.05$	0.4622	0.4741	0.5042	0.1420	0.2319	0.3469	0.0228	0.1042	0.2527
$k = 0.10$	0.7478	0.6910	0.6463	0.0656	0.1566	0.2628	0.0000	0.0061	0.1028
CIR process									
$k = 0.05$	0.4709	0.4873	0.5286	0.1293	0.2245	0.3615	0.0072	0.0685	0.2378
$k = 0.10$	0.7517	0.7082	0.6755	0.0533	0.1446	0.3048	0.0000	0.0000	0.0413
IGBM process									
$k = 0.05$	0.4677	0.4933	0.5497	0.1152	0.2134	0.3750	0.0012	0.0406	0.2249
$k = 0.10$	0.7448	0.7046	0.6936	0.0421	0.1350	0.3173	0.0000	0.0000	0.0048

This table values the probabilities of entry in the market during a time horizon T for the mean-reverting CEV. In each simulation, $100,000$ paths and $1,000$ time steps are used. θ is the long-run output price level, k is the speed of mean reversion, σ is the volatility of the output price process, and ρ is the correlation between the equilibrium output price and the market portfolio. In all cases, $P_0^* = (\overline{P} + \underline{P})/2$, the time horizon is $T = 10$ years, the market price of risk is $\lambda = 0.4$, the variable flow cost is $C = 1$, and the entry and exit sunk costs are $\overline{K} = 3$ and $\underline{K} = -2$, respectively.

6 Conclusions

In this article, we consider the optimal entry and exit policy of a firm in the presence of output price uncertainty and subject to costly reversibility of investment under a generalized class of one-dimensional diffusions. We derive explicit solutions for the value functions for options of reversible investments under CEV and mean-reverting CEV processes.

We compare the different stochastic processes studied by doing an analysis of optimal entry-exit policy. This analysis includes both numerical and graphical illustrations, where we have concluded that the hysteretic band increases when: i) β and γ parameters increase; ii) when both investment and divestment equal costs increase, and iii) the volatility parameter increases.

We have also computed the *ex ante* probabilities of entry and exit and try to show that the choice of the stochastic process for the output price has a significant impact on investment and divestment decisions.

Acknowledgements

Dias gratefully acknowledges the financial support from the Fundação para a Ciência e Tecnologia (FCT) grant number PTDC/EGE-ECO/099255/2008, while Larguinho and Braumann acknowledge the financial support from the Centro de Investigação em Matemática e Aplicações (CIMA) financed by the FCT.

Appendix A: Monte Carlo Methods

This appendix shows in detail the Monte Carlo simulation methods used for computing the *ex ante* probabilities of entry and exit. First, we compute the upper and lower thresholds for each considered stochastic process. Then, a large number M of possible future output prices P at dates $0 < t_1 < t_2 < ... < t_N = T$ are generated via the following methods:

A) The simple Euler discretisation method was the scheme used to compute the CEV output prices. This method discretises the continuous time stochastic differential (7.1) to:

$$P_{i+1} = P_i + \mu(P_i)(t_{i+1} - t_i) + \sigma(P_i)\sqrt{t_{i+1} - t_i}Z_{i+1}, \tag{A.1}$$

where $i = 0, 1, ..., N-1$ and $Z_{i+1} \sim i.i.d.\mathcal{N}(0,1)$ random variables.

A detailed explanation of the methods set out below, B) and C), can be found in Kahl and Jäckel (2006).

B) The Pathwise Adapted Linearization method was the scheme used to compute the IGBM output prices. With this method the IGBM output prices are calculated as

$$P_{i+1} = P_i e^{-\delta_i(t_{i+1} - t_i)} + k\theta\left(\frac{1 - e^{-\delta_i(t_{i+1} - t_i)}}{\delta_i}\right), \tag{A.2}$$

where $i = 0, 1, ..., N-1$ and $Z_{i+1} \sim i.i.d.\mathcal{N}(0,1)$ random variables, and with

$$\delta_i = k + \frac{1}{2}\delta^2 - \delta\beta_i, \tag{A.3}$$

and

$$\beta_i = \frac{Z_{i+1} - Z_i}{t_{i+1} - t_i}. \tag{A.4}$$

C) The Pathwise Adapted Linearization Quadratic method was the scheme used to compute the CIR output prices. With this method the CIR output prices are calculated as

$$P_{i+1} \approx P_i + \big(k(\widetilde{\theta} - P_i) + \delta\beta_i\sqrt{P_i}\big)(t_{i+1} - t_i)\left[1 + \frac{\delta\beta_i - 2k\sqrt{P_i}}{4\sqrt{P_i}}(t_{i+1} - t_i)\right], \tag{A.5}$$

where $i = 0, 1, ..., N-1$ and $Z_{i+1} \sim i.i.d.\mathcal{N}(0,1)$ random variables, and with

$$\widetilde{\theta} = \theta - \frac{\delta^2}{4k}. \tag{A.6}$$

Estimates of the probabilities of entry and exit are calculated via

$$Pr(\overline{\tau} \leq T) = \frac{\overline{M}}{M} \tag{A.7}$$

and

$$Pr(\underline{\tau} \leq T) = \frac{\underline{M}}{M} \tag{A.8}$$

where $\overline{M} = \{P_{i+1} \in M : P_{i+1} \geq \overline{P}\}$ and $\underline{M} = \{P_{i+1} \in M : P_{i+1} \leq \overline{P}\}$, $i = 0, 1, ..., N-1$ are the number of simulated paths with at least one price P_{i+1} exceeding or falling below the entry and exit price thresholds, respectively.

In all results reported, M is set to $100,000$ (an exception is made for the CEV process, due to lack of PC memory, where we have used $70,000$) and $N = 1,000$. All computations were made in Matlab R2008b, whose default random number generator was used for the random variables Z_{i+1}.

References

Abadie, Luis M. and José M. Chamorro, 2008, Valuing Flexibility: The Case of an Integrated Gasification Combined Cycle Power Plant, *Energy Economics* 30, 1850–1881.

Abel, Andrew B. and Janice C. Eberly, 1996, Optimal Investment with Costly Reversibility, *Review of Economic Studies* 63, 581–593.

Abel, Andrew B., Avinash K. Dixit, Janice C. Eberly, and Robert S. Pindyck, 1996, Options, the Value of Capital, and Investment, *Quarterly Journal of Economics* 111, 753–777.

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Akerlof, George A., 1970, The Market for "Lemons": Quality Uncertainty and the Market Mechanism, *Quarterly Journal of Economics* 84, 488–500.

Alvarez, Luis H. R., 2011, Optimal Capital Accumulation under Price Uncertainty and Costly Reversibility, *Journal of Economic Dynamics and Control* 35, 1769–1788.

Andersen, Leif B. G. and Vladimir V. Piterbarg, 2007, Moment Explosions in Stochastic Volatility Models, *Finance and Stochastics* 11, 29–50.

Arrow, Kenneth J., 1968, Optimal Capital Policy with Irreversible Investment, in J. N. Wolfe, ed.: *Value, Capital, and Growth: Papers in Honour of Sir John Hicks* (Edinburgh University Press, Edinburgh), 1–19.

Bar-Ilan, Avner and William C. Strange, 1996, Investment Lags, *American Economic Review* 86, 610–622.

Bekaert, Geert and Guojun Wu, 2000, Asymetric Volatility and Risk in Equity Markets, *Review of Financial Studies* 13, 1–42.

Bhattacharya, Sudipto, 1978, Project Valuation with Mean-Reverting Cash Flow Streams, *Journal of Finance* 33, 1317–1331.

Brennan, Michael J. and Eduardo S. Schwartz, 1980, Analyzing Convertible Bonds, *Journal of Financial and Quantitative Analysis* 15, 907–929.

Chan, K. C., G. Andrew Karolyi, Francis A. Longstaff, and Anthony B. Sanders, 1992, An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *Journal of Finance* 47, 1209–1227.

Choi, Jin W. and Francis A. Longstaff, 1985, Pricing Options on Agricultural Futures: An Application of the Constant Elasticity of Variance Option Pricing Model, *Journal of Futures Markets* 5, 247–258.

Constantinides, George M. and Scott F. Richard, 1978, Existence of Optimal Simple Policies for Discounted-Cost Inventory and Cash Management in Continuous Time, *Operations Research* 26, 620–636.

Cox, John C., 1975, Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working Paper, Stanford University. Reprinted in *Journal of Portfolio Management*, 23 (1996), 15-17.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Davydov, Dmitry and Vadim Linetsky, 2001, Pricing and Hedging Path-Dependent Options under the CEV Process, *Management Science* 47, 949–965.

Dennis, Patrick and Stewart Mayhew, 2002, Risk-Neutral Skewness: Evidence from Stock Options, *Journal of Financial and Quantitative Analysis* 37, 471–493.

Dias, José Carlos and João Pedro Nunes, 2011, Pricing Real Options under the Constant Elasticity of Variance Diffusion, *Journal of Futures Markets* 31, 230–250.

Dias, José Carlos and Mark B. Shackleton, 2011, Hysteresis Effects under CIR Interest Rates, *European Journal of Operational Research* 211, 594–600.

Dixit, Avinash K., 1989a, Entry and Exit Decisions under Uncertainty, *Journal of Political Economy* 97, 620–638.

Dixit, Avinash K., 1991, A Simplified Treatment of the Theory of Optimal Regulation of Brownian Motion, *Journal of Economic Dynamics and Control* 15, 657–673.

Dumas, Bernard, 1991, Super Contact and Related Optimality Conditions, *Journal of Economic Dynamics and Control* 15, 675–685.

Emanuel, David C. and James D. MacBeth, 1982, Further Results on the Constant Elasticity of Variance Call Option Pricing Model, *Journal of Financial and Quantitative Analysis* 17, 533–554.

Geman, Hélyette and Yih Fong Shih, 2009, Modeling Commodity Prices Under the CEV Model, *Journal of Alternative Investments* 11, 65–84.

Glasserman, Paul, 2004, *Monte Carlo Methods in Financial Engineering,* (Springer, New York).

Harrison, J. Michael, Thomas M. Sellke, and Allison J. Taylor, 1983, Impulse Control of Brownian Motion, *Mathematics of Operations Research* 8, 454–466.

Insley, Margaret, 2002, A Real Options Approach to the Valuation of a Forestry Investment, *Journal of Environmental Economics and Management* 44, 471–492.

Jackwerth, Jens Carsten and Mark Rubinstein, 2001, Recovering Stochastic Processes from Option Prices. Working Paper, London Business School.

Jeanblanc, Monique, Marc Yor, and Marc Chesney, 2009, *Mathematical Methods for Financial Markets* (Springer, New York).

Jorgenson, Dale W., 1963, Capital Theory and Investment Behavior, *American Economic Review* 53, 247–259.

Kahl, Christian and Peter Jäckel, 2006, Fast Strong Approximation Monte Carlo Schemes for Stochastic Volatility Models, *Quantitative Finance* 6, 513–536.

Karatzas, Ioannis and Steven E. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, 2nd ed. (Springer, New York).

Keswani, Aneel and Mark B. Shackleton, 2006, How Real Option Disinvestment Flexibility Augments Project NPV, *European Journal of Operational Research* 168, 240–252.

Lee, Youah and Eunnyeong Heo, 2008, Analysis of Crude Oil Price Dynamics Using CKLS Stochastic Models. Working Paper, Seoul National University.

Lord, Roger, Remmert Koekkoek, and Dick Van Dijk, 2010, A Comparison of Biased Simulation Schemes for Stochastic Volatility Models, *Quantitative Finance* 10, 177–194.

Majd, Saman and Robert S. Pindyck, 1987, Time to Build, Option Value, and Investment Decisions, *Journal of Financial Economics* 18, 7–27.

McDonald, Robert L. and Daniel R. Siegel, 1984, Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note, *Journal of Finance* 39, 261–265.

McKean, Jr., Henry P., 1965, Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem of Mathematical Economics, *Industrial Management Review* 6, 32–39.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, Robert C., 1973b, An Intertemporal Capital Asset Pricing Model, *Econometrica* 41, 867–887.

Milne, Alistair and A. Elizabeth Whalley, 2000, ‘Time to Build, Option Value and Investment Decisions‘: A Comment, *Journal of Financial Economics* 56, 325–332.

Samuelson, Paul A., 1965, Rational Theory of Warrant Pricing, *Industrial Management Review* 6, 13–32.

Sarkar, Sudipto, 2000, On the Investment-Uncertainty Relationship in a Real Options Model, *Journal of Economic Dynamics and Control* 24, 219–225.

Sarkar, Sudipto, 2003, The Effect of Mean Reversion on Investment Under Uncertainty, *Journal of Economic Dynamics and Control* 28, 377–396.

Shackleton, Mark B. and Sigbjørn Sødal, 2005, Smooth Pasting as Rate of Return Equalization, *Economics Letters* 89, 200–206.

Smith, James E. and Kevin F. McCardle, 1999, Options in the Real World: Lessons Learned in Evaluating Oil and Gas Investments, *Operations Research* 47, 1–15.

Tsekrekos, Andrianos E., 2010, The Effect of Mean Reversion on Entry and Exit Decisions Under Uncertainty, *Journal of Economic Dynamics and Control* 34, 725–742.

Chapter 8

Conclusion

This thesis provides important results concerning the valuation of standard European-style options (in four separate articles), and concerning the valuation of real options (in two other articles).

The most important theoretical contribution of the first part of this thesis, chapters 2 and 3, is the derivation of closed-form solutions for computing sensitivity measures of European-style options under the unrestricted CEV model for both $\beta < 2$ and $\beta > 2$, and for the particular case of the absolute diffusion, where the elasticity parameter β of the local volatility is zero. The knowledge of these analytical expressions for determining Greeks allows us to better understand the role, genesis, and relationships between the various Greeks. As we have demonstrated, the computation time needed for computing Greeks with the novel analytical solutions will diminish substantially, which is extremely relevant when one needs to design hedging strategies through time. Finally, the existence of analytical solutions allows that they can be coded in any desired computer language. Another aim of the paper in Chapter 3 is to provide comparative results in terms of speed and accuracy of existing alternative algorithms for computing the noncentral chi-square distribution function to be used for option pricing and hedging under the CEV model. We have concluded that the considered iterative procedures are accurate, but the computational results show that the algorithm proposed by Benton and Krishnamoorthy (2003) is the best choice, since it is the most accurate one and the fastest. We have also concluded that the analytic approximations run quickly but have an accuracy that varies significantly over the considered parameter space.

The second part of the thesis, chapters 4 and 5, is concerned with the CIR model diffusion. The principal theoretical contribution of this part is the derivation of closed-form solutions for determining sensitivity measures of both pure discount and coupon-paying bond options under the CIR model. We have shown that these measures are accurate, easy to implement and computationally very efficient. Furthermore, armed with these formulae, we are able to extend the Bacinello et al. (1996) approach by analyzing, in closed-form, the comparative statics properties of a default-free sinking-fund bond in the CIR model. We have also analyzed the performance in terms of accuracy and computational time of alternative iterative procedures for computing the noncentral chi-square distribution function associate with the calculation of the bond options under the CIR framework. We have concluded, again, that the method proposed by Benton and Krishnamoorthy (2003) is the most efficient.

Finally, in the third part, chapters 6 and 7, which is concerned with the real options literature, we have analyzed the decision of a firm to switch from the idle state to the operating state, and vice-versa, under uncertainty, for a generalized class of one-dimensional diffusions. We start by considering the situation of a firm that can invest a lump-sum cost at any time and receive a perpetuity (a project) with constant cash flow rate, where the perpetuity value is stochastic because the interest rate used to discount the perpetual flows is assumed to follow a CIR diffusion process. We have analyzed alternative methods proposed to compute the options to invest and divest from a project in a CIR economy, as well as different methods of determining CIR perpetuities. We have concluded that the solution proposed by Dias and Shackleton (2011) is the easiest to apply and turns the economic hysteresis problem simpler to analyze and understand. From the examination of the alternative methods to compute the perpetuities, we have concluded that the Method D_2 performs better in terms of accuracy, and the Method D_1 is the most efficient in terms of computation time, both methods proposed by Delbaen (1993). Next, we analyzed how output price uncertainty and costly reversibility affect the optimal entry and exit policy of a competitive price-taking firm, and how the hysteretic band is affected by the choice of the stochastic process, namely, by the CEV and mean-reverting CEV processes. We have concluded that the scalar parameter (β in CEV model and γ in mean-reverting CEV model) and the hysteretic band have a direct relation. The range of inaction also has a direct relation with the initial instantaneous volatility. We have also analyzed the *ex ante* probabilities of entry and exit for an idle and an active firm, respectively. Here, and for the CEV process, we have concluded that there is an inverse relation between *ex ante* probabilities of entry and exit and the scale parameter, while for the mean-reverting CEV, there is no monotonic relation between the *ex ante* probabilities and the scale parameter.

Global References

Abadie, Luis M. and José M. Chamorro, 2008, Valuing Flexibility: The Case of an Integrated Gasification Combined Cycle Power Plant, *Energy Economics* 30, 1850–1881.

Abel, Andrew B. and Janice C. Eberly, 1996, Optimal Investment with Costly Reversibility, *Review of Economic Studies* 63, 581–593.

Abel, Andrew B., Avinash K. Dixit, Janice C. Eberly, and Robert S. Pindyck, 1996, Options, the Value of Capital, and Investment, *Quarterly Journal of Economics* 111, 753–777.

Abramowitz, Milton and Irene A. Stegun, 1972, *Handbook of Mathematical Functions* (Dover, New York).

Akerlof, George A., 1970, The Market for “Lemons”: Quality Uncertainty and the Market Mechanism, *Quarterly Journal of Economics* 84, 488–500.

Allegretto, Walter, Yanping Lin, and Hongtao Yang, 2003, Numerical Pricing of American Put Options on Zero-Coupon Bonds, *Applied Numerical Mathematics* 46, 113–134.

Alvarez, Luis H. R., 2001, On the Form and Risk-Sensitivity of Zero Coupon Bonds for a Class of Interest Rate Models, *Insurance: Mathematics and Economics* 28, 83–90.

Alvarez, Luis H. R., 2011, Optimal Capital Accumulation under Price Uncertainty and Costly Reversibility, *Journal of Economic Dynamics and Control* 35, 1769–1788.

Andersen, Leif B. G. and Vladimir V. Piterbarg, 2007, Moment Explosions in Stochastic Volatility Models, *Finance and Stochastics* 11, 29–50.

Arrow, Kenneth J., 1968, Optimal Capital Policy with Irreversible Investment, in J. N. Wolfe, ed.: *Value, Capital, and Growth: Papers in Honour of Sir John Hicks* (Edinburgh University Press, Edinburgh), 1–19.

Ağca, Senay and Don M. Chance, 2003, Speed and Accuracy Comparison of Bivariate Normal Distribution Approximations for Option Pricing, *Journal of Computational Finance* 6, 61–96.

Bacinello, Anna Rita and Fulvio Ortu, 1999, Arbitrage Valuation and Bounds for Sinking-Fund Bonds with Multiple Sinking-Fund Dates, *Applied Mathematical Finance* 6, 293–312.

Bacinello, Anna Rita, Fulvio Ortu, and Patrizia Stucchi, 1996, Valuation of Sinking-Fund Bonds in the Vasicek and CIR Frameworks, *Applied Mathematical Finance* 3, 269–294.

Bajo, Emanuele and Massimiliano Barbi, 2010, The Risk-Shifting Effect and the Value of a Warrant, *Quantitative Finance* 10, 1203–1213.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical Performance of Alternative Option Pricing Models, *Journal of Finance* 52, 2003–2049.

Bar-Ilan, Avner and William C. Strange, 1996, Investment Lags, *American Economic Review* 86, 610–622.

Beaglehole, David R. and Mark S. Tenney, 1991, General Solutions of Some Interest Rate-Contingent Claim Pricing Equations, *Journal of Fixed Income* 1, 69–83.

Bekaert, Geert and Guojun Wu, 2000, Asymetric Volatility and Risk in Equity Markets, *Review of Financial Studies* 13, 1–42.

Ben-Ameur, Hatem, Michèle Breton, Lotfi Karoui, and Pierre L'Ecuyer, 2007, A Dynamic Programming Approch for Pricing Options Embedded in Bonds, *Journal of Economic Dynamics and Control* 31, 2212–2233.

Benton, Denise and K. Krishnamoorthy, 2003, Computing Discrete Mixtures of Continuous Distributions: Noncentral Chisquare, Noncentral *t* and the Distribution of the Square of the Sample Multiple Correlation Coefficient, *Computational Statistics and Data Analysis* 43, 249–267.

Bergman, Yaacov Z., Bruce D. Grundy, and Zvi Wiener, 1996, General Properties of Option Prices, *Journal of Finance* 51, 1573–1610.

Bhattacharya, Sudipto, 1978, Project Valuation with Mean-Reverting Cash Flow Streams, *Journal of Finance* 33, 1317–1331.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Breeden, Douglas T. and Robert H. Litzenberger, 1978, Prices of State-Contingent Claims Implicit in Option Prices, *Journal of Business* 51, 621–651.

Brennan, Michael J. and Eduardo S. Schwartz, 1980, Analyzing Convertible Bonds, *Journal of Financial and Quantitative Analysis* 15, 907–929.

Brigo, Damiano and Fabio Mercurio, 2001, A Deterministic-Shift Extension of Analytically-Tractable and Time-Homogeneous Short-Rate Models, *Finance and Stochastics* 5, 369–387.

Broadie, Mark and Jérôme Detemple, 1996, American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods, *Review of Financial Studies* 9, 1211–1250.

Brown, Roger H. and Stephen M. Schaefer, 1994, The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model, *Journal of Financial Economics* 35, 3–42.

Brown, Stephen J. and Philip H. Dybvig, 1986, The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates, *Journal of Finance* 41, 617–630.

Carmona, Julio and Angel León, 2007, Investment Option under CIR Interest Rates, *Finance Research Letters* 4, 242–253.

Carr, Peter, 2001, Deriving Derivatives of Derivative Securities, *Journal of Computational Finance* 4, 5–29.

Carr, Peter and Vadim Linetsky, 2006, A Jump to Default Extended CEV Model: An Application of Bessel Processes, *Finance and Stochastics* 10, 303–330.

Chan, K. C., G. Andrew Karolyi, Francis A. Longstaff, and Anthony B. Sanders, 1992, An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *Journal of Finance* 47, 1209–1227.

Chen, Ren-Raw and Louis Scott, 1992, Pricing Interest Rate Options in a Two-Factor Cox-Ingersoll-Ross Model of the Term Structure, *Review of Financial Studies* 5, 613–636.

Chen, Ren-Raw and Louis Scott, 1993, Maximum Likelihood Estimators for a Multifactor Equilibrium Model of the Term Structure of Interest Rates, *Journal of Fixed Income* 3, 14–31.

Chen, Ren-Raw and Louis Scott, 1995, Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure, *Journal of Derivatives* 3, 53–72.

Chen, Ren-Raw and Louis Scott, 2003, Multi-Factor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests from a Kalman Filter Model, *Journal of Real Estate Finance and Economics* 27, 143–172.

Choi, Jin W. and Francis A. Longstaff, 1985, Pricing Options on Agricultural Futures: An Application of the Constant Elasticity of Variance Option Pricing Model, *Journal of Futures Markets* 5, 247–258.

Chung, San-Lin and Pai-Ta Shih, 2009, Static Hedging and Pricing American Options, *Journal of Banking and Finance* 33, 2140–2149.

Cohen, Jonathan D., 1988, Noncentral Chi-Square: Some Observations on Recurrence, *American Statistician* 42, 120–122.

Constantinides, George M. and Scott F. Richard, 1978, Existence of Optimal Simple Policies for Discounted-Cost Inventory and Cash Management in Continuous Time, *Operations Research* 26, 620–636.

Cox, John C., 1975, Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working Paper, Stanford University. Reprinted in *Journal of Portfolio Management*, 23 (1996), 15-17.

Cox, John C. and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics* 3, 145–166.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1979, Duration and the Measurement of Basis Risk, *Journal of Business* 52, 51–61.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–408.

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229–263.

Davydov, Dmitry and Vadim Linetsky, 2001, Pricing and Hedging Path-Dependent Options under the CEV Process, *Management Science* 47, 949–965.

Delbaen, Freddy, 1993, Consols in the CIR Model, *Mathematical Finance* 3, 125–134.

Dennis, Patrick and Stewart Mayhew, 2002, Risk-Neutral Skewness: Evidence from Stock Options, *Journal of Financial and Quantitative Analysis* 37, 471–493.

Dias, J. C., "Essays in real options models under interest rate uncertainty." PhD dissertation, ISCTE Business School 2006.

Dias, José Carlos and João Pedro Nunes, 2011, Pricing Real Options under the Constant Elasticity of Variance Diffusion, *Journal of Futures Markets* 31, 230–250.

Dias, José Carlos and Mark B. Shackleton, 2011, Hysteresis Effects under CIR Interest Rates, *European Journal of Operational Research* 211, 594–600.

Ding, Cherng G., 1992, Algorithm AS 275: Computing the Non-Central χ^2 Distribution Function, *Applied Statistics* 41, 478–482.

Dixit, Avinash K., 1989a, Entry and Exit Decisions under Uncertainty, *Journal of Political Economy* 97, 620–638.

Dixit, Avinash K., 1991, A Simplified Treatment of the Theory of Optimal Regulation of Brownian Motion, *Journal of Economic Dynamics and Control* 15, 657–673.

Dixit, Avinash K. and Robert S. Pindyck, 1994, *Investment under Uncertainty* (Princeton University Press, Princeton, New Jersey).

Dumas, Bernard, 1991, Super Contact and Related Optimality Conditions, *Journal of Economic Dynamics and Control* 15, 675–685.

Dyrting, S., 2004, Evaluating the Noncentral Chi-Square Distribution for the Cox-Ingersoll-Ross Process, *Computational Economics* 24, 35–50.

Emanuel, David C. and James D. MacBeth, 1982, Further Results on the Constant Elasticity of Variance Call Option Pricing Model, *Journal of Financial and Quantitative Analysis* 17, 533–554.

Farebrother, R. W., 1987, Algorithm AS 231: The Distribution of a Noncentral χ^2 Variable with Nonnegative Degrees of Freedom, *Applied Statistics* 36, 402–405.

Feller, William, 1951, Two Singular Diffusion Problems, *Annals of Mathematics* 54, 173–182.

Fraser, D. A. S., J. Wu, and A. C. M. Wong, 1998, An Approximation for the Noncentral Chi-Squared Distribution, *Communications in Statistics - Simulation and Computation* 27, 275–287.

Garman, Mark, 1992, Charm School, *Risk* 5, 53–56.

Geman, Hélyette and Marc Yor, 1993, Bessel Processes, Asian Options, and Perpetuities, *Mathematical Finance* 3, 349–375.

Geman, Hélyette and Yih Fong Shih, 2009, Modeling Commodity Prices Under the CEV Model, *Journal of Alternative Investments* 11, 65–84.

Glasserman, Paul, 2004, *Monte Carlo Methods in Financial Engineering,* (Springer, New York).

Gorovoi, Viatcheslav and Vadim Linetsky, 2004, Black's Model of Interest Rates as Options, Eigenfunction Expansions and Japanese Interest Rates, *Mathematical Finance* 14, 49–78.

Grünbichler, Andreas and Francis A. Longstaff, 1996, Valuing Futures and Options on Volatility, *Journal of Banking and Finance* 20, 985–1001.

Harrison, J. Michael, Thomas M. Sellke, and Allison J. Taylor, 1983, Impulse Control of Brownian Motion, *Mathematics of Operations Research* 8, 454–466.

Haug, Espen Gaarder, 2006, *The Complete Guide to Option Pricing Formulas*, 2nd ed. (McGraw-Hill, New York, NY).

Hull, John and Alan White, 1990, Pricing Interest-Rate-Derivative Securities, *Review of Financial Studies* 3, 573–592.

Hull, John C., 2008, *Options, Futures, and Other Derivatives*, 7th ed. (Prentice Hall, Upper Saddle River, NJ).

Insley, Margaret, 2002, A Real Options Approach to the Valuation of a Forestry Investment, *Journal of Environmental Economics and Management* 44, 471–492.

Jackwerth, Jens Carsten and Mark Rubinstein, 1996, Recovering Probability Distributions from Option Prices, *Journal of Finance* 51, 1611–1631.

Jackwerth, Jens Carsten and Mark Rubinstein, 2001, Recovering Stochastic Processes from Option Prices. Working Paper, London Business School.

Jamshidian, Farshid, 1989, An Exact Bond Option Formula, *Journal of Finance* 44, 205–209.

Jamshidian, Farshid, 1995, A Simple Class of Square-Root Interest-Rate Models, *Applied Mathematical Finance* 2, 61–72.

Jeanblanc, Monique, Marc Yor, and Marc Chesney, 2009, *Mathematical Methods for Financial Markets* (Springer, New York).

Johnson, Norman L., Samuel Kotz, and N. Balakrishnan, 1995, *Continuous Univariate Distributions*, Vol. 2, 2nd ed. (John Wiley & Sons, New York, NY).

Jorgenson, Dale W., 1963, Capital Theory and Investment Behavior, *American Economic Review* 53, 247–259.

Kahl, Christian and Peter Jäckel, 2006, Fast Strong Approximation Monte Carlo Schemes for Stochastic Volatility Models, *Quantitative Finance* 6, 513–536.

Karatzas, Ioannis and Steven E. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, 2nd ed. (Springer, New York).

Kawai, Reiichiro and Atsushi Takeuchi, 2011, Greeks Formulas for an Asset Price Model with Gamma Processes, *Mathematical Finance* 21, 723–742.

Keswani, Aneel and Mark B. Shackleton, 2006, How Real Option Disinvestment Flexibility Augments Project NPV, *European Journal of Operational Research* 168, 240–252.

Knüsel, L. and B. Bablok, 1996, Computation of the Noncentral Gamma Distribution, *SIAM Journal of Scientific Computing* 17, 1224–1231.

Larguinho, Manuela, José Carlos Dias, and Carlos A. Braumann, 2013, On the Computation of Option Prices and Greeks under the CEV Model, *Quantitative Finance* 13, 907–917.

Lee, Youah and Eunnyeong Heo, 2008, Analysis of Crude Oil Price Dynamics Using CKLS Stochastic Models. Working Paper, Seoul National University.

Longstaff, Francis A., 1993, The Valuation of Options on Coupon Bonds, *Journal of Banking and Finance* 17, 27–42.

Longstaff, Francis A. and Eduardo S. Schwartz, 1992, Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model, *Journal of Finance* 47, 1259–1282.

Lord, Roger, Remmert Koekkoek, and Dick Van Dijk, 2010, A Comparison of Biased Simulation Schemes for Stochastic Volatility Models, *Quantitative Finance* 10, 177–194.

Maghsoodi, Yoosef, 1996, Solution of the Extended CIR Term Structure and Bond Option Valuation, *Mathematical Finance* 6, 89–109.

Majd, Saman and Robert S. Pindyck, 1987, Time to Build, Option Value, and Investment Decisions, *Journal of Financial Economics* 18, 7–27.

Mallier, R. and G. Alobaidi, 2004, Interest Rate Swaps under CIR, *Journal of Computational and Applied Mathematics* 164-165, 543–554.

McDonald, Robert L. and Daniel R. Siegel, 1984, Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note, *Journal of Finance* 39, 261–265.

McKean, Jr., Henry P., 1965, Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem of Mathematical Economics, *Industrial Management Review* 6, 32–39.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, Robert C., 1973b, An Intertemporal Capital Asset Pricing Model, *Econometrica* 41, 867–887.

Milne, Alistair and A. Elizabeth Whalley, 2000, 'Time to Build, Option Value and Investment Decisions': A Comment, *Journal of Financial Economics* 56, 325–332.

Nunes, João Pedro, 2009, Pricing American Options under the Constant Elasticity of Variance Model and Subject to Bankruptcy, *Journal of Financial and Quantitative Analysis* 44, 1231–1263. Correction: Nunes, J. P., 2012, American-style Puts under the

JDCEV Model: A Correction. JFQA web site at http://depts.washington.edu/jfqa/ in Unpublished Appendixes, Notes, Comments, and Corrections.

Nunes, João Pedro and José Carlos Dias, 2012, The Valuation of Double Barrier Options under Multifactor Pricing Models. Working Paper, BRU-UNIDE and ISCTE-IUL Business School.

Pearson, Neil D. and Tong-Sheng Sun, 1994, Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model, *Journal of Finance* 49, 1279–1304.

Pelsser, Antoon and Ton Vorst, 1994, The Binomial Model and the Greeks, *Journal of Derivatives* 1, 45–49.

Penev, Spiridon and Tenko Raykov, 2000, A Wiener Germ Approximation of the Noncentral Chi Square Distribution and of its Quantiles, *Computational Statistics* 15, 219–228.

Posten, Harry O., 1989, An Effective Algorithm for the Noncentral Chi-Squared Distribution Function, *American Statistician* 43, 261–263.

Ruas, J.P., J.C. Dias, and J.P.V. Nunes, 2013, Pricing and static hedging of American-style options under the jump to default extended CEV model, *Journal of Banking and Finance* 37, 4059–4072.

Samuelson, Paul A., 1965, Rational Theory of Warrant Pricing, *Industrial Management Review* 6, 13–32.

Sankaran, Munuswamy, 1963, Approximations to the Non-Central Chi-Square Distribution, *Biometrika* 50, 199–204.

Sarkar, Sudipto, 2000, On the Investment-Uncertainty Relationship in a Real Options Model, *Journal of Economic Dynamics and Control* 24, 219–225.

Sarkar, Sudipto, 2003, The Effect of Mean Reversion on Investment Under Uncertainty, *Journal of Economic Dynamics and Control* 28, 377–396.

Schroder, Mark, 1989, Computing the Constant Elasticity of Variance Option Pricing Formula, *Journal of Finance* 44, 211–219.

Shackleton, Mark B. and Sigbjørn Sødal, 2005, Smooth Pasting as Rate of Return Equalization, *Economics Letters* 89, 200–206.

Shadwick, Brad A. and William F. Shadwick, 2002, On the Computation of Option Prices and Sensitivities in the Black-Scholes-Merton Model, *Quantitative Finance* 2, 158–166.

Shaw, William, 1998, *Modelling Financial Derivatives with Mathematica* (Cambridge University Press, Cambridge, UK).

ShuJin, Li and Li ShengHong, 2006, Pricing American Interest Rate Option on Zero-Coupon Bond Numerically, *Applied Mathematics and Computation* 175, 834–850.

Smith, James E. and Kevin F. McCardle, 1999, Options in the Real World: Lessons Learned in Evaluating Oil and Gas Investments, *Operations Research* 47, 1–15.

Tsekrekos, Andrianos E., 2010, The Effect of Mean Reversion on Entry and Exit Decisions Under Uncertainty, *Journal of Economic Dynamics and Control* 34, 725–742.

Vasicek, Oldrich, 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* 5, 177–188.

Wei, Jason Z., 1997, A Simple Approach to Bond Option Pricing, *Journal of Futures Markets* 17, 131–160.

Zhou, Hong Jun, Ka Fai Cedric Yiu, and Leong Kwan Li, 2011, Evaluating American Put Options on Zero-Coupon Bonds by a Penalty Method, *Journal of Computational and Applied Mathematics* 235, 3921–3931.