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Geometric Conditions for Regularity in a Time-Minimum Problem with Constant Dynamics*

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Continuing the earlier research [16] on local well-posedness of a time-minimum problem associated to a closed target set $C \subset H$ (H is a Hilbert space) and a convex constant dynamics $F \subset H$ we study the Lipschitz (or, in general, Hölder) regularity of the (unique) point $\pi_C^F(x)$ in C achieved from x for a minimal time. As a consequence, smoothness of the value function is proved, and an explicit formula for its derivative is given.

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1. Introduction

Let us consider a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. For a nonempty closed convex bounded set $F \subset H$ (further called *dynamics*) containing the origin in its interior and for some nonempty closed *target set* $C \subset H$ we denote by $\mathfrak{T}_C^F(x)$, $x \in H$, the value function in the following time optimal control problem

$$\min\left\{T > 0 : \exists x(\cdot),\ x(T) \in C,\ x(0) = x, \text{ and } \dot{x}(t) \in F \text{ a.e. in } [0, T]\right\}. \tag{1}$$

The set of terminal points $x(T)$ for all functions $x(\cdot)$, which are minimizers in (1) (or, in other words, the set of points in C attainable from x in a minimal time), is said to be the *time-minimum projection* of x onto C (with respect to F) and is

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denoted by $\pi_C^F(x)$. We use the same name and notation also for the unique element of $\pi_C^F(x)$ if it is a singleton. Due to the convexity of F the problem (1) is equivalent to the mathematical programming problem

$$\min\left\{\rho_F(y-x) : y \in C\right\},$$

where $\rho_F(\cdot)$ is the *Minkowski functional* (*gauge function*) associated to F,

$$\rho_F(\xi) := \inf\left\{\lambda > 0 : \xi \in \lambda F\right\}, \ \ \xi \in H. \tag{2}$$

Therefore,

$$\mathfrak{T}_C^F(x) = \inf_{y \in C} \rho_F(y-x)$$

and

$$\pi_C^F(x) = \left\{y \in C : \rho_F(y-x) = \mathfrak{T}_C^F(x)\right\}.$$

If $F = \overline{B}$ is the closed unit ball in H then $\rho_F(\cdot)$, $\mathfrak{T}_C^F(x)$ and $\pi_C^F(x)$ are reduced to the norm $\|\cdot\|$, to the distance $\mathrm{d}_C(x)$ from x to the set C and to the usual *metric projection* $\pi_C(x)$ of x onto C, respectively. Observe that unlike this particular case we do not suppose the set F to be either symmetric or smooth, or strictly convex.

It is well known that for each convex C the projection $\pi_C(x)$ is a singleton continuously depending on $x \in H$ (it is even Lipschitzean with the Lipschitz constant 1), and the distance $\mathrm{d}_C(\cdot)$ is continuously Fréchet differentiable outside C. Some generalization of the convexity leads to validity of the properties above not on the whole space but only on an open neighbourhood of the target set. Namely, in past many authors (see, e.g., [15, 25, 4, 5, 24, 9, 21, 11, 1] and others) studied the class of the so-called *φ-convex* or *proximally smooth sets* (the exact definition will be given in the sequel). Roughly speaking, these sets could be characterized by the following geometric property: given $\overline{x} \in \partial C$, for any $x, y \in C$ near $\overline{x}$ the convex combination $\lambda x + (1-\lambda) y$ (not necessarily belonging to C) is distant from C not more than of the order $O\left(\|x-y\|^2\right)$, and this proximity is controlled by some (continuous) function $\varphi(\cdot)$. We know that a closed set $C \subset H$ is φ-convex iff for some open neighbourhood $\mathcal{U} \supset C$ depending on $\varphi(\cdot)$ the map $\pi_C(\cdot)$ is well defined, single-valued and (Lipschitz) continuous on $\mathcal{U}$; or, equivalently, the distance function $\mathrm{d}_C(\cdot)$ is Fréchet differentiable on $\mathcal{U}$ (outside of C) that justifies the other name given to these sets. Moreover, the gradient $\nabla \mathrm{d}_C(\cdot)$ turns out to be Lipschitzean near each point $x \in \mathcal{U} \diagdown C$ with Lipschitz constant tending to infinity as x verges on the boundary of $\mathcal{U}$.

Notice that the continuous differentiability of the distance function is strictly related to the well-posedness (i.e., existence, uniqueness and continuity) of the metric projection (see, e.g., [9, Proposition 3.6] and [18, Section 1.3.3]). As follows from Theorem 22 [20] formulated in a general Banach space setting these properties are equivalent in a Hilbert space. Therefore, if the well-posedness of the projection is violated in a neighbourhood of some point $x \in H$ then there is no differentiability of the distance, and one is led to study various types of subdifferentials of $\mathrm{d}_C(\cdot)$ at x and the relationships between them. In this direction one usually distinguishes two cases: $x \in \partial C$ and $x \notin C$. In the first case (the distance, certainly, is never differentiable at that point) the relationships between (proximal, Fréchet, limiting or Clarke) subdifferential and the respective normal cone to C at the point x are well known (see [2, 3] and the bibliography therein). If $x \notin C$ instead then there are two kinds of formulas. Some of them relate subdifferentials of $\mathrm{d}_C(\cdot)$ to the normal cones to the sublevel set $C(r) := \{y \in H : \mathrm{d}_C(y) \leq r\}$ where $r := \mathrm{d}_C(x)$ (see, e.g., [9, Theorem 3.4] and [2, Theorem 3.6]), while others are written in terms of the normal cones to the set C itself at a metric projection $\overline{x} \in \pi_C(x)$ (if any). Taking into account these formulas the questions of regularity of the distance function (in the sense of coincidence of various subdifferentials) were studied. For details we refer to the book [18] and to the extensive bibliography therein.

During the last ten years the majority of the results above were generalized to an arbitrary constant convex dynamics F (see, e.g., [26, 13, 14, 27, 19, 12]). In the simplest case of the convex target set C the subdifferential of $\mathfrak{T}_C^F(\cdot)$ at $x \notin C$ (in the sense of Convex Analysis) was represented in terms of the so-called *separating cone* (see [14, Theorem 4.2]), which involves both normal cones (to C at $\overline{x} \in \pi_C^F(x)$ and to F at some associated point). In the same paper, considering an arbitrary closed C, the authors expressed both proximal and Fréchet subdifferentials of the value function at a point $x \notin C$ through the respective normal cones to the "*enlargement set*" $\left\{y \in H : \mathfrak{T}_C^F(y) \leq \mathfrak{T}_C^F(x)\right\}$ similarly as for the distance (see [14, Theorem 3.1]), while in [27] these formulas were generalized to Banach spaces, and the similar representation for the Clarke subdifferential was given. The latter work contains also various formulas for subdifferentials of the minimum time function at $x \in \partial C$. In [19] the limiting subdifferentials are considered as well.

If $x \notin C$ and one assumes existence of a point $\overline{x} \in \pi_C^F(x)$ then it is natural to link a subdifferential of $\mathfrak{T}_C^F(\cdot)$ at x to the respective normal cone to the target set at $\overline{x}$. The case when C is convex suggests that such relationships should contain the normal cone to the convex set F at some associated point (but this is not of the essence for the distance because in the case $F = \overline{B}$ there is a unique exterior normal to F at a boundary point coinciding with the radius-vector). However, without any supplementary assumptions they admit the form of the one-sided inclusions only (see, e.g., [14, Theorem 3.3]). In order to have equalities one should suppose the set C or F to be regular in some sense (as, e.g., in Theorems 5.10 and 5.12 from [14]). On the other hand, such equalities are the main tools in studying the smoothness of the value function. Notice that in general the function $\mathfrak{T}_C^F(\cdot)$ can be very nice near a point $x \notin C$ when $\pi_C^F(x)$ is not a singleton (or even when it is the empty set, see [12, Example 4.3]), which never happens in the case of distance. Nevertheless, here we consider a more regular situation imposing conditions, which guarantee the well-posedness of the time-minimum projection as well (similarly as in the case $F = \overline{B}$). Moreover, we are interested in the validity of the latter property not at a single point out of C but on a neighbourhood of the target set (compare with the equivalent characterizations of φ-convex sets).

The global well-posedness conditions have been proposed in [14] as the generalization of the respective hypothesis for the metric projection. Namely, the authors required the φ-convexity of C (with a constant function $\varphi(x) \equiv \varphi$) and some kind of uniform strict convexity of the set F controlled by a parameter $\gamma > 0$. Under these assumptions the function $x \mapsto \pi_C^F(x)$ is well defined, single-valued and Hölder continuous with the exponent $1/2$ on an open neighbourhood of C determinable through the balance between two constants φ and γ (see [14, Theorems 5.6 and 5.7]). Later, in [16] we localized these (uniform) hypotheses, assuming the set C to be φ-convex with an arbitrary continuous function $\varphi(\cdot)$ and writing the (local) rotundity condition for F in terms of some kind of curvature. Besides that, we proposed an alternative (first order) condition based on mutual regularity of the proximal normals to C (near a fixed point) and of the duality mapping associated to F. As a result, an open neighbourhood $\mathcal{U} \supset C$, on which the (single-valued) time-minimum projection $\pi_C^F(\cdot)$ is well defined and enough regular, can be constructed by employment of both type of local conditions (see [16, Theorem 6.1]). Nevertheless, in contrast to the case $F = \overline{B}$, the Fréchet differentiability of $\mathfrak{T}_C^F(\cdot)$ does not follow immediately from this well-posedness (in particular, from our hypotheses), but it needs some more regularity assumptions on the target set or on the dynamics such as those given, for instance, in Theorem 5.14 [14].

Our paper continues research [14, 16] concerning the well-posedness and the regularity of the time-minimum projection in a neighbourhood of the target set. The main hypotheses (A) and (B) we use, studied in detail in [16], are formulated in Section 2, while the goal of Section 3 is to show how one can increase the degree of the Hölderianity of $\pi_C^F(\cdot)$ by increasing regularity of one of the sets C or F. It is interesting to observe that under the assumption (A) we always have the Lipschitz continuity of the projection, while under (B) we can obtain gratis (without any supplementary hypothesis on C and F) the Hölder continuity with the exponent $1/2$ only (as in [14, Theorem 5.7(i)]). However, if the target set and the dynamics are "compatible" near a given point in a Hölderian way (in particular, if one of these sets admits a Hölder continuous normal vector) then the Hölder regularity of $\pi_C^F(\cdot)$ can be essentially improved (see Theorems 3.5 and 3.8 below). In Section 4 we consider a special case when the target set is strictly convex in a neighbourhood of a given point. Then (in Section 5) we obtain first a general result on the Clarke regularity of the value function $\mathfrak{T}_C^F(\cdot)$ (see Proposition 5.1), which is used further for proving the continuous differentiability of $\mathfrak{T}_C^F(\cdot)$ near the target C. The formulas for the gradient $\nabla\mathfrak{T}_C^F(x)$ under various assumptions are also given, and the Hölder regularity of $\nabla\mathfrak{T}_C^F(\cdot)$ is studied. Finally, in the last section we set two examples, which illustrate the obtained results.

2. Preliminaries. Basic hypotheses

Throughout the paper we consider a pair of subsets (F, C) of a Hilbert space H assuming F to be closed convex bounded with $0 \in \operatorname{int} F$ ("int" means *interior* of F), and C to be just nonempty and closed. Let us introduce some concepts relating to both sets, which will be used in the sequel.

We denote by F^0 the *polar set*, i.e.,

$$F^0 := \{\xi^* \in H : \langle \xi, \xi^* \rangle \leq 1 \ \ \forall \xi \in F\},$$

and by $\mathfrak{J}_F : \partial F^0 \to \partial F$ the *duality mapping* that associates to each $\xi^* \in \partial F^0$ the set

$$\mathfrak{J}_F(\xi^*) := \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\}.$$

In other words, $\mathfrak{J}_F(\xi^*)$ is the set of all points ξ from the *boundary* ∂F, in which the functional $\eta \mapsto \langle \xi^*, \eta \rangle$ *supports* F. We say that (ξ, ξ^*) is a *dual pair* if $\xi^* \in \partial F^0$ and $\xi \in \mathfrak{J}_F(\xi^*)$. The *Minkowski functional* (*gauge function*) $\rho_F(\xi)$ (see (2)) can be also represented through the *support function* to the polar set, namely,

$$\rho_F(\xi) = \sigma_{F^0}(\xi) := \sup\left\{\langle \xi, \xi^* \rangle : \xi^* \in F^0\right\}, \tag{3}$$

and satisfies the inequalities

$$\frac{1}{\|F\|}\|\xi\| \le \rho_F(\xi) \le \left\|F^0\right\| \|\xi\|, \quad \xi \in H, \tag{4}$$

where $\|F\| := \sup\{\|\xi\| : \xi \in F\}$. Hence

$$|\rho_F(\xi_1) - \rho_F(\xi_2)| \le \left\|F^0\right\| \|\xi_1 - \xi_2\|, \quad \xi_1, \xi_2 \in H.$$

Let us denote by $\mathbf{N}_F(\xi)$ the *normal cone* to F at $\xi \in \partial F$ and by $\partial\rho_F(\xi)$ the *subdifferential* of the function $\rho_F(\cdot)$ in the sense of Convex Analysis. Notice that $\mathfrak{J}_F(\xi^*)$, $\xi^* \in \partial F^0$, is nothing else than $\partial\rho_{F^0}(\xi^*)$, while the set $\mathbf{N}_F(\xi) \cap \partial F^0$ coincides with the pre-image $\mathfrak{J}_F^{-1}(\xi) = \mathfrak{J}_{F^0}(\xi)$, $\xi \in \partial F$.

Recall some notions studied in detail in [16] that quantitatively characterize the rotundity of convex sets. Given a dual pair (ξ, ξ^*) we introduce the *modulus of strict convexity* (*rotundity*) of F as the real function

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) := \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F,\ \|\xi - \eta\| \ge r\}, \quad r > 0. \tag{5}$$

The set F is said to be *strictly convex* (*rotund*) at the point ξ with respect to (w.r.t.) ξ^* if $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) > 0$ for all $r > 0$. Since in this case ξ is the unique element of $\mathfrak{J}_F(\xi^*)$, we could speak just about the strict convexity w.r.t. the vector ξ^*. We keep the same agreement for all the similar concepts introduced below. Given a set $U \subset \partial F^0$, we say that F is *uniformly strictly convex* (*uniformly rotund*) w.r.t. U if

$$\beta_U(r) := \inf\left\{\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) : \xi^* \in U\right\} > 0 \tag{6}$$

for all $r > 0$. If in the definition above U is a neighbourhood of a point $\xi_0^* \in \partial F^0$ then we say that F is uniformly strictly convex w.r.t. ξ_0^*. Observe that there is a strong connection between the rotundity properties of the set F and the smoothness of its polar F^0. Namely (see [16, Proposition 3.3]), F is strictly convex at $\xi \in \partial F$ w.r.t. $\xi^* \in \partial F^0$ if and only if the dual Minkowski functional $\rho_{F^0}(\cdot)$ is Fréchet differentiable at ξ^* with $\nabla\rho_{F^0}(\xi^*) = \xi$. In this case we say also that F^0 is *smooth* at the point ξ^*. To formulate the uniform version of this property let us introduce the so called *modulus of smoothness* of F^0 at $\xi^* \in \partial F^0$ w.r.t. $\xi \in \partial F$,

$$\mathfrak{S}_{F^0}(t, \xi^*, \xi) := \sup\left\{\rho_{F^0}(\xi^* + t\eta^*) - \rho_{F^0}(\xi^*) - t\langle \xi, \eta^* \rangle : \eta^* \in F^0\right\}.$$

Proposition 2.1. *Let U be an arbitrary subset of ∂F^0. Then the following assertions are equivalent:*

(*i*) *the set F is uniformly rotund w.r.t. U;*

(*ii*) *the mapping $\mathfrak{J}_F(\cdot)$ is single-valued on U and uniformly continuous in the following sense:*

$$\sup_{\eta\in\mathfrak{J}_F(\eta^*)} \|\mathfrak{J}_F(\xi^*)-\eta\| \to 0 \quad \text{as } \|\xi^*-\eta^*\|\to 0,\ \xi^*\in U,\ \eta^*\in\partial F^0;$$

(*iii*) $\lim_{t\to 0+}\frac{\mathfrak{S}_{F^0}(t,\xi^*,\xi)}{t}=0$, *and the limit is uniform in $\xi^*\in U$, $\xi\in\mathfrak{J}_F(\xi^*)$.*

Proof. The implication $(i) \Longrightarrow (ii)$ can be shown similarly as in [16, Proposition 3.4]. Indeed, assuming that the second part of the assertion (ii) does not hold (single-valuedness of $\mathfrak{J}_F(\cdot)$ on U follows from (i) immediately) let us take $\varepsilon>0$ and sequences $\{\xi_n^*\}\subset U$, $\{\eta_n^*\}\subset\partial F^0$ such that $\|\xi_n^*-\eta_n^*\|\to 0$ as $n\to\infty$ but $\|\xi_n-\eta_n\|\ge\varepsilon$ for some $\eta_n\in\mathfrak{J}_F(\eta_n^*)$, $n=1,2,\ldots$. Here $\xi_n\in\mathfrak{J}_F(\xi_n^*)$. By definition of the rotundity modulus (5) we obviously have

$$\langle \xi_n-\eta_n,\xi_n^*-\eta_n^*\rangle \ge \widehat{\mathfrak{C}}_F(\varepsilon,\xi_n,\xi_n^*)+\widehat{\mathfrak{C}}_F(\varepsilon,\eta_n,\eta_n^*)\ge\beta_U(\varepsilon)>0,$$

which is a contradiction.

To prove $(ii) \Longrightarrow (iii)$ we follow the same line as [17, Lemma 5.5.9]. Observing that $\frac{\xi^*+t\eta^*}{\rho_{F^0}(\xi^*+t\eta^*)}\to\xi^*$ as $t\to 0+$ uniformly in $\xi^*,\eta^*\in\partial F^0$ and using the uniform continuity of $\mathfrak{J}_F(\cdot)$, for given $\varepsilon>0$ we choose $\delta>0$ such that

$$\|\mathfrak{J}_F(\xi^*)-\zeta\|\le\varepsilon \tag{7}$$

for all $\xi^*\in U$, $\zeta\in\mathfrak{J}_F\left(\frac{\xi^*+t\eta^*}{\rho_{F^0}(\xi^*+t\eta^*)}\right)$, $\eta^*\in\partial F^0$ and $0<t\le\delta$. Let us estimate now the difference $\rho_{F^0}(\xi^*+t\eta^*)-\rho_{F^0}(\xi^*)$. On one hand, by (3) and by definition of the duality mapping we successively have

$$\begin{aligned}
&\rho_{F^0}(\xi^*+t\eta^*)-\rho_{F^0}(\xi^*)\\
&=\rho_{F^0}(\xi^*+t\eta^*)\left[\left\langle\zeta,\frac{\xi^*+t\eta^*}{\rho_{F^0}(\xi^*+t\eta^*)}\right\rangle-\rho_{F^0}\left(\frac{\xi^*}{\rho_{F^0}(\xi^*+t\eta^*)}\right)\right]\\
&\le\rho_{F^0}(\xi^*+t\eta^*)\left\langle\zeta,\frac{\xi^*+t\eta^*}{\rho_{F^0}(\xi^*+t\eta^*)}-\frac{\xi^*}{\rho_{F^0}(\xi^*+t\eta^*)}\right\rangle\\
&=t\langle\zeta,\eta^*\rangle.
\end{aligned}\tag{8}$$

On the other hand,

$$\begin{aligned}
\rho_{F^0}(\xi^*+t\eta^*)-\rho_{F^0}(\xi^*)\ge{}&\langle\mathfrak{J}_F(\xi^*),\xi^*+t\eta^*\rangle-\langle\mathfrak{J}_F(\xi^*),\xi^*\rangle\\
={}&t\langle\mathfrak{J}_F(\xi^*),\eta^*\rangle.
\end{aligned}\tag{9}$$

Setting $\xi:=\mathfrak{J}_F(\xi^*)$, we obtain from (9), (8) and (7) that

$$\begin{aligned}
0&\le\frac{\rho_{F^0}(\xi^*+t\eta^*)-\rho_{F^0}(\xi^*)}{t}-\langle\xi,\eta^*\rangle\\
&\le\|\mathfrak{J}_F(\xi^*)-\zeta\|\,\|\eta^*\|\le\varepsilon\left\|F^0\right\|
\end{aligned}$$

whenever $\eta^* \in \partial F^0$, $\xi^* \in U$ and $0 < t \le \delta$, and (iii) follows.

Finally, in order to prove the implication $(iii) \Longrightarrow (i)$ let us recall the *Lindenstrauss type formula* (see [16, Proposition 4.2])

$$\mathfrak{S}_{F^0}(t, \xi^*, \xi) = \sup\left\{tr - \mathfrak{C}_F^+(r, \xi, \xi^*) : r > 0\right\}, \tag{10}$$

where

$$\mathfrak{C}_F^+(r, \xi, \xi^*) := \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F,\ \rho_F(\eta - \xi) \ge r\}$$

is a slightly modificated rotundity modulus. Since for each $r > 0$

$$\mathfrak{C}_F^+\left(\frac{r}{\|F\|}, \xi, \xi^*\right) \le \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) \le \mathfrak{C}_F^+\left(r\left\|F^0\right\|, \xi, \xi^*\right),$$

in definition of the uniform rotundity (6) the modulus $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*)$ can be obviously substituted by $\mathfrak{C}_F^+(r, \xi, \xi^*)$. Let us fix now $r > 0$ and by (iii) choose $\delta > 0$ such that

$$\frac{\mathfrak{S}_{F^0}(t, \xi^*, \xi)}{t} \le \frac{r}{2}$$

for all $\xi^* \in U$, $\xi \in \mathfrak{J}_F(\xi^*)$ and $0 < t \le \delta$. Taking into account (10) we have

$$r - \frac{\mathfrak{C}_F^+(r, \xi, \xi^*)}{t} \le \frac{r}{2},$$

or, equivalently,

$$\mathfrak{C}_F^+(r, \xi, \xi^*) \ge \frac{r}{2} t,$$

which proves inequality (6). □

From Proposition 2.1 it follows immediately that the uniform rotundity of the set $F \subset H$ w.r.t. $U \subset \partial F^0$ implies the uniform continuity of the *Fréchet derivative* (or *gradient*) $\nabla \rho_{F^0}(\cdot)$ on U. If the latter property takes place then we say that F^0 is *uniformly smooth* on U. In the case when U is a neighbourhood of some point $\xi_0^* \in \partial F^0$ the set F^0 is said to be also uniformly smooth at ξ_0^*. By duality the similar definitions (of smoothness and uniform smoothness) can be certainly applied to the set F itself.

For a dual pair (ξ, ξ^*) the (*square*) *curvature* of F at ξ w.r.t. ξ^* is defined by

$$\hat{\varkappa}_F(\xi, \xi^*) := \frac{1}{\|\xi^*\|} \liminf_{\substack{(r,\eta,\eta^*) \to (0+,\xi,\xi^*) \\ \eta \in \mathfrak{J}_F(\eta^*),\, \eta^* \in \partial F^0}} \frac{\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^2}, \tag{11}$$

and the set F is said to be *strictly convex* (*rotund*) *of second order* (at the point $\xi \in \partial F$) w.r.t. the normal vector $\xi^* \in \partial F^0$ if $\hat{\varkappa}_F(\xi, \xi^*) > 0$. Geometrically, the curvature can be characterized through the so called *curvature radius*

$$\widehat{\mathfrak{R}}_F(\xi, \xi^*) := \|\xi^*\| \limsup_{\substack{(\varepsilon,\eta,\eta^*) \to (0+,\xi,\xi^*) \\ \eta \in \mathfrak{J}_F(\eta^*),\, \eta^* \in \partial F^0}} \inf\left\{r > 0 : F \cap \left(\eta + \varepsilon \overline{B}\right) \subset \eta - r\eta^* + r\left\|\eta^*\right\| \overline{B}\right\}. \tag{12}$$

Namely (see [16, Proposition 3.8]),

$$\widehat{\mathfrak{R}}_F(\xi,\xi^*) = \frac{1}{2\hat{\varkappa}_F(\xi,\xi^*)}.$$

In the past similar concepts of curvature (but without passage to limits as $(\eta,\eta^*) \to (\xi,\xi^*)$) were studied for $\mathcal{C}^1$ manifolds embedded into a Banach space (see, e.g., [23, 6, 7] and the bibliography therein). For our objectives instead such passage in formulas (11) and (12) is very important. In particular, it guarantees lower semicontinuity of the curvature. Moreover, we put these notions into the general duality settings of Convex Analysis (in an arbitrary Hilbert space). For instance, Proposition 4.4 and inequalities (21) from [16] link the curvature (11) with the second order derivative $\nabla^2\rho_{F^0}(\cdot)$. So, we can consider $\hat{\varkappa}_F(\xi,\xi^*)$ as a quantitative characteristics of the polar set F^0 at the point ξ^* as well.

However, in what follows a slightly bigger radius

$$\mathfrak{R}_F(\xi,\xi^*) := \|\xi^*\| \limsup_{\substack{(\eta,\eta^*)\to(\xi,\xi^*)\\ \eta\in\mathfrak{I}_F(\eta^*),\,\eta^*\in\partial F^0}} \inf\left\{r>0 : F\subset \eta - r\eta^* + r\|\eta^*\|\,\overline{B}\right\} \tag{13}$$

is more relevant. Since it depends not only on the local structure of the boundary near ξ but also on the size of the set F, the numbers (13) and $\varkappa_F(\xi,\xi^*) := \frac{1}{2\mathfrak{R}_F(\xi,\xi^*)}$ are sometimes called the *scaled curvature radius* and the *scaled curvature*, respectively. Notice that $\mathfrak{R}_F(\xi,\xi^*)$ can not be too small unlike the "true" curvature radius $\widehat{\mathfrak{R}}_F(\xi,\xi^*)$. Namely,

$$\mathfrak{R}_F(\xi,\xi^*) \geq \|\xi^*\|\,\mathfrak{r}_F > 0, \tag{14}$$

where $\mathfrak{r}_F$ is the *Chebyshev radius* of the convex set F.

Passing now to an arbitrary closed set C (*target*) observe that there are various notions of normal cones to C, which can be given through the respective subdifferentials of the *indicator function* $\mathbf{I}_C(\cdot)$, equal to zero on C and to $+\infty$ elsewhere. Since in what follows we deal also with various kinds of subdifferentials of the time-minimum function, it is convenient to give here the general definitions.

For a lower semicontinuous function $f : H \to \mathbb{R}\cup\{+\infty\}$ and $x \in \operatorname{dom} f := \{x\in H : f(x) < +\infty\}$ we define

- the *proximal subdifferential* $\partial^p f(x)$ as the set of all $\zeta \in H$ such that there exist $\sigma \geq 0$ and $\varepsilon > 0$ with

$$f(y) \geq f(x) + \langle \zeta, y-x\rangle - \sigma\|y-x\|^2 \quad \forall y,\ \|y-x\|\leq\varepsilon;$$

- the *Fréchet subdifferential*

$$\partial^f f(x) := \left\{\zeta\in H : \liminf_{x\neq y\to x} \frac{f(y)-f(x)-\langle\zeta,y-x\rangle}{\|y-x\|} \geq 0\right\}; \tag{15}$$

- the *limiting* (*Mordukhovich*) *subdifferential*

$$\partial^l f(x) := \left\{w\text{-}\lim_{n\to\infty}\zeta_n : \zeta_n\in\partial^p f(x_n),\ x_n\to x,\ f(x_n)\to f(x)\right\}, \tag{16}$$

  where "$w\text{-}\lim_{n\to\infty}$" means the *weak limit*.

For the properties of the subdifferentials above and for the relationships between them we refer to [10, 2, 18] and to the bibliography therein. Let us observe only that $\partial^p f(x)$ is always convex (possibly empty), that $\partial^f f(x)$ is convex and closed and that $\partial^l f(x) \neq \varnothing$ whenever the function $f(\cdot)$ is Lipschitzean around x (see [18, Corollary 2.25]). In the latter case one can introduce the *Clarke subdifferential* by the formula

$$\partial^c f(x) := \left\{ \zeta \in H : \lim_{y\to x,\, t\to 0+} \sup \frac{f(y+tv) - f(y)}{t} \geq \langle \zeta, v \rangle \ \forall v \in H \right\}.$$

It can be represented also as the closed convex hull of the limiting subdifferential $\partial^l f(x)$ (see [10, p. 88] and [18, p. 317]). Therefore, $\partial^c f(x)$ is a nonempty closed convex bounded set and, consequently, a weakly compact set. Moreover, the mapping $x \mapsto \partial^c f(x)$ *has* $s \times w$*-closed graph*, i.e., given sequences $\{x_n\}$ and $\{\zeta_n\}$ in H such that $x_n \to x$ and $\{\zeta_n\}$ converges to ζ weakly as $n \to \infty$, the relations $\zeta_n \in \partial^c f(x_n)$, $n = 1, 2, \dots$, imply that $\zeta \in \partial^c f(x)$ (see [8, Proposition 2.1.5]). There are various generalizations of the Clarke subdifferential to non (locally) Lipschitzean functions, but in the sequel we use only the case $f = \mathbf{I}_C$ and define its Clarke subdifferential (called also the *Clarke normal cone* to C at the point $x \in C$) by

$$\partial^c \mathbf{I}_C(x) = \mathbf{N}_C^c(x) := \overline{\text{co}}\ \mathbf{N}_C^l(x),$$

where "$\overline{\text{co}}$" stands for the *closed convex hull*, and $\mathbf{N}_C^l(x) := \partial^l \mathbf{I}_C(x)$ means the *limiting* (*Mordukhovich*) *normal cone*. In finite dimensions the cone $\mathbf{N}_C^l(x)$ is closed but may be not convex, $\mathbf{N}_C^l(x) \neq \{0\}$ for all $x \in \partial C$, and the mapping $x \mapsto \mathbf{N}_C^l(x)$ has closed graph. However, the latter property fails in general for the Clarke cone $\mathbf{N}_C^c(\cdot)$ (see [22, Counterexample 2]) in contrast with the Clarke subdifferential of a locally Lipschitzean function (see above). In an infinite dimensional Hilbert space $\mathbf{N}_C^l(x)$ can be trivial and non closed for some $x \in \partial C$ (see [18, p. 11]).

Given $x \in C$ the *proximal* and *Fréchet normal cones* to C at the point x are defined, respectively, by $\mathbf{N}_C^p(x) := \partial^p \mathbf{I}_C(x)$ and by $\mathbf{N}_C^f(x) := \partial^f \mathbf{I}_C(x)$, or, equivalently, by

$$\mathbf{N}_C^p(x) := \left\{ v \in H : \exists \sigma \geq 0 \text{ such that } \langle v, y - x \rangle \leq \sigma \|y - x\|^2 \ \forall y \in C \right\}$$

and by

$$\mathbf{N}_C^f(x) := \left\{ v \in H : \limsup_{y\to x,\, x\neq y\in C} \left\langle v, \frac{y-x}{\|y-x\|} \right\rangle \leq 0 \right\}.$$

The following inclusions take place:

$$\mathbf{N}_C^p(x) \subseteq \mathbf{N}_C^f(x) \subseteq \mathbf{N}_C^l(x) \subseteq \mathbf{N}_C^c(x). \tag{17}$$

Analogously,

$$\partial^p f(x) \subseteq \partial^f f(x) \subseteq \partial^l f(x) \subseteq \partial^c f(x) \tag{18}$$

for each $x \in \text{dom} f$. If the set C (the function f) is convex then all the cones (all the subdifferentials) above coincide with the normal cone $\mathbf{N}_C(x)$ (respectively, with the subdifferential $\partial f(x)$) in the sense of Convex Analysis. On the other hand, if at the point x some of the inclusions in (17) (in (18)) can be reversed then we are led to various kinds of regularity of the set C (respectively, of the function $f(\cdot)$) at x. Namely, C is said to be

- *proximally regular* at x if $\mathbf{N}_C^p(x) = \mathbf{N}_C^l(x)$;
- *normally regular* at x if $\mathbf{N}_C^f(x) = \mathbf{N}_C^l(x)$;
- *Fréchet regular* at x if $\mathbf{N}_C^p(x) = \mathbf{N}_C^f(x)$;
- *Clarke regular* at x if $\mathbf{N}_C^f(x) = \mathbf{N}_C^c(x)$.

The notions of *proximal* (*Fréchet* or *Clarke*) *regularity* of a function $f : H \to \mathbb{R} \cup \{+\infty\}$ at $x \in \mathrm{dom} f$ are introduced by substituting in the definitions above the normal cones by the respective subdifferentials. If $\partial^f f(x) = \partial^l f(x)$ then the function $f(\cdot)$ is said to be *lower regular* at the point x. A little asymmetry in terminology suggests another name "*upper regularity*" in the case of functions, which is reserved for the respective superdifferentials.

Notice that the proximal regularity is a very strong property. Simple examples in $\mathbb{R}$ show that even a continuously differentiable function $f(\cdot)$ may be not proximally regular in general (moreover, the subdifferential $\partial^p f(x)$ may be empty). However, this does not occur if the derivative $f'(\cdot)$ (or the Fréchet gradient $\nabla f(\cdot)$ in the case of an arbitrary Hilbert space) is Lipschitz continuous near x. Following the tradition we denote the class of functions satisfying the latter property for all $x \in U$ ($U \subset H$ is an open set) by $\mathcal{C}_{loc}^{1,1}(U)$. Slightly extending this family we denote by $\mathcal{C}_{loc}^{1,\alpha}(U)$ the class of functions $f(\cdot) \in \mathcal{C}^1(U)$, whose gradient $\nabla f(\cdot)$ is Hölder continuous near each $x \in U$ with the exponent $0 < \alpha \leq 1$, i.e., given $x \in U$ there exist a constant $K = K(x) > 0$ and $\varepsilon > 0$ such that

$$\|\nabla f(x_1) - \nabla f(x_2)\| \leq K \|x_1 - x_2\|^\alpha$$

whenever $\|x_i - x\| \leq \varepsilon$, $i = 1, 2$. In what follows we distinguish the (global) *Lipschitzeanity* or *Hölderianity* on some set U from the *Lipschitz* (respectively, *Hölder*) *continuity*, which is always local property.

The closed set $C \subset H$ is said to be φ-*convex* (or *proximally smooth*) if there exists a continuous function $\varphi : C \to \mathbb{R}^+$ such that

$$\langle v, y - x \rangle \leq \varphi(x) \|v\| \|y - x\|^2$$

for all $x, y \in C$ and $v \in \mathbf{N}_C^p(x)$. This is equivalent to a kind of *external sphere condition*: for each $x \in \partial C$ and $v \in \mathbf{N}_C^p(x)$, $\|v\| = 1$, there exists a sphere with the center placed on $x + v\mathbb{R}^+$ and with locally uniform radius, which touches the set C at the point x only. If this property takes place only in a neighbourhood of $x_0 \in \partial C$, or, in other words, there exist $\delta = \delta(x_0) > 0$ and $M = M(x_0) \geq 0$ such that

$$\langle v, y - x \rangle \leq M \|v\| \|y - x\|^2 \tag{19}$$

for all $x \in C$, $\|x - x_0\| \leq \delta$, $v \in \mathbf{N}_C^p(x)$ and all $y \in C$ then we say that C is φ-*convex* (*proximally smooth*) *near* x_0 (on the δ-neighbourhood of this point). Notice that each φ-convex set is regular at every $x \in \partial C$ close to x_0 in the sense of all the definitions above, i.e., all the normal cones to C at $x \in \partial C$, $\|x - x_0\| \leq \delta$, coincide.

We say that a closed set $C \subset H$ *has smooth* (or $\mathcal{C}^1$) *boundary at* $x_0 \in \partial C$ if for each $x \in \partial C$ enough close to x_0 there exists a unique normal vector $\mathfrak{n}_C(x) \in \mathbf{N}_C^l(x)$ with $\|\mathfrak{n}_C(x)\| = 1$, which depends continuously on x. In what follows we use also a stronger property assuming, in addition, the Hölder continuity of $\mathfrak{n}_C(\cdot)$ with an exponent $0 < \alpha \leq 1$. In this case we say that C *has* $\mathcal{C}^{1,\alpha}$*-boundary at* x_0. Observe that each set with $\mathcal{C}^{1,1}$-boundary at $x_0 \in \partial C$ is φ-convex near this point. If, instead, $\alpha < 1$ then the set C may be even not proximally regular, and the proximal normal cone $\mathbf{N}_C^p(x)$ may be trivial. However, it is well known (see [10, p. 49]) that the *"reduced" boundary*

$$\partial^* C := \{x \in \partial C : \mathbf{N}_C^p(x) \neq \{0\}\}$$

is always dense in ∂C.

We are ready now to introduce the basic (local) hypotheses for the pair of sets (F, C), under which the regularity results are obtained. Notice that the first one uses only the linear approximation of the sets (duality mapping and normal vectors), while the second deals with their square characteristics (namely, with the internal and external curvatures). Therefore, they can be named *hypotheses of the first* and *of the second order*, respectively.

We say that (F, C) satisfies *the condition* (A) at a point $x_0 \in \partial C$ if there exists $\delta = \delta(x_0) > 0$ such that

(A$_1$) the composed mapping $x \mapsto \mathfrak{J}_F(-\mathbf{N}_C^p(x) \cap \partial F^0)$ is *single-valued* and *Lipschitzean* on

$$C_\delta(x_0) := \{x \in \partial^* C : \|x - x_0\| \leq \delta\};$$

(A$_2$) the set F is *uniformly rotund* w.r.t.

$$U_\delta(x_0) := \bigcup_{x \in C_\delta(x_0)} \left(-\mathbf{N}_C^p(x) \cap \partial F^0\right). \tag{20}$$

According to Proposition 2.1 in the place of the hypothesis (A$_2$) one can require the uniform continuity of the single-valued mapping $\mathfrak{J}_F(\cdot)$ (or of the gradient $\nabla \rho_{F^0}(\cdot)$) on a slightly larger set than $U_\delta(x_0)$, which is sometimes easier to verify.

Alternatively, we say that (F, C) satisfies *the condition* (B) at $x_0 \in \partial C$ if for some $\delta = \delta(x_0) > 0$

(B$_1$) the set C is φ*-convex* on $C_\delta(x_0)$, i.e., there exists $M \geq 0$ such that inequality (19) holds for all $x \in C_\delta(x_0)$, $v \in \mathbf{N}_C^p(x)$ and $y \in C$;

(B$_2$) there exist $\delta' = \delta'(x_0) > 0$ and $K = K(x_0) > 0$ such that

$$\varkappa_F(\mathfrak{J}_F(\xi^*), \xi^*) \geq K$$

for all $\xi^* \in \hat{U}_{\delta,\delta'}(x_0)$, where

$$\hat{U}_{\delta,\delta'}(x_0) := \partial F^0 \cap \bigcup_{x \in C_\delta(x_0) \setminus \{x_0\}} \left[-\mathbf{N}_C^p(x) \cap \partial F^0 + \delta' \overline{\mathbf{B}}\right].$$

Some simplifications of the conditions (A$_2$) and (B$_2$) can be made in finite dimensions. For instance, for validity of (A$_2$) it is enough to assume only the *strict convexity* of F w.r.t. each direction from the closure of $U_\delta(x_0)$. Notice that the latter set is closed, e.g., when C is *proximally regular* near x_0. Furthermore, (A_2) is reduced to the rotundity of F w.r.t. each $\xi^* \in \partial F^0$ close to

$$\xi_0^* := -\frac{\mathfrak{n}_C(x_0)}{\rho_{F^0}(-\mathfrak{n}_C(x_0))} \tag{21}$$

when C *has smooth boundary* at x_0 (with the normal vector $\mathfrak{n}_C(\cdot)$).

Observe that due to the condition (B_2) if $\mathbf{N}_C^p(x_0) \neq \{0\}$ then the set F may be even not strictly convex w.r.t. some vector $v_0 \in \mathbf{N}_C^p(x_0)$ (for instance, (B_2) is trivially fulfilled for the pair of sets (F, C),

$$F := \left\{(x_1, x_2) \in \mathbb{R}^2 : \max(|x_1|, |x_2|) \leq 1\right\};$$
$$C := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\right\},$$

at the point $(0,1) \in \partial C$). On the other hand, in finite dimensions for validity of (B_2) it is enough to require that F is *rotund of second order* w.r.t. each $\xi^* \in U_\delta(x_0)$ (see (20)). This follows from both lower semicontinuity of the curvature and compactness of the set (20) (the latter is guaranteed by the hypothesis (B_1)). If, moreover, C has $\mathcal{C}^{1,1}$-boundary at x_0 then (B_1) holds automatically, while (B_2) is reduced to the condition $\varkappa_F(\mathfrak{J}_F(\xi_0^*), \xi_0^*) > 0$ (see (21)).

For the convenience of the further references let us formulate here a local well-posedness result (see [16]).

Theorem 2.2. *Given $x_0 \in \partial C$, assume that the pair of sets (F, C) satisfies the condition* (A) *or* (B) *at the point x_0. Then there exists an open neighbourhood $\mathcal{U}(x_0)$ of x_0 such that for each $x \in \mathcal{U}(x_0)$ the time-minimum projection $\pi_C^F(x)$ is a singleton, and the mapping $x \mapsto \pi_C^F(x)$ is continuous on $\mathcal{U}(x_0)$.*

Remark 2.3. Originally, the hypothesis (A_2) in [16] required the uniform strict convexity of F w.r.t. a neighbourhood of the set $U_\delta(x_0)$ (denoted as $U_{\delta,\delta'}(x_0)$). But applying Proposition 2.1 (more fine than Proposition 3.4 from [16]) such a slightly stronger version of Theorem 6.1 [16] can be proved. The respective simple modifications will be seen in the proof of Theorem 3.1 below.

Remark 2.4. In the case of the condition (A) a neighbourhood $\mathcal{U}(x_0)$ can be chosen as (see (50) in [16])

$$\mathcal{U}(x_0) := \left\{x \in H : \|x - x_0\| < \frac{\delta}{\|F\| \|F^0\| + 1}, \mathfrak{T}_C^F(x) < \frac{1}{L}\right\}, \tag{22}$$

where $L > 0$ is the Lipschitz constant of $x \mapsto \mathfrak{J}_F(-\mathbf{N}_C^p(x) \cap \partial F^0)$ on $C_\delta(x_0)$. If, instead, (F, C) satisfies the condition (B) then we set (see (59) in [16])

$$\mathcal{U}(x_0) := \left\{x \in H : \|x - x_0\| < \frac{\delta}{\|F\| \|F^0\| + 1}, \mathfrak{T}_C^F(x) < \frac{K}{M}\right\}. \tag{23}$$

3. Hölder regularity of the time-minimum projection

We start from the case when the condition (A) holds.

Theorem 3.1. *Let us fix a point $x_0 \in \partial C$. Assume that the pair of sets (F, C) satisfies the condition* (A) *at x_0. Then the time-minimum projection $\pi_C^F(\cdot)$ is Lipschitz continuous on the neighbourhood* (22).

Proof. In virtue of Theorem 2.2 the mapping $x \mapsto \pi_C^F(x)$ is well defined, single-valued and continuous on $\mathcal{U}(x_0)$. We want to prove that it is in fact locally Lipschitzean on this neighbourhood, i.e., given $x \in \mathcal{U}(x_0)$ there exist $\varepsilon = \varepsilon(x) > 0$ and a constant $\mathfrak{L} = \mathfrak{L}(x) > 0$ such that

$$\left\|\pi_C^F(z_1) - \pi_C^F(z_2)\right\| \leq \mathfrak{L}\,\|z_1 - z_2\| \tag{24}$$

whenever $z_i \in \mathcal{U}(x_0)$, $\|z_i - x\| \leq \varepsilon$, $i = 1, 2$.

Let us choose $\tau > 0$ and $0 < \varepsilon \leq \frac{\tau}{2\|F^0\|}$ such that

$$1 - \left(\mathfrak{T}_C^F(x) + \tau\right) L > 0 \tag{25}$$

and

$$\left(\|F\|\left\|F^0\right\| + 1\right)(\|x - x_0\| + \varepsilon) < \delta. \tag{26}$$

Given $z_1, z_2 \in x + \varepsilon\overline{\mathbf{B}}$ for the sake of simplicity we denote by $\bar{z}_i := \pi_C^F(z_i)$, $i = 1, 2$, and by $\beta := \|z_1 - z_2\|/2$ assuming naturally that $\beta > 0$. Let us suppose first that both points z_1 and z_2 are out of C. By using the condition (A_2) and Proposition 2.1(ii) we find $0 < \nu \leq \min\{\beta, \varepsilon\}$ such that

$$\|\eta - \mathfrak{J}_F(u)\| \leq \beta \tag{27}$$

for all $\eta \in \mathfrak{J}_F(\eta^*)$, $\eta^* \in \partial F^0$, $u \in U_\delta(x_0)$ with $\|\eta^* - u\| \leq \nu$. Moreover, we assume $\nu > 0$ to be so small that

$$\nu + \left(\|F\|\left\|F^0\right\| + 1\right)(\|x - x_0\| + \varepsilon) < \delta \tag{28}$$

(see (26)) and that

$$\left(z_i + \nu\overline{\mathbf{B}}\right) \cap C = \varnothing, \quad i = 1, 2. \tag{29}$$

Since $\bar{z}_i$ minimizes the functional $F_i(y) := \rho_F(y - z_i) + \mathbf{I}_C(y)$, $y \in H$, we have $0 \in \partial^p F_i(\bar{z}_i)$, and by using the fuzzy sum rule for the proximal subdifferentials (see [10, p. 56]) we find $z_i', z_i'' \in \bar{z}_i + \nu\overline{\mathbf{B}}$ with $z_i' \in \partial C$ and vectors $v_i \in \mathbf{N}_C^p(z_i')$, $\xi_i^* \in \partial\rho_F(z_i'' - z_i)$ such that

$$\|v_i + \xi_i^*\| \leq \nu. \tag{30}$$

Notice that $z_i' \neq z_i$ by (29), and $z_i' \in \partial^* C$ by (30) taking into account that $\xi_i^* \in \partial F^0$. Moreover, multiplying v_i if necessary by some positive number, we arrive at $-v_i \in \partial F^0$.

The Lipschitz continuity of $\mathfrak{T}_C^F(\cdot)$ implies

$$\mathfrak{T}_C^F(z_i) \leq \left\|F^0\right\|\|z_i - x_0\|,$$

and hence (see (28))

$$\begin{aligned}\|z_i' - x_0\| \le\ & \|z_i' - \bar{z}_i\| + \|F\| \rho_F(\bar{z}_i - z_i) + \|z_i - x_0\| \\ \le\ & \nu + \left(\|F\| \left\|F^0\right\| + 1\right)(\|z_i - x\| + \|x - x_0\|) < \delta,\end{aligned}$$

i.e., $z_i' \in C_\delta(x_0)$.

It follows from Corollary 2.3 [14] that

$$\xi_i^* \in \partial F^0 \cap \mathbf{N}_F\left(\frac{z_i'' - z_i}{\rho_F(z_i'' - z_i)}\right),$$

and hence $\xi_i := \frac{z_i'' - z_i}{\rho_F\left(z_i'' - z_i\right)} \in \mathfrak{J}_F(\xi_i^*)$. Since $-v_i \in U_\delta(x_0)$, by (27) and (30) we obtain

$$\|\xi_i - \mathfrak{J}_F(-v_i)\| \le \beta. \tag{31}$$

Joining together the inequalities (31) for $i = 1, 2$ and taking into account the hypothesis (A$_1$), we have

$$\begin{aligned}\|\xi_1 - \xi_2\| &\le 2\beta + \|\mathfrak{J}_F(-v_1) - \mathfrak{J}_F(-v_2)\| \\ &\le 2\beta + L\|z_1' - z_2'\| \\ &\le 2\beta + L(2\nu + \|\bar{z}_1 - \bar{z}_2\|).\end{aligned} \tag{32}$$

On the other hand,

$$\begin{aligned}|\rho_F(z_1'' - z_1) - \rho_F(z_2'' - z_2)| &\le 2\left\|F^0\right\|\nu + \left|\mathfrak{T}_C^F(z_1) - \mathfrak{T}_C^F(z_2)\right| \\ &\le \left\|F^0\right\|(2\nu + \|z_1 - z_2\|),\end{aligned} \tag{33}$$

and by the choice of $\varepsilon > 0$

$$\begin{aligned}\rho_F(z_i'' - z_i) \le\ & \left\|F^0\right\|\nu + \mathfrak{T}_C^F(z_i) \\ \le\ & \left\|F^0\right\|\nu + \mathfrak{T}_C^F(x) + \left\|F^0\right\|\|x - z_i\| \le \mathfrak{T}_C^F(x) + \tau.\end{aligned} \tag{34}$$

Consequently, by using representation of ξ_i and estimates (34), (32) and (33) we successively obtain

$$\begin{aligned}\|\bar{z}_1 - \bar{z}_2\| \le\ & \|\bar{z}_1 - z_1''\| + \|z_1'' - z_2''\| + \|\bar{z}_2 - z_2''\| \\ \le\ & 2\nu + \|\xi_1\rho_F(z_1'' - z_1) - \xi_2\rho_F(z_2'' - z_2)\| + \|z_1 - z_2\| \\ \le\ & 2\nu + \rho_F(z_1'' - z_1)\|\xi_1 - \xi_2\| \\ & \quad + \|\xi_2\|\,|\rho_F(z_1'' - z_1) - \rho_F(z_2'' - z_2)| + \|z_1 - z_2\| \\ \le\ & 2\nu + \left(\mathfrak{T}_C^F(x) + \tau\right)(2\beta + L(2\nu + \|\bar{z}_1 - \bar{z}_2\|)) \\ & \quad + \|F\|\left\|F^0\right\|(2\nu + \|z_1 - z_2\|) + \|z_1 - z_2\|.\end{aligned} \tag{35}$$

Recalling definitions of β and ν we can rewrite inequality (35) in the form

$$\left(1 - \left(\mathfrak{T}_C^F(x) + \tau\right)L\right)\|\bar{z}_1 - \bar{z}_2\| \le \hat{\mathfrak{L}}\|z_1 - z_2\|,$$

where

$$\hat{\mathfrak{L}} := 2\left(\|F\|\left\|F^0\right\|+1\right)+\left(\mathfrak{T}_C^F(x)+\tau\right)(1+L).$$

Thus, it follows from (25) that inequality (24) holds with the Lipschitz constant

$$\mathfrak{L}=\mathfrak{L}(x):=\frac{\hat{\mathfrak{L}}}{1-\left(\mathfrak{T}_C^F(x)+\tau\right)L}. \tag{36}$$

Assuming now that one of the points z_i (say z_2) belongs to C, we immediately have

$$\begin{aligned}\|\bar{z}_1-\bar{z}_2\| = & \ \|\bar{z}_1-z_2\| \le \|\bar{z}_1-z_1\|+\|z_1-z_2\| \\ \le & \ \|F\|\left|\mathfrak{T}_C^F(z_1)-\mathfrak{T}_C^F(z_2)\right|+\|z_1-z_2\| \\ \le & \ \left(1+\|F\|\left\|F^0\right\|\right)\|z_1-z_2\|,\end{aligned} \tag{37}$$

and (24) also follows. □

Remark 3.2. If the condition (A) is fulfilled at each point $x_0 \in \partial C$ then $\pi_C^F(\cdot)$ is locally Lipschitzean on the open set $\mathfrak{A} := C \cup \bigcup_{x_0\in\partial C}\mathcal{U}(x_0)$, and the Lipschitz constant $\mathfrak{L}(x)$ increases when x tends to the boundary $\partial\mathfrak{A}$. Moreover, $\mathfrak{L}(x) \to +\infty$ if the radii $\delta(x_0)$, $x_0 \in \partial C$, are large enough (see (36)).

In the case when a balance between the curvatures takes place we have the following weaker regularity result.

Theorem 3.3. *Let us fix $x_0 \in \partial C$. If the pair of sets (F, C) satisfies the condition* (B) *at x_0 then $\pi_C^F(\cdot)$ is Hölder continuous on the neighbourhood* (23) *with the exponent* $1/2$.

Proof. Existence, uniqueness of the projection $\pi_C^F(x)$ for each $x \in \mathcal{U}(x_0)$ and continuity of the mapping $x \mapsto \pi_C^F(x)$ on $\mathcal{U}(x_0)$ follow also from Theorem 2.2. Here $\mathcal{U}(x_0)$ is the neighbourhood of x_0 given by the formula (23). For a fixed $x \in \mathcal{U}(x_0)$ we should find $\varepsilon = \varepsilon(x) > 0$ and $\mathfrak{L} = \mathfrak{L}(x) > 0$ such that

$$\left\|\pi_C^F(z_1)-\pi_C^F(z_2)\right\| \le \mathfrak{L}\|z_1-z_2\|^{1/2} \tag{38}$$

for all $z_i \in \mathcal{U}(x_0)$ with $\|z_i - x\| \le \varepsilon$, $i = 1, 2$ (compare with (24)).

Let us choose $\tau > 0$ such that

$$2\nu := \frac{1}{\|F\|}\left(K-\left(\mathfrak{T}_C^F(x)+\tau\right)M\right) > 0 \tag{39}$$

and set $\varepsilon := \frac{\tau}{\|F^0\|}$ assuming also that

$$\left(1+\|F\|\left\|F^0\right\|\right)(\|x-x_0\|+\varepsilon) < \delta. \tag{40}$$

Setting as earlier $\bar{z}_i := \pi_C^F(z_i)$, $i = 1, 2$, by the latter inequality we have

$$\begin{aligned}\|\bar{z}_i-x_0\| \le & \ \|F\|\,\mathfrak{T}_C^F(z_i)+\|z_i-x_0\| \\ \le & \ \left(1+\|F\|\left\|F^0\right\|\right)(\|x-x_0\|+\|z_i-x\|) < \delta,\end{aligned} \tag{41}$$

i.e., $\bar{z}_i \in C_\delta(x_0)$ if $z_i \in x + \varepsilon\overline{\mathbf{B}}$ (notice that $\bar{z}_i \in \partial^* C$ due to the hypothesis (B_1)).

In the case $z_1, z_2 \notin C$ we apply the necessary condition of minimum for the functional $F_i(y) := \rho_F(y - z_i) + \mathbf{I}_C(y)$, $y \in H$, at $\bar{z}_i$ in the limiting form: $0 \in \partial^l F_i(\bar{z}_i)$. By the calculus of the limiting subdifferentials (see, e.g., [10, p. 62]) and by the representation formula for $\partial\rho_F(\bar{z}_i - z_i)$ (see [14, Corollary 2.3]) we conclude that the set

$$-\mathbf{N}_C^p(\bar{z}_i) \cap \mathbf{N}_F\left(\frac{\bar{z}_i - z_i}{\rho_F(\bar{z}_i - z_i)}\right) \cap \partial F^0, \tag{42}$$

$i = 1, 2$, is nonempty. We use here also the fact that the limiting and proximal normal cones to C at the point $\bar{z}_i$ coincide (see (B_1)). Let us denote by ξ_i^* an arbitrary element of the set (42), and assume that at least one of the points $\bar{z}_i$ (say $\bar{z}_1$) is different from x_0. Then $\xi_1^* \in \hat{U}_{\delta,\delta'}(x_0)$ and $\varkappa_F(\mathfrak{J}_F(\xi_1^*), \xi_1^*) \geq K$ due to the hypothesis (B_2).

Recalling the scaled curvature radius (see (13)) and setting

$$R := \frac{\mathfrak{R}_F(\mathfrak{J}_F(\xi_1^*), \xi_1^*)}{\|\xi_1^*\|} = \frac{1}{2\|\xi_1^*\|\,\varkappa_F(\mathfrak{J}_F(\xi_1^*), \xi_1^*)},$$

by the choice of $\varepsilon > 0$ and by the Lipschitz continuity of $\mathfrak{T}_C^F(\cdot)$ we deduce from (39) that

$$\frac{1}{2R} - \mathfrak{T}_C^F(z_1)\,M\,\|\xi_1^*\| \geq \|\xi_1^*\|\left(K - \mathfrak{T}_C^F(z_1)\,M\right) \geq 2\nu.$$

Let us choose $\nu' > 0$ so small that

$$\frac{1}{2(R+\nu')} - \mathfrak{T}_C^F(z_1)\,M\,\|\xi_1^*\| \geq \nu. \tag{43}$$

Taking into account that $\mathfrak{J}_F(\xi_1^*) = \frac{\bar{z}_1 - z_1}{\rho_F(\bar{z}_1 - z_1)}$, by the formula (13) we have

$$F \subset \frac{\bar{z}_1 - z_1}{\rho_F(\bar{z}_1 - z_1)} - (R+\nu')\,\xi_1^* + (R+\nu')\,\|\xi_1^*\|\,\overline{\mathbf{B}}.$$

From the latter inclusion it follows, in particular, that

$$\left\|\frac{\bar{z}_1 - z_1}{\rho_F(\bar{z}_1 - z_1)} - \frac{\bar{z}_2 - z_2}{\rho_F(\bar{z}_2 - z_2)} - (R+\nu')\,\xi_1^*\right\| \leq (R+\nu')\,\|\xi_1^*\|. \tag{44}$$

Let us devide the reminder of the proof into two steps.

Step 1. Assume that $\rho_F(\bar{z}_1 - z_1) = \rho_F(\bar{z}_2 - z_2)$ and denote their common value by ρ. Then (44) gives

$$\|(z_1 - z_2) - (\bar{z}_1 - \bar{z}_2) + \rho\,(R+\nu')\,\xi_1^*\| \leq \rho\,(R+\nu')\,\|\xi_1^*\|,$$

and after simple transformations we have

$$\begin{aligned} &\|(z_1 - z_2) - (\bar{z}_1 - \bar{z}_2)\|^2 \\ &\leq 2\rho\,(R+\nu')\,\langle -\xi_1^*, (z_1 - z_2) - (\bar{z}_1 - \bar{z}_2)\rangle \\ &\leq 2\rho\,(R+\nu')\left(\langle \xi_1^*, z_2 - z_1\rangle + M\,\|\xi_1^*\|\,\|\bar{z}_1 - \bar{z}_2\|^2\right). \end{aligned} \tag{45}$$

Here we used the hypothesis (B_1) and the inequality (19) recalling that $-\xi_1^* \in \mathbf{N}_C^p(\bar{z}_1)$. Combining (45) with

$$\|(z_1 - z_2) - (\bar{z}_1 - \bar{z}_2)\|^2 \geq \|\bar{z}_1 - \bar{z}_2\|^2 - 2\langle z_1 - z_2, \bar{z}_1 - \bar{z}_2\rangle,$$

we obtain

$$\begin{aligned} &(1 - 2\rho(R + \nu') M \|\xi_1^*\|) \|\bar{z}_1 - \bar{z}_2\|^2 \\ &\leq 2\rho(R + \nu')\langle \xi_1^*, z_2 - z_1\rangle + 2\langle z_1 - z_2, \bar{z}_1 - \bar{z}_2\rangle. \end{aligned}$$

Hence, by (43) and (14),

$$\nu \|\bar{z}_1 - \bar{z}_2\|^2 \leq \rho\langle \xi_1^*, z_2 - z_1\rangle + \frac{1}{\mathfrak{r}_F}\|z_1 - z_2\| \|\bar{z}_1 - \bar{z}_2\|, \tag{46}$$

where $\mathfrak{r}_F > 0$ is the Chebyshev radius of the set F. Since $\xi_1^* \in F^0$, we deduce from (46) that

$$\|\bar{z}_1 - \bar{z}_2\| \left(\nu \|\bar{z}_1 - \bar{z}_2\| - \frac{1}{\mathfrak{r}_F}\|z_1 - z_2\|\right) \leq \rho \left\|F^0\right\| \|z_1 - z_2\|.$$

Therefore, one of the inequalities

$$\|\bar{z}_1 - \bar{z}_2\| \leq \sqrt{\rho \|F^0\|}\, \|z_1 - z_2\|^{1/2} \tag{47}$$

or

$$\nu \|\bar{z}_1 - \bar{z}_2\| \leq \sqrt{\rho \|F^0\|}\, \|z_1 - z_2\|^{1/2} + \frac{1}{\mathfrak{r}_F}\|z_1 - z_2\|, \tag{48}$$

clearly, takes place. Observing that $\rho = \mathfrak{T}_C^F(z_i) \leq \|F^0\| \delta$, $\|z_1 - z_2\| \leq \delta$ (the latter follows from (23) and from the obvious inequality $\|F\| \|F^0\| \geq 1$), and that $\nu > 0$ can be always chosen less than 1, we obtain from (47) and (48):

$$\|\bar{z}_1 - \bar{z}_2\| \leq \hat{\mathfrak{L}}\, \|z_1 - z_2\|^{1/2}, \tag{49}$$

where

$$\hat{\mathfrak{L}} = \hat{\mathfrak{L}}(x) := \sqrt{\delta} \cdot \frac{1}{\nu}\left(\left\|F^0\right\| + \frac{1}{\mathfrak{r}_F}\right). \tag{50}$$

Step 2. In the general case we slightly diminish ε by setting

$$\varepsilon' := \varepsilon / \left(1 + 2\|F\| \left\|F^0\right\|\right).$$

Then, given $z_1, z_2 \in \left(x + \varepsilon' \overline{\mathbf{B}}\right) \diagdown C$ let us denote by $\rho_i := \rho_F(\bar{z}_i - z_i)$, $i = 1, 2$, assuming without loss of generality that $\rho_1 < \rho_2$.

Choose now $u \in \partial F$ with $\bar{z}_2 = z_2 + \rho_2 u$ and set $z_2' := z_2 + (\rho_2 - \rho_1) u$. We obviously have

$$\mathfrak{T}_C^F(z_2') \leq \rho_F(\bar{z}_2 - z_2') = \rho_F(\rho_2 u - (\rho_2 - \rho_1) u) = \rho_1, \tag{51}$$

while, on the other hand,

$$\rho_F\left(y - z_2'\right) \geq \rho_F\left(y - z_2\right) - \left(\rho_2 - \rho_1\right) \tag{52}$$

for each $y \in C$. Passing to infimum in (52) and comparing with (51) we conclude that $\mathfrak{T}_C^F\left(z_2'\right) = \rho_1 = \mathfrak{T}_C^F\left(z_1\right)$. Moreover, it follows from (51) that $\pi_C^F\left(z_2'\right) = \bar{z}_2$. Observe also that

$$\begin{aligned}
\left\|z_2' - x\right\| \leq\ & \left\|z_2 - x\right\| + \left|\mathfrak{T}_C^F\left(z_1\right) - \mathfrak{T}_C^F\left(z_2\right)\right| \left\|F\right\| \\
\leq\ & \left(1 + 2\left\|F\right\| \left\|F^0\right\|\right)\varepsilon' = \varepsilon
\end{aligned}$$

and $\left\|z_1 - x\right\| \leq \varepsilon' < \varepsilon$. Thus, we can apply to the points z_1 and z_2' the inequality (49) obtained in the Step 1. Finally, for the original points we have

$$\begin{aligned}
\left\|\bar{z}_1 - \bar{z}_2\right\| &\leq \hat{\mathfrak{L}}\left\|z_1 - z_2'\right\|^{1/2} \leq \hat{\mathfrak{L}}\left(\left\|z_1 - z_2\right\| + \left\|z_2 - z_2'\right\|\right)^{1/2} \\
&\leq \hat{\mathfrak{L}}\left(\left\|z_1 - z_2\right\| + \left|\mathfrak{T}_C^F\left(z_1\right) - \mathfrak{T}_C^F\left(z_2\right)\right| \left\|u\right\|\right)^{1/2} \\
&\leq \hat{\mathfrak{L}}\sqrt{1 + \left\|F\right\| \left\|F^0\right\|}\left\|z_1 - z_2\right\|^{1/2},
\end{aligned} \tag{53}$$

and (38) is proved with $\mathfrak{L} = \mathfrak{L}\left(x\right) := \hat{\mathfrak{L}}\sqrt{1 + \left\|F\right\| \left\|F^0\right\|}$.

When one of the points z_i, $i = 1, 2$, belongs to C, similarly as in the proof of Theorem 3.1 even the Lipschitz inequality follows (see (37)). □

Notice that the argument allowing us to reduce the general situation to the case when both points z_1 and z_2 have the same minimal time value was proposed ealier in [14, Theorem 5.7]. It essentially simplifies the respective estimates and will be used more in the sequel.

Remark 3.4. The assertion similar to Remark 3.2 is valid under the hypothesis (B) as well. In particular, if (B) holds for each $x_0 \in \partial C$ and the numbers $\delta\left(x_0\right) > 0$ are large enough then $\pi_C^F\left(\cdot\right)$ is Hölder continuous on $\mathfrak{A} := C \cup \bigcup_{x_0 \in \partial C} \mathcal{U}\left(x_0\right)$ with the exponent $1/2$ and with the (local) Hölder constant $\mathfrak{L}\left(x\right)$ tending to $+\infty$ as x approximates to $\partial\mathfrak{A}$ (see (50) and (39)). On the other hand, $\mathfrak{L}\left(x\right) \to 0$ as x tends to ∂C because it is proportional to $\sqrt{\delta}$.

The next natural question is to impose some supplementary hypotheses, which would permit to improve the regularity of $\pi_C^F\left(\cdot\right)$. These hypotheses concern, obviously, the behaviour of the time-minimum projection outside the target because if one of the points belongs to C then always the Lipschitz inequality (37) holds. We deal also with the second order condition (B) because under the assumption (A) the time-minimum projection is Lipschitz continuous in virtue of Theorem 3.1.

Theorem 3.5. *Under the hypotheses of Theorem* 3.3 *let us suppose, in addition, that the mapping*

$$\Phi : x \mapsto -\mathbf{N}_C^p\left(\bar{x}\right) \cap \mathbf{N}_F\left(\frac{\bar{x} - x}{\rho_F\left(\bar{x} - x\right)}\right) \cap \partial F^0 \tag{54}$$

is single-valued and satisfies the Hölder inequality with an exponent $0 < \alpha \le 1$ *on* $\mathcal{U}(x_0) \setminus C$. *Here as usual* $\bar{x} := \pi_C^F(x)$, *and the neighbourhood* $\mathcal{U}(x_0)$ *is of the form* (23). *Then* $\pi_C^F(\cdot)$ *is Hölder continuous on* $\mathcal{U}(x_0)$ *with the exponent* $\frac{1+\alpha}{2}$. *In particular, the Lipschitzeanity of the mapping* $\Phi(\cdot)$ *implies the Lipschitz continuity of the time-minimum projection on* $\mathcal{U}(x_0)$.

Proof. We start by proceeding as in the proof of Theorem 3.3. Namely, given $x \in \mathcal{U}(x_0)$ we find $\tau > 0$ and $\varepsilon > 0$ satisfying (39) and (40), take arbitrary $z_1, z_2 \in \left(x + \varepsilon\overline{\mathbf{B}}\right) \setminus C$ with $\bar{z}_1 \ne x_0$ and denote by ξ_i^*, $i = 1, 2$, the unique element of the set (42). So, $\xi_i^* = \Phi(z_i)$. Considering first the simpler case when $\mathfrak{T}_C^F(z_1) = \mathfrak{T}_C^F(z_2) = \rho$, we arrive at the inequality (46).

Let us assume also that $\bar{z}_2 \ne x_0$. Then by changing the places of the points z_1 and z_2 the same reasoning as above gives:

$$\nu \left\| \bar{z}_1 - \bar{z}_2 \right\|^2 \le \rho \left\langle \xi_2^*, z_1 - z_2 \right\rangle + \frac{1}{\mathfrak{r}_F} \left\| z_1 - z_2 \right\| \left\| \bar{z}_1 - \bar{z}_2 \right\|. \tag{55}$$

After adding inequalities (46) and (55) we have

$$\begin{aligned} 2\nu \left\| \bar{z}_1 - \bar{z}_2 \right\|^2 &\le \rho \left\langle \Phi(z_2) - \Phi(z_1), z_1 - z_2 \right\rangle + \frac{2}{\mathfrak{r}_F} \left\| z_1 - z_2 \right\| \left\| \bar{z}_1 - \bar{z}_2 \right\| \\ &\le \rho\mathfrak{h} \left\| z_1 - z_2 \right\|^{1+\alpha} + \frac{2}{\mathfrak{r}_F} \left\| z_1 - z_2 \right\| \left\| \bar{z}_1 - \bar{z}_2 \right\|, \end{aligned} \tag{56}$$

where $\mathfrak{h} > 0$ is the Hölder constant of $\Phi(\cdot)$ (in the ε-neighbourhood of x). In turn, (56) written in the form

$$\left\| \bar{z}_1 - \bar{z}_2 \right\| \left(2\nu \left\| \bar{z}_1 - \bar{z}_2 \right\| - \frac{2}{\mathfrak{r}_F} \left\| z_1 - z_2 \right\| \right) \le \rho\mathfrak{h} \left\| z_1 - z_2 \right\|^{1+\alpha}$$

splits into two inequalities:

$$\left\| \bar{z}_1 - \bar{z}_2 \right\| \le \sqrt{\rho\mathfrak{h}} \left\| z_1 - z_2 \right\|^{\frac{1+\alpha}{2}} \tag{57}$$

and

$$2\nu \left\| \bar{z}_1 - \bar{z}_2 \right\| \le \sqrt{\rho\mathfrak{h}} \left\| z_1 - z_2 \right\|^{\frac{1+\alpha}{2}} + \frac{2}{\mathfrak{r}_F} \left\| z_1 - z_2 \right\|. \tag{58}$$

Since $\rho \le \left\| F^0 \right\| \delta$, $\left\| z_1 - z_2 \right\| \le \delta$ (see the proof of Theorem 3.3) and one can always assume that $2\nu < 1$ and $\mathfrak{h} \ge \max\left\{ \left\| F^0 \right\|, \frac{2}{\mathfrak{r}_F} \right\}$, we deduce from the alternative inequalities (57) and (58) that

$$\left\| \bar{z}_1 - \bar{z}_2 \right\| \le \hat{\mathfrak{L}} \left\| z_1 - z_2 \right\|^{\frac{1+\alpha}{2}},$$

where

$$\hat{\mathfrak{L}} = \hat{\mathfrak{L}}(x) := \delta^{\frac{1-\alpha}{2}} \frac{\mathfrak{h}}{\nu}. \tag{59}$$

If $\mathfrak{T}_C^F(z_1) \neq \mathfrak{T}_C^F(z_2)$ then we proceed as in the Step 2 of the proof of the previous theorem, substituting the point with the larger minimum time value (say z_2) for another one z_2' whose projection $\pi_C^F(z_2')$ coincides with $\bar{z}_2 = \pi_C^F(z_2)$ and $\rho_F(\bar{z}_2 - z_2') = \rho_F(\bar{z}_1 - z_1)$. Thus, employing the Hölder inequality already proved for the latter case we have

$$\|\bar{z}_1 - \bar{z}_2\| \leq \hat{\mathfrak{L}} \|z_1 - z_2'\|^{\frac{1+\alpha}{2}},$$

and similarly to (53) we conclude that

$$\|\bar{z}_1 - \bar{z}_2\| \leq \hat{\mathfrak{L}} \left(1 + \|F\| \left\|F^0\right\|\right)^{\frac{1+\alpha}{2}} \|z_1 - z_2\|^{\frac{1+\alpha}{2}}.$$

Notice that the points z_1 and z_2 are supposed to belong to a smaller neighbourhood $x + \varepsilon' \overline{\mathbf{B}}$ where $\varepsilon' := \varepsilon / (1 + 2\|F\| \|F^0\|)$.

It remains to treat only the case when $\bar{z}_2 = x_0$. To this end let us define $z_\lambda := \lambda z_1 + (1-\lambda) z_2$ and $\lambda^* := \sup\left\{\lambda \in [0,1] : \pi_C^F(z_\lambda) = x_0\right\}$. Since $\bar{z}_1 = \pi_C^F(z_1) \neq x_0$ and the mapping $z \mapsto \pi_C^F(\cdot)$ is continuous on $\mathcal{U}(x_0)$, we have $\lambda^* < 1$, $\pi_C^F(z_{\lambda^*}) = x_0$ and $\pi_C^F\left(z_{\lambda^*+1/n}\right) \neq x_0$ for $n = 1, 2, ...$ large enough. Applying the Hölder inequality for these points we have

$$\left\|\pi_C^F(z_1) - \pi_C^F\left(z_{\lambda^*+1/n}\right)\right\| \leq \mathfrak{L} \left\|z_1 - z_{\lambda^*+1/n}\right\|^{\frac{1+\alpha}{2}}. \tag{60}$$

Passing now to limit in (60) as $n \to \infty$ and taking into account that $\pi_C^F(z_{\lambda^*}) = \pi_C^F(z_2)$ and that $\|z_1 - z_{\lambda^*}\| \leq \|z_1 - z_2\|$ we finish the proof. □

Remark 3.6. The assertion of Remark 3.4 is applicable also under the hypotheses of Theorem 3.5 (certainly, with the larger Hölder exponent $\frac{1+\alpha}{2}$). Notice only that the Hölder constant here is proportional to $\delta^{\frac{1-\alpha}{2}}$ (see (59)), and this dependence vanishes in the case of Lipschitzeanity ($\alpha = 1$).

Remark 3.7. Observe that proving Theorems 3.3 and 3.5 we used the condition (B_2) in a simpler form. Namely, we needed only that

$$\varkappa_F\left(\mathfrak{J}_F(\xi^*), \xi^*\right) \geq K \tag{61}$$

for all $\xi^* \in \Phi(x)$, where $x \in \mathcal{U}(x_0) \diagdown C$ are such that $\pi_C^F(x) \neq x_0$. Nevertheless, in order to have the time-minimum projection well defined on $\mathcal{U}(x_0)$ we should require validity of the inequality (61) on a larger set $\hat{U}_{\delta,\delta'}(x_0)$.

An essential disadvantage of Theorem 3.5 is that the time-minimum projection $\pi_C^F(\cdot)$ itself entries into the definition of $\Phi(\cdot)$, and, therefore, the Hölder condition for this mapping (w.r.t. x) is practically unverifiable. However, we can overcome this difficulty by setting another type of hypotheses, which involve the Hölder regularity of the sets C and F separately.

Theorem 3.8. *Let us fix $x_0 \in \partial C$ and assume that (F, C) satisfies the condition* (B) *at x_0. Moreover, suppose that one of the following conditions holds:*

(*i*) *the target set* C *has smooth boundary at the point* x_0*, and the function* $\mathfrak{n}_C(\cdot)$*, which associates to each* $x \in C_\delta(x_0)$ *the respective (unique) normal vector* $\mathfrak{n}_C(x) \in \mathbf{N}^l_C(x) \cap \partial\overline{B}$*, satisfies the Hölder inequality on* $C_\delta(x_0)$ *with an exponent* $0 < \alpha \le 1$*;*

(*ii*) *the set* F *is smooth at each* $\xi \in F_\delta(x_0)$*, where*

$$F_\delta(x_0) := \bigcup_{x \in C_\delta(x_0) \setminus \{x_0\}} \mathfrak{J}_F\left(-\mathbf{N}^p_C(x) \cap \partial F^0\right), \tag{62}$$

and the unit normal vector $\mathfrak{n}_F(\cdot)$ *to* F*,* $\mathfrak{n}_F(\xi) = \frac{\nabla\rho_F(\xi)}{\|\nabla\rho_F(\xi)\|}$*, satisfies the Hölder inequality on the set* $F_\delta(x_0)$ *with an exponent* $0 < \alpha \le 1$*.*

Then the time-minimum projection $\pi^F_C(\cdot)$ *is Hölder continuous on the neighbourhood* $\mathcal{U}(x_0)$ *with the exponent* $\frac{1}{2-\alpha}$*. In particular,* $\pi^F_C(\cdot)$ *is Lipschitz continuous on* $\mathcal{U}(x_0)$ *whenever either* $\mathfrak{n}_C(\cdot)$ *or* $\mathfrak{n}_F(\cdot)$ *is Lipschitzean on the respective set.*

Proof. Given $x \in \mathcal{U}(x_0)$ let us choose $\tau > 0$ and $0 < \varepsilon < \frac{\tau}{\|F^0\|}$ such that inequalities (39) and (40) hold. Taking arbitrary $z_1, z_2 \in \left(x + \varepsilon\overline{B}\right) \setminus C$ we consider only the case when $\bar{z}_i := \pi^F_C(z_i) \ne x_0$, $i = 1, 2$, and $\mathfrak{T}^F_C(z_1) = \mathfrak{T}^F_C(z_2) = \rho$, while the other cases can be treated similarly as in the Step 2 of Theorem 3.3 and in the last part of the proof of Theorem 3.5.

Assume that the regularity condition (i) is fulfilled. It was already shown (see (41)) that $\bar{z}_i \in C_\delta(x_0)$, and due to φ-convexity of the set C (the condition (B_1))

$$\mathbf{N}^p_C(\bar{z}_i) \cap \partial\overline{B} = \{\mathfrak{n}_C(\bar{z}_i)\}.$$

Consequently, the sets $\Phi(z_1)$ and $\Phi(z_2)$ (see (54)) are singletons, and

$$\Phi(z_i) = -\frac{\mathfrak{n}_C(\bar{z}_i)}{\rho_{F^0}(-\mathfrak{n}_C(\bar{z}_i))}, \tag{63}$$

$i = 1, 2$. Then, as follows from the first inequality in (56),

$$2\nu\|\bar{z}_1 - \bar{z}_2\|^2 \le \rho\|\Phi(z_2) - \Phi(z_1)\|\|z_1 - z_2\| + \frac{2}{\mathfrak{r}_F}\|z_1 - z_2\|\|\bar{z}_1 - \bar{z}_2\|. \tag{64}$$

Denoting by $\mathfrak{h} > 0$ the Hölder constant of $\mathfrak{n}_C(\cdot)$ and taking into account the inequalities $\frac{1}{\|F^0\|} \le \rho_{F^0}(-\mathfrak{n}_C(\bar{z}_i)) \le \|F\|$ (see (4)), $i = 1, 2$, we obtain from (63) that

$$\|\Phi(z_2) - \Phi(z_1)\| \le 2\mu\|\bar{z}_1 - \bar{z}_2\|^\alpha,$$

where $\mu := \mathfrak{h}\|F\|\|F^0\|^2$, and write (64) in the form

$$\|\bar{z}_1 - \bar{z}_2\|^{1-\alpha}\left(\nu\|\bar{z}_1 - \bar{z}_2\| - \frac{1}{\mathfrak{r}_F}\|z_1 - z_2\|\right) \le \mu\rho\|z_1 - z_2\|. \tag{65}$$

Representing the right-hand side of (65) as

$$\left[\mu\rho\|z_1 - z_2\|^{\frac{1}{2-\alpha}}\right]^{1-\alpha} \cdot (\mu\rho)^\alpha\|z_1 - z_2\|^{\frac{1}{2-\alpha}},$$

we see that (65) splits into two alternative inequalities:

$$\|\bar{z}_1 - \bar{z}_2\| \leq \mu\rho \|z_1 - z_2\|^{\frac{1}{2-\alpha}}$$

and

$$\|\bar{z}_1 - \bar{z}_2\| \leq \frac{(\mu\rho)^\alpha}{\nu} \|z_1 - z_2\|^{\frac{1}{2-\alpha}} + \frac{1}{\mathfrak{r}_F} \|z_1 - z_2\|.$$

Since $\|z_1 - z_2\| \leq \delta$, $\rho := \mathfrak{T}_C^F(z_i) \leq \|F^0\| \delta$ as we observed earlier, and $\nu > 0$ can be chosen enough small, it follows that in any case

$$\left\|\pi_C^F(z_1) - \pi_C^F(z_2)\right\| \leq \mathfrak{L} \|z_1 - z_2\|^{\frac{1}{2-\alpha}},$$

where

$$\mathfrak{L} = \mathfrak{L}(x) := \frac{1}{\nu}\left(\mathfrak{h} \|F\| \left\|F^0\right\|^3 \delta\right)^\alpha + \frac{1}{\mathfrak{r}_F}\delta^{\frac{1-\alpha}{2-\alpha}}.$$

Observe that here only the number ν depends on x becoming smaller whenever x tends to a part of $\partial\mathcal{U}(x_0)$ determinable by the condition $\mathfrak{T}_C^F(x) = K/M$.

Let us prove now the statement of the theorem under the condition (ii).

First of all, due to the various characterizations of the duality mapping we have

$$\mathbf{N}_F(\xi) \cap \partial F^0 = \{\nabla\rho_F(\xi)\}, \quad \xi \in F_\delta(x_0).$$

In particular, $\rho_{F^0}(\mathfrak{n}_F(\xi)) = \frac{1}{\|\nabla\rho_F(\xi)\|}$, and in the place of the Hölder condition for $\mathfrak{n}_F(\cdot)$ we can use the same property for the gradient $\nabla\rho_F(\cdot)$ (with the exponent $0 < \alpha \leq 1$ and with some Hölder constant $\mathfrak{h} > 0$).

At the beginning of the proof of Theorem 3.3 by using the necessary condition of optimality it was shown that the sets (42), which are nothing else than $\Phi(z_i)$, $i = 1, 2$, are nonempty. Since $\bar{z}_i \in C_\delta(x_0) \setminus \{x_0\}$, we directly have that (the singletons) $\mathfrak{J}_F(\xi_i^*)$ belong to $F_\delta(x_0)$ (see (62)) for each $\xi_i^* \in \Phi(z_i)$, $i = 1, 2$. On the other hand,

$$\mathbf{N}_F\left(\frac{\bar{z}_i - z_i}{\rho_F(\bar{z}_i - z_i)}\right) \cap \partial F^0 = \mathfrak{J}_F^{-1}\left(\frac{\bar{z}_i - z_i}{\rho_F(\bar{z}_i - z_i)}\right) \tag{66}$$

$$= \partial\rho_F\left(\frac{\bar{z}_i - z_i}{\rho_F(\bar{z}_i - z_i)}\right). \tag{67}$$

From (66) it follows that

$$\mathfrak{J}_F(\xi_i^*) = \frac{\bar{z}_i - z_i}{\rho_F(\bar{z}_i - z_i)},$$

while equality (67) implies that

$$\xi_i^* = \nabla\rho_F\left(\frac{\bar{z}_i - z_i}{\rho_F(\bar{z}_i - z_i)}\right)$$

is the unique element of $\Phi(z_i)$, $i = 1, 2$. Hence, by the Hölder condition for $\nabla\rho_F(\cdot)$ we have

$$\|\Phi(z_1) - \Phi(z_2)\| \le \mathfrak{h} \left\| \frac{\bar{z}_1 - z_1}{\rho_F(\bar{z}_1 - z_1)} - \frac{\bar{z}_2 - z_2}{\rho_F(\bar{z}_2 - z_2)} \right\|^{\alpha} \\ \le \mathfrak{h}\rho^{-\alpha}\left(\|\bar{z}_1 - \bar{z}_2\|^{\alpha} + \|z_1 - z_2\|^{\alpha}\right), \tag{68}$$

where we use the numerical inequality $(a+b)^{\alpha} \le a^{\alpha} + b^{\alpha}$ valid for all $a, b > 0$ and $0 < \alpha \le 1$ (to see this it is enough to integrate the obvious inequality $(t+1)^{\alpha-1} \le t^{\alpha-1}$ on the interval $\left[0, \frac{a}{b}\right]$). Combining (68) with (64), which takes place also in this case, we obtain

$$\|\bar{z}_1 - \bar{z}_2\|^{\alpha} \Big[2\nu \|\bar{z}_1 - \bar{z}_2\|^{2-\alpha} - \mu \|z_1 - z_2\| \\ - \frac{2}{\mathfrak{r}_F} \|z_1 - z_2\| \|\bar{z}_1 - \bar{z}_2\|^{1-\alpha} \Big] \le \mu \|z_1 - z_2\|^{\alpha+1}, \tag{69}$$

where $\mu := \rho^{1-\alpha}\mathfrak{h}$. Applying the same argument as in the first part of the proof we obtain from (69) that either

$$\|\bar{z}_1 - \bar{z}_2\| \le \mu \|z_1 - z_2\| \tag{70}$$

or

$$2\nu \|\bar{z}_1 - \bar{z}_2\|^{2-\alpha} \le \left(\mu + \mu^{1-\alpha}\right) \|z_1 - z_2\| + \frac{2}{\mathfrak{r}_F} \|z_1 - z_2\| \|\bar{z}_1 - \bar{z}_2\|^{1-\alpha}.$$

In turn, the second inequality can be rewritten as

$$2\|\bar{z}_1 - \bar{z}_2\|^{1-\alpha} \left(\nu \|\bar{z}_1 - \bar{z}_2\| - \frac{1}{\mathfrak{r}_F} \|z_1 - z_2\| \right) \\ \le \mu^{1-\alpha-\beta} \|z_1 - z_2\|^{\frac{1-\alpha}{2-\alpha}} \cdot \mu^{\beta}(\mu^{\alpha} + 1) \|z_1 - z_2\|^{\frac{1}{2-\alpha}}$$

for an arbitrary fixed β with $0 < \beta < 1 - \alpha$. So, one of the following two inequalities

$$\|\bar{z}_1 - \bar{z}_2\| \le \mu^{1-\frac{\beta}{1-\alpha}} \|z_1 - z_2\|^{\frac{1}{2-\alpha}}; \tag{71}$$

$$\|\bar{z}_1 - \bar{z}_2\| \le \frac{1}{2\nu}\mu^{\beta}(\mu^{\alpha} + 1) \|z_1 - z_2\|^{\frac{1}{2-\alpha}} + \frac{1}{\nu\mathfrak{r}_F} \|z_1 - z_2\| \tag{72}$$

holds. Comparing inequalities (70), (71) and (72) we conclude that

$$\|\bar{z}_1 - \bar{z}_2\| \le \mathfrak{L} \|z_1 - z_2\|^{\frac{1}{2-\alpha}} \tag{73}$$

with some constant $\mathfrak{L} = \mathfrak{L}(x)$ whenever z_1 and z_2 are enough close to x. Since $\nu > 0$ can be chosen as small as we want, recalling the value of μ and the inequalities $\|z_1 - z_2\| \le \delta$, $\rho \le \|F^0\| \delta$ we see that the contribution of the right-hand side of (72) into the Hölder inequality (73) is more essential, and the constant $\mathfrak{L}$ is of the order $O\left(\delta^{(1-\alpha)\beta}\right)$ that can be controlled by the suitable choice of $\beta \in \,]0, 1-\alpha[$. Theorem is proved. □

Remark 3.9. In comparison with Theorem 3.5 assuming the Hölder condition either on the set C or on F we lose a little regularity of the time-minimum projection ($\frac{1}{2-\alpha} < \frac{\alpha+1}{2}$ if $0 < \alpha < 1$). It is due to the fact that the mapping (54) is the composition of the normal vectors to C (or to F) and the projection $\pi_C^F(\cdot)$ itself.

Although Theorem 3.8 is more appropriate for applications, Theorem 3.5 has a certain theoretical meaning because it connects the regularity of the time-minimum projection with the similar property of the *(Fréchet) gradient* of the value function (remind that this connection is immediate in the case of distance). Indeed, if we assume that $\mathfrak{T}_C^F(\cdot)$ is Fréchet differentiable on $\mathcal{U}(x_0)$ and the set $\Phi(x)$ is a singleton for $x \in \mathcal{U}(x_0)$ (see (54)) then $\Phi(x) = -\nabla\mathfrak{T}_C^F(x)$ (see [14, Theorem 3.3]). Therefore, the Hölder condition for the gradient $\nabla\mathfrak{T}_C^F(\cdot)$ on $\mathcal{U}(x_0)$ (with an exponent $0 < \alpha \leq 1$) implies the Hölder continuity of $\pi_C^F(\cdot)$ with the exponent $\frac{\alpha+1}{2} \geq \alpha$. In particular, we have always the 1/2-Hölderianity (see Theorem 3.3), while the time-minimum projection is Lipschitz continuous whenever the gradient of $\mathfrak{T}_C^F(\cdot)$ is Lipschitzean. In Section 5 we study the reverse question: which conditions on $\pi_C^F(\cdot)$ should we impose to have certain regularity of the value function (of its gradient), while now let us complete the previous results with some special case where the gauge F can be arbitrary.

4. The case of a locally convex target

Recall first a well-posedness result from [16, Proposition 7.7(i)]: if the target set C has smooth boundary at the point x_0, and it is strictly convex of second order near x_0, then the mapping $\pi_C^F(\cdot)$ is single-valued and continuous in some neighbourhood of x_0.

Notice that due to the invariantness of the curvature w.r.t. translations (see [16, Proposition 3.7]) the second order rotundity does not depend on the position of the origin in the interior of the set, and, consequently, this property is applicable also in the case when the origin is out of the interior (it is important only that the interior is not empty). Then, we have the following regularity result, which is a mixture of Theorems 3.3 and 3.8 under the (very strong) convexity hypothesis on C.

Theorem 4.1. *Let us assume that for a fixed $x_0 \in \partial C$ the boundary of C is smooth at x_0, and there exists $\delta > 0$ such that the set $G := C \cap \left(x_0 + \delta\overline{B}\right)$ with* int $G \neq \emptyset$, *is strictly convex of second order at x_0 (w.r.t. respective normal vector). Then $\pi_C^F(\cdot)$ is Hölder continuous with the exponent $1/2$ on some neighbourhood $\mathcal{U}(x_0)$ of x_0. If, moreover, we suppose that the unit normal vector $\mathfrak{n}_C(x)$ satisfies near x_0 the Hölder inequality with an exponent $0 < \alpha \leq 1$ then the time-minimum projection $\pi_C^F(\cdot)$ is Hölder continuous on $\mathcal{U}(x_0)$ with the exponent $\frac{1}{2-\alpha}$. In particular, Lipschitzeanity of $\mathfrak{n}_C(\cdot)$ implies the Lipschiz continuity of $\pi_C^F(\cdot)$.*

Proof. Without loss of generality (translating the set C if necessary) we can suppose that $0 \in$ int G. Considering G and G^0 as an usual dual pair of sets, we can apply to them all the notations and definitions from Section 2. In particular, setting

$$v(x) := \frac{\mathfrak{n}_C(x)}{\rho_{G^0}(\mathfrak{n}_C(x))} \tag{74}$$

for $x \in \partial C$, $\|x - x_0\| < \delta$, we observe that $v(x) \in \partial G^0 \cap \mathbf{N}_G(x)$ and $\mathfrak{J}_G(v(x)) = \{x\}$, and define $\nu := \varkappa_G(x_0, v_0) > 0$ where $v_0 := v(x_0)$. By the formula (22) from [16] there exist $\theta \geq \nu/2$ and $0 < \delta' \leq \delta$ such that

$$\widehat{\mathfrak{C}}_G(r, x, v) \geq \theta r^2 \tag{75}$$

whenever $\|x - x_0\| \leq \delta'$, $\|v - v_0\| \leq \delta'$ with $\mathfrak{J}_G(v) = \{x\}$, $x \in \partial G^0$ and $r > 0$. By using continuity of the mapping $x \mapsto \mathfrak{n}_C(x)$ (consequently, of $v(\cdot)$) we find $0 < \varepsilon \leq \delta'$ so small that $\|v(x) - v_0\| \leq \delta'$ whenever $\|x - x_0\| \leq \varepsilon$.

Let us set now $\mathcal{U}(x_0) := x_0 + \frac{\varepsilon}{2\|F^0\|\|F\|}\overline{B}$ and take arbitrary points $z_1, z_2 \in \mathcal{U}(x_0)$. Due to [16, Proposition 7.7(i)] there exist (unique) time-minimum projections $\bar{z}_1 := \pi_C^F(z_1)$ and $\bar{z}_2 := \pi_C^F(z_2)$, which belong to the boundary of G, and, moreover, $\|\bar{z}_i - x_0\| \leq \varepsilon$, $i = 1, 2$. Applying inequality (75) with the points $\bar{z}_i$ and taking into account (74) we obtain from definition (5):

$$\frac{\nu}{2}\rho_{G^0}(\mathfrak{n}_C(\bar{z}_1))\|\bar{z}_1 - \bar{z}_2\|^2 \leq \langle \bar{z}_1 - \bar{z}_2, \mathfrak{n}_C(\bar{z}_1)\rangle \tag{76}$$

and

$$\frac{\nu}{2}\rho_{G^0}(\mathfrak{n}_C(\bar{z}_2))\|\bar{z}_1 - \bar{z}_2\|^2 \leq \langle \bar{z}_2 - \bar{z}_1, \mathfrak{n}_C(\bar{z}_2)\rangle . \tag{77}$$

On the other hand, from the necessary condition of optimality applied to the function $y \mapsto \rho_F(y - z_i) + \mathbf{I}_C(y)$ it follows that

$$0 \in \partial\rho_F(\bar{z}_i - z_i) + \mathbf{N}_C^p(\bar{z}_i)$$

(observe that $\mathbf{N}_C^p(\bar{z}_i)$ is the normal cone to the convex set G, i.e., the semiline generated by the vector $\mathfrak{n}_C(\bar{z}_i)$). Consequently, there exists $\lambda_i > 0$ with

$$-\lambda_i \mathfrak{n}_C(\bar{z}_i) \in \partial\rho_F(\bar{z}_i - z_i) = N_F\left(\frac{\bar{z}_i - z_i}{\rho_i}\right) \cap \partial F^0 \tag{78}$$

(in fact, $\lambda_i = \frac{1}{\rho_{F^0}(-\mathfrak{n}_C(\bar{z}_i))}$) where $\rho_i := \mathfrak{T}_C^F(z_i) = \rho_F(\bar{z}_i - z_i)$, $i = 1, 2$.

In the remainder of the proof we consider only the case when $\rho_1 = \rho_2 = \rho$ because the general situation can be easily treated as in the proof of Theorem 3.3 (see Step 2). Then by the convexity of F it follows from (78) that

$$\langle \mathfrak{n}_C(\bar{z}_1), (\bar{z}_2 - z_2) - (\bar{z}_1 - z_1)\rangle \geq 0 \tag{79}$$

and

$$\langle \mathfrak{n}_C(\bar{z}_2), (\bar{z}_1 - z_1) - (\bar{z}_2 - z_2)\rangle \geq 0. \tag{80}$$

In particular, combining inequalities (79) and (76) we obtain that

$$\frac{\nu}{2}\rho_{G^0}(\mathfrak{n}_C(\bar{z}_1))\|\bar{z}_1 - \bar{z}_2\|^2 \leq \langle \mathfrak{n}_C(\bar{z}_1), z_1 - z_2\rangle \leq \|z_1 - z_2\| .$$

Hence, applying (4) we have

$$\|\bar{z}_1 - \bar{z}_2\| \le \mathfrak{L} \|z_1 - z_2\|^{1/2}, \tag{81}$$

where $\mathfrak{L} := \left[\frac{2}{\nu} \|G^0\|\right]^{1/2}$.

In order to prove the second part of theorem let us denote by $\mathfrak{h} > 0$ a Hölder constant of $\mathfrak{n}_C(\cdot)$ on $\partial C \cap \left(x_0 + \delta \overline{B}\right)$. So,

$$\|\mathfrak{n}_C(\bar{z}_1) - \mathfrak{n}_C(\bar{z}_2)\| \le \mathfrak{h} \|\bar{z}_1 - \bar{z}_2\|^{\alpha}. \tag{82}$$

Now, adding (76) and (77) we have

$$\nu \|\bar{z}_1 - \bar{z}_2\|^2 \le \left\|G^0\right\| \langle \bar{z}_1 - \bar{z}_2, \mathfrak{n}_C(\bar{z}_1) - \mathfrak{n}_C(\bar{z}_2)\rangle, \tag{83}$$

while inequalities (79) and (80) imply that

$$\langle \mathfrak{n}_C(\bar{z}_1) - \mathfrak{n}_C(\bar{z}_2), \bar{z}_1 - \bar{z}_2\rangle \le \langle \mathfrak{n}_C(\bar{z}_1) - \mathfrak{n}_C(\bar{z}_2), z_1 - z_2\rangle. \tag{84}$$

Combining (83) with (84) and applying (82) we obtain:

$$\begin{aligned} \nu \|\bar{z}_1 - \bar{z}_2\|^2 \le & \left\|G^0\right\| \|\mathfrak{n}_C(\bar{z}_1) - \mathfrak{n}_C(\bar{z}_2)\| \|z_1 - z_2\| \\ & \le \mathfrak{h} \left\|G^0\right\| \|\bar{z}_1 - \bar{z}_2\|^{\alpha} \|z_1 - z_2\|, \end{aligned}$$

or, after simple transformations,

$$\|\bar{z}_1 - \bar{z}_2\| \le \mathfrak{L}' \|z_1 - z_2\|^{\frac{1}{2-\alpha}}$$

where the Hölder constant $\mathfrak{L}' := \left[\frac{\mathfrak{h}}{\nu} \|G^0\|\right]^{\frac{1}{2-\alpha}}$. □

With minor modifications the latter proof can be adapted to the case when in place of Hölder regularity of the normal vector to the target set one requires the same property for the dynamics (in a neighbourhood of some given point).

Theorem 4.2. *Let us fix $x_0 \in \partial C$ and assume that the following hypotheses hold:*

(*i*) *the boundary ∂C is smooth at x_0 with the normal vector $\mathfrak{n}_C(\cdot)$;*

(*ii*) *the set $G := C \cap \left(x_0 + \delta \overline{B}\right)$ has nonempty interior and is strictly convex of second order for some $\delta > 0$;*

(*iii*) *the vector $\mathfrak{v}_0 := -\frac{\mathfrak{n}_C(x_0)}{\rho_{F^0}(-\mathfrak{n}_C(x_0))} \in \partial F^0$ supports F at a unique point $\xi_0 \in \partial F$ (in other words, ξ_0 is a strongly exposed point of F w.r.t. $\mathfrak{v}_0$);*

(*iv*) *the set F is smooth at ξ_0, and its (unit) normal vector $\mathfrak{n}_F(\cdot)$ satisfies the Hölder inequality in a neighbourhood of this point with an exponent $0 < \alpha \le 1$.*

Then $\pi_C^F(\cdot)$ is Hölder continuous near x_0 with the exponent $\frac{1}{2-\alpha}$.

Proof. Keeping all the notations from the proof of Theorem 4.1 and repeating its steps let us indicate some changements, which should be made.

Similarly, as in the proof of Theorem 3.8(ii) we can use here the Hölder inequality for the gradient $\nabla\rho_F(\cdot)$ (in the place of $\mathfrak{n}_F(\cdot)$). Denote by $\mathfrak{h}>0$ the respective Hölder constant. So, for some $\delta''>0$ we have

$$\|\nabla\rho_F(\xi_1)-\nabla\rho_F(\xi_2)\|\leq\mathfrak{h}\,\|\xi_1-\xi_2\|^\alpha$$

whenever $\xi_i\in\partial F$, $\|\xi_i-\xi_0\|\leq\delta''$, $i=1,2$. Let us show that for a suitable choice of $\varepsilon>0$ each point $z\in\mathcal{U}(x_0):=x_0+\frac{\varepsilon}{2\|F^0\|\|F\|}\overline{B}$ has a (unique) projection $\bar z:=\pi_C^F(z)$ such that $\frac{\bar z-z}{\rho_F(\bar z-z)}$ belongs to the δ''-neighbourhood of ξ_0. Existence of the unique projection $\bar z$ follows from [16, Proposition 7.7(i)], while the relation (78) (applied to an arbitrary $z\in\mathcal{U}(x_0)$ in the place of z_i) means, in particular, that

$$\frac{\bar z-z}{\rho_F(\bar z-z)}\in\mathfrak{J}_F(\nabla\rho_F(\bar z-z)) \tag{85}$$

and that

$$\nabla\rho_F\left(\frac{\bar z-z}{\rho_F(\bar z-z)}\right)=\nabla\rho_F(\bar z-z)=-\frac{\mathfrak{n}_C(\bar z)}{\rho_{F^0}(-\mathfrak{n}_C(\bar z))}. \tag{86}$$

Since the duality mapping $\mathfrak{J}_F(\cdot)$ is Hausdorff continuous at $\mathfrak{v}_0$ (see [16, Proposition 3.3(ii)]), by (85) there exists $\varepsilon'>0$ such that

$$\left\|\frac{\bar z-z}{\rho_F(\bar z-z)}-\xi_0\right\|\leq\delta''$$

whenever $\|\nabla\rho_F(\bar z-z)-\mathfrak{v}_0\|\leq\varepsilon'$. On the other hand, the latter inequality is equivalent to

$$\left\|\frac{\mathfrak{n}_C(\bar z)}{\rho_{F^0}(-\mathfrak{n}_C(\bar z))}-\frac{\mathfrak{n}_C(x_0)}{\rho_{F^0}(-\mathfrak{n}_C(x_0))}\right\|\leq\varepsilon' \tag{87}$$

(see (86)), and surely there exists $\varepsilon>0$ such that (87) holds for all $z\in\mathcal{U}(x_0)$.

Taking now arbitrary $z_1,z_2\in\mathcal{U}(x_0)$, denoting by $\bar z_i:=\pi_C^F(z_i)$ and assuming as earlier that $\rho_i:=\mathfrak{T}_C^F(z_i)=\rho$, $i=1,2$, by the convexity of F we have (see (78) and (86)):

$$\left\langle\nabla\rho_F\left(\frac{\bar z_1-z_1}{\rho}\right)-\nabla\rho_F\left(\frac{\bar z_2-z_2}{\rho}\right),(\bar z_1-z_1)-(\bar z_2-z_2)\right\rangle\geq 0. \tag{88}$$

Furthermore, multiplying the inequalities (76) and (77), respectively, by λ_1 and λ_2, adding them and taking into account (78), we obtain that

$$\frac{\nu}{\|F\|\,\|G^0\|}\|\bar z_1-\bar z_2\|^2\leq\left\langle\bar z_2-\bar z_1,\nabla\rho_F\left(\frac{\bar z_1-z_1}{\rho}\right)-\nabla\rho_F\left(\frac{\bar z_2-z_2}{\rho}\right)\right\rangle.$$

Hence, comparing with (88) and recalling (82),

$$\begin{aligned}\frac{\nu}{\|F\|\,\|G^0\|}\|\bar z_1-\bar z_2\|^2&\leq\left\|\nabla\rho_F\left(\frac{\bar z_1-z_1}{\rho}\right)-\nabla\rho_F\left(\frac{\bar z_2-z_2}{\rho}\right)\right\|\|z_1-z_2\|\\&\leq\mathfrak{h}\rho^{-\alpha}\|(\bar z_1-\bar z_2)+(z_2-z_1)\|^\alpha\|z_1-z_2\|\\&\leq\mathfrak{h}\rho^{-\alpha}\left(\|\bar z_1-\bar z_2\|^\alpha+\|z_1-z_2\|^\alpha\right)\|z_1-z_2\|.\end{aligned}$$

Representing the latter inequality in the form

$$\|\bar{z}_1 - \bar{z}_2\|^{\alpha} \left[\frac{\nu}{\|F\| \|G^0\|} \|\bar{z}_1 - \bar{z}_2\|^{2-\alpha} - \mathfrak{h}\rho^{-\alpha} \|z_1 - z_2\| \right] \le \mathfrak{h}\rho^{-\alpha} \|z_1 - z_2\|^{\alpha+1},$$

and using the same argument as in the previous theorems, we arrive at

$$\frac{\nu}{\|F\| \|G^0\|} \|\bar{z}_1 - \bar{z}_2\|^{2-\alpha} \le 2\mathfrak{h}\rho^{-\alpha} \|z_1 - z_2\|,$$

and everything is proved. □

Remark 4.3. Notice that the Hölder constants of $\pi_C^F(\cdot)$ found in Theorems 4.1 and 4.2 depend only on the point x_0. So, we have in fact the global Hölderianity of the time-minimum projection on $\mathcal{U}(x_0)$. Furthermore, these constants grow up whenever the curvature $\varkappa_G(x_0, v_0)$ becomes smaller.

5. Smoothness of the time-minimum function

At the beginning of this section we establish an auxiliary result showing that the Clarke regularity (and, consequently, lower regularity) of the function $\mathfrak{T}_C^F(\cdot)$ at a point $x \notin C$ takes place under some enough strong hypothesis on the *modulus of continuity* of the projection $\pi_C^F(\cdot)$ (assuming, certainly, that $\pi_C^F(y)$ is a singleton for y close to x). Observe that the Clarke regularity is necessary for existence of the continuous (Fréchet) gradient $\nabla\mathfrak{T}_C^F(x)$, and it is needed to prove the statements below.

For $x \notin C$ and $r > 0$ we set

$$\omega(x; r) := \sup\left\{ \left\|\pi_C^F(y) - \pi_C^F(x)\right\| : \|y - x\| \le r \right\}.$$

Proposition 5.1. *Let us fix $x \in H \diagdown C$ and assume that the mapping $z \mapsto \pi_C^F(z)$ is well defined and single-valued in some neighbourhood $U(x)$ of the point x. Suppose also that*

$$\lim_{r \to 0+} \frac{\omega(x; r)}{\sqrt{r}} = 0 \tag{89}$$

and that the target set C is proximally regular at $\bar{x} := \pi_C^F(x)$. Then the function $\mathfrak{T}_C^F(\cdot)$ is Clarke (and lower) regular at x. Furthermore, the following equalities hold:

$$\partial^c \mathfrak{T}_C^F(x) = \partial^f \mathfrak{T}_C^F(x) = -\partial\rho_F(\bar{x} - x) \cap \mathbf{N}_C^p(\bar{x}). \tag{90}$$

Proof. First of all, since the (unique) projection $\bar{x} := \pi_C^F(x)$ exists, by using [14, Theorem 3.3] and the proximal regularity of C at $\bar{x}$ we have

$$\partial^f \mathfrak{T}_C^F(x) \subset -\partial\rho_F(\bar{x} - x) \cap \mathbf{N}_C^p(\bar{x}). \tag{91}$$

Let us prove the opposite inclusion. To this end we fix a vector ζ from the right-hand side of (91) and a number $\sigma = \sigma(\bar{x}, \zeta) > 0$ such that

$$\langle \zeta, y - \bar{x} \rangle \le \sigma \|y - \bar{x}\|^2 \tag{92}$$

whenever $y \in C$. On the other hand, by the definition of subdifferential of a convex function

$$\rho_F(z) \geq \rho_F(\bar{x} - x) + \langle -\zeta, z - \bar{x} + x \rangle \quad \forall z \in H. \tag{93}$$

In particular, substituting $z = \bar{y} - y$, $\bar{y} := \pi_C^F(y)$, into (93) we have that

$$\rho_F(\bar{y} - y) - \rho_F(\bar{x} - x) - \langle \zeta, y - x \rangle \geq \langle -\zeta, \bar{y} - \bar{x} \rangle \tag{94}$$

for all $y \in H$ enough close to x. Comparing (94) with (92) and passing to $\liminf$ as $y \to x$ we successively obtain

$$\begin{aligned}
&\liminf_{x \neq y \to x} \frac{\mathfrak{T}_C^F(y) - \mathfrak{T}_C^F(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \\
&= \liminf_{x \neq y \to x} \frac{\rho_F(\bar{y} - y) - \rho_F(\bar{x} - x) - \langle \zeta, y - x \rangle}{\|y - x\|} \\
&\geq \liminf_{x \neq y \to x} \left(-\sigma \frac{\left\| \pi_C^F(y) - \pi_C^F(x) \right\|^2}{\|y - x\|} \right) \\
&= -\sigma \left[\limsup_{x \neq y \to x} \frac{\left\| \pi_C^F(y) - \pi_C^F(x) \right\|}{\sqrt{\|y - x\|}} \right]^2 \geq -\sigma \left[\limsup_{r \to 0+} \frac{\omega(r)}{\sqrt{r}} \right]^2 .
\end{aligned}$$

Hence, by using the assumption (89) and the definition of Fréchet subgradients (see (15)) we conclude that $\zeta \in \partial^f \mathfrak{T}_C^F(x)$. Thus, the equality

$$\partial^f \mathfrak{T}_C^F(x) = -\partial \rho_F(\bar{x} - x) \cap \mathbf{N}_C^p(\bar{x}) \tag{95}$$

is proved. It is enough to show now that

$$\partial^c \mathfrak{T}_C^F(x) \subset \partial^f \mathfrak{T}_C^F(x), \tag{96}$$

because the opposite inclusion always holds (see (18)).

By the definition (16) due to the Lipschitz continuity of the function $\mathfrak{T}_C^F(\cdot)$ each vector $\zeta \in \partial^l \mathfrak{T}_C^F(x)$ can be represented as the weak limit $w\text{-}\lim_{n \to \infty} \zeta_n$, where $\zeta_n \in \partial^p \mathfrak{T}_C^F(x_n)$, $x_n \to x$ as $n \to \infty$. Since for $n = 1, 2, ...$ large enough the (unique) projection $\bar{x}_n := \pi_C^F(x_n)$ exists, applying again [14, Theorem 3.3] we have that $\zeta_n \in \mathbf{N}_C^p(\bar{x}_n)$ and $-\zeta_n \in \partial \rho_F(\bar{x}_n - x_n)$. From the first relation taking into account that $\bar{x}_n \to \bar{x}$ we have $\zeta \in \mathbf{N}_C^l(\bar{x})$, while from the second and from the $s \times w$-closedness of the graph of the subdifferential $\partial \rho_F(\cdot)$ it follows that $-\zeta \in \partial \rho_F(\bar{x} - x)$. Therefore, by the regularity of the set C the vector ζ belongs to $-\partial \rho_F(\bar{x} - x) \cap \mathbf{N}_C^f(\bar{x})$. The latter set being convex and closed, we conclude that

$$\partial^c \mathfrak{T}_C^F(x) = \overline{\text{co}}\, \partial^l \mathfrak{T}_C^F(x) \subset -\partial \rho_F(\bar{x} - x) \cap \mathbf{N}_C^f(\bar{x}),$$

and the inclusion (96) follows immediately from (95). □

If we suppose that the time-minimum projection $\pi_C^F(\cdot)$ is Hölder continuous near x with an exponent larger than $1/2$ then the condition (89) will be fulfilled not only at the point x itself but in its neighbourhood. Therefore, under the supplementary assumption of proximal regularity of C, equalities (90) also take place near x, and in order to have smoothness of the time-minimum function it is enough to require that the intersection in (90) is a continuous singleton. So, we have the first smoothness result.

Theorem 5.2. *Let $x \in H \diagdown C$ be such that $z \mapsto \pi_C^F(z)$ is single-valued and Hölder continuous with an exponent $1/2 < \beta \leq 1$ on a neighbourhood $U(x)$. Assume, moreover, that the target set C is proximally regular at each point near $\bar{x} := \pi_C^F(x)$, and that the mapping $\Phi : U(x) \to \partial F^0$, $\Phi(y) := -\partial\rho_F(\bar{y} - y) \cap \mathbf{N}_C^p(y)$, is single-valued and continuous. Then the function $\mathfrak{T}_C^F(\cdot)$ is (Fréchet) continuously differentiable at x, and $\nabla\mathfrak{T}_C^F(x) = -\Phi(x)$.*

Remark 5.3. In a finite dimensional space the mapping $\Phi(\cdot)$, obviously, admits a closed graph. So, in Theorem 5.2 it is enough to require only that the sets $\Phi(y)$, $y \in U(x)$, are singletons.

Taking into account that the subdifferential $\partial\rho_F(\cdot)$ is represented through the normal cone to the (convex) set F, we can express the last assumption of Theorem 5.2 in terms of the smoothness properties of the involved sets and obtain nice formulas for the gradient of the time-minimum function.

Corollary 5.4. *Assume as in Theorem 5.2 that the time-minimum projection $\pi_C^F(\cdot)$ is single-valued and Hölder continuous with an exponent $1/2 < \beta \leq 1$ in a neighbourhood $U = U(x)$ of a point $x \in H \diagdown C$, and that the set C is proximally regular near $\bar{x} := \pi_C^F(x)$. Then the value function $\mathfrak{T}_C^F(\cdot)$ is (Fréchet) continuously differentiable at x if at least one of the two following conditions holds:*

(i) *C has smooth boundary at $\bar{x}$ with the unit normal vector $\mathfrak{n}_C(\cdot)$;*
(ii) *F is smooth at $\xi := \frac{\bar{x}-x}{\rho_F(\bar{x}-x)}$.*

Furthermore, in the first case we have

$$\nabla\mathfrak{T}_C^F(x) = \frac{\mathfrak{n}_C(\bar{x})}{\rho_{F^0}(-\mathfrak{n}_C(\bar{x}))}, \tag{97}$$

while in the second

$$\nabla\mathfrak{T}_C^F(x) = -\nabla\rho_F\left(\frac{\bar{x}-x}{\rho_F(\bar{x}-x)}\right). \tag{98}$$

Proof. It is enough to observe that the set $\Phi(y)$ from Theorem 5.2 is contained either in $\left\{-\frac{\mathfrak{n}_C(\bar{y})}{\rho_{F^0}(-\mathfrak{n}_C(\bar{y}))}\right\}$ or in $\left\{\nabla\rho_F\left(\frac{\bar{y}-y}{\rho_F(\bar{y}-y)}\right)\right\}$ whenever, respectively, the condition (i) or the condition (ii) is fulfilled. On the other hand, $\Phi(y) \neq \varnothing$ for $y \in U(x)$ due to equalities (90). □

Combining the previous statement with the Hölder continuity we have

Corollary 5.5. *Under the assumptions and notations of Corollary 5.4 suppose, in addition, that the normal vector $\mathfrak{n}_C(\cdot)$ (respectively, $\nabla\rho_F(\cdot)$) is Hölderian near $\bar{x}$ (respectively, near ξ) with the exponent $0 < \alpha \leq 1$. Then the value function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\alpha\beta}(U)$.*

Proof. In the first case it is enough to observe that the gradient $\nabla\mathfrak{T}_C^F(\cdot)$ given by the formula (97) is the composition of the normal vector $\mathfrak{n}_C(\cdot)$ and the projection $\pi_C^F(\cdot)$. Therefore, taking into account the bounds of $\rho_{F^0}(-\mathfrak{n}_C(\cdot))$, for all $x_1, x_2 \in U(x)$ we can write

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \le 2\mathfrak{h}_C \|F\| \left\|F^0\right\|^2 \mathfrak{h}^\alpha \|x_1 - x_2\|^{\alpha\beta},$$

where $\mathfrak{h}_C > 0$ is the Hölder constant of $\mathfrak{n}_C(\cdot)$ near $\bar{x}$ and $\mathfrak{h} > 0$ is the Hölder constant of $\pi_C^F(\cdot)$ on $U(x)$.

If, instead, F is smooth at ξ with the Hölderian gradient $\nabla\rho_F(\cdot)$ then assuming without loss of generality that $U(x) \subset x + \frac{\mathfrak{T}_C^F(x)}{2\|F^0\|}\overline{B}$, taking $x_1, x_2 \in U(x)$ and setting $\rho_i := \rho_F(\bar{x}_i - x_i)$, $\xi_i := \frac{\bar{x}_i - x_i}{\rho_i}$, $i = 1, 2$, we obtain that

$$\begin{aligned}
\|\xi_1 - \xi_2\| &= \frac{1}{\rho_1\rho_2} \|(\bar{x}_1 - x_1)\rho_2 - (\bar{x}_2 - x_2)\rho_1\| \\
&\le \frac{1}{\rho_1\rho_2} \left(\|\bar{x}_1 - x_1\| |\rho_1 - \rho_2| + \rho_1 \|\bar{x}_1 - \bar{x}_2 + x_2 - x_1\|\right) \\
&\le \frac{1}{\rho_2} \left(\|F\| \left|\mathfrak{T}_C^F(x_1) - \mathfrak{T}_C^F(x_2)\right| + \|x_1 - x_2\| + \left\|\pi_C^F(x_1) - \pi_C^F(x_2)\right\|\right) \\
&\le \frac{1}{\mathfrak{T}_C^F(x_2)} \left[\left(\|F\| \left\|F^0\right\| + 1\right) \|x_1 - x_2\| + \mathfrak{h} \|x_1 - x_2\|^\beta\right] \qquad (99)
\end{aligned}$$

and

$$\mathfrak{T}_C^F(x_2) \ge \mathfrak{T}_C^F(x) - \left\|F^0\right\| \|x - x_2\| \ge \frac{1}{2}\mathfrak{T}_C^F(x). \qquad (100)$$

Since the gradient $\nabla\mathfrak{T}_C^F(\cdot)$ in this case is given by formula (98), combining inequalities (99), (100) and

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \le h_F \|\xi_1 - \xi_2\|^\alpha,$$

where $\mathfrak{h}_F > 0$ is the Hölder constant of $\nabla\rho_F(\cdot)$, we find that

$$\left\|\nabla\mathfrak{T}_C^F(x_1) - \nabla\mathfrak{T}_C^F(x_2)\right\| \le \bar{\mathfrak{h}} \|x_1 - x_2\|^{\alpha\beta}$$

for some $\bar{\mathfrak{h}} > 0$ depending on the point x only. □

The previous statements are proved under an *a priori* assumption that the time-minimum projection $\pi_C^F(\cdot)$ is enough regular (Hölder continuous with an exponent $\beta > 1/2$) in a neighbourhood of a fixed point outside the target set. Let us return now to the hypotheses (A) and (B), which, according to the results of Section 3, guarantee such regularity in proximity of the target.

Observe that the hypothesis (A) gives automatically the Hölder continuity of $\pi_C^F(\cdot)$ with $\beta = 1$ (see Theorem 3.1), while (B) guarantees *a priori* the Hölder continuity only with the exponent $\beta = 1/2$ (Theorem 3.3), which is not enough even for the Clarke regularity of the value function. However, imposing a smoothness assumption either on the target set or on the dynamics (Hölderianity of the respective normals with an arbitrary exponent $\alpha > 0$) we improve the Hölder regularity of the projection up to the exponent $\beta = \frac{1}{2-\alpha} > \frac{1}{2}$ (Theorem 3.8) and can apply Corollary 5.5 to treat the smoothness of $\mathfrak{T}_C^F(\cdot)$. Let us formulate the respective theorems.

Theorem 5.6. *Let $x_0 \in \partial C$. Suppose that*

(a) *the pair of sets (F, C) satisfies the condition* (A) *at the point x_0;*
(b) *the target set C is proximally regular near x_0;*
(c) *for each $x \notin C$ close to x_0 one of the properties below is fulfilled:*
  (i) *C has smooth boundary at $\bar{x} := \pi_C^F(x)$, and the unit normal $\mathfrak{n}_C(\cdot)$ is Hölderian near $\bar{x}$ with an exponent $\alpha > 0$;*
  (ii) *F is smooth at $\xi := \frac{\bar{x}-x}{\rho_F(\bar{x}-x)}$, and $\nabla\rho_F(\cdot)$ is Hölderian near ξ with an exponent $\alpha > 0$.*

Then the value function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\alpha}$ on $\mathcal{U}(x_0)\diagdown C$ where $\mathcal{U}(x_0)$ is defined by (22).

Theorem 5.7. *Let $x_0 \in \partial C$. Suppose that*

(a) *the pair of sets (F, C) satisfies the condition* (B) *at the point x_0;*
(b) *for each $x \notin C$ enough close to x_0 one of the properties (i) or (ii) from Theorem* 5.6 *holds.*

Then the time-minimum function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,\frac{\alpha}{2-\alpha}}$ on $\mathcal{U}(x_0)\diagdown C$ where $\mathcal{U}(x_0)$ here is given by (23).

Remark 5.8. In accordance with Theorems 4.1 and 4.2 the condition (B) in the previous statement can be substituted for the assumption that the set $C \cap \left(x_0 + \delta\overline{B}\right)$ (with nonempty interior) is strictly convex of second order, and the boundary ∂C is smooth near x_0 (F is assumed to be an arbitrary gauge here). Below we refer to the latter property as to the condition (C) imposed on the pair (F, C) at the point x_0.

Remark 5.9. Observe more that the conditions (*i*) and (*ii*) of Theorem 5.6 are formulated in terms of the time-minimum projection $\bar{x} := \pi_C^F(x)$, which a priori is not assumed to be known. However, in the place of (*i*) we can require clearly the Hölder regularity of ∂C near $x_0 \in \partial C$ (due to continuity of the projection), while (*ii*) should be substituted for the condition (*ii*) of Theorem 3.8, i.e., for the Hölder regularity of ∂F on the total image of a neighbourhood of x_0 (eventually, without the point x_0 itself) under the mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^0\right)$. The latter assertion follows from the fact that $\frac{\bar{x}-x}{\rho_F(\bar{x}-x)} \in \mathfrak{J}_F(y)$ for some $y \in -\mathbf{N}_C^p(x) \cap \partial F^0$ (this is a direct consequence of the necessary condition of optimality applied to the function $y \mapsto \rho_F(y-x) + \mathbf{I}_C(y)$).

In conclusion let us formulate a global version of Theorems 5.6 and 5.7 assuming that at each point $x_0 \in \partial C$ at least one of the conditions (A), (B) or (C) is fulfilled. Since the results above give different exponents for the Hölder regularity of $\nabla\mathfrak{T}_C^F(\cdot)$ we need to reduce them to the common one. Namely, fixed $0 < \beta \leq 1$ we determine $0 < \alpha \leq 1$ from the condition $\beta = \frac{\alpha}{2-\alpha}$, i.e., $\alpha = \frac{2\beta}{\beta+1}$. So, as an immediate

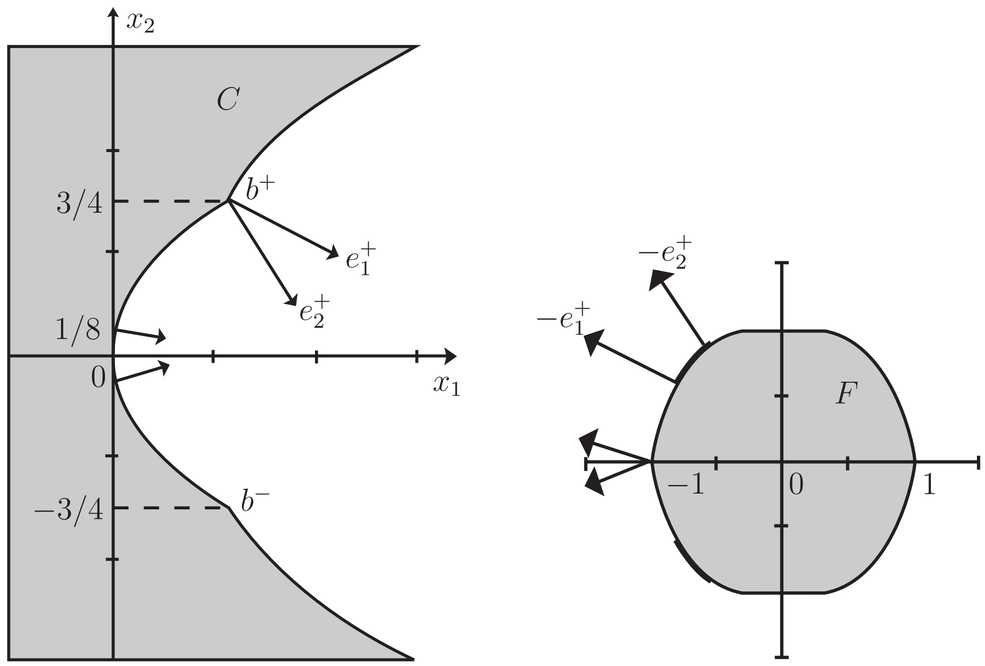

Figure 6.1: Example 6.1

consequence of the two previous theorems (and of Remark 5.8 as well) we have

Theorem 5.10. *Let us suppose that the pair of sets* (F, C) *satisfies at each point* $x_0 \in \partial C$ *one of the conditions* (A), (B) *or* (C), *and that* C *is proximally regular. Moreover, we assume that the hypothesis* (c) *of Theorem* 5.6 *is fulfilled with* $\alpha = \beta$ *if the condition* (A) *holds, and with* $\alpha = \frac{2\beta}{\beta+1}$ *under the condition* (B) *or* (C). *Then there exists an open set* $\mathfrak{A} \supset C$ *such that the time-minimum function* $\mathfrak{T}_C^F(\cdot)$ *is of class* $\mathcal{C}_{loc}^{1,\beta}(\mathfrak{A} \diagdown C)$.

6. Examples

For the sake of geometric vizualization we give here some simple examples in finite dimensions (precisely, $H = \mathbb{R}^2$). Notice that the results obtained in the previous sections are new also in this case.

Example 6.1. We set

$$F := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \le 1 - \xi_1^4, \ -1 \le \xi_1 \le 1 \right\};$$

$$C := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \le \min \left(x_2^2, \left(x_2 - \frac{1}{2} \right)^2 + \frac{1}{2}, \left(x_2 + \frac{1}{2} \right)^2 + \frac{1}{2} \right) \right\}.$$

The same dynamics and a similar target set were considered already in [16] for the scope to illustrate the well-posedness results. Let us supplement inferences made there with the regularity properties of both the time-minimum projection and the value function.

Here F is closed convex bounded with $0 \in \mathrm{int} F$, and the set C is closed and φ-convex with $\varphi(\cdot)$ given by

$$\varphi(x) = \max\left\{ \frac{1}{\sqrt{1+4x_2^2}}, \frac{1}{\sqrt{1+(2x_2-1)^2}} \right\}, \quad x = (x_1, x_2) \in C.$$

Furthermore, C has smooth boundary at each point except $b^{\pm} = \left(\frac{9}{16}, \pm\frac{3}{4}\right)$ with the (unit) normal vector

$$\mathfrak{n}_C(x) = \begin{cases} \frac{1}{\sqrt{1+4x_2^2}}(1, -2x_2) & \text{if } |x_2| < \frac{3}{4}; \\ \frac{1}{\sqrt{1+(2x_2-1)^2}}(1, -2x_2+1) & \text{if } x_2 > \frac{3}{4}; \\ \frac{1}{\sqrt{1+(2x_2+1)^2}}(1, -2x_2-1) & \text{if } x_2 < -\frac{3}{4}, \end{cases} \tag{101}$$

while at the points $b^{\pm}$ we have

$$\mathbf{N}_C^p\left(b^{\pm}\right) = \left\{\lambda_1 e_1^{\pm} + \lambda_2 e_2^{\pm} : \lambda_1 \geq 0, \lambda_2 \geq 0\right\},$$

where $e_1^{\pm} = \left(1, \mp\frac{1}{2}\right)$ and $e_2^{\pm} = \left(1, \mp\frac{3}{2}\right)$ (see Figure 1). Proceeding as in [16, Example 8.4] we conclude that the condition (B) holds at each point $x_0 = (x_1^0, x_2^0) \in \partial C$.

On the other hand, the gauge set F is smooth at every $\xi = (\xi_1, \xi_2) \in \partial F \setminus \{(\pm 1, 0)\}$ with the respective unit normal

$$\mathfrak{n}_F(\xi) = \frac{\nabla \rho_F(\xi)}{\|\nabla \rho_F(\xi)\|} = \frac{1}{\sqrt{16\xi_1^6 + 1}}\left(4\xi_1^3, \operatorname{sgn} \xi_2\right), \tag{102}$$

which is, moreover, locally Lipschitzean on that domain. It is easy to see that given $x_0 \in \partial C$ with $|x_2^0| > 1/8$ there exists $\delta > 0$ such that

$$\bigcup_{x \in C_\delta(x_0)} \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^0\right) \subset \partial F \setminus \{(\pm 1, 0)\}. \tag{103}$$

Indeed, choosing an arbitrary $x \in \partial C$ close to x_0 and assuming $x_2 > 0$ (the other case is symmetric) we consider three situations. In the first one $1/8 < x_2 < 3/4$, and by direct calculations we obtain from (101) that

$$\mathfrak{v}(x) := -\frac{\mathfrak{n}_C(x)}{\rho_{F^0}(-\mathfrak{n}_C(x))} = \frac{8x_2^{1/3}}{3 + 16x_2^{4/3}}(-1, 2x_2).$$

Hence, for a (unique) $\xi = (\xi_1, \xi_2) \in \mathfrak{J}_F(\mathfrak{v}(x))$ we have

$$\xi_1 = -\frac{1}{2x_2^{1/3}} \in \left]-1, 0\right[.$$

If instead $x_2 > 3/4$ then similarly

$$\mathfrak{v}(x) = \frac{3 + 4^{4/3}(2x_2-1)^{4/3}}{4^{4/3}(2x_2-1)^{1/3}}(-1, 2x_2-1),$$

and comparing with (102) we find, in particular, that

$$\xi_1 = \frac{1}{4^{1/3}\left(1-2x_2\right)^{1/3}} \in \left[-\frac{1}{2^{1/3}}, 0\right[.$$

Finally, although at the points $x = b^{\pm}$ the boundary of C is not smooth, the image $\mathfrak{J}_F\left(-\mathbf{N}_C^p\left(b^{\pm}\right)\cap\partial F^0\right)$ (the bold part of ∂F on Figure 1) is far from $(\pm 1, 0)$, and (103) follows.

Thus the condition (ii) of Theorem 3.8 is fulfilled for each $x_0 \in \partial C$, $|x_0| > 1/8$, with $\alpha = 1$, and we can affirm that the time-minimum projection $\pi_C^F(\cdot)$ is Lipschitz continuous on a neighbourhood $\mathcal{U}(x_0)$. Furthermore, from Theorem 5.7 (see the hypothesis (ii) of Theorem 5.6) and from Remark 5.9 we can infer that the value function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,1}$ on the open set $\mathcal{U}(x_0)\diagdown C$. Notice that in the case $x_0 \neq b^{\pm}$ we can apply also either the condition (i) of Theorem 3.8 (because C has smooth boundary near x_0 with Lipschitzean normal vector $\mathfrak{n}_C(\cdot)$ (see (101))) or Theorem 3.1 (due to Lipschitz continuity of the composed function $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$).

On the other hand, if $|x_2^0| \leq 1/8$ then to guarantee the Lipschitz continuity of $\pi_C^F(\cdot)$ we can use only the last two hypotheses (i.e., either the local regularity of ∂C or the condition (A)), since in this case $\xi_0 := \mathfrak{J}_F(\mathfrak{v}(x_0)) = (-1, 0)$, and the smoothness of ∂F at ξ_0 fails. However, the condition (A) is not sufficient for the Lipschitz continuity of the gradient $\nabla\mathfrak{T}_C^F(\cdot)$, and in accordance with Theorem 5.6 we should anyway use the Lipschitzeanity of $\mathfrak{n}_C(\cdot)$ near this point.

Thus, summarizing everything said above we conclude that the projection $\pi_C^F(\cdot)$ is Lipschitz continuous on some open set $\mathfrak{A} \supset C$, and that the time-minimum function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,1}(\mathfrak{A}\diagdown C)$.

Example 6.2. Let

$$F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^2 + (\xi_2 - 1)^2 \leq 4,\ \xi_2 \leq 1\right\};$$
$$C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq f(x_2)\right\},$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(t) = \begin{cases} -\frac{t}{3} - \frac{1}{3} & \text{if } t < -1; \\ -\sqrt{1-t^2} & \text{if } -1 \leq t \leq \frac{\sqrt{3}}{2}; \\ -\frac{1}{2} & \text{if } t > \frac{\sqrt{3}}{2}. \end{cases}$$

Observe that F is closed convex bounded with $0 \in \operatorname{int} F$, and C is closed and φ-convex with $\varphi(x) \equiv 1/2$. Furthermore, C has smooth boundary at each $x = (x_1, x_2) \in \partial C$ except $(0, -1)$ and $\left(-1/2, \sqrt{3}/2\right)$ with the unit normal vector

$$\mathfrak{n}_C(x) = \begin{cases} -x & \text{if } -1 < x_2 < \frac{\sqrt{3}}{2}; \\ (1, 0) & \text{if } x_2 > \frac{\sqrt{3}}{2}; \\ \frac{1}{\sqrt{10}}(3, 1) & \text{if } x_2 < -1. \end{cases}$$

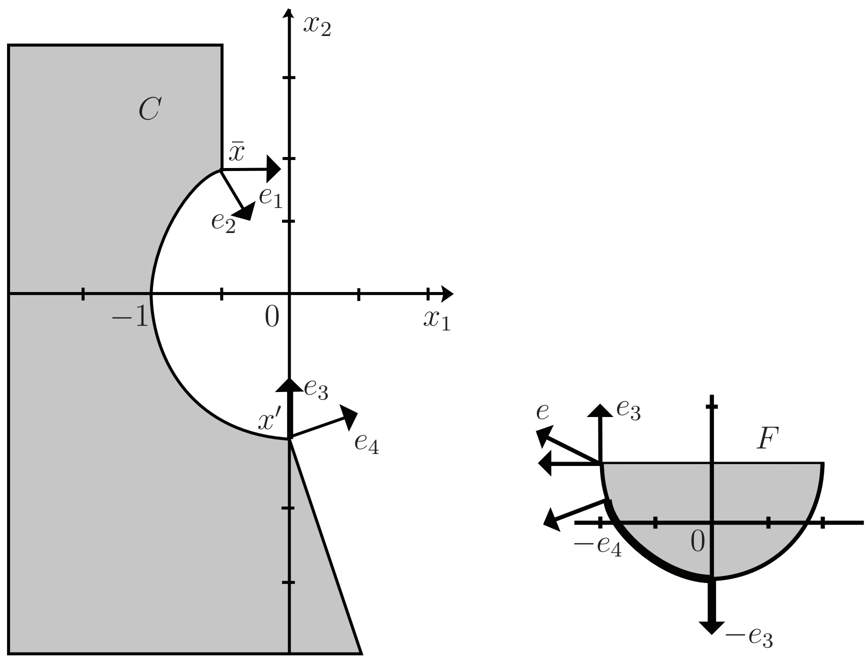

Figure 6.2: Example 6.2

As concerns the dynamics F, resolving a simple mathematical programming problem we obtain that

$$\rho_{F^0}(\xi^*) = \begin{cases} 2|\xi_1^*| + \xi_2^* & \text{if } \xi_2^* \geq 0; \\ 2\sqrt{(\xi_1^*)^2 + (\xi_2^*)^2} + \xi_2^* & \text{if } \xi_2^* < 0. \end{cases}$$

It is obvious that the (scaled) curvature $\varkappa_F(\xi, \xi^*)$ is equal to zero at each $\xi = (\xi_1, 1)$, $-2 < \xi_1 < 2$ (here $\xi^* = (0,1) \in \partial F^0$ is the unique normal to F at ξ), while $\varkappa_F(\xi, \xi^*) = \frac{1}{4\|\xi^*\|}$ for each $\xi^* \in \partial F^0$, $\xi^* = (\xi_1^*, \xi_2^*)$, with $\xi_2^* \leq 0$ and for the respective (unique) $\xi \in \mathfrak{J}_F(\xi^*)$ (see (13)). When both $\xi_2^* > 0$ and $\xi_1^* < 0$ we have instead that the (finite) scaled curvature $\varkappa_F(\xi, \xi^*) \geq \frac{1}{4\|\xi^*\|}$, while clearly $\hat{\varkappa}_F(\xi, \xi^*) = +\infty$. In this case $\xi = (-2, 1)$.

Let us check now that the pair (F, C) verifies the condition (B) at each point $x_0 = (x_1^0, x_2^0) \in \partial C$. To this end it is enough to estimate the scaled curvature of F w.r.t. every vector associated to $x = (x_1, x_2) \in \partial C$. If $-1 < x_1 < -1/2$ and $x_2 > 0$ then the vector $\mathfrak{v}(x) = -\frac{\mathfrak{n}_C(x)}{\rho_{F^0}(-\mathfrak{n}_C(x))}$ supports the set F at the point $(-2, 1)$ and as observed above

$$\varkappa_F((-2,1), \mathfrak{v}(x)) \geq \frac{1}{4\|\mathfrak{v}(x)\|} \geq \frac{1}{4\|F^0\|}. \tag{104}$$

Similarly, if $-1 \leq x_1 < 0$ and $x_2 \leq 0$ then the vector $\mathfrak{v}(x)$ supports F at a unique point ξ belonging to the circumference, and, consequently,

$$\varkappa_F(\xi, \mathfrak{v}(x)) = \frac{1}{4\|\mathfrak{v}(x)\|} \geq \frac{1}{4\|F^0\|} \tag{105}$$

as well. It is easy to see that the same inequalities for the curvatures take place if x belongs to the affine pieces of ∂C. Finally, for each $\xi^* \in -\mathbf{N}_C^p(\bar{x}) \cap \partial F^0$ (see Figure 2) we have $\mathfrak{J}_F(\xi^*) = \{(-2,1)\}$ and $\varkappa_F((-2,1),\xi^*) \geq \frac{1}{4\|F^0\|}$, while the image $\mathfrak{J}_F(-\mathbf{N}_C^p(x') \cap \partial F^0)$ (the bold part of ∂F on Figure 2) is strictly contained in the circumference. So, the estimates on the curvature hold also for the points $\bar{x}$ and x'.

Thus the condition (B) is fulfilled at each $x_0 \in \partial C$, and the projection $\pi_C^F(\cdot)$ is well defined on an open set $\mathfrak{A} \supset C$. Moreover, by (23) taking into account that $\|F^0\| = 1$ the estimates (104) and (105) imply that

$$\mathfrak{A} = \left\{x \in \mathbb{R}^2 : 2\mathfrak{T}_C^F(x) < 1\right\}.$$

By Theorem 3.3 we have also that $\pi_C^F(\cdot)$ is Hölder continuous on $\mathfrak{A}$ with the exponent $1/2$. In order to improve the Hölder regularity of the projection as well as to show the smoothness of the value function (if any) we should exploit the smoothness either of the target set or of the dynamics. Since ∂C is smooth at each $x_0 \neq x', \bar{x}$, and the normal vector $\mathfrak{n}_C(\cdot)$ is Lipschitzean near x_0, we apply first Theorem 3.8(i) and then Corollary 5.5(i) to conclude that $\pi_C^F(\cdot)$ is Lipschitz continuous near x_0, and the time-minimum function $\mathfrak{T}_C^F(\cdot)$ is of class $\mathcal{C}_{loc}^{1,1}$ in a neighbourhood of that point (out of C).

Considering the point $x' = (0,-1)$ we see that F is smooth at each ξ from

$$\bigcup_{x \in C_\delta(x')} \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^0\right)$$

with some $\delta > 0$, and the normal vector $\nabla\rho_F(\xi) / \|\nabla\rho_F(\xi)\|$ is Lipschitz continuous on the latter set, which is little larger than the bold part on Figure 2. Therefore, the Lipschitz continuity of $\pi_C^F(\cdot)$ (of the gradient $\nabla\mathfrak{T}_C^F(\cdot)$) holds also in a neighbourhood of x' by Theorem 3.8(ii) (Corollary 5.5(ii), respectively).

Let us consider now the point $\bar{x} = \left(-1/2, \sqrt{3}/2\right)$. Since the cone $-\mathbf{N}_C^p(\bar{x})$ being contained in $\mathbf{N}_C^p(-2,1)$ does not touch the vertical semiline $\mathbb{R}^+ e_3$, the condition (A) is fulfilled at $\bar{x}$ as well. In turn, this implies (see Theorem 3.1) that the projection $\pi_C^F(\cdot)$ is, in fact, Lipschitz continuous near $\bar{x}$, and we can apply Proposition 5.1, which states that the value function $\mathfrak{T}_C^F(\cdot)$ is Clarke (and lower) regular at each $x \notin C$, close to $\bar{x}$. On the other hand, taking the vector $e = (-2,1)$ we directly see that $\pi_C^F(\bar{x} - \varepsilon e) = \{\bar{x}\}$ for all $\varepsilon > 0$ small enough. Then for $x_\varepsilon = \bar{x} - \varepsilon e$ the intersection

$$-\mathbf{N}_C^p(\bar{x}) \cap \mathbf{N}_F\left(\frac{\bar{x} - x_\varepsilon}{\rho_F(\bar{x} - x_\varepsilon)}\right) \cap \partial F^0 \tag{106}$$

is reduced to $\{\lambda_1 e_1 + \lambda_2 e_2 : \lambda_1 \leq 0, \lambda_2 \leq 0\} \cap \partial F^0$, which is not a singleton. On the other hand, due to formula (90) the set (106) is nothing else than $-\partial^f \mathfrak{T}_C^F(x)$. So, we have no even Fréchet differentiability of the value function near the point $\bar{x}$. This justifies importance of the supplementary regularity hypothesis on F or C (see Theorems 5.6 and 5.7) for differentiability of the value function.

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