Modeling Longevity Risk using Extreme Value Theory: An Empirical Investigation using Portuguese and Spanish Population Data

Jorge Miguel Bravo* Pedro Corte-Real†

March 2012

Abstract

Extreme value theory (EVT) provides a framework to formalize the study of behaviour in the tails of a distribution. In this paper we use EVT to model the statistical behaviour of mortality rates over a given high threshold age and to estimate the significance of rare longevity risk in a given population. We adopt a piecewise approach in estimating the optimal threshold age using an iterative algorithm of maximum likelihood estimation.that statistically determines the cut-off between the central (Gompertz) part of the distribution and the upper tail modelled using the generalized Pareto distribution. The model is empirically tested using the most recent period mortality data for the total, male and female populations of Portugal and Spain. We use some classical results from EVT to estimate the evolution of the theoretical maximum life span over time and to derive confidence intervals for the central estimates. We then use time series methods to forecast the highest attained age. We observe a good fit of the model in all populations and subperiods analysed and on the whole life span considered. We estimate an increase in the theoretical maximum life span over time for all populations, more significant in the male subpopulations.

JEL Code: G22, J11.

Keywords: longevity risk, extreme value theory, life tables, maximum life span.

*Corresponding author: University of Évora - Department of Economics and CEFAGE-UE (Center for Advanced Studies in Management and Economics), Largo dos Colegiais, N.º 2, 7000-803, Évora/Portugal, e-mail: jbravo@uevora.pt. The author gratefully acknowledges partial financial support from FCT, program POCTI.

†Pedro Corte Real. Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal. E-mail: parcr@fct.unl.pt

1 Introduction

Increasing life expectancy at all ages in the developed world is one of the success stories of the last century. Improvements in survival are pushing new limits: today more than half of all males and two thirds of all females born in Western countries may reach their 80th birthday. The proportion of centenarians increased about ten times over the last thirty years, and more and more people celebrate their 100^{th} birthday (Robine & Vaupel, 2001).

These mortality improvements are a clear evidence of how far society and science have come in improving general living conditions, promoting healthier lifestyles, offering better medical and healthcare services that helped prolong our lives. As a result, the demographic structure of the population has changed significantly, with an increasing proportion to the overall mortality improvement in developed countries arising from a faster than expected decrease in mortality rates at advanced ages. Developments in the treatment of heart diseases,greater awareness of the dangers of smoking are just some of the reasons behind this trend that is reflected in the rapidly increasing number of centenarians in the industrialized world (Vaupel, 2010). Unless radical breakthroughs are achieved, humans will continue to suffer senescence although improvements in life expectancy are expected to continue in the near future due to advances in the treatment of cancer or dementia.

These positive news create significant financial challenges to governments that require governments, insurers, pension fund sponsors, individuals and actuaries to understand the drivers in order to estimate future trends. Governments have to predict longevity in order to provide for sustainable pension and healthcare systems. Pension fund sponsors must appropriately fund pensions and other benefits promised to their employees in retirement. Individuals know that they will be increasingly subject to longevity risk and are expected to resume to private solutions in order to supplement their retirement income. Life insurance companies need to meet their customers' requirements by providing annuities and other retirement products, by pricing products in fair and appropriate manner and by guaranteeing they have enough capital to meet future liabilities and uncertainties.

As demand for individual annuity increases, insurers will have to manage the potential longevity risk in ongoing/new new annuity policies. Longevity risk is defined as the uncertainty of mortality improvement in the future. It exists in both individual (idiosyncratic) and aggregate (systematic) levels. In aggregate terms, longevity risk is defined as the risk that members of some reference population

might live longer, on average, than anticipated. The risk of systematic deviations is different in nature from that of random fluctuations around the trend, a well-known type of risk in the insurance business, breaks down the risk pooling mechanism and becomes non-diversifiable, making the provision of risk management tools increasingly difficult. Effectively, the risk of systematic deviations arises from either a "model" or a "parameter" risk, which are unquestionably non-pooling risks.

Individual longevity risk refers to the risk that individuals might live longer than their own expected life expectancy and thus face the risk of outliving their resources. They can reduce the risk of exhausting assets before passing away by consuming less per year, but such strategy then increases the chance that they might die with too much wealth left unconsumed. In other words, dying with too little wealth is undesirable, but having too much wealth is also undesirable, since it represents foregone consumption opportunities. In this scenario, individuals will have to become more self-reliant and will wish to diversify their sources of income in retirement, insuring against the risk through the social security systems, defined benefits plans and, increasingly, through private annuity products.

For actuaries, it has long been crucial to have a reliable model of old-age mortality for pricing and reserve calculations and risk management (e.g., alternative risk transfer mechanisms such as the longevity bond), particularly in products whose cash flows are contingent on survival. In this area, life tables are the most popular instrument used to represent the underlying distribution of future lifetime variable and their construction relies on reliable mortality data. In this regard, although data on population estimates (and corresponding exposure to risk) and death counts at advanced ages is normally available, except for censuses years their quality is considered poor leading to large sampling errors and highly volatile crude death rates. These problems are exacerbated in life insurance or pension fund populations given their smaller dimension.

In the past, in the absence of appropriate mortality data at advanced ages actuaries very often neglected the importance of this phenomena and arbitrarily adopted an ad-hoc procedure by selecting an ultimate age and setting the death probability at that age equal to 1 without any changes to other mortality rates. This creates a discontinuity at the ultimate age compared to the penultimate and prior ages. Given the nature of mortality dynamics at these ages and the increasing importance of survival payments in life insurance and pension fund portfolios, it is inaccurate to close life tables this way. Measuring and managing mortality and longevity risks is a huge challenge for risk managers. The financial

effect of underestimating the life table limiting age can be substantial, not only in terms of expected losses but particularly in terms of risk measures such as VaR or Expected-Shortfall since these quantities heavily rely on the tail of the population survival distribution.

Various methodologies have been proposed for estimating mortality rates at oldest ages and for closing life tables within the insurance industry (for a more detailed review see, for instance, Thatcher *et al.* (1998), Boleslawski and Tabeau (2001), Buettner (2002), Pitacco (2004), Bravo *et al.* (2007) and references therein). Ad-hoc methods include the forced method described above, a blended method consisting in selecting an ultimate age and blend the death probabilities from some earlier age to converge smoothly into 1.0 at the ultimate age, a pattern extrapolation method that simply consists in letting the pattern of mortality continue until the mortality quotient approaches or hits 1 and in setting that as the life table ultimate age and, finally, methods that involve selecting an ultimate age but end the table at whatever rate is produced by the extrapolation procedure at that age given that the ultimate death probability is less than 1.0. Other methods generate population numbers from death registrations, which for the purpose of estimating the number of very old people are considered to be more reliable than population estimates derived from censuses. The most popular methods included in this category are the method of extinct generations and the survivor ratio method.

Other methodologies include fitting mortality curves over a certain age range, for which crude mortality rates may be calculated directly from data, followed by extrapolation. The Coale-Kisker method, named after Coale and Guo (1989) and Coale and Kisker (1990), assumes that the exponential rate of mortality increase at very old ages is not constant, as stipulated by the classical Gompertz and Heligman-Pollard (1980) models, but declines linearly with age. Himes, Preston, and Condran (1994) presented a standard life table model standard for adult ages combined with a Brass-type logit relational model for mortality at the oldest-old ages. Denuit and Goderniaux (2005) developed a log-quadratic regression model on death probabilities observed at advanced ages combined with closure constraints on life tables. These constraints are set to ensure the existence of a horizontal tangent at a pre-defined ultimate age and to prevent an eventual decrease of mortality quotients at very old ages. Kannisto (1992) and Thatcher *et al.* (1998), among others, consider that the logistic function has a convenient asymptotic behaviour (decelerating increase in mortality rates) when it comes to model mortality rates at advanced ages.

In recent years several papers have been published using extreme value theory (EVT) to model human mortality at extremely high ages as an attractive solution for the problems of inaccuracy and unavailability of mortality data at very old ages. Aarssen and De Haan (1994) estimated a finite upper bound on the distribution of human life spans, while Galambos and Macri (2000) argued that such an upper bound could not exist. Thatcher (1999) modeled the highest attainable age by using classical extreme value theory. Watts *et al.* (2006) modeled the highest attained age by using the Generalized Extreme Value (GEV) distribution. Beelders and Colarossi (2004) use EVT to model mortality risk and apply the results to the pricing of the Swiss Re mortality bond issued in 2003.

Han (2005) uses EVT to model the mortality rate for the elderly. Li *et al.* (2008, 2010) use some classical results from EVT to develop a model named threshold life table that can be used to extrapolate survival distributions to extreme ages and to estimate the appropriate end point of a life table. Chen and Cummins (2010) employ extreme value theory to model rare longevity events in the context of longevity risk securitization.

In this paper, we use the threshold life table model proposed by Li *et al.* (2008) to model a 30-year series of recent period life tables produced for the Portuguese and Spanish male, female and total populations. The model integrates EVT with the classical approach of parametric modeling of mortality. More specifically, the model combines the classical Gompertz mortality law for the mortality rates before the threshold and assumes mortality exceedances over the threshold follow a generalized Pareto distribution. The optimal threshold age is obtained through an iterative method of maximum likelihood estimation. This allows us to not only model the mortality data within our samples but also make statistically significant extrapolations of more extreme out of sample longevity events. We observe a good fit of the model in all populations and subperiods analysed and on the whole life span considered. Using some classical results from EVT, we then test for the existence of a finite upper bound for the survival distribution, i.e., for the existence of an end point of a life table and analyse the evolution of the highest attainable age in the sample populations. Confidence intervals for the life table end point are estimated using the classical delta-method. We observe an increasing trend in the limiting age in all populations over time, with slight gender differences. The results of the models are not considered by their own but are benchmarked to results obtained in similar studies.

Finally, we go a step further and use standard time series methods to model the evolution of the limiting age over time and to derive point forecasts and

correspondent confidence intervals for the highest attained age in the future.

The outline of the paper is as follows. Section 2 summarizes some results from EVT and describes the methodology used for mortality modeling. Section 3 presents the data used in this study and applies the threshold life table model to the male, female and total populations of Portugal and Spain. Section 4 considers the use of EVT to estimate the theoretical maximum life span attained in the populations under consideration over time. Section 5 uses time series methods to forecast the theoretical maximum life span. Section 6 concludes.

2 Mortality Modeling

2.1 The tail distribution: Extreme value theory

In this section we summarize some results from EVT which underlie this study. For a detailed review of this subject see, e.g., Embrechts, Klüppelberg and Mikosch (2008). Extreme value theory provides a framework to formalize the study of behaviour in the tails of a distribution. Broadly speaking, there are two types of models for extreme values. Block maxima models apply to maxima of a sequence of observations and the Peaks-Over-Threshold (POT) models deal with exceedances over a given high threshold. In our case, we are interested in the exceedances in the tail distribution of human life span since we focus on the behaviour of mortality rates at advanced ages, i.e., over a given high threshold age u.

Suppose we have a sequence of *iid* random variables $X_1, ..., X_n$, representing risks or losses, from an unknown common distribution function F and let $M_n = \max\{X_1, ..., X_n\}$. A natural measure of extreme events are the values of X_i that exceed a high threshold u. Let x_0 be the finite or infinite right endpoint of the distribution F. That is to say, $x_0 = sup\{x \in R : F(x) < 1\} \leq \infty$. We define the excess distribution above the threshold u as the conditional probability

$$F_u(x) = \mathbb{P}\{X - u \leq x | X > u\} = \frac{F(x+u) - F(u)}{1 - F(u)}, \tag{1}$$

for $0 \leq x < x_0 - u$. $F_u(x)$ is thus the probability that X exceeds the threshold u by no more than an amount x, given that the threshold is exceeded.

According to the Pickand-Balkema-de Haan Theorem (Balkema and de Haan, 1974), for a sufficiently high threshold u, the excess distribution function $F_u(x)$ may be approximated by the generalized Pareto distribution (GDP), $G_{\xi,\theta}(x)$, for

some value of ξ and θ. The GPD, $G_{\xi,\theta}(x)$, is defined here as

$$G_{\xi,\theta}(x) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\theta}\right)^{-1/\xi}, & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{x}{\theta}\right), & \text{if } \xi = 0 \end{cases} \tag{2}$$

where $\theta > 0$, and the support is $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\theta/\xi$ when $\xi < 0$. ξ represents the shape parameter of the distribution or *tail index* and θ is an additional scaling parameter. when $\xi > 0$ we have a reparameterized version of the ordinary Pareto distribution. The case $\xi = 0$ corresponds to the exponential distribution and $\xi < 0$ is usually known as a type II Pareto distribution.

We can extend the GPD family by adding a location parameter γ. The GPD $G_{\xi,\gamma,\theta}(x)$ is then defined to be $G_{\xi,\theta}(x - \gamma)$. Therefore, for $x - u \geq 0$, the distribution function of the ground-up exceedances $F_u(x - u)$ may be approximated by $G_{\xi,\theta}(x - u) = G_{\xi,u,\theta}(x)$.

From the many applications of GPD, in this paper we focus on the behaviour of mortality rates over a given high threshold age u. In this sense, $G_{\xi,u,\theta}(x)$ can be interpreted as follows: Let X represent the time-to-death random variable for a person aged 0. Then, for some high age u, $G_{\xi,u,\theta}(x)$ represents the probability that the person will die before age $u + x$, given survival to age u, that is, in actuarial terms, ${}_xq_u$. Fortunately, the GP distribution provides us with closed expressions for yearly death probabilities q_x and mortality forces μ_x. Let T_x denote the remaining lifetime for an individual aged x at time $t = 0$. For a given high age $x \geq u$, we can derive

$$q_x = 1 - \left(1 + \frac{\xi}{\theta + \xi(x - u)}\right)^{-\frac{1}{\xi}} \tag{3}$$

and

$$\mu_x = \frac{1}{\theta + \xi(x - u)} \tag{4}$$

The above results have been given in terms of stationary sequences of random variables. Interestingly, they can be adapted for use with data from non-stationary sequences, in which characteristics of the stochastic process change with modifications in some related random variable. For instance, the distribution of life spans might shift upward/downward over time due to medical breakthroughs, pandemic episodes or cohort specific covariates. In this case, the GPD parameters can be expressed as functions of time and that information can be used to project the evolution of extreme life spans over time.

2.2 Model Specification

Traditionally, the modeling of an age pattern of mortality is performed in a two-stage piecewise approach. First, parametric or non-parametric graduation techniques are applied to the death probabilities (rates) at young and adult ages in order to smooth out data irregularities and, eventually, to resume the mortality curve by a few parameters. Then, the graduated rates are extrapolated to advanced ages using an appropriate life table closing procedures (e.g., Brass relational model, Heligman-Pollard model, 1980; Coale-Kisker method, 1990; Denuit and Goderniaux, 2005).

This approach has, however, two main drawbacks: first, although some methods (e.g., Denuit and Goderniaux, 2005) provide statistical algorithms for the choice of the threshold age, all the methods require a more or less subjective decision on the age at which the extrapolation begins. Second, all the methods demand an assumption on either the age at which the life table is closed (e.g., Coale-Kisker method, Denuit and Goderniaux, 2005) or, in other cases, an ad-hoc constraint on the profile of mortality rates at advanced ages, e.g., the Gompertz law in the Heligman-Pollard model.

The application of EVT entails the choice of an adequate cut-off between the central part of the distribution and the upper tail, i.e., a point separating ordinary realizations of the random variable considered from extreme realizations of the same variable. When working with threshold exceedances, the cut-off is induced by the threshold age u. This is a very delicate issue concerning statistical methods of EVT, since the choice of the threshold age u entails a trade-off between bias and variance.of the parameter estimates.

On the one hand, we need to choose a high enough u so that the GPD can be applied asymptotically to mortality data (reduce bias). An excessively high threshold leaves us with scant extreme observations, not enough to obtain efficient estimates, yielding imprecise upper quantile estimates. On the contrary, if u is set too low many ordinary data are taken as extreme ones, thus yielding biased estimates. A value of u too small implies that the generalized Pareto character does not hold for the moderate observations and it yields biased quantiles estimates. In both cases, the resulting estimates are flawed and may lead to erroneous conclusions when assessing risk.

To identify the optimal threshold value we can resort to (i) graphical tools, namely to an empirical mean excess function plot or to a plot of the index maximum likelihood estimators resulting from using increasing thresholds, (ii) to common sense-based choices of the cut-off (e.g., choose u in such a way that about

5%-15% of the data are thought of as extreme observations), (iii) to Monte Carlo simulation methods and, finally, (iv) to algorithms (based for instance on the bootstrap method) that endogenously pick out the cut-off that is most suited to the data at hand.

The threshold life table model developed by Li *et al.* (2008) addresses this issue by adopting a piecewise approach in which the threshold age is chosen in a statistical way without the need of any subjective decision and in which the fitted statistical distribution determines, if exists, the appropriate end point of the life table. More specifically, in a threshold life table death rates at earlier adult ages are graduated by means of a parametric function, namely the classical Gompertz (1825) mortality law. At advanced ages, instead of the traditional mathematical extrapolation, the threshold life table model assumes a given extreme value statistical distribution, namely the GPD.

Let $z = \{ X - u | \, X > u \}$ be the conditional exceedances of the age at death X over a given threshold age u. Based on the above extreme value theory results, the threshold life table model is defined as

$$F(x) = \begin{cases} 1 - \exp\left(-\frac{B}{\ln C}\left(C^x - 1\right)\right), & \text{if } x \leq u \\ 1 - \left(1 + \frac{\xi(x-u)}{\theta}\right)^{-1/\xi}, & \text{if } x > u \end{cases} \tag{5}$$

In other words, the threshold life table model assumes that the survival distribution is Gompertzian before the threshold age, and the exceedances over the threshold age u follow a Generalized Pareto distribution. To ensure that $F(x)$ is a proper distribution function, the following constraints must be met: $B > 0$, $C > 1$ and $\theta < 0$. By construction, the model guarantees that $F(x)$ is continuous at the threshold age, but that the smoothness of $F(x)$ around u is not guaranteed and should be carefully addressed in empirical applications. In fact, standard parametric or non-parametric graduation (Splines, Loess,...) methods may be needed to smooth the mortality curve around the threshold age.

2.3 Algorithm for Parameter Estimation

Li *et al.* (2008) propose two methods for choosing the threshold age: a maximum likelihood estimation method, that we adopt here, and a weighted least-squares estimation method. In describing the method, we assume that we are provided with period (static) mortality data for individual ages $x_{\min}$ to $x_{\max} - 1$ and the open age group $x_{\max}$ and above.

Let l_x denote the number of survivors to age x. The number of deaths between ages x and $x + 1$ is therefore $d_x = l_x - l_{x+1}$. The likelihood contribution for each

age $x = x_{\min}, x_{\min}+1, ..., x_{\max}-1$ is the probability of dying between age x and age $x+1$, raised to the number of deaths, or

$$\left(\frac{s(x) - s(x+1)}{s(x_{\min})}\right)^{d_x}.$$

where $s(x) = 1 - F(x)$ is the survival function. The likelihood contribution for the survivors to age $x_{\max}$ is the probability of survival to age $x_{\max}$, raised to the number of survivors, or

$$\left(\frac{s(x_{\max})}{s(x_{\min})}\right)^{l_{x_{\max}}}$$

The resulting likelihood function is therefore

$$\mathcal{L}(B, C, \xi, \theta, u) = \left[\prod_{x=x_{\min}}^{x_{\max}-1}\left(\frac{s(x) - s(x+1)}{s(x_{\min})}\right)^{d_x}\right] \times \left(\frac{s(x_{\max})}{s(x_{\min})}\right)^{l_{x_{\max}}} \quad (6)$$

The logarithm of $\mathcal{L}(B, C, \xi, \theta, u)$ can be decomposed, after some algebra, into the sum of two components, $l_1(B, C, u) + l_2(\xi, \theta, u)$, where

$$l_1(B, C, u) = \sum_{x=x_{\min}}^{u-1}\left(d_x \ln(s(x) - s(x+1))\right) + l_u \ln(s(u)) - l_{x_{\min}} \ln(s(x_{\min})) \quad (7)$$

where $s(x) = \exp\left(-\frac{B}{\ln C}(C^x - 1)\right)$, and

$$l_2(\xi, \theta, u) = \sum_{x=u}^{x_{\max}}\left(d_x\left(\frac{s(x)}{s(u)} - \frac{s(x+1)}{s(u)}\right)\right) + l_{x_{\max}} \ln\left(\frac{s(x_{\max})}{s(u)}\right) \quad (8)$$

where $\frac{s(x)}{s(u)} = \left(1 + \frac{\xi(x-u)}{\theta}\right)^{-1/\xi}$.

For a fixed u, parameter estimation for the parametric (Gompertz) modeling part and the generalized Pareto part can be done separately by maximizing l_1 and l_2, respectively. The choice of u depends on the maximization of the profile log-likelihood function l_p:

$$l_p(u) = l\left(\hat{B}(u), \hat{C}(u), \hat{\xi}(u), \hat{\theta}(u), u\right) \quad (9)$$

where $l = \ln(\mathcal{L})$, $\hat{B}(u)$, $\hat{C}(u)$, $\hat{\xi}(u)$, and $\hat{\theta}(u)$ are the maximum likelihood estimates of B, C, ξ and θ for a fixed u, respectively.

The algorithm for estimating the optimal threshold age u and other parameters in our model can be summarized as follows:

1. for $u = x_{\max} - 1$,

(a) find the values of B and C that maximize $l_1(B, C, u)$;

(b) find the values of ξ and θ that maximize $l_2(\xi, \theta, u)$;

(c) compute the value of the profile log-likelihood, l_p;

2. repeat Step (1) for $u = x_{\max} - 2, x_{\max} - 3, ..., x_{\min}$;

3. Find the value of u that yields the maximum profile log-likelihood.

The value of u obtained in step (3) is the optimal threshold age. The maximum likelihood estimates $\hat{B}$, $\hat{C}$, $\hat{\xi}$ and $\hat{\theta}$ under the optimal threshold age are then considered the optimal model parameter values.

Suppose the threshold excess $X - u_0$ follows a GPD with estimated parameter $\hat{\xi} < 1$. For any $u > u_0$, we can estimate the mean excess lifetime function as

$$e(u) = E[X - u | X > u] = \frac{\hat{\theta}(u_0) + \hat{\xi}(u - u_0)}{1 - \hat{\xi}}. \tag{10}$$

or, alternatively, for a given $y > 0$

$$e(u_0 + y) = E[X - (u_0 + y) | X > u_0 + y] = \frac{\hat{\theta}(u_0) + \hat{\xi} y}{1 - \hat{\xi}} \tag{11}$$

3 Data and Results

In this section we use the threshold life table model to model the most recent period (contemporaneous) mortality rates for the total, male and female populations of Portugal and Spain. The data are period life table functions (deaths, number of survivors,...) by single years of age up to 110 and over provided by the Human Mortality Database (2011), hereafter referred to as the HMD. For both countries, the latest available data on HMD is for year 2009. For Portugal data is available since 1940, whereas for Spain data is available since 1908. For both countries, the use of data prior to 1970 is questioned due to age heaping problems, as detailed in the HMD background and documentation papers. For that reason, we have considered for both populations life table data only for ages $x \in [65, 110+]$ and calendar years $x \in [1980, 2009]$.

In Figures 1 and 2 we take a first look at the evolution of death rates by age and calendar year in the overall populations of both countries. As can be observed, in the last thirty years the death rates have been declining steadily at all ages, with greater speed at ages between 65 and 85. The behaviour of death rates for the male and female populations in both countries shows a similar pattern.

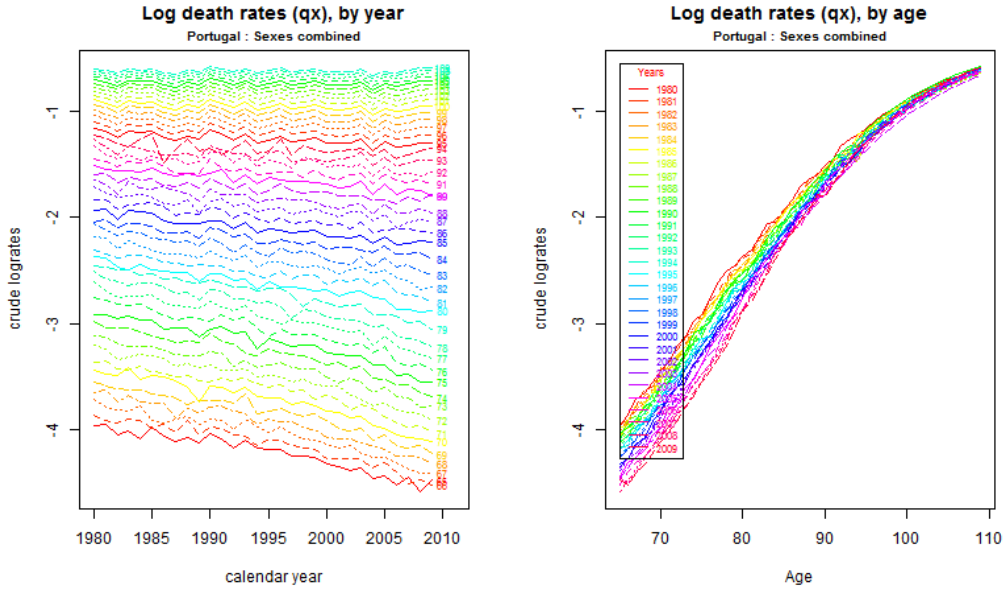

Figure 1: Death rates q_x by year and age, Portugal, sexes combined

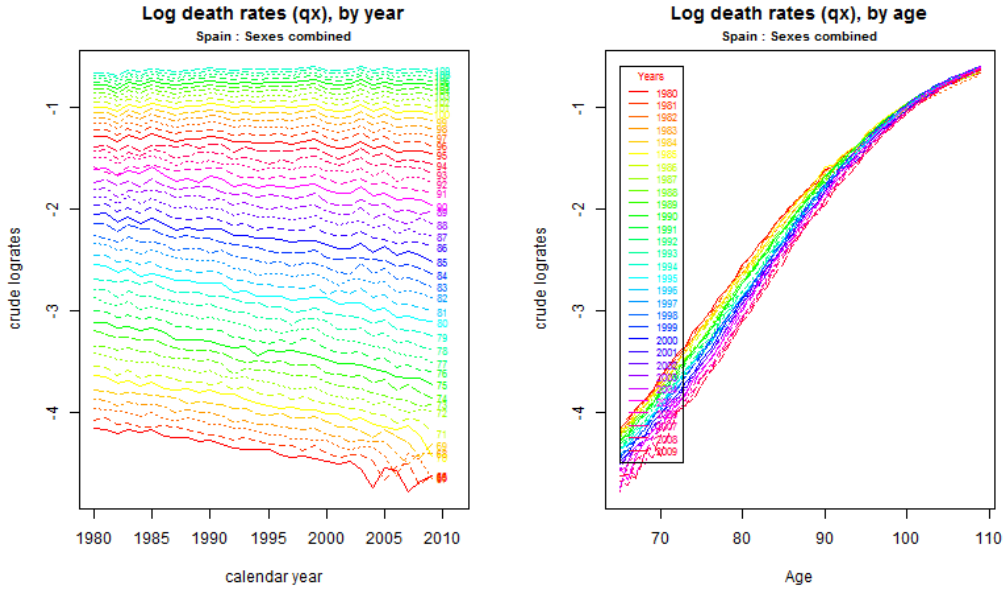

Figure 2: Death rates q_x by year and age, Spain, sexes combined

Next, for each year in the sample period and for both populations and corre-

sponding gender specific sub-populations we implemented the algorithm detailed in 2 using the R language (R Development Core Team, 2011). The search for the optimal threshold age u and corresponding maximum likelihood parameter estimates $\hat{B}$, $\hat{C}$, $\hat{\xi}$ and $\hat{\theta}$ was conducted using an iterative procedure that maximizes the overall fitness of the combined model, rather than the fitness at the extreme ages only. In estimating the optimal threshold age we have considered integer ages in the age interval $[85, 102]$.

Figures 3, 4, 5 and 6 display for each year in the estimation period the optimal threshold age u^*, the maximum likelihood estimates of GPD parameters (ξ, θ) and the mean excess life time over u^*.

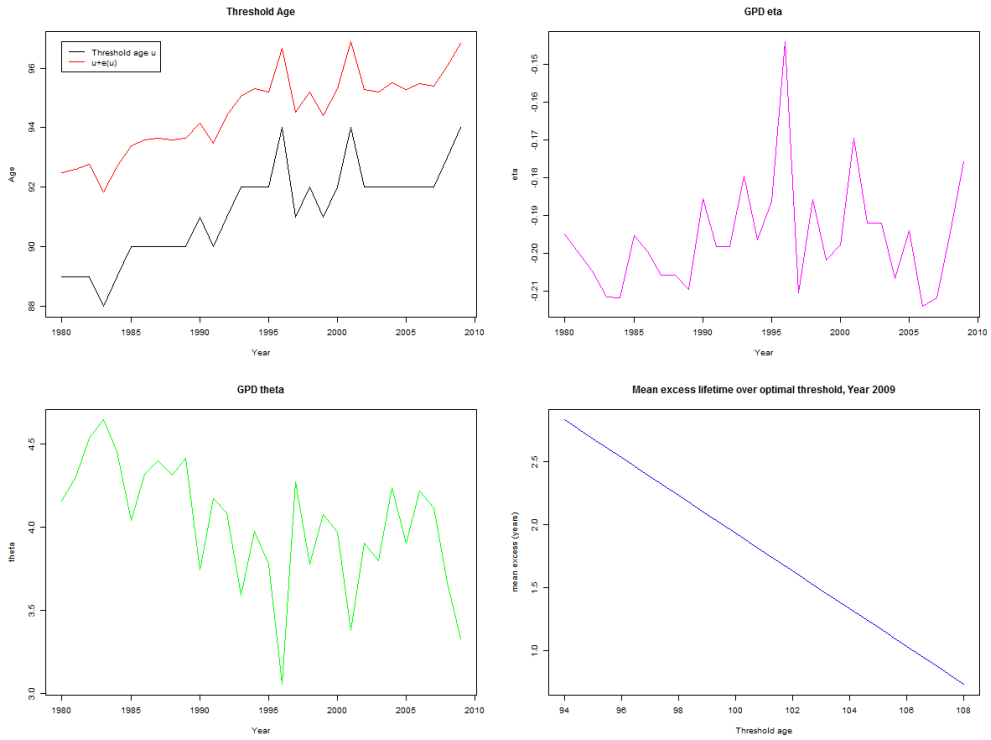

Figure 3: Optimal threshold age u^*, mean excess life time over threshold u^* and GPD parameters by year $t \in [1989, 2009]$, and mean excess life time over threshold u^* in 2009, Portugal, total population.

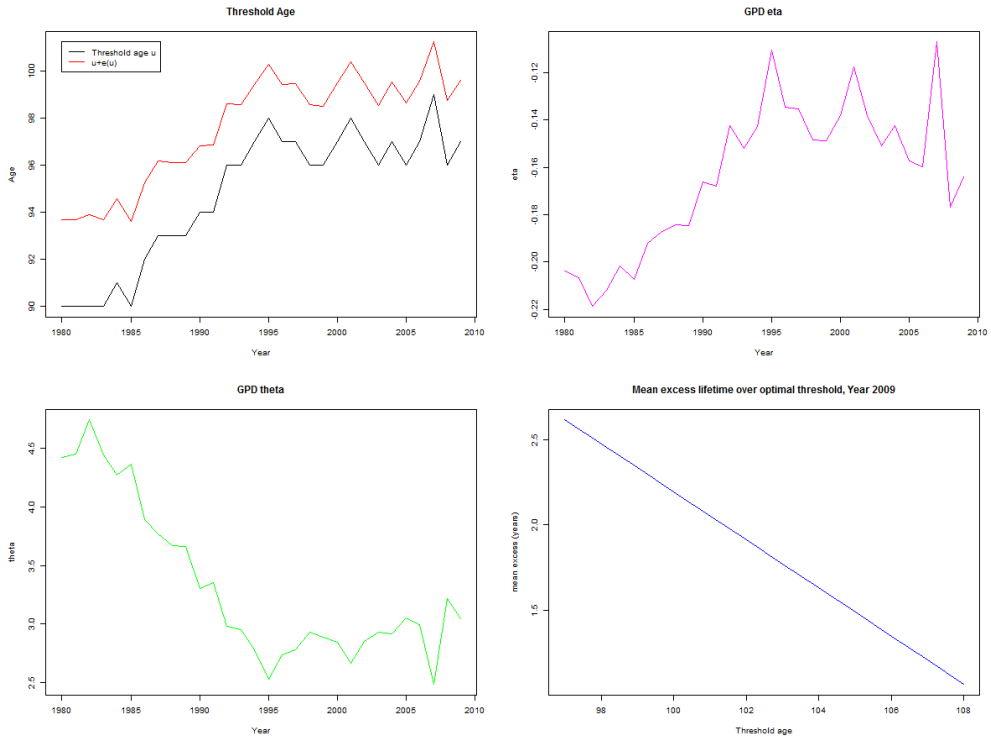

Figure 4: Optimal threshold age u^*, mean excess life time over threshold u^* and GPD parameters by year $t \in [1989, 2009]$, and mean excess life time over threshold u^* in 2009, Spain, total population.

We can observe that the optimal threshold age has been increasing steadily over time in all populations under analysis and that u^* is normally slightly higher for the male population than for the female population. Not surprisingly, the mean excess life time seems to be decreasing over time, particularly for the Spanish population. The scale parameter θ exhibits a declining trend whereas the shape parameter ξ exhibits a volatile pattern over time with no clear trend.

To illustrate the model's performance, Figure 7 shows the maximum likelihood estimated threshold life tables for the Portuguese and Spanish male, female and total populations in 2009 at the optimal threshold age. We observe a good fit for the entire life span, despite the fact that the original model definition doesn't ensure a smooth transition between the mortality rates by the Gompertz component and GPD segment. Once again, this is a consequence of the iterative estimation procedure that favours the overall goodness-of-fit instead of partial performance.

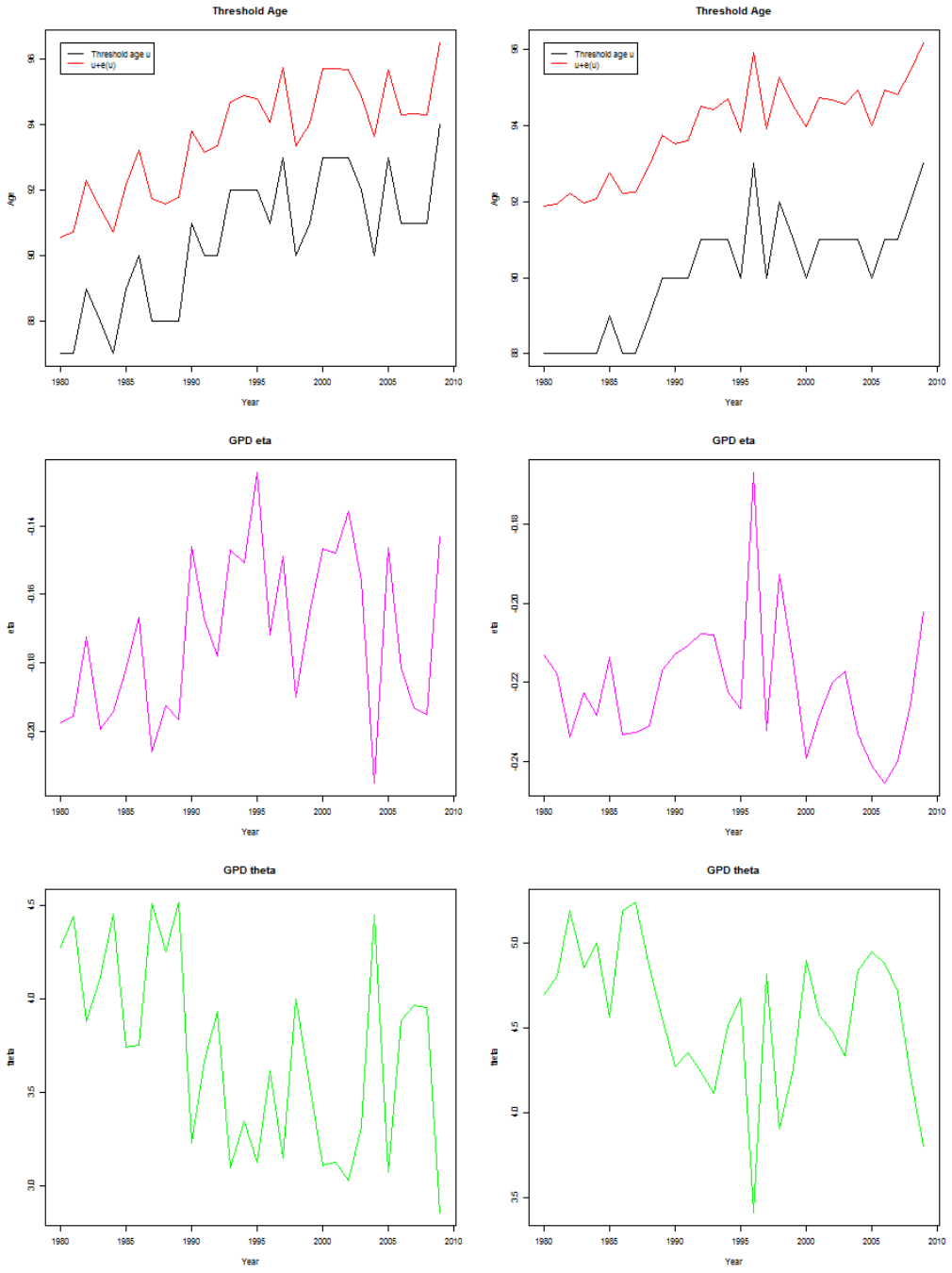

Figure 5: Optimal threshold age u^*, mean excess life time over u^* and GPD parameters for the male (left panel) and female (right panel) Portuguese populations.

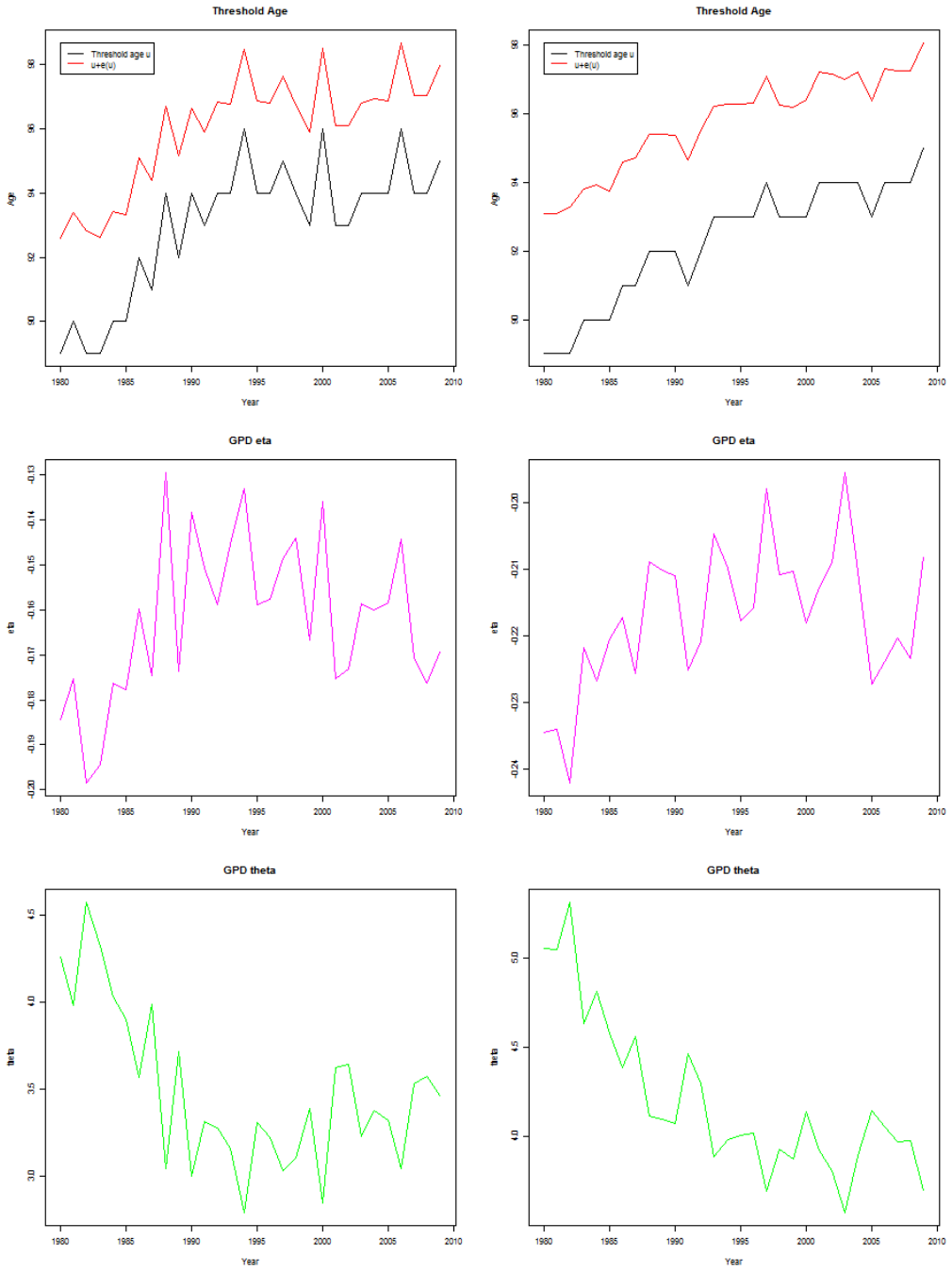

Figure 6: Optimal threshold age u^*, mean excess life time over u^* and GPD parameters for the male (left panel) and female (right panel) Spanish populations.

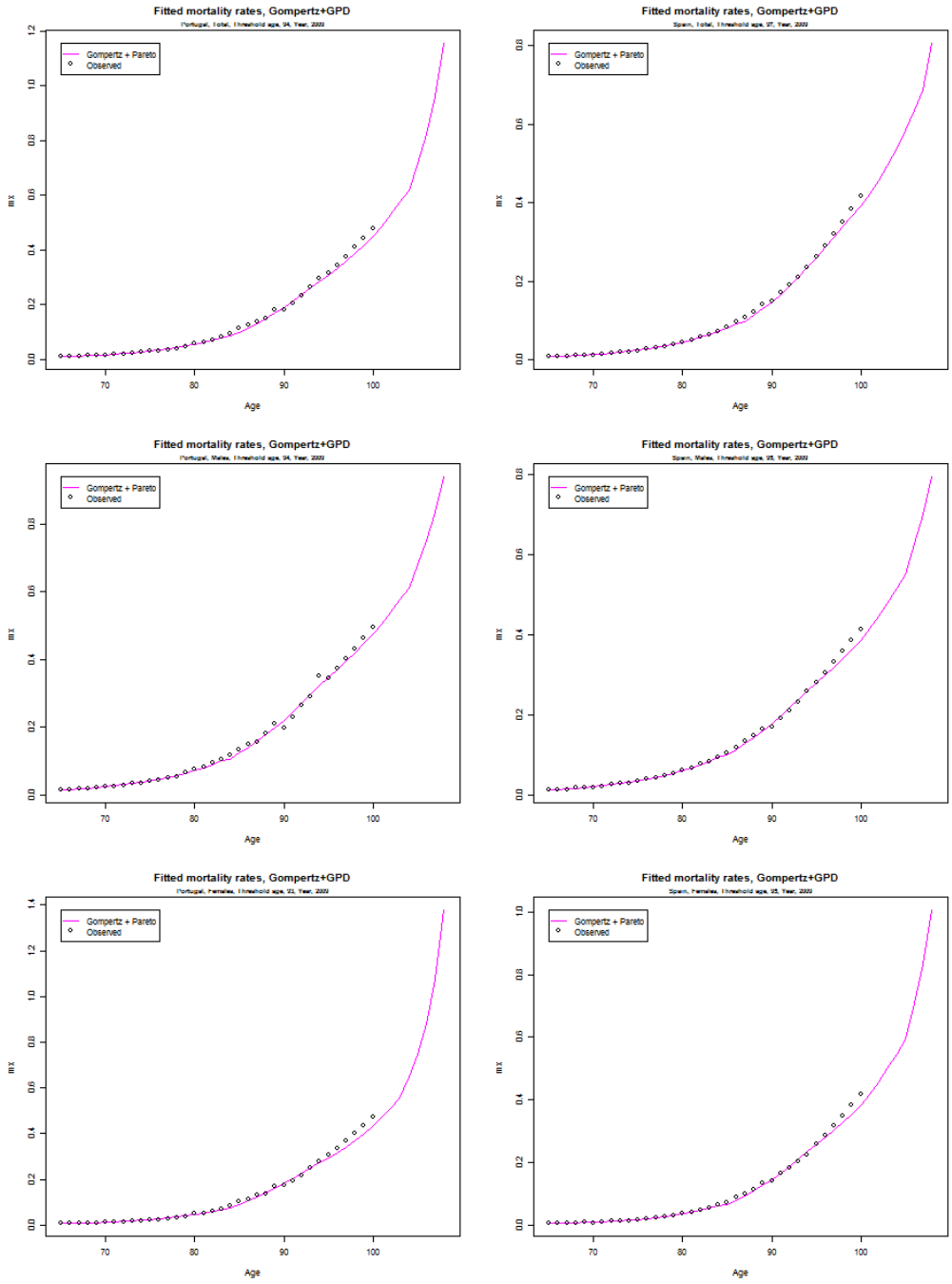

Figure 7: Maximum likelihood estimated threshold life tables for the Portuguese and Spanish male, female and total populations in 2009 at the optimal threshold age.

Beyond the threshold age, the mortality rates increase progressively up to an end point, which contrary to previous methods is determined statistically.

Table 1 exhibits, for all populations under consideration, the parameter estimates and corresponding standard errors (in brackets) for the Gompertz and GPD components of the optimal threshold life table, the optimal threshold age and the mean excess life time at the optimal threshold age in 2009.

Parameter	Portugal			Spain		
Estimates	Total	Male	Female	Total	Male	Female
$\ln B$	-12.4264	-11.3836	-13.9247	-12.3482	-11.14593	-14.3180
se	(0.145972)	(0.119124)	(0.156839)	(0.137046)	(0.127534)	(0.150765)
p-value	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001
$\ln C$	0.119307	0.109497	0.135526	0.115845	0.104500	0.137273
se	(0.001927)	(0.001583)	(0.002059)	(0.001785)	(0.001683)	(0.001957)
p-value	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001
ξ	−0.17589	−0.14312	−0.20422	−0.16413	−0.16943	−0.20827
se	(0.00831)	(0.013931)	(0.005590)	(0.011046)	(0.010421)	(0.006355)
p-value	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001
θ	3.32856	2.856573	3.789288	3.045186	3.463071	3.700971
se	(0.045071)	(0.062403)	(0.037106)	(0.050734)	(0.05549)	(0.038234)
p-value	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001	<0.00001
u	94	94	93	97	95	95
$e(u)$	2.83	2.50	3.15	2.62	2.96	3.06

Table 1: Parameter estimates, standard errors and p-values for the Gompertz and GPD parameters, optimal threshold age and mean excess life time; Male, female and total Portuguese and Spanish populations, 2009

We can observe that the parameter estimates of the Gompertz mortality law and of the GPD function are all statistically significant at the optimal threshold age. The empirical results also show that the shape parameter ξ exhibits a volatile pattern over time but is always negative, which means that according to classical results on EVT the support of the distribution has a finite upper bound. In the following section, we elaborate on this topic and test for the existence of a theoretical end point for the life table and its dynamics over time.

4 Estimating the maximum life span

As we saw on Section 2, in the threshold life table the exceedance $z = \{X - u | X > u\}$ over the threshold age follows a GPD with parameters θ and ξ, the value of which determines the precise distribution of z. In Section 3 we concluded that the estimated parameters for the GPD are (statistically) significantly less than zero. In these cases, i.e., when $\xi < 0$, we know from classical results on EVT that the support of the distribution $G_{\xi,\theta}(x)$ is $0 \leq x \leq -\theta/\xi$. In other words, we can test for the existence of a finite upper bound given by $\omega = u - \theta/\xi$ by testing the hypothesis $\xi < 0$. Our empirical results suggest that life tables for both populations and subpopulations have a theoretical end point. Moreover, since $\xi < 0$ and $\theta > 0$ this theoretical end point ω will be greater than the optimal threshold age u.

Following Li *et al.* (2008), we now consider a group of n individuals who survive to the threshold age u. We let X_i be the age at death for the ith individual, $z_i = \{X_i - u | X_i > u\}$, and denote $M_n = \max\{z_i, i = 1, 2, ..., n\}$ the highest exceedance over the threshold age. In this case, $u + M_n$ will be the highest attained age for this group of n individuals.

Assuming that the exceedances z_i and z_j for two different individuals are independent and that z_i, $i = 1, 2, ..., n$ follows a GPD with $\xi < 0$, the distribution function of M_n can be expressed as:

$$F_{M_n}(y) = \begin{cases} 0, & y < 0 \\ (F_Y(y))^n, & 0 \leq y < -\theta/\xi \\ 1, & y \geq -\theta/\xi \end{cases} \tag{12}$$

From 12 it follows that when n tends to infinity the distribution of M_n asymptotically degenerates at $-\theta/\xi$, i.e.,

$$\lim_{n\to\infty} F_{M_n}(y) = \begin{cases} 0, & y < -\theta/\xi \\ 1, & y \geq -\theta/\xi \end{cases} \tag{13}$$

If the number of survivors at the threshold age is sufficiently large, the highest attained age of these n survivors converges (in probability) to $\omega = u - \theta/\xi$, the theoretical end point of the threshold life table, which is strictly greater than u since $\theta > 0$ and $\xi < 0$.

To derive confidence intervals for end point of the threshold life table, we recall first that the uncertainty of the estimate of ω arises from the variability of the maximum likelihood estimates of the GPD parameters θ and ξ. This means that we can resort to some classical results on the asymptotic properties of maximum

likelihood estimates. In particular, we follow Li *et al.* (2008) and Han (2005) and compute confidence intervals for ω using the classical delta method.

The delta method is a method for deriving an approximate probability distribution for a function of an asymptotically normal statistical estimator from knowledge of the asymptotic variance of that estimator. More formally, we recall a general result for maximum likelihood theory that states that under standard regularity conditions (see, e.g., Rohatgi, 1976), if $\hat{\beta}$ is a vector of ML estimates, then

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \xrightarrow{d} \mathcal{N}\left(0, Var\left(\hat{\beta}\right)\right)$$

Let $G(\beta)$ be some function. To estimate the variance, we evaluate the partial derivatives at the ML estimates, $\left(\frac{\partial G(\beta|x)}{\partial \beta'}\right)\Big|_{\beta=\hat{\beta}}$, which leads to

$$Var\left(G(\hat{\beta})\right) = \frac{\partial G(\hat{\beta})}{\partial \hat{\beta}'} Var\left(\hat{\beta}\right) \frac{\partial G(\hat{\beta})}{\partial \hat{\beta}}$$

Given these results, the asymptotic variance of the estimate of ω can be estimated by

$$Var\left(\hat{\omega}\right) = \begin{pmatrix} \frac{\partial \omega}{\partial \xi} & \frac{\partial \omega}{\partial \theta} \end{pmatrix} \left(\mathcal{I}\left(\xi,\theta\right)\right)^{-1} \begin{pmatrix} \frac{\partial \omega}{\partial \xi} \\ \frac{\partial \omega}{\partial \theta} \end{pmatrix} \tag{14}$$

where $\mathcal{I}\left(\xi,\theta\right)$ is the information matrix of the estimators $\hat{\theta}$ and $\hat{\xi}$.

Using the asymptotic normality property of maximum likelihood estimates, the approximate $100\% \times (1-\alpha)$ confidence interval for ω can be written using the classic Wald (1939) formula

$$\left(\hat{\omega} - z_{\alpha/2}\sqrt{Var\left(\hat{\omega}\right)}, \hat{\omega} + z_{\alpha/2}\sqrt{Var\left(\hat{\omega}\right)}\right) \tag{15}$$

where $z_{\alpha/2}$ denotes the corresponding value from the standardized cumulative distribution function for the significance level α.

In figure 8 we display the mean and 95% confidence interval estimates of ω for the Portuguese and Spanish male, female and total populations in the period from 1980 to 2009. In all populations, we can observe an increasing trend in the estimated highest attainable age, more pronounced in the Spanish populations. The mean estimates of ω are consistently higher for the Spanish population, for all subpopulations. The Portuguese male and female populations exhibit mean estimates of ω that follow similar paths and are close in absolute terms, although it is clear that the mean estimates for the male population are more volatile over time.

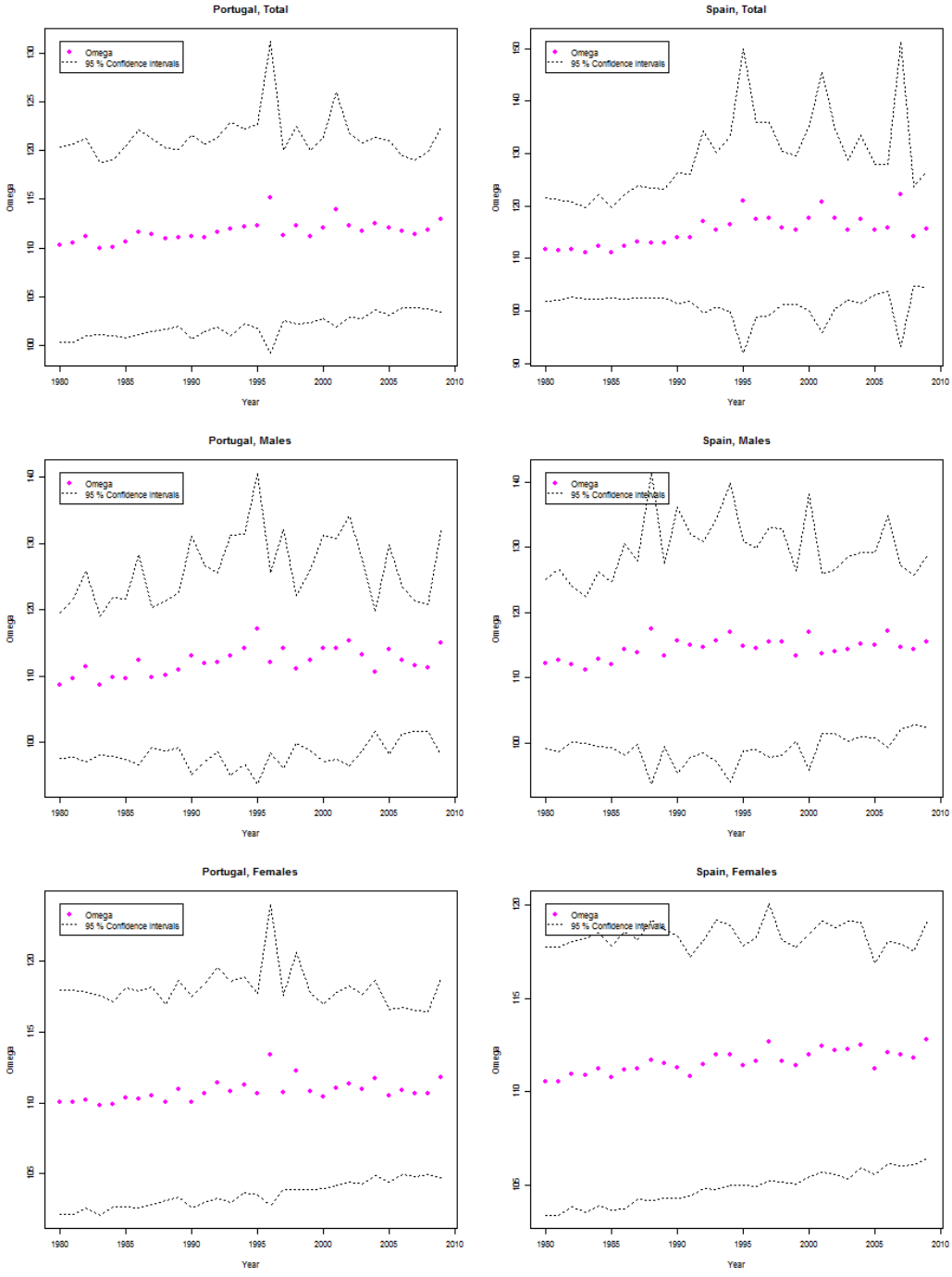

Figure 8: Point estimates and corresponding 95% confidence intervals for the highest attainable age ω in Portugal and Spain, disaggregated by sex and calendar year.

In Spain, the mean estimates of ω are clearly higher and more volatile over time for the male population when compared to their female counterparts.

The mean and 95% confidence interval estimates of ω for all populations under consideration in 2009 are displayed in Table 2.

Country	Population	Central Estimate	95% Confidence Interval	Threshold Age
Portugal	Total	112.92	(103.49; 120.32)	94
	Male	114.96	(98.05, 131.87)	94
	Female	111.78	(104.73, 118.84)	93
Spain	Total	115.55	(104.41, 126.69)	97
	Male	115.44	(102.41, 128.47)	95
	Female	112.77	(106.42, 119.12)	95

Table 2: Predicted limiting ages considering the GP distribution estimated parameters at the optimal threshold age, Year 2009

We can observe that the theoretical end point of the life table is consistently higher for the Spanish total, male and female populations when compared with their Portuguese counterparts. In both populations, the rate of improvement in the estimated highest attained age has been higher for the male sub-population, whereas for the Spanish population the rate of improvement has been similar. Over the whole period of analysis, the average increase in the estimated highest attained age is approximately 0.09 (0.11) year per annum for the total Portuguese (Spanish) population, 0.21 (0.11) per annum for the male Portuguese (Spanish) population and 0.06 (0.07) for the female Portuguese (Spanish) population.

The average increase in the maximum life span reported here is in line with the related literature.[1] The central estimates of ω for the Portuguese and Spanish total populations are in line with those obtained by Li *et al.* (2010) for the Australian (112.20), New Zealand (109.43) populations and by Li (2007) for the Japanese (112.46), Danish (108.24), Finish (109.01), Norwegian (112.56) and Swedish (111.98) populations, in this later case considering this case the 1897 birth cohort. In the Spanish population, the maximum life span is consistently higher in the male population when compared with the female counterpart, whereas in Portuguese population male and female populations interchange positions over time.

[1]For example, Li *et al.* (2008) estimate an average increase in the estimated highest attained age of 0.15 year per annum for the Japanese population.

In Portugal there are no official life tables designed for use within the Portuguese insurance and pension fund industry. On the contrary, in Spain life tables commonly used for valuing benefits for pension fund retirees and life insurance policies are the PERM/F 2000C (for policies existing at 3/10/2000) and PERM/F 2000 P (for policies issued after 3/10/2000). Both cohort life tables assume that no one will survive to age 115, an age in line with mean estimates for the maximum life span derived in this study.

The amplitude of the confidence intervals is higher in the Spanish population, particularly in male subpopulation, and is relatively stable in period analysed. Confidence intervals are significantly wider for the male population when compared with their female counterparts. In Table 3 we exhibit a list of validated supercentenarians for the Spanish population, obtained from the Human Longevity Database (HLD).[2]

We observe that the list of confirmed supercentenarians exhibits a maximum life span well within the confidence intervals for Spanish population.

It should be noted, however, that maximum observed life spans are not necessarily synonymous with theoretical maximums for at least two reasons. First, maximum longevity is an inappropriate general concept because there is no specific identifiable age for each species to which some select individuals can survive but beyond which none can live. Second, the record age of a species is heavily influenced by the number of individuals observed (Carey *et al.*, 2003).

[2]Up to date, the Human Longevity Database doesn't provide data for the Portuguese population.

ID Number	Age	Sex	Birth date	Death date	Status
115	114	F	10/06/1881	16-01-1996	Death
116	114	M	03/04/1883	03-07-1997	Death
117	114	M	15/12/1889	05-03-2004	Death
118	113	F	28/10/1883	25-05-1997	Death
119	112	M	23/02/1881	20-01-1994	Death
120	112	F	01/09/1889	15-02-2002	Death
121	111	F	20/07/1885	21-06-1997	Death
122	111	M	18/02/1888	15-11-1999	Death
123	111	F	27/08/1895	08-04-2007	Death
124	111	F	18/08/1890	25-03-2002	Death
125	111	F	28/07/1891	16-11-2002	Death
126	111	F	04/05/1888	09-08-1999	Death
127	111	F	13/12/1894	02-02-2006	Death
128	111	F	26/03/1884	12-04-1995	Death
129	110	F	19/06/1891	21-04-2002	Death
130	110	M	22/04/1879	23-01-1990	Death
131	110	F	01/03/1885	24-11-1995	Death
132	110	M	11/04/1879	01-01-1990	Death
133	110	F	15/02/1886	06-11-1996	Death
134	110	M	24/12/1886	09-09-1997	Death
135	110	M	16/10/1878	13-06-1989	Death
136	110	F	18/03/1887	27-10-1997	Death
137	110	F	08/08/1886	01-12-1996	Death
138	110	F	24/05/1879	08-09-1989	Death
139	110	F	06/11/1887	19-01-1998	Death
140	110	F	21/04/1895	02-06-2005	Death
141	110	F	21/11/1883	25-11-1993	Death
142	110	F	22/04/1894	26-04-2004	Death

Table 3: Validated Supercentenarians in Spain as January 2011, HLD

5 Forecasting the maximum life span

In this section we assume that mean estimates of the limiting age ω can intrinsically be viewed as stochastic process. In this sense, standard Box-Jenkins techniques are used to estimate and forecast ω_t within an ARIMA(p,d,q) time series model. Let $\{\hat{\omega}_t, t = t_{\min}, \ldots, t_{\max}\}$ denote a realization of the finite chronologic time series $\mathcal{K} = \{\omega_t, t \in \mathbb{N}\}$. The model takes the general form

$$(1-B)^d \omega_t = \mu + \frac{\Theta_q(B)\,\epsilon_t}{\Phi_q(B)}$$

where B is the delay operator (i.e., $B(\omega_t) = \omega_{t-1}$, $B^2(\omega_t) = \omega_{t-2}, \ldots$), $1 - B$ is the difference operator (i.e., $(1-B)\omega_t = \omega_t - \omega_{t-1}$, $(1-B)_t^2 \omega = \omega_t - 2\omega_{t-1} + \omega_{t-2}, \ldots$), $\Theta_q(B)$ is the Moving Average polynomial, with coefficients $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q)$, $\Phi_q(B)$ is the Autoregressive polynomial, with coefficients $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_p)$, and ϵ_t is white noise with variance σ_ϵ^2.

Before proceeding to estimate the model, we tested for the existence of unit roots in order to determine if the variables were non-stationary, using the adjusted Dickey-Fuller (ADF) test (Dickey and Fuller, 1979). We used two approaches to determine the number of lags to be included in the model in order to determine the order of the autoregression and to eliminate correlation in the residuals. More specifically, we determine the optimal number of lags by examining the BIC criterion and by examining the $t-$values on coefficients and use the Durbin Watson statistic (Durbin and Watson, 1950) to check for first order autocorrelation in the residuals. We tested for the inclusion of a trend and/or a drift term in the specification of the model in those cases where it was significant.

Table 4 reports the ADF test results of the variables in levels and in first differences. The results for the variables in levels indicate that we cannot reject the existence of a unit root in all series $\mathcal{K}$. The results of the ADF test on the first difference of these variables indicate that they are $I(1)$ in levels.

Country	Population	Variables in levels		1st Difference series	
		Statistic	*p-value*	Statistic	*p-value*
Portugal	Total	-3.0975	0.12503	-5.237	0.00019
	Male	-2.8432	0.19402	-4.9208	0.00044
	Female	-2.3763	0.38356	-5.4952	0.00009
Spain	Total	-2.5343	0.31067	-6.2326	0.00001
	Male	-2.3169	0.41276	-5.9486	0.00003
	Female	-3.541	0.05290	-6.3728	0.00001

Table 4: Augmented Dickey-Fuller (ADF) tests for the existence of a unit root in the time series of the limiting age

The results of the ADF tests indicate that the series under analysis are $I(1)$ so we stationarize them by estimating the model in first annual differences. Next, we followed the standard identification, estimation, diagnostic checking and forecasting stages of the Box-Jenkins methodology to estimate the ARIMA$((p, d, q)$

models for each time series. To determine whether AR or MA terms are needed to correct any autocorrelation that remains in the differenced series we looked at the autocorrelation function (ACF) and partial autocorrelation (PACF) plots of the differenced series and selected the model that performs better under the BIC Information criterion. Table 5 resumes the results of this procedure and exhibits the best $ARIMA(p,d,q)$ model selected for each population and corresponding BIC criterion.

	Portugal			Spain		
Model/Criterion	Total	Male	Female	Total	Male	Female
$ARIMA(p,d,q)$	$(0,1,1)$	$(0,1,1)$	$(1,1,0)$	$(0,1,1)$	$(0,1,1)$	$(0,1,2)$
BIC	86.39	123.46	67.36	134.99	109.53	52.88
LB test statistic	12.7193	8.7496	9.4621	15.9864	26.8666	8.4879
LB test $p-value$	0.8891	0.9856	0.9769	0.7175	0.1391	0.9881

Table 5: Best ARIMA(p,d,q) model and BIC Information criterion

Finally, to investigate whether the forecast errors of the ARIMA models are normally distributed with mean zero and constant variance, and whether the are correlations between successive forecast errors we checked the correlogram of the residuals and tested formally the null hypothesis of zero autocorrelation considering the Ljung-Box (LB) test. The results exhibited in Table 5 indicate that we cannot reject the null hypothesis of null autocorrelation between the model residuals.

In Figure we show the estimated and forecasted values of the maximum life span ω_t and corresponding 50-99 % confidence intervals.

We forecast an increase in the theoretical maximum life span over time for all populations, more pronounced in the male subpopulation. However, it should be stressed that the amplitude of the confidence intervals gives a clear measure of the significant uncertainty in the forecasted values.

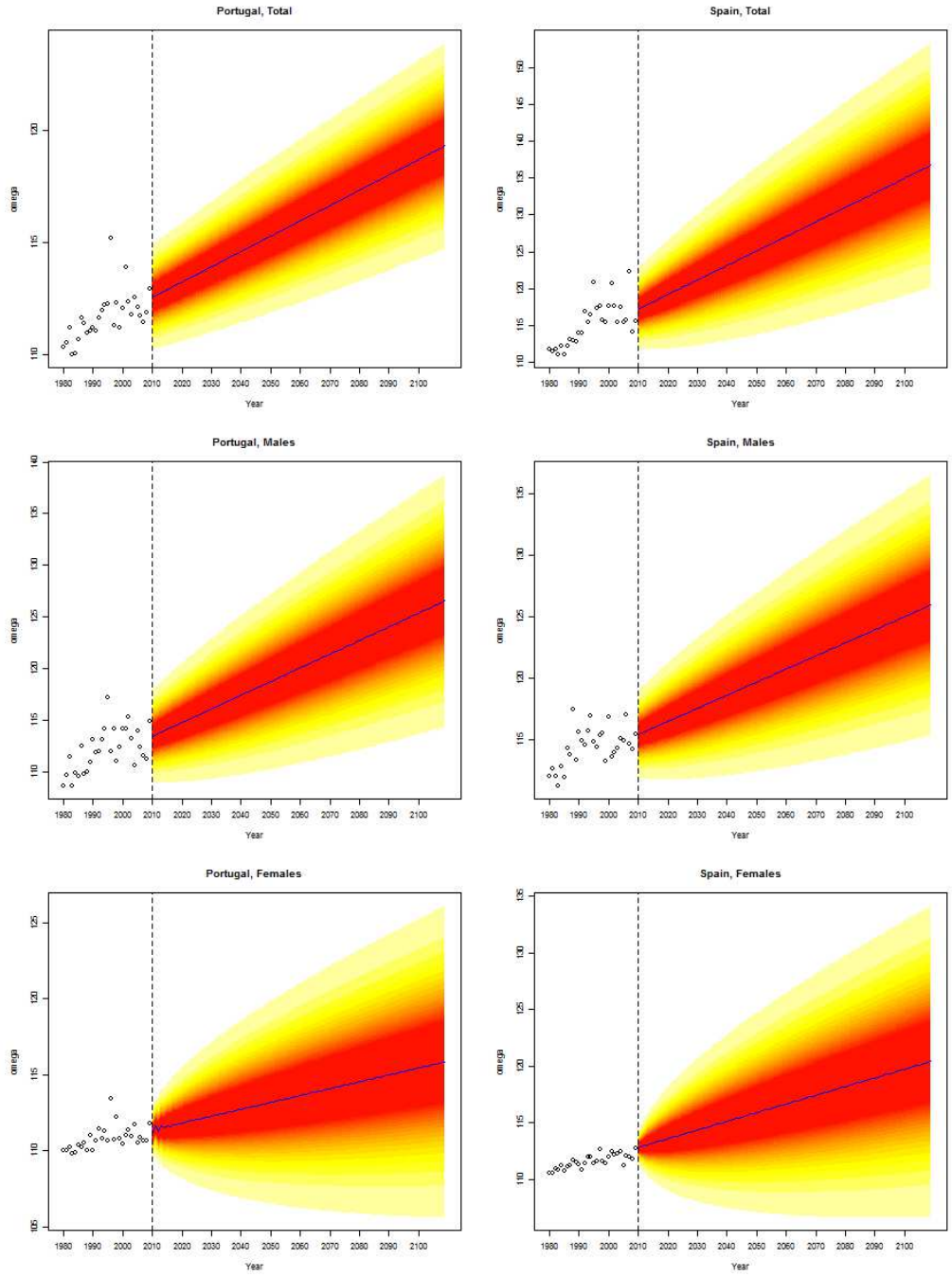

Figure 9: Estimated and forecasted values of the maximum life span ω_t and corresponding 50-99 % confidence intervals.

6 Conclusion

Old-age mortality for populations of developed countries has been improving rapidly over the last decades. Mortality improvements are naturally viewed as a positive change for individuals and as a substantial social achievement for societies, but they challenge actuaries, economists, demographers and policy planners in obtaining a reliable estimate of old-age mortality in preparing their demographic and financial projections. In this paper, we use EVT to model the statistical behaviour of mortality rates over a given high threshold age and to estimate the theoretical maximum life span in the Portuguese and Spanish male, female and total populations. We observe a good fit of the model in all populations and subperiods analysed and on the whole life span considered. The empirical results derived in this study suggest the existence of an increasing over time finite upper bound for the survival distribution, i.e., indicate the existence of a theoretical end point of a life table. This result is crucial for actuaries that need a reliable model of old-age mortality for pricing and reserve calculations and for implement longevity risk management techniques. Using standard time series methods we forecast an increase in the highest attained age in all populations, more significant in the male subpopulation.

References

[1]Aarssen K., de Haan L. (1994). On the maximal lifespan of humans. *Mathematical Population Studies* 4, 259–281

[2]Balkema, A., and De Haan, L. (1974). Residual Life Times at Great Age. *Annals of Probability* 2, 792–804.

[3]Beelders, O., Colarossi, D. (2004). Modeling mortality risk with extreme value theory: The case of swiss Re's mortality-indexed bond. *Global Association of Risk Professionals* 4, 26 30.

[4]Boleslawski, L. and Tabeau, E. (2001). Comparing theoretical age patterns of mortality beyond the age of 80. In E. Tabeau et al. (Eds). *Forecasting Mortality in Developed Countries: insights from a statistical, demographical and epidemiological perspective*. Kluwer Academic Publishers, pp. 127-155.

[5]Bravo, J. M., Magalhães, M. G., Coelho, E. (2007). Mortality and Longevity Projections for the Oldest-Old in Portugal. in Working Session on Demographic Projections, Bucharest, Romania, 10-12 October, 2007, EUROSTAT (*Series: Population and social conditions, Collection: Methodologies and working papers*), pp. 117-132.

[6]Buettner, T. (2002). Approaches and experiences in projecting mortality patterns for the oldest-old. *North American Actuarial Journal*, 6(3), 14-25.

[7]Carey, James R. and Shripad Tuljapurkar (eds.). Life Span: A Conceptual Overview. in *Life Span: Evolutionary, Ecological, and Demographic Perspectives, Supplement to Population and Development Review*, vol. 29, 2003. New York: Population Council.

[8]Chen, H. and Cummins, D. (2010). Longevity bond premiums: The extreme value approach and risk cubic pricing. *Insurance: Mathematics and Economics*, 46, 150-161.

[9]Coale, A. and Guo, G. (1989). Revised regional model life tables at very low levels of mortality. *Population Index* 55, 613–43.

[10]Coale, A., and Kisker, E. (1990). Defects in Data on Old-Age Mortality in the United States: New Procedures for Calculating Mortality Schedules and Life Tables at the Highest Ages. *Asian and Pacific Population Forum* 4, 1-31.

[11]Denuit, M. e Goderniaux, A. (2005). Closing and projecting lifetables using log-linear models. *Bulletin de l'Association Suisse des Actuaries*, 1, 29-49.

[12]Embrechts, P., Klüppelberg, C., Mikosch, T., 2008. *Modeling Extremal Events: For Insurance and Finance*. Springer-Verlag, Berlin.

[13]Galambos, J. and Macri, N. (2000). The Life Length of Humans Does Not Have a Limit. *Journal of Applied Statistical Science* 9(4): 253–64.

[14]Gompertz, B (1825). On the Nature of the Function Expressive of the Law of Human Mortality and on a New Mode of Determining Life Contingencies. Royal Society of London, *Philosophical Transactions*, Series A 115, 513–85.

[15]Han, Z. (2005). Living to 100 and Beyond: An Extreme Value Study. Living to 100 andBeyond Symposium Monograph. Available at: http://www.soa.org/ccm/content/research-publications/library-publications/monographs/life-monographs/*living-to-100-and-beyond-monograph/*

[16]Heligman, L., and Pollard, J. (1980). The Age Pattern of Mortality. *Journal of the Institute of Actuaries* 107, 437–55.

[17]Himes, C. L., Preston, S. H. and Condran, G. A. (1994). A Relational Model of Mortality at Older Ages in Low Mortality Countries. *Population Studies* 48, 269–91.

[18]HUMAN MORTALITY DATABASE (2011). University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). www.mortality.org or www.humanmortality.de. Accessed December 15, 2011.

[19]Kannistö, V. (1992). *Development of oldest-old mortality, 1950-1990: Evidence from 28 developed countries*. Odense University Press.

[20]Li S. H., and Hardy M. R. and Tan K. S. (2008). Threshold Life Tables and Their Applications. *North American Actuarial Journal*, Vol 12, 2, 99-115.

[21]Li, J., Ng, A. and Chan, W. (2010). Modeling old-age mortality risk for the populations of Australia and New Zealand: An extreme value approach. *Mathematics and Computers in Simulation*, Volume 81 Issue 7, pp. 1325-1333.

[22]Li, S. H. (2007). Stochastic Mortality Models with Applications in Financial Risk Management. PhD Thesis, University of Waterloo, 189 pages.

[23]Pitacco, E. (2004). Survival models in a dynamic context: a survey. *Insurance: Mathematics and Economics*, 35, 279-298.

[24]Population Index, 55, 613-643.

[25]R Development Core Team (2011). *R: A language and environment for statistical computing*. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org/.

[26]Robine, J.-M. and J. W. Vaupel (2001). Supercentenarians: slower ageing individuals or senile elderly? *Experimental Gerontology*, Vol. 36, pp. 915-930.

[27]Rohatgi, V. (1976). *An Introduction to Probability Theory and Mathematical Statistics*. 1st Edition, John Wiley & Sons.

[28]Thatcher, A. R. (1999). The long-term pattern of adult mortality and the highest attained age. *Journal of the Royal Statistical Society* 162 (1), 5-43.

[29]Thatcher, A., Kannisto, V. and Vaupel, J. (1998). *The force of mortality at ages 80 to 120*. Odense Monographs on Population Aging, Odense University Press, Odense.

[30]Vaupel, J. M. (2010). Biodemography of human ageing. *Nature* 464, 536-542.

[31]Wald, A., & Wolfowitz, J. (1939). Confidence limits for continuous distribution functions. *The Annals of Mathematical Statistics*, 10, 105–118.

[32]Watts, K.A., D.J. Dupuis and B.L. Jones (2006). An Extreme Value Analysis of Advanced Age Mortality Data, *North American Actuarial Journal*, 10, 4, p. 162-177.