The first eigenvalue of the Laplacian and the ground flow of a compact surface

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Abstract— *We present some results whose central theme is that the phenomenon of the first eigenvalue of the Laplacian and the ground flow of the compact surface (bitorus). Our main tool is a method for studying how the hyperbolic metric on a Riemann surface behaves under deformation of the surface. With this model, we show that there are variation of the first eigenvalue of the Laplacian and the ground flow with the Fenchel-Nielsen coordinates, that characterize the surface.*

I. Introduction

The Selberg trace formula has been of great interest to mathematicians for almost 50 years. It was discovered by Selberg, who also defined the Selberg zeta function, by analogy with the Riemann zeta function, to be a product over prime geodesics in a compact Riemann surface.

The Selberg trace formula has been of great interest to mathematicians for almost 50 years. It was discovered by Selberg [13], who also defined the Selberg zeta function, by analogy with the Riemann zeta function, to be a product over prime geodesics in a compact Riemann surface. But the analogue of the Riemann hypothesis is provable for the Selberg zeta function. The trace formula shows that there is a relation between the length spectrum of these prime geodesics and the spectrum of the Laplace operator on the surface.

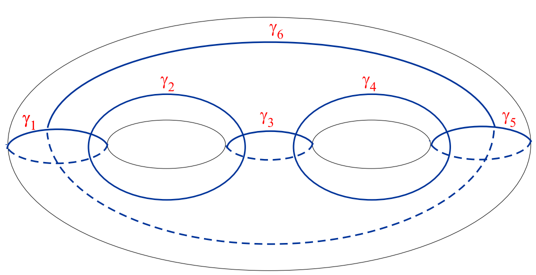

Fig. 1. Bitorus

More recently quantum physicists (specifically those working on quantum chaos theory) have been investigating the Selberg trace formula and its generalizations because it provides a connection between classical and quantum physics. In fact, of late there has been much communication between mathematicians and physicists on this and matters related to the statistics of spectra and zeta zeros.

For the computation of the length geodesic spectrum see [11]

In this note, we announce a collection of results connected to the behavior of the first eigenvalue $\lambda_1(S)$ of a compact Riemann surface S of genus 2, endowed with a metric of constant curvature -1.

The idea of studying the first eigenvalue of a Riemann surface via the study of eigenvalues of 3-regular graphs comes from the work of Buser [5], [7]. In effect, our approach here is a variation on his idea, where we first study the behavior of λ_1 on finite-area Riemann surfaces connected to 3-regular graphs, and then see how λ_1 changes when we deformed the surface. It is interesting to pass back and forth between the geometric and graph-theoretic pictures. One reason for doing this is that a problem which appears difficult from one point of view may be relatively easy, or even solved, from the other point of view. Another reason is that attitudes towards various results may differ markedly in two areas, and comparing them may be an important source of insight.

Our main analytic tool is a method for studying how the hyperbolic metric of a finite-area Riemann surface behaves under such a deformation.

The first eigenvalue of the Laplacian on a surface can be viewed as a functional on the space of Riemannian metrics of a given area.

Every such surface is represented by a quotient space H^2/Γ of the upper half-plane H^2 by a Fuchsian group Γ which is isomorphic to a fundamental group of $\mathcal{M}$. The discrete group Γ is identified with the corresponding system of generators. A fundamental domain $\mathcal{F}$ is defined. The method is to decompose Riemann surface into a set of 2 pairs of pants by simple closed geodesics. Then the Fenchel-Nielsen coordinates are defined by geodesic length functions of three simple closed geodesics and twist angles along these geodesics.

Here we use a real-analytic embedding of the Teichmüller space $\mathcal{T}$ of closed Riemann surfaces of genus 2 onto an explicitly defined region $\mathcal{R} \subset \mathcal{R}$/ The parameters are explicitly defined in terms of the underlying hyperbolic geometry. The parameters are elementary functions of lengths of simple closed geodesics, angles and distances between simple closed geodesics. The embedding is accomplished by writing down four matrices in $PSL(2, \mathbb{R})$, where the entries in these matrices are explicit algebraic functions of the parameters. With explicit constructions and side pairing transformations (see [12]) we define the Fuchsian group Γ representing the closed Riemann surface of genus 2.

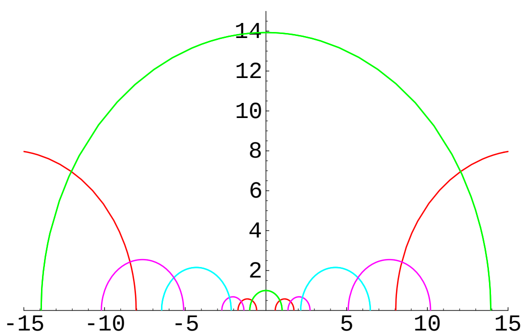

Fig. 2. Hyperbolic plane

Cheeger's celebrated inequality relates the first non-trivial eigenvalue of a compact manifold to an isoperimetric constant, the Cheeger constant, defined as follows: let S be a closed Riemannian manifold (compact, no boundary) and let X be a hypersurface dividing S into two parts, A and B. Then

$$h(S) = \inf_{S} \frac{area(X)}{\min(vol(A), vol(B))}$$

where the in infimum runs over all such hypersurfaces X. Let λ_1 denote the first nontrivial eigenvalue of S. Cheeger in [8] proved:

Theorem 1: Let S be a closed Riemannian manifold. Then $\lambda_1 \geq \frac{1}{4}h^2$.

This bound is remarkable for its universal character. It is natural to ask whether there is an upper bound for the first eigenvalue in terms of Cheeger's constant. With an added hypothesis on curvatures, such an upper bound was proved by [7]:

Theorem 2: Suppose that S is a smooth Riemannian manifold with $Ricc(S) \geq -c$. Then there are constants c_1 and c_2 depending on c so that $\lambda_1 \leq c_1 h + c_2 h^2$.(see [7]).

It is not surprising that there are graph-theoretic analogues of these notions and results. As usual, there are different conventions and normalizations which are essentially equivalent. It convenient to stick to the case of k-regular graphs G, and define the Laplacian as

$$\Delta(f)(x) = \frac{1}{k} \sum_{y \sim x} (f(x) - f(y))$$

and the Cheeger constant to be

$$h(G) = \inf_{E} \frac{\#(X)}{\min(\#(A), \#(B))}$$

where E runs over collections of edges such that $G - E$ decomposes into two pieces A and B, and $\#(A)$ (resp. $\#(B)$) denotes the number of vertices in A (resp. B).

Cheeger's inequality then becomes:

Theorem 3: $\lambda_1 \geq \frac{1}{2k^2}h^2$,(see [1]).

The analogue of Buser's inequality is:

Theorem 4: $\lambda_1 \leq 2h$.(see [1]).

The importance of the study of the first eigenvalue of a Riemann surface is fairly well established by now in the literature. But with the introduction of the notion of conductance of a dynamical system, particularly in the graph theory, and with the relationship with the smallest non-zero eigenvalue of the Laplacian this importance get bigger. Another field is beginning.

II. Variations with Fenchel-Nielsen coordinates

In a general way, to each discrete dynamical system (I, f) defined by the iterates of a map f on the interval I, we associate a Markov matrix, which is representable by a non-regular, oriented graph G_f (the elements of E are now ordered pairs). So we have systems defined by the adjacency matrix $A_f = (a_{ij})$ of G_f, that is, the $0 - 1$ matrix where $a_{ij} = 1$ iff ij is an edge. We can now establish the notion of conductance of a discrete dynamical system. For more details see [10].

Definition 1: Let $A_f = (a_{ij})_{i,j=1}^n$ be the adjacency matrix associated to (I, f) and G_f the Markov graph. Define the diagonal matrix $D_f = (d_{ij})_{i,j=1}^n$, putting in the diagonal d_{ii} the number of edges that incide (in and out) in the vertex i (loops contribute with 2). We will call the matrix

$$\Delta_f = D_f - (A_f + A_f^T)$$

the Laplacian matrix of the graph G_f.

As we will see, the smallest non-zero eigenvalue of the Laplacian is closely related with the conductance of the system. This result can be proved by symbolic dynamic methods.

To explicit construction of a fundamental domain we consider the geodesics in the hyperbbolic plane H^2 given the surface $\mathcal{M} = \mathcal{H}^{\in}/-$, see Fig. 2. When we cut the surface $\mathcal{M}$ along these geodesics then we divide it into four equilateral hexagons. The sides are are obtained by the intersection of the axis, they are geodesics segments. These geodesics are the shortest geodesics in the free homotopy class of loops corresponding to some elements h_i ($i = 1, ..., 6$) of $\Gamma = \pi_1(\mathcal{M})$, the fundamental group of $\mathcal{M}$. We have the hexagon H_1 whose sides s_i are the arcs of γ_i and these arcs are contained in the axes of the hyperbolic transformations h_i ($i = 1, ..., 6$). Their translation length in the positive direction along these axis is $2l_i$ where l_i denote the length of $\gamma_i = l(\gamma_i)$. They are four of the parameters. The other parameters are the gluing angles. So:

$$\begin{array}{l} c_1 = l(\gamma_1), c_2 = l(\gamma_2), c_3 = l(\gamma_3), c_4 = l(\gamma_4), \\ \sigma = |P - P_2|\,, \tau = arc\tanh\left(\cos(\theta_2)\right)\,, \\ \rho = arc\tanh\left(\cos(\theta_3)\right) \end{array}$$

But c_4 is determined by the others parameters, so with this parametrization, each point t_i of the Teichmüller space $\mathcal{T}$ is $t_i = t_i(c_1, c_2, c_3, \sigma, \tau, \rho)$. This construction is dependent from de choice of the original geodesics γ_i, the chain, thus the dependence from the parameters $c_i = l(\gamma_i)$.

The sides are labelled $s_1, ..., s_{12}$ reading counterclockwise from the zero.

Let H^2/Γ our compact surface of genus $g = 2$. The fundamental domain its a bounded fundamental polygon whose boundary ∂F consists of the 12 geodesics segments $s_1, ..., s_{12}$.

Each side s_i of F is identified with s_j , by an element $g \in \Gamma$ and so each $g \in \Gamma$ produces a unique side s , namely, $s = \overline{F} \cap g(\overline{F})$ There is a bijection between the set of the sides of F and the set of elements g in Γ for which $\overline{F} \cap g(\overline{F})$ is a side of F.

We construct a map from the set of the sides of F onto itself , $g : s_i \to s_j$ where .s_i is identified with s_j. This is called a *side-pairing* of F. The *side-pairing* elements of Γ generate Γ.

In this construction we choose the side rule for the pairing

$$\begin{array}{rcl} s_1 & \to & s_7,\ s_2 \to s_{12},\ s_3 \to s_5, \\ s_4 & \to & s_{10},\ s_6 \to s_8,\ s_9 \to s_{11} \end{array}$$

With this choice we explicitly calculate formulas for the side pairing transformations $g_1, ..., g_6$,$g_7 = g_1^{-1}, ...$,$g_{12} = g_6^{-1}$. This mean that $s_7 = g_1(s_1)$,...,$s_9 = g_6(s_{11})$, $s_1 = g_7(s_7)$,...,$s_{11} = g_{12}(s_9)$, thus we obtain explicitly the generators $g_i = g_i(c_1, c_2, c_3, \sigma, \tau, \rho)\,, i = 1, ..., 12$.

With the linear fractional transformations defined above it is possible to obtain the boundary map: $f_\Gamma : \partial F \to \partial F$, defined by piecewise linear fractional transformations in the partition $\mathcal{P} = \{\mathcal{I}_\rangle = [\sqrt{\rangle}, \sqrt{\rangle+\infty}), i = 1, ..., 11, [p_{12}, p_1)\}$, which is orbit equivalent to the action of the fundamental group Γ on ∂F. The boundary map is represented by

$$f_\Gamma : \bigcup_{i=1,...,12} I_i \to \bigcup_{i=1,...,12} I_i$$

$$f_\Gamma(x)_{|I_i} = g_i(x), \;\; i = 1, ..., 12$$

We are able to define a map that codifies the expansion of boundary points of F. And we determine the Markov matrix A_Γ associated to Γ. Let be A_Γ the matrix

$$a_{ij} = \begin{cases} 1 & \text{if} \quad J_j \subset f_\Gamma(J_i) \\ 0 & \text{otherwise} \end{cases}$$

In [11] we have introduced a Markov partition for the Bowen-Series boundary map f_Γ associated with the fundamental group Γ and we defined the 24×24 Markov matrix A_Γ. It is known that there are the correspondence between the closed geodesics of the surface and the conjugacy classes of the group so with the list above we identify each closed geodesic. We obtained the length spectrum of the closed geodesics by computing $l(g) = 2\ Arc\cos h[tr(g)/2]$.

Returning to our original question, we note that a closed surface of genus at least 2 has uncountably many hyperbolic structures up to homotopy relative to the boundary, and these may be parametrized by Fenchel-Nielsen coordinates in Teichmüller space, so they have the remarkable property, known as flexibility. But, in spite of this flexibility, there is the following property, that state the existence of constants, who guarantees a certain boundedness of hyperbolic structures under deformation of a hyperbolic surface.

Theorem 5: Let $\mathcal{M}$ a hyperbolic surface of genus $g = 2$ (bitorus). Then smallest non-zero eigenvalue of the Laplacian is bounded from above by a constant $c_0 < 1/4$ (see [14]).

In this comunication we determine explicitly this constant. First we need to be clear this constant.

Definition 2: Let be a geodesic chain $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ where the four geodesics have equal length and the twist parameters are zero. We call regular domain of the genus $g = 2$ closed Riemann surface $\mathcal{M}$ with this Fenchel-Nielsen coordinates choice.

Here, the Cheeger constant is $h(G_{f_\Gamma})$ is the minimum of the conductances on $\mathcal{M}_\sqcup$ where $t = t(c_1, c_2, c_3, \sigma, \tau, \rho)$ is a point of the Teichmüller space $\mathcal{T}$.

We are able to introduced our main result. We denote by $\Phi(\Delta_{f_\Gamma})$ the eigenvector associated to the smallest non-zero eigenvalue of the Laplacian Δ_{f_Γ}, and we designate it by *ground flow eigenvector.*

Theorem 6: Let $\Phi(\Delta_{f_\Gamma})$ be the *ground flow eigenvector associated to the compact surface $\mathcal{M}$, then the total sum of its components is zero.*

Proof: The prove can be obtained using the fact that the Laplacian matrix Δ_{f_Γ} is symmetric. ∎

See [9] and [10] for the relation with the mixing rate.

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