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Symbolic Dynamics and Chaotic Synchronization in Coupled Duffing Oscillators

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Abstract

In this work we discuss the complete synchronization of two identical double-well Duffing oscillators unidirectionally coupled, from the point of view of symbolic dynamics. Working with Poincaré cross-sections and the return maps associated, the synchronization of the two oscillators, in terms of the coupling strength, is characterized. We obtained analytically the threshold value of the coupling parameter for the synchronization of two unimodal and two bimodal piecewise linear maps, which by semi-conjugacy, under certain conditions, gives us information about the synchronization of the Duffing oscillators.

1 Introduction

Symbolic dynamics is a fundamental tool available to describe complicated time evolution of a chaotic dynamical system. Instead of representing a trajectory by numbers, one uses sequences of symbols of a finite alphabet which symbols correspond to the intervals defined by the turning points of a map f. We define the concept of symbolic synchronization in terms of a distance between the symbolic sequences generated by the iterates of the map f on its turning points. We show numerically that this distance may be considered a good measure of the synchronizability, since it converges to zero as the coupling parameter grows.

The analysis of synchronization phenomena of dynamical systems started in the 17th century with the finding of Huygens that two very weakly coupled pendulum clocks become synchronized in phase. Recently, the search for synchronization has evolved to chaotic systems. Two or more, identical, separated, chaotic systems starting from slightly different initial conditions would evolve in time, with completely different behaviour, but if they are coupled, we may see that after some time they exhibit exactly the same behaviour. Synchronization is a process wherein two or more systems adjust a given property of their motion to a common behaviour, due to coupling or forcing. Various types of synchronization have been studied. This includes complete synchronization (CS), phase synchronization (PS), lag synchronization (LS) generalized synchronization

(GS), anticipated synchronization (AS), and so on [2]. The coupled systems might be identical or different, the coupling might be unidirectional, (master-slave or drive-response), or bidirectional (mutual coupling) and the driving force might be deterministic or stochastic.

In [6], Kyprianidis et al. observed numerically the synchronization of two identical single-well Duffing oscillators as a function of the coupling parameter. In [5], Vincent and Kenfack studied numerically the bifurcation structure of a double-well Duffing oscillator coupled with a single-well one. To detect synchronization they use the quantity p defined by the time during which the distance between two trajectories $x(t)$ and $y(t)$ is not greater then ε divided by the total time span on the considered trajectory. They consider that there is synchronization if $\varepsilon \leqslant 0.01$ and obtain, for several values of ω a region in the parameter β, k plane where the oscillators are synchronized. This is not a complete synchronization because the two oscillators are not identical.

In this work we investigate the synchronization of two identical unimodal and bimodal piecewise linear maps. We obtain, analytically, the value of the coupling parameter for which the complete synchronization is achieved. Then, we examine numerically the synchronization of two identical unidirectionally coupled double-well Duffing oscillators for fixed values of the parameters α and β in terms of the coupling parameter k. We verify that the synchronization threshold agrees with the theoretical results obtained for the unimodal and bimodal piecewise linear maps.

2 Main Results

Consider the coupling of two identical maps $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$, denoting by k the coupling parameter. To be able to say if the two systems are synchronized we must examine the difference $z_n = y_n - x_n$ and see if this difference converges to zero, as $n \to \infty$.

If the coupling is unidirectional

$$\begin{cases} x_{n+1} = f(x_n) \\ y_{n+1} = f(y_n) + k\left[f(x_n) - f(y_n)\right] \end{cases}$$

then

$$z_{n+1} = (1-k)\left[f(y_n) - f(x_n)\right]. \tag{2.1}$$

These two systems are said to be in complete synchronization if there is an identity between the trajectories of the two systems. In [9] and [10] it was establish that this kind of synchronization can be achieved provided that all the conditional Lyapunov exponents are negative. Since then, some authors [12] have reported theirs computational experiments showing that apparently, it is possible to achieve synchronization without the negativity of all conditional Lyapunov exponents and some others (see [3] and references there in) have reported that sometimes there is brief lack of synchronization in the region where all the conditional Lyapunov exponents are negative. How to explain these situations? In fact, there is a numerical trap when coupling identical systems. Near the synchronization manifold, the two identical systems looks like in complete synchronization due to finite precision of numerical calculations (see [11]). As a matter of fact, the negativity of the conditional Lyapunov exponents is a necessary condition for the stability of the synchronized state (see [2]). The negativity of the conditional or transverse Lyapunov exponents is a mathematical expression of the decreasing to zero of the logarithm average of the distance of the solutions on the transverse manifold to the solutions on the synchronization manifold. So, if there is a strong convergence of this distance to zero, this average must decrease to zero. But the converse is not true. Indeed, even when all the conditional Lyapunov exponents are negative, it is possible that some orbits escape from the synchronization manifold. This is only a weak synchronization, in the sense of Milnor (see [11]). Only Lyapunov functions give necessary and sufficient conditions for the stability of the synchronization manifold. Besides the fact that the stability based on the negativity of the conditional Lyapunov exponents be only a weak stability, there are other phenomena like bubbling, riddling and blowout bifurcation that can explain the brief and persistent events of desynchronization in the region where all the conditional Lyapunov exponents are negative, see [2] and references therein. When coupling two systems, as the coupling parameter k grows from zero, we may find first a region $(k_c, k_{c,\max})$ where there is weak (in the sense of Milnor) synchronization and then a region $(k_{c,\max}, +\infty)$ where there is strong (in the topological sense) synchronization, see [11] and references therein. Note that, if the coupling involves more then two systems (network [3]) the synchronization interval may be bounded.

Nevertheless, if the coupled systems are defined by piecewise linear maps, which is the case we are going to study in the next section, the weak and the strong concepts of synchronization coincides and it occurs for $k > k_c$. The synchronization threshold k_c obtained from the assumption that all conditional Lyapunov exponents are negative, is expressed in terms of the Lyapunov exponent of the local map f. If this local map f is a piecewise linear map with slope $\pm s$ everywhere, then its Lyapunov exponent is exactly $\log|s|$.

2.1 Synchronization of Unimodal Maps

Consider the tent map f_s: $[0,1] \to [0,1]$ defined by

$$f_s(x) = \begin{cases} sx - s + 2, & \text{if } 0 \leqslant x < 1 - \frac{1}{s}, \\ s - sx & \text{if } 1 - \frac{1}{s} \leqslant x \leqslant 1. \end{cases}$$

Recall that any piecewise monotonic map of positive entropy and growth number s is topologically semi-conjugated to a piecewise linear map with slope $\pm s$ everywhere, see [8]. This map can be written as

$$f_s(x) = sx - s + 2 + 2\theta\,(s - sx - 1)\,, \tag{2.2}$$

with

$$\theta(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1 - \frac{1}{s}, \\ 1 & \text{if } 1 - \frac{1}{s} \leqslant x \leqslant 1. \end{cases}$$

The symbols $\{0,1\}$ correspond to the usual alphabet $\{L,R\}$ in the symbolic dynamics, see [8] and [7]. As the value of k grows, the number of initial equal symbols in the x and y symbolic sequences (see section 4), grows also. This is a numerical evidence that the two systems will be synchronized and can be expressed by the following definition of distance. Compare with [4].

Definition 1. Let $S = S_1 S_2 \ldots$ be a symbolic sequence, using symbols S_i belonging to some alphabet $\mathscr{A}$. Define a distance between two symbolic sequences $S_x = S_{x_1} S_{x_2} \ldots S_{x_p} \ldots$ and $S_y = S_{y_1} S_{y_2} \ldots S_{y_q} \ldots$ by

$$d(S_x, S_y) = e^{-n}, \text{ where } n = \min\{n \geqslant 1 : S_{x_n} \neq S_{y_n}\}.$$

We may say that two symbolic sequences S_x and S_y are synchronized if their distance $d(S_x, S_y)$, as defined above, converges to zero and two systems $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ are symbolic synchronized if the symbolic sequences S_x and S_y, of each turning point, are synchronized.

Theorem 1. *Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two identical coupled systems, with f given by (2.2), s the growth number of (2.2), with $1 < s \leqslant 2$ and $k \in [0,1]$ the coupling parameter. If*

$$\exists n \in \mathbb{N} : d\left(\theta(y_{n+j}), \theta(x_{n+j})\right) \leqslant e^{-n},\ \forall j \geqslant 0,$$

then, the unidirectional coupled systems (2.1) are completely synchronized if $k > \frac{s-1}{s}$.

Proof. Considering (2.1) and (2.2),
$z_{n+1} = (1-k)\left[sy_n - s + 2 + 2\theta_{y_n}(s - sy_n - 1) - sx_n + s - 2 - 2\theta_{x_n}(s - sx_n - 1)\right].$
If $\theta_{y_{n+j}} = \theta_{x_{n+j}}, \forall j \geqslant 0$, then $z_{n+1} = (1-k)(1-2\theta)sz_n$.
It follows that, $z_{n+m} = \left[(1-k)(1-2\theta)s\right]^m z_n$.
Thus, if $\theta = 0$, then $z_{n+m} = \left[(1-k)s\right]^m z_n$ and if $\theta = 1$, $z_{n+m} = \left[(1-k)(-s)\right]^m z_n$. In both cases, $z_{n+m} = r^m z_n$, with $|r| = (1-k)s$.
So, letting $m \to \infty$, we have $\lim_{m\to\infty} r^m z_n = 0$, iff $|r| < 1$, i.e., $|(1-k)s| < 1 \Rightarrow k > \frac{s-1}{s}$, for $k \in [0,1]$, as desired. ■

2.2 Synchronization of Bimodal Maps

Consider the bimodal piecewise linear map $f_{s,r}\colon [0,1] \to [0,1]$, with slopes $\pm s$, and $s > 1$, defined by

$$f_{s,r}(x) = \begin{cases} -sx + 1 & \text{if } 0 \leqslant x < c_1 \\ sx + r - 1 & \text{if } c_1 \leqslant x < c_2 \\ -sx + s & \text{if } c_2 \leqslant x \leqslant 1 \end{cases}$$

with $r = \frac{3+s}{2} - s(c_1 + c_2)$ and critical points $c_1 = \frac{2-r}{2s}$ and $c_2 = \frac{1+s-r}{2s}$. Recall that any transitive bimodal map is semi-conjugated to such a map. This map can be written as

$$f_{s,r}(x) = -sx + 1 + \theta_{c_1}(2sx + r - 2) + \theta_{c_2}(-2sx + s - r + 1), \tag{2.3}$$

with

$$\theta_{c_i}(x) = \begin{cases} 0, \text{ if } 0 \leqslant x < c_i \\ 1, \text{ if } c_i \leqslant x \leqslant 1 \end{cases} \qquad (i = 1, 2).$$

Note that, in this case, we may define $\theta(x) = \theta_{c_1}(x) + \theta_{c_2}(x)$, *i.e.*,

$$\theta(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < c_1 \\ 1 & \text{if } c_1 \leqslant x < c_2 \\ 2 & \text{if } c_2 \leqslant x \leqslant 1 \end{cases}.$$

The symbols $\{0, 1, 2\}$ correspond to the usual alphabet $\{L, M, R\}$ in the symbolic dynamics, see [8] and [7].

Theorem 2. *Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two identical coupled systems, with f given by (2.3), s the growth number of (2.3), with $1 < s \leqslant 2$ and $k \in [0,1]$ the coupling parameter. If*

$$\exists n \in \mathbb{N} : d\left(\theta(y_{n+j}), \theta(x_{n+j})\right) \leqslant e^{-n},\ \forall j \geqslant 0,$$

then, the unidirectional coupled systems (2.1) are completely synchronized if $k > \frac{3s-1}{3s}$.

Proof. Considering (2.1) and (2.3)

$$\begin{aligned} z_{n+1} = \quad & (1-k)\left\{\left[-sy_n + 1 + \theta_{c_{1_{y_n}}}(2sy_n + r - 2) + \theta_{c_{2_{y_n}}}(-2sy_n + s - r + 1)\right]\right. \\ & \left. -\left[-sx_n + 1 + \theta_{c_{1_{x_n}}}(2sx_n + r - 2) + \theta_{c_{2_{x_n}}}(-2sx_n + s - r + 1)\right]\right\}. \end{aligned}$$

If $\theta_{c_{1_{y_n}}} = \theta_{c_{1_{x_n}}} = \theta_{c_1}$ and $\theta_{c_{2_{y_n}}} = \theta_{c_{2_{x_n}}} = \theta_{c_2}$, then

$$z_{n+1} = (1-k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s z_n.$$

It follows that $z_{n+m} = \left[(1-k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s\right]^m z_n$.

Denoting $r = (1-k)\left(-1 + 2\left(\theta_{c_1} - \theta_{c_2}\right)\right) s$, we have $z_{n+m} = r^m z_n$. Thus, if $\theta_{c_1} - \theta_{c_2} = 0$ or 1, then $|r| = (1-k)s < 1 \Rightarrow k > \frac{s-1}{s}$. If $\theta_{c_1} - \theta_{c_2} = -1$, then $|r| = 3(1-k)s < 1 \Rightarrow k > \frac{3s-1}{3s}$. So, as $\frac{3s-1}{3s} > \frac{s-1}{s}$, to have synchronization it suffices that $k > \frac{3s-1}{3s}$. ■

3 Duffing Oscillator Application

Consider two identical unidirectionally coupled Duffing oscillators

$$\begin{cases} x''(t) = x(t) - x^3(t) - \alpha x'(t) + \beta\, Cos(wt) \\ y''(t) = y(t) - y^3(t) - \alpha y'(t) + k\left[x(t) - y(t)\right] + \beta\, Cos(wt) \end{cases} \tag{3.1}$$

where k is the coupling parameter, see [5] and references therein. We will choose parameter values for which each uncoupled oscillator exhibits a chaotic behaviour, so if they synchronize, that will be a chaotic synchronization.

3.1 The Uncoupled Case

The system (3.1) with $k = 0$ (uncoupled) reduces to

$$x''(t) = x(t) - x^3(t) - \alpha x'(t) + \beta Cos(wt).$$

Attending to the complexity of the above equation, a basic tool is to do an appropriate Poincaré section to reduce the dimensionality. In our case, we did a section defined by $y = 0$, since it is transversal to the flow, it contains all fixed points and captures most of the interesting dynamics. In order to see how the first return Poincaré map change with the parameters we did bifurcation diagrams. See in Fig. 1 the variation of the coordinate x_n of the first return Poincaré map, versus the parameter $\beta \in [0.15, 0.5]$, for a fixed value of $\alpha = 0.25$. It is clear the growing of complexity as the parameter β increases.

We found in the parameter plane (α, β), a region $\mathscr{U}$ where the first return Poincaré map behaves like a unimodal map and a region $\mathscr{B}$ where the first return Poincaré map behaves like a bimodal map, see Fig. 2.

Consider parameter values and initial conditions for which each uncoupled system exhibits a chaotic behaviour and its first return Poincaré map is like a unimodal or like a bimodal map. We choose, for example, $w = 1.18$, $x_0 = 0.5$, $x'_0 = -0.3$, $y_0 = 0.9$, $y'_0 = -0.2$ and $\alpha = 0.4$, $\beta = 0.3578$, for the unimodal case and $\alpha = 0.5$, $\beta = 0.719$, for the bimodal case.

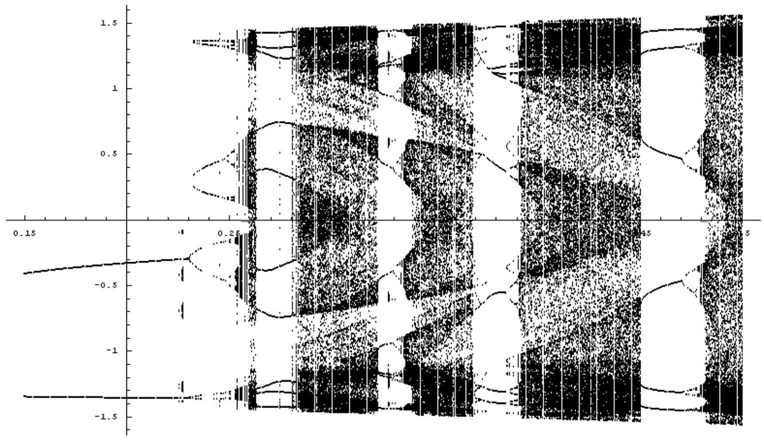

Figure 1: Bifurcation diagram for x_n as a function of $\beta \in [0.15, 0.5]$, for a fixed $\alpha = 0.25$.

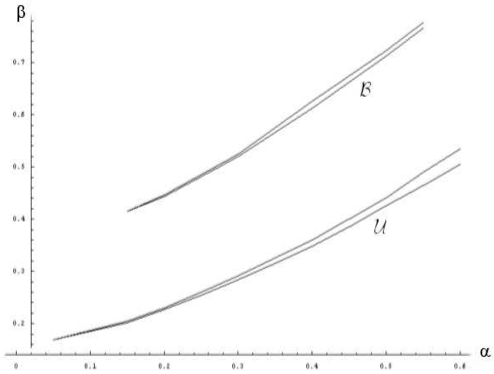

Figure 2: Unimodal and bimodal region in the parameter (α, β) plane.

3.2 The Coupled Case

In Fig. 3 the bifurcation diagram for the unidirectional coupled system (3.1) with $\alpha = 0.4$, $\beta = 0.3578$ and the coupling parameter, $k \in [0.001, 0.03]$, shows several kind of regions. In section 4, we will compute the topological entropy in some points of these regions.

We choose for example the values $k = 0$, $k = 0.05$, $k = 0.13$, $k = 0.5$ and $k = 0.301$. We will see that for k larger than $k \approx 0.13$ the topological entropy remains constant, but positive. Meanwhile we find values, of the k parameter, where the entropy is zero, that is, where there is chaos-destroying synchronization, see [11]. Numerically we can also see the evolution of the difference $z = y - x$ with k. The synchronization will occur when $x = y$. See some examples in Fig.4 to the unimodal case and in Fig.5 to the bimodal case.

Notice that, the pictures in Fig.4 and Fig.5 confirms numerically the theoretical results given by theorems 2 and 3, respectively. For $\alpha = 0.4$ and $\beta = 0.3578$ (Fig.4) which correspond to $s = 1.272\ldots$, the synchronization occurs for $k > 0.214\ldots$. For the bimodal case, $\alpha = 0.5$ and $\beta = 0.719$, (Fig.5), which correspond to $s = 2,618\ldots$, the synchronization occurs for $k > 0.873\ldots$.

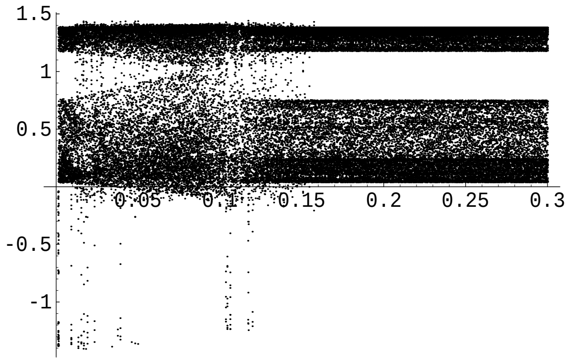

Figure 3: Bifurcation diagram for x_n as a function of $k \in [0.001, 0.3]$, for fixed values of $\alpha = 0.4$ and $\beta = 0.3578$.

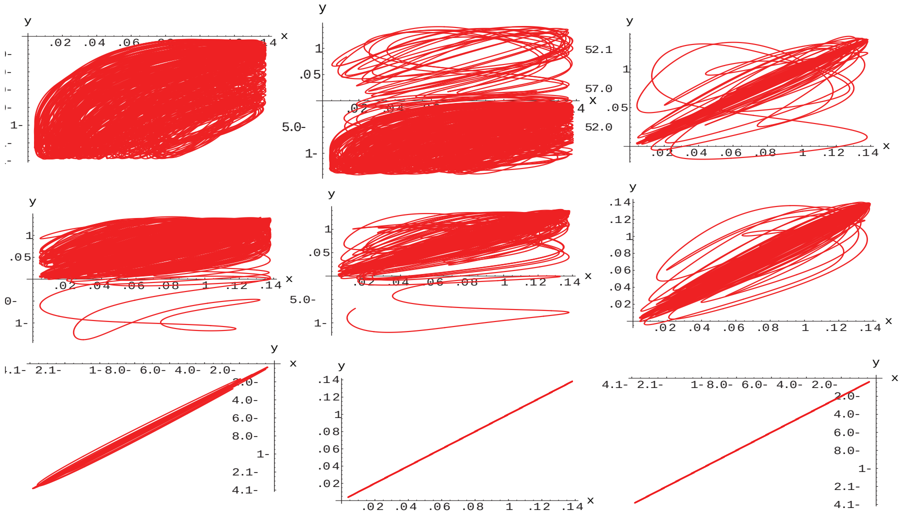

Figure 4: Evolution of x versus y for the unimodal case ($\alpha = 0.4$, $\beta = 0.3578$) for some values of k: 0.003, 0.005, 0.022, 0.103, 0.111, 0.12, 0.136, 0.195 and 0.306.

4 Symbolic Dynamics

Using techniques from Symbolic Dynamics, see [8] and [7], we compute the topological entropy h_{top} for some values of the coupling parameter. Considering the return map for the second equation of system (3.1), with $\alpha = 0.4$ and $\beta = 0.3578$ (unimodal case), we obtain for $k = 0.13$, the kneading sequence $(CRLRRRLRLR)^{\infty}$. Using kneading theory we obtain the kneading matrix

$$N(t) = \left[\frac{(1-t^2)\left(-1+t^2+t^4-t^6+t^8\right)}{1-t^{20}} \quad \frac{-(1-t)^2\left(-1+t^2+t^4-t^6+t^8\right)}{1-t^{20}} \right]$$

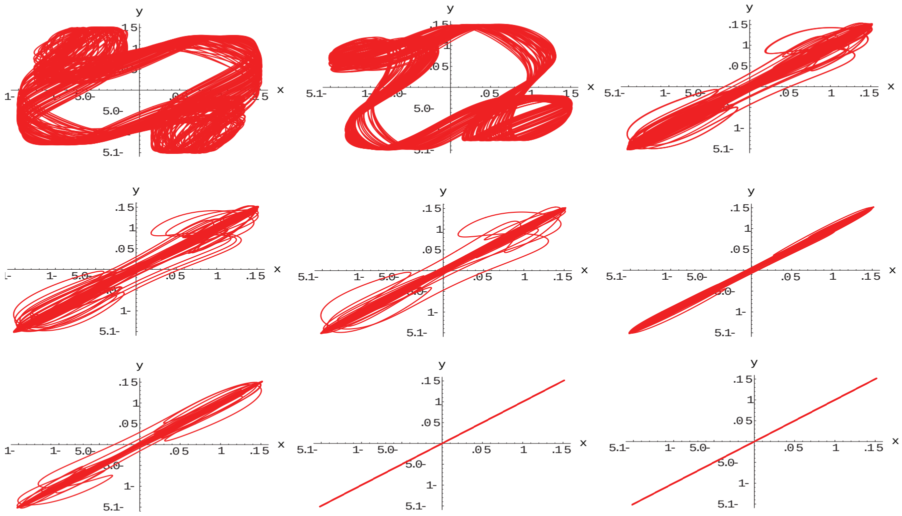

Figure 5: Evolution of x versus y for the bimodal case ($\alpha = 0.5$, $\beta = 0.719$) for some values of k: 0.003, 0.014, 0.08, 0.095, 0.1, 0.113, 0.126, 0.875 and 0.916.

and the kneading determinant

$$D(t) = \frac{(-1+t)\left(-1+t^2+t^4-t^6+t^8\right)}{1-t^{20}}.$$

Denoting $t^* = \min\{t \in [0,1] : D(t) = 0\} = 0.813014\ldots$ and s the growth number of f, then $s = \frac{1}{t^*}$ and the topological entropy of the map f is $h_{top}(f) = \log s \approx 0.20701$. By this method, we compute some values of the topological entropy h_{top} for other values of the coupling parameter k. Notice the correspondence of these values for the topological entropy with the evolution of k in the bifurcation diagram, see Fig. 3.

k	S_x	$D(t)$	h_{top}
0	$(CRLRRR)^\infty$	$\frac{(-1+t)\left[\left(-1+t^2\right)+t^4\right]}{1-t^{12}}$	$0.24061\ldots$
0.05	$(CRLR)^\infty$	$\frac{-(1+t)\left(-1+t^2\right)}{1-t^8}$	0
0.13	$(CRLRRRLRLR)^\infty$	$\frac{(-1+t)\left[\left(-1+t^2\right)(1-t^4)+t^8\right]}{1-t^{20}}$	$0.20701\ldots$
0.5	$(CRLRRRLRLR)^\infty$	$\frac{(-1+t)\left[\left(-1+t^2\right)(1-t^4)+t^8\right]}{1-t^{20}}$	$0.20701\ldots$
0.301	$(CRLRRRLRLR)^\infty$	$\frac{(-1+t)\left[\left(-1+t^2\right)(1-t^4)+t^8\right]}{1-t^{20}}$	$0.20701\ldots$

As the value of k grows, the number of initial equal symbols in the x and y symbolic sequences, grows also. This can be expressed by the distance defined above and it is a numerical evidence that the two systems will be synchronized.

	k	n
S_x : *RLRRRLRLRRRLRRR*	0.006	
S_y : *RLRRRLRRRLRRRLRRRLRRRLRRRLRLRR*		7
S_x : *RLRRRLRLRLRLRRRLRLRLR*	0.03	
S_y : *RLRRRLRRRLRLRLRRRLRRRLRLRRRLRR*		7
S_x : *RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL*	0.045	
S_y : *RLRLRLRRRLRLRLRLRRRLRRRRRLRRRL*		3
S_x : *RLRRRLRLRRRLRRRLRLRLRLRRRLRRRL*	0.066	
S_y : *RLRRRLRLRLRRRLRLRLRLRRRLRLRRRL*		9
S_x : *RLRRRLRRRLRRRLRRRLRRRLRRRLRLRR*	0.1	
S_y : *RLRRRLRRRRRLRLRRRLRRRRRLRLRRRR*		9
S_x : *RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL*	0.1577	
S_y : *RLRRRLRLRRRLRRRLRLRLRLRLRLRLRL*		19
S_x : *RLRRRLRLRLRLRRRLRRRLRRRLRLRRRL*	0.1595	
S_y : *RLRRRLRLRLRLRRRLRRRLRRRLRLRRRL*		30
S_x : *RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL*	0.3	
S_y : *RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL*		30

5 Conclusions

When doing Poincaré sections with $y = 0$, we obtained regions $\mathscr{U}$ and $\mathscr{B}$ where the Poincaré map behaves like a unimodal and bimodal map respectively. By a result from Milnor and Thurston [8] and Parry it is known that every m-modal map $f\colon I = [a,b] \subset \mathbb{R} \to I$, with growth rate s and positive topological entropy h_{top} ($\log s = h_{top}(f)$) is topologically semi-conjugated to a $p+1$ piecewise linear map T, with $p \leqslant m$, defined on the interval $J = [0,1]$, with slope $\pm s$ everywhere and $h_{top}(T) = h_{top}(f) = \log s$, i.e., there exist a function h continuous, monotone and onto, $h : I \to J$, such that $T \circ h = h \circ f$. If, in addition, h is a homeomorphism, then f and T are said topologically conjugated. It is proved that in the case of topological conjugacy the synchronization of the two piecewise linear maps T implies the synchronization of the two conjugated m-modal maps f. Furthermore, by a result of Preston [1], if f is topologically transitive, then the mentioned semi-conjugacy is in fact a conjugacy. By a result of Blokh [1], a sufficient condition for a map f to be topologically transitive is $h_{top}(f) \geqslant (1/2)\log 2$. So, the study and conclusions about synchronization of piecewise linear unimodal and bimodal maps, expressed in theorems 2 and 3, can be applied to guarantee the synchronization of more general maps.

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