ON THE EMBEDDING OF A (P-1)-DIMENSIONAL NON INVERTIBLE MAP INTO A P-DIMENSIONAL INVERTIBLE MAP (P=2,3)

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This paper concerns the description of some properties of p-dimensional invertible real maps T_b, turning into a $(p-1)$-dimensional non invertible ones T_0, $p = 2, 3$, when a parameter b of the first map is equal to a critical value, say b=0. Then it is said that the noninvertible map is embedded into the invertible one. More particularly properties of the stable, and the unstable manifolds of a saddle fixed point are considered in relation with this embedding. This is made by introducing the notion of *folding* as resulting from the crossing through a *commutation curve* when $p = 2$, or a *commutation surface* when $p = 3$.

1. Introduction

This paper is devoted to the description of some properties of p-dimensional invertible real maps T_b, which turn into a $(p-1)$-dimensional non invertible one T_0, $p = 2, 3$, when a parameter b of the first map is equal to a critical value, say b=0. Then it is said that the noninvertible map is embedded into the invertible one. The properties considered here are essentially related to the manifolds of a saddle fixed point q_i having at least one of its multipliers (eigenvalues) with a modulus less than one.

The first steps dealing with both stable and unstable manifolds of a saddle point were obtained for $p = 2$ in [Mira, 1978, 1979, 1987] [Gumowski & Mira, 1980] . It was about two-dimensional invertible maps T_b (Jacobian $J = -b$) (written here in the recurrence form):

$$\begin{matrix} x_{n+1} = f(x_n, a) + y_n \\ y_{n+1} = bx_n \end{matrix} , \qquad n = 0, 1, ..., \tag{1}$$

which from the initial condition (x_0, y_0), $y_0 \neq 0$, turns into the one-dimensional noninvertible map T_0:

$$x_{n+1} = f(x_n, a), \qquad n = 1, 2, ..., \tag{2}$$

with the initial condition x_1 for $n = 1$,when $b = 0$.

The formulation of the problem for $p = 3$ is given in [Mira et al., 1996] (pages 565-567). The same reference (pages 8-13) considers the general problem of the embedding of $(p - k)$ noninvertible map into a p-dimensional invertible one, $p = 3, 4, ...$, $k = 2, 3, ...$, from the Valiron's results [Valiron, 1948]. Considering an embedding of a noninvertible map into a p-dimensional invertible one with $p = 2$, and 3, the purpose of this paper concerns some properties of the unstable manifold and the stable one related to saddle fixed point, associated with some *limit* properties when the embedding parameter $b \to 0$. It is a question of completing the previous results for $p = 2$, which in the past were only deduced from numerical observations in some cases, and indicating some extensions for $p = 3$. It appears from these considerations that the $(p - 1)$-dimensional noninvertible map gives a *germinal* situation of the phase space, which permits to understand the p-dimensional invertible map behavior for sufficiently small $| b |$-values. We remark that the properties appear in a more evident way in the case of piecewise linear maps (cf. the pioneering results of [Lozi, 1978]), because the stable and unstable manifolds of saddle points are also piecewise linear.

This paper does not intend to present a general theory of the embedding of a $(p - 1)$-dimensional noninvertible map into a p-dimensional invertible map, but only to give a comprehensive view of this question from some examples. In this framework it must not be considered as a mathematical text in the classical sense. It is only a question to give some key directions for future more elaborated texts. So, in order to facilitate the matter exposition, the simplest situations are considered. They are related to the following hypotheses called (H).

- (a) The maps are defined by continuous functions.

- (b) The embedded (p-1)-dimensional noninvertible map is of $(Z_0 - Z_2)$ type, that is a critical set CS separates the phase space into two regions : Z_0 whose points have no real preimages, and Z_2 each point having two real preimages. CS is a *critical point* C (image of the unique extremum of $f(x, a)$ in (2)) for $p = 2$. When $p = 3$, let T_0 be the embedded two dimensional noninvertible map. CS is a *critical curve* LC, image of the set defined by $J = 0$, J being the Jacobian of T_0 if the map is smooth, or image of the set of non-smoothness in the other case. The critical set CS separates a region $J > 0$ from a region $J < 0$, and CS is the set of points having two coincident rank-one preimages. The set $CS_n = T_0^n(CS)$ is called critical set of rank $n + 1$. Here T_0^n indicates n successive applications of the map T_0. When $p = 3$, it is supposed that LC is connected, made up of only one branch.

- (c) The maps and their inverses are without denominator (for avoiding difficulties described in [Bischi *et al.*, 1999]).

The properties of a saddle unstable manifold and of a stable one, when the embedding parameter b tends toward zero, have the following important consequences:

- A saddle unstable manifold in the p-dimensional space of the invertible map is crushed in the $(p - 1)$-dimensional space of the noninvertible map. Then the set of bending back (or "fold") points of the saddle unstable manifold tend toward arcs of the *critical sets* of rank 1, 2, 3, ..., related to the embedded noninvertible map, i.e. arcs of the set CS of points, and their successive images (pages 291-296 of [Mira, 1987]). In particular, a chaotic attractor generated by a noninvertible map, the boundary of which is made up of arcs of CS and CS_n, $n = 1, 2, ..., k$, can be considered as the result of such a crushing.

- The limit set of a saddle stable manifold in the p-dimensional space of the invertible map tends toward a $(p-1)$-dimensional non connected set of this space, which intersects the $(p-1)$-dimensional space of the noninvertible map at the $(p - 2)$-dimensional saddle stable set of T_0, and its successive preimages.

After this introduction, the second section gives some definitions and reminds useful properties of two-dimensional noninvertible maps. The third section is devoted to the case $p = 2$. Considering piecewise linear maps and smooth maps, properties of the stable, or the unstable, manifold of a saddle point are described, from the notion of *commutation sets* defined in sec. 2. The fourth section concerns the case $p = 3$. Due to difficulties related to the dimension increase of the phase space, this last part will be limited with respect to the case $p = 2$. In particular the two-dimensional stable manifold of a saddle point will be considered for a smooth map, and the two-dimensional unstable manifold of a saddle in the case of a piecewise linear map.

2. Formulation of the problem. Definitions. Properties

2.1. Formulation of the problem

For $p = 2$ in a recurrence form, the embding is given by (1) which from the initial condition (x_0, y_0) turns into the one-dimensional noninvertible map T_0 given by (2).In a map form, T_b is written:

$$\begin{aligned} x' &= f(x, a) + y \\ y' &= bx \end{aligned} \tag{3}$$

The corresponding noninvertible map form T_0 is written :

$$x' = f(x, a) \tag{4}$$

When the dimension of the embedding is $p = 3$, the recurrence form is :

$$\begin{aligned} x_{n+1} &= f(x_n, y_n, a) \\ y_{n+1} &= g(x_n, y_n, c) + z_n \qquad n = 0, 1, .., \\ z_{n+1} &= b(dx_n + ey_n) \end{aligned} \tag{5}$$

The map form T_b is:

$$\begin{aligned} x' &= f(x, y, a) \\ y' &= g(x, y, c) + z \\ z' &= b(dx + ey) \end{aligned} \tag{6}$$

With the initial condition (x_0, y_0, z_0), $z_0 \neq 0$, from $n = 1$, $T_{b=0}$ turns into the two-dimensional recurrence T_0:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n, a) \\ y_{n+1} &= g(x_n, y_n, c) \end{aligned} \quad , \qquad n = 1, 2, ..., \tag{7}$$

with the initial condition (x_1, y_1), supposed to be a noninvertible recurrence. In a noninvertible map form T_0 is written :

$$\begin{aligned} x' &= f(x, y, a) \\ y' &= g(x, y, c) \end{aligned} \tag{8}$$

The following assumptions are made for the functions $f(x, y, a)$ and $g(x, y, c)$:

- (i) the map T_0 is noninvertible of $(Z_0 - Z_2)$ type,

- (ii) the Jacobian of T_b does not change its sign for $b \neq 0$, that is the three-dimensional map is invertible.

The condition (ii) ensures that the map T_b is invertible whether the map is smooth, or nonsmooth. The paper concerns only maps family having the forms (3) and (6).

2.2. Definitions

Definition 2.1. Let T a continuous noninvertible map $X' = TX$, $\dim X = p$. The critical set of rank-one, say CS , is the geometrical locus of points X having at least two coincident preimages. The critical set CS_i of rank-$(i{+}1)$, $i > 0$, is the rank-i image of the set $CS_0 \equiv CS$.

Noninvertible maps, giving rise to regions Z_i are classified into types related to the nature of the regions Z_i characterizing the considered map. The class of maps having the simplest properties is that of the $(Z_0 - Z_2)$ maps. For this class CS separates the space into two open regions Z_0 and Z_2, a point $X \in Z_2$ having two real distinct preimages of rank-one, and a point $X \in Z_0$ being without real preimages.

Definition 2.2. A closed and invariant set A is called an attracting set if some neighborhood U of A exists such that $T(U) \subset U$, and $T^n(X) \to A$ as $n \to \infty, \forall X \in U$.

Definition 2.3. The set $D = \cup_{n \geq 0} T^{-n}(U)$ is the total basin (or simply: basin of attraction, or influence domain) of the attracting set A (definition 2).

Definition 2.4. Let p a saddle fixed point and U a neighborhood of p. The local unstable set $W^u_{loc}(p)$ of $p \in U$ is given by: $W^u_{loc}(p) = \{x \in U : x_{-n} \in T^{-n}(x) \to p,\ x_{-n} \in U, \forall n\}$, and the global unstable set $W^u(p)$ of p is given by $W^u(p) = \bigcup_{n \geq 0} T^n[W^u_{loc}(p)]$. The local stable set $W^s_{loc}(p)$ of $p \in U$ is given by: $W^s_{loc}(p) = \{x \in U : x_n \in T^n(x) \to p,\ x_n \in U, \forall n\}$, and the global stable set $W^s(p)$ of p is given by $W^s(p) = \bigcup_{n \geq 0} T^n[W^s_{loc}(p)]$.

Definition 2.5. The point q is said homoclinic to the non-attracting fixed point p (or homoclinic point of p) iff $q \in W^u(p) \cap W^s(p)$. Let r another non-attracting fixed point. A point $q \in U(p)$ is said heteroclinic from p to r, if $T^n(q) \to r$, when n increases, and q belongs to the local unstable set

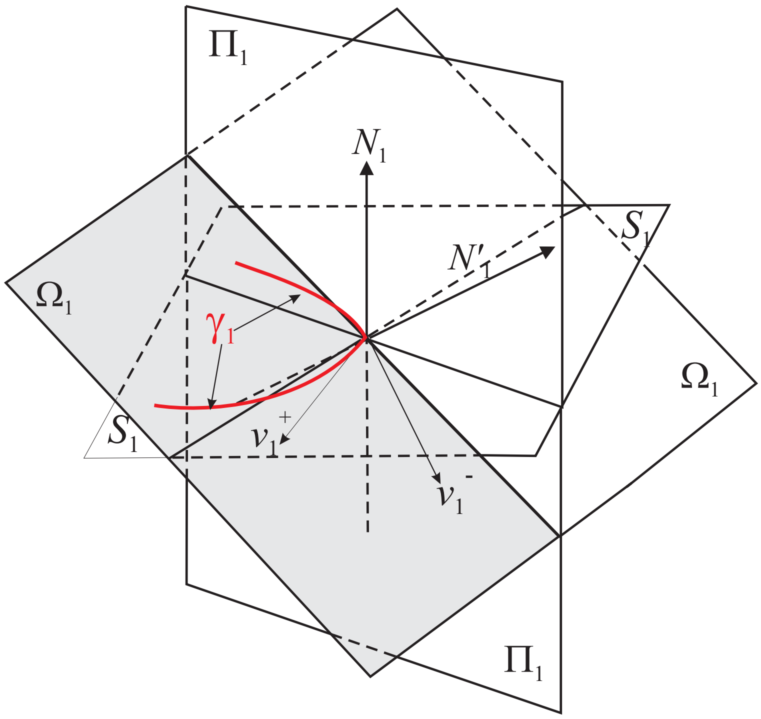

Fig. 1. Folding of an arc, generated by the commutation set (surface) S_0, in the case $p = 3$. Case of a nonsmooth map.

$W^u_{loc}(p)$ of $p \in U(p)$.

Definition 2.6. *Commutation sets (S) of the maps (3) and (6).*

(a) *Case of a continuous nonsmooth map* : S_0 is the set of non smoothness in the p-dimensional space. When $b = 0$, in the (p-1)-dim. space, S_0, turns into a set separating two regions where the noninvertible map Jacobian has opposite signs.

(b) *Case of a smooth map.* Let J" be the Jacobian matrix of T_0, that is the (p-1)×(p-1) minor of the Jacobian matrix of T_b, defined by the (p-1) first linesrows and columns in the p-dimensional space. S_0 is the set $J" = 0$ in the p-dimensional space. When $b = 0$, in the (p-1)-dim space, S_0, turns into a set separating two regions where J" has opposite signs.

(c) S_0 is called rank-0 commutation set. $S_n = T_b^n(S_0)$ is called rank-n commutation set.

Definition 2.7. *Folding of an arc, generated by the commutation set S_0, in the case $p = 2$.*

Let γ_0 be a sufficiently small arc crossing through S_0, $\gamma_0 \cap S_0 = a_0$, $a_1 = T_b(a_0)$, $a_1 \in S_1$, $\gamma_1 = T_b(\gamma_0)$. Let N_1 be the normal to S_1 at a_1.

(a) *Case of a nonsmooth map on S_0.* The commutation set (curve) S_0 is said to give rise to a folding, if the two arcs of γ_1, joining at a_1 (fold point), are located on the same side of N_1.

(b) *Case of a smooth map on S_0.* S_0 gives rise to a folding near the point a_1, when the two arcs of γ_1, joining at a_1, are locally settled on the same side of a line M_1 parallel to N_1, and located in a neighborhood of N_1. The point a'_1 of γ_1 on M_1 is the fold point.

Definition 2.8. *Folding of an arc, generated by the commutation set (surface) S_0, in the case $p = 3$.*

Let γ_0 be a sufficiently small oriented arc crossing through S_0, $\gamma_0 \cap S_0 = a_0$, $a_1 = T_b(a_0)$, $a_1 \in S_1$, $\gamma_1 = T_b(\gamma_0)$, v_1^- and v_1^+ being the tangents to γ_1 just before and just after a_1. Let Ω_1 the plane defined by the vectors v_1^- and v_1^+, N_1 the normal to S_1 at a_1, N'_1 the normal to Ω_1 at a_1. Let Π_1 be the plane defined by N_1 and N'_1.

(a) *Case of a nonsmooth map on S_0.* S_0 is said to give rise to a folding at the point a_1, when v_1^- and v_1^+ are located on the same side of the plane Π_1 (see Fig. 1).

(b) *Case of a smooth map on S_0.* S_0 gives rise

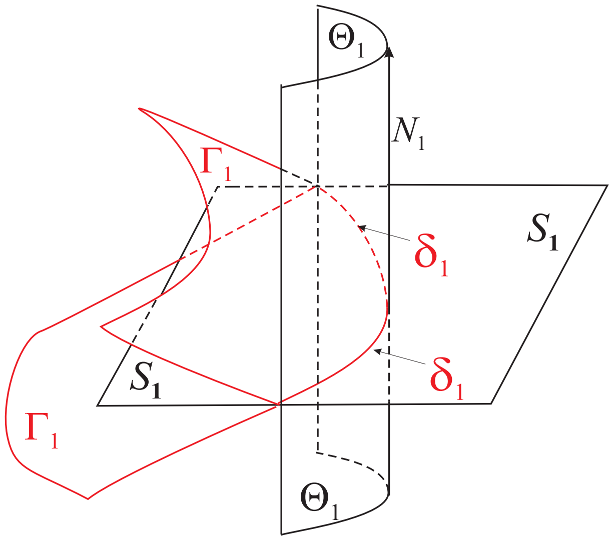

Fig. 2. Folding of a piece of surface, generated by the commutation set (surface) S_0, in the case $p = 3$. Case of a nonsmooth map.

to a folding near the point a_1, when v_1^- and v_1^+ are located on the same side of a plane Π_1' located in a neighborhood of Π_1, and parallel to this plane. The point a_1' of γ_1 on Π_1' is the fold point.

Definition 2.9. *Folding of a piece of surface, generated by the commutation set (surface) S_0, in the case $p = 3$.*

Let Γ_0 be a piece of surface crossing through S_0, $\Gamma_0 \cap S_0 = \delta_0$, $\Gamma_1 = T_b(\Gamma_0)$, $\Gamma_1 \cap S_1 = \delta_1$.and an arc $\eta_1 \subseteq \delta_1$. Let Θ_1 be the piece of surface made up of the set of the normals N_1 to S_1 at each point $a_1 \in \eta_1$.

(a) *Case of a nonsmooth map on S_0.* S_0 gives rise to a folding along η_1, when Γ_1 is locally situated on the same side of Θ_1(see Fig. 2).

(b) *Case of a smooth map on S_0.* S_0 gives rise to a folding along an arc η_1' near η_1, when Γ_1 is locally situated on the same side of a piece Θ_1' located in a neighborhood of Θ_1. The arc η_1' of Γ on Π_1' is the fold arc.

Remark. If S_0 gives rise to a folding in the situations of the definitions 7, 8 and 9, it is not always ensured that the arc $\gamma_n = T_b^n(\gamma_0)$ undergoes a fold on the set $S_n = T_b^n(S_0)$, in the cases of definitions 7 and 8, and the piece $\Gamma_n = T_b^n(\Gamma_0)$ undergoes a fold in the case of definition 9. The folding is characterized by properties equivalent to those related to N_1 in the case of definition 7, Π_1 in the case of definition 8, and Θ_1 in the case of definition 9, by considering the tangent set to the rank-n iterate of each of these three sets locally.

2.3. Some reminders and characteristic features

2.3(A). Critical sets

Most of results obtained till now concern one-dimensional noninvertible maps (CS is a point C) and two-dimensional ones (CS is a curve LC) [Mira, 1980; Mira *et al.*, 1996], i.e. $p = 2$ for the invertible maps (3), and $p = 3$ for the invertible maps (6). For the general class of noninvertible maps of the plane $T : R^2 \to R^2$, LC is made up of several branches separating the plane into regions, the points of a region having a number of rank-one preimages (or antecedents) different from the one related to the adjoining region. The plane R^2 can be subdivided into open regions Z_i, $R^2 = \bigcup_i \overline{Z_i}$, ($\overline{Z_i}$ being the closure of Z_i), the points of which have i distinct rank-one preimages. The boundaries of these regions are branches of the rank-one critical curve LC, locus of points such that at least two determinations of the inverse map are merging. In general, a critical curve LC is made up of several branches. The locus of these "*coincident rank-one preimages*" is a curve LC_{-1}, called rank-one curve of merging preimages. As in any neighborhood of a point of LC there are points for which at least two distinct inverses are defined, LC_{-1} is a set of points for which the Jacobian determinant of the noninvertible map T vanishes when it is smooth, or for which T is not differentiable, when it is nonsmooth. The curve LC satisfies the relations $T^{-1}(LC) \supseteq LC_{-1}$, T^{-1} being the backward iteration of T, and $T(LC_{-1}) = LC$.

The simplest case of two-dimensional noninvertible maps corresponds to LC made up of only one branch, which separates the plane into two open regions Z_0 and Z_2 (case of this paper). A point belonging to Z_2 has two distinct real preimages (or antecedents) of rank-one, and a point of Z_0 has no real preimages. The corresponding maps are said of $(Z_0 - Z_2)$ type. In more complex cases a classification of noninvertible maps from the structure of the set of Z_i regions can be made [Mira *et al.*, 1996].

2.3(B). Basins of invertible maps and noninvertible ones

Let T be the p-dimensional map: $X' = TX$. The total basin D of an attracting set A is the open set of points X whose forward trajectories (set of successive images of X) converge towards A. When the map is invertible the basin D is simply connected, and D is invariant by application of T, and under backward iteration T^{-1} of T. If the map is noninvertible D is invariant under backward iteration T^{-1} of T, but not necessarily invariant by T: $T^{-1}(D) = D$, $T(D) \subseteq D$. Here, the strict inclusion holds iff D contains points of Z_0, i.e. points without rank-one preimages. Such a basin may be *simply connected* as in the invertible case, but also *nonconnected*, and *multiply connected*, in the noninvertible case. These relations hold also for the closure of D. ∂D denotes the boundary of D. This boundary satisfies: $T^{-1}(\partial D) = \partial D$ and $T(\partial D) \subseteq \partial D$.

2.3(C). Absorbing areas and chaotic areas in two-dimensional noninvertible maps

The notion of *chaotic area* generated by a noninvertible map needs previously the definition of *absorbing area*. These notions were firstly introduced in [Gumowski & Mira, 1977] (cf. also p.356 [Gumowski & Mira, 1980]), and developed in [Barugola A., Cathala J.C. & Mira, 1986, 1995; Barugola A., & Cathala J.C., 1992; Cathala J.C., 1989; Gardini, 1992; Mira *et al.*, 1996c].

Critical curves are at the basis of the definition of absorbing area, and chaotic area (cf. more information in [Mira *et al.*, 1996] and its references). Roughly speaking an *absorbing area* (d') (of nonmixed type) is a region bounded by critical curves segments of different rank, such that from a finite number of iterations the successive images of all points of a neighborhood $U(d')$ enter into (d') and cannot get away after entering. Except for some bifurcation cases, a *chaotic area* is an invariant absorbing area, the points of which give rise to iterated sequences (or orbits) having the property of sensitivity to initial conditions. In general it contains infinitely many unstable cycles of increasing period, the corresponding limit sets, and the preimages of increasing rank of all these points.

In numerical simulations of a chaotic area, a critical set appears as a place of higher concentration of iterated points if the map is smooth, or as a separation of regions with different densities of iterated points if the map is not smooth. This is directly related to properties of local extrema of the map.

The notion of absorbing area, and chaotic area is extended to *mixed absorbing area*, *mixed chaotic area* (cf. [Mira *et al.*, 1996] and its references). These last areas differ from the non mixed ones by the fact that their boundaries are now made up of the union of critical curves segments, and segments of the unstable set of a saddle fixed point, or a saddle cycle (periodic point), or even segments of several unstable sets associated with different cycles. With respect to a "*simple*" (non mixed) absorbing, or chaotic area, these areas are such that successive images of *almost all points* of a neighborhood enter into the area from a finite number of iterations and cannot get away after entering. The successive images of the points, which do not enter into the area, are those belonging to one of the two segments of the stable set of saddle points on the area boundary. Though not entering the area, these images tend toward the boundary saddle point.

About chaotic areas, or mixed chaotic areas, it is important to emphasize that the study of such areas has only the purpose of obtaining the *macroscopic* properties of the attracting set leading to the considered area. In particular, these properties are those appearing in a first step from a numerical simulation of the iterated sequences generated by the map. The *microscopic* properties, i.e. the nature of closed invariant sets generated by such maps, or the internal structure of an attractor (if it exists), implies further studies and are more difficult to identify. So results in this field are rare and concern degenerated cases of two-dimensional noninvertible maps.

For $p = 3$, the hypotheses (H) of sec.1 is completed by: *the embedded two-dimensional noninvertible map generates only absorbing, or chaotic, areas of nonmixed type, and not mixed ones.*

Considering the *microscopic point of view*, it is worth noting that in 1979 Newhouse stated a very important result. It states that in any neighborhood of a C^r-smooth ($r \geq 2$) dynamical system, in the space of dynamical systems (or a parameter space), there exist regions for which systems with homoclinic tangencies (then with structurally unstable, or nonrough homoclinic orbits) are dense.

Domains having this property are called *Newhouse regions*. This result is completed in [Gonchenko *et al.*, 1993] which asserts that systems with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This has the following important consequence : *systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible.* Indeed in many smooth cases, due to the finite time of a simulation, what numerically appears as a chaotic (strange) attractor contains a "large" hyperbolic subset in presence of a finite or an infinite number of stable periodic solutions. Generally such stable solutions have large periods, and narrow "oscillating" tangled basins, which are impossible to exhibit numerically due to the finite time of observation, and unavoidable numerical errors. So it is only possible to consider some of the characteristic properties of the system, their interest depending on the problem nature [Shilnikov, 1997]. Such complex behaviors occur for p-dimensional flows, $p > 2$, and thus for $p \geq 2$ invertible and noninvertible maps.

From a *macroscopic point of view* the union of the numerous, and even infinitely many stable solutions, which are stable cycles for a map, forms an attracting set A in the sense of definition 2. A numerical simulation of the map iterated sequences is always made from a limited number of iterations. Consider the case of a noninvertible map numerically giving rise to a chaotic area, that is after elimination of a transient, made up of a sufficiently large set of initial iterations. Then the numerical simulation (a) either reproduces points belonging to a *strict* strange attractor in the mathematical sense, (b) or represents a transient toward an attracting set A including many stable cycles of large period. The first case (a) for example is that of some piecewise smooth maps (i.e. with isolated points of nonsmoothness), not permitting stable cycles (i.e. the Jacobian determinant cannot be sufficiently small). In the second case (b), supposing numerical iterations without error, the transient would be that toward a stable cycle having a period either larger than the number of iterations, or less than this number but this transient occurring inside a very narrow basin, tangled with similar basins of the other stable cycles of large period. In presence of unavoidable numerical errors, the iterate points cannot remain inside the same narrow basin. They sweep across the narrow tangled basins of cycles of the attracting set A. Then they reproduce a chaotic area bounded by segments of critical curves. This means that the chaotic area boundary is that observed numerically, in the *smooth case* as a transient toward an attracting set located inside the area, in the *nonsmooth case* as the boundary of a strange attractor. Such a property constitutes an important characteristic of the system dynamics. This shows the high interest of the notion of chaotic area and critical sets, even if in the smooth case it is impossible to discriminate numerically a situation of a strange attractor in the mathematical sense, from that of an attracting set made up of many stable cycles with very large period.

Then a chaotic area is considered in a *strict sense*, when the noninvertible maps is nonsmooth in a situation which does not permit the existence of stable cycles. This means that the Jacobian determinant cannot be sufficiently small. In the smooth case, as indicated above, generally it is impossible to prove that inside the chaotic area, stable cycles cannot exist. This "*chaotic area*" (chaotic transient toward a cycle of very large period and narrow basin) is said defined in a *nonstrict sense*.

2.3(D). Strange attractor induced by the folding generated by a p-dimensional invertible map

Strange attractors are characterized by a dimension that is a noninteger number. This situation results from a stretching in some directions, accompanied by a contraction in another ones, applied to a p-dimensional set (area if $p = 2$, volume if $p = 3$). These directions are given by the eigenvectors of saddle fixed points, or cycles. The eigenvalues having a modulus less (resp. higher) than one give the contraction (resp. stretching) effect. The p-dimensional set is stretched and at the same time folded over itself by successive iterations of the map. Then the existence of a chaotic behavior generated by the map (3), or (6), implies that the commutation set S_0 gives rise to a folding in the sense of definitions 7, 8, and. 9.

Consider the map (3) (i.e. $p = 2$), with the parameter b tending toward zero, which implies that one of the two multipliers of saddle points tends toward zero, because the Jacobian of (3) is $J = -b$. Then the unstable manifold of each saddle point tends toward $y = 0$, i.e. it is crushed on this line at the limit. The fold points of this manifold tend toward the critical points of rank 1, 2, 3, ... (pages 291-296 of [Mira, 1987]). For the map (6) (i.e. $p = 3$) an equivalent behavior occurs. If a saddle unstable manifold is two-dimensional, it tends toward $z = 0$, and crushed on this plane at the limit. The folded arcs tend toward the critical arcs of rank 1, 2, 3, ... So *strange attractors of the p-dimensional invertible map turn into chaotic areas of the* $(p-1)$ *embedded noninvertible map when* $b \to 0$, if some conditions of chaotic behavior preservation are satisfied, which is obtained by choosing the other parameters correctly. Such properties will appear in the examples of secs. 3 & 4.

3. Embedding of a one-dimensional noninvertible map into a two-dimensional invertible one ($p = 2$)

3.1. Reminders of results on stable and unstable manifolds of a fixed point

The first results dealing with both stable and unstable manifolds of a saddle point were obtained for $p = 2$ in [Mira, 1978, 1979, 1987] [Gumowski & Mira, 1980]. They can be summarized as follows. $T_{b=0}$ is regarded as a degenerate two-dimensional map ($J = 0$), obtained from (1), which behaves as T_0 for $n \geq 1$, but is different from T_0 with the initial condition (x_0, y_0), $y_0 \neq 0$ for $n = 0$. Let q be a saddle fixed point of the map T_b with $b \neq 0$, q_0 ($x = \alpha$, $y = 0$) the corresponding fixed point of the map $T_{b=0}$ defined by (1), and the initial condition $(x_0, y_0 \neq 0)$. After one iteration, the iterated sequence (x, y) belongs to $y = 0$. Suppose that q_0 is unstable for the one-dimensional noninvertible map T_0. Then q_0 is a saddle ($S_2 = 0$, $|S_1| > 1$).

Let (c_0) be the curve $y = \alpha - f(x, a)$ in the(x, y) plane. Using (1) with $b = 0$, i.e. $T_{b=0}$, it is easy to verify that the set of the points belonging to (c_0) has the same point q_0 as rank-one image. Let α_{-1} be the rank-one preimage of q_0, different from q_0, $T_0^{-1}(q_0) = q_0 \bigcup \alpha_{-1}$. Let $\alpha^{-p}_{i_1,\ldots,i_{p-1}}$, p= 2,3,..., be the sequence of rank-n preimages of q_0, obtained from the inverse map T_0^{-p}. Let $(c^{p}_{i_1,\ldots,i_{p-1}})$ be the curve: $y = \alpha^{-p}_{i_1,\ldots,i_{p-1}} - f(x, a)$. The set (Γ)= $c_0 \bigcup_{p>1,i_s} (c^{p}_{i_1,\ldots,i_{p-1}})$ is the degenerate stable manifold $W^s(q_0)$ of the fixed point q_0 of the map (1) with $b = 0$, [Mira, 1978, 1979, 1987] [Gumowski & Mira, 1980]. This degenerate stable manifold, made up of non connected arcs, is associated with the multiplier $S_2 = 0$. The other multiplier S_1 is related to the fixed point of the one-dimensional map (2). In the case of the map (1) with $b \neq 0$ and $f(x, a) \equiv 1 - ax^2$, (Γ) is used as a *germ* in [Mira *et al.*, 1999] for obtaining an approximate implicit analytical representation of the stable manifold $W^s(q)$ related to the saddle fixed point q ($S_2 \neq 0$) belonging to the basin boundary of the attracting set generated by this map.

Remind that a rank-n critical point of T_0, defined by (2) is the rank-n image of an extremum of the function $f(x, a)$. For simplifying consider $f(x, a) \equiv 1 - ax^2$, for $a > -1/4$ T_0 has two fixed points: q_1 ($S > 1$), q_2 ($S < 1$). Let C be the rank-one critical point of T_0. The *degenerate unstable manifold* $W^u(q_1)$ of the saddle q_1 of $T_{b=0}$ belongs to $y = 0$, and it is made up of the segment $-\infty \leq x < x(C)$. When $a > 3/4$, for $T_{b=0}$ the *degenerate unstable manifold* $W^u(q_2)$ of the saddle q_2 ($S < -1$) is made up of a segment bounded by C and the rank-two critical point C_1, that is defined by $1 - a \leq x \leq 1$, $y = 0$. When $b \to 0$, each of these two segments is the limit of the unstable manifold $W^u(q_i)$, $i = 1, 2$. It was seen numerically that the bending back points of $W^u(q_i)$ with $b \neq 0$ (considered roughly as points with a local minimum of curvature radius) tends toward the set of the critical points C_n of T_0 ($C_0 \equiv C$) with increasing ranks, when $b \to 0$. This means that each of the degenerate unstable manifolds of q_i, located on $y = 0$, results from the crushing of $W^u(q_i)$ loops on $y = 0$ for $b \to 0$. Each unstable manifold can be considered as a piling up of segments bounded by increasing ranks critical points on $y = 0$ [Mira, 1978, 1979, 1987] [Gumowski & Mira, 1980]. As indicated in [Sharkovskij, 1969], for the one-dimensional noninvertible map T_0, the stable set $W^s(q_i)$ of q_i is made up of q_i and its arborescent sequences of preimages. Then the homoclinic points $W^u(q_i) \bigcap W^s(q_i)$, generated by T_0 and related to q_i supposed unstable, $i = 1, 2$, naturally appear as the limit of those generated by T_b when $b \to 0$.

It is known that $W^s(q_i)$ may belong to the basin boundary of an attractor (when it exists). Due to the fact that the one-dimensional map T_0 is noninvertible, this basin may be non connected, but for the two-dimensional map T_b with $b \neq 0$, the invertibility implies that the basin (and so $W^s(q_i)$) must be connected. This means that the disconnected arcs of the set (Γ), defined above, must join. As indicated in [Mira, 1979] and [Mira, 1987], for $b \neq 0$, but small in absolute value, these arcs join at absolute y-values, the larger as $\mid b \mid$ is small, the saddle multiplier S_2 becoming different from zero. The saddle unstable set is a curve of the two-dimensional space, turning into a curve of the (x, y) plane.

These ancient results were presented without using the notion of commutation set. Such a notion is now used in order to improve and extend them.

3.2. Conditions of folding

We consider the map T (1) in the recurrence form, that is: $x_{n+1} = f(x_n, a) + y_n$, $y_{n+1} = bx_n$.

Let $p_n = dy_n/dx_n$, be the slope of an arc γ_n at the point (x_n, y_n). Let $[df((x)/dx]_n$ be the value of $df((x)/dx$ at the point (x_n, y_n). From (1) the slope $p_{n+1} = dy_{n+1}/dx_{n+1}$ of γ_{n+1} at the point (x_{n+1}, y_{n+1}), point belonging to the image γ_{n+1} of γ_n, is given by the recurrence relationship:

$$p_{n+1} = b/\{[df((x)/dx]_n + p_n\} \tag{9}$$

Let $q_n = d^2y_n/d^2x_n$, related to the arc γ_n at the point (x_n, y_n). From (1) q_{n+1}, at the point (x_{n+1}, y_{n+1}), is given by the recurrence relationship:

$$q_{n+1} = -b\{[d^2f((x)/dx^2]_n + 2q_n]/\{[df((x)/dx]_n + p_n\}^2 \tag{10}$$

Under the hypotheses (H) the equation of S_0 is $x = x_{0e}$, abscissa of the extremum of $f((x, a)$, the equation of S_1 is $y = bx_{0e}$. Let γ_0 be a sufficiently small arc crossing through S_0, $\gamma_0 \cap S_0 = a_0$, $a_1 = T_b(a_0)$, $a_1 \in S_1$, $\gamma_1 = T_b(\gamma_0)$.

(a) *Nonsmooth map at* S_0

Let $[df((x)/dx]_0^+$ and $[df((x)/dx]_0^-$ be respectively the values of $df((x)/dx$ for $x = x_{0e} + \varepsilon$ and $x = x_{0e} - \varepsilon, \varepsilon > 0$ being infinitely small. The normal N_1 to S_1 ($y = bx_{0e}$) at a_1 being a vertical line, then from (9) the necessary and sufficient condition of folding is:

Condition C1 For the nonsmooth map (1) consider an arc γ_0 crossing through S_0, $\gamma_0 \cap S_0 = a_0$. Let p_0 be the slope of γ_0 at a_0. If $[df((x)/dx]_0^+ + p_0$ and $[df((x)/dx]_0^- + p_0$ have opposite signs, then a folding occurs for $\gamma_1 = T_b(\gamma_0)$ at the point $a_1 = T_b(a_0)$, $a_1 \in S_1$.

(b) *Smooth map at* S_0

Let $p_0 = dy_0/dx_0$ be the slope of an arc γ_0 at the point (x_{0e}, y_0). At the point $\gamma_0 \bigcap S_0 = a_0$ one has $[df((x)/dx]_{0e} = 0$. Now a folding does not occur for $\gamma_1 = T_b(\gamma_0)$ at the point $a_1 = T_b(a_0)$, $a_1 \in S_1$. It takes place at a point r_1 belonging to the line M_1 parallel to N_1, and located in a neighborhood of N_1, mentioned in definition 7. This line M_1 is the vertical tangent to γ_1, that is the slope of γ_1 at r_1 is $p_{1f} = b/\{[df((x)/dx]_{0f} + p_{0f}\} = \infty$. Here $[df((x)/dx]_{0f}$ is the value of $df((x)/dx$ at the point $r_0 \in \gamma_0$, $r_0 \notin S_0$, $r_1 = T_b(r_0)$, p_{0f} is the slope of γ_0 at r_0.

The following condition results from above:

Condition C2 Consider a smooth map (1), and an arc γ_0 crossing through the commutation set S_0, $\gamma_0 \bigcap S_0 = a_0$. If in a neighborhood of a_0 there exists a point $r_0 \in \gamma_0$ giving rise to a folding at $r_1 = T_b(r_0)$, $r_1 \in \gamma_1 = T_b(\gamma_0)$, then this point is defined by the relation: $[df((x)/dx]_{0f} = -p_{0f}(x)$, the lower index "$_{0f}$" being related to the value at r_0.

3.3. Nonsmooth maps as piecewise linear ones.

3.3(A). Properties of commutation curves related to the map

A simple example is sufficient to show the basic properties of T_b, $f(x, a)$ being such as the hypotheses (H) are satisfied. Let the map T_b the map (3)given by:

$$\begin{aligned} &x' = f(x, a) + y, \quad y' = bx \\ &f(x, a) = 2x, \quad x \leq 1/2 \\ &f(x, a) = -ax + 1 + a/2, \quad x \geq 1/2, \quad a > 0 \end{aligned} \tag{11}$$

The commutation set S_0 is given by $x = 1/2$, its image S_1 by $y = b/2$. The commutation set S_2, made up of two segments of straight lines is defined by the relations:

$y = b(x - b/2)/2$, for $y \leq b/2$, $\quad y = b(1 + a/2 + b/2 - x)/a$, for $y \geq b/2$

$S_1 \cap S_2 = E(x = 1 + b/2, \quad y = b/2)$.

Property 1 Consider the map (11) with $a > 0$, $b > 0$. Let γ_0 be an arc crossing through the commutation straight line S_0 ($x = 1/2$), $\gamma_0 \bigcap S_0 = a_0$, $a_1 = T_b(a_0)$, $a_1 \in S_1$, $\gamma_1 = T_b(\gamma_0)$. The arc γ_1 undergoes a fold by crossing through S_1, if the γ_0 slope p_0 at a_0 is $-2 < p_0 < a$.

From the condition (9) the slope p_1^- of the image of the γ_0 arc ($x < \frac{1}{2}$) at the point a_1 is $p_1^- = b/(2 + p_0)$. The slope p_1^+ of the image of the γ_0 arc ($x > \frac{1}{2}$) at the point a_1 is $p_1^+ = b/(-a + p_0)$. Condition (C1) gives a folding of γ_1 at a_1 when $-2 < p_0 < a$. The boundaries of folding are: $p_0 = -2$ (S_1 is perpendicular to p_1^-), $p_0 = a$ (S_1 is perpendicular to p_1^+).

Definition 3.1. Let $E = S_1 \cap S_2$, then $E_n = T_b^n(E) = S_{n+1} \cap S_{n+2}$, $n = 0, 1, ...,$ is called *principal S-fold point* of S_{n+2}. The other fold points of S_{n+2} resulting from $T_b[S_{n+1} \cap S_k]$, $k = 0, 1, ..., n$, are called *secondary S-fold points* of S_{n+2}, if the segment of S_{n+2}, in the neighborhood of a point $[S_{n+2} \cap S_{k+1}]$, is located on the same side of the normal to S_{k+1}at this point.

Property 2 Consider the map (11) with $a > 1$, $0 < b < 1$. The commutation set $S_2 = T_b(S_1)$ undergoes a *principal folding* at the point E belonging to S_1. The commutation set $S_n = T_b^n(S_1)$ undergoes a *principal folding* at the point $E_{n-2} = T_b^{n-2}(E)$ belonging to S_{n-1}. S_3 undergoes two *secondary foldings* on S_1 at $T_b(S_2 \cap S_0)$. S_n undergoes *secondary foldings* on S_1 at points $T_b(S_0 \cap S_{n-1})$. S_n does not always undergo *secondary foldings* on S_j, $j < n$, for all the points $T_b(S_{j-1} \cap S_{n-1})$.

The proof of this property implies very long analytical developments. It is established from the relation (9) with the condition $-2 < p_0 < a$, the hypothesis $a > 1$, $0 < b < 1$, taking into account the slope of S_1 ($p_0 = 0$), a S_2 folding occurring at E because $-2 < p_0 < a$. It can be verified numerically in the worst parametric conditions ($a = b = 1$). Analytically the first secondary foldings of $S_n = T_b^n(S_1)$ on S_1 can be proved. So the slope of S_2 at the two points $S_2 \cap S_0$ is $p_0^1 = b/2$ (for $y \leq b/2$), $p_0^2 = -b/a$ (for $y \geq b/2$). Due to the relations $-2 < p_0^i < a$, $i = 1, 2$, a folding occurs at the points $T_b(S_2 \cap S_0)$ on S_1. The slopes of S_3 at $S_3 \cap S_0$ are $p_0^{1-} = 2b/(4 + b) > 0$, $p_0^{1+} = 2b/(b - 2a) < 0$, $p_0^{2-} = ab/(2a - b) > 0$, $p_0^{2+} = -ab/(a^2 + b) < 0$. The relation $-2 < p_0^{i\pm} < a$, $i = 1, 2$, is satisfied.

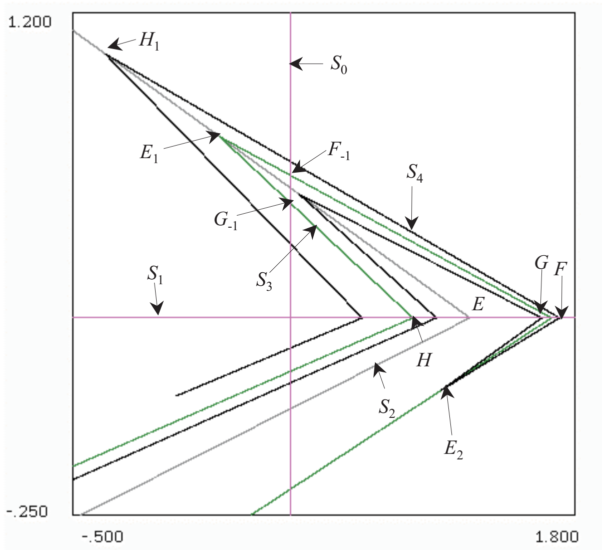

Fig. 3. Map (11), $a = 1.4$, $b = 0.6384$. Commutation sets S_k, $k = 0, ..., 4$. E_n ($E_0 \equiv E$), $n = 0, 1, 2$, are principal fold points. Points of $S_1 \cap S_m$, $m > 2$, different from E, and their successive images are secondary fold points. Points F_{-1}, G_{-1} are the rank-one preimages of F and G belonging to S_1.

Let C be the rank-one critical point of the one-dimensional non invertible map T_0 obtained from (11) with $b = 0$, that is the rank-one image of the extremum ($x = \frac{1}{2}$) of $f(x, a)$, $x(C) = 1$.

Property 3 When $b \to 0$, (a) the commutation sets S_n, $n \geq 2$, are crushed on the axis $y = 0$, and $\lim(S_2) = [y = 0, -\infty < x \leq x(C)] = \lim(S_n)$, (b) the principal S-fold points are such that $\lim(E_n) = C_n$, $C_n = T_b^n(C)$ rank-$(n-1)$ critical point ($C \equiv C_0$) of T_0 on $y = 0$.

From the equation of the two S_2 segments (given above) and $S_1 \cap S_2 = E(x = 1 + b/2, y = b/2)$, one has $\lim(S_2) = [y = 0$, in the interval $-\infty < x \leq x(C)]$. The other points of the proposition results from the iteration of S_2.

Figure 3 ($a = 1.4$, $b = 0.6384$) shows the commutation sets S_k, $k = 0, ..., 4$, with the principal fold points E_n ($E_0 \equiv E$), $n = 0, 1, 2$. Points of $S_1 \cap S_m$, $m > 2$, different from E, and their successive images are secondary fold points. Points F_{-1},

G_{-1} are the rank-one preimages of F and G belonging to S_1.

3.3(B). Properties of commutation curves related to the inverse map

The inverse map T_b^{-1} is defined by the relations

$$\begin{aligned} x &= y'/b, \qquad y = x' - f(y'/b) \\ f &= 2y'/b \quad if\ y' \le b/2 \\ f &= 1 + a/2 - ay'/b \quad if\ y' \ge b/2 \end{aligned} \tag{12}$$

The commutation set S'_0 is given by $y = b/2$, its image S'_1 by $x = 1/2$. The commutation set S'_2, made up of two segments of straight lines is defined by the relations:

$y = 1/2 - 2x$, for $x \le 1/2$, $\quad y = a(x - 1/2) - 1/2$, for $x \ge 1/2$, with $S'_1 \cap S'_2 = E^-(x = 1/2, y = -1/2)$.

The points $E_n^- = T_b^n(E^-) = S'_{n+1} \cap S'_{n+2}$, $n = 0, 1, ...$, are the *principal* S'*-fold point* of S'_{n+2}. The other fold points of S'_{n+2} resulting from $T_b[S'_{n+1} \cap S'_k]$, $k = 0, 1, ..., n$, are *secondary* S'*-fold points* of S'_{n+2}, if the segment of S'_{n+2}, in the neighborhood of a point $[S'_{n+2} \cap S'_{k+1}]$, is located on the same side of the normal to S'_{k+1}at this point.

Figure 4 ($a = 1.4$, $b = 0.6384$) shows the commutation sets S'_k, $k = 0, ..., 4$, with the principal fold points E_n^- ($E_0 \equiv E$), $n = 0, 1, 2$. Points of $S'_1 \cap S'_m$, $m > 2$, different from E^-, and their successive images are secondary fold points.

3.3(C). Unstable manifolds of the fixed points

The fixed points of (11) are q_1 ($x = y = 0$), and q_2 $[x = (1 + a/2)/(1 - b + a)$, $y = bx]$. The multipliers (eigen values) of $q_1$are: $S_1(q_1) = 1 - \sqrt{1 + b}$, $S_2(q_1) = 1 + \sqrt{1 + b}$. The slope of the eigenvector related to the unstable manifold $W^u(q_1)$ of q_1 is $s_2(q_1) = \sqrt{1 + b} - 1$. The multipliers of q_2 are: $S_1(q_2) = [-a - a\sqrt{1 + 4b/a^2}]/2$, $S_2(q_2) = [-a + a\sqrt{1 + 4b/a^2}]/2$. The slope $s_1(q_2)$ of the eigenvector related to unstable manifold $W^u(q_2)$ of q_2 is $s_1(q_2) = [a - a\sqrt{1 + 4b/a^2}]/2$.

We define the *local unstable manifold* $W^u_{loc}(q_1)$ of q_1 as the segment $y = s_2(q_1)x$, $-\infty < x \le x(B)$, bounded by the point $B = [y = s_1(q_1)x] \cap S_1$, $x(B) = b/[2s_1(q_1)]$, $y(B) = b/2$. The *local unstable manifold* $W^u_{loc}(q_2)$ of q_2 is a segment of the straight line (d_2) $y = s_1(q_2)x + q$, $q = (1 + a/2)[b - s_1(q_2)]/(1 + a - b)$, bounded by the point $D = (d_2) \cap S_1$, and the point $D_1 = T_b(D)$, $x(D) = (b/2 - q)/p$, $y(D) = b/2$, $p = s_1(q_2)$. The unstable manifold of the fixed point q_i is $W^u(q_i) = \lim T_b^n[W^u_{loc}(q_i)]$, when $n \to \infty$, $i = 1, 2$.

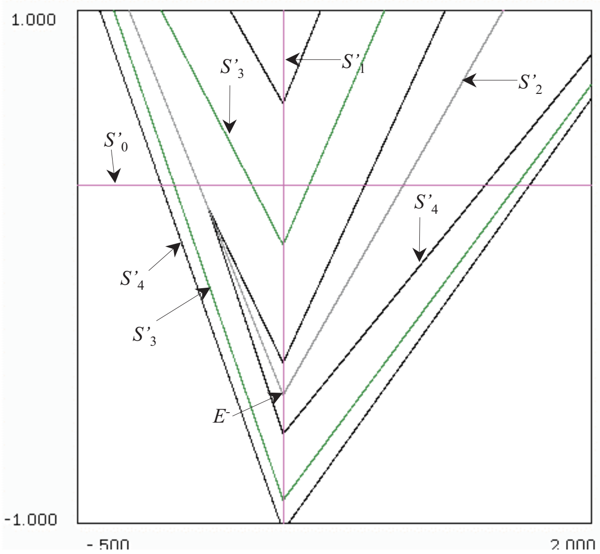

Fig. 4. Inverse of the map (11), with $a = 1.4$, $b = 0.6384$. Commutation sets S'_k, $k = 0, ..., 4$, with the principal fold points E_n^- ($E_0 \equiv E$), $n = 0, 1, 2$. Points of $S'_1 \cap S'_m$, $m > 2$, different from E^-, and their successive images are secondary fold points.

Definition 3.2. The points $B_n = T_b^n(B) \in W^u(q_1)$, $B_n \in S_{n+1}$, and $D_n = T_b^n(D) \in W^u(q_2)$, $D_n \in S_{n+1}$, $n = 0, 1, ...$, ($B_0 \equiv B$, $D_0 \equiv D$), are called *principal fold points* of each unstable manifold. The other fold points resulting from $T_b[W^u(q_i) \cap S_k]$, $k = 0, 1, ...$, $i = 1, 2$, are called *secondary fold points* of $W^u(q_i)$.

Property 4 When $b \to 0$, the principal fold points of $W^u(q_i)$, $i = 1, 2$, are such that $\lim(B_n) = \lim(D_n) = C_n$, $C_n = T_b^n(C)$ rank-$(n-1)$ critical point of T_0 on $y = 0$, with $B_0 \equiv B$, $D_0 \equiv D$, $C_0 \equiv C$. Moreover $\lim[S_1 \cap W^u(q_i)] = C$.

This results from the above equations of $W^u_{loc}(q_i)$, the coordinates of the points B and D and their iterates.

Figure 5 ($a = 1.4$, $b = 0.6384$) shows $W^u(q_1)$ and the commutation sets S_k, $k = 0, ..., 4.$. In this figure $B \in S_1 \cap W^u(q_2)$ (the first intersection of this

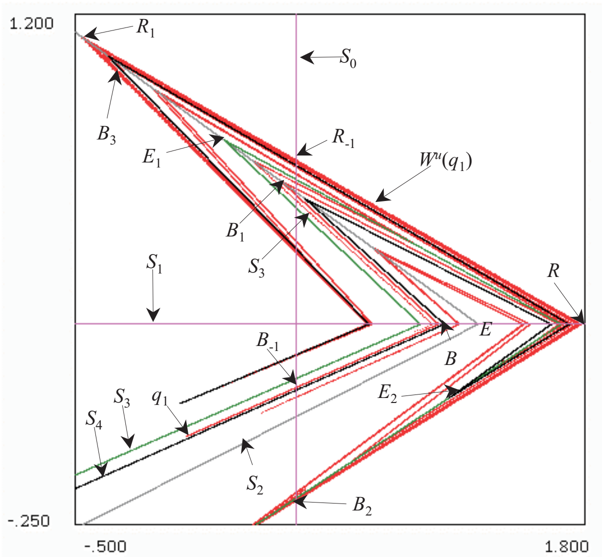

Fig. 5. Map (11), $a = 1.4$, $b = 0.6384$. Unstable manifolds $W^u(q_1)$ (red colored) with the commutation sets S'_k, $k = 0, ..., 4$.

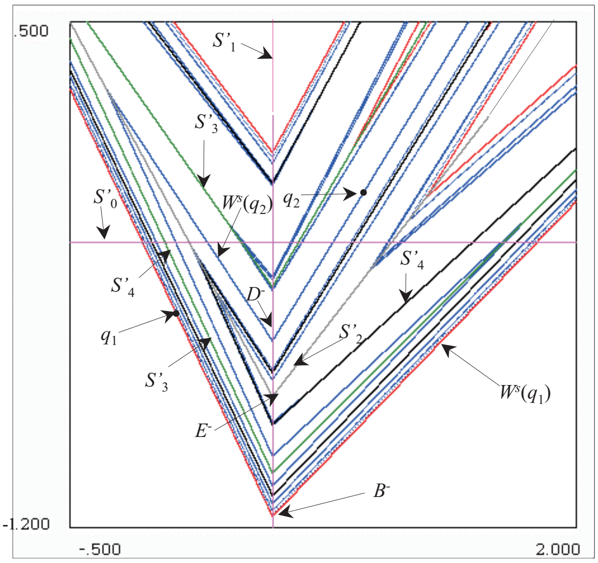

Fig. 6. Map (11), $a = 1.4$, $b = 0.6384$. Stable manifolds $W^s(q_1)$ (red colored), and $W^s(q_2)$ (blue colored), with the commutation sets S'_k, $k = 0, ..., 4$.

unstable manifold from q_1), and $B_n = T_b^n(B)$. The points set $S_1 \cap W^u(q_2)$ different from B, and their successive images are secondary fold points. Points B_{-1}, R_{-1} are the rank-one preimages of B and R belonging to S_1.

3.3(D). Stable manifolds of the fixed points

The slope of the eigenvector related to the stable manifold $W^s(q_1)$ of q_1 is $s_2(q_1) = -\sqrt{1+b} - 1$. The slope $s_2(q_2)$ of the eigenvector related to the unstable manifold $W^u(q_2)$ of q_2 is $s_1(q_2) = [a + a\sqrt{1 + 4b/a^2}]/2$.

We define the *local stable manifold* $W^s_{loc}(q_1)$ of q_1 as the segment $y = s_1(q_1)x$, $-\infty < x \leq x(B^-)$, bounded by the point $B^- = [y = s_1(q_1)x] \cap S'_1$, $x(B^-) = 1/2$, $y(B) = s_1(q_1)/2$. The *local stable manifold* $W^s_{loc}(q_2)$ of q_2 is a segment of the straight line (d'_2) $y = s_2(q_2)x + q$, $q = (1 + a/2)[b - s_2(q_2)]/(1 + a - b)$, bounded by the point $D^- = (d'_2) \cap S'_1$, and the point $D^-_1 = T_b^{-1}(D^-)$, $x(D^-) = 1/2$, $y(D^-) = p/2 + q$, $p = s_2(q_2)$. The stable manifold of the fixed point q_i is $W^s(q_i) = \lim T_b^{-n}[W^u_{loc}(q_i)]$, when $n \to \infty$, $i = 1, 2$.

Figure 6 ($a = 1.4$, $b = 0.6384$) shows $W^s(q_1)$ (red colored) $W^s(q_2)$ (blue colored), and the commutation sets S'_k, $k = 0, ..., 4$.. The points $B^- \in S'_1 \cap W^s(q_1)$ (the first intersection of this stable manifold from q_1), $B^-_n = T_b^{-n}(B^-)$, $D^- \in S'_1 \cap W^s(q_1)$ (the first intersection of this stable manifold from q_1), and $D^-_n = T_b^{-n}(D^-)$ are principal fold points. The points set $S'_1 \cap W^u(q_1)$ different from B^-, $S'_1 \cap W^s(q_2)$ different from D^-, and their successive images are secondary fold points.

Property 5 When $b \to 0$, the stable manifolds $W^s(q_i)$, $i = 1, 2$, tends toward the *degenerate stable manifolds* defined in sec. 3.1. These manifolds are made up of non connected arcs, associated with the multiplier $S_2 = 0$ of the fixed point q_i related to the map $T_{b=0}$.

3.4. Quadratic map

3.4(A). Properties of the commutations sets related to the map

We consider the map (3) with: $f(x, a) = ax(1 - x)$:

$$\begin{aligned} x' &= ax(1 - x) + y \\ y' &= bx, \quad 2 < a < 4,\ 0 < b < 1 \end{aligned} \tag{13}$$

The extremum of $f(x, a)$ is obtained for $x = 1/2$. The commutation set S_0 is $x = 1/2$, S_1 is

$y = b/2$, S_2 is the parabola: $x = ay/b - ay^2/b^2 + b/2$, and $S_1 \cap S_2 = E$, the coordinates of E being $x(E) = (a/2 + b)/2$, $y(E) = b/2$, $E = T_b(E_{-1})$, $E_{-1} = S_0 \cap S_1$, $x(E_{-1}) = 1/2$, $y(E_{-1}) = b/2$].

Let $E = S_1 \cap S_2$, the points $E_n = T_b^n(E) = S_{n+1} \cap S_{n+2}$, $n = 1, 2, 3, \ldots$, give rise to *principal S-fold points* (in the sense of part (b) of the definition 7) E'_n ($E \equiv E'$, $E_n \neq E'_n$) of S_{n+2} in the neighborhood of E_n. The other fold points of S_{n+2} resulting from $T_b[S_{n+1} \cap S_k]$, $k = 0, 1, \ldots, n-1$, are the *secondary S-fold points* of S_{n+2}.

Property 6 Consider the part (b) of the definition 7 with $b \to 0$. (a) The commutation sets S_n, $n \geq 2$, are crushed on the axis $y = 0$, and $\lim(S_2) = [y = 0, -\infty < x \leq x(C)] = \lim(S_n)$. (b) The principal S-fold points are such that $\lim(E_n) = C_n$, $C_n = T_b^n(C)$ rank-$(n-1)$ critical point ($C \equiv C_0$) of T_0 on $y = 0$. (c) The secondary fold points on S_1 tend toward the rank-one critical point C.

The axis of the parabola S_2 is $y = b/2$, and its vertex is the point E ($x = a/4 + b/2$, $y = b/2$), with $x \leq a/4 + b/2$. The rank-one critical point C of the one-dimensional noninvertible map T_0, $x' = ax(1-x)$, is located on $y = 0$, with $x(C) = T_0(1/2) = a/4$. From the equation of S_2 one has $\lim(S_2) = [y = 0, -\infty < x \leq x(C)]$, when $b \to 0$, and the same for $\lim(S_n)$. When $b \to 0$ the coordinates of the point $E_1 = T_b(E)$ tend toward $[x = a^2/4 - a^3/16$, $y = 0]$, which are the coordinates of the point $C_1 = T_b(C)$, and the point (b) occurs for the iterates of C. The point (c) is satisfied considering the parametric equation of S_3, $x = af(t)[1 - f(t)] + t$, $y = bf(t)$, $f(t) = at/b - at^2/b^2 + b/2$, and the two points $S_1 \cap S_3$, which are such than they tend toward the point C on $y = 0$.

Figure 7 ($a = 2.2$, $b = 0.7$) shows the commutation sets S_k, $k = 0, \ldots, 4$, the *folding* occurring according the part (b) of definition 7. The tangent to S_2 at the point $E = S_1 \cap S_2$ is vertical, so that this point is a principal fold, but the successive images of E give rise to principal fold points E'_n, $n > 0$, only in a neighborhood of these images. In this figure the points set $S_1 \cap S_n$, $n > 2$, has secondary fold points in its neighborhood if the condition (C2) of section 3.2 is satisfied. It is the case of the points M and N, and L (fold points in their neighborhood: M', N', L'), but not of the point K, considering the slopes of the related arcs when they intersect S_0.

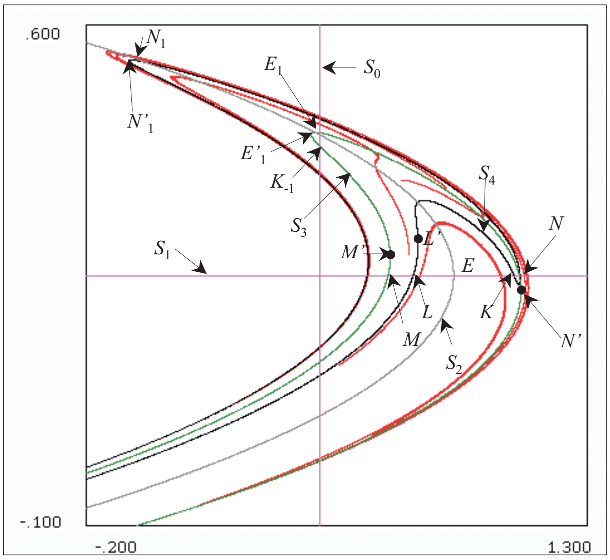

Fig. 7. Map (13), $a = 2.2$, $b = 0.7$. Commutation sets S_k, $k = 0, \ldots, 4$, the *folding* occurring according to the part (b) of definition 7.

3.4(B). Unstable manifolds of the fixed points

The fixed points of (13) are q_1 ($x = y = 0$) and q_2: $x(q_2) = (a + b - 1)/a$, $y(q_2) = bx(q_2)$. The parameters (a, b) are chosen such that the two fixed points are saddles, and $W^s(q_1) \cap W^u(q_1) = \emptyset$, i.e. no homoclinic point from q_1(but heteroclinic points from q_1 to q_2 may exist). When $b \to 0$ this condition is satisfied for $3 < a < 4$.The multipliers of $q_1$are: $S_1(q_1) = [a - \sqrt{a^2 + 4b}]/2$, $S_2(q_1) = [a + \sqrt{a^2 + 4b}]/2$, due to $3 < a \leq 4$, $0 < b < 1$, it is a saddle with $-1 < S_1(q_1) < 0$, $S_2(q_1) > 1$. The slope of the eigenvector related to $W^u(q_1)$ at q_1 is $s_1(q_1) = 2b/S_2(q_1) = 2b/[a + \sqrt{a^2 + 4b}]$. The multipliers of q_2 are $S_1(q_2) = [2 - 2b - a - \sqrt{\Delta}]/2$, $S_2(q_2) = [2 - 2b - a + \sqrt{\Delta}]/2$, $\Delta = (a + 2b)^2 + 4(1 - a - b)$, it is a saddle with $S_1(q_2) < -1$, $0 < S_2(q_2) < 1$. The slope of the eigenvector related to $W^u(q_2)$ at q_2 is : $s_1(q_2) = 2b/S_1(q_2)$.

The unstable manifolds $W^u(q_i)$, $i = 1, 2$, are determined as particular solutions of the functional equation $\Lambda[ax(1-x) + y, bx] = \Lambda(x, y)$. It is possible to obtain a series expansion of such a solution

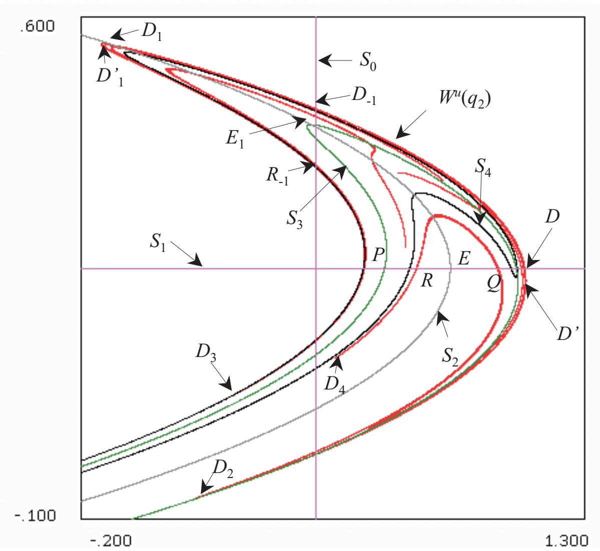

Fig. 8. Map (13), $a = 2.2$, $b = 0.7$. Unstable manifold $W^u(q_2)$ and the commutation sets S_k, $k = 0, ..., 4$, the *folding* occurring according to the part (b) of definition 7.

in a parametric form $x = \varphi(t)$, $y = \psi(t)$, using an iterative method inspired from a Picard's result [Gumowski & Mira, 1980a], which permits to attain high degrees in the expansion. Another method, based on Lattes' results, locally gives the series expansion in the form $y = \theta(x) = \beta_1 x + \beta_2 x^2 + ...$, when the fixed point is at the origin. So for $W^u(q_1)$ β_1 is the slope s_1 of the eigenvector, and β_2 is given by $\beta_2 = as_1/[s_1 + (a + s_1)^2 - b]$. The first terms of the series expansion gives a germ on a sufficiently small segment, the iteration of which leads to a numerically computed points of the unstable manifold with the corresponding slope and second derivative, and precision checks [Gumowski & Mira, 1980b].

Figure 8 ($a = 2.2$, $b = 0.7$) shows $W^u(q_2)$ and the commutation sets S_k, $k = 0, ..., 4$, the *folding* occurring according the part (b) of definition 7. In the figure near $D \in S_1 \bigcap W^u(q_2)$ (the first intersection of this unstable manifold from q_2) is located a principal fold point D' (the tangent to $W^u(q_2)$ is normal to S_1). Principal fold points D'_n in a neighborhood of $D_n = T_b^n(D)$ are defined (at the figure scale D'_n cannot be distinguished from D_n for $n > 1$). A neighborhood of a subset of $S_1 \bigcap W^u(q_2)$ contains secondary fold points. It is the case of the points set P and the set Q resulting from infinitely many $W^u(q_2)$ arcs intersecting S_1, because the condition (C2) of sec. 3.2 is satisfied. It is not the case for the points set R ($R = T(R_{-1})$, $R_{-1} \in S_0$) belonging to S_1 (no point with a vertical tangent exists in its neighborhood). For $W^u(q_1)$ an equivalent figure is obtained. Near the point $B \in S_1 \bigcap W^u(q_1)$ (the first intersection of this unstable manifold from q_1) is located a principal fold point B' (the tangent to $W^u(q_1)$ is normal to S_1). Principal fold points B'_n in a neighborhood of $B_n = T_b^n(B)$ are defined. When $b \to 0$ properties equivalent to those of the piecewise linear map (11) are obtained from numerical computations leading to the determination of $W^u(q_i)$, $i = 1, 2$.

Property 7 Consider the piecewise linear map T_b (11). When $b \to 0$ the principal fold points of $W^u(q_i)$, $i = 1, 2$, are such that $\lim(B_n) = \lim(B'_n) = \lim(D_n) = \lim(D'_n) = C_n$, $C_n = T_b^n(C)$ rank-$(n-1)$ critical point of T_0 on $y = 0$, with $B_0 \equiv B$, $D_0 \equiv D$, $C_0 \equiv C$. Moreover $\lim[S_1 \cap W^u(q_1)] = C$.

3.4(C). Commutations sets related to the inverse map and stable manifolds

The inverse map T_b^{-1} is defined by the relations:

$$x = y'/b, \qquad y = x' + ay'(1 - y'/b)/b \tag{14}$$

Let S'_0 be the line $y = b/2$, which is given by $J" = 0$ as indicated in the definition 6 applied to (14). The commutation curves are defined as $S'_n = T_b^{-n}(S'_0)$. They give rise to fold points if the part (b) of definition 7 is satisfied. The stable manifolds $W^s(q_i)$, $i = 1, 2$, behave like for the piecewise linear map, but with fold points in a neighborhood of the S'_n. In particular when $b \to 0$, the stable manifolds $W^s(q_i)$, $i = 1, 2$, tend toward the *degenerate stable manifolds* defined in sec. 3.1. These manifolds are made up of non connected arcs of parabola, associated with the multiplier $S_2 = 0$ of the fixed point q_i related to the map $T_{b=0}$.

4. Embedding of a two-dimensional noninvertible map into a three-dimensional invertible one (p=3)

4.1. Three-dimensional invertible map and its degenerate form

Let T_b be the invertible map defined by:

$$\begin{aligned} x' &= y, \\ y' &= f(x, y, \lambda) + z, \\ z' &= b(x + y) \end{aligned} \tag{15}$$

The Jacobian determinant is constant, $J = b$. It is supposed that, when the parameter of embedding is $b = 0$, the function $f(x, y, \lambda)$ is such as (15) turns into a two-dimensional noninvertible map T_0:

$$x' = y, \qquad y' = f(x, y, \lambda), \tag{16}$$

Starting from an initial condition $(x, y, z \neq 0)$, after one iteration step the iterated sequences belong to the plane (x, y). The map (15), with $b = 0$, and an initial condition $z_0 \neq 0$, is considered as a "*degenerate three-dimensional map*". It is easy to verify that the locus of points $(x, y, z \neq 0)$, having a given rank-one image $(\xi, \eta, 0)$, i.e. located in the (x, y) plane, is the curve:

$$y = \xi, \qquad z = \eta - f(x, y, \lambda) \tag{17}$$

4.2. Commutation sets

The commutation sets S_n are surfaces characterized by definition 9, and satisfy properties equivalent to those of the case $p = 2$. S_2 undergoes a folding along (resp. in a neighborhood of) ζ_E, if the map is nonsmooth (resp. smooth). The symbolism is equivalent, so $\zeta_E = S_1 \cap S_2$, and $(\zeta_E)_n = T_b^n(\zeta_E) = S_{n+1} \cap S_{n+2}$, $n = 0, 1, ...$, is called *principal S-fold curve* of S_{n+2}, $(\zeta_E)_0 \equiv \zeta_E$. S_n undergoes a folding along the curve $(\zeta_E)^{n-2} = T_b^{n-2}(\zeta_E)$. The other fold curves of S_{n+2}, resulting from $T_b[S_{n+1} \cap S_k]$, $k = 0, 1, ..., n$, are called *secondary S-fold curves* of S_{n+2}, if the surface piece of S_{n+2} is locally located on the same side of the surface normal to S_{k+1} along a fold arc in the neighborhood of $[S_{n+2} \cap S_{k+1}]$.

Let LC be the critical curve of the two-dimensional noninvertible map T_0, separating the (x, y) plane into the regions Z_0 and Z_2. When $b \to 0$: (a) the commutation sets S_n, $n \geq 2$, are crushed on the plane $z = 0$, and $\lim(S_2) = [z = 0, (x, y) \in Z_2] = \lim(S_n)$; (b) the principal S-fold arcs are such that $\lim(\zeta_E)_n = LC_n$, $LC_n = T_b^n(LC)$ rank-$(n-1)$ critical sets ($LC \equiv C_0$) of T_0 on $z = 0$. A subset of the secondary S-fold lines tends toward $(\zeta_E)_n$ and thus toward LC_n. Contrary to the case $p = 2$, due to difficulties of representation in the three-dimensional space, the commutation sets will not be represented by figures.

When $p = 3$, the commutation sets of definition 6 permit to obtain the folding properties of two-dimensional stable (or unstable) manifolds of saddle fixed points, and the one-dimensional stable (or unstable) manifolds. The one-dimensional manifolds are characterized by definition 8, the two-dimensional manifods by definition 9. In the case of a two-dimensional stable manifold of saddle, the fixed point is characterized by two multipliers (eigenvalues) $|S_1| < 1$, $|S_2| < 1$, the third multiplier being such that $|S_3| > 1$. More precisely the fixed point is either of *saddle-node* (S_1 and S_2 being real) or *saddle-focus* (S_1 and S_2 being complex conjugate) type. The unstable manifold of the saddle is one-dimensional. In the case of a one-dimensional stable manifold of saddle, the fixed point is characterized by two multipliers (eigenvalues) $|S_1| > 1$, $|S_2| > 1$, the third multiplier being such that $|S_3| < 1$. The fixed point is also either of *saddle-node* (S_1 and S_2 being real) or *saddle-focus* (S_1 and S_2 being complex conjugate) type. The stable manifold of the saddle is one-dimensional.

4.3. Stable set of a saddle fixed point

4.3(A). One-dimensional stable set

With the hypotheses (H) and $b = 0$, the function $f(x, y, \lambda)$ leads to a map (16) having the type $(Z_0 - Z_2)$. Consider an unstable focus, or an unstable node, fixed point A (u, v) generated by the two-dimensional noninvertible map (16), and A^{-1} (u^{-1}, v^{-1}) its rank one preimage different from A. For the degenerate three-dimensional map, this fixed point becomes a *focus-saddle* (or a *node-saddle*), the unstable manifold $W^u(A)$ of A is two-dimensional (it belongs to the plane $z = 0$). If $A^{-1} \in Z_0$, the one dimensional degenerate stable manifold $W^s(A)$ is made up of two non connected arcs: the curve given by (17) with $\xi = u$, $\eta = v$ (which intersects the plane $z = 0$ in two

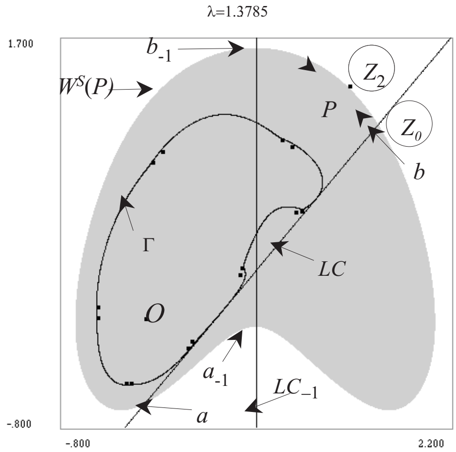

Fig. 9. Embedded two-dimensional map (16). Stable set $W^s(P)$ of the map T_0 with $f(x, y, \lambda) \equiv y - \lambda x + x^2$, $\lambda = 1.3785$. The domain of bounded orbits is simply connected. $W^s(P)$ is the boundary ∂D between the blue and white regions.

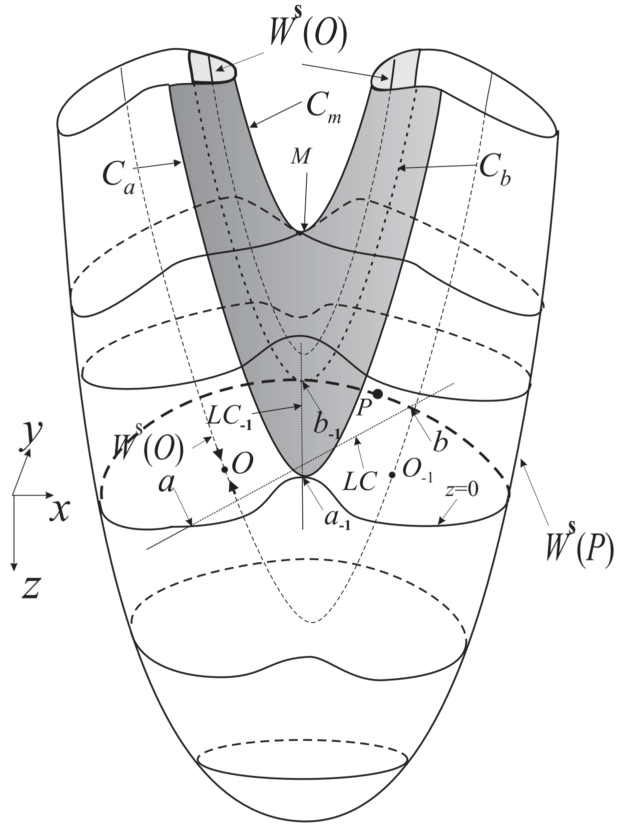

Fig. 10. Map (15), with $f(x, y, \lambda) \equiv y - \lambda x + x^2$, $\lambda = 1.3785$. Qualitative representation of the stable sets $W^s(P)$ and $W^s(O)$ related to the degenerated three-dimensional map $T_{b=0}$.

points), and the curve with $\xi = u^{-1}$, $\eta = v^{-1}$ (which does not intersect $z = 0$). If $A^{-1} \in Z_2$, the degenerate stable set $W^s(A)$ is made up of infinitely non connected arcs given by (17) with $(\xi, \eta) = T^{-n}(u^{-1}, v^{-1})$, $n = 1, 2, 3, \ldots$. The multiplier associated with this stable set is $S_3 = 0$, those (S_1, S_2) related to the unstable set are such as $\mid S_1 \mid > 1$, $\mid S_2 \mid > 1$.

The map (15) is invertible for $b \neq 0$, then $W^s(A)$ must be connected. This means that the non connected arcs of $W^s(A)$, obtained when $b = 0$, must communicate for $b \neq 0$. When $\mid b \mid$ is small, these arcs join for $\mid z \mid$-values, the larger as $\mid b \mid$ is small, the multiplier S_3 becoming different from zero.

In order to illustrate the preceding points, we consider the map (15) with $f(x, y, \lambda) \equiv y - \lambda x + x^2$, with $\lambda = 1.3785$ [Mira *et al.*, 1996]. The embedded two-dimensional non invertible map (16) has a fixed point: O $(x = y = 0)$ (an unstable focus point) inside the domain bounded by a stable closed invariant curve (Γ). The rank-one preimage O_{-1} $(\lambda; 0)$ different from O, belongs to Z_0 the region without preimage. The map has a second fixed point P $(\lambda; \lambda)$, a saddle with its multiplier related to $W^s(P)$ being $-1 < S_1 < 0$, if $-1/4 < \lambda < 2$, the multiplier related to $W^u(P)$ being $S_2 > 1$. The unstable manifold $W^u(O)$ of O is the closed domain bounded by (Γ). The stable manifold $W^s(P)$ is the boundary ∂D of the basin D of (Γ), blue colored in Fig. 9.

This figure also represents the critical curve LC, and the locus of merging preimages LC_{-1}, $T(LC_{-1}) = LC$, $LC \bigcap \partial D = a \bigcup b$, $LC_{-1} \bigcap \partial D = a_{-1} \bigcup b_{-1}$. Considering (17) with $\xi = 0$, $\eta = 0$, the stable manifold $W^s(O)$ of the three-dimensional degenerate map $T_{b=0}$, is the union of two parabolae $[y = 0,\ z = \lambda x - x^2 - y]$ and $[y = \lambda,\ z = \lambda x - x^2 - y]$. The map (15) is invertible for $b \neq 0$, then the saddle stable set $W^s(O)$ must be connected. This means that the two arcs of $W^s(O)$ must join. For $b \neq 0$, but small in absolute value, they join with $z < 0$ at absolute z-values, the larger as $\mid b \mid$ is small, and the third saddle multiplier becomes different from zero.

4.3(B). Two-dimensional stable set

Consider the two-dimensional noninvertible map (16) generating an attractor with a basin D which can be either simply connected, or non connected, or multiply connected [Mira *et al.*, 1996]. The basin boundary ∂D contains the stable set of saddle fixed (or periodic) points, when they exist. Let ∂D_0 be the boundary of the immediate basin (that contains the attractor) if this basin is simply connected, or the external boundary if it is multiply connected. Suppose that ∂D_0 is parametrized by $\xi = \xi(t)$, $\eta = \eta(t)$. Then (17) generates a *surface* in the (x, y, z) space, when t describes the (ξ, η) points of ∂D_0. This surface belongs to the stable set of a saddle when it exists. Doing the same for each non connected part (*island*) if the basin is not connected, and each hole (*lake*) of D_0 if D_0 is multiply connected, (17) generates a well-defined *tube* for each of these components. The surface generated from ∂D_0 and all these tubes are non-intersecting, and do not communicate. Their union, associated with the given total basin (the boundary of which contains a saddle P), constitutes the *two-dimensional degenerate stable set* of S, and one of the three multipliers of P is equal to zero (the one related to the z-axis). All these non connected sets have an asymptotic direction $x = y = 0$ (the z-axis).

The map (15) being invertible, for $b \neq 0$ the saddle stable set of T_b must be connected. This means that the surface generated from ∂D_0 and all the tubes generated by the islands and the lakes must communicate. For $b \neq 0$, but small in absolute value, the surface and the tubes of a given basin, generated by $T_{b=0}$, now must join, and communicate, the third saddle multiplier becoming different from zero. Such a communication occurs at absolute z-values, the larger as $\mid b \mid$ is small. The saddle unstable set is a curve of the three-dimensional space, turning into a curve of the (x, y) plane when $b = 0$.

With such a process it is possible to interpret some of the complex structures of saddle sets generated by three-dimensional invertible maps, from those of two-dimensional noninvertible maps [Mira *et al.*, 1996]. So a basin boundary generated by a two-dimensional non invertible map, whatever the complexity of the basin structure, associated with the union of the above surface and tubes (as two-dimensional degenerated saddle stable set), is a *germ* of the saddle stable set of T_b when $\mid b \mid$ is sufficiently small. In order to illustrate the preceding points, let (15) be the three-dimensional invertible map T_b, $b \neq 0$, with $f(x, y, \lambda) \equiv y - \lambda x + x^2$. Consider the parameter value $\lambda = 1.3785$, giving the Fig. 9 situation for the embedded two-dimensional non invertible map T_0. The blue colored region is the basin D of the stable closed invariant curve (Γ), the white one being the domain of divergence, i.e. $W^s(P)$ is the boundary ∂D between these two regions. The stable set $W^s(P)$ of the degenerate three-dimensional map $T_{b=0}$ is made up of all the parabolae (17) generated when the point (ξ, η) moves along ∂D (Fig. 9). The arc $\partial D \bigcap Z_2$ gives rise to a set of parabolae, which intersect the plane $z = 0$, the limit situation corresponding to the two parabolae C_a and C_b (Fig. 10), tangent to $z = 0$, at the points $a_{-1} \bigcup b_{-1} = LC_{-1} \bigcap \partial D$, $(a \bigcup b = LC \bigcap \partial D)$. The arc $\partial D \bigcap Z_0$ gives rise to a set of parabolae (grey colored), which does not intersect the plane $z = 0$. One of them C_m has a maximum with the smallest possible value of z, $z_m = \lambda^2/4 - \max(\eta - \xi)$, $(\xi, \eta) \in \partial D \bigcap Z_0$.

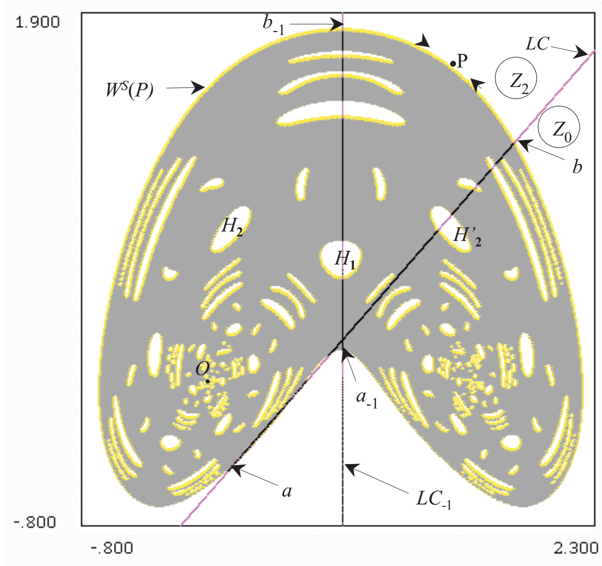

Fig. 11. Embedded two-dimensional map (16). Stable set $W^s(P)$ of the map T_0 with $f(x, y, \lambda) \equiv y - \lambda x + x^2$, $\lambda = 1.5475$. The domain of bounded orbits is multiply connected. $W^s(P)$ (yellow colored)is the boundary ∂D between the blue and white regions.

Now consider the parameter value $\lambda = 1.5475$, giving the multiply connected basin of Fig. 11 related to the attracting set generated by the embedded two-dimensional non invertible map T_0. The lakes H_1, H_2 and H_2', $T^{-1}(H_1) = H_2 \bigcup H_2'$, are the first of an infinite sequence [Mira *et al.*, 1996], the limit points of which are the fixed point O (0;0) (an unstable focus point) and its rank-one preimage O_{-1} $(\lambda; 0)$, which is different from O, and belongs to Z_0 the region without preimage. When $b = 0$, $W^s(P)$, i.e. the basin boundary of the attracting set generated by $T_{b=0}$, has an "external" part generated by the boundary of the Fig. 11 blue region deprived of the "tubes" generated by the above infinite sequence of lakes. The "internal" part of $W^s(P)$ is made up of all the "tubes" generated from this infinite sequence of lakes. When $b \neq 0$, all these tubes are connected, giving rise to an unique "oscillating" tube ("internal" part of $W^s(P)$).

Fig. 12 ($b > 0$) shows $W^s(P)$, i.e. the basin boundary of the attracting set generated by T_b. The "external" part of this manifold (i.e. the part without the "tube" resulting from the connection of infinitely many ones) of this stable manifold is blue colored, and an opening permits to see inside it. So this opening shows how the non connected lakes of T_0 give rise to the "internal oscillating tube" belonging to $W^s(P)$, permitting a connected communication with the domain of divergence. The arrow F indicates that the following part of the tube issued from H_2, non represented (for avoiding a too complex figure), goes forward intersecting $z = 0$ infinitely many times. The limit set of the "internal oscillating tube" is the stable manifold $W^s(O)$. Figure 12 shows some sections, and the part inside the tube which is yellow colored. The intersection of the "tube" and the "external" part of $W^s(O)$, i.e. the "opening" corresponds to the yellow colored region noted (V). When $b \to 0$, this intersection is associated with the *bay* H_0 [Mira *et al.*, 1996] in Fig. 11. We remind that a bay is the region, bounded by a segment of LC and an arc of the "external" basin boundary ∂D, such as $T(H_1) = H_0$. The *lakes* are generated by the successive preimages (with increasing rank) of the bay H_0, and when $b \to 0$, one has $z(V) \to \infty$.

Important remark. With the parameter value $\lambda = 1.5475$, the multiply connected basin of Fig. 11 is such that the lakes structure, generated by T_0, is not fractal. Nevertheless they are map examples leading to a fractalization of the lakes structure (see. Fig. 5.66 of [Mira *et al.*, 1996] for example). For T_b, $b \neq 0$, from this situation results a fractal structure of the set made by the "internal tube" with its "loops". This "internal" fractal structure belongs to the stable set of a saddle, this point being located on the "external" part of the stable set. It also may occur that a simply connected basin, generated by T_0, has a fractal boundary ∂D (see Fig.5.21 of [Mira *et al.*, 1996] for example). In this case, for T_b, $b \neq 0$, the corresponding basin boundary in the three-dimensional space is fractal in the sense that the sections $z =$ constant give a fractal set. The same situation occurs for islands, i.e. non connected parts of a total basin, out of the immediate basin. One has an higher degree of complexity for the tubes organization, when the noninvertible map T_0 generates embedded structures of "*islands in a lake*" type [Mira *et al.*, 1996], these islands containing a lake, which contains islands, etc ... (for example see [Kitajima *et al.*, 2000]).

4.4. Unstable saddle manifolds in the case of piecewise linear maps

4.4(A). One-dimensional unstable manifold of a fixed point

Consider the map (6) and a saddle fixed point Q, the situation $\dim[W^u(Q)] = 1$ takes place when the multipliers (eigenvalues) of Q are such as $|S_1| < 1$, $|S_2| < 1$, with the third multiplier satisfying $|S_3| > 1$. The unstable manifold $W^u(Q)$ is defined as in the case $p = 2$, i.e. from a segment of straight line $W^u_{loc}(Q)$ corresponding to the eigen vector related to the multiplier S_3. One end of this segment is B, the first intersection of $W^u(Q)$ with the commutation set S_0. The points $B_n = T_b^n(B) \in W^u(Q)$, $B_n \in S_{n+1}$, $n = 0, 1, ...$, are called principal fold points of $W^u(Q)$. The other fold points of $W^u(Q)$, resulting from $T_b[W^u(Q) \cap S_k]$, $k = 0, 1, ..., n$, are called secondary fold points of $W^u(Q)$. When $b \to 0$, the principal fold points of $W^u(Q)$ are such that $\lim(B_n) \in LC_n$, $LC_n = T_b^n(LC)$ rank-$(n-1)$ critical line of T_0, located on $z = 0$. A subset of the secondary fold points tends toward each B_n and thus toward points of LC_n.

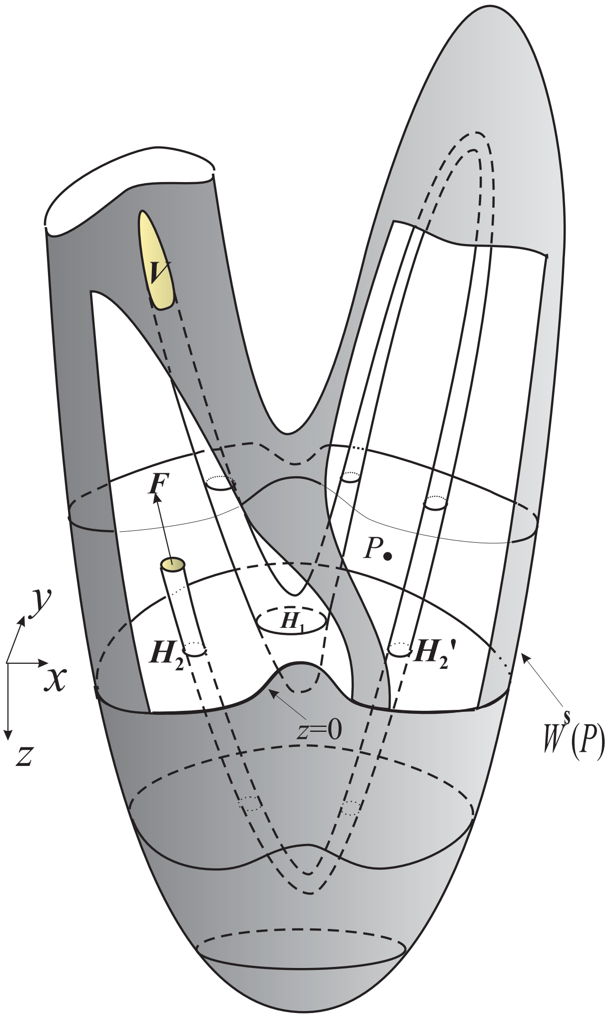

Fig. 12. Map (15), with $f(x, y, \lambda) \equiv y - \lambda x + x^2$, $\lambda = 1.5475$. Qualitative representation of the stable set $W^s(P)$ of the three-dimensional map T_b, b being sufficiently small.The white part is a window opened toward the inside of the volume.

4.4(B). Two-dimensional unstable manifold of a fixed point

Let T_b be the map (6) and Q a saddle fixed point with $\dim[W^u(Q)] = 2$. This situation takes place when the Q multipliers (eigenvalues) are such as $|S_1| > 1$, $|S_2| > 1$, with the third multiplier satisfying $|S_3| < 1$. Now the determination of $W^u(Q)$ presents higher difficulties with respect to the case of a one-dimensional unstable manifold. This is due to the fact that the local unstable manifold $W^u_{loc}(Q)$ cannot be defined from a maximum of two intersections with the first commutation sets (S_1) and (S_2). Consider the case when $W^u(Q)$ is bounded (i.e. its points cannot tend toward infinity). Let $P^0_W \supset W^u_{loc}(Q)$ be the plane defined by the eigenvectors related to $|S_1| > 1$, $|S_2| > 1$, it can be limited into a half plane $P^1_W \supset W^u_{loc}(Q)$ from $P^0_W \bigcap (S_1) = \delta^1$. Continuing the consideration of intersections of P^0_W with correctly chosen pieces of the commutation sets (S_n), $n = 1, 2, ..., k$, it may be possible to determine a subset P^k_W of P^0_W, bounded by a closed curve made up of segments $\delta^n = P^0_W \bigcap (S_n)$, $n = 1, ..k$, such as P^k_W represents $W^u_{loc}(Q)$. This process looks like the one defining an invariant absorbing area generated by a two-dimensional noninvertible map [Mira *et al.*, 1996], but without the invariance, because T^h_b, $h = 1, 2, ...$, maps $W^u_{loc}(Q)$ with foldings in directions which vary at each iteration.

Example

In order to illustrate the above points, let T_b be another form of (6), the piecewise linear invertible map ($d = 1$, $e = -1$):

$$\begin{aligned} x' &= y \\ y' &= g(x, y, \lambda) + z \\ z' &= b(x - y) \end{aligned} \tag{18}$$

with

$$\begin{aligned} g(x, y, \lambda) &= \alpha y - \lambda x, \qquad x \leq 6 \\ g(x, y, \lambda) &= \alpha y + 2x - 6(2 + \lambda), \qquad x \geq 6/\lambda \end{aligned}$$

The commutation set S_0 is the plane $x = 6$, S_1 is the plane $z = b(6 - x)$, $S_0 \bigcap S_1$ is the straight line $[z = 0, x = 6]$. S_2 is made up of two half-planes intersecting on S_1,:$S_1 \bigcap S_2$ is the straight line ζ_E defined by $[z = b(6 - x), y = \alpha x - 6\lambda]$.

The fixed points are O ($x = y = z = 0$), and P ($z = 0, x = y = 6(2 + \lambda)/(1 + \alpha)$. With the parameter values $\alpha = 0.1$, $\lambda = 1.5$, $\mid b \mid$ being sufficiently small, O is a saddle-focus, its stable manifold being one-dimensional, the unstable one being two-dimensional.

The two-dimensional noninvertible map T_0 generates a chaotic area (d) (red colored part in Fig. 13), bounded by six segments of critical curves LC_n, $n = 0, 1, 2, 3, 4$, $LC_0 \equiv LC$, $LC_q = T_0^q(LC)$, $q = 1, 2, \ldots.$ The chaotic area (d) basin is yellow colored. The equation of LC is $y = \alpha x - 6\lambda$, i.e. $y = x/10 - 9$. In order to be more clear, Fig. 13 limits the critical curves until those of rank four, but infinitely many of them intersect inside and outside (d). Each LC_q is a line of folding for sheets of the (x, y) plane, each sheet being associated with a well defined rank-$(q+1)$ preimage [Mira *et al.*, 1996].

When $b \to 0$, $S_1 \cap S_2 = \zeta_E$, defined by $[z = b(6-x),\ y = \alpha x - 6\lambda]$, tends toward the critical curve LC, and thus $(\zeta_E)_n = T_b^n(\zeta_E) = S_{n+1} \cap S_{n+2}$, $n = 0, 1, \ldots$, principal S-fold curve of S_{n+2}, tends toward LC_n. It results that the commutation sets (S_n) crush on the plane $z = 0$, and tend toward the sheets, associated with a well defined preimage of a given rank, these sheets being folded along critical curves

When $b \to 0$ the equation of the plane P_W^0 tends toward $(15 - 10b)z + 10by + 14bx = 0$. Then $P_W^0(O) \bigcap (S_1) \to [y = x/10 - 9]$, which is the LC equation. It is the same for $P_W^0(O) \bigcap (S_q)$, $q = 2, 3, 4$, which tends toward segments of LC_q. It results that $W^u(O)$, made up of infinitely many sheets folded in directions fixed by the intersections with the commutation sets (S_n), $n = 1, 2, \ldots$, crushes on the plane $z = 0$, with $W^u(O) \to W^u_{loc} \to (d)$, when $b \to 0$. So the critical curves LC_q appear as the limits of the fold lines (definition 9 and the remark which follows) of the commutation sets. Fig. 13 gives a view of the first folds.

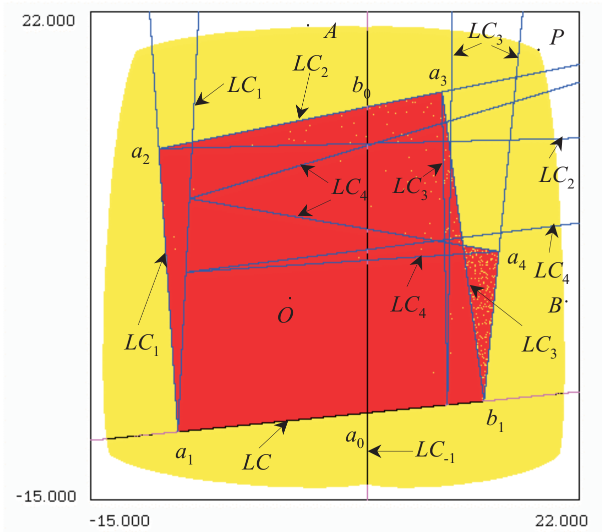

Fig. 13. Map (18). The two-dimensional noninvertible map T_0 generates a chaotic area (d) (red colored part), bounded by six segments of critical curves LC_n, $n = 0, 1, 2, 3, 4$, $LC_0 \equiv LC$, $LC_q = T_0^q(LC)$, $q = 1, 2, \ldots.$ The chaotic area (d) basin is yellow colored.

5. Conclusion

The points presented in this paper can be summarized as follows:

(a) When $b \to 0$, the stable manifold of a saddle point of the p-dimensional invertible map T_b tends toward a non connected set, the *degenerate stable set* of this saddle (with a multiplier equal to zero), generated by the map $T_{b=0}$. After one iteration the map $T_{b=0}$ turns into the $(p-1)$-dimensional noninvertible map T_0. The degenerate saddle stable set (of the p-dimensional space) intersects the $(p-1)$-dimensional space at the$(p-2)$-dimensional saddle stable set (the fixed point and its successive preimages if $p = 2$) of T_0.

(b) When $b \to 0$, the unstable manifold of a saddle point, generated by the two-dimensional invertible map T_b, tends toward segments bounded by increasing rank *critical points* of the map T_0. The two-dimensional unstable manifold of a saddle point, generated by a three-dimensional invertible map T_b, tends toward an area (belonging to $z = 0$) bounded by segments of increasing rank *critical curves* of the map T_0. The one-dimensional unstable manifold of a saddle point, generated by a three-dimensional invertible map T_b, tends toward arcs bounded by points belonging to increasing rank *critical curves* of the map T_0.

(c) When $b \to 0$, the intersections of two commutation sets of the p-dimensional invertible map T_b tends toward a *critical set* with a fixed rank, generated by the $(p-1)$-dimensional noninvertible map T_0.

(d) By embedding, the $(p-1)$-dimensional noninvertible map T_0 generates a *germinal* view of the singularities structure, and of the bifurcations, produced by the p-dimensional invertible map, for sufficiently small values of $\mid b \mid$.

Even if these results are shown here for particular simple examples, they might be verified for more general cases. So the elementary view of the embedding problem of a $(p-1)$-dimensional non invertible map into a p-dimensional invertible one, given in this paper, has the purpose to furnish the bases of more elaborate and more complete studies. Such studies concern not only a better understanding of the $(Z_0 - Z_2)$ particular case when the two-dimensional non invertible map is embedded into a three-dimensional invertible map, but also extensions by consideration of situations which do not satisfy the hypotheses (H) of sec. 1. In this context, the publication [Mira *et al.*, 1996], with the description of many situations of complex basins, invariant absorbing areas, and their bifurcations, gives the *germs* of a lot problems to be solved. So it is the case of *contact bifurcations* (between a critical curve and a stable, or unstable set) which may give rise to a tangled fan of various homoclinic, or heteroclinic bifurcations. More particularly the fundamental global contact bifurcation described in [Millerioux & Mira, 1997], and in sec. 4.3 of [Mira *et al.*, 1996], which separates two very different structures of stable sets of the map (15) with $f(x, y, \lambda) \equiv y - \lambda x + x^2$, leads to interesting problems by embedding into a three-dimensional invertible map.

Another general problem concerns the embedding of $(p-k)$ noninvertible map into a p-dimensional invertible one, $p = 3, 4, ..., k = 2, 3, ...$, from the Valiron's results [Valiron, 1948], which can be also dealt in the same way. This is a source of open problems.

The embedding problem of a $(p-1)$-dimensional non invertible map, into a p-dimensional noninvertible one, can be studied in a similar way, the $(p-1)$-dimensional situation giving a germinal state for understanding the p-dimensional case. In the case $p = 3$, this is a first step to understand the structure of two-dimensional critical sets, invariant *absorbing volumes*, *chaotic volumes* (natural extension of the notions of absorbing area and chaotic area, for $p = 2$), of basins, and their bifurcations, from an embedding of a two-dimensional noninvertible map into a three-dimensional noninvertible map.

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