Positive Mathematical Programming: a Comparison of Different Specification Rules

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Abstract— **In this paper, the prescriptive capacity of different types of positive mathematical programming models applied to the Alentejo agricultural sector is analysed. Model results are compared for 2000 and 2004 agricultural price and subsidies scenarios, regarding optimal combination of activities. Thus, it is tested, on one hand, models capacity to reproduce Alentejo agricultural sector behaviour, and by the other hand, their response and adjustment capacities to changes in prices and in agricultural policy.**

Keywords — **Positive mathematical programming, agricultural supply, Alentejo.**

I. INTRODUCTION

Mathematical programming (MP) models have been largely utilized in the area of agricultural economics, because their structure can easily suit to the economic production theory. Based on an optimisation criterion, these models allow representing agricultural production conditions and the analysis of the adjustments from technical, economic and institutional changes [1].

Early applications of MP to agricultural economics aimed to solve and to analyse problems dealing with farm planning [2 and 3]. These models are simple to formulate and very useful for understanding reality, but have some limitations in supporting decision and evaluation of agricultural policy and rural development measures. These limitations are principally due to the need of detailed information to obtain suitable coefficients describing the production technologies, and to the deviations in optimal from observed values [4].

In order to approximate the results of the MP models to the observed behaviour, it is usual to add arbitrary constraints which limit their analysis potential. In this context, Positive Mathematical Programming (PMP) made up a feasible alternative that allows to automatically calibrating the models without additional constraints [5]. The resulting model is able to respond more smoothly to changes in parameters, so that it is more consistent with changes on observed behaviour. This technique can be understood as a compromise between econometric models and MP models, because parameterization is done based on observed behaviour, as for econometrics, and primal solution exhibits an explicit specification of technology, as done in any MP model. Recently, the PMP methodology has been often used in the study of economic, social and environmental problems, like those of modelling the Common Agricultural Policy.

The objective of this paper is to evaluate the calibration and prescription capacity of a supply response PMP model for the Alentejo region. The model will be calibrated for prices and agricultural subsidies of the base year (2000 scenario), using different specification rules of the cost function. Then, the model is utilized for the prescription of the results for the scenario of prices and subsidies of 2004.

The paper is organised in more four sections regarding the PMP and cost function specification rules, the development of an agricultural model supply response for the Alentejo, results and finally conclusions.

II. POSITIVE MATHEMATICAL PROGRAMMING AND COST FUNCTION SPECIFICATION RULES

Even before its formal presentation [5], PMP had been employed in modelling economic problems applied to the agricultural sector [6, 7, 8, 9]. After the article of Howitt [5], it was clear the interest with its use, and new developments have intensified its interest [10, 11, 12, 13, 14, 15, 16].

PMP uses the information contained in dual variables of the constraints of a profit maximization LP problem, which bound activities to observed levels. These dual variables are used to specify a non-linear objective function such that the optimal solution will reproduce the observed activity levels. The empirical procedures of the PMP problem consist of two phases, comprising the estimation of the calibration parameters (phase I), and the specification of a non-linear objective function (phase II).

In phase I the calibration constraints are used in order to force the LP model solution to the observed activity levels:

$$\begin{aligned} &\max Z = p'x - c'x \\ &\text{s.t.} \\ &Ax \leqslant b\ [\lambda] \\ &x \leqslant \left(x^o + \varepsilon\right)[\rho] \\ &x \geqslant 0 \end{aligned} \quad (1)$$

Where: Z = objective function value representing the farm profit; p = (nx1) vector of product prices; c = (nx1) vector of variable costs per unit of activity; x = (nx1) vector of production activity levels; A =(mxn) matrix of coefficients in resource constraints; b = (mx1) vector of available resources; λ = (mx1) vector of dual variables associated with the resource constraints; x^0 = (nx1) vector of observed activity levels; ε = (nx1) vector of a small positive numbers to avoid a degenerate solution; ρ = dual variables associated with calibration constraints.

The level of at least one of the activities in the LP model is not bounded by its calibration constraint, but for one of the fixed resources constraint. In this way, the vector x can be divided into a vector of preferable activities (x^p) bounded by the calibration constraints, and a vector of marginal activities (x^m), which are constrained by the resource constraints. The Kuhn-Tucker conditions are:

$$\rho^p = p^p - c^p - A^p\lambda \quad (2)$$

$$\rho^m = [0] \quad (3)$$

$$\lambda = \left(p^m - c^m\right)\left(A^m\right)^{-1} \quad (4)$$

Dual value of the calibration constrains for preferable activities, for marginal activities and for resource constraints area given by the equations (2), (3) e (4), respectively.

In phase II, the dual values of the calibration constraints, ρ^p, are used to specify a non-linear objective function, such that the marginal cost of the preferable activities are equal to the respective price at the base year observed activity levels, x^0. Given these conditions, the model should reproduce exactly the vector, x^0.

The quadratic cost function is often utilized for computational simplicity and because it fits well to the hypothesis of decreasing returns in agricultural production:

$$c' = d'x + \frac{1}{2}x'Qx \quad (5)$$

Where d = (nx1) vector of parameters associated with the linear term; and Q = (nxn) symmetric, positive definite matrix of parameters associated with the quadratic term.

The linear marginal variable cost function is the sum of linear costs, c, and marginal costs, ρ:

$$Cm^V = \frac{\partial C^V\left(x^o\right)}{\partial x} = d + Qx^o = c + \rho \quad (6)$$

Given d and Q, the non-linear programming problem that reproduces the observed activity levels is:

$$\begin{aligned} &\max Z = p'x - d'x - \frac{1}{2}x'Qx \\ &\text{s.t.} \\ &Ax \leqslant b\ [\lambda] \\ &x \geqslant 0 \end{aligned} \quad (7)$$

The condition Cm = c + ρ implies an undetermined system associated to an infinite response patterns. Trying to avoid arbitraries simulations on response behaviour, several methods for specification of the parameters d and Q of the variable cost function have been developed [4]. A short overview of some of these methods is given.

In the early utilizations of PMP, the specification problem of the quadratic cost function was solved by doing d=c and setting equal to zero all off-diagonal elements of Q matrix. In this approach called *standard* specification the diagonal elements of Q, q_{ij}, were calculated as:

$$q_{jj} = \frac{\rho_j}{x_j^0} \quad j = 1,2,\ldots,n \quad (8)$$

Since ρ^m=0, the *standard* specification rule leads to a cost function which is linear in marginal activity.

This implies that a price change of a preferable activity only leads to a substitution of the marginal activity. The advantages of this method are basically on the simplicity of the specification and on ease computational, mainly, when available information is shortened.

Paris [17] used an alternative specification rule (*Paris standard*) where the parameter d of the cost function is equal to zero and the elements of the Q matrix are calculated as a function of the observed explicit costs in the base year, c, and of the dual values of the calibration constraints, ρ.

$$d = 0$$

$$q_{jj} = \frac{c_j + \rho_j}{x_j^0} \quad j = 1,2,...,n \qquad (9)$$

Diagonal elements of Q for marginal activities are all positive. So, a change of a preferable activity is done not at the expense of the marginal activities, but of the other preferable activities.

Other specification of the cost function, named by *average cost,* assumes that the observed vector of the accounting cost per activity unit in the base year, c, is equal to the average cost of quadratic variable cost function:

$$q_{jj} = \frac{2\rho_j}{x_j^0} \quad j = 1,2,...,n \qquad (10)$$

$$d_j = c_j \quad \rho_j \quad j = 1,2,...,n$$

In this approach, the diagonal elements of Q are larger than those obtained from the standard rule in (8), what implies smaller implicit elasticities, but the problem of the marginal activities with constant returns remains.

Another approach that allows the incorporation of prior information is the *exogenous supply elasticities*. Being $\partial x/\partial p$ equal to q_{jj}^{-1}, then price elasticity for activity j is calculated by:

$$\varepsilon_{jj} = \frac{1}{q_{jj}} \frac{p_j^0}{x_j^0} \quad j = 1,2,...,n$$

The parameters q_{jj} and d_j of the cost function are determined as:

$$q_{jj} = \frac{1}{\varepsilon_{jj}} \frac{p_j^0}{x_j^0} \quad j = 1,2,...,n \qquad (11)$$

$$d_j = c_j + \rho_j \quad q_{jj} x_j^0 \quad j = 1,2,...,n$$

III. REGIONAL MODEL OF AGRICULTURE SUPPLY FOR ALENTEJO REGION

In order to analyse the prescription capacity of the considered specification rules for the cost function, a PMP model adapted to the regional characteristics of the Alentejo region was developed.

The simplified formulation of this model is

$$\text{Max } Z = \sum_j p_j X_j + \sum_j a_j X_j + \sum_i p_i Y_i + \sum_i a_i Y_i$$

$$- \sum_j c_j X_j - \frac{1}{2} \sum_j q_{jj} X_j^2 - \sum_i c_i Y_i - \frac{1}{2} \sum_i q_{ii} Y_i^2 \qquad (12)$$

$$- phT - pi\,E$$

Subject To

$$\sum_i e_{jf} Y_i \leq X_{jf} \qquad (13)$$

$$\sum_{js} X_{js} * 0{,}1 \leq X_{set} \qquad (14)$$

$$\sum_j X_j \leq b_s \qquad (15)$$

$$\sum_j h_j X_j + \sum_i h_i Y_i \leq b_t + T \qquad (16)$$

$$\sum_j c_j X_j + \sum_i c_i Y_i \leq b_c + E \qquad (17)$$

Where: X_j and Y_i are the decision variables concerning the area of crop activities j in hectares (ha) and the size of livestock activities i in livestock units; T and E are the overtime working units and the additional operation capital units; p, a, c, and h are, respectively, the output value, subsidies, variable costs and work needs per unit of activity j end i; ph and pi are the hour cost of T and the annual loan interest rate of E; e_{if} are the livestock stocking rates; and bs, bt and bc are the fixed resources land, work and capital availability.

The objective function (12) maximizes the gross margin in euros and it is calculated by the difference between revenue and total variable costs. The revenue includes agricultural output value and the direct subsidies. The variable costs comprehend short time linear input costs (cj and ci), costs with overtime working (ph) and operating capital (pi) and also marginal costs coefficients of activities (qjj and qii).

Decision variables in the model include eighteen agricultural activities of the Alentejo Region between crops and livestock activities. Crop activities comprise cereals and oil seeds, horticulture and fruit culture, fruit trees, vineyards, olive tree, permanent pastures, forage, compulsory set-aside, fallow and an activity regarding land occupied by forests. Livestock activities comprehend beef cattle, sheep and extensive swine.

Permanent pastures and forages are intermediate activities because they are not sold but are an input for livestock activities. So, these activities only have costs, being their profits indirectly obtained from animal activities. The profit transfer between activities is done essentially by equation (13), which defines the balance between forage areas (X_{jf}) and the total number of animals.

Equation (14) models the set-aside (X_{set}) imposed by CAP. This equation states that 10% of the crop area (X_{js}) has to be retired from production and put in set-aside.

Equations (15) to (17) stay for the use of land, labour and capital. These equations state that the resource demand is less than or equal to their availability.

In spite of the objective function represent the return to land, labour and capital, model solution is limited only by land availability in (15). Labour (16) and capital (17) demand can exceed their availabilities by purchasing additional hours of labour at an hour cost of €3.5 and additional units of capital at an annual loan interest rate of 7%.

IV. RESULTS

The results of PMP model of agriculture supply of the Alentejo is obtained for each one of the specification rules of the cost function. First, the PMP model is calibrated for the base year (2000). Then, prices and subsidies vectors are changed and the model is used for prescription of results for 2004 scenario.

In both scenarios, results are compared to available data for the Alentejo region, concerning crop areas and the number of livestock units.

For the base year the model reproduces exactly the observed level of the activities, whatever the specification rule of the cost function used. The different specification approach of the cost function give the same results, because the condition Cm = c + ρ constitutes an undetermined system. So, there are an infinite number of values for the parameters q_{jj} e q_{ii} satisfying the conditions of the PMP problem.

Table 1 presents the absolute deviation to activities observed levels in 2004. This table also presents the total weighted absolute deviation, which have in account the relative weight of each crop on the total land and of each animal activity on the total livestock unit.

Table 1 Absolute deviation on the activity levels for 2004 (%)

Crop activities	Standard	Paris Standard	Average Cost	Exogen. Elasticit.
Common Wheat	-26.6	109.9	81.8	-7.3
Durum Wheat	-12.6	-11.2	-10.5	-11.5
Maize	-11.4	-8.3	-9.1	-9.9
Rice	-65.2	-19.5	-36.6	-23.5
Horticulture	18.4	11.0	11.4	10.4
Sunflower	51.9	63.7	62.5	76.4
Olive trees	100.0	-0.8	-53.4	11.1
Vineyard	-36.2	-35.3	-33.2	-48.6
Fruits	-7.3	-8.7	0.0	-5.5
Permanent pastures	44.3	2.0	23.9	-3.4
Forage	44.3	2.0	23.9	-3.4
Fallow	-79.9	-10.7	-49.7	6.3
Forests	4.1	4.1	4.1	4.1
Set-aside	-5.4	10.5	8.1	0.6
Beef cattle	88.7	20.9	38.1	4.2
Sheep	5.5	4.7	7.0	11.3
Swine	271.8	12.6	258.5	15.8
WADC	39.2	8.3	26.2	7.3
WADA	88.0	14.4	63.4	7.8

Source: Results of PMP models

The results obtained for the 2004 scenario show that the rule of *exogenous elasticities* is superior to the others. The weighted absolute deviations are smaller on crop activities (7.3%) (WADC), and on animal activities (7.8%) (WADA). For *Paris Standard* rule the deviations are 8.3% on crop activities and 14.4% on animal activities. *Standard* and *average cost* rules have weighted absolute deviations respectively of 39.2% and 26.2% on crop activities, and 88% and 63.4% on animal activities. These results show the poor prescription capacity of these two methods.

When the *exogenous elasticities* approach is used only three activities present an absolute deviation above the 15% indicated by Hazell & Norton [18], as the maximum value for a desirable calibration. These activities are rice (-23.5%), sunflower (76.4%) and vine (-48.6%). The observed values, in terms of area, for 2000 and 2004, of those activities do not change, only the area of vineyard had a light increase.

For *Paris standard* rule there are six activities presenting absolute deviations above the 15%, four crop activities and two livestock activities. Particularly big are the absolute deviation registered on the area of common wheat (109.9%) and of sunflower (63.7%). Concerning livestock activities, the absolute deviation of 20.9% on beef cattle determines an increase of this activity bigger than that have actually happened in the beef cattle sector.

Regarding *standard* and *average cost* specification rules, ten activities present absolute deviations above the 15%, being particularly big on animal activities. For instance, extensive swine production registered an absolute deviation of more than 200%. Along with these deviations, there are also big absolute deviations on intermediate of pasture and forage. The variability of the obtained results with the different specification rules of the cost function can be explained by the implicit supply elasticities in each one of the crop or animal activity (Table 2).

In general, the results obtained from the specification rule of *exogenous elasticities* and of *Paris standard* present smaller values, in average, in implicit supply elasticity, such that these are the rules that exhibit the best prescription capacity of the results on 2004 scenario.

The specification rules *standard* and *average cost* presenting the biggest, in average, implicit supply elasticities on activities, and showing results far from observed reality, have the poorest prescription capacity.

Table 2 - Supply elasticity of agricultural activities

Crop activities	Standard	Paris Standard	Average Cost	Exogen. Elasticit.
Common wheat	4.68	1.77	1.33	4.05
Durum wheat	11.22	3.71	7.42	7.42
Maize	0.11	0.00	0.00	0.00
Rice	6.04	2.26	0.00	0.00
Horticulture and Fruit culture	4.08	5.10	0.00	0.00
Sunflower	8.33	5.32	2.66	0.00
Olive tree	51.00	14.19	1.15	1.81
Vine	0.00	0.00	0.00	8.50
Fruit culture	14.47	14.47	14.47	14.47
Set-aside	1.79	1.51	1.51	1.51
Forests	0.00	0.00	0.00	0.00
Beef cattle	2.63	1.66	0.99	0.37
Sheep	1.11	0.77	1.27	0.16
Swine	∞	∞	∞	∞

Source: PMP model results

V. CONCLUSION

Mathematical programming (MP) models have been largely utilized in the area of agricultural economics, because their structure can easily suit to the economic production theory.

In general, MP models area aimed to evaluate economic, technical and institutional scenarios, implying changes in prices, technologies and available inputs. Their quality is checked by the sensitivity and post-optimal analysis to changes in their coefficients.

In this context, this paper evaluates the calibration and prescription capacities of a PMP model, developed for the agriculture supply conditions of the Alentejo region. The considered cost function specification rules were *standard*, *Paris standard*, *average cost* and *exogenous elasticities.*

The results showed that the PMP model reproduces exactly the observed activity levels on the base year, whatever the rule used to specify the cost function. This property is due to the condition $Cm = c + \rho$ and to the functional form of the cost function. There are an infinite number of parameters satisfying the conditions of a non-linear PMP problem.

Regarding the prescription capacity of future results, PMP revealed being a feasible methodological option, mainly if *exogenous elasticities* or *Paris* *standard* approaches were used to specify the cost function. Specification rules of cost function based on *standard* method or *average cost* method showed a smaller prescription capacity of future results. These methods present, in average, big implicit supply elasticities on agricultural activities.

We can conclude that the properties of PMP do not only exhaust just in the exact calibration of the agriculture supply models. Those properties also respect the prescription capacity of future results. In this case, the *exogenous elasticities* approach showed being superior to the others, even though *Paris Standard* method be also a good alternative.

REFERENCES

1. Mccarl B. And Spreen T. (1980) Price Endogenous Mathematical Programming as a Tool for Sector Analysis. American Journal of Agricultural Economics, pp. 87-102.
2. Throsby C.D. (1974) New methodologies in agricultural prodution economics: A review. In The Future of Agriculture. Papers and reports, 15 th international Conferece of Agricultural Economics, S. Paulo, Brazil, 1974, pp. 150-169.
3. Martin L.R. (1977) A Survey of Agricultural Economics Literature vol 2. University of Minnesota Press, Minneapolis.
4. Heckelei T., Britz W. (2005) Models Based on Positive Mathematical Programming: State of the Art and Further Extensions, EAAE Proc. 89th European Seminar of the European Association of Agricultural Economists, Parma, Italy, 2005, pp 48-73.
5. Howitt R. (1995) Positive Mathematical Programming. American Journal of Agricultural Economics, 77 (2): 329-342.
6. House R. (1987) USMP Regional Agricultural Model. Washington DC: USDA. National Economics Division Report, ERS, 30.
7. Kasnakoglu H., Bauer S. (1988) Concept and Application of an Agricultural Sector Model for Policy Analysis in Turkey. In: Agricultural Sector Modelling. S. Bauer and W. Henrichsmeyer (Eds.), Vauk Verlag: Kiel.
8. Bauer S., Kanakoglu H. (1990) Non Linear Programming Models for Sector Policy Analysis. Economic Modelling 7: 272-90.
9. Horner G., Corman J., Howitt R, Carter C., Macgregor R. (1992) The Canadian Regional Agriculture Model: Structure, Operation and Development. Agriculture, Canada. Technical Report 1/92, Ottawa.
10. Arfini F., Donati M., Zuppiroli M., Paris Q. (2005) Ex-post evaluation of set-aside using Symmetric Positive Equilibrium Problem. EAAE Proc. 89th Symposium of the European Association of Agricultural Economists, Parma, Italy, 2005.
11. Cypris C. (2000) Positiv Mathematische Programmierung (PMP) im Agrarsektormodells RAUMIS. Dissertation, University of Bonn.
12. Gohn A., Chantreuil F. (1999) La programmation mathématique positive dans les modèles d'exploitation agricole. Principes et l'importance du calibrage. Cahiers d'Economie et de Sociologie Rurales 52: 59-79
13. Barkaqui A., Butault J. (1999) Positive Mathematical Programming and Cereals and Oilseeds Supply with EU under Agenda 2000. Proc. 9th European Congress of Agricultural Economists, Warsaw, 1999.
14. Baskaqui A., Butault J., Rousselle J. (2001) Positive Mathematical Programming and Agricultural Supply within EU under Agenda 2000. EAAE Proc. 65th European Association of Agricultural Economists, Bonn, Vauk Verlag Kiel, 2000: pp 200.
15. Graindorge C, Henry B., Howitt R. (2001) Analysing the effects of Agenda 2000 using a CES Calibrated Model of Belgian Agriculture. EAAE Proc. 65th European Association of Agricultural Economists, Bonn, Vauk Verlag Kiel, 2000, pp: 176-186.
16. Helming J., Peeters L., Veendendaal P. (2001) Assessing the Consequences of Environmental Policy Scenarios in Flemish Agriculture. EAAE Proc. 65th European Association of Agricultural Economists, Bonn, Vauk Verlag Kiel, 2000, pp: 237-245.
17. Paris Q. (1988) PQP, PMP, Parametric Programming and Comparative Static. Chapter 11 in Notes for AE 253. Department of Agricultural Economics, University of California, Davis.
18. Hazell P., Norton R. (1986) Mathematical Programming for Economic Analysis in Agriculture. Mac Millan Publishing Company, New York.