



#G3

INTEGERS 12 (2012)

**A RECURSIVE PROCESS RELATED TO A PARTIZAN
VARIATION OF WYTHOFF¹****Alda Carvalho***High Institute of Engineering of Lisbon & CEMAPRE-ISEG, Lisboa, Portugal***Carlos P. Santos²***High Institute of Education and Science, Lisboa & CIMA-UE, Evora, Portugal*`carlos.santos@isec.universitas.pt`**Cátia Lente Dias***High Institute of Engineering of Lisbon & CIMA-UE, Evora, Portugal***Francisco Coelho***University of Evora & LabMAG, Lisboa, Portugal***João Pedro Neto***University of Lisbon & LabMAG, Lisboa, Portugal***Sandra Vinagre***University of Evora & CIMA-UE, Evora, Portugal**Received: 8/6/11, Revised: 2/24/12, Accepted: 5/15/12, Published: 5/25/12***Abstract**

WYTHOFF QUEENS is a classical combinatorial game related to very interesting mathematical results. An amazing one is the fact that the \mathcal{P} -positions are given by $(\lfloor \varphi n \rfloor, \lfloor \varphi^2 n \rfloor)$ and $(\lfloor \varphi^2 n \rfloor, \lfloor \varphi n \rfloor)$ where $\varphi = \frac{1+\sqrt{5}}{2}$. In this paper, we analyze a different version where one player (Left) plays with a chess bishop and the other (Right) plays with a chess knight. The new game (call it CHESSFIGHTS) lacks a Beatty sequence structure in the \mathcal{P} -positions as in WYTHOFF QUEENS. However, it is possible to formulate and prove some general results of a general recursive law which is a particular case of a PARTIZAN SUBTRACTION game. ³

¹Research in *Play mathematics - A Workshop in Combinatorial Game Theory*, Center for Mathematics and Fundamental Applications, Lisbon, November, 2010.

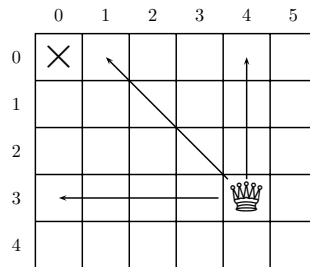
http://ptmat.fc.ul.pt/arquivo/docs/seminarios/confwork/2010/Mini_workshop.pdf

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1. Introduction

WYTHOFF QUEENS is played on a quarter-infinite chessboard, extending downwards and to the right. A chess queen is placed in some cell of the board. On each turn, a player moves the queen as in chess, except that the queen can only move left, up, or diagonally up-left. The player who moves the queen to the corner (0, 0) wins.

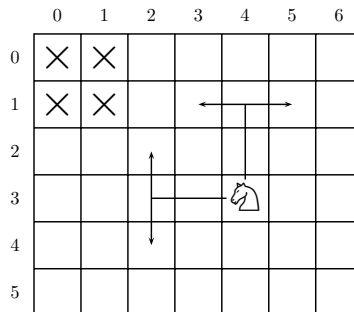


We can also interpret WYTHOFF QUEENS as a pile game. There are two piles of stones and, on each turn, a player either removes an arbitrary number of stones from one pile, or the same number of stones from both piles. The player who makes the last move wins.

A nice result about WYTHOFF QUEENS is the following one (first proved in [6]): The \mathcal{P} -positions of WYTHOFF QUEENS are given by $(\lfloor \varphi n \rfloor, \lfloor \varphi^2 n \rfloor)$ and $(\lfloor \varphi^2 n \rfloor, \lfloor \varphi n \rfloor)$ where $\varphi = \frac{1+\sqrt{5}}{2}$.

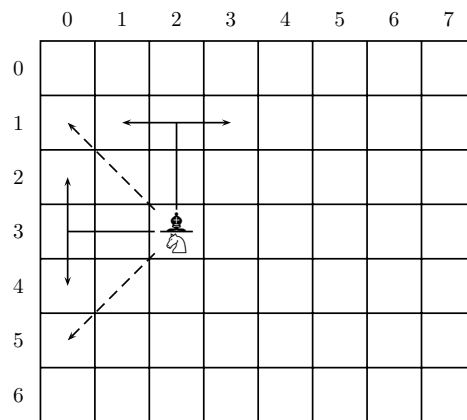
There are some variations of the game. One very interesting, analyzed in [2] (page 56), is the game WHITE KNIGHT. In this variation, instead of a queen, the players move a chess knight. The legal moves are the following (row x and column y):

$$(x, y) \rightarrow (x - 1, y - 2) \text{ or } (x, y) \rightarrow (x + 1, y - 2) \text{ or } (x, y) \rightarrow (x - 2, y - 1) \text{ or } (x, y) \rightarrow (x - 2, y + 1)$$



We consider a variation of WYTHOFF QUEENS, the game CHESSFIGHTS. The rules of this variation are the following ones:

- The board is as in WYTHOFF QUEENS and WHITE KNIGHT;
- Right plays with the knight as in WHITE KNIGHT;
- Left plays with the bishop: $(x, y) \rightarrow (x - i, y - i)$ or $(x, y) \rightarrow (x + i, y - i)$ (in the first case, we must have $x - i \geq 0 \wedge y - i \geq 0$ and, in the second case, we must have $x + i \geq 0 \wedge y - i \geq 0$, in other words, the move must be made inside the board).



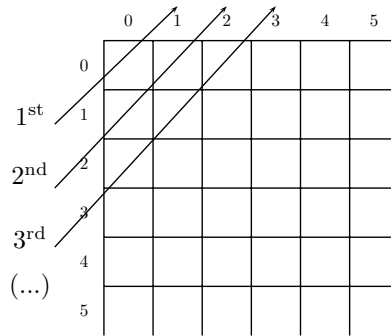
CHESSFIGHTS is a *partizan* game. For ease, the game with the piece in the cell (x, y) will be represented by the pair (x, y) .

The game converges to the end because, after two moves, $(x, y) \mapsto (x', y') \mapsto (x'', y'')$, we have $x'' + y'' < x + y$.

2. Some Theorems of CHESSFIGHTS

The *options* of a game are all those positions which can be reached in one move. In combinatorial game theory, games can be expressed recursively as $G = \{\mathcal{G}^L | \mathcal{G}^R\}$ where \mathcal{G}^L are the Left options and \mathcal{G}^R are the Right options of G . The *followers* of G are all the games that can be reached by all the possible sequences of moves from G (this is the usual notation of [3], [2], and [1]).

In the particular case of CHESSFIGHTS, we can compute the values of the cells (or, rather, the games corresponding to the placement of a single piece in a cell). The best way to do it is to choose a diagonal path:



With this procedure, we get an organized table (the following example corresponds to 9×9):

0	1	{1 0}	$\frac{1}{2}$	1	{1 ↑}	$\frac{1}{2}$	1	{1 ↑3*}
0	1	{1 0}	↑	1	{1 $\frac{1}{2}$, {1 *}}	↑3*	1	{1 $\frac{1}{2}$ }
0	*	{1 0}	{1 *}	↑	{1 ↑}	{1 1, {1 *2}}	↑3	{1 ↑3*}
0	*	↑*	{1 *}	{1 *2}	↑*	{1 ↑, {1 *2}}	{1 {1 ↑}, ↑3*}	↑3*3
0	*	*2	↑	{1 *2}	{1 ↑}	↑*2	{1 ↑*, {1 0 *, *2}}	{1 ↑3, {1 ↑*3}}
0	*	*2	↑	↑*3	{1 0 *, *2}	{1 ↑*3}	{0 0 *, *2}	{1 ↑*2}
0	*	*2	{0 *, *2}	↑*3	{0 0 *, *2}	{1 ↑*3}	{1 0 0 *, *2}	{0 0 *2, {0 *, *2}}
0	*	*2	{0 *, *2}	↑*3	{0 0 *, *2}	{0 0 *2, {0 *, *2}}	{1 0 0 *, *2}	{1 0 0 *2, {0 *, *2}}
0	*	*2	{0 *, *2}	{0 *2, {0 *, *2}}	{0 0 *, *2}	{0 0 *2, {0 *, *2}}	{0 0 *, *2}	{1 0 0 *2, {0 *, *2}}

The same table just with the reduced canonical forms:

0	1	{1 0}	$\frac{1}{2}$	1	{1 0}	$\frac{1}{2}$	1	{1 0}
0	1	{1 0}	0	1	{1 $\frac{1}{2}$ }	0	1	{1 $\frac{1}{2}$ }
0	0	{1 0}	{1 0}	0	{1 0}	1	0	{1 0}
0	0	0	{1 0}	{1 0}	0	{1 0}	{1 0}	0
0	0	0	0	{1 0}	{1 0}	0	{1 0}	{1 0}
0	0	0	0	0	{1 0}	{1 0}	0	{1 0}
0	0	0	0	0	0	{1 0}	{1 0}	0
0	0	0	0	0	0	0	{1 0}	{1 0}
0	0	0	0	0	0	0	0	{1 0}

A visual inspection of the table allows us to guess some patterns. In fact, it is possible to prove some results.

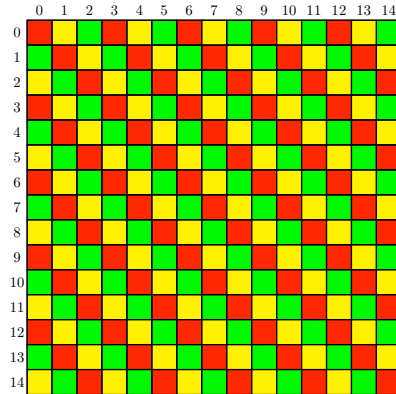
Proposition 1. $(x, 0) = 0$.

Proof. Left has no options. Right has no options (cases $(0, 0)$ and $(1, 0)$) or Right has just one option to $(x - 2, 1)$. If so, Left plays to $(x - 1, 0)$ and, by induction, Right loses. \square

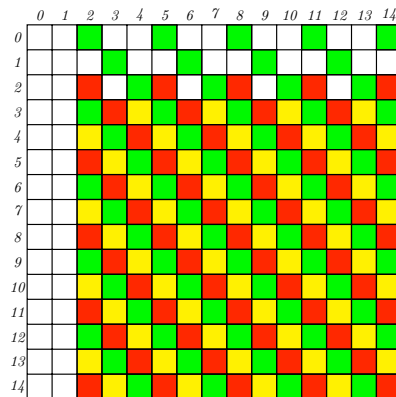
In the next results, it is important to consider the following groups of cells:

- Red $\rightarrow (x, y) : y - x \equiv 0 \pmod{3}$

- Yellow $\rightarrow (x, y) : y - x \equiv 1 \pmod{3}$
- Green $\rightarrow (x, y) : y - x \equiv 2 \pmod{3}$



Lemma 1. *From the games in the following region (call it \mathfrak{R}),*



Right to move, has a strategy allowing, at all times, if the sub-position is still not zero, a Right move to a green cell or a Right move to zero.

Proof. Let us analyze all the possible sub-positions (a, b) (Right moving).

- If $b = 0$ then the position $(a, b) = 0$ (Proposition 1).
- If $(a, b) \in \mathfrak{R}$ is green ($b - a \equiv 2 \pmod{3}$) then Right moves to $(a + 1, b - 2)$. We can see that $(a + 1, b - 2)$ remains green because $(b - 2) - (a + 1) \equiv 2 \pmod{3}$.
- If $(a, b) \in \mathfrak{R}$ is red ($b - a \equiv 0 \pmod{3}$) then Right moves to $(a - 1, b - 2)$. We can see that $(a - 1, b - 2)$ turns green because $(b - 2) - (a - 1) \equiv 2 \pmod{3}$.

- If $(a, b) \in \mathfrak{R}$ is yellow ($b - a \equiv 1 \pmod{3}$) then Right moves $(a - 2, b - 1)$. We can see that $(a - 2, b - 1)$ turns green because $(b - 1) - (a - 2) \equiv 2 \pmod{3}$.
- The only possible Left moves to $(a, b) \notin \mathfrak{R}$ are $(a, 0)$ (item 1) and $(a, 1) \wedge a > 1$ (in this case, the Right option to $(a - 2, b - 1) = 0$ is available). The moves indicated in the previous items never allow other options $(a, b) \notin \mathfrak{R}$ for Left.

□

Proposition 2. $(0, 3k + 1) = 1$ ($k \geq 0$) and $(0, 3k) = \frac{1}{2}$ ($k \geq 1$).

Proof. Let us prove that $(0, 3k + 1) = 1$ ($k \geq 0$).

The base case $(0, 1) = 1$ is calculated by hand. We want to prove that, for $k \geq 1$, $(0, 3k + 1) + \{ |0\} = 0$, i.e., $(0, 3k + 1) + \{ |0\}$ is in \mathcal{P} .

If Right plays to $(0, 3k + 1)$, Left replies to $(3k + 1, 0) = 0$ (Proposition 1).

If Right plays to $(1, 3k - 1) + \{ |0\}$, Left replies to $(0, 3k - 2) + \{ |0\} = (0, 3(k - 1) + 1) + \{ |0\} = 1 - 1$ (induction).

So, if Right plays, Right loses.

If Left plays first to $(a, b) + \{ |0\}$ then $(a, b) \in \mathfrak{R}$ or $(a, b) = (a, 0)$ or $(a, b) = (a, 1) \wedge a > 1$. The last two cases are trivial. For the first case, Right just plays in (a, b) with the strategy of the Lemma 1 eventually ending in $0 - 1$. So, playing first, Left loses.

Let us prove that $(0, 3k) = \frac{1}{2}$ ($k \geq 1$). The base case $(0, 3) = \frac{1}{2}$ is calculated by hand. We want to prove that, for $k > 1$, $(0, 3k) + \{ -1|0\} = 0$, i.e., $(0, 3k) + \{ -1|0\}$ is in \mathcal{P} .

If Right plays to $(0, 3k)$, Left replies to $(3k, 0) = 0$ (Proposition 1).

If Right plays to $(1, 3k - 2) + \{ -1|0\}$, Left replies to $(0, 3k - 3) + \{ -1|0\} = (0, 3(k - 1)) + \{ -1|0\} = \frac{1}{2} - \frac{1}{2}$ (induction).

So, if Right plays, Right loses.

If Left plays first to $(1, 3k - 1) + \{ -1|0\}$, Right replies to $(0, 3k - 3) + \{ -1|0\} = (0, 3(k - 1)) + \{ -1|0\} = \frac{1}{2} - \frac{1}{2}$ (induction).

If Left plays to $(a, b) + \{ -1|0\}$ with $a > 1$ then $(a, b) \in \mathfrak{R}$ or $(a, b) = (a, 0)$ or $(a, b) = (a, 1) \wedge a > 1$. The last two cases are trivial. For the first case, Right just plays in (a, b) with the strategy of the Lemma 1 eventually ending in $0 - \frac{1}{2}$. So, playing first, Left loses. □

The next proposition is a useful inequality. With this result it will be possible to make some arguments of domination and reversibility.

We will write $\underline{(x, y)}$ to represent the game (x, y) , but Left playing with the Knight and Right with the Bishop. We have $-(x, y) = \underline{(x, y)}$. This is a nice tool to perform

proofs on the board with two different pieces. Also, we call *principal diagonal* to the set of cells such that $x = y$.

Lemma 2. *If $k \geq 2$ and $x' > y - k$ then $(x, y) + (x', y - k) \not\leq 0$ (if the second component is below the principal diagonal and the components are separated by more than one column, Left wins playing first).*

Proof. If $y - k = 0$ then $(x', y - k) = 0$ (Proposition 1). So, Left plays in the other component to $(x + y, 0) + (x', 0)$ going to zero.

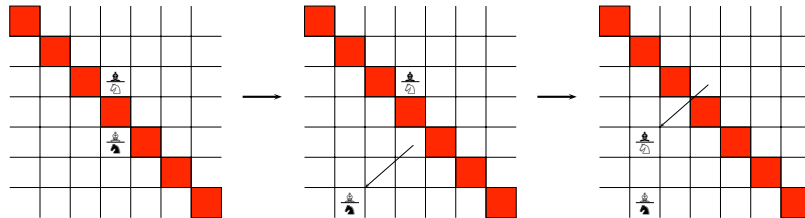
If $y - k = 1$, Left moves to $(x, y) + (x' - 2, 0)$ which is equal to (x, y) (Proposition 1). Following, after a move by Right in (x, y) , Left moves this component to the column 0.

If $y - k > 1$, Left moves to $(x, y) + (x' + 1, y - k - 2)$. Following, all the possible moves by Right maintain the Lemma conditions. So, by induction, Left wins. \square

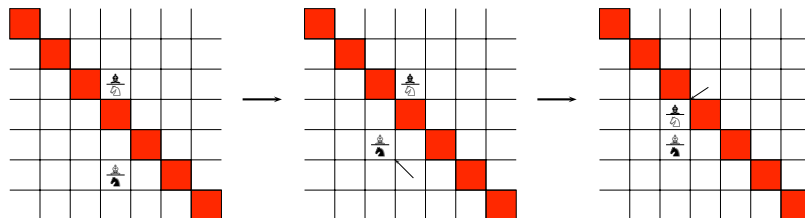
Proposition 3. *If $x > y$ then $(x - k, y) \geq (x, y)$ ($k \geq 0$, positions inside the board).*

Proof. We want to prove that, if $x > y$, $(x - k, y) - (x, y) \geq 0$. So, we want to prove that Right loses playing first in the game $(x - k, y) + (x, y)$. We will analyze all the Right options (consider the principal diagonal, red cells such that $x = y$).

- Right plays to $(x - k, y) + (x + i, y - i)$.
Left moves to $(x - k + i, y - i) + (x + i, y - i)$ and, by induction, Left wins.

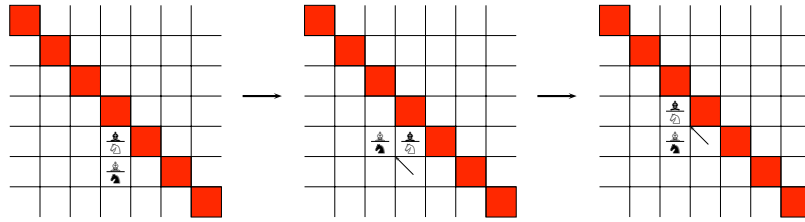


- Right plays to $(x - k, y) + (x - 1, y - 1)$ (and $k \geq 1$).
Left moves to $(x - k + 1, y - 1) + (x - 1, y - 1)$ and, by induction, Left wins.

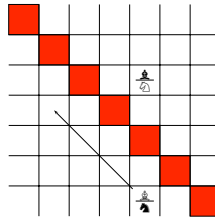


- Right plays to $(x - k, y) + (x - 1, y - 1)$ (and $k \leq 1$).
Left moves to $(x - k - 1, y - 1) + (x - 1, y - 1)$ (available) and, by induction,

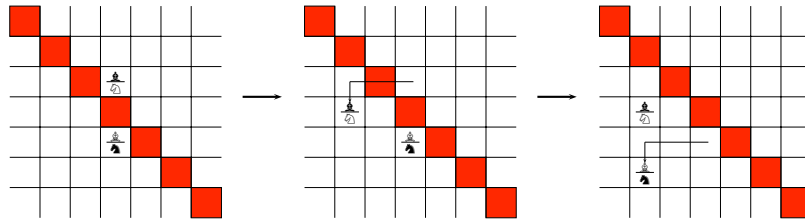
Left wins.



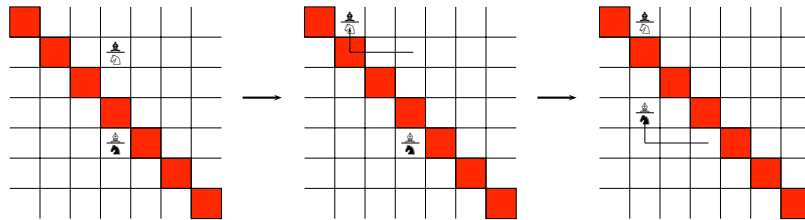
- Right plays to $(x - k, y) + \underline{(x - i, y - i)}$ ($i > 1$).
By Lemma 2, Left wins.



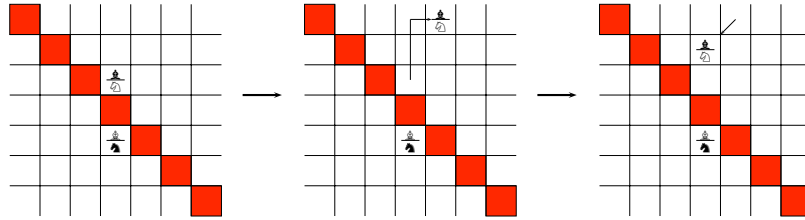
- Right plays to $(x - k + 1, y - 2) + \underline{(x, y)}$.
Left moves to $(x - k + 1, y - 2) + \underline{(x + 1, y - 2)}$ and, by induction, Left wins.



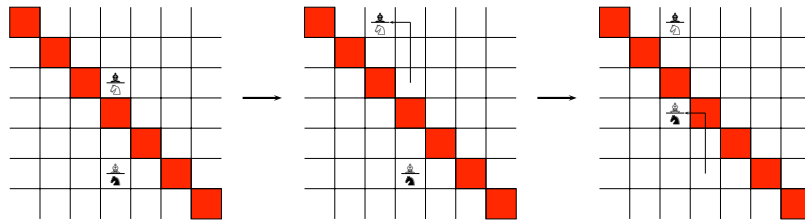
- Right plays to $(x - k - 1, y - 2) + \underline{(x, y)}$.
Left moves to $(x - k - 1, y - 2) + \underline{(x - 1, y - 2)}$ and, by induction, Left wins.



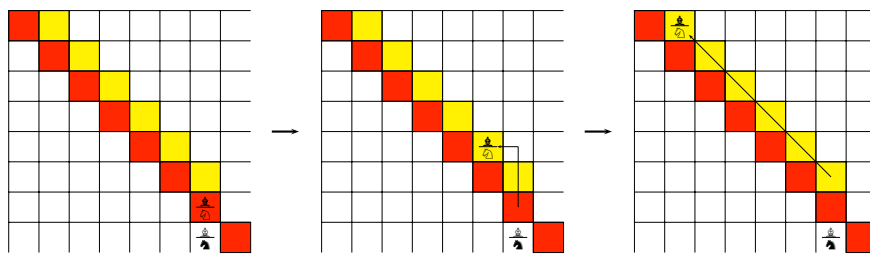
- Right plays to $(x - k - 2, y + 1) + \underline{(x, y)}$.
Left moves to $(x - k - 1, y) + \underline{(x, y)}$ and, by induction, Left wins.



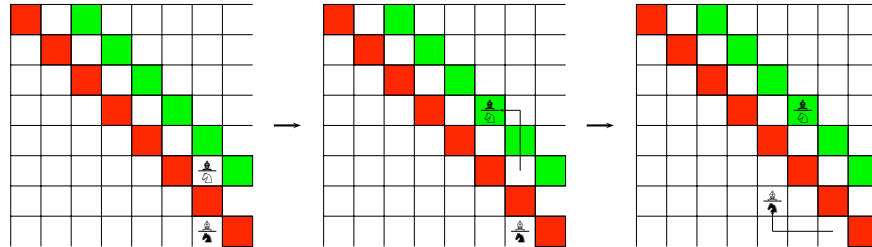
- Right plays to $(x - k - 2, y - 1) + \underline{(x, y)}$ and $(x - 1 > y$ or $k = 0)$.
Left moves to $(x - k - 2, y - 1) + \underline{(x - 2, y - 1)}$ and, by induction, Left wins.



- Right plays to $(x - k - 2, y - 1) + \underline{(x, y)}$ and, using the previous notation, $(x - k - 2, y - 1)$ is a red or a yellow cell.
Left moves to $(0, y - x + k + 1) + \underline{(x, y)}$ and, because $(0, y - x + k + 1) = 1$ or $(0, y - x + k + 1) = \frac{1}{2}$ (Proposition 2), Left wins maintaining the second component below the principal diagonal.



- Right plays to $(x - k - 2, y - 1) + \underline{(x, y)}$ and $(x - k - 2, y - 1)$ is a green cell.
Left moves to $(x - k - 2, y - 1) + \underline{(x - 1, y - 2)}$ and, if Right wants to avoid the induction, must move to $(x - k - 4, y - 2) + \underline{(x - 1, y - 2)}$. After this pair of moves, $(x - k - 4, y - 2)$ turns red or yellow and Left chooses the strategy of the previous item.



□

Proposition 4. *If $x \geq 2$ then $(x, 1) = *$.*

Proof. We can calculate by hand $(2, 1) = *$. Now we prove the theorem by induction in x . The Left options of $(x, 1)$ are 0 (Proposition 1). The Right options are $(x-2, 0) = 0$ and $(x-2, 2)$. Against a Right's move to $(x-2, 2)$, Left can immediately reply to $(x-1, 1)$. By Proposition 3, $(x-1, 1) \geq (x, 1)$. So, by reversibility, the Right option $(x-2, 2)$ can be replaced by Right options of $(x-1, 1)$. But, by induction, $(x-1, 1) = *$ and $(x-2, 2)$ can be replaced by 0. □

Lemma 3. *If $x > y$ then $1 \geq (x, y)$.*

Proof. Let us analyze $1 + (x, y)$ to see that Right, playing first, loses. Against a Right move (if he has one)

- To $1 + (x', 0)$. In that case, the game turned $1 + 0$.
- To $1 + (x', 1)$. In that case, the game turned $1*$.
- To $1 + (x', k)$ ($k \geq 2$). In that case, Left answers to $1 + (x' + 1, k - 2)$ reaching the same kind of position as before.

In all cases, Left wins. □

Lemma 4. *If $x > y$ then $(x - 2, y + 1) \geq (x + 1, y - 2)$.*

Proof. Let us analyze $(x - 2, y + 1) + (x + 1, y - 2)$ to see that Right, playing first, loses. If Right plays in the component $(x - 2, y + 1)$, Left replies in the same component to the column $y - 2$ and wins (Proposition 3).

If Right plays to $(x - 2, y + 1) + (x + 1 - i, y - 2 - i)$, Left replies to $(x - 2 - i, y + 1 - i) + (x + 1 - i, y - 2 - i)$ maintaining the situation. If the Left answer was not available, that was because Right's move was to $(x - 2, y + 1) + (k, 1)$ ($k \geq 2$) or to $(x - 2, y + 1) + (k, 0)$ ($k \geq 1$). Against the first, Left moves the component $(x - 2, y + 1)$ to the column 1 and against the second, Left moves the component $(x - 2, y + 1)$ to the column 0.

In both cases, Left wins. □

Theorem 1. *The games (x, y) for $x > y$ are all-small.*

Proof. Let us consider $y \geq 2$ (the cases $y = 0$ and $y = 1$ are already known). By induction, Left options are all-small. Right has 4 options. By induction, $(x+1, y-2)$ and $(x-1, y-2)$ are all-small.

- Right option to $(x-2, y-1)$.
 If $(x-2, y-1)$ is not in the principal diagonal, by induction, $(x-2, y-1)$ is all-small.
 If $(x-2, y-1)$ is in the principal diagonal, Left can answer to $(1, 1) = 1$. By Lemma 3, $1 \geq (x, y)$. So, the Right option is reversible to \emptyset .
- Right option to $(x-2, y+1)$.
 By Lemma 4 $(x+1, y-2)$ dominates $(x-2, y+1)$. Because we are thinking for columns with index $y \geq 2$, $(x+1, y-2)$ is available.

□

3. The General Recursive Process

As we saw in the previous section, the Right option $(x-2, y+1)$ is dominated (see Theorem 1). For the sensible options, the column number is decreased by one or two. This strongly motivates the analysis of the recursion

$$g(n) = \{g(0), \dots, g(n-1) \mid g(n-1), g(n-2)\}.$$

This is a special case of a partizan subtraction game (see [4]). The first elements of the sequence are

0	1	2	3	4
0	*	*2	{0 *, *2}	{0 *2, {0 *, *2}}
5			6	
{0 {0 *, *2}, {0 *2, {0 *, *2}}}			{0 {0 *2, {0 *, *2}}, {0 {0 *, *2}, {0 *2, {0 *, *2}}}	

We can generalize the recursive law for similar chess knights (capable of making “larger” moves):

$$g_k(n) = \{g_k(0), \dots, g_k(n-1) \mid g_k(n-k), g_k(n-2k)\} \quad (n \geq 0).$$

There is no problem with the $g_k(i)$ not previously defined. The empty set is available for the construction of the games.

For impartial subtraction games, it is well-known that $\text{SUBTRACTION}(ms_1, \dots, ms_k)$ is the m -plicate of $\text{SUBTRACTION}(s_1, \dots, s_k)$ ([2], page 98 and a proof in [5], page 36). We will prove that the general g_k is also a kind of “dilation” of g_1 . Just for intuition, we list the first elements of $g_2(n)$ and $g_3(n)$:

0	1	2	3	4	5	6
0	1	{1 0}	1*	{1, 1* 0, {1 0}}	1*2	{1 {1 0}, {1, 1* 0, {1 0}}}

7	8
{1 1*, 1*2}	{1 {1, 1* 0, {1 0}}, {1 {1 0}, {1, 1* 0, {1 0}}}

0	1	2	3	4	5	6	7
0	1	2	{2 0}	{2 1}	2*	{2, 2* 0, {2 0}}	{2, 2* 1, {2 1}}

8	9	10
2*2	{2 {2 0}, {2, 2* 0, {2 0}}}	{2 {2 1}, {2, 2* 1, {2 1}}}

We start with a result about the left options of $g_k(n)$.

Lemma 5. For $k \geq 1$, we have

$$g_k(n) = \{g_k(0), \dots, g_k(n-1) \mid g_k(n-k), g_k(n-2k)\}$$

$$= \begin{cases} n & n \leq k-1 \\ \{k-1, (k-1)* \mid g_k(n-k), g_k(n-2k)\} & 2k \leq n \leq 3k-1 \\ \{k-1 \mid g_k(n-k), g_k(n-2k)\} & \text{other cases} \end{cases} .$$

Proof. **Case (a)** $n \leq k-1$. By definition,

$$g_k(0) = \{ \mid \} = 0$$

$$g_k(1) = \{g_k(0) \mid \} = \{0 \mid \} = 1$$

(...)

$$g_k(k-1) = \{g_k(k-2) \mid \} = \{k-2 \mid \} = k-1.$$

Case (b) $k \leq n \leq 2k-1$. We already know that $g_k(0) = 0, g_k(1) = 1, \dots, g_k(k-1) = k-1$. Therefore, by definition (and domination),

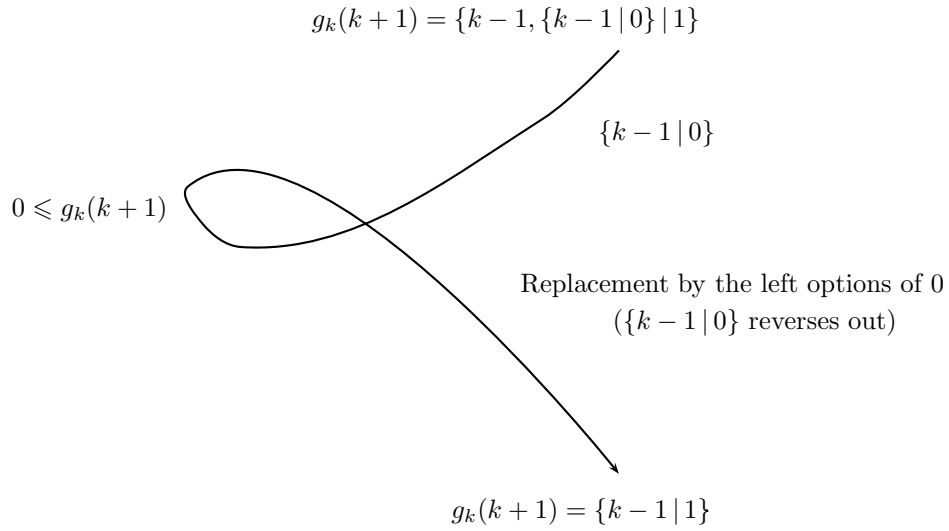
$$g_k(k) = \{k-1 \mid 0\}$$

$$g_k(k+1) = \{k-1, \{k-1 \mid 0\} \mid 1\}$$

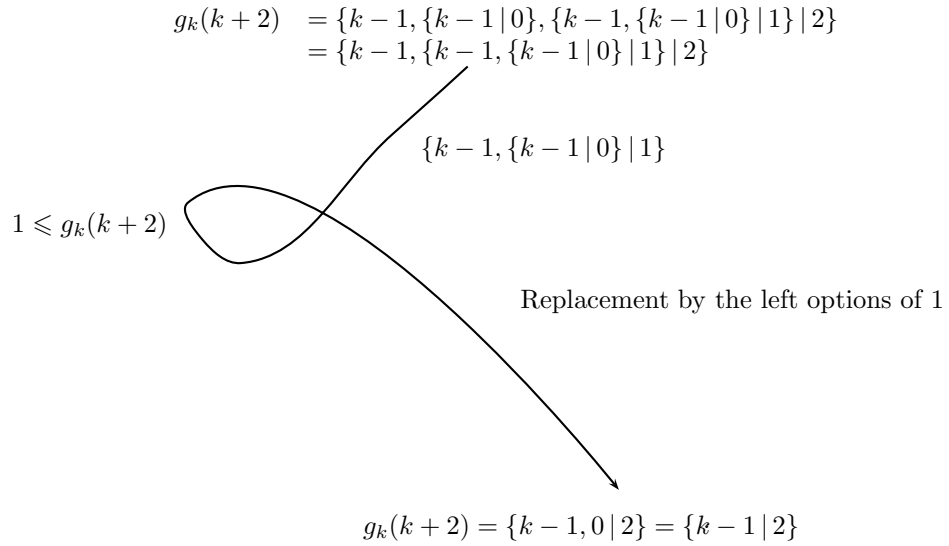
$$g_k(k+2) = \{k-1, \{k-1 \mid 0\}, \{k-1, \{k-1 \mid 0\} \mid 1\} \mid 2\}$$

(:).

We can use reversibility arguments:



Similarly,



In general, for $0 \leq j \leq k-1$,

$$g_k(k+j) = \{k-1, g_k(k), g_k(k+1), \dots, g_k(k+j-1) | j\}$$

and

$g_k(k)$ reverses out through 0;

$g_k(k+1)$ reverses through 1 to 0 which is dominated by $k-1$;

($\dot{}$)
 $g_k(k + j - 1)$ reverses through $j - 1$ to $j - 2$ which is dominated by $k - 1$.

The reversibility effects are justified by the inequality

$$\{k - 1, g_k(k), g_k(k + 1), \dots, g_k(k + j - 1) | j\} \geq j - 1$$

We can conclude that the property is true for $k \leq n \leq 2k - 1$.

Case (c) $2k \leq n \leq 3k - 1$. We have,

$$\begin{aligned} g_k(2k) &= \{k - 1, (k - 1) * | 0, \{k - 1 | 0\}\} \\ g_k(2k + 1) &= \{k - 1, (k - 1)*, g_k(2k) | 1, \{k - 1 | 1\}\} \\ g_k(2k + 2) &= \{k - 1, (k - 1)*, g_k(2k), g_k(2k + 1) | 2, \{k - 1 | 2\}\} \\ &(\dots) \end{aligned}$$

As the previous cases, it is easy to check that only the left options $k - 1$ and $(k - 1)*$ don't reverse. In fact, in general, for $0 \leq j \leq k - 1$,

$$g_k(2k + j) = \{k - 1, (k - 1)*, g_k(2k), g_k(2k + 1), \dots, g_k(2k + j - 1) | j, \{k - 1 | j\}\}$$

and

$g_k(2k)$ reverses out through 0;

$g_k(2k + 1)$ reverses through 1 to 0 which is dominated by $k - 1$;

($\dot{}$)
 $g_k(2k + j - 1)$ reverses through $j - 1$ to $j - 2$ which is dominated by $k - 1$.

The reversibility effects are justified by the inequality

$$\{k - 1, (k - 1)*, g_k(2k), g_k(2k + 1), \dots, g_k(2k + j - 1) | j, \{k - 1 | j\}\} \geq j - 1.$$

We can conclude that the property is true for $2k \leq n \leq 3k - 1$.

Case d) Other cases. In the other cases, also $(k - 1)*$ reverses. This is true because, in these cases, we have

$$\{k - 1, (k - 1) * | g_k(n - k), g_k(n - 2k)\} \geq k - 1.$$

We can see that, in the game

$$\{k - 1, (k - 1) * | g_k(n - k), g_k(n - 2k)\} + 1 - k,$$

if Right begins, Right loses. This happens because the Left option $k - 1$ is available in the games $g_k(n - k)$ and $g_k(n - 2k)$. □

Now, we are able to prove a kind of “dilation” theorem.

Theorem 2. Consider $n \geq 0$ and $k \geq 1$.

1. If $n \leq k - 1$, $g_k(n) = n$.
2. If $n > k - 1$, we obtain $g_k(n)$ from $g_1(n)$ as indicated: consider $i \in \{0, \dots, k - 1\}$ such that $n \equiv i \pmod{k}$. Let G be the game $g_1(\lfloor \frac{n}{k} \rfloor)$ (the form of the game according to its initial definition) and J the game constructed from G executing the following:
 - (a) Add $k - 1$ to the games $G^L, G^{RL}, G^{RRL}, \dots$
 - (b) Add i to the games $G^R, G^{RR}, G^{RRR}, \dots$ not affected by the first step.

We have $g_k(n) = J$.

Proof. The theorem is compatible with Lemma 5 because adding $k - 1$ to the Left options of the game $g_1(\lfloor \frac{n}{k} \rfloor)$ generates exactly the same Left options for $g_k(n)$ indicated in the Lemma 5. So, we just have to analyze the Right options.

Just the induction step is non-trivial. Consider the game

$$g_k(n + 1) = \{g_k(0), \dots, g_k(n) \mid g_k(n + 1 - k), g_k(n + 1 - 2k)\}.$$

By induction, we have to add $k - 1$ and i in the games $g_1(\lfloor \frac{n+1-k}{k} \rfloor)$ and $g_1(\lfloor \frac{n+1-2k}{k} \rfloor)$ where $n + 1 - 2k \equiv n + 1 - k \equiv i \pmod{k}$.

But this is exactly the same as adding $k - 1$ and i in the Right options of $g_1(\lfloor \frac{n+1}{k} \rfloor)$. This is true because the Right options of $g_1(\lfloor \frac{n+1}{k} \rfloor)$ are $g_1(\lfloor \frac{n+1}{k} \rfloor - 1) = g_1(\lfloor \frac{n+1-k}{k} \rfloor)$ and $g_1(\lfloor \frac{n+1}{k} \rfloor - 2) = g_1(\lfloor \frac{n+1-2k}{k} \rfloor)$ and $n + 1 \equiv i \pmod{k}$. \square

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