

Symbolic dynamics and chaotic synchronization

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Abstract: Chaotic communications schemes based on synchronization aim to provide security over the conventional communication schemes. Symbolic dynamics based on synchronization methods has provided high quality synchronization [5]. Symbolic dynamics is a rigorous way to investigate chaotic behavior with finite precision and can be used combined with information theory [13]. In previous works we have studied the kneading theory analysis of the Duffing equation [3] and the symbolic dynamics and chaotic synchronization in coupled Duffing oscillators [2] and [4]. In this work we consider the complete synchronization of two identical coupled unimodal and bimodal maps. We relate the synchronization with the symbolic dynamics, namely, defining a distance between the kneading sequences generated by the map iterates in its critical points and defining n -symbolic synchronization. We establish the synchronization in terms of the topological entropy of two unidirectional or bidirectional coupled piecewise linear unimodal and bimodal maps. We also give numerical simulations with coupled Duffing oscillators that exhibit numerical evidence of the n -symbolic synchronization.

Keywords: Chaotic synchronization, Symbolic dynamics, Symbolic synchronization, Kneading theory.

1. Introduction

Synchronization of two or more systems, is a process wherein the systems adjust a given property of their motion to a common behaviour, due to coupling or forcing. Chaos synchronization is the agreement or correlation of different chaotic processes in time. A great deal of researches has been undertaken on this subject, since the pioneering works of Pecora and Carroll ([9], [10]). Various types of synchronization have been studied. This includes complete synchronization, phase synchronization, lag synchronization, generalized synchronization, anticipated synchronization, and so on [1]. The coupled systems might be identical or different, the coupling might be unidirectional, (master-slave or drive-response), or bidirectional (mutual coupling) and the driving force might be deterministic or stochastic [11].

In this paper we relate the synchronization with symbolic dynamics, namely, defining a distance between the kneading sequences generated by the map iterates in its critical points and defining n -symbolic synchronization. We will see that this distance is a good measure of synchronization, since it decreases to zero, as the coupling parameter increases and the oscillators begin to synchronize. Moreover we establish in two theorems the synchronization in terms of the topological entropy of two unidirectional or bidirectional coupled piecewise linear unimodal and bimodal maps. The n -symbolic synchronization is numerically showed for Duffing oscillators.

2. The Synchronization Problem

Consider two coupled systems $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$. The two identical systems are in complete synchronization, if the difference $z_n = y_n - x_n$ converges to zero, as $n \rightarrow \infty$. The coupling may be unidirectional

$$\begin{cases} x_{n+1} = f(x_n) \\ y_{n+1} = f(y_n) + k[f(x_n) - (y_n)] \\ z_{n+1} = (1-k)[f(y_n) - f(x_n)], \end{cases} \quad (1)$$

or bidirectional

$$\begin{cases} x_{n+1} = f(x_n) - k[f(x_n) - (y_n)] \\ y_{n+1} = f(y_n) + k[f(x_n) - (y_n)] \\ z_{n+1} = (1-2k)[f(y_n) - f(x_n)], \end{cases} \quad (2)$$

where k is the coupling parameter. In [9] and [10] it is establish that this kind of synchronization can be achieved provided that all the Lyapunov exponents are negative.

Motivated by the fact that any transitive m -modal map with positive topological entropy and growth number $\pm s$, is topologically semi-conjugated to a piecewise linear map with slope $\pm s$ everywhere, see [8], we will study the synchronization of unimodal and bimodal piecewise linear maps.

2.1. Coupling two unimodal piecewise linear maps

Consider the tent map $f_s : [0,1] \rightarrow [0,1]$ defined by $f_s(x) = sx - s + 2$,

if $0 \leq x \leq 1 - 1/s$ and $f_s(x) = s - sx$, if $1 - 1/s \leq x \leq 1$. This map f_s can be written as

$$f_s(x) = sx - s + 2 + 2\theta(x)(s - sx - 1) \quad (3)$$

with $\theta(x) = 1$, if $0 \leq x \leq 1 - 1/s$ and $\theta(x) = 0$, if $1 - 1/s \leq x \leq 1$.

Note that, there exist a one-to-one correspondence between the images of the map θ , $\{0,1\}$, and the usual alphabet $A=\{L, R\}$, associated to the unimodal map. With the kneading theory we construct symbolic sequences associated with the critical points of the map, see [3], [7] and [8]. Noticing that, as the coupled systems became synchronized, the number of equal initial symbols in the kneading sequences grows (see Table 1), we are led to the following.

Definition 2.1. Let $S_z = S_{z_1} S_{z_2} \dots S_{z_n} \dots$, be a symbolic sequence, associated to the orbit of a point z of the map f , using in position i a symbol S_{z_i} belonging to some alphabet A . Define a distance between two sequences $S_x = S_{x_1} S_{x_2} \dots S_{x_n} \dots$ and $S_y = S_{y_1} S_{y_2} \dots S_{y_n} \dots$ by $d(S_x, S_y) = e^{-n}$, where $n \triangleq \min\{n \geq 1 : S_{x_n} \neq S_{y_n}\}$.

Definition 2.2. In the conditions of the previous definition, two sequences $S_x = S_{x_1} S_{x_2} \dots S_{x_n} \dots$ and $S_y = S_{y_1} S_{y_2} \dots S_{y_n} \dots$ are said n -symbolic synchronized iff $\exists n \in \mathbb{N} : d(S_x, S_y) \leq e^{-n/2}$.

Note that $d(S_x, S_y) = e^{-n}$, means that, $S_{x_i} = S_{y_i}$ ($1 \leq i < n$) and $n = \min\{n \geq 1 : S_{x_n} \neq S_{y_n}\}$. For the unimodal map f_s , we used $\theta(x)$ to represent the symbolic sequence S_x . In this case, the symbol $S_{x_i} = \theta(x_i)$, in the position i of S_x , can only be 0 or 1. As we have $x_{n+1} = f(x_n)$, we must consider $\theta(x_n)$ and $\theta(x_{n+1})$.

Theorem 2.1. Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two identical coupled systems, with f given by (3) and $1 < s \leq 2$. Let h be the topological entropy of (3) ($h = \log s$) and $k \in [0, 1]$ the coupling parameter. If $\theta(y)$ and $\theta(x)$ are n -symbolic synchronized, then

- 1) The unidirectional coupled systems (1) are synchronized if $k > 1 - e^{-h}$.
- 2) The bidirectional coupled systems (2) are synchronized if

$$(1/2)(1 - e^{-h}) < k < (1/2)(1 + e^{-h}). \tag{4}$$

Proof. Attending to (1) and (3), we have for the unidirectional case

$z_{n+1} = (1 - k)[sy_n + 2\theta(y_n)(s - sy_n - 1) - sx_n - 2\theta(x_n)(s - sx_n - 1)]$. From the considered hypothesis, it follows that, for n sufficiently large, $\theta(y_n) = \theta(x_n) = \theta$, and $z_{n+1} = (1 - k)(1 - 2\theta)sz_n$.

Then, $z_{n+m} = [(1 - k)(1 - 2\theta)s]^m z_n$. We may write $z_{n+m} = q^m z_n$, with $q = (1 - k)(1 - 2\theta)s$. So, we have $\lim_{m \rightarrow \infty} q^m = 0$, iff $|q| < 1$. From the definition of θ , $|1 - 2\theta| = 1$, then, to have complete synchronization of the unidirectionally coupled maps it suffices $k > 1 - s^{-1}$, i.e., $k > 1 - e^{-h}$.

On the other hand, attending to (2) and (3), we have for the bidirectional case $z_{n+1} = (1 - 2k)[sy_n + 2\theta(y_n)(s - sy_n - 1) - sx_n - 2\theta(x_n)(s - sx_n - 1)]$. From the considered hypothesis, it follows that, for n large enough,

$\theta(y_n) = \theta(x_n) = \theta$, and $z_{n+1} = (1 - 2k)(1 - 2\theta)sz_n$.

Then, $z_{n+m} = [(1 - 2k)(1 - 2\theta)s]^m z_n$. So, we have $\lim_{m \rightarrow \infty} z_{n+m} = 0$, iff $|1 - 2k| |1 - 2\theta| s < 1$. From the definition of θ , $|1 - 2\theta| = 1$, then, to have complete synchronization of the bidirectionally coupled maps it suffices (4). □

2.2. Coupling two bimodal piecewise linear maps

Consider the bimodal piecewise linear map $f_{s,r} : [0,1] \rightarrow [0,1]$, with slopes $\pm s$, defined by

$$f_{s,r} = \begin{cases} -sx + 1 & \text{if } 0 \leq x < c_1 \\ sx + r - 1 & \text{if } c_1 \leq x < c_2 \\ -sx + s & \text{if } c_2 \leq x \leq 1 \end{cases}$$

with $r = (3 + s)/2 - s(c_1 + c_2)$ and critical points $c_1 = (2 - r)/2s$ and $c_2 = (1 + s - r)/2s$, see [7]. Recall that any transitive bimodal map is topologically semi-conjugated to such a map. The map $f_{s,r}$ can be written as

$$f_{s,r}(x) = -sx + 1 + \theta_{c_1}(x)(2sx + r - 2) + \theta_{c_2}(y)(-2sx + s - r - 1), \quad (5)$$

with
$$\theta_{c_i}(x) = \begin{cases} 0, & \text{if } 0 \leq x < c_i \\ 1, & \text{if } c_i \leq x \leq 1 \end{cases} \quad (i = 1, 2).$$

In this case, we may define $\theta(x) = \theta_{c_1}(x) + \theta_{c_2}(x)$, i.e.,

$$\theta(x) = \begin{cases} 0 & \text{if } 0 \leq x < c_1 \\ 1 & \text{if } c_1 \leq x < c_2 \\ 2 & \text{if } c_2 \leq x \leq 1 \end{cases}$$

Note the existence of a one-to-one correspondence between the images of the map θ , $\{0, 1, 2\}$, and the usual alphabet associated to the bimodal map $A = \{L, M, R\}$, see [3], [8] and [6]. For the bimodal map $f_{s,r}$, the symbol $S_{x_i} = \theta(x_i)$, in the position i of $S_x = \theta(x)$, can be 0, 1, or 2. Now, we can state a theorem for bimodal maps similar to the one to unimodal maps. The conclusions are the same, although for a different local map f .

Theorem 2.2. Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two identical coupled systems, with f given by (5) and $1 < s \leq 2$. Let h be the topological entropy of (5) ($h = \log s$) and $k \in [0, 1]$ the coupling parameter. If $\theta_{c_i}(x)$ and $\theta_{c_i}(y)$ for $i=1,2$, are n -symbolic synchronized, then

- 1) The unidirectional coupled systems (1) are synchronized if $k > 1 - e^{-h}$.
- 2) The bidirectional coupled systems (2) are synchronized if $(1/2)(1 - e^{-h}) < k < (1/2)(1 + e^{-h})$.

Prof. Attending to (1) and (5), we have for the unidirectional case

$$z_{n+1} = (1 - k) \{ [-sy_n + 1 + \theta_{c_1}(y_n)(2sy_n + r - 2) + \theta_{c_2}(y_n)(-2sy_n + s - r + 1)] - [-sx_n + 1 + \theta_{c_1}(x_n)(2sx_n + r - 2) + \theta_{c_2}(x_n)(-2sx_n + s - r + 1)] \}.$$

From the considered hypothesis, it follows that, for n large enough,

$$\theta_{c_1}(y_n) = \theta_{c_1}(x_n) = \theta_{c_1} \text{ and } \theta_{c_2}(y_n) = \theta_{c_2}(x_n) = \theta_{c_2} \text{ and}$$

$$z_{n+1} = (1-k)[-1-2(\theta_{c_1} - \theta_{c_2})]sz_n.$$

Then, $z_{n+m} = [(1-k)[-1-2(\theta_{c_1} - \theta_{c_2})]s]^m z_n$. We may write $z_{n+m} = q^m z_n$, with $q = (1-k)[-1-2(\theta_{c_1} - \theta_{c_2})]s$. So, we have $\lim_{m \rightarrow \infty} z_{n+m} = 0$, iff $|q| = |1-k| |-1-2(\theta_{c_1} - \theta_{c_2})| s < 1$. Attending that,

$$\theta_{c_1} - \theta_{c_2} = \begin{cases} 0 & \text{if } x < c_1 \quad \text{or} \quad c_2 \leq x \\ 1 & \text{if } c_1 \leq x < c_2 \end{cases}$$

then, $q = \begin{cases} -(1-k)s & \text{if } x < c_1 \quad \text{or} \quad c_2 \leq x \\ (1-k)s & \text{if } c_1 \leq x < c_2 \end{cases}$.

So, $|q| = (1-k)s$, and to have complete synchronization it suffices, $k > 1 - s^{-1}$, i.e., $k > 1 - e^{-h}$.

For the bidirectional case and attending to (2) and (5), we have

$$z_{n+1} = (1-2k)\{-sy_n + 1 + \theta_{c_1}(y_n)(2sy_n + r - 2) + \theta_{c_2}(y_n)(-2sy_n + s - r + 1) - [-sx_n + 1 + \theta_{c_1}(x_n)(2sx_n + r - 2) + \theta_{c_2}(x_n)(-2sx_n + s - r + 1)]\}.$$

From the hypothesis, $\theta_{c_1}(y_n) = \theta_{c_1}(x_n) = \theta_{c_1}$ and $\theta_{c_2}(y_n) = \theta_{c_2}(x_n) = \theta_{c_2}$,

for n large enough. Then, $z_{n+m} = [(1-2k)[-1+2(\theta_{c_1} - \theta_{c_2})]s]^m z_n$. So,

$\lim_{m \rightarrow \infty} z_{n+m} = 0$, iff $|1-2k| |-1+2(\theta_{c_1} - \theta_{c_2})| s < 1$. As in the previous case, $|-1+2(\theta_{c_1} - \theta_{c_2})| = 1$, so the complete synchronization of the bidirectionally coupled maps happens if (6) is verified. □

3. Duffing application

Consider two identical bidirectionally coupled Duffing oscillators, [12].

$$\begin{cases} x''(t) = x(t) - x^3(t) - \alpha x'(t) + k[y(t) - x(t)] + \beta \cos(\omega t) \\ y''(t) = y(t) - y^3(t) - \alpha y'(t) - k[y(t) - x(t)] + \beta \cos(\omega t) \end{cases} \quad (7)$$

We consider parameter values for which each uncoupled ($k = 0$) oscillator exhibits a chaotic behavior, so if they synchronize, that will be a chaotic synchronization. A basic tool is to do an appropriate Poincaré section. In our case, we did a section defined by $y = 0$, since it is transversal to the flow, it contains all fixed points and captures most of the interesting dynamics. In a previous work we have found in the parameter plane (α, β) regions \mathbb{U} and \mathbb{B} where the first return Poincaré map behaves like a unimodal map and like a bimodal map respectively, see [4].

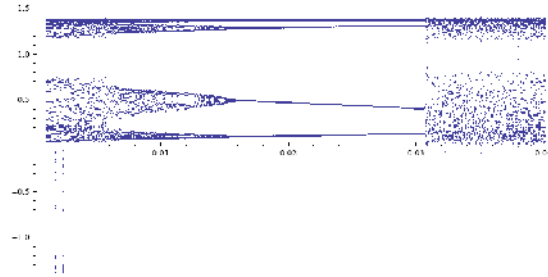


Figure 1. Bifurcation diagram for x_n as a function of $k \in [0.001, 0.04]$, and parameter values, in the unimodal region.

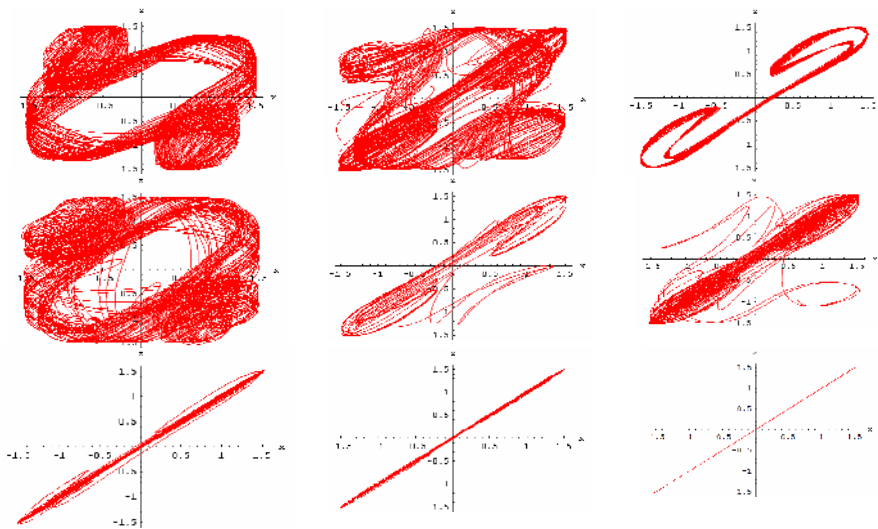


Figure 2. x versus y in the bimodal case for $(\alpha, \beta) = (0.5, 0.719)$ for increasing k .

Fig. 1 shows the bifurcation diagram for x_n as a function of $k \in [0.001, 0.04]$, for parameter values for which the Poincaré map behaves like a unimodal.

We computed the kneading sequences, the kneading determinant and the topological entropy for some values of the parameters (α, β) , see [2] and [4]. The synchronization will occur when $x = y$. See in Fig.2 the evolution of x versus y for $k=0.008, 0.014, 0.016, 0.018, 0.022, 0.023, 0.048, 0.061, 0.097$.

4. Symbolic Dynamics

From Table 1 we verify that, as the coupling parameter k grows and the two coupled maps begin to synchronize, the number of initial equal symbols in the x and y symbolic sequences, grows also and the distance previously

defined, decreases. We have verified the same situation for others values of the (α, β) parameters. This gives numerical evidence of the quality of the synchronization between the two oscillators.

Kneading sequences for x and y	k	n
$S_x : \text{RLRRRLRLRRRLRRRLRRRLRLRLRRRLRL}$	0.00601	
$S_y : \text{RLRRRLRRRLRRRLRRRLRRRLRRRLRRRL}$		7
$S_x : \text{RLRRRRRLRRRLRRRLRRRLRRRLRRRLRL}$	0.05	
$S_y : \text{RLRRRLRLRLRRRLRRRLRRRLRRRLRRRL}$		5
$S_x : \text{RLRRRLRLRLRRRLRRRLRRRLRLRLRLRL}$	0.06	
$S_y : \text{RLRRRLRLRRRLRRRLRRRLRLRLRLRL}$		9
$S_x : \text{RLRRRLRLRLRRRLRRRLRRRLRRRLRRRL}$	0.064	
$S_y : \text{RLRRRLRLRLRLRRRLRRRLRRRLRRRLRR}$		11
$S_x : \text{RLRRRLRLRRRLRRRLRLRLRRRLRRRL}$	0.065	
$S_y : \text{RLRRRLRLRRRLRRRLRRRLRLRLRLRL}$		17
$S_x : \text{RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL}$	0.07	
$S_y : \text{RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL}$		30
$S_x : \text{RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL}$	0.08	
$S_y : \text{RLRRRLRLRRRLRRRLRLRRRLRRRLRRRL}$		30

Table 1: Symbolic sequences S_x and S_y , with n identical initial symbols, for the bidirectional coupled Duffing oscillators, for some values of the coupling parameter k , with $(\alpha, \beta) = (0.4, 0.3578)$ fixed.

The number of symbols used in information transmission depends on the limitation of the channel capacity [13]. Only finitely precise synchronization is achievable when oscillators are joined by channel with limited capacity. On this basis, we established in Definition 2.1 and 2.2. finitely precise synchronization with symbolic dynamics.

5. Conclusions

Doing Poincaré sections in the system of two coupled Duffing oscillators, we obtained regions U and B where the Poincaré map behaves like a unimodal and bimodal map respectively. By a result from Milnor and Thurston [8] it is

known that every m -modal map $f: I = [a, b] \subset \mathbb{R} \rightarrow I$, with growth rate s and

positive topological entropy $h_{top}(f)$ ($\log s = h_{top}(f)$) is topologically semi-conjugated to a $p+1$ piecewise linear map T , with $p \leq m$, defined on the interval $J = [0, 1]$, with slope $\pm s$ everywhere and $h_{top}(f) = h_{top}(T) = \log s$. Motivated by this fact, we study the synchronization of piecewise linear unimodal and bimodal maps and obtained the results in theorems 2.1 and 2.2 expressing the synchronization interval in terms of the topological entropy of the coupled maps. From the previous theorems we may also verify that the bidirectional coupled maps begin to synchronize at half the coupling

parameter for the unidirectional case. We relate the synchronization with symbolic dynamics, namely, defining a distance between the kneading sequences and defining n -symbolic synchronization.

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