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PERIODIC SOLUTIONS FOR SOME FULLY NONLINEAR FOURTH ORDER DIFFERENTIAL EQUATIONS

Feliz Minhós

Department of Mathematics. School of Sciences and Technology. University of Évora
Research Center in Mathematics and Applications of University of Évora (CIMA-UE)
Rua Romão Ramalho, 59
7000-671 Évora. PORTUGAL

Abstract. In this paper we present sufficient conditions for the existence of solutions to the periodic fourth order boundary value problem

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u'(x), u''(x), u'''(x)) \\ u^{(i)}(a) &= u^{(i)}(b), \quad i = 0, 1, 2, 3, \end{aligned}$$

for $x \in [a, b]$, and $f : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ a continuous function. To the best of our knowledge it is the first time where this type of general nonlinearities is considered in fourth order equations with periodic boundary conditions.

The difficulties in the odd derivatives are overcome due to the following arguments: the control on the third derivative is done by a Nagumo-type condition and the bounds on the first derivative are obtained by lower and upper solutions, not necessarily ordered.

By this technique, not only it is proved the existence of a periodic solution, but also, some qualitative properties of the solution can be obtained.

1. **Introduction.** In this paper we get sufficient conditions for the existence of solutions of the periodic fourth order boundary value problem composed by the fully equation

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)) \tag{1}$$

for $x \in [a, b]$, and $f : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ a continuous function, and the boundary conditions

$$u^{(i)}(a) = u^{(i)}(b), \quad i = 0, 1, 2, 3. \tag{2}$$

Higher order periodic boundary value problems have been studied by several authors in last decades, using different types of arguments and techniques. However, as far as we know, these methods were not able to deal with fully nonlinear differential equations, in particular with both odd derivatives. As examples, see [7, 8, 16] for variational methods, [1, 2, 3, 5, 14, 15, 17, 18] for second and higher order equations, and [12] for a linear n^{th} order periodic problem.

The above difficulties are overcome applying lower and upper solutions technique and topological degree, like it is suggested in [4, 6, 11]. In short, the method is based in three key points:

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- A Nagumo-type condition ([13]) assumed on the nonlinearity, which is useful to obtain an *a priori* estimation for the third derivative and to define an open and bounded set where the topological degree is well defined.
- A new kind of definition of lower and upper solutions, required to deal with the absence of a definite order for the first derivatives. In fact lower and upper solutions, not necessarily ordered, are associated by translations with some ordered functions, used to define the sets where the solutions and their first derivatives are contained.
- An adequate auxiliary and perturbed problem, where the truncations and the homotopy are extended to some boundary conditions of mixed type, allowing an invertible linear operator and the evaluation of the Leray-Schauder degree.

Remark that, by [9], for periodic second order problems, the existence of lower and upper solutions is not sufficient to obtain a solution in the absence of Nagumo condition. At the moment this is an open issue for higher order boundary value problems.

This method is particularly well adapted to boundary value problems (see [10]) because it provides not only the existence of a periodic solution, without monotone assumptions on the nonlinearity, but also, some qualitative properties of this solution and its derivatives.

The last section contains an example where the location part is used to prove that the solution is nontrivial and to obtain some of its values.

2. **Definitions and *a priori* bounds.** In this section it is precise how to control the odd derivatives. For the third derivative it is obtained an *a priori* estimate via a Nagumo-type growth condition.

Definition 2.1. A continuous function $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ is said to satisfy the Nagumo-type condition in

$$E = \left\{ (x, y_0, y_1, y_2, y_3) \in [a,b] \times \mathbb{R}^4 : \gamma_i(x) \le y_i \le \Gamma_i(x), i = 0,1,2 \right\}, \tag{3}$$

with $\gamma_i(x)$ and $\Gamma_i(x)$ continuous functions such that,

$$\gamma_i(x) \le \Gamma_i(x), \text{ for } i = 0,1,2 \text{ and every } x \in [a,b], \tag{4}$$

if there exists a real continuous function $h_E : [0,+\infty[\to]0,+\infty[$ such that

$$|f(x, y_0, y_1, y_2, y_3)| \le h_E(|y_3|), \quad \forall (x, y_0, y_1, y_2, y_3) \in E, \tag{5}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{6}$$

Lemma 2.2. *([11], Lemma 2) Suppose that f satisfies Nagumo-type condition as in Definition 2.1. Then, there exists $R > 0$ (depending on γ_2, Γ_2 and h_E) such that every solution $u(x)$ of (1) with*

$$\gamma_i(x) \le u^{(i)}(x) \le \Gamma_i(x),$$

for $i = 0,1,2$ and $x \in [a,b]$ verifies $\|u'''\|_\infty < R$.

Non ordered upper and lower solutions will be useful to define such set E and the strips where solutions and the derivatives will be located.

In the periodic case, lower and upper solutions and their first derivatives are translate to some auxiliary functions, which are well ordered:

Definition 2.3. *The function $\alpha \in C^4([a,b])$ is a lower solution of problem (1)–(2) if:*

(i) $\alpha^{(iv)}(x) \geq f(x, \alpha_0(x), \alpha_1(x), \alpha''(x), \alpha'''(x))$
with

$$\alpha_0(x) := \alpha(x) - \alpha(a) - ||\alpha'||_\infty \, (x-a), \tag{7}$$
$$\alpha_1(x) := \alpha'(x) - ||\alpha'||_\infty; \tag{8}$$

(ii) $\alpha'(a) \geq \alpha'(b)$, $\alpha''(a) = \alpha''(b)$, $\alpha'''(a) \geq \alpha'''(b)$.

The function $\beta \in C^4([a,b])$ is an upper solution of problem (1)–(2) if:

(i) $\beta^{(iv)}(x) \leq f(x, \beta_0(x), \beta_1(x), \beta''(x), \beta'''(x))$
with

$$\beta_0(x) := \beta(x) - \beta(a) + ||\beta'||_\infty \, (x-a), \tag{9}$$
$$\beta_1(x) := \beta'(x) + ||\beta'||_\infty; \tag{10}$$

(ii) $\beta'(a) \leq \beta'(b)$, $\beta''(a) = \beta''(b)$, $\beta'''(a) \leq \beta'''(b)$.

3. **Existence of periodic solutions.** Next theorem provides an existence and location result for problem (1)–(2) in presence of non ordered lower and upper solutions.

Theorem 3.1. *Assume that α, $\beta \in C^4([a,b])$ are lower and upper solutions of (1)–(2), respectively, such that*

$$\alpha''(x) \leq \beta''(x), \; \forall x \in [a,b]. \tag{11}$$

Let $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function verifying a Nagumo-type condition in

$$E_* := \{(x, y_0, y_1, y_2, y_3) : \alpha_j \leq y_j \leq \beta_j, \; j = 0, 1, \; \alpha'' \leq y_2 \leq \beta''\}$$

and

$$f(x, \alpha_0, \alpha_1, y_2, y_3) \geq f(x, y_0, y_1, y_2, y_3) \geq f(x, \beta_0, \beta_1, y_2, y_3), \tag{12}$$

for fixed $(x, y_2, y_3) \in [a,b] \times \mathbb{R}^2$, $\alpha_j \leq y_j \leq \beta_j$, $j = 0, 1$.
Then problem (1)–(2) has at least a periodic solution $u \in C^4([a,b])$ such that

$$\alpha_j(x) \leq u^{(j)}(x) \leq \beta_j(x), \; j = 0, 1, \; \text{and } \alpha''(x) \leq u''(x) \leq \beta''(x),$$

for $x \in [a,b]$.

Remark 1. These associated functions, $\alpha_j(x) \leq 0 \leq \beta_j$, $j = 0, 1$, are not unique. In Definition 2.3 we consider the optimal cases in the sense that $\alpha_0(a) = 0 = \beta_0(a)$. So, the solution found by Theorem 3.1 satisfies additionally $u(a) = u(b) = 0$.
However it is always possible to find "well ordered" functions $\alpha_j(x)$ and $\beta_j(x)$. The "easiest way" is by translation $\alpha_0(x) - k$ and $\beta_0(x) + k$, for some $k > 0$ such that $\alpha_0(x) \leq \beta_0(x)$ and $\alpha_1(x) \leq \beta_1(x)$. Another way, for example, is integrating (11) in $[x, b]$ to obtain

$$\alpha_0(x) := \alpha(x) - \alpha(b) - ||\alpha'||_\infty \, (b-x),$$
$$\alpha_1(x) := \beta'(x) - ||\beta'||_\infty,$$

$$\beta_0(x) := \beta(x) - \beta(b) + ||\beta'||_\infty \, (b-x),$$
$$\beta_1(x) := \alpha'(x) + ||\alpha'||_\infty.$$

Note that, in this case, lower and upper solutions must be defined like a pair of functions.

Remark 2. The above remark shows that the order between lower and upper solutions and their first derivatives is not important to guarantee the solvability of the periodic problem (1)–(2).

Remark 3. The proof that the solution is nontrivial can be obtained by the location part of the theorem. For example, if the lower solution is chosen such that $\alpha''(x) > 0$, or the upper solution such that $\beta''(x) < 0$, for some $x \in [a, b]$, then the periodic solution of (1)–(2) is nontrivial.
The last section contains an example where this issue is applied.

Proof. By Remark 1 we can consider, for $\lambda \in [0, 1]$, the homotopic and truncated auxiliary equation

$$\begin{aligned} u^{(iv)}(x) &= \lambda f(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), u'''(x)) \\ &\quad + u''(x) - \lambda \delta_2(x, u''(x)), \end{aligned} \tag{13}$$

where the continuous functions $\delta_j, \delta_2 : \mathbb{R}^2 \to \mathbb{R}$ $(j = 0, 1)$ are given by

$$\delta_j(x, y_j) = \begin{cases} \beta_j(x) & , \quad y_j > \beta_j(x) \\ y_j & , \quad \alpha_j(x) \leq y_j \leq \beta_j(x) \\ \alpha_j(x) & , \quad y_j < \alpha_j(x) \end{cases}, \; j = 0, 1,$$

with α_j defined in (7) and (8), and β_j in (9) and (10),

$$\delta_2(x, y_2) = \begin{cases} \beta''(x) & , \quad y_2 > \beta''(x) \\ y_2 & , \quad \alpha''(x) \leq y_2 \leq \beta''(x) \\ \alpha''(x) & , \quad y_2 < \alpha''(x), \end{cases}$$

coupled with the boundary conditions

$$\begin{aligned} u(a) &= \lambda \, \eta_0 \left(u(b) \right) \\ u'(a) &= \lambda \, \eta_1 \left(u'(b) \right) \\ u^{(i)}(a) &= u^{(i)}(b), \; i = 2, 3, \end{aligned} \tag{14}$$

where $\eta_k : \mathbb{R} \to \mathbb{R}$ $(k = 0, 1)$ are defined by

$$\eta_k(u^{(k)}(b)) = \begin{cases} \beta_k(a) & , \quad u^{(k)}(b) > \beta_k(a) \\ u^{(k)}(b) & , \quad \alpha_k(a) \leq u^{(k)}(b) \leq \beta_k(a) \\ \alpha_k(a) & , \quad u^{(k)}(b) < \alpha_k(a) \end{cases}, \; k = 0, 1. \tag{15}$$

Take $r_2 > 0$ such that, for every $x \in [a, b]$,

$$-r_2 < \alpha''(x) \leq \beta''(x) < r_2, \tag{16}$$

$$f(x, \alpha_0(x), \alpha_1(x), \alpha''(x), 0) - r_2 - \alpha''(x) < 0$$

and

$$f(x, \beta_0(x), \beta_1(x), \beta''(x), 0) + r_2 - \beta''(x) > 0. \tag{17}$$

Step 1: *Every solution u of the problem (13)-(14) satisfies, in $[a, b]$,*

$$|u''(x)| < r_2, \; |u'(x)| < r_1 \; \text{ and } \; |u(x)| < r_0,$$

independently of $\lambda \in [0, 1]$, with r_2 given above, $r_1 = \xi + r_2(b - a)$, where

$$\xi := \max \left\{ ||\alpha'||_\infty - \alpha'(a), \; \beta'(a) + ||\beta'||_\infty \right\}, \tag{18}$$

and $r_0 = \xi(b - a) + r_2(b - a)^2$.

Let u be a solution of problem (13)-(14).

Assume, by contradiction, that there exists $x \in [a,b]$ such that $|u''(x)| \geq r_2$. In the case $u''(x) \geq r_2$ define

$$\max_{x\in[a,b]} u''(x) = u''(x_0) \ (\geq r_2 > 0).$$

If $x_0 \in \,]a,b[\,$, then $u'''(x_0) = 0$ and $u^{(iv)}(x_0) \leq 0$. By (12), (16) and (17), for $\lambda \in [0,1]$, the following contradiction holds

$$\begin{aligned} 0 &\geq u^{(iv)}(x_0) \\ &= \lambda f(x_0, \delta_0(x_0,u(x_0)), \delta_1(x_0,u(x_0)), \beta''(x_0), 0) + u''(x_0) - \lambda\beta''(x_0) \\ &\geq \lambda\left[f(x_0,\beta_0(x_0),\beta_1(x_0),\beta''(x_0),0) + r_2 - \beta''(x_0)\right] + u''(x_0) - \lambda r_2 > 0. \end{aligned}$$

If $x_0 = a$ then

$$\max_{x\in[a,b]} u''(x) = u''(a) \quad (\geq r_2 > 0).$$

By (14),

$$0 \geq u'''(a) = u'''(b) \geq 0, \tag{19}$$

$u'''(a) = 0$ and $u^{(iv)}(a) \leq 0$. From the computations above, with x_0 replaced by a, a similar contradiction is achieved.

The case $x_0 = b$ is analogous. Thus, $u''(x) < r_2$, for every $x \in [a,b]$. In a similar way it can be proved that the inequality $u''(x) > -r_2$ holds for every $x \in [a,b]$. And so,

$$|u''(x)| < r_2, \ \forall x \in [a,b].$$

By integration, (14) and (15) the following relations are achieved:

$$\begin{aligned} u'(x) &< u'(a) + r_2(x-a) = \lambda\ \eta_1\left(u'(b)\right) + r_2(x-a) \\ &\leq \lambda\beta_1(a) + r_2(b-a) \leq \beta_1(a) + r_2(b-a) \\ &\leq \beta'(a) + ||\beta'||_\infty + r_2(b-a) \leq \xi + r_2(b-a) \end{aligned}$$

and

$$\begin{aligned} u'(x) &> u'(a) - r_2(x-a) \geq \lambda\alpha_1(a) - r_2(b-a) \geq \alpha_1(a) - r_2(b-a) \\ &\geq \alpha'(a) - ||\alpha'||_\infty - r_2(b-a) \geq -\xi - r_2(b-a). \end{aligned}$$

Therefore, for ξ given by (18),

$$|u'(x)| < r_1 \text{ with } r_1 := \xi + r_2(b-a).$$

The same technique leads, by (7) and (9), to

$$|u(x)| < r_0 \text{ where } r_0 := \xi(b-a) + r_2(b-a)^2.$$

Step 2: *There exists $R > 0$ such that every solution u of the problem (13)-(14) satisfies*

$$|u'''(x)| < R, \quad \forall x \in [a,b],$$

independently of $\lambda \in [0,1]$.

Consider, for r_0, r_1 and r_2 defined above, the set

$$E_1 := \left\{(x,y_0,y_1,y_2,y_3) \in [a,b]\times\mathbb{R}^4 : -r_i \leq y_i \leq r_i, \ i = 0,1,2\right\}$$

and the function $F_\lambda : E_1 \to \mathbb{R}$ given by

$$\begin{aligned} F_\lambda\left(x,y_0,y_1,y_2,y_3\right) &= \lambda f(x,\delta_0(x,y_0),\delta_1(x,y_1),\delta_2(x,y_2),y_3) \\ &\quad + y_2 - \lambda\delta_2(x,y_2). \end{aligned} \tag{20}$$

As f satisfies a Nagumo-type condition in E_*, consider the function $\varphi \in C\left(\mathbb{R}_0^+, [k, +\infty[\right)$, for some $k > 0$, such that (5) and (6) hold with E replaced by E_*. Thus, for $(x, y_0, y_1, y_2, y_3) \in E_1$, we have, by (13) and (16),

$$F_\lambda\left(x, y_0, y_1, y_2, y_3\right) \leq \varphi(|y_3|) + 2\ r_2.$$

For $\overline{\varphi}(w) := \varphi(w) + 2r_2$ then

$$\int_0^{+\infty} \frac{s}{\overline{\varphi}(s)} ds = \int_0^{+\infty} \frac{s}{\varphi(s) + 2r_2} ds \geq \frac{1}{1 + \frac{2r_2}{k}} \int_0^{+\infty} \frac{s}{\varphi(s)} ds,$$

and so $\overline{\varphi}(w)$ verifies (6). Therefore, F_λ satisfies Nagumo condition in E_1 with $\varphi(w)$ replaced by $\overline{\varphi}(w)$, independently of λ.

Defining

$$\gamma_i(x) := -r_i, \quad \Gamma_i(x) := r_i, \quad i = 0, 1, 2,$$

the assumptions of Lemma 2.2 are satisfied with E replaced by E_1. So there exists $R > 0$, depending only on r_i, $\ i = 0, 1, 2$, and φ, such that $|u'''(x)| < R$, for every $x \in [a, b]$. Therefore, the *a priori* bound $|u'''(x)| < R$ is independent of λ.

Step 3: *For $\lambda = 1$ problem (13)-(14) has a solution $u_1(x)$.*

Define the operators

$$\mathcal{L} : C^4\left([a, b]\right) \subset C^3\left([a, b]\right) \longmapsto C\left([a, b]\right) \times \mathbb{R}^4$$

and, for $\lambda \in [0, 1]$,

$$\mathcal{N}_\lambda : C^3\left([a, b]\right) \longmapsto C\left([a, b]\right) \times \mathbb{R}^4$$

by

$$\mathcal{L}u = (u^{(iv)}, u(a), u'(a), u''(a), u'''(a))$$

and

$$\mathcal{N}_\lambda\ u = \begin{pmatrix} \lambda f(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), u'''(x)) + u''(x) - \lambda\delta_2(x, u''(x)), \\ \lambda\ \eta_0\left(u(b)\right), \lambda\ \eta_1\left(u'(b)\right),\ u''(b),\ u'''(b) \end{pmatrix}.$$

As $\mathcal{L}$ has a compact inverse then it can be considered the completely continuous operator

$$\mathcal{T}_\lambda : \left(C^3\left([a, b]\right), \mathbb{R}\right) \longmapsto \left(C^3\left([a, b]\right), \mathbb{R}\right)$$

defined by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

For R given by Step 2, consider the set

$$\Omega = \left\{y \in C^3\left([a, b]\right) : \left\|y^{(i)}\right\|_\infty < r_i,\ i = 0, 1, 2, \|y'''\|_\infty < R\right\}.$$

By Steps 1 and 2, for every u solution of (13)-(14), $u \notin \partial\Omega$ and so the degree $d(I - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$. By the invariance under homotopy

$$d(I - \mathcal{T}_0, \Omega, 0) = d(I - \mathcal{T}_1, \Omega, 0).$$

Since the equation $\mathcal{T}_0(x) = x$, equivalent to the problem

$$\left\{\begin{array}{c} u^{(iv)}(x) - u''(x) = 0 \\ u(a) = 0 \\ u'(a) = 0 \\ u''(a) = u''(b) \\ u'''(a) = u'''(b) \end{array}\right.,$$

has only the null solution then, by degree theory,

$$d(I-\mathcal{T}_0,\Omega,0)=\pm 1.$$

So, the equation $\mathcal{T}_1(x)=x$ has at least a solution, that is the corresponding problem

$$\begin{cases} u^{(iv)}(x)=f(x,\delta_0(x,u(x)),\delta_1(x,u'(x)),\delta_2(x,u''(x)),u'''(x))+u''(x)-\delta_2(x,u''(x)), \\ u(a)=\eta_0\left(u(b)\right) \\ u'(a)=\eta_1\left(u'(b)\right) \\ u''(a)=u''(b) \\ u'''(a)=u'''(b) \end{cases}$$

has at least a solution $u_1(x)$ in Ω.

Step 4: *$u_1(x)$ is a solution of (1)–(2)*

This solution $u_1(x)$ will be also a solution of problem (1)–(2) since it verifies

$$\begin{array}{ccccc} \alpha_0(x) & \leq & u_1(x) & \leq & \beta_0(x), \\ \alpha_1(x) & \leq & u_1'(x) & \leq & \beta_1(x), \\ \alpha''(x) & \leq & u_1''(x) & \leq & \beta''(x),\ \forall x\in[a,b]. \end{array}$$

Suppose, by contradiction, that there is $x\in[a,b]$ such that

$$\alpha''(x)>u_1''(x)$$

and define

$$\min_{x\in[a,b]}\left[u_1''(x)-\alpha''(x)\right]:=u_1''(x_1)-\alpha''(x_1)<0.$$

If $x_1\in\,]a,b[$ then $u_1'''(x_1)=\alpha'''(x_1)$ and $u_1^{(iv)}(x_1)\geq\alpha^{(iv)}(x_1)$. Therefore, by (12) and Definition 2.3, we have the contradiction

$$\begin{array}{rcl} 0 & \leq & u_1^{(iv)}(x_1)-\alpha^{(iv)}(x_1) \\ & \leq & f(x_1,\delta_0(x_1,u_1(x_1)),\delta_1(x_1,u_1'(x_1)),\alpha''(x_1),\alpha'''(x_1)) \\ & & +u_1''(x_1)-\alpha''(x_1)-f(x_1,\alpha_0(x_1),\alpha_1(x_1),\alpha''(x_1),\alpha'''(x_1)) \\ & \leq & u_1''(x_1)-\alpha''(x_1)<0. \end{array} \tag{21}$$

If $x_1=a$ then

$$\min_{x\in[a,b]}\left[u_1''(x)-\alpha''(x)\right]:=u_1''(a)-\alpha''(a)<0.$$

By Definition 2.3 (ii),

$$0\leq u_1'''(a)-\alpha'''(a)\leq u_1'''(b)-\alpha'''(b)\leq 0$$

and, therefore,

$$u_1'''(a)=\alpha'''(a),\quad u_1^{(iv)}(a)\geq\alpha^{(iv)}(a).$$

Arguing as in (21) a similar contradiction is achieved. Analogously for $x_1=b$. Then

$$\alpha''(x)\leq u_1''(x),\ \forall x\in[a,b].$$

Applying the same arguments, it can be proved that $u_1''(x)\leq\beta''(x)$, for every $x\in[a,b]$, and so

$$\alpha''(x)\leq u_1''(x)\leq\beta''(x),\ \forall x\in[a,b]. \tag{22}$$

Integrating (22) in $[a,x]$, for the first inequality it is obtained, by (15) and (8),

$$\begin{aligned} u_1'(x) &\geq u_1'(a)+\alpha'(x)-\alpha'(a) \\ &\geq \alpha_1(a)+\alpha'(x)-\alpha'(a) \\ &= \alpha'(x)-\|\alpha'\|_\infty \\ &= \alpha_1(x). \end{aligned}$$

For the second inequality in (22), we have by (15) and (10),

$$\begin{aligned} u_1'(x) &\leq u_1'(a)+\beta'(x)-\beta'(a) \\ &\leq \beta_1(a)+\beta'(x)-\beta'(a) \\ &= \beta'(x)+\|\beta'\|_\infty \\ &= \beta_1(x). \end{aligned}$$

Therefore,

$$\alpha_1(x) \leq u_1'(x) \leq \beta_1(x),\ \forall x \in [a,b]. \tag{23}$$

By integration of (23) in $[a,x]$, by (15) and (7) one has

$$\begin{aligned} u_1(x) &\geq u_1(a)+\alpha(x)-\alpha(a)-\|\alpha'\|_\infty(x-a) \\ &\geq \alpha_0(a)+\alpha(x)-\alpha(a)-\|\alpha'\|_\infty(x-a) \\ &= \alpha(x)-\alpha(a)-\|\alpha'\|_\infty(x-a) \\ &= \alpha_0(x). \end{aligned}$$

With the same technique,

$$\begin{aligned} u_1(x) &\leq u_1(a)+\beta(x)-\beta(a)+\|\beta'\|_\infty(x-a), \\ &\leq \beta_1(a)+\beta(x)-\beta(a)+\|\beta'\|_\infty(x-a) \\ &= \beta(x)-\beta(a)+\|\beta'\|_\infty(x-a) \\ &= \beta_0(x) \end{aligned}$$

and, therefore,

$$\alpha_0(x) \leq u_1(x) \leq \beta_0(x),\ \forall x \in [a,b].$$

□

4. **Example.** Consider the fourth order differential equation

$$u^{(4)}(x) = -u(x)^3 - u'(x)^5 + 150\ u''(x) + \sqrt[3]{u'''(x)+1} - 130, \tag{24}$$

for $x \in]0,1[$, with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0,1,2,3.$$

The continuous functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} \alpha(x) &= -\frac{x^4}{4}+\frac{x^3}{2}-\frac{x^2}{4}-x-1, \\ \beta(x) &= \frac{x^4}{12}-\frac{x^3}{6}+x^2-\frac{x}{42}+1 \end{aligned}$$

are lower and upper solutions, respectively, of problem (24)–(2) verifying (11) with

$$\begin{aligned}
\alpha_0(x) &= -\frac{x^4}{4}+\frac{x^3}{2}-\frac{x^2}{4}-2.048\ x,\\
\alpha_1(x) &= -x^3+\frac{3}{2}x^2-\frac{1}{2}x-2.048,\\
\beta_0(x) &= \frac{x^4}{12}-\frac{x^3}{6}+x^2+\frac{x}{12},\\
\beta_1(x) &= \frac{x^3}{3}-\frac{x^2}{2}+2x+\frac{1}{12}.
\end{aligned}$$

The function

$$f\left(x,y_0,y_1,y_2,y_3\right)=-y_0^3-y_1^5+150\ y_2+\sqrt[3]{y_3+1}-130$$

is continuous, verifies Nagumo-type conditions (5) and (6) in

$$E=\left\{\begin{array}{c}(x,y_0,y_1,y_2,y_3)\in[0,1]\times\mathbb{R}^4:\\ \alpha_j\leq y_j\leq\beta_j,\ j=0,1,\ \alpha''\leq y_2\leq\beta''\end{array}\right\}$$

and satisfies (12).

By Theorem 3.1 there is a solution $u(x)$ of problem (24)–(2), such that

$$\begin{aligned}
-\tfrac{x^4}{4}+\tfrac{x^3}{2}-\tfrac{x^2}{4}-2.048\ x &\leq u(x) \leq \tfrac{x^4}{12}-\tfrac{x^3}{6}+x^2+\tfrac{x}{12},\\
-x^3+\tfrac{3}{2}x^2-\tfrac{1}{2}x-2.048 &\leq u'(x) \leq \tfrac{x^3}{3}-\tfrac{x^2}{2}+2x+\tfrac{1}{12},
\end{aligned}$$

and

$$-3x^2+3x-0.5\leq u''(x)\leq x^2-x+2, \tag{25}$$

for $x\in[0,1]$.

The proof that this solution is a nontrivial periodic solution can be done applying the location part of Theorem 3.1 in two ways:

The only constant solution allowed by the equation (24) would be $\sqrt[3]{129}$, but this value is not in the set

$$[\alpha_0,\beta_0]:=\{y(x):\alpha_0(x)\leq y(x)\leq\beta_0(x),x\in[0,1]\}.$$

Another possible argument is to remark that the solution $u(x)$ can not be a constant because, by (25),

$$0<\frac{1}{4}\leq u''\left(\frac{1}{2}\right).$$

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E-mail address: fminhos@uevora.pt