Stability and Unobstructedness of Syzygy Bundles

Laura Costa[a,1], Pedro Macias Marques[a,b,2], Rosa María Miró-Roig[a,1]

[a] *Departament d'Àlgebra i Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E – 08007 Barcelona, Spain*

[b] *Departamento de Matemática, Universidade de Évora, Rua Romão Ramalho, 59, P – 7000–671 Évora, Portugal*

Abstract

It is a longstanding problem in Algebraic Geometry to determine whether the syzygy bundle $E_{d_1,\dots,d_n}$ on $\mathbb{P}^N$ defined as the kernel of a general epimorphism

$$\phi : \mathcal{O}(-d_1) \oplus \cdots \oplus \mathcal{O}(-d_n) \longrightarrow \mathcal{O}$$

is (semi)stable. In this note we restrict our attention to the case of syzygy bundles $E_{d,n}$ on $\mathbb{P}^N$ associated to n generic forms $f_1, \dots, f_n \in K[X_0, X_1, \dots, X_N]$ of the same degree d. Our first goal is to prove that $E_{d,n}$ is stable if $N + 1 \leq n \leq \binom{d+2}{2} + N - 2$ and $(N, n, d) \neq (2, 5, 2)$. This bound improves, in general, the bound $n \leq d(N + 1)$ given by G. Hein in [2], Appendix A.

In the last part of the paper, we study moduli spaces of stable rank $n-1$ vector bundles on $\mathbb{P}^N$ containing syzygy bundles. We prove that if $N + 1 \leq n \leq \binom{d+2}{2} + N - 2$, $N \neq 3$ and $(N, n, d) \neq (2, 5, 2)$, then the syzygy bundle $E_{d,n}$ is unobstructed and it belongs to a generically smooth irreducible component of dimension $n\binom{d+N}{N} - n^2$, if $N \geq 4$, and $n\binom{d+2}{2} + n\binom{d-1}{2} - n^2$, if $N = 2$.

Keywords: Moduli spaces, stability, vector bundles

Contents

[1] Partially supported by MTM2007-61104.

[2] Partially supported by Fundação para a Ciência e Tecnologia, under grant SFRH/BD/27929/2006, and by CIMA – Centro de Investigação em Matemática e Aplicações, Universidade de Évora.

1. Introduction

Let $R = K[X_0, X_1, \dots, X_N]$, $\mathbb{P}^N = \mathrm{Proj}(R)$ be the N-dimensional projective space over an algebraically closed field K of characteristic 0. Set $\mathfrak{m} = (X_0, X_1, \dots, X_N)$. It is a classical and difficult problem in Algebraic Geometry, as well in Commutative Algebra, to understand the syzygy bundle $E_{d_1,\dots,d_n}$ on $\mathbb{P}^N$ defined as the kernel of a general epimorphism

$$\phi = (f_1, \dots, f_n) : \mathcal{O}_{\mathbb{P}^N}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^N}(-d_n) \longrightarrow \mathcal{O}_{\mathbb{P}^N},$$

where $(f_1, \dots, f_n) \subset R$ is an $\mathfrak{m}$-primary ideal, and f_i is an homogeneous polynomial of degree $d_i = deg(f_i)$. We would like to know the cohomology of $E_{d_1,\dots,d_n}$, its splitting type on a generic line, and whether it is simple, exceptional or stable. In particular, we are led to consider the following problem:

Problem 1.1. Let $f_1, \dots, f_n \in R$ be a family of $\mathfrak{m}$-primary homogeneous polynomials of degree $deg(f_i) = d_i$, $1 \leq i \leq n$. Let $E_{d_1,\dots,d_n}$ be the syzygy bundle on $\mathbb{P}^N$ associated to $f_1, \dots, f_n$. Is $E_{d_1,\dots,d_n}$ a (semi)stable vector bundle on $\mathbb{P}^N$?

In the last few years, Problem 1.1 has been extensively studied and surprisingly only a few partial results have been obtained. We refer to [2] and [3] for precise information. In this paper we restrict our attention to the case $d_1 = d_2 = \ldots = d_n = d$ and we address the following problem, which should be viewed as a particular case of Problem 1.1.

Problem 1.2. Let $f_1, \dots, f_n \in R$ be a family of $\mathfrak{m}$-primary forms of the same degree d and let $E_{d,n}$ be the syzygy bundle associated to them. Is $E_{d,n}$ a (semi)stable vector bundle on $\mathbb{P}^N$?

Note that since $(f_1, \dots, f_n)$ is an $\mathfrak{m}$-primary ideal, we always have $N+1 \leq n \leq \binom{d+N}{N}$. Problem 1.2 turns out to be true for a set of n general $\mathfrak{m}$-primary forms of the same degree d, provided

- d and N are arbitrary and $n = \binom{N+d}{N}$ [11];
- d and N are arbitrary and $n = N + 1$ [1];
- d and N are arbitrary and $n \leq d(N + 1)$ [2].

The first goal of this paper is to give an affirmative answer to Problem 1.2 for the case of n general $\mathfrak{m}$-primary forms of the same degree d, provided

(1) $N = 2$ and $3 \leq n \leq \binom{d+2}{2}$ (see Theorem 3.5);

(2) $N \geq 2$ and $N + 1 \leq n \leq \binom{d+2}{2} + N - 2$ (see Theorem 4.2).

We want to point out that the result (1) was announced by Brenner in [2] but no proof was included and the result (2) strongly improves, in general, the bound $N + 1 \leq n \leq d(N + 1)$ given by G. Hein in [3], Theorem A1.

In the last section of this work, we also study the unobstructedness of stable syzygy bundles on $\mathbb{P}^N$. There exists a beautiful theorem due to Maruyama establishing the existence of the moduli space $M = M(r; c_1, \dots, c_s)$ of rank r, stable vector bundles E on $\mathbb{P}^N$ with fixed Chern classes $c_i(E) = c_i$ for $i = 1, \dots, s = \min(r, N)$ (see [9] and [10]). Unfortunately, in general, very little is known about its local and global structure. In this paper we prove that points $[E_{d,n}]$ of $M = M(r; c_1, \dots, c_s)$ parameterizing stable syzygy bundles $E_{d,n}$ on $\mathbb{P}^N$, $N \neq 3$, $N+1 \leq n \leq \binom{d+2}{2} + N - 2$ and $(N, n, d) \neq (2, 5, 2)$, are smooth and we compute the dimension of the irreducible component of $M = M(r; c_1, \dots, c_s)$ passing through $[E_{d,n}]$ in terms of d, n and N (see Theorem 4.4).

Notation: We work over an algebraically closed field K of characteristic zero. We set $\mathbb{P}^N = \mathrm{Proj}(K[X_0, X_1, \dots, X_N])$ and $\mathfrak{m} = (X_0, X_1, \dots, X_N)$. Given coherent sheaves E and F on $\mathbb{P}^N$, we write $h^i(E)$ (resp. $ext^i(E, F)$) to denote the dimension of the ith cohomology group $H^i(\mathbb{P}^N, E) = H^i(E)$ (resp. ith Ext group $Ext^i(E, F)$) as a K-vector space.

For any $x \in \mathbb{R}$, we set $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid x \leq n\}$.

2. Stability of syzygy bundles. Generalities

In this section we recall the notion of (semi)stability of torsion free sheaves on projective spaces and its basic properties. We review the useful cohomological characterization of (semi)stability due to Hoppe as well as its applications to the problem of determining the (semi)stability of syzygy bundles.

Let us start by fixing the notation and some basic definitions.

Definition 2.1. Let E be a torsion free sheaf on $\mathbb{P}^N$ and set

$$\mu(E) := \frac{c_1(E)}{rk(E)}.$$

The sheaf E is said to be *semistable* in the sense of Mumford-Takemoto if

$$\mu(F) \leq \mu(E)$$

for all non-zero subsheaves $F \subset E$ with $rk(F) < rk(E)$; if strict inequality holds then E is *stable.*

Note that for rank r, torsion free sheaves E on $\mathbb{P}^N$, with $(c_1(E), r) = 1$, the concepts of stability and semistability coincide.

Notation 2.2. Let E be a rank r vector bundle on $\mathbb{P}^N$. We set $E_{norm} := E(k_E)$ where k_E is the unique integer such that $c_1(E(k_E)) \in \{-r+1, \dots, 0\}$.

For rank 2 vector bundles on $\mathbb{P}^N$ we have the following useful stability criterion: a rank 2 vector bundle E on $\mathbb{P}^N$ is stable (resp. semistable) if and only if $H^0(\mathbb{P}^N, E_{norm}) = 0$ (resp. $H^0(\mathbb{P}^n, E_{norm}(-1)) = 0$). This criterion was generalized by Hoppe in [4], Lemma 2.6. We have

Proposition 2.3. *Let E be a rank r vector bundle on $\mathbb{P}^N$. The following hold:*

(a) If $H^0(X, (\wedge^q E)_{norm}) = 0$ for $1 \leq q \leq r-1$, then E is stable.

(b) $H^0(X, (\wedge^q E)_{norm}(-1)) = 0$ *for* $1 \leq q \leq r-1$ *if and only if* E *is semistable.*

Remark 2.4. The conditions of Proposition 2.3(a) are not necessary. The simplest counterexamples are the nullcorrelation bundles E on $\mathbb{P}^N$ (N odd) where by a nullcorrelation bundle we mean a rank $N-1$ vector bundle E on $\mathbb{P}^N$ (N odd) defined by an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow \Omega^1_{\mathbb{P}^N}(1) \longrightarrow E \longrightarrow 0.$$

E is a stable vector bundle of rank $N-1$ on $\mathbb{P}^N$ (N odd) and $H^0(\mathbb{P}^N, (\wedge^2 E)_{norm}) \neq 0$ (in fact, $(\wedge^2 E)_{norm}$ contains $\mathcal{O}_{\mathbb{P}^N}$ as a direct summand).

Definition 2.5. A *syzygy sheaf* $E_{d_1,\dots,d_n}$ on $\mathbb{P}^N$ is a coherent sheaf defined as the kernel of a morphism

$$\phi : \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-d_i) \xrightarrow{f_1,\dots,f_n} \mathcal{O}_{\mathbb{P}^N},$$

where $f_1, \dots, f_n \in K[X_0, X_1, \dots, X_N]$ are forms of degree $d_i = deg(f_i)$. If $(f_1, \dots, f_n) \subset K[X_0, X_1, \dots, X_N]$ is an $\mathfrak{m}$-primary ideal, this sheaf is locally free, and we call it *syzygy bundle.* When $d_1 = d_2 = \cdots = d_n = d$, we write $E_{d,n}$ instead of $E_{d_1,\dots,d_n}$.

Let $E_{d_1,\dots,d_n}$ be a syzygy sheaf on $\mathbb{P}^N$. By construction, $E_{d_1,d_2,\dots,d_n}$ is a torsion-free sheaf of rank $n-1$, locally free on $\cup_{i=1}^n D_+(f_i) \subset \mathbb{P}^N$. Moreover, we have $c_1(E_{d_1,\dots,d_n}) = d - \sum_{i=1}^n d_i$, where d is the degree of the highest common factor of $f_1, \dots, f_n$ and hence the slope of $E_{d_1,d_2,\dots,d_n}$ is

$$\mu(E_{d_1,d_2,\dots,d_n}) = \frac{d - \sum_{i=1}^n d_i}{n-1}.$$

Note that when $E_{d_1,d_2,\dots,d_n}$ be a syzygy bundle, we get $c_1(E_{d_1,d_2,\dots,d_n}) = -\sum_{i=1}^n d_i$ and

$$\mu(E_{d_1,d_2,\dots,d_n}) = -\frac{\sum_{i=1}^n d_i}{n-1}.$$

Note also that since $(f_1, \dots, f_n)$ is an $\mathfrak{m}$-primary ideal, we have $n \geq N+1$.

In this paper we address problems 1.1 and 1.2. As far as we know, there exist very few contributions to these problems, and we summarize all of them, as well as the techniques that have been used to prove these results.

First of all, we observe that, as an easy application of Hoppe's Theorem, we obtain the following result, which also follows from [1], Theorem 2.7.

Proposition 2.6. *Let* $E_{d,N+1}$ *be the syzygy bundle on* $\mathbb{P}^N$ *associated to* $N+1$ *generic forms of degree* d*. Then,* $E_{d,N+1}$ *is stable.*

Proof. Since stability is preserved by duality, it is enough to check that $F = E^\vee_{d,N+1}$ is stable. According to Proposition 2.3, it is enough to prove that $H^0\left(\mathbb{P}^N, (\wedge^q F)_{norm}\right) = 0$ for $1 \leq q \leq N-1$. First of all, note that since $c_1(\wedge^q F) = \binom{N-1}{q-1}(N+1)q$, we have

$(\wedge^q F)_{norm} = (\wedge^q F)(k_F)$ with $k_F < -dq$. Twisting by $\mathcal{O}_{\mathbb{P}^N}(k_F)$ the qth wedge power of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(d)^{N+1} \longrightarrow F \longrightarrow 0,$$

we get the long exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(k_F) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(k_F) \otimes \mathcal{O}_{\mathbb{P}^N}(d)^{N+1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(k_F) \otimes \wedge^2 \left(\mathcal{O}_{\mathbb{P}^N}(d)^{N+1}\right) \longrightarrow \cdots$$
$$\longrightarrow \mathcal{O}_{\mathbb{P}^N}(k_F) \otimes \wedge^{q-1} \left(\mathcal{O}_{\mathbb{P}^N}(d)^{N+1}\right) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(k_F) \otimes \wedge^q \left(\mathcal{O}_{\mathbb{P}^N}(d)^{N+1}\right) \longrightarrow \wedge^q F(k_F) \longrightarrow 0.$$

Cutting it into short exact sequences, for $2 \leq j \leq q-1$, we get:

$$0 \longrightarrow K_{q+1-i} \longrightarrow \wedge^i \left(\mathcal{O}_{\mathbb{P}^N}(d)^{N+1}\right)(k_F) \longrightarrow K_{q+2-i} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(k_F) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(d+k_F)^{N+1} \longrightarrow K_{q-1} \longrightarrow 0.$$

Since line bundles on $\mathbb{P}^N$ have no intermediate cohomology, taking cohomology on the above exact sequences we obtain

$$h^1(K_1) = h^2(K_2) = \cdots = h^{q-1}(K_{q-1}) = h^q(\mathcal{O}_{\mathbb{P}^N}(k_F)) = 0,$$

where the last equality follows from the fact that $q < N$. On the other hand, since $k_F < -qd$,

$$H^0\left(\mathcal{O}_{\mathbb{P}^N}(k_F) \otimes \wedge^q \left(\mathcal{O}_{\mathbb{P}^N}(d)^{N+1}\right)\right) = H^0\left(\mathcal{O}_{\mathbb{P}^N}(qd+k_F)^{\binom{N+1}{q}}\right) = 0.$$

Putting all together we get that for $1 \leq q \leq N-1$,

$$H^0\left(\mathbb{P}^N, (\wedge^q F)(k_F)\right) = H^0\left(\mathbb{P}^N, (\wedge^q F)_{norm}\right) = 0,$$

which proves that F, and hence $E_{d,N+1}$, is stable. □

Using the fact that the syzygy bundle $E_{d,\binom{d+N}{d}}$ on $\mathbb{P}^N$ is a homogeneous bundle, to prove the stability of $E_{d,\binom{d+N}{d}}$ it is enough to check that the slope of any homogeneous sub-bundle of $E_{d,\binom{d+N}{d}}$ is less than the slope of $E_{d,\binom{d+N}{d}}$. In [11], the author described all the homogenous sub-bundles of $E_{d,\binom{d+N}{d}}$ and she proved

Proposition 2.7. *Let $E_{d,\binom{d+N}{d}}$ be the syzygy bundle on $\mathbb{P}^N$ associated to $\binom{d+N}{d}$ K-linearly independent homogeneous forms of degree d. Then, $E_{d,\binom{d+N}{d}}$ is stable.*

Proof. See [11], Theorem 2.8. □

Using Klyachko results on toric bundles ([5], [6] and [7]), Brenner deduced the following nice combinatoric criterion for the (semi)stability of the syzygy bundle $E_{d_1,\ldots,d_n}$ in the case where the associated forms $f_1, \ldots, f_n$ are all monomials. Indeed, we have

Proposition 2.8. *Let $f_i = {X_0}^{i_0}{X_1}^{i_1}\cdots{X_N}^{i_N}$, $i \in I$, be a set of $\mathfrak{m}$-primary monomials of degree $d_i = \sum_{j=0}^{N} i_j$. Then the syzygy bundle $E_{d_1,\dots,d_n}$ on $\mathbb{P}^N$ associated to the f_i, $i \in I$, is semistable (resp. stable) if and only if for every $J \subsetneq I$, $|J| \geq 2$, the inequality*

$$\frac{d_J - \sum_{i\in J} d_i}{|J|-1} \leq \frac{-\sum_{i\in I} d_i}{|I|-1} \qquad \textit{(resp. <)} \tag{1}$$

holds, where d_J is the degree of the greatest common factor of the f_i, $i \in J$.

Proof. See [2], Proposition 2.2 and Corollary 6.4. $\square$

Example 2.9. (1) If we consider the set $I := \{{X_0}^5,\ {X_1}^5,\ {X_2}^5,\ {X_0}^2{X_1}^2X_2\}$ of $\mathfrak{m}$-primary monomials, inequality (1) is strictly fulfilled for any proper subset $J \subsetneq I$. Therefore the syzygy bundle E associated to I is stable.

(2) If we consider the set $I := \{{X_0}^5,\ {X_1}^5,\ {X_2}^5,\ {X_0}^4X_1\}$ of $\mathfrak{m}$-primary monomials, then for the subset $J := \{{X_0}^5,\ {X_0}^4X_1\}$ inequality (1) is not fulfilled. Therefore the syzygy bundle E associated to I is not stable. In fact, the slope of E is $\mu(E) = -20/3$ and the syzygy sheaf F associated to J is a subsheaf of E with slope $\mu(F) = -6$. Since $\mu(F) \not\leq \mu(E)$, we conclude that E is not stable.

Remark 2.10. (a) Let I be a set of n $\mathfrak{m}$-primary monomials of degree d. It easily follows from the above proposition that the syzygy bundle $E_{d,n}$ on $\mathbb{P}^N$ associated I is (semi)stable if and only if for every subset $J \subset I$ with $k := |J| \geq 2$,

$$(d - d_J)n + d_J - dk > 0 \quad \text{(resp. } \geq 0\text{)}, \tag{2}$$

where d_J is the degree of the greatest common factor of the monomials in J.

(b) If we use the notation $a_{d,j} := -\frac{jd}{j-1}$, inequality (1) can be written

$$\tfrac{d_J}{k-1} + a_{d,k} \leq a_{d,n}.$$

The fact that, once d is fixed, the sequence $(a_{d,j})_{j\geq 2}$ is monotonically increasing will be useful in many arguments.

Due to Proposition 2.8, to decide whether a syzygy bundle on $\mathbb{P}^N$ associated to a set of $\mathfrak{m}$-primary monomials of degree d is semistable or not is a purely combinatorial problem but not yet solved, even when all monomials f_i have the same degree. In [2], Question 7.8, Brenner asks

Question 2.11. Does there exist for every d and every $n \leq \binom{N+d}{N}$ a family of n monomials in $K[X_0,\dots,X_N]$ of degree d such that their syzygy bundle is semistable?

Remark 2.12. For $N = 1$, $d = 9$ and $n = 3$ the answer to this question is negative. In fact, if we consider a family $I := \{X^9,\ Y^9,\ X^\alpha Y^{9-\alpha}\}$, with $\alpha \geq 9-\alpha$, i.e. $\alpha \geq 5$, the subset $J \subset I$ with a greatest common factor of highest degree is $\{X^9,\ X^\alpha Y^{9-\alpha}\}$, its greatest common factor is X^α, but inequality 1 fails, since $(9-\alpha)\cdot 3 + \alpha - 9\cdot 2 = 9 - 2\alpha < 0$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & {X_2}^d & & & & & & \\
 & & & & & X_0{X_2}^{d-1} & & X_1{X_2}^{d-1} & & & & & \\
 & & & & {X_0}^2{X_2}^{d-2} & & X_0X_1{X_2}^{d-2} & & {X_1}^2{X_2}^{d-2} & & & & \\
 & & & \iddots & & \iddots & & \ddots & & \ddots & & & \\
 & & {X_0}^{d-2}{X_2}^2 & & \iddots & & & & \ddots & & {X_1}^{d-2}{X_2}^2 & & \\
 & {X_0}^{d-1}X_2 & & {X_0}^{d-2}X_1X_2 & & \cdots & & \cdots & & X_0{X_1}^{d-2}X_2 & & {X_1}^{d-1}X_2 & \\
{X_0}^d & & {X_0}^{d-1}X_1 & & {X_0}^{d-2}{X_1}^2 & & \cdots & & {X_0}^2{X_1}^{d-2} & & X_0{X_1}^{d-1} & & {X_1}^d
\end{array}
$$

Figure 1: Monic monomials in $K\left[X_0, X_1, X_2\right]$ of degree d.

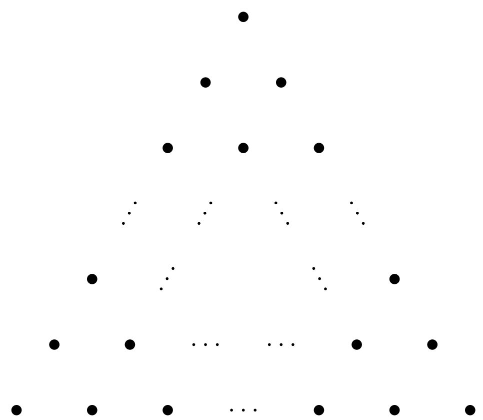


Figure 2: Simpler sketch of the monomials in figure 1.

3. The case $N = 2$. Stability

The goal of this section is to solve Problem 1.2 and Question 2.11, when $N = 2$. As a main tool, we use the criterion given in Proposition 2.8. Let us sketch our strategy. Monic monomials in $K[X_0, X_1, X_2]$ of a given degree d can be sketched in a triangle as in figure 1. For the sake of simplicity, we can sketch the triangle in figure 1 as shown in figure 2. Once arranged in this manner, the closer two monomials are, the higher the degree of their greatest common factor is.

Proposition 3.1. *For any integer $3 \leq n \leq 18$ and any integer $d \geq n-2$ there is a set $I_{d,n}$ of n $\mathfrak{m}$-primary monomials in $K[X_0, X_1, X_2]$ of degree d such that the corresponding syzygy bundle $E_{d,n}$ is stable.*

Proof. We apply Proposition 2.8 and Remark 2.10. So, for any integer $3 \leq n \leq 18$ and any integer $d \geq n - 2$ we explicitly give a set $I_{d,n}$ of n $\mathfrak{m}$-primary monomials in $K[X_0, X_1, X_2]$ of degree d such that for every subset $J \subset I_{d,n}$ with $k := |J| \geq 2$, we have

$$(d - d_J)n + d_J - dk > 0, \tag{3}$$

where d_J is the degree of the greatest common factor of the monomials in J.

Let e_0, e_1 and e_2 be integers such that

$$e_0 + e_1 + e_2 = d, \qquad e_0 \geq e_1 \geq e_2 \qquad \text{and} \qquad e_0 - e_2 \leq 1.$$

In particular, $e_0 = \left\lceil \frac{d}{3} \right\rceil$. Let $i := \left\lceil \frac{d}{2} \right\rceil$.

In cases $3 \leq n \leq 8$, we consider the sets

$$I_{d,3} := \left\{X_0{}^d,\ X_1{}^d,\ X_2{}^d\right\},$$

$$I_{d,4} := \left\{X_0{}^d,\ X_1{}^d,\ X_2{}^d,\ X_0{}^{e_0} X_1{}^{e_1} X_2{}^{e_2}\right\},$$

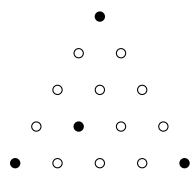
$$I_{d,5} := \left\{X_0{}^d,\ X_1{}^d,\ X_2{}^d,\ X_0{}^{e_0} X_1{}^{e_1} X_2{}^{e_2},\ X_1{}^{d-i} X_2{}^i\right\},$$

$$I_{d,6} := \{{X_0}^d,\, {X_1}^d,\, {X_2}^d,\, {X_0}^{e_0}{X_1}^{d-e_0},\, {X_0}^{d-e_0}{X_2}^{e_0},\, {X_1}^{e_0}{X_2}^{d-e_0}\},$$

$$I_{d,7} := \{{X_0}^d,\, {X_1}^d,\, {X_2}^d,\, {X_0}^{e_0}{X_1}^{e_1}{X_2}^{e_2},\, {X_0}^{e_0}{X_1}^{d-e_0},\, {X_0}^{d-e_0}{X_2}^{e_0},\, {X_1}^{e_0}{X_2}^{d-e_0}\},$$

and

$$\begin{aligned} I_{d,8} := \{&{X_0}^d,\, {X_1}^d,\, {X_2}^d,\, {X_0}^{e_0}{X_1}^{e_1}{X_2}^{e_2},\\ &{X_0}^{e_0+e_1}{X_1}^{e_2},\, {X_0}^{e_2}{X_2}^{e_0+e_1},\, {X_1}^{e_0+e_1}{X_2}^{e_2},\, {X_1}^{e_0}{X_2}^{e_1+e_2}\}. \end{aligned}$$



$I_{4,4}$

In case $n = 9$, we shall look at two cases separately: if $d = 8$, we consider the set

$$\begin{aligned} I_{8,9} := \{&{X_0}^8,\, {X_1}^8,\, {X_2}^8,\, {X_0}^3{X_1}^3{X_2}^2,\, {X_0}^6{X_1}^2,\\ &{X_0}^2{X_2}^6,\, {X_0}^5{X_2}^3,\, {X_1}^6{X_2}^2,\, {X_1}^3{X_2}^5\}; \end{aligned}$$

if $d \neq 8$, let $d = 3m + t$, with $0 \leq t < 3$, and for each $l \in \{1, 2\}$, let $i_l := lm + \min(l, t)$ and consider the set

$$\begin{aligned} I_{d,9} := \{&{X_0}^d,\, {X_1}^d,\, {X_2}^d,\, {X_0}^{i_1}{X_1}^{d-i_1},\, {X_0}^{i_2}{X_1}^{d-i_2},\\ &{X_0}^{d-i_1}{X_2}^{i_1},\, {X_0}^{d-i_2}{X_2}^{i_2},\, {X_1}^{i_1}{X_2}^{d-i_1},\, {X_1}^{i_2}{X_2}^{d-i_2}\}. \end{aligned}$$

In case $n = 10$, we shall distinguish two cases: if $d = 9$, we consider the set

$$\begin{aligned} I_{9,10} := \{&{X_0}^9,\, {X_1}^9,\, {X_2}^9,\, {X_0}^3{X_1}^3{X_2}^3,\, {X_0}^6{X_1}^3,\, {X_0}^3{X_1}^6,\\ &{X_0}^6{X_2}^3,\, {X_0}^3{X_2}^6,\, {X_1}^6{X_2}^3,\, {X_1}^3{X_2}^6\}; \end{aligned}$$

if $d \neq 9$, let $d = 5m + t$, where $0 \leq t < 5$, and for each $l \in \{1, 2, 3, 4\}$, let $i_l := lm + \min(l, t)$; consider the set

$$\begin{aligned} I_{d,10} := \{&{X_0}^d,\, {X_1}^d,\, {X_2}^d,\, {X_0}^{i_2}{X_1}^{i_1}{X_2}^{d-i_1-i_2},\, {X_0}^{i_4}{X_1}^{d-i_4},\, {X_0}^{i_2}{X_1}^{d-i_2},\\ &{X_0}^{i_3}{X_2}^{d-i_3},\, {X_0}^{i_1}{X_2}^{d-i_1},\, {X_1}^{i_2}{X_2}^{d-i_2},\, {X_1}^{i_4}{X_2}^{d-i_4}\}. \end{aligned}$$

In case $n = 11$, we shall distinguish two cases: if $d = 12$, we consider the set

$$\begin{aligned} I_{12,11} := \{&{X_0}^{12},\, {X_1}^{12},\, {X_2}^{12},\, {X_0}^9{X_1}^3,\, {X_0}^6{X_1}^6,\, {X_0}^3{X_1}^9,\\ &{X_0}^9{X_2}^3,\, {X_0}^6{X_2}^6,\, {X_0}^3{X_2}^9,\, {X_1}^9{X_2}^3,\, {X_1}^6{X_2}^6\}; \end{aligned}$$

if $d \neq 12$, let us write $d = 5m + t$, where $0 \leq t < 5$, and for each $l \in \{1, 2, 3, 4\}$, let $i_l := lm + \min(l, t)$; consider the set

$$\begin{aligned} I_{d,11} := \{&{X_0}^d,\, {X_1}^d,\, {X_2}^d,\, {X_0}^{i_2}{X_1}^{i_1}{X_2}^{d-i_1-i_2},\, {X_0}^{i_4}{X_1}^{d-i_4},\, {X_0}^{i_3}{X_1}^{d-i_3},\\ &{X_0}^{i_2}{X_1}^{d-i_2},\, {X_0}^{i_3}{X_2}^{d-i_3},\, {X_0}^{i_1}{X_2}^{d-i_1},\, {X_1}^{i_2}{X_2}^{d-i_2},\, {X_1}^{i_4}{X_2}^{d-i_4}\}. \end{aligned}$$

In case $n = 12$, we shall distinguish two cases: if $d = 11$, we consider the set

$$I_{11,12} := \{X_0{}^{11},\, X_1{}^{11},\, X_2{}^{11},\, X_0{}^8X_1{}^3,\, X_0{}^8X_2{}^3,\, X_0{}^5X_1{}^2X_2{}^4,\, X_0{}^4X_1{}^4X_2{}^3,$$
$$X_0{}^3X_1{}^8,\, X_0{}^3X_2{}^8,\, X_0{}^2X_1{}^5X_2{}^4,\, X_1{}^8X_2{}^3,\, X_1{}^3X_2{}^8\};$$

if $d \neq 11$, let $d = 4m+t$, where $0 \leq t < 4$, and for each $l \in \{1, 2, 3\}$, let $i_l := lm+\min(l, t)$; consider the set

$$I_{d,12} := \{X_0{}^d,\, X_1{}^d,\, X_2{}^d,\, X_0{}^{i_3}X_1{}^{d-i_3},\, X_0{}^{i_2}X_1{}^{d-i_2},\, X_0{}^{i_1}X_1{}^{d-i_1},\, X_0{}^{i_3}X_2{}^{d-i_3},$$
$$X_0{}^{i_2}X_2{}^{d-i_2},\, X_0{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_2}X_2{}^{d-i_2},\, X_1{}^{i_3}X_2{}^{d-i_3}\}.$$

In case $13 \leq n \leq 15$, let $d = 4m + t$, where $0 \leq t < 4$, and for each $l \in \{1, 2, 3\}$, let $i_l := lm + \min(l, t)$. Consider the sets

$$\begin{aligned}I_{d,13} := \{&X_0{}^d,\, X_1{}^d,\, X_2{}^d,\, X_0{}^{i_2}X_1{}^{d-i_3}X_2{}^{i_3-i_2},\\ &X_0{}^{i_3}X_1{}^{d-i_3},\, X_0{}^{i_2}X_1{}^{d-i_2},\, X_0{}^{i_1}X_1{}^{d-i_1},\\ &X_0{}^{i_3}X_2{}^{d-i_3},\, X_0{}^{i_2}X_2{}^{d-i_2},\, X_0{}^{i_1}X_2{}^{d-i_1},\\ &X_1{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_2}X_2{}^{d-i_2},\, X_1{}^{i_3}X_2{}^{d-i_3}\},\\ I_{d,14} := \{&X_0{}^d,\, X_1{}^d,\, X_2{}^d,\, X_0{}^{i_2}X_1{}^{d-i_3}X_2{}^{i_3-i_2},\, X_0{}^{i_1}X_1{}^{d-i_2}X_2{}^{i_2-i_1},\\ &X_0{}^{i_3}X_1{}^{d-i_3},\, X_0{}^{i_2}X_1{}^{d-i_2},\, X_0{}^{i_1}X_1{}^{d-i_1},\\ &X_0{}^{i_3}X_2{}^{d-i_3},\, X_0{}^{i_2}X_2{}^{d-i_2},\, X_0{}^{i_1}X_2{}^{d-i_1},\\ &X_1{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_2}X_2{}^{d-i_2},\, X_1{}^{i_3}X_2{}^{d-i_3}\},\end{aligned}$$

and

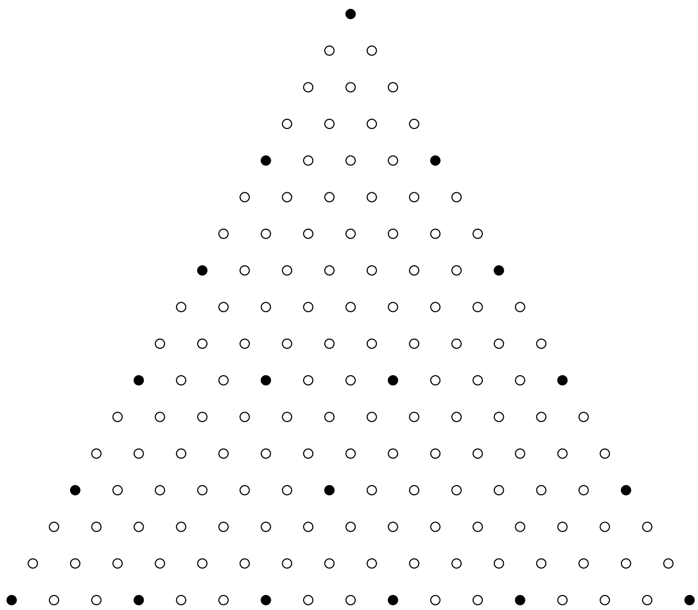
$$\begin{aligned}I_{d,15} := \{&X_0{}^d,\, X_1{}^d,\, X_2{}^d,\\ &X_0{}^{i_2}X_1{}^{d-i_3}X_2{}^{i_3-i_2},\, X_0{}^{i_1}X_1{}^{d-i_2}X_2{}^{i_2-i_1},\, X_0{}^{i_1}X_1{}^{d-i_3}X_2{}^{i_3-i_1},\\ &X_0{}^{i_3}X_1{}^{d-i_3},\, X_0{}^{i_2}X_1{}^{d-i_2},\, X_0{}^{i_1}X_1{}^{d-i_1},\\ &X_0{}^{i_3}X_2{}^{d-i_3},\, X_0{}^{i_2}X_2{}^{d-i_2},\, X_0{}^{i_1}X_2{}^{d-i_1},\\ &X_1{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_2}X_2{}^{d-i_2},\, X_1{}^{i_3}X_2{}^{d-i_3}\}.\end{aligned}$$

In case $16 \leq n \leq 18$, let $d = 5m + t$, where $0 \leq t < 5$, and for each $l \in \{1, 2, 3, 4\}$, let $i_l := lm + \min(l, t)$. Consider the sets

$$\begin{aligned}I_{d,16} := \{&X_0{}^d,\, X_1{}^d,\, X_2{}^d,\, X_0{}^{i_2}X_1{}^{d-i_3}X_2{}^{i_3-i_2},\, X_0{}^{i_4}X_1{}^{d-i_4},\, X_0{}^{i_3}X_1{}^{d-i_3},\, X_0{}^{i_2}X_1{}^{d-i_2},\\ &X_0{}^{i_1}X_1{}^{d-i_1},\, X_0{}^{i_4}X_2{}^{d-i_4},\, X_0{}^{i_3}X_2{}^{d-i_3},\, X_0{}^{i_2}X_2{}^{d-i_2},\, X_0{}^{i_1}X_2{}^{d-i_1},\\ &X_1{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_2}X_2{}^{d-i_2},\, X_1{}^{i_3}X_2{}^{d-i_3},\, X_1{}^{i_4}X_2{}^{d-i_4}\},\\ I_{d,17} := \{&X_0{}^d,\, X_1{}^d,\, X_2{}^d,\, X_0{}^{i_2}X_1{}^{d-i_3}X_2{}^{i_3-i_2},\, X_0{}^{i_2}X_1{}^{d-i_4}X_2{}^{i_4-i_2},\\ &X_0{}^{i_4}X_1{}^{d-i_4},\, X_0{}^{i_3}X_1{}^{d-i_3},\, X_0{}^{i_2}X_1{}^{d-i_2},\, X_0{}^{i_1}X_1{}^{d-i_1},\\ &X_0{}^{i_4}X_2{}^{d-i_4},\, X_0{}^{i_3}X_2{}^{d-i_3},\, X_0{}^{i_2}X_2{}^{d-i_2},\, X_0{}^{i_1}X_2{}^{d-i_1},\\ &X_1{}^{i_1}X_2{}^{d-i_1},\, X_1{}^{i_2}X_2{}^{d-i_2},\, X_1{}^{i_3}X_2{}^{d-i_3},\, X_1{}^{i_4}X_2{}^{d-i_4}\},\end{aligned}$$

and

$$I_{d,18} := \{ {X_0}^d, {X_1}^d, {X_2}^d, {X_0}^{i_2}{X_1}^{d-i_3}{X_2}^{i_3-i_2}, {X_0}^{i_2}{X_1}^{d-i_4}{X_2}^{i_4-i_2}, {X_0}^{i_1}{X_1}^{d-i_3}{X_2}^{i_3-i_1},$$
$$ {X_0}^{i_4}{X_1}^{d-i_4}, {X_0}^{i_3}{X_1}^{d-i_3}, {X_0}^{i_2}{X_1}^{d-i_2}, {X_0}^{i_1}{X_1}^{d-i_1},$$
$$ {X_0}^{i_4}{X_2}^{d-i_4}, {X_0}^{i_3}{X_2}^{d-i_3}, {X_0}^{i_2}{X_2}^{d-i_2}, {X_0}^{i_1}{X_2}^{d-i_1},$$
$$ {X_1}^{i_1}{X_2}^{d-i_1}, {X_1}^{i_2}{X_2}^{d-i_2}, {X_1}^{i_3}{X_2}^{d-i_3}, {X_1}^{i_4}{X_2}^{d-i_4} \}.$$



$I_{16,18}$

For any $3 \leq n \leq 18$ and $d \geq n-2$, we consider the described set $I_{d,n}$ and for any subset $J \subset I_{d,n}$ with $k := |J| \geq 2$, we have to check that inequality (3) is satisfied. We check the case $n = 18$ and we leave the other cases to the reader.

So, assume $n = 18$. In this case we use the fact that no monomial of degree d_J divides a greater number of monomials in $I_{d,n}$ than ${X_0}^{d_J}$.

If $0 < d_J \leq i_1$, the multiples of ${X_0}^{d_J}$ in $I_{d,18}$ are the monomials in the set

$$J := \{ {X_0}^d, {X_0}^{i_2}{X_1}^{d-i_3}{X_2}^{i_3-i_2}, {X_0}^{i_2}{X_1}^{d-i_4}{X_2}^{i_4-i_2}, {X_0}^{i_1}{X_1}^{d-i_3}{X_2}^{i_3-i_1},$$
$$ {X_0}^{i_4}{X_1}^{d-i_4}, {X_0}^{i_3}{X_1}^{d-i_3}, {X_0}^{i_2}{X_1}^{d-i_2}, {X_0}^{i_1}{X_1}^{d-i_1},$$
$$ {X_0}^{i_4}{X_2}^{d-i_4}, {X_0}^{i_3}{X_2}^{d-i_3}, {X_0}^{i_2}{X_2}^{d-i_2}, {X_0}^{i_1}{X_2}^{d-i_1} \}.$$

Therefore we have $k = 12$ and

$$(d-d_J)n + d_J - dk \geq 18(d-d_J) + d_J - 12d = 6d - 17d_J \geq 6d - 17i_1 \geq$$
$$\geq 13m + 6t - 17\min(1,t) \geq 13m - 11 > 0.$$

If $i_1 < d_J \leq i_2$, the multiples of ${X_0}^{d_J}$ in $I_{d,18}$ are the monomials in the set

$$J := \big\{ {X_0}^d,\ {X_0}^{i_2}{X_1}^{d-i_3}{X_2}^{i_3-i_2},\ {X_0}^{i_2}{X_1}^{d-i_4}{X_2}^{i_4-i_2},\\ {X_0}^{i_4}{X_1}^{d-i_4},\ {X_0}^{i_3}{X_1}^{d-i_3},\ {X_0}^{i_2}{X_1}^{d-i_2},\\ {X_0}^{i_4}{X_2}^{d-i_4},\ {X_0}^{i_3}{X_2}^{d-i_3},\ {X_0}^{i_2}{X_2}^{d-i_2} \big\}.$$

Therefore we have $k = 9$ and

$$(d-d_J)n + d_J - dk = 18(d-d_J) + d_J - 9d = 9d - 17d_J \geq 9d - 17i_2 \geq \\ \geq 11m + 9t - 17\min(2,t) \geq 11m - 16 > 0.$$

If $i_2 < d_J \leq i_3$, the multiples of ${X_0}^{d_J}$ in $I_{d,18}$ are the monomials in the set

$$J := \big\{ {X_0}^d,\ {X_0}^{i_4}{X_1}^{d-i_4},\ {X_0}^{i_3}{X_1}^{d-i_3},\ {X_0}^{i_4}{X_2}^{d-i_4},\ {X_0}^{i_3}{X_2}^{d-i_3} \big\}.$$

Therefore we have $k = 5$ and

$$(d-d_J)n + d_J - dk = 18(d-d_J) + d_J - 5d = 13d - 17d_J \geq 13d - 17i_3 \geq \\ \geq 14m + 13t - 17\min(3,t) \geq 14m - 12 > 0.$$

If $i_3 < d_J \leq i_4$, the multiples of ${X_0}^{d_J}$ in $I_{d,18}$ are the monomials in the set

$$J := \big\{ {X_0}^d,\ {X_0}^{i_4}{X_1}^{d-i_4},\ {X_0}^{i_4}{X_2}^{d-i_4} \big\}.$$

Therefore we have $k = 3$ and

$$(d-d_J)n + d_J - dk = 18(d-d_J) + d_J - 3d = 15d - 17d_J \geq 15d - 17i_4 \geq \\ \geq 7m - 2t \geq 7m - 8 > 0.$$

If $i_4 < d_J < d$, the only multiple of ${X_0}^{d_J}$ in $I_{d,18}$ is ${X_0}^d$ and we have nothing to check.

Thus, we conclude that the stability is guaranteed in all the cases. □

Proposition 3.2. *For any integers n and d such that $18 < n \leq d+2$, there is a set $I_{d,n}$ of n $\mathfrak{m}$-primary monomials in $K[X_0, X_1, X_2]$ of degree d such that the corresponding syzygy bundle $E_{d,n}$ is stable.*

Proof. For each integer $j \geq 1$, let $T_j := \binom{j+1}{2}$ be the jth triangular number. Choose j such that $T_{j+2} \leq n < T_{j+3}$, and write $n = T_{j+2} + r$, with $0 \leq r \leq j+2$. Since $n > 18$, we have $j \geq 3$. Since $n \leq d+2$, we get $T_{j+2} \leq d+2$, and therefore $2d - j^2 - 5j - 2 \geq 0$.

From now until the end of this proof we shall adopt the following strategy:

Strategy: For each given d and n, we choose a set of n monomials $I_{d,n}$ such that for $0 < d_J < d$, no monomial of degree d_J divides a greater number of monomials in $I_{d,n}$ than ${X_0}^{d_J}$.

This strategy will make it easier to check the conditions of Remark 2.10 (a) for $I_{d,n}$, since for each value of d_J we only have to consider the multiples of ${X_0}^{d_J}$.

We write $d = m(j+1)+t$, where $0 \le t < j+1$. Note that, since $2d \ge j^2+5j+2$, we get $d \ge 3(j+1)+1$, and therefore $m \ge 3$. For each $l \in \{1,\dots,j\}$, we define

$$i_l := lm + \min(l,t).$$

We have $0 < i_1 < \cdots < i_j < d$,

$$d - i_j \le i_j - i_{j-1} \le \cdots \le i_2 - i_1 \le i_1,$$

and $i_1 - (d - i_j) \le 1$. Set $e := \lceil \frac{m}{2} \rceil$. Consider the set

$$\begin{aligned}
I' := \{&{X_0}^d,\, {X_0}^{i_j}{X_1}^{d-i_j},\, {X_0}^{i_j}{X_2}^{d-i_j},\\
&{X_0}^{i_{j-1}}{X_1}^{d-i_{j-1}},\, {X_0}^{i_{j-1}}{X_1}^{d-i_j}{X_2}^{i_j-i_{j-1}},\, {X_0}^{i_{j-1}}{X_2}^{d-i_{j-1}},\\
&{X_0}^{i_{j-2}}{X_1}^{d-i_{j-2}},\, {X_0}^{i_{j-2}}{X_1}^{d-i_{j-1}}{X_2}^{i_{j-1}-i_{j-2}},\, {X_0}^{i_{j-2}}{X_1}^{d-i_j}{X_2}^{i_j-i_{j-2}},\, {X_0}^{i_{j-2}}{X_2}^{d-i_{j-2}},\\
&\dots\\
&{X_0}^{i_1}{X_1}^{d-i_1},\, {X_0}^{i_1}{X_1}^{d-i_2}{X_2}^{i_2-i_1},\\
&\quad {X_0}^{i_1}{X_1}^{d-i_3}{X_2}^{i_3-i_1},\dots,{X_0}^{i_1}{X_1}^{d-i_j}{X_2}^{i_j-i_1},\, {X_0}^{i_1}{X_2}^{d-i_1},\\
&{X_1}^d,\, {X_1}^{i_j}{X_2}^{d-i_j},\dots,{X_1}^{i_1}{X_2}^{d-i_1},\, {X_2}^d\},
\end{aligned}$$

and the sequence

$$\begin{aligned}
&({X_0}^{i_j+e}{X_1}^{d-i_j-e},\, {X_0}^{e}{X_2}^{d-e},\, {X_1}^{i_j+e}{X_2}^{d-i_j-e},\\
&\ {X_0}^{i_{j-1}+e}{X_1}^{d-i_{j-1}-e},\, {X_0}^{i_1+e}{X_2}^{d-i_1-e},\, {X_1}^{i_{j-1}+e}{X_2}^{d-i_{j-1}-e},\\
&\ \dots,\\
&\ {X_0}^{i_{j-q}+e}{X_1}^{d-i_{j-q}-e},\, {X_0}^{i_q+e}{X_2}^{d-i_q-e},\, {X_1}^{i_{j-q}+e}{X_2}^{d-i_{j-q}-e}),
\end{aligned}$$

where $q := \lceil \frac{j-1}{3} \rceil$. Let I'' be the set of the first r monomials in this sequence, and let $I_{d,n} = I' \cup I''$. Since I' has T_{j+2} monomials, the number of monomials in $I_{d,n}$ is n.

For $1 \le l \le j$, let J_l be the set of monomials in I' that are multiples of ${X_0}^{i_l}$. We have

$$\begin{aligned}
J_l := \{&{X_0}^d,\, {X_0}^{i_j}{X_1}^{d-i_j},\, {X_0}^{i_j}{X_2}^{d-i_j},\\
&{X_0}^{i_{j-1}}{X_1}^{d-i_{j-1}},\, {X_0}^{i_{j-1}}{X_1}^{d-i_j}{X_2}^{i_j-i_{j-1}},\, {X_0}^{i_{j-1}}{X_2}^{d-i_{j-1}},\\
&{X_0}^{i_{j-2}}{X_1}^{d-i_{j-2}},\, {X_0}^{i_{j-2}}{X_1}^{d-i_{j-1}}{X_2}^{i_{j-1}-i_{j-2}},\, {X_0}^{i_{j-2}}{X_1}^{d-i_j}{X_2}^{i_j-i_{j-2}},\, {X_0}^{i_{j-2}}{X_2}^{d-i_{j-2}},\\
&\dots\\
&{X_0}^{i_l}{X_1}^{d-i_l},\, {X_0}^{i_l}{X_1}^{d-i_{l+1}}{X_2}^{i_{l+1}-i_l},\\
&\quad {X_0}^{i_l}{X_1}^{i_{l+2}}{X_2}^{i_{l+2}-i_l},\dots,{X_0}^{i_l}{X_1}^{d-i_j}{X_2}^{i_j-i_l},\, {X_0}^{i_l}{X_2}^{d-i_l}\}
\end{aligned}$$

and $|J_l| = T_{j-l+2}$.

We distinguish two cases.

Case 1: $n = T_{j+2}$. Since we are following the strategy mentioned above, for $0 < d_J < d$, we only have to check inequality (3) for multiples of ${X_0}^{d_J}$.

If $0 < d_J \le i_1$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are among the monomials in the set J_1. Therefore if k is the number of multiples of ${X_0}^{d_J}$, we have $k = T_{j+1}$ and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)T_{j+2} + d_J - dT_{j+1} = \\ &= d(j+2) - d_J T_{j+2} + d_J \ge \\ &\ge (m(j+1)+t)(j+2) - i_1(T_{j+2}-1) = \\ &= (m(j+1)+t)(j+2) - (m+\min(1,t))(T_{j+2}-1).\end{aligned}$$

This last expression takes the following forms, depending on the different values of t:

- for $t = 0$, $\quad \frac{m}{2} j(j+1)$;
- for $t > 0$, $\quad \frac{(j+2)}{2}\big(2t + (m-1)(j-1) - 4\big) + m + 1$.

These expressions are positive in both cases because $j \ge 3$ and $m \ge 3$. So inequality (3) is strictly satisfied.

If $i_l < d_J \le i_{l+1}$, for $1 \le l \le j-1$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set J_{l+1}. Therefore we have $k = T_{j+1-l}$ and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)T_{j+2} + d_J - dT_{j+1-l} = \\ &= d(T_{j+2} - T_{j+1-l}) - d_J(T_{j+2}-1) \ge \\ &\ge d(T_{j+2} - T_{j+1-l}) - i_{l+1}(T_{j+2}-1).\end{aligned}$$

This last expression takes the following forms, depending on the different values of t:

- for $t \le l+1$,
$$\begin{aligned}&\tfrac{1}{2}(m-1)lj(j-l) + \tfrac{1}{2}(m-1)j(j-l) + \tfrac{1}{2}(m-2)l(j-l) + \\ &\quad + \tfrac{1}{2}(m-2)(j-l) + \tfrac{1}{2}(l-1)^2(j-l) + (l-1)(j-l) \\ &\quad + \tfrac{1}{2}\left((j-l)^2 + 3(j-l)\right)(l+1-t) > 0;\end{aligned}$$
- for $t > l+1$,
$$\begin{aligned}&\tfrac{1}{2}(m-1)lj(j-l) + \tfrac{1}{2}(m-1)j(j-l) + \tfrac{1}{2}(m-2)l(j-l) + \\ &\quad + \tfrac{1}{2}(m-3)(j-l) + \tfrac{1}{2}l^2(j-l) + \\ &\quad + \tfrac{1}{2}\left(2l(j-l) + l^2 + 2j + 3l + 4\right)(t-l-1) > 0.\end{aligned}$$

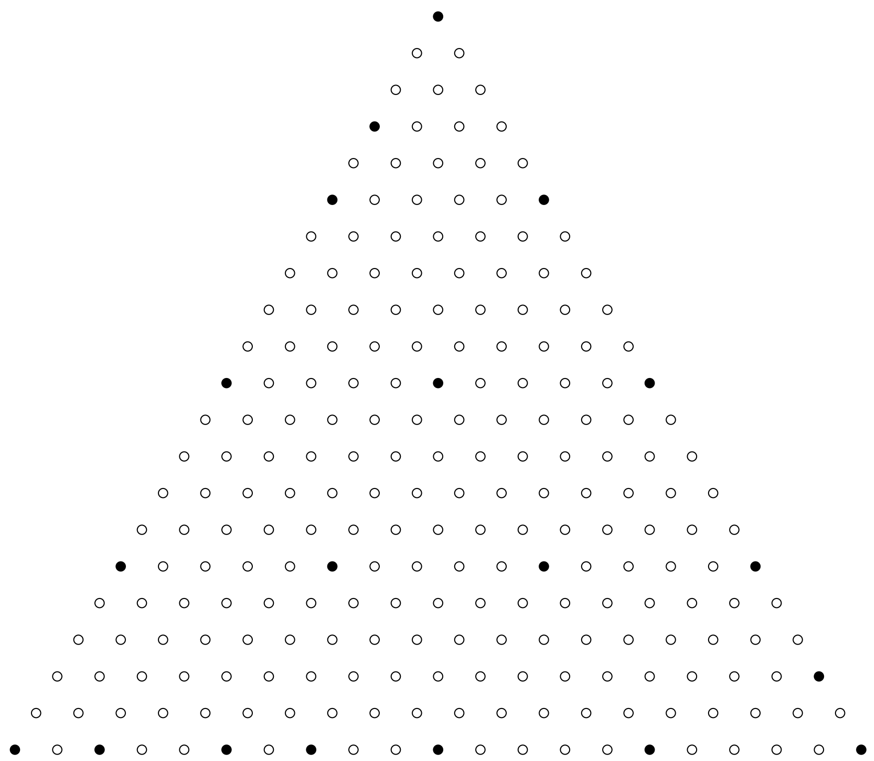
Therefore inequality (3) is strictly satisfied.

If $i_j < d_J < d$, the only multiple of ${X_0}^{d_J}$ in $I_{d,n}$ is ${X_0}^d$ and there is nothing to check.

Therefore all possible values of d_J are verified, and hence the syzygy bundle $E_{d,n}$ is stable.

Case 2: $n > T_{j+2}$.

Here is a picture of $I_{d,n}$ in case $n = 19$ and $d = 20$. In this case, we get $j = 3$ and $d = 5(j+1)$, therefore $m = 5$, $t = 0$ and $e = 3$.



$I_{20,19}$

Since $n = T_{j+2} + r \leq d + 2$, with $0 < r \leq j + 2$, we have $d + 2 \geq T_{j+2} + 1$. From this, if $j > 3$, we get

$$2d \geq j^2 + 5j + 4 \geq 9j + 4 \geq 8(j + 1).$$

In case $j = 3$, since $d \geq 17$, we have $d \geq 4(j + 1) + 1$. In any case, $m \geq 4$.

We distinguish three subcases.

Case 2.1: $r = 3s + 1$, with $s \geq 0$.

If $0 < d_J \leq e$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned} J := J_1 \cup \big\{ & {X_0}^{i_j+e}{X_1}^{d-i_j-e}, \ldots, {X_0}^{i_{j-s}+e}{X_1}^{d-i_{j-s}-e}, \\ & {X_0}^{e}{X_2}^{d-e}, \ldots, {X_0}^{i_{s-1}+e}{X_2}^{d-i_{s-1}-e} \big\}. \end{aligned}$$

Therefore if k is the number of multiples of ${X_0}^{d_J}$, we have $k = T_{j+1} + 2s + 1$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2}+3s+1) + d_J - d(T_{j+1}+2s+1) = \\ &= d(j+2+s) - d_J(T_{j+2}+3s) \geq \\ &\geq (m(j+1)+t)(j+2+s) - e(T_{j+2}+3s) \geq \\ &\geq (m(j+1)+t)(j+2+s) - \tfrac{m+1}{2}(T_{j+2}+3s) = \\ &= \tfrac{1}{4}(3m-1)j^2 + \tfrac{1}{4}(7m-5)(j-2) + 4(m-1) + \\ &\qquad + \tfrac{1}{2}(2m(j-2)+3(m-1))s + t(j+2+s) > 0.\end{aligned}$$

If $e < d_J \leq i_1$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned}J := J_1 \cup \big\{&{X_0}^{i_j+e}{X_1}^{d-i_j-e}, \ldots, {X_0}^{i_{j-s}+e}{X_1}^{d-i_{j-s}-e}, \\ &{X_0}^{i_1+e}{X_2}^{d-i_1-e}, \ldots, {X_0}^{i_{s-1}+e}{X_2}^{d-i_{s-1}-e}\big\}.\end{aligned}$$

Therefore we have $k = T_{j+1} + \max(2s, 1)$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2}+3s+1) + d_J - d(T_{j+1}+\max(2s,1)) = \\ &= d(j+3+3s-\max(2s,1)) - d_J(T_{j+2}+3s) \geq \\ &\geq d(j+3+3s-\max(2s,1)) - i_1(T_{j+2}+3s) = \\ &= (m(j+1)+t)(j+3+3s-\max(2s,1)) - \\ &\qquad - (m+\min(1,t))(T_{j+2}+3s).\end{aligned}$$

This last expression takes the following forms, depending on the different values of s and t:

- for $s = t = 0$, $\quad \frac{m}{2}(j+2)(j-1)$;
- for $s = 0$ and $t > 0$, $\quad \frac{1}{2}\big((m-1)(j-2)^2 + (5m-7)(j-2) + 4(m-3)\big) + (t-1)(j+2)$;
- for $s > 0$ and $t = 0$, $\quad m\left(\frac{1}{2}j(j+3) + (j-2)s\right)$;
- for $s > 0$ and $t > 0$, $\quad \frac{1}{2}\big((m-1)j^2 + (3m-5)(j-2)\big) + 3(m-3) + (j+3+s)(t-1) + m(j-2)s + j + 4 - 2s$.

These expressions are positive in all cases because $m \geq 4$, $j \geq 3$ and $s \leq \frac{j+1}{3}$. So, inequality (2) is strictly satisfied.

If $i_l < d_J \leq i_l + e$, for $1 \leq l \leq j-2$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned}J := J_{l+1} \cup \big\{&{X_0}^{i_j+e}{X_1}^{d-i_j-e}, \ldots, {X_0}^{i_a+e}{X_1}^{d-i_a-e}, \\ &{X_0}^{i_l+e}{X_2}^{d-i_l-e}, \ldots, {X_0}^{i_{s-1}+e}{X_2}^{d-i_{s-1}-e}\big\},\end{aligned}$$

where $a = \max(j-s, l)$ and the second line is understood to be empty if $s \leq l$. Therefore we have $k = T_{j+1-l} + \min(s+1, j+1-l) + \max(s-l, 0)$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2}+3s+1) + d_J - \\ &\quad - d\big(T_{j+1-l} + 1 + \min(s, j-l) + \max(s-l,0)\big) = \\ &= d\big(T_{j+2} - T_{j+1-l} + 3s - \min(s, j-l) - \max(s-l,0)\big) - \\ &\quad - d_J(T_{j+2}+3s) \geq \\ &\geq d\big(T_{j+2} - T_{j+1-l} + 3s - \min(s, j-l) - \max(s-l,0)\big) - \\ &\quad - \left(i_l + \tfrac{m+1}{2}\right)(T_{j+2}+3s).\end{aligned}$$

We can rewrite this last expression in the following forms, depending on the different values of j, l, s and t, so that they become sums of non-negative numbers (and at least one of these is strictly positive):

- for $s \leq j-l,\ s \leq l,\ t \leq l$,

$$\begin{aligned}&\tfrac{1}{2}(m-2)lj(j-l-1) + \tfrac{3}{4}(m-1)j(j-l) + \tfrac{1}{4}(m-4)l(j-l) + \\ &\quad + \tfrac{3}{4}(m-4)l^2 + \tfrac{7}{4}(m-3)(j-l) + \tfrac{1}{4}(m-4)l + \tfrac{1}{2}m + \\ &\quad + \tfrac{1}{2}(l-1)(j-l-2)^2 + (l-1)^2(j-l-2) + j(j-l-2) + \\ &\quad + \tfrac{3}{4}l(j-l-2) + \tfrac{15}{4}l(l-1) + 3(j-l) + \tfrac{5}{2} + \\ &\quad + \tfrac{1}{2}(4m(j-l) + m - 3)s + l(m+1)(j-l-s) \\ &\quad + \tfrac{1}{2}(l-t)\big((j-l)^2 + 3(j-l) + 2 + 2s\big) > 0;\end{aligned}$$

- for $s \leq j-l,\ s \leq l,\ t > l$,

$$\begin{aligned}&\tfrac{1}{2}(m-4)jl(j-l-1) + \tfrac{3}{4}(m-1)j(j-l) + \tfrac{1}{4}(m-4)lj + \tfrac{1}{2}ml^2 + \\ &\quad + \tfrac{7}{4}(m-1)(j-l) + \tfrac{1}{4}ml + \tfrac{1}{2}(m-4) + \tfrac{3}{2}lj(j-l-2) + \\ &\quad + \tfrac{1}{2}(l-1)^2(j-l) + \tfrac{1}{2}j(j-l-2) + \tfrac{1}{4}l(j-l-2) + j + \\ &\quad + \tfrac{7}{4}l(l-1) + \tfrac{1}{2} + \tfrac{1}{2}(t-l)\big((2j-l)l + 2j + 3l + 4 + 4s\big) + \\ &\quad + \tfrac{1}{2}(4m(j-l) + m - 3)s + (m+1)l(j-l-s) > 0;\end{aligned}$$

- for $s \leq j-l, s > l, t \leq l$,

$$\begin{aligned}&\tfrac{1}{2}(m-1)lj(j-l) + \tfrac{3}{4}(m-1)j(j-l) + \tfrac{7}{4}(m-3)l(j-l) + \tfrac{1}{4}ml^2 + \\ &\quad + \tfrac{1}{4}(m-2)(3j+4l+2) + \tfrac{1}{2}l^2(j-l) + \tfrac{1}{2}j^2 + l(j-l) + \\ &\quad + \tfrac{1}{4}l(l-1) + \tfrac{1}{4}(j-l-2) + \tfrac{1}{2}(2m(j-2l) + m - 3)(s-l) + \\ &\quad + \big((m+2)l + m\big)(j-l-s) + \\ &\quad + \tfrac{1}{2}\left((j-l)^2 + 3(j-2l) + l + 2 + 4s\right)(l-t) > 0;\end{aligned}$$

- for $s \leq j-l,\ s > l,\ t > l$,

$$\begin{aligned}&\tfrac{1}{2}(m-1)lj(j-l) + \tfrac{3}{4}(m-1)j(j-l) + \tfrac{7}{4}(m-3)l(j-l) + \tfrac{1}{4}ml^2 + \\ &\quad + \tfrac{1}{4}(m-1)(3j+4l+2) + \tfrac{1}{2}l^2(j-l) + \tfrac{1}{2}j(j-2l-1) + \\ &\quad + \tfrac{1}{2}(2m(j-2l) + m - 3)(s-l) + + \tfrac{7}{4}(l-1) + \tfrac{3}{4} \\ &\quad + \big((m+2)l + m\big)(j-l-s) + 2l(j-2l) + \tfrac{13}{4}l(l-1) + \\ &\quad + \tfrac{1}{2}\left(2(l+1)(j-l) + l^2 + 7l + 4 + 2s\right)(t-l) > 0;\end{aligned}$$

- for $s > j-l,\ s \leq l,\ t \leq l$,

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l-2) + \tfrac{11}{4}(m-2)j(j-l) + \tfrac{5}{4}(m-1)l(2l-j)+\\&\quad + \tfrac{9}{4}m(j-l) + \tfrac{1}{4}ml + \tfrac{1}{2}m + \tfrac{3}{2}(l-1)(j-l-2)^2 + \tfrac{1}{4}l + \tfrac{15}{2}+\\&\quad + 2(l-1)^2(j-l-2) + \tfrac{27}{4}j(j-l-2) + \tfrac{1}{2}l(j-l-2)+\\&\quad + (l-1)^2 + \tfrac{11}{4}(j-l) + \tfrac{1}{2}(s-j+l)(6m(j-l)+3m-3)+\\&\quad + \tfrac{1}{2}(l-t)\big((j-l)^2 + 5(j-l) + 2\big) > 0;\end{aligned}$$

- for $s > j-l,\ s \leq l,\ t > l$,

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l-2) + \tfrac{11}{4}(m-2)j(j-l) + \tfrac{5}{4}(m-1)l(2l-j)+\\&\quad + \tfrac{9}{4}m(j-l) + \tfrac{1}{4}ml + \tfrac{1}{2}m + \tfrac{3}{2}(l-1)(j-l-2)^2+\\&\quad + 2(l-1)^2(j-l-2) + \tfrac{27}{4}j(j-l-2) + \tfrac{1}{2}l(j-l-2) + (l-1)^2+\\&\quad + \tfrac{11}{4}(j-l) + \tfrac{1}{4}l + \tfrac{15}{2}+\\&\quad + \tfrac{1}{2}(s-j+l)(6m(j-l)+3m-3)+\\&\quad + \tfrac{1}{2}(t-l)\big(2l(j-l) + l^2 + 5l + 4 + 6s\big) > 0.\end{aligned}$$

Since $s \leq \frac{j+1}{3}$, if $l < s$, we get $j-l > \frac{2j-1}{3} \geq \frac{j+1}{3} \geq s$. Therefore all possible cases are checked, and inequality (3) is strictly satisfied.

If $i_l + e < d_J \leq i_{l+1}$, for $1 \leq l \leq j-2$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned}J := J_{l+1} \cup \big\{&{X_0}^{i_j+e}{X_1}^{d-i_j-e}, \ldots, {X_0}^{i_a+e}{X_1}^{d-i_a-e},\\&{X_0}^{i_{l+1}+e}{X_2}^{d-i_{l+1}-e}, \ldots, {X_0}^{i_{s-1}+e}{X_2}^{d-i_{s-1}-e}\big\},\end{aligned}$$

where $a = \max(j-s, l+1)$, and the second line is understood to be empty if $s \leq l+1$. Therefore we have $k = T_{j+1-l} + \min(s+1, j-l) + \max(s-l-1, 0)$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2} + 3s + 1) + d_J -\\&\quad - d\big(T_{j+1-l} + \min(s+1, j-l) + \max(s-l-1,0)\big) =\\&= d\big(T_{j+2} - T_{j+1-l} + 3s + 1 - \min(s+1, j-l) -\\&\quad - \max(s-l-1,0)\big) - d_J(T_{j+2} + 3s) \geq\\&\geq d\big(T_{j+2} - T_{j+1-l} + 3s + 1 - \min(s+1, j-l) -\\&\quad - \max(s-l-1,0)\big) - i_{l+1}(T_{j+2} + 3s).\end{aligned}$$

As before, we can rewrite this last expression in the following forms, depending on the different values of j, l, s and t:

- for $s+1 \leq j-l,\ s \leq l+1,\ t \leq l+1$,

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l) + \tfrac{1}{2}(m-4)(j-l)(j-l-1) + (m-4)(j-l-1)+\\&\quad + \tfrac{3}{2}lj(j-l-2) + \tfrac{1}{2}l^2(j-l) + \tfrac{3}{2}(j-l)(j-l-2) + \tfrac{1}{2}l(j-l)+\\&\quad + \tfrac{5}{2}(j-l-2) + 3l^2 + 1 + m(2(j-l-1)+1)s+\\&\quad + (ml+t)(j-l-1-s) + \tfrac{1}{2}\left((j-l)^2 + 5(j-l)\right)(l+1-t) > 0;\end{aligned}$$

- for $s+1 \le j-l,\ s \le l+1,\ t > l+1,$

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l) + \tfrac{1}{2}(m-4)(j-l)(j-l-1) + (m-4)(j-l-1)+\\ &\quad + \tfrac{3}{2}l(j-l-2)^2 + 2l(l-1)(j-l) + \tfrac{3}{2}(j-l)(j-l-2)+\\ &\quad + \tfrac{7}{2}l(j-l-2) + \tfrac{1}{2}(j-l-2) + 3(l-1) + 2+\\ &\quad + (2m(j-l-1) + m + 2t)s + (ml + 3l + 3)(j-l-1-s)+\\ &\quad + \tfrac{1}{2}\left(2l(j-l) + l^2 + 2(j-l) + 5l + 4\right)(t-l-1) > 0;\end{aligned}$$

- for $s+1 \le j-l, s > l+1, t \le l+1,$

$$\begin{aligned}&\tfrac{1}{2}(m-1)lj(j-l) + \tfrac{1}{2}(m-4)\left(j^2+l^2\right) + \tfrac{5}{2}(m-1)(j-l-1) + ml+\\ &\quad + \tfrac{1}{2}m + \tfrac{1}{2}(l-1)^2(j-l) + \tfrac{3}{2}j(j-l-1) + 4l^2 + \tfrac{7}{2}(l-1) + 3+\\ &\quad + m(j-l-2)(s-l-1) + 2(ml+t)(j-l-1-s)+\\ &\quad + \tfrac{1}{2}\left((j-l-2)^2 + 11(j-l-2) + 14\right)(l+1-t) + lt > 0;\end{aligned}$$

- for $s+1 \le j-l,\ s > l+1,\ t > l+1,$

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l) + \tfrac{1}{2}(m-1)\left(j^2+l^2\right) + \tfrac{5}{2}(m-2)(j-l-1) + ml+\\ &\quad + \tfrac{1}{2}m + \tfrac{3}{2}lj(j-l-2) + \tfrac{1}{2}l(l-1)(j-l) + 6l^2 + \tfrac{1}{2}j + \tfrac{11}{2}(l-1) + \tfrac{9}{2}+\\ &\quad + (m(j-l-2)+t)(s-l-1) + \big((2m+3)l+3\big)(j-l-1-s)+\\ &\quad + \tfrac{1}{2}\left(2l(j-l) + l^2 + 2j + 7l + 8\right)(t-l-1) > 0;\end{aligned}$$

- for $s+1 > j-l,\ s \le l+1,\ t \le l+1,$

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l) + \tfrac{5}{2}(m-1)(j-l)^2 + \tfrac{7}{2}(m-1)(j-l) + \tfrac{3}{2}lj(j-l-2)+\\ &\quad + \tfrac{1}{2}l^2(j-l) + 2(j-l)^2 + \tfrac{1}{2}lj + \tfrac{5}{2}l^2 + j - l+\\ &\quad + 3m(j-l)(s-j+l-1)+\\ &\quad + \tfrac{1}{2}\left((j-l)^2 + 5(j-l)\right)(l+1-t) > 0;\end{aligned}$$

- for $s+1 > j-l,\ s \le l+1,\ t > l+1,$

$$\begin{aligned}&\tfrac{1}{2}(m-4)lj(j-l) + \tfrac{5}{2}(m-1)(j-l)^2 + \tfrac{7}{2}(m-1)(j-l)+\\ &\quad + \tfrac{3}{2}lj(j-l-2) + \tfrac{1}{2}l^2(j-l) + 2(j-l)^2 + \tfrac{1}{2}lj+\\ &\quad + \tfrac{5}{2}l^2 + j - l + 3m(j-l)(s-j+l-1)+\\ &\quad + \tfrac{1}{2}\left(2l(j-l) + l^2 + 5l + 6 + 6s\right)(t-l-1) > 0.\end{aligned}$$

Since $s \le \frac{j+1}{3}$, if $l+1 < s$, we get $j-l-1 > \frac{2j-1}{3} \ge \frac{j+1}{3} \ge s$. Therefore all possible cases are checked, and inequality (3) is strictly satisfied.

If $i_{j-1} < d_J \le i_{j-1} + e$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$J := J_j \cup \left\{ {X_0}^{i_j+e}{X_1}^{d-i_j-e},\ {X_0}^{i_{j-a}+e}{X_1}^{d-i_{j-a}-e} \right\},$$

where $a = \min(s,1)$. Therefore we have $k = T_2 + \min(s+1,2)$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2} + 3s + 1) + d_J - d\big(T_2 + \min(s+1,2)\big) =\\ &= d\big(T_{j+2} - T_2 + 3s - \min(s,1)\big) - d_J(T_{j+2} + 3s) \ge\\ &\ge d\big(T_{j+2} - 3 + 3s - \min(s,1)\big) -\\ &\qquad - \left(i_{j-1} + \tfrac{m+1}{2}\right)(T_{j+2} + 3s).\end{aligned}$$

This last expression takes the following forms, depending on the different values of j, s and t:

- for $s \leq 1,\ t \leq j-1$,

$$\tfrac{3}{4}(m-2)j(j-1) + \tfrac{1}{2}(m-4)j + \tfrac{9}{2}(m-1) + \tfrac{5}{4}(j-3)^2 + \tfrac{11}{4}(j-3) + 4+ \\ + \big((m+1)(j-3)+2\big)(1-s) + \tfrac{1}{2}(m-3)s + (s+3)(j-1-t) > 0;$$

- for $s \leq 1,\ t = j$,

$$\tfrac{3}{4}(m-1)j(j-1) + \tfrac{1}{2}mj + \tfrac{9}{2}(m-1) + (j-3)^2 + \tfrac{5}{2}j+ \\ + (m+1)(j-3)(1-s) + \tfrac{1}{2}(m-3)s > 0;$$

- for $s > 1,\ t \leq j-1$,

$$\tfrac{3}{4}(m-2)j(j-1) + \tfrac{1}{2}(m-4)j + 5(m-1) + \tfrac{5}{4}(j-3)^2 + \tfrac{11}{4}(j-3) + 3+ \\ + \tfrac{1}{2}(9(m-1)+6)(s-1) + 4(j-1-t) > 0;$$

- for $s > 1,\ t = j$,

$$\tfrac{3}{4}(m-1)j(j-1) + \tfrac{1}{2}mj + 5(m-1) + (j-3)^2 + \tfrac{5}{2}(j-1) + \tfrac{3}{2}+ \\ + \tfrac{1}{2}(9m+3)(s-1) > 0.$$

Therefore inequality (3) is strictly satisfied.

If $i_{j-1} + e < d_J \leq i_j$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$J := J_j \cup \big\{ {X_0}^{i_j+e} {X_1}^{d-i_j-e} \big\}.$$

Therefore we have $k = T_2 + 1 = 4$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2}+3s+1) + d_J - 4d = \\ &= d(T_{j+2}+3s-3) - d_J(T_{j+2}+3s) \geq \\ &\geq d(T_{j+2}+3s-3) - i_j(T_{j+2}+3s) = \\ &= (m(j+1)+t)(T_{j+2}+3s-3) - \\ &\quad - (mj+t)(T_{j+2}+3s) = \\ &= \tfrac{1}{2}\big((m-3)j(j-1) + 3j(j-3)\big) + 3(j-t) + 3ms > 0.\end{aligned}$$

If $i_j < d_J \leq i_j + e$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$J := \big\{ {X_0}^d,\ {X_0}^{i_j+e} {X_1}^{d-i_j-e} \big\}.$$

Therefore we have $k = 2$, and

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2}+3s+1) + d_J - 2d = \\ &= d(T_{j+2}+3s-1) - d_J(T_{j+2}+3s) \geq \\ &\geq d(T_{j+2}+3s-1) - (i_j+e)(T_{j+2}+3s) \geq \\ &\geq (m(j+1)+t)(T_{j+2}+3s-1) - \\ &\quad - \left(mj+t+\tfrac{m+1}{2}\right)(T_{j+2}+3s) = \\ &= \tfrac{1}{2}\big((m-1)j^2 + 3(m-1)j + 3(m-1)s\big) + (j-t)+ \\ &\quad + 2(m-1) + 1 > 0.\end{aligned}$$

If $i_j + e < d_J < d$, the only multiple of ${X_0}^{d_J}$ in $I_{d,n}$ is ${X_0}^d$, and there is nothing to prove.

Therefore all possible values of d_J are verified, and hence the syzygy bundle $E_{d,n}$ is stable.

Case 2.2: $r = 3s + 2$, with $s \geq 0$. The difference between this case and the previous one is that we are adding the monomial ${X_0}^{i_s+e}{X_2}^{d-i_s-e}$ to $I_{d,n}$. Therefore we should only worry with the cases $0 < d_J \leq i_s + e$, since for degrees greater than $i_s + e$ the set J of multiples of ${X_0}^{d_J}$ has the same number of elements as in the corresponding sets of the previous case, whereas the set $I_{d,n}$ has one more element. Given the fact that the sequence $(a_{d,j})_{j\geq 2}$ is monotonically increasing, inequality (3) is strictly satisfied.

If $0 < d_J \leq e$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are among the monomials in the set

$$\begin{aligned} J := J_1 \cup \big\{&{X_0}^{i_j+e}{X_1}^{d-i_j-e}, \dots, {X_0}^{i_{j-s}+e}{X_1}^{d-i_{j-s}-e},\\ &{X_0}^{e}{X_2}^{d-e}, \dots, {X_0}^{i_s+e}{X_2}^{d-i_s-e}\big\}. \end{aligned}$$

Therefore if k is the number of multiples of ${X_0}^{d_J}$, we have $k = T_{j+1} + 2s + 2$, and

$$\begin{aligned} (d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2} + 3s + 2) + d_J - d(T_{j+1} + 2s + 2) = \\ &= d(j+2+s) - d_J(T_{j+2} + 3s + 1) \geq \\ &\geq (m(j+1)+t)(j+2+s) - e(T_{j+2} + 3s + 1) \geq \\ &\geq (m(j+1)+t)(j+2+s) - \tfrac{m+1}{2}(T_{j+2} + 3s + 1) = \\ &= \tfrac{1}{4}(3m-1)j^2 + \tfrac{1}{4}(7m-5)(j-2) + \tfrac{7}{2}(m-2) + \tfrac{5}{2} + \\ &\qquad + \tfrac{1}{2}(2m(j-2) + 3(m-1))s + t(j+2+s) > 0. \end{aligned}$$

If $e < d_J \leq i_1$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned} J := J_1 \cup \big\{&{X_0}^{i_j+e}{X_1}^{d-i_j-e}, \dots, {X_0}^{i_{j-s}+e}{X_1}^{d-i_{j-s}-e},\\ &{X_0}^{i_1+e}{X_2}^{d-i_1-e}, \dots, {X_0}^{i_s+e}{X_2}^{d-i_s-e}\big\}. \end{aligned}$$

Therefore we have $k = T_{j+1} + 2s + 1$ and

$$\begin{aligned} (d-d_J)n + d_J - dk &= (d-d_J)(T_{j+2} + 3s + 2) + d_J - d(T_{j+1} + 2s + 1) = \\ &= d(j+3+s) - d_J(T_{j+2} + 3s + 1) \geq \\ &\geq d(j+3+s) - i_1(T_{j+2} + 3s + 1) = \\ &= (m(j+1)+t)(j+3+s) - \\ &\qquad - \big(m + \min(1,t)\big)(T_{j+2} + 3s + 1). \end{aligned}$$

This last expression takes the following forms, depending on the different values of t:

- for $t = 0$, $\quad \frac{m}{2}(j+2)(j-1) + m(j-2)s + mj$;
- for $t > 0$, $\quad \frac{1}{2}\big((m-1)(j-2)^2 + (7m-7)(j-2) + 8(m-3) + 12\big) + $
  $+ (m(j-3) + m - 3 + t)s + (t-1)(j+3)$.

These expressions are both positive, so inequality (3) is strictly satisfied.

If $i_l < d_J \le i_l + e$, for $1 \le l \le s$, we get $j - s \ge \frac{2}{3}j > l$, since $3s + 2 \le j + 2$. Therefore the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned} J := J_{l+1} \cup \big\{ & {X_0}^{i_j+e}{X_1}^{d-i_j-e}, \ldots, {X_0}^{i_{j-s}+e}{X_1}^{d-i_{j-s}-e}, \\ & {X_0}^{i_l+e}{X_2}^{d-i_l-e}, \ldots, {X_0}^{i_s+e}{X_2}^{d-i_s-e} \big\}. \end{aligned}$$

Therefore we have $k = T_{j+1-l} + 2s + 2 - l$, and

$$\begin{aligned} (d - d_J)n + d_J - dk = & (d - d_J)(T_{j+2} + 3s + 2) + d_J - \\ & - d(T_{j+1-l} + 2s + 2 - l) = \\ & = d(T_{j+2} - T_{j+1-l} + s + l) - d_J(T_{j+2} + 3s + 1) \ge \\ & \ge d(T_{j+2} - T_{j+1-l} + s + l) - \left(i_l + \tfrac{m+1}{2}\right)(T_{j+2} + 3s + 1). \end{aligned}$$

This last expression takes the following forms, depending on the different values of t:

- for $t \le l$,

$$\begin{aligned} & \tfrac{1}{2}(m-1)lj(j-l) + \tfrac{3}{4}(m-1)j(j-l) + \tfrac{7}{4}(m-3)l(j-l) + \tfrac{1}{4}ml^2 + \\ & + \tfrac{3}{4}(m-4)j + \tfrac{1}{2}l^2(j-l) + \tfrac{1}{2}j(j-2l) + 2l(j-2l) + \tfrac{13}{4}l(l-1) + \\ & + \tfrac{7}{4}(j-l-2) + \tfrac{3}{2}(l+1) + \tfrac{1}{2}(2m(j-2l) + m - 3)(s-l) + \\ & + \big((m+2)l + m\big)(j-l-s) + \\ & + \tfrac{1}{2}\left((j-l)^2 + 3(j-2l) + l + 4 + 4s\right)(l-t) > 0; \end{aligned}$$

- for $t > l$,

$$\begin{aligned} & \tfrac{1}{2}(m-1)lj(j-l) + \tfrac{3}{4}(m-1)j(j-l) + \tfrac{7}{4}(m-3)l(j-l) + \tfrac{1}{4}ml^2 + \\ & + \tfrac{3}{4}(m-1)j + \tfrac{1}{2}l^2(j-l-2) + \tfrac{1}{2}j(j-2l-1) + \\ & + 2l(j-2l-1) + \tfrac{17}{4}l(l-1) + \tfrac{11}{4}(l-1) + \tfrac{3}{4} + \\ & + \tfrac{1}{2}(2m(j-2l) + m - 3)(s-l) + \\ & + \big((m+2)l + m\big)(j-l-s) + \\ & + \tfrac{1}{2}\left(2(l+1)(j-l) + l^2 + 7l + 4 + 2s\right)(t-l) > 0. \end{aligned}$$

If $i_l + e < d_J \le i_{l+1}$, for $1 \le l \le s-1$, the multiples of ${X_0}^{d_J}$ in $I_{d,n}$ are the monomials in the set

$$\begin{aligned} J := J_{l+1} \cup \big\{ & {X_0}^{i_j+e}{X_1}^{d-i_j-e}, \ldots, {X_0}^{i_{j-s}+e}{X_1}^{d-i_{j-s}-e}, \\ & {X_0}^{i_{l+1}+e}{X_2}^{d-i_{l+1}-e}, \ldots, {X_0}^{i_s+e}{X_2}^{d-i_s-e} \big\}. \end{aligned}$$

Therefore we have $k = T_{j+1-l} + 2s + 1 - l$, and

$$\begin{aligned} (d - d_J)n + d_J - dk = & (d - d_J)(T_{j+2} + 3s + 2) + d_J - \\ & - d(T_{j+1-l} + 2s + 1 - l) = \\ & = d(T_{j+2} - T_{j+1-l} + s + 1 + l) - d_J(T_{j+2} + 3s + 1) \ge \\ & \ge d(T_{j+2} - T_{j+1-l} + s + 1 + l) - i_{l+1}(T_{j+2} + 3s + 1). \end{aligned}$$

This last expression takes the following forms, depending on the different values of t:

- for $t \leq l+1$,

$$\begin{aligned}
&\tfrac{1}{2}(m-1)lj(j-l) + \tfrac{1}{2}(m-4)\left(j^2+l^2\right) + \tfrac{5}{2}(m-1)(j-l-2) + 2m+ \\
&\quad + \tfrac{1}{2}(l-1)^2(j-l) + \tfrac{3}{2}j(j-l-1) + 4l(l-1) + \tfrac{13}{2}(l-1) + \tfrac{5}{2}+ \\
&\quad + m(j-l-2)(s-l-1) + 2(ml+t)(j-l-1-s)+ \\
&\quad + \tfrac{1}{2}\left((j-l-2)^2 + 11(j-l-2) + 16\right)(l+1-t) + lt > 0;
\end{aligned}$$

- for $t > l+1$,

$$\begin{aligned}
&\tfrac{1}{2}(m-4)lj(j-l) + \tfrac{1}{2}(m-1)\left(j^2+l^2\right) + \tfrac{5}{2}(m-2)(j-l-2) + 2m+ \\
&\quad + \tfrac{3}{2}lj(j-l-2) + \tfrac{1}{2}l(l-1)(j-l) + 6l(l-1) + \tfrac{1}{2}j + \tfrac{21}{2}(l-1) + \tfrac{7}{2}+ \\
&\quad + (m(j-l-2)+t)(s-l-1) + \left((2m+3)l+3\right)(j-l-1-s)+ \\
&\quad + \tfrac{1}{2}\left(2l(j-l) + l^2 + 2j + 7l + 8\right)(t-l-1) > 0.
\end{aligned}$$

Therefore inequality (3) is strictly satisfied.

Case 2.3: $r = 3s$, with $s \geq 1$. The difference between this case and the previous one is that we are adding the monomial ${X_1}^{i_s+e}{X_2}^{d-i_s-e}$ to $I_{d,n}$. Since this is no multiple of ${X_0}^{d_J}$, the set J of multiples of ${X_0}^{d_J}$ has the same number of elements as in the corresponding sets of the previous case, whereas the set $I_{d,n}$ has one more element. Given the fact that the sequence $\left(a_{d,j}\right)_{j\geq 2}$ is monotonically increasing (see Remark 2.10(b)), inequality (3) is strictly satisfied.

We can conclude that stability is guaranteed in all cases. $\square$

Proposition 3.3. *For any integers n and d such that $d+2 < n \leq 3d$ and $(n,d) \neq (5,2)$, there is a set $I_{d,n}$ of n $\mathfrak{m}$-primary monomials in $K[X_0,X_1,X_2]$ of degree d such that the corresponding syzygy bundle $E_{d,n}$ is stable. For $(n,d) = (5,2)$, there are 5 $\mathfrak{m}$-primary monomials in $K[X_0,X_1,X_2]$ of degree 2 such that the corresponding syzygy bundle $E_{2,5}$ is semistable.*

Proof. Suppose that $d \geq 4$, and consider the set

$$I' := \left\{{X_0}^d,\ {X_0}^{d-1}X_1, \ldots, X_0{X_1}^{d-1},\ {X_1}^d,\ {X_2}^d\right\}.$$

and the sequence

$$\begin{aligned}
\big(&X_0{X_2}^{d-1},\ {X_0}^2{X_2}^{d-2}, \ldots, {X_0}^{d-2}{X_2}^2, \\
&{X_1}^{d-2}{X_2}^2,\ {X_0}^{d-1}X_2, \\
&X_1{X_2}^{d-1},\ {X_1}^2{X_2}^{d-2}, \ldots, {X_1}^{d-3}{X_2}^3,\ {X_1}^{d-1}X_2\big).
\end{aligned}$$

If $1 \leq i \leq 2d-2$, let I'' be the set of the first i monomials in this sequence and let $I_{d,n} = I' \cup I''$. The number of monomials in $I_{d,n}$ is $n = d+2+i$.

For $0 < d_J < d$, since we are again following the strategy mentioned in Proposition 3.2, it is enough to count, in each case, the number of multiples of ${X_0}^{d_J}$ which are in $I_{d,n}$.

If $i \leq d-2$, the set of multiples of ${X_0}^{d_J}$ in $I_{d,n}$ is

$$\left\{{X_0}^d, \ldots, {X_0}^{d_J}{X_1}^{d-d_J},\ {X_0}^{d_J}{X_2}^{d-d_J}, \ldots, {X_0}^e{X_2}^{d-e}\right\},$$

where $e := \max\{i, d_J - 1\}$ and the list ${X_0}^{d_J}{X_2}^{d-d_J}, \ldots, {X_0}^e {X_2}^{d-e}$ is understood to be empty if $e = d_J - 1$. The number of monomials in this set is $k = d - 2d_J + e + 2$, and we get

$$(d - d_J)n + d_J - dk = i(d - d_J) + dd_J - d_J - de > 0.$$

If $i = d - 1$, the set of multiples of ${X_0}^{d_J}$ is

$$\big\{{X_0}^d, \ldots, {X_0}^{d_J}{X_1}^{d-d_J},\ {X_0}^{d_J}{X_2}^{d-d_J}, \ldots, {X_0}^{d-2}{X_2}^2\big\}.$$

The list ${X_0}^{d_J}{X_2}^{d-d_J}, \ldots, {X_0}^{d-2}{X_2}^2$ is again understood to be empty if $d_J = d - 1$. The number of monomials in this set is $k = 2d - 2d_J$, and we get

$$(d - d_J)n + d_J - dk = d > 0.$$

If $i \geq d$, the set of multiples of ${X_0}^{d_J}$ is

$$\big\{{X_0}^d, \ldots, {X_0}^{d_J}{X_1}^{d-d_J},\ {X_0}^{d_J}{X_2}^{d-d_J}, \ldots, {X_0}^{d-1}X_2\big\}.$$

The number of monomials in this set is $k = 2d - 2d_J + 1$, and we get

$$(d - d_J)n + d_J - dk \geq d - d_J > 0.$$

In all cases, inequality (3) is strictly satisfied, and the corresponding syzygy bundle is stable.

For cases $d = 2$ and $d = 3$, we consider the sets

$$
\begin{aligned}
I_{2,5} &:= \big\{{X_0}^2,\ {X_1}^2,\ {X_2}^2,\ X_0X_1,\ X_0X_2\big\}\\
I_{2,6} &:= \big\{{X_0}^2,\ {X_1}^2,\ {X_2}^2,\ X_0X_1,\ X_0X_2,\ X_1X_2\big\}\\
I_{3,6} &:= \big\{{X_0}^3,\ {X_1}^3,\ {X_2}^3,\ {X_0}^2X_1,\ X_0{X_2}^2,\ {X_1}^2X_2\big\}\\
I_{3,7} &:= \big\{{X_0}^3,\ {X_1}^3,\ {X_2}^3,\ {X_0}^2X_1,\ X_0{X_2}^2,\ {X_1}^2X_2,\ X_0X_1X_2\big\}\\
I_{3,8} &:= \big\{{X_0}^3,\ {X_1}^3,\ {X_2}^3,\ {X_0}^2X_1,\ X_0{X_1}^2,\ {X_0}^2X_2,\ X_0{X_2}^2,\ {X_1}^2X_2\big\}\\
I_{3,9} &:= \big\{{X_0}^3,\ {X_1}^3,\ {X_2}^3,\ {X_0}^2X_1,\ X_0{X_1}^2,\ {X_0}^2X_2,\ X_0{X_2}^2,\ {X_1}^2X_2,\ X_1{X_2}^2\big\}
\end{aligned}
$$

In all cases but $(n, d) = (5, 2)$, the corresponding syzygy bundle is stable. $\square$

Proposition 3.4. *For any integers n and d such that $3d < n \leq \binom{d+2}{2}$, there is a set $I_{d,n}$ of n $\mathfrak{m}$-primary monomials in $K[X_0, X_1, X_2]$ of degree d such that the corresponding syzygy bundle $E_{d,n}$ is stable.*

Proof. We divide the proof in three cases. Let $j \geq 1$ be such that $3j < d$ and suppose that

$$\binom{d+2}{2} - \binom{d+2-3j}{2} < n \leq \binom{d+2}{2} - \binom{d+2-3(j+1)}{2}.$$

Note that as j varies, we get all values of n mentioned, except $\binom{d+2}{2}$ when d is a multiple of 3. However, for this highest possible value of n, the result follows from Proposition 2.7.

Case 1. Suppose that

$$n = \binom{d+2}{2} - \binom{d+2-3j}{2} + i = 3dj - \tfrac{9j(j-1)}{2} + i,$$

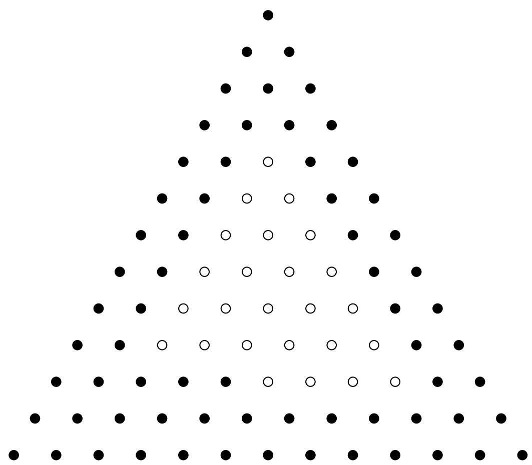
with $1 \leq i \leq d - 3j + 1$ and consider the set

$$I' := \left\{ {X_0}^{i_0} {X_1}^{i_1} {X_2}^{i_2} : i_0 + i_1 + i_2 = d \text{ and } (i_0 < j \vee i_1 < j \vee i_2 < j) \right\}.$$

Consider the sequence

$$\left({X_0}^{d-2j} {X_1}^{j} {X_2}^{j},\ {X_0}^{d-2j-1} {X_1}^{j+1} {X_2}^{j}, \ldots, {X_0}^{j} {X_1}^{d-2j} {X_2}^{j} \right).$$

Let I'' be the set of the first i monomials in this sequence and let $I_{d,n} = I' \cup I''$. Then $I_{d,n}$ has n monomials and we verify that it strictly satisfies inequality (3).



$I_{12,66}$

For $0 < d_J < d$, all we have to do is to count, in each case the number of multiples of ${X_0}^{d_J}$ which are present in $I_{d,n}$, since we are again applying the strategy mentioned in Proposition 3.2.

For $d - 2j \leq d_J < d$, all monomials of degree d of type ${X_0}^{i_0} {X_1}^{i_1} {X_2}^{i_2}$, with $i_0 \geq d_J$, are in $I_{d,n}$. Therefore the number of multiples of ${X_0}^{d_J}$ in $I_{d,n}$ is

$$k = \binom{d-d_J+2}{2}$$

and we get

$$(d - d_J)n + d_J - dk = (d - d_J)\left(3dj - \tfrac{9j(j-1)}{2} + i\right) + d_J - d\binom{d-d_J+2}{2}.$$

This expression can be rewritten in the two following ways:

$$\begin{aligned} &\tfrac{1}{2} d(d - d_J)(d_J + j - d) + \tfrac{5}{2}(d - 3j)(d - d_J)(j - 1) + 3(d - d_J)j(j - 1) + \\ &\quad + d(d - d_J) + (i - 1)(d - d_J) \end{aligned}$$

and

$$\begin{aligned}
&\tfrac{1}{2}d(d-d_J-j)(d_J+2j-d)+\tfrac{3}{2}(d-d_J-j)^2(j-1)+\\
&\quad+\tfrac{3}{2}(d-d_J-j)d_J(j-1)+\tfrac{1}{2}(d_J+2j-d)j^2+\\
&\quad+\tfrac{5}{2}(d_J-j)j(j-1)+\tfrac{3}{2}(d-d_J-j)j+d_Jj+\tfrac{1}{2}j^2+(i-1)(d-d_J).
\end{aligned}$$

From the first one, we can see that the expression above is positive for $d-j \le d_J < d$, and the second shows us positivity for $d-2j \le d_J < d-j$ (since $3j < d$, we get in this case $j < d_J$).

For $j \le d_J < d-2j$, the monomials in $I_{d,n}$ that are multiples of ${X_0}^{d_J}$ are the ones in the set

$$\begin{aligned}
J := &\left\{ {X_0}^{i_0}{X_1}^{i_1}{X_2}^{i_2} \in I' : i_0 \ge d_J \right\} \cup \\
&\cup \left\{ {X_0}^{d-2j}{X_1}^{j}{X_2}^{j},\ {X_0}^{d-2j-1}{X_1}^{j+1}{X_2}^{j}, \dots, {X_0}^{d-2j-e}{X_1}^{j+e}{X_2}^{j} \right\},
\end{aligned}$$

where $e := \min(i-1, d-2j-d_J)$. Therefore their number is

$$k = \tbinom{d-d_J+2}{2} - \tbinom{d-2j-d_J+2}{2} + e.$$

If $i-1 \le d-2j-d_J$, we get

$$\begin{aligned}
(d-d_J)n+d_J-dk &= (d-d_J)\left(3dj-\tfrac{9j(j-1)}{2}+i\right)+d_J-\\
&\quad -d\left(\tbinom{d-d_J+2}{2}-\tbinom{d-2j-d_J+2}{2}+i-1\right)=\\
&= (d-j-d_J)(d-2j-d_J)j+(d-2j-d_J)d_J(j-1)+\\
&\quad+\tfrac{1}{2}dj^2+\tfrac{7}{2}(d_J-j)j(j-1)+\tfrac{3}{2}j^2(j-1)+\\
&\quad+\tfrac{3}{2}(d-2j-d_J)j+\tfrac{1}{2}d_Jj+j^2+d+\\
&\quad+(d-2j-d_J+1-i)d_J > 0
\end{aligned}$$

since $d-2j-d_J > 0$ and $j \ge 1$. If $i-1 > d-2j-d_J$, we get

$$\begin{aligned}
(d-d_J)n+d_J-dk &= (d-d_J)\left(3dj-\tfrac{9j(j-1)}{2}+i\right)+d_J-\\
&\quad -d\left(\tbinom{d-d_J+2}{2}-\tbinom{d-2j-d_J+2}{2}+d-2j-d_J\right)=\\
&= (d-d_J-2j)(d-j)(j-1)+\tfrac{1}{2}dj^2+\tfrac{7}{2}(d_J-j)j(j-1)+\\
&\quad+\tfrac{3}{2}j^3+(d-d_J-2j)^2+\tfrac{5}{2}(d-d_J-2j)j+\tfrac{1}{2}(d_J-j)j+d+\\
&\quad+(i-1-d+2j+d_J)(d-d_J) > 0
\end{aligned}$$

since $d-2j-d_J > 0$ and $j \ge 1$.

For $0 < d_J < j$, the number of monomials in $I_{d,n}$ that are multiples of ${X_0}^{d_J}$ is

$$k = \tbinom{d-d_J+2}{2} - \tbinom{d-3j+2}{2} + i$$

and we get

$$\begin{aligned}(d-d_J)n+d_J-dk &= (d-d_J)\left(3dj-\tfrac{9j(j-1)}{2}+i\right)+d_J- \\ &\quad -d\left(\tbinom{d-d_J+2}{2}-\tbinom{d-3j+2}{2}+i\right)= \\ &= (d-3j)^2d_J+3(d-3j)(j-d_J)d_J+\tfrac{5}{2}(d-3j)d_J^2+ \\ &\quad +\tfrac{9}{2}j(j-d_J)d_J+3jd_J^2+\tfrac{1}{2}(d-3j)d_J+(d-3j+1-i)d_J>0.\end{aligned}$$

In all cases, inequality (3) is strictly satisfied, and hence the corresponding syzygy bundle is stable.

Case 2. Now suppose that

$$n=\tbinom{d+2}{2}-\tbinom{d+1-3j}{2}+i=3dj+d+1-\tfrac{3j(3j-1)}{2}+i,$$

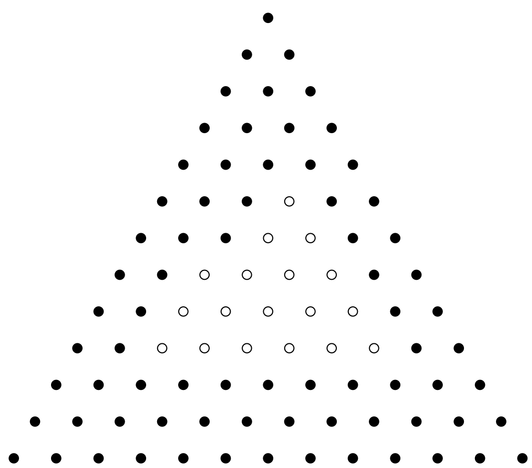
with $1 \leq i \leq d-3j$ and consider the set

$$I' := \left\{{X_0}^{i_0}{X_1}^{i_1}{X_2}^{i_2} : i_0+i_1+i_2=d \text{ and } (i_0<j \vee i_1<j \vee i_2 \leq j)\right\}.$$

Consider the sequence

$$\left({X_0}^j{X_1}^j{X_2}^{d-2j},\ {X_0}^{j+1}{X_1}^j{X_2}^{d-2j-1},\dots,{X_0}^{d-2j-1}{X_1}^j{X_2}^{j+1}\right).$$

Let I'' be the set of the first i monomials in this sequence and let $I_{d,n}=I'\cup I''$. Then $I_{d,n}$ has n monomials and we verify that it strictly satisfies inequality (3).



$I_{12,73}$

As in the previous step, for $0<d_J<d$, no monomial of degree d_J divides a greater number of monomials in $I_{d,n}$ than ${X_0}^{d_J}$. Therefore all we have to do is count, in each case the number of multiples of ${X_0}^{d_J}$ which are present in $I_{d,n}$.

For $d-2j \leq d_J < d$, all monomials of degree d of type ${X_0}^{i_0}{X_1}^{i_1}{X_2}^{i_2}$, with $i_0 \geq d_J$, are in $I_{d,n}$. Therefore the number of multiples of ${X_0}^{d_J}$ in $I_{d,n}$ is

$$k = \binom{d-d_J+2}{2},$$

as it was in step 1, and we can claim that since all values are the same except for n, which is bigger, inequality (2) is strictly satisfied, due to the fact that sequence $(a_{d,j})_{j\geq 2}$ is monotonically increasing (see Remark 2.10(b)).

For $j \leq d_J < d-2j$, the monomials in $I_{d,n}$ that are multiples of ${X_0}^{d_J}$ are the ones in the set

$$\begin{aligned} J := & \left\{ {X_0}^{i_0}{X_1}^{i_1}{X_2}^{i_2} \in I' : i_0 \geq d_J \right\} \cup \\ & \cup \left\{ {X_0}^{d_J}{X_1}^{j}{X_2}^{d-j-d_J},\, {X_0}^{d_J+1}{X_1}^{j}{X_2}^{d-j-d_J-1}, \dots, {X_0}^{j+i-1}{X_1}^{j}{X_2}^{d-2j-i+1} \right\}, \end{aligned}$$

where this last set is understood to be empty if $j+i-1 < d_J$. Therefore their number is

$$k = \binom{d-d_J+2}{2} - \binom{d-2j-d_J+1}{2} + \max(0, j+i-d_J).$$

If $j+i \leq d_J$, we get (keeping in mind that $i \geq 1$)

$$\begin{aligned} (d-d_J)n + d_J - dk = {} & (d-d_J)\left(3dj + d + 1 - \tfrac{3j(3j-1)}{2} + i\right) + d_J - \\ & - d\left(\binom{d-d_J+2}{2} - \binom{d-2j-d_J+1}{2}\right) = \\ = {} & (d-j)(d-2j-d_J)j + \tfrac{1}{2}dj^2 + \tfrac{7}{2}(d_J-j)j(j-1) + \tfrac{3}{2}j^3 + \\ & + \tfrac{1}{2}(d-2j-d_J)j + \tfrac{5}{2}(d_J-j)j + i(d-d_J) > 0. \end{aligned}$$

If $j+i > d_J$, we get

$$\begin{aligned} (d-d_J)n + d_J - dk = {} & (d-d_J)\left(3dj + d + 1 - \tfrac{3j(3j-1)}{2} + i\right) + d_J - \\ & - d\left(\binom{d-d_J+2}{2} - \binom{d-2j-d_J+1}{2} + j + i - d_J\right) = \\ = {} & (d-j)(d-2j-d_J)j + \tfrac{1}{2}(d-2j-d_J)j(j-1) + \\ & + 4(d_J-j)j^2 + 3j^3 + (d_J-j)j + (d-3j-i)d_J > 0. \end{aligned}$$

For $0 < d_J < j$, the number of monomials in $I_{d,n}$ that are multiples of ${X_0}^{d_J}$ is

$$k = \binom{d-d_J+2}{2} - \binom{d-3j+1}{2} + i$$

and we get

$$\begin{aligned} (d-d_J)n + d_J - dk = {} & (d-d_J)\left(3dj + d + 1 - \tfrac{3j(3j-1)}{2} + i\right) + d_J - \\ & - d\left(\binom{d-d_J+2}{2} - \binom{d-3j+1}{2} + i\right) = \\ = {} & (d-j)(d-3j)d_J + d(j-d_J)d_J + \tfrac{1}{2}dd_J(d_J-1) + \\ & + \tfrac{3}{2}j^2d_J + \tfrac{3}{2}jd_J + (d-3j-i)d_J > 0. \end{aligned}$$

Again in all cases, inequality (3) are strictly satisfied and the associated syzygy bundle is stable.

Case 3. If $d = 3j + 1$, case 2 has exhausted all possible monic monomials of degree d, and this proof is ended.

If $d > 3j + 1$, then suppose that

$$n = \binom{d+2}{2} - \binom{d-3j}{2} + i = 3dj + 2d + 1 - \tfrac{3j(3j+1)}{2} + i,$$

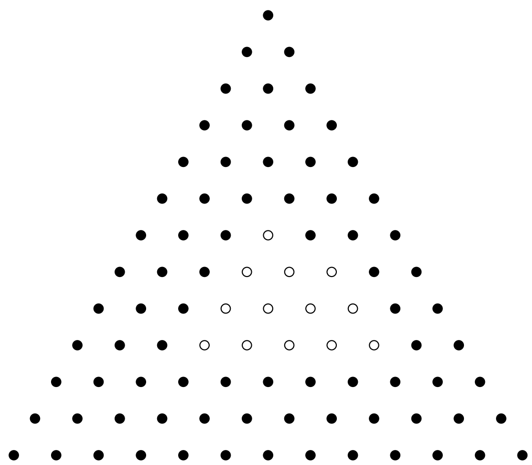
with $1 \leq i \leq d - 3j - 1$, and consider the set

$$I' := \left\{ {X_0}^{i_0} {X_1}^{i_1} {X_2}^{i_2} : i_0 + i_1 + i_2 = d \text{ and } (i_0 < j \vee i_1 \leq j \vee i_2 \leq j) \right\}.$$

Consider the ordered multiple

$$\left({X_0}^{j} {X_1}^{j+1} {X_2}^{d-2j-1}, \; {X_0}^{j} {X_1}^{j+2} {X_2}^{d-2j-2}, \ldots, {X_0}^{j} {X_1}^{d-2j-1} {X_2}^{j+1} \right).$$

Let I'' be the set of the first i monomials in this ordered multiple and let $I_{d,n} = I' \cup I''$. Then $I_{d,n}$ has n monomials and we shall verify that it strictly satisfies inequality (3).



$I_{12,78}$

As in the previous cases, all we have to do is count, in each case the number of multiples of ${X_0}^{d_J}$ which are present in $I_{d,n}$.

For $d - 2j \leq d_J < d$, all monomials of degree d of type ${X_0}^{i_0} {X_1}^{i_1} {X_2}^{i_2}$, with $i_0 \geq d_J$, are in $I_{d,n}$. Therefore the number of multiples of ${X_0}^{d_J}$ in $I_{d,n}$ is

$$k = \binom{d - d_J + 2}{2},$$

as it was in cases 1 and 2, and we can claim that since all values are the same except for n, which is bigger, inequality (3) are strictly satisfied, due to the fact that sequence $(a_{d,j})_{j \geq 2}$ is monotonically increasing (see Remark 2.10(b)).

For $j < d_J < d - 2j$, an analogous argument based on calculations for case 2 allows us to claim that inequality (3) is strictly satisfied.

For $0 < d_J \le j$, the number of monomials in $I_{d,n}$ that are multiples of ${X_0}^{d_J}$ is

$$k = \binom{d-d_J+2}{2} - \binom{d-3j}{2} + i$$

and we get (keeping in mind that $i \le d - 3j - 1$)

$$\begin{aligned}(d-d_J)n + d_J - dk &= (d-d_J)\left(3dj + 2d + 1 - \tfrac{3j(3j+1)}{2} + i\right) + d_J - \\ &- d\left(\binom{d-d_J+2}{2} - \binom{d-3j}{2} + i\right) = \\ &= (d-2j)(d-3j)d_J + 2(d-j)(j-d_J)d_J + \tfrac{3}{2}(d-j)d_J(d_J-1) + \\ &+ \tfrac{1}{2}j(j-d_J)d_J + 3jd_J + d_J + (d-3j-1-i)d_J > 0.\end{aligned}$$

Again in all cases, inequality (3) is strictly satisfied, which makes the syzygy bundle stable, and concludes the proof. $\square$

Putting all together we have got

Theorem 3.5. *For any integers $d, n \ge 1$ with $(n, d) \ne (5, 2)$ and $3 \le n \le \binom{d+2}{2}$, there is a family of n $\mathfrak{m}$-primary monomials in $K[X_0, X_1, X_2]$ of degree d such that the corresponding syzygy bundle is stable. For $(n, d) = (5, 2)$, there are 5 $\mathfrak{m}$-primary monomials in $K[X_0, X_1, X_2]$ of degree 2 such that the corresponding syzygy bundle is semistable.*

As an immediate consequence of Theorem 3.5 we obtain

Corollary 3.6. *Let $E_{d,n}$ be the syzygy bundle on $\mathbb{P}^2$ associated to n general $\mathfrak{m}$-primary forms of the same degree d. Suppose that $3 \le n \le \binom{d+2}{2}$. Then $E_{d,n}$ is stable when $(n, d) \ne (5, 2)$ and $E_{2,5}$ is semistable.*

Proof. It follows from Theorem 3.5, taking into account that stability is an open property. $\square$

4. Moduli spaces of syzygy bundles

In this section we study the moduli space of syzygy bundles on $\mathbb{P}^N$. We denote by $M = M(r; c_1, \dots, c_s)$ the moduli space of rank r, stable vector bundles E on $\mathbb{P}^N$ with fixed Chern classes $c_i(E) = c_i$, for $i = 1, \dots, s = min(r, N)$. The existence of the moduli space $M(r; c_1, \dots, c_s)$ was established by Maruyama in 1977 (see [9] and [10]) and once the existence of the moduli space is established, the question arises as what can be said about its local and global structure. More precisely, what does the moduli space look like as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look like as a topological space? What is its geometry? Until now, there is no general answer to these questions. The goal of this section is to determine the unobstructedness of stable syzygy bundles $E_{d,n}$ on $\mathbb{P}^N$ and to compute the dimension of the irreducible component of the corresponding moduli space.

Let us start by analyzing whether a syzygy bundle on $\mathbb{P}^N$ is stable and to state our contribution to study (semi)stability properties of syzygy bundles on $\mathbb{P}^N$. This will improve all previous known results, which we quickly recall now.

Let $C \subset \mathbb{P}^N$ be a smooth, projective, elliptic curve embedded by a complete system of degree $N+1$. Using the fact that the restriction of a general syzygy bundle $E_{d,n}$ on $\mathbb{P}^N$ to C is (semi)stable, Hein proved:

Proposition 4.1. *Let $f_1, \dots, f_n \in K[X_0, X_1, \dots, X_N]$, $N \geq 2$, denote generic homogeneous forms of degree d. Suppose that $N+1 \leq n \leq d(N+1)$. Then the syzygy bundle $E_{d,n}$ on $\mathbb{P}^N$ is semistable.*

Proof. See [2], Theorem 8.6 and Theorem A.1. □

As another application of Theorem 3.5, we can improve the above proposition and we get

Theorem 4.2. *Let $f_1, \dots, f_n \in K[X_0, X_1, \dots, X_N]$, $N \geq 2$, denote generic homogeneous polynomials of degree d. Suppose $N+1 \leq n \leq \binom{d+2}{2} + N - 2$. Then the syzygy bundle $E_{d,n}$ on $\mathbb{P}^N$ is stable when $(N,n,d) \neq (2,5,2)$, and $E_{2,5}$ is semistable on $\mathbb{P}^2$.*

Proof. Since stability is an open property in a flat family of torsion free sheaves, it is enough to prove the stability property for a single choice of homogeneous forms $f_1, \dots, f_n$ of degree d.

If $(n,d) \neq (5,2)$ we proceed by induction on N. Case $N=2$ follows from Theorem 3.5. Assume $N > 2$ and take an integer n such that $N+1 \leq n \leq \binom{d+2}{2} + N - 2$. By hypothesis of induction on N, there exists a family of $n-1$ $(X_0, X_1, \dots, X_{N-1})$-primary forms $g_1, \dots, g_{n-1} \in K[X_0, X_1, \dots, X_{N-1}]$ of degree d such that the kernel $F_{d,n-1}$ of the epimorphism

$$\phi := (g_1, \dots, g_{n-1}) : \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^{N-1}}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^{N-1}}$$

is a rank $n-2$ stable syzygy bundle. Consider the family of $n-1$ forms $f_1, \dots, f_{n-1} \in K[X_0, X_1, \dots, X_N]$ of degree d such that $f_i(X_0, X_1, \dots, X_{N-1}, X_N) = g_i(X_0, X_1, \dots, X_{N-1})$, $1 \leq i \leq n-1$, and add $f_n(X_0, \dots, X_N) = {X_N}^d$. Note that $f_1, \dots, f_n$ is a set of n $(X_0, X_1, \dots, X_N)$-primary forms and we denote by $E_{d,n}$ the syzygy bundle on $\mathbb{P}^N$ associated to them. Identifying $\mathbb{P}^{N-1}$ with the hyperplane $\{X_N = 0\}$ in $\mathbb{P}^N$, we have $E_{d,n}|_{\mathbb{P}^{N-1}} = F_{d,n-1}$ and since $F_{d,n-1}$ is stable, $E_{d,n}$ is also stable. Indeed, if there is a sub-bundle G destabilizing $E_{d,n}$, then $G|_{\mathbb{P}^{N-1}}$ destabilizes $F_{d,n-1}$ contradicting its stability.

Assume $(n,d) = (5,2)$. Note that in that case $2 \leq N \leq 4$. If $N=2$, $E_{5,2}$ is a semistable bundle on $\mathbb{P}^2$ by Theorem 3.5. If $N=4$, $E_{5,2}$ is a stable bundle on $\mathbb{P}^4$ by Proposition 2.6. Finally if $N=3$, we consider the set $I := \{{X_0}^2, {X_1}^2, {X_2}^2, {X_3}^2, X_0X_1\}$. The associated syzygy bundle is stable and hence, by the openness of the stability, $E_{5,2}$ is stable on $\mathbb{P}^3$. □

Remark 4.3. In general, the bound $N+1 \leq n \leq \binom{d+2}{2} + N - 2$ generalizes the bound $N+1 \leq n \leq d(N+1)$ given by Hein in [3], Theorem A.1.

We are now ready to state the unobstructedness of stable syzygy bundles on $\mathbb{P}^N$.

Theorem 4.4. *Assume $N+1 \leq n \leq \binom{d+2}{2}+N-2$, $N \neq 3$ and $(N,n,d) \neq (2,5,2)$. Then the syzygy bundle $E_{d,n}$ is unobstructed and it belongs to a generically smooth irreducible component of dimension $n\binom{d+N}{N} - n^2$, if $N \geq 4$, and $n\binom{d+2}{2} + n\binom{d-1}{2} - n^2$, if $N = 2$.*

Proof. Let us denote by $c_i = c_i(E_{d,n})$, $i = 1, \dots, \min(n-1, N)$ the ith Chern class of $E_{d,n}$ and let $M = M(n-1; c_1, \dots, c_{\min(n-1,N)})$ be the moduli space of rank $n-1$, stable vector bundles on $\mathbb{P}^N$ with Chern classes c_i. From deformation theory, we know that the Zariski tangent space of M at $[E_{d,n}]$ is canonically given by

$$T_{[E_{d,n}]}M \cong Ext^1(E_{d,n}, E_{d,n}) \cong H^1(E_{d,n} \otimes E_{d,n}^\vee);$$

and the obstruction space of the local ring $\mathcal{O}_{M,[E_{d,n}]}$ is a subspace of $Ext^2(E_{d,n}, E_{d,n})$. Thus, if

$$Ext^2(E_{d,n}, E_{d,n}) \cong H^2(E_{d,n} \otimes E_{d,n}^\vee) = 0,$$

then the moduli space M is smooth at $E_{d,n}$ and in this last case

$$\dim_K Ext^1(E_{d,n}, E_{d,n}) = \dim_{[E_{d,n}]} M(n-1; c_1, \dots, c_{\min(n-1,N)})$$

(see [9] and [10]).

To compute $Ext^i(E_{d,n}, E_{d,n})$, we consider the exact sequence

$$0 \longrightarrow E_{d,n} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-d)^n \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0 \tag{4}$$

and its dual

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(d)^n \longrightarrow E_{d,n}^\vee \longrightarrow 0. \tag{5}$$

First of all, note that by the cohomological exact sequence associated to the exact sequence (4), we get

$$\begin{aligned} &h^0(E_{d,n}) = 0;\\ &h^1(E_{d,n}) = 1;\\ &h^2(E_{d,n}) = \begin{cases} 0, & \text{if } \ N \geq 4 \\ n\binom{d-1}{2}, & \text{if } \ N = 2; \end{cases}\\ &h^3(E_{d,n}) = 0. \end{aligned} \tag{6}$$

Denote by $F = E_{d,n} \otimes E_{d,n}^\vee$. Consider the cohomological exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(E_{d,n}) \longrightarrow H^0(E_{d,n}(d))^n \longrightarrow H^0(F) \longrightarrow \\ \longrightarrow H^1(E_{d,n}) \longrightarrow H^1(E_{d,n}(d))^n \longrightarrow H^1(F) \longrightarrow \\ \longrightarrow H^2(E_{d,n}) \longrightarrow H^2(E_{d,n}(d))^n \longrightarrow H^2(F) \longrightarrow \\ \longrightarrow H^3(E_{d,n}) \longrightarrow H^3(E_{d,n}(d))^n \longrightarrow H^3(F) \longrightarrow \cdots \end{array} \tag{7}$$

associated to the exact sequence

$$0 \longrightarrow E_{d,n} \longrightarrow E_{d,n}(d)^n \longrightarrow F \longrightarrow 0.$$

Since $E_{d,n}$ is stable, it is simple, i.e. $H^0(F) = K$. Thus, from the exact sequence (7), and the fact that by (6), $H^0(E_{d,n}) = 0$, we get $H^0(E_{d,n}(d)) = 0$.

Twisting by $\mathcal{O}_{\mathbb{P}^N}(d)$ the exact sequence (4), and taking cohomology, we deduce

$$\begin{aligned} &h^2(E_{d,n}(d)) = 0, \\ &h^3(E_{d,n}(d)) = 0, \\ &h^1(E_{d,n}(d)) = \tbinom{N+d}{d} - n. \end{aligned} \tag{8}$$

In particular, from (7) we get $H^2(F) \cong Ext^2(E_{d,n}, E_{d,n}) = 0$ and the exact sequence

$$0 \longrightarrow K \longrightarrow K \longrightarrow H^1(E_{d,n}(d))^n \longrightarrow H^1(F) \longrightarrow H^2(E_{d,n}) \longrightarrow 0.$$

Therefore

$$h^1(F) = ext^1(E_{d,n}, E_{d,n}) = \begin{cases} n\binom{N+d}{d} - n^2, & \text{if } \quad N \geq 4 \\ n\binom{d+2}{2} + n\binom{d-1}{2} - n^2, & \text{if } \quad N = 2, \end{cases}$$

which finishes the proof. □

[1] G. Bohnhorst and H. Spindler, *The stability of certain vector bundles on* $\mathbb{P}^n$, Lecture Notes in Math., **1507** (1992), 39–50.

[2] H. Brenner, *Looking out for stable syzygy bundles*, Advances in Mathematics, **219** (2008), 401–427.

[3] H. Brenner *Tight closure and vector bundles. Three lectures on commutative algebra.*, Univ. Lecture Ser., **42**, Amer. Math. Soc., Providence, RI, (2008), 1–71.

[4] H.J. Hoppe, *Generischer Spaltungstyp und zweite Chernklasse stabiler Vektorraumbündel vom Rang 4 auf* $\mathbb{P}^4$, Math. Z., **187** (1984), 345–360.

[5] A. Klyachko, *Equivariant bundles over toric varieties* Math. USSR-Izv., **35** (1990), 337–375.

[6] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math., **4** (1998), 419–445.

[7] A. Klyachko, *Vector bundles, linear representations, and spectral problems*, Proceedings of the ICM, Vol. II (Beijing, 2002), 599–613.

[8] P. Macias Marques, *Stability and moduli spaces of syzygy bundles*, PhD Thesis, University of Barcelona. In preparation.

[9] M. Maruyama, *Moduli of stable sheaves, I*, J. Math. Kyoto Univ., **17** (1977), 91–126.

[10] M. Maruyama, *Moduli of stable sheaves, II*, J. Math. Kyoto Univ., **18** (1978), 557–614.

[11] R. Paoletti, *Stability of a class of homogeneous vector bundles on* $\mathbb{P}^n$, Boll. Un. Mat. Ital., **9** (1995), 329–343.