Weighted metrics on tangent sphere bundles

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December 21, 2010

Abstract

Natural metric structures on the tangent bundle and tangent sphere bundles S_rM of a Riemannian manifold M with radius function r enclose many important unsolved problems. Admitting metric connections on M with torsion, we deduce the equations of induced metric connections on those bundles. Then the equations of reducibility of TM to the almost Hermitian category. Our purpose is the study of the natural contact structure on S_rM and the G_2-twistor space of any oriented Riemannian 4-manifold.

Key Words: tangent sphere bundle, metric connection, complex, symplectic and contact structures.

MSC 2010: Primary: 53C05, 53D35, 58A05; Secondary: 53C05, 53C17

The author acknowledges the support of Fundação Ciência e Tecnologia, Portugal, through Centro de Investigação em Matemática e Aplicações da Universidade de Évora (CIMA-UE) and the sabbatical grant SFRH/BSAB/895/2009.

1.1 Introduction

This article is the first part of a study of the geometry of tangent sphere bundles $S_rM = \{u \in TM : \ \|u\| = r\}$ of a Riemannian manifold (M, g) with variable radius and weighted Sasaki metric.

It is today well established that any oriented Riemannian 4-manifold M gives rise to a canonical G_2 structure on S_1M. This was discovered in [7, 9, 10] partly recurring to twistor methods; so we call it the G_2-twistor bundle of M. Indeed, the pull-back of the volume form coupled with each point $u \in S_1M$, say a 3-form α, induces a quaternionic

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structure which is reproduced twice in horizontal and vertical parts of T_uS_1M. Then the Cayley-Dickson process gives the desired $G_2 = \mathrm{Aut}\,\mathbb{O}$-structure over S_1M. Some properties of the so called *gwistor space* have been discovered, namely that it is cocalibrated if and only if the 4-manifold is Einstein. The first variation of that structure, which may yield interesting features, is by choosing both any metric connection (i.e. with torsion) or a different weigh on both the horizontal and vertical sides of the Sasaki metric. Another open problem in the theory resides in understanding a certain tensor $\mathcal{R}^\xi\alpha$ which consists of a derivation of α by the curvature of M. The G_2-representation theory on the gwistor space adds further perspectives upon the well known $SO(4)$ theory of metric compatible tensors on M. Henceforth we were led to a study of isometries of tangent sphere bundles ([8]). A study of the curvature of S_rM will appear as the second part of this work.

Throughout, we assume that M is an m-dimensional manifold. We start by viewing a rather personal construction of the tangent bundle $TM \stackrel{\pi}{\to} M$, i.e. the $2m$-dimensional point-vector manifold which governs most of the differential geometry of M.

Next we assume a Riemannian metric g and a compatible metric connection ∇ on M. The latter induces a splitting of $TTM = H \oplus V$ with both H, V parallel and isometric to π^*TM, the pull-back bundle. We review the known classification of g-natural metrics on TM by [1, 2, 3] and continue our study assuming metrics of the kind $f_1\pi^*g \oplus f_2\pi^*g + f_3\mu\otimes\mu$ on $H \oplus V$, where f_1, f_2, f_3 are certain $\mathbb{R}$-valued functions on M and μ is a canonical 1-form.

We have in view the study of the induced metric on the tangent sphere bundle S_rM with variable radius function $r \in \mathrm{C}^\infty_M$. There exist Einstein metrics in some examples, precisely with those metrics for which $f_3 \neq 0$.

We proceed with the weighted metric $g^{f_1,f_2} = f_1\pi^*g \oplus f_2\pi^*g$ with $f_1, f_2 > 0$. Recall the Sasaki metric is just $g^S = g^{1,1}$ with H induced by the Levi-Civita connection. We construct an almost complex structure I^G and the associated symplectic structure on TM, first announced in [8] without proofs. In studying the equations of integrability, the roles of the functions f_1, f_2 are clearly distinguished. We deduce the torsion T^∇ must be of a precise vectorial type. As a corollary we find that the functions only have to be both constant, the curvature flat and the torsion zero, if and only if we require the structure on TM to be Kähler.

The canonical symplectic structure of T^*M arising from the Liouville form is here related, implying further understanding of the contact geometry of the (co-)tangent sphere bundle. Long before G_2-twistor space, Y. Tashiro showed S_1M admits a canonical metric contact structure. We present here a complete generalization of this result.

Parts of this article were written during a sabbatical leave at Philipps-Universität Marburg. The author wishes to thank their hospitality and expresses his gratitude to Ilka Agricola, from Philipps-Universität, also for her careful reading of many parts of this article.

This article is dedicated to Ilda Figueiredo, member of the European Parliament and Teacher.

1.2 Differential geometry of the tangent bundle

Let M be an m-dimensional smooth manifold. Suppose we are given two charts (U_1,ϕ_1) and (U_2,ϕ_2), two points $x\in U_1,\ y\in U_2$ and two vectors $v_1,v_2\in\mathbb{R}^m$. Then we may define an equivalence relation between these objects:

$$(U_1,\phi_1,x,v_1)\sim(U_2,\phi_2,y,v_2)\ \Leftrightarrow\ \begin{cases} x=y, \\ \mathrm{d}(\phi_2\circ{\phi_1}^{-1})_{\phi_1(x)}(v_1)=v_2 \end{cases}. \tag{1}$$

Notice that it might happen $U_1\cap U_2\neq\emptyset$. This equivalence relation gives place to a new finite dimensional manifold

$$TM=\frac{\bigsqcup U\times\mathbb{R}^m}{\sim} \tag{2}$$

by gluing all the charts $U\times\mathbb{R}^m$ (the set of charts arising from a covering of M in the same atlas being sufficient, within the same differentiable structure). Charts of lower differentiable class clearly induce the same TM, which is called the tangent vector bundle.

We define a tangent vector X_x at the point x to be the class $X_x=[(\phi,x,v)]$ given a chart ϕ on a neighbourhood of x and a $v\in\mathbb{R}^m$. In particular we have $TM=\bigcup_{x\in M}T_xM$, where T_xM is the tangent space at x, the set of tangent vectors at x, naturally endowed with the structure of Euclidean space. In particular, one usually denotes ∂_i or $\frac{\partial}{\partial\phi^i}(x)=[\phi,x,e_i]$ when e_i is a vector from the canonical basis of $\mathbb{R}^m$. We also write shortly $u\in TM$ to refer to a vector, without mentioning the base point x to which it corresponds. There is a bundle projection $\pi:TM\rightarrow M$ which stands for this relation, $\pi(u)=x$. We may also see $\pi^{-1}(x)=T_xM$.

Given two manifolds M,N and a smooth map f between them, the classes defined above are known to correctly transform under a map, called differentiation and denoted $\mathrm{d}f$. It is defined from the tangent bundle TM into f^*TN and essentially described by

$$\mathrm{d}f([\phi,x,v])=[\psi,f(x),\mathrm{d}(\psi\circ f\circ\phi^{-1})_{\phi(x)}(v)] \tag{3}$$

with the obvious notation.

As a manifold, TM has its own tangent vector bundle $TTM\rightarrow TM$. If we differentiate π, then $V=\ker\mathrm{d}\pi$ is the vertical bundle tangent to TM. There are canonical induced charts around a point $u=[\phi,x,v]$; we may then write T_uTM vectors as $[\phi\times 1_m,[\phi,x,v],(v_1,v_2)]$, which demonstrates the existence of a canonical embedding of T_xM as $V_u\subset T_uTM$ (the set of all tangent vectors such that $v_1=0$ does not vary with the charts). Hence we have a canonical identification of $V=\pi^*TM$ and a short exact sequence

$$0\longrightarrow V\longrightarrow TTM\xrightarrow{\mathrm{d}\pi}\pi^*TM\longrightarrow 0. \tag{4}$$

As it is also well known, a vector field X is a section of the tangent bundle. The tangent bundle TTM has a canonical vector field denoted ξ. It is defined by $\xi_u=u$, thus defined as a vertical vector field.

If we want to differentiate X in various ways and directions and compare the results, then it is useful to have a *linear connection* in order to respect the vector bundles in which the derivatives appear. We thence suppose we have a connection ∇ on M. Then

$$H=\{X\in TTM:\ \pi^*\nabla_X\xi=0\} \tag{5}$$

is a complement for V. Indeed, picking a chart such that $\nabla_i\partial_j=\sum_l\Gamma^l_{ij}\partial_l$, defining the Christoffel symbols of the connection, and writing shortly $X_u=[(x,v),(v_1,v_2)],\ \xi=[(x,v),(0,v)]$, with $v=\sum v^je_j,\ v_1=\sum a^j_1e_j,\ v_2=\sum a^j_2e_j$, we find

$$\pi^*\nabla_X\xi=\sum_i \mathrm{d}v^i(X)\partial_i+v^i\nabla_{\mathrm{d}\pi(X)}\partial_i=\sum_i(a^i_2+\sum_{j,k}v^ja^k_1\Gamma^i_{kj})\partial_i.$$

Notice that, if $X\in V$, then $a^k_1=0,\ \forall k$, so that $\pi^*\nabla_X\xi=\sum a^i_2\partial_i=X$ (we have abbreviated ∂_i for $\pi^*\partial_i$). Thus clearly the m-dimensional kernel H is a complement for V. Moreover, $\pi^*\nabla_{\cdot}\xi$ is the vertical projection onto V. For any vector field X over TM we may always find the unique decomposition (∇^* denotes the pull-back connection)

$$X=X^h+X^v=X^h+\nabla^*_X\xi. \tag{6}$$

As a corollary to these observations, for a parametrized curve $\gamma\subset M$ we have that

$$\gamma \text{ is a geodesic of } \nabla\ \Leftrightarrow\ \dot\gamma \text{ is horizontal, i.e. } \ddot\gamma\in H. \tag{7}$$

Indeed, $\pi\dot\gamma=\gamma$, so the chain rule gives $\mathrm{d}\pi(\ddot\gamma)=\dot\gamma$ and thence we have $\ddot\gamma=[(\gamma,\dot\gamma),(\dot\gamma,\ddot\gamma)]$ in any given chart. Finally the equation $\nabla^*_{\ddot\gamma}\xi=0$, taking from above and introducing the chart components, becomes $\ddot\gamma^i+\sum_{j,k}\dot\gamma^j\dot\gamma^k\Gamma^i_{kj}=0$, which is the equation of geodesics.

Now, $\mathrm{d}\pi$ induces an isomorphism between H and π^*TM, cf. (4), and we have $V=\pi^*TM$, this being by definition the kernel of $\mathrm{d}\pi$. Hence we may define a vector bundle endomorphism

$$\theta:TTM\longrightarrow TTM \tag{8}$$

sending X^h to the respective $\theta X^h\in V$ and sending V to 0. We also define an endomorphism, denoted θ^t, which gives $\theta^tX^v\in H$ and which annihilates H. In particular $\theta^t\theta X^h=X^h$ and $\theta^2=0$. We remark that the role of the morphism θ is not considered by other authors studying the tangent bundle. Sometimes we call θX^h the mirror image of X^h in V. The map θ was first used in [7, 9, 10].

Another main instrument to use in our study, adapted from the theory of twistor spaces, is given as follows. We endow TTM with the direct sum connection $\nabla^*\oplus\nabla^*$, which we denote simply by ∇^* or even just ∇. We have that $\nabla^*\theta=\nabla^*\theta^t=0$.

Remark. Away from the zero section, i.e. on $TM\backslash M$, we have a line bundle $\mathbb{R}\xi\subset\pi^*TM$. Notice the canonical section can be mirrored by θ^t to give another canonical vector field $\theta^t\xi$ and therefore a line bundle too, sub-bundle of H. This canonical horizontal vector field $\theta^t\xi$ is called the *spray* of the connection in [16, 23] or called *geodesic field* in the more recent [17]. It has the further property that $\mathrm{d}\pi_u(\theta^t\xi)=u,\ \forall u\in TM$.

1.3 Natural metrics on TM

Suppose the previous manifold M,∇ is also furnished with a Riemannian metric g. We also use $\langle\ ,\ \rangle$ in place of the symmetric tensor g; this same remark on notation is valid for the pull-back metric on π^*TM. We recall from [25] the now called Sasaki metric in $TTM = H\oplus V$: it is given by $g^S=\pi^*g\oplus\pi^*g$. With it, $\theta_| : H\rightarrow V$ is an isometric morphism and θ^t corresponds with the adjoint endomorphism of θ. We stress that $\langle\ ,\ \rangle$ on tangent vectors to the tangent bundle, with $V\perp H$, always refers to the Sasaki metric.

With the canonical vector field ξ we may produce other symmetric tensors over TM: first the linear forms

$$\xi^\flat \qquad \text{and} \qquad \mu=\xi^\flat\circ\theta \tag{9}$$

and then the three symmetric products of these.

Remark. In fact one may see that the 1-form μ does not depend on the chosen metric connection (it is the pull-back of the Liouville form on the co-tangent bundle under the musical isomorphism, cf. section 1.6).

The classification of all natural metrics on TM induced from g may be found in [1, 2, 3]. An analysis of the convexity properties has shown that the metrics correspond with six weight functions $f_1,\ldots,f_6$ which depend only on $\|u\|^2_g$, $u\in TM$. So we assume the $f_i:[0,+\infty)\rightarrow\mathbb{R}$ below are composed with the squared norm. First let, $\forall X,Y\in TTM$,

$$\hat{g}(X,Y)=g^S(\theta X,Y)+g^S(X,\theta Y)=\langle\theta X,Y\rangle+\langle\theta Y,X\rangle. \tag{10}$$

This is a metric of signature (n,n). Also let

$$g^{f_1,f_2}=f_1\pi^*g\ \oplus\ f_2\pi^*g \tag{11}$$

so $g^S=g^{1,1}$.

The referred classification may be written quite easily in the present setting. Following [3, Corollary 2.4], the statement is that every natural metric on TM is given by

$$G=g^{f_1,f_2}+f_3\hat{g}+f_4\xi^\flat\otimes\xi^\flat+f_5\xi^\flat\odot\mu+f_6\mu\otimes\mu \tag{12}$$

with further conditions, inequalities, on those functions to assure G is positive definite. The interested reader may see properties of G in general in [1, 3, 11, 12, 16, 18, 21, 22] and other references therein. One of the peculiar natural metrics is the Cheeger-Gromoll metric: $G^{\text{C-G}}=g^{1,f_2}+f_2\xi^\flat\otimes\xi^\flat$ with $f_2=\frac{1}{1+\|u\|^2},\ u\in TM$, and this has been studied by quite a few authors, cf. [12, 19, 20].

1.4 Some connections on TM

Let M be a Riemannian manifold of dimension $m=n+1$ with $n\geq1$ and let us continue to denote the metric by $g=\langle\ ,\ \rangle$ and the linear connection by ∇. From now on we assume the connection is metric, which implies $\nabla^* g^S = 0$. Let $r \in \mathrm{C}^\infty_M$ be a function on M. Then we may consider the tangent sphere bundle of radius r

$$S_rM = \{u \in TM : \ \|u\|^2_g = r^2\}. \tag{13}$$

It is a $2n+1$-dimensional submanifold of TM, which carries a canonical contact structure for certain metrics. This was found by Y. Tashiro in [26] and will be dealt with later. We refer the reader to [14] for a state of the art on this development.

We shall be interested in the case of r constant and thus on the metrics G defined in (12) for which we may write $T_uS_rM = u^\perp$, the G orthogonal subspace. Since now $\xi^\flat = 0$ on the hypersurface and r is constant, it is not hard to see that we are referring only to metrics of the form

$$G = g^{f_1,f_2} + f_3\mu\otimes\mu \qquad \text{with} \quad f_1, f_2, f_3 \in \mathrm{C}^\infty_M \ \text{ and} \tag{14}$$

with the functions f_1, f_2, f_3 (obviously we let these functions be composed with π on the right hand side when used on the manifold TM) such that $f_1, f_2 > 0$ and $f_1 + f_3 > 0$. We thus assume

$$f_1 = \mathrm{e}^{2\varphi_1}, \qquad f_2 = \mathrm{e}^{2\varphi_2} \tag{15}$$

for some functions φ_1, φ_2 on M.

If ∇ is a metric connection for g, i.e. makes g parallel, then it is well known that $\nabla^{f_1} = \nabla + C_1$, with

$$C_1(X,Y) = X(\varphi_1)Y + Y(\varphi_1)X - \langle X,Y\rangle \operatorname{grad}\varphi_1, \tag{16}$$

is a metric connection for $f_1 g$ on M with the same torsion as ∇ (cf. [13, Theorem 1.159]). We denote

$$X(\varphi) = \mathrm{d}\varphi(X) = \langle \operatorname{grad}\varphi, X\rangle. \tag{17}$$

On TM we define the function $\partial\varphi(u) = \mathrm{d}\varphi_{\pi(u)}(u)$, $\forall u$. In other words,

$$\partial\varphi = \langle \theta\pi^*\operatorname{grad}\varphi, \xi\rangle \tag{18}$$

where θ is the map introduced in (8). In particular, $\nabla^{*,f_1} = \nabla^* + \pi^*C_1$ makes $f_1\pi^*g$ parallel on H and $\nabla^{*,f_2}_X Y = \nabla^*_X Y + \theta\pi^*C_2(X, \theta^t Y)$ makes $f_2\pi^*g$ parallel on V. Now the analysis of $\nabla^{*,f_1}(f_3\mu\otimes\mu)$ gives a quite complicated expression. It simplifies if we assume $f_3 = 0$ or both f_1, f_3 are constant.

Proposition 1.1. *Consider the linear connection $\tilde{D}^* = \nabla^{*,f_1} \oplus \nabla^{*,f_2}$ over TM.*
(i) If $f_3 = 0$, then g^{f_1,f_2} is parallel for the connection $\tilde{D}^*$.
(ii) If f_1, f_3 are constants, then $G = g^{f_1,f_2} + f_3\mu\otimes\mu$ is parallel for $\tilde{\tilde{D}}^* = \tilde{D}^* + K$ where

$$K_XY = \left(\frac{f_3}{f_1}\theta^tX - \frac{r^2f_3^2}{(r^2f_3+f_1)f_1}\Omega(X)\theta^t\xi\right)\mu(Y) \tag{19}$$

and where $r^2 = \|\xi\|^2$ and $\Omega(X) = \frac{1}{r^2}\langle\xi, X^v\rangle$ so that $\mu(\theta^tX) = \Omega(X)r^2$.
(iii) The connection $\nabla^{*,f_2,\prime}_X Y = \nabla^*_X Y + X(\varphi_2)Y$ is also metric on $(V, f_2\pi^*g)$.

Proof. (i) The first assertion was proved earlier.
(ii) Since

$$\begin{aligned}\nabla^{*,f_1}_X\mu\, Y &= X(\mu Y)-\mu(\nabla^{*,f_1}_X Y)\\ &= \langle\nabla^*_X(\theta Y),\xi\rangle+\langle\theta Y,\nabla^*_X\xi\rangle-\langle\theta\nabla^{*,f_1}_X Y,\xi\rangle\\ &= \langle\theta Y,X\rangle-X(\varphi_1)\mu(Y)-Y(\varphi_1)\mu(X)+\langle X^h,Y\rangle\partial\varphi_1\end{aligned}$$

we find $\tilde{D}^*_X(f_3\mu\otimes\mu)=X(f_3)\mu\otimes\mu+f_3\nabla^{*,f_1}_X\mu\odot\mu=X(f_3)\mu\otimes\mu+f_3\big(X^\flat\circ\theta-X(\varphi_1)\mu-\mu(X)\mathrm{d}\varphi_1+\partial\varphi_1.(X^h)^\flat\big)\odot\mu$. So if f_3 and f_1 are constant, this derivative becomes $f_3X^\flat\circ\theta\odot\mu$. Notice $X^\flat\theta=(\theta^tX)^\flat$. Now writing $\tilde{\tilde{D}}^*=\tilde{D}^*+K$ we find

$$\begin{aligned}-\tilde{\tilde{D}}^*_X G(Y,Z) &= -(\tilde{D}^*_X G+K_X\cdot G)(Y,Z)\\ &= f_1\langle K^h_XY,Z^h\rangle+f_2\langle K^v_XY,Z^v\rangle-f_3\langle\theta^tX,Y\rangle\mu(Z)\\ &\quad -f_3\langle\theta^tX,Z\rangle\mu(Y)+f_3\mu(K_XY)\mu(Z)+f_3\mu(Y)\mu(K_XZ)+\\ &\quad +f_1\langle Y^h,K^h_XZ\rangle+f_2\langle Y^v,K^v_XZ\rangle.\end{aligned}$$

which has the solution given in (19); notice in particular $K^v=0$.
(iii) We have for any vector X

$$\begin{aligned}\nabla^{*,f_2,'}_X f_2\pi^*g &= X(f_2)\pi^*g+f_2\nabla^{*,f_2,'}_X\pi^*g\\ &= X(f_2)\pi^*g+f_2\nabla^*_X\pi^*g-2f_2X(\varphi_2)\pi^*g=0\end{aligned}$$

using $X(f_2)=2f_2X(\varphi_2)$. ∎

As the reader shall see, the last connection of the three is more relevant than the other given on V. We remark

$$\nabla^{*,f_2}_X Y-\nabla^{*,f_2,'}_X Y=\langle\theta\mathrm{grad}\,\varphi_2,Y\rangle\theta X-\langle\theta X,Y\rangle\theta\mathrm{grad}\,\varphi_2. \tag{20}$$

We have now a metric connection for each of the two cases mentioned above. With some extra work it is possible to find the Levi-Civita connection. However, for the moment, it seems rather cumbersome to study the analogous metric of the Cheeger-Gromoll metric, referring here to the extra weight $f_3\mu\otimes\mu$ on H instead of V. Although this new metric is non-trivial on the tangent sphere bundles.

1.5 The weighted metric g^{f_1,f_2}

We shall proceed with the metric $G=g^{f_1,f_2}$. Recall this metric is supported by the decomposition $H\oplus V$ and H depends on ∇. Moreover the projections $\cdot^h$ and $\cdot^v$ act accordingly. We now give a generalization of [6, Theorem 3.1].

First, we recall the metric connection on V:

$$\nabla^{*,f_2,'}_X Y=\nabla^*_X Y+X(\varphi_2)Y \tag{21}$$

and thus we define D^* as

$$D^* = \nabla^{*,f_1} \oplus \nabla^{*,f_2,\prime} \tag{22}$$

and define a tensor $B \in \Omega^1(\mathrm{End}\, TTM)$ by

$$B(X,Y) = Y(\varphi_2)X^v - \frac{f_2}{f_1}\langle X^v, Y^v\rangle \mathrm{grad}\,\varphi_2, \tag{23}$$

where $\mathrm{grad}\,\varphi_2$ is the horizontal lift of the gradient. Also we let $R^* = \pi^* R^\nabla = R^{\pi^*\nabla}$ denote the curvature tensor of ∇^* and let $\mathcal{R}^\xi = R^*\xi$. Notice $\mathcal{R}^\xi(X,Y) = \mathcal{R}^\xi(X^h, Y^h)$ and that we have $\mathcal{R}^\xi \in \Omega^2(V)$. Then we let A and τ be H-valued tensors defined respectively by

$$f_1\langle A_X Y, Z\rangle = \frac{f_2}{2}\big(\langle R^*_{X^h,Z^h}\xi, Y^v\rangle + \langle R^*_{Y^h,Z^h}\xi, X^v\rangle\big) \tag{24}$$

and

$$\tau(X,Y,Z) = \langle \tau_X Y, Z^h\rangle = \frac{1}{2}\big(T(Y,X,Z) + T(X,Z,Y) + T(Y,Z,X)\big), \tag{25}$$

with $T(X,Y,Z) = \langle \pi^* T^\nabla(X,Y), Z\rangle$ for any vector fields X, Y, Z over TM, cf. (27).

Notice $\mathcal{R}^\xi(X,Y)$ and $\tau(X,Y,Z)$ vanish if one of the directions X, Y or Z is vertical, whereas with $A(X,Y)$ the same happens if both X, Y are vertical or both are horizontal. Hence, A could be defined simply by $f_1\langle A_X Y, Z\rangle = \frac{f_2}{2}(\langle \mathcal{R}^\xi{}_{X,Z}, Y\rangle + \langle \mathcal{R}^\xi{}_{Y,Z}, X\rangle)$.

Proposition 1.2. *The torsion of $\nabla^* \oplus \nabla^*$ is $\pi^* T^\nabla + \mathcal{R}^\xi$.*

The proof of this essential equation is within the lines of the following result.

Theorem 1.1. *The Levi-Civita connection ∇^G of TM with metric $G = g^{f_1,f_2}$ is given by*

$$\nabla^G_X Y = D^*_X Y - \frac{1}{2}\mathcal{R}^\xi(X,Y) + A(X,Y) + B(X,Y) + \tau(X,Y) \tag{26}$$

$\forall X, Y$ vector fields over TM.

Proof. Let us assume the identity and first see the horizontal part of the torsion:

$$\begin{aligned} \mathrm{d}\pi(T^{\nabla^G}(X,Y)) &= D^*_X Y^h + A_X Y + B^h_X Y + \tau_X Y \\ &\qquad - D^*_Y X^h - A_Y X - B^h_X Y - \tau_Y X - \mathrm{d}\pi[X,Y] \\ &= \pi^* T^\nabla(X,Y) + \tau_X Y - \tau_Y X, \end{aligned}$$

since this is how the torsion tensor of ∇ lifts to $\pi^* TM$ and since A and B^h are symmetric tensors. Also recall C_1 is symmetric, so the torsion $T^\nabla = T^{\nabla^{f_1}}$. Now we check the vertical part:

$$\begin{aligned} (T^{\nabla^G}(X,Y))^v &= D^*_X Y^v - \frac{1}{2}R^*_{X,Y}\xi + B^v_X Y - D^*_Y X^v + \frac{1}{2}R^*_{Y,X}\xi - B^v_Y X - [X,Y]^v \\ &= \nabla^*_X \nabla^*_Y \xi + X(\varphi_2)Y^v - R^*_{X,Y}\xi + Y(\varphi_2)X^v \\ &\quad - \nabla^*_Y \nabla^*_X \xi - Y(\varphi_2)X^v - X(\varphi_2)Y^v - \nabla^*_{[X,Y]}\xi \;=\; 0. \end{aligned}$$

∇^G is a metric connection if and only if the difference with D^* is skew-adjoint. Then, on one hand,

$$\begin{aligned}
G((\nabla^G - D^*)_X Y, Z) \ &= \\
&= -\frac{f_2}{2}\langle \mathcal{R}^\xi{}_{X,Y}, Z^v\rangle + f_1\langle A_X Y + \tau_X Y, Z^h\rangle + G(B_X Y, Z) \\
&= -\frac{f_2}{2}\langle R^*_{X,Y}\xi, Z^v\rangle + \frac{f_2}{2}\langle R^*_{X^h,Z^h}\xi, Y^v\rangle + \frac{f_2}{2}\langle R^*_{Y^h,Z^h}\xi, X^v\rangle + \\
&\quad + f_1\tau(X,Y,Z) + f_2 Y(\varphi_2)\langle X^v, Z^v\rangle - f_2\langle X^v, Y^v\rangle\langle \mathrm{grad}\,\varphi_2, Z^h\rangle
\end{aligned}$$

and, on the other,

$$\begin{aligned}
G((\nabla^G - D^*)_X Z, Y) \ &= \\
&= -\frac{f_2}{2}\langle \mathcal{R}^\xi{}_{X,Z}, Y^v\rangle + f_1\langle A_X Z + \tau_X Z, Y^h\rangle + G(B_X Z, Y) \\
&= -\frac{f_2}{2}\langle R^*_{X,Z}\xi, Y^v\rangle + \frac{f_2}{2}\langle R^*_{X^h,Y^h}\xi, Z^v\rangle + \frac{f_2}{2}\langle R^*_{Z^h,Y^h}\xi, X^v\rangle + \\
&\quad + f_1\tau(X,Z,Y) + f_2 Z(\varphi_2)\langle X^v, Y^v\rangle - f_2\langle X^v, Z^v\rangle\langle \mathrm{grad}\,\varphi_2, Y^h\rangle
\end{aligned}$$

hence the condition is expressed simply by $\tau(X,Y,Z) = -\tau(X,Z,Y)$. This, together with $\pi^*T^\nabla(X,Y) + \tau_X Y - \tau_Y X = 0$, determines τ uniquely as the form given by (25). ■

It is clear that H corresponds to an integrable distribution if and only if the connection ∇ is flat. Indeed, the vertical part of $[X,Y] = \nabla^G_X Y - \nabla^G_Y X$, for any pair of horizontal vector fields, is $\mathcal{R}^\xi(X,Y) = R^*_{X,Y}\xi$.

A first geometric consequence is at hand.

Corollary 1.1. *The fibres T_xM, $x \in M$, are totally geodesic submanifolds of TM if and only if f_2 is a constant.*

The zero section of TM, i.e. the embedding $M \subset TM$, is totally geodesic if and only if $R^\nabla = 0$.

Proof. In view of the observations prior to the theorem, if X,Y are two vertical vector fields, then $\nabla^G_X Y = \nabla^*_X Y - \frac{f_2}{f_1}\langle X^v, Y^v\rangle \mathrm{grad}\,\varphi_2$. Having this again in $\Gamma(V)$ is equivalent to the condition of each fibre being a totally geodesic submanifold. We immediately see that $\nabla^G_X Y$ is a vertical vector field if and only if $\mathrm{grad}\,\varphi_2 = 0$. The question for the trivial horizontal section is solved analogously. ■

It is important to understand when the tensor τ vanishes. We have the following result:

$$\tau = 0 \qquad \text{if and only if} \qquad T^\nabla = 0. \tag{27}$$

Indeed, if $\tau = 0$, then $T(Y,X,Z) = T(Z,X,Y) + T(Z,Y,X)$; by the symmetries in X,Y this tensor vanishes.

Remark. By a result of É. Cartan, cf. [4], it is known that the space of torsion tensors $\Lambda^2 TM \otimes TM$ of a metric connection decomposes into irreducible subspaces like

$$\mathcal{A} \oplus \Lambda^3 TM \oplus TM, \tag{28}$$

where Λ^3 is the one for which $\langle T^\nabla(X,Y), Z\rangle$ is completely skew-symmetric and where TM is the subspace of vectorial type torsions, i.e. for which there exists a vector field V such that $T^\nabla(X,Y) = \langle V, X\rangle Y - \langle V, Y\rangle X$. The invariant subspace $\mathcal{A}$ is the orthogonal to those two. We also remark that, in dimension 4, under the special orthogonal group the space $\mathcal{A}$ is further decomposable in two 8 dimensional subspaces. Since $\Lambda^3 TM^4$ is 4 dimensional, there is a second type of both vectorial and skew-symmetric torsion. This result has had consequences in [7].

1.6 Almost Hermitian structure

We continue the study of TM with the metric $G = g^{f_1,f_2}$ where $f_1 = \mathrm{e}^{2\varphi_1}$ and $f_2 = \mathrm{e}^{2\varphi_2}$. We let ∇ denote a metric connection on M with torsion T^∇. Some authors have studied an almost complex structure over TM compatible with the Sasaki metric g^S which was first discovered by Sasaki, cf. [16, 25]. It may be written as the bundle endomorphism $I^S = \theta^t - \theta$, see (8). We call (g^S, I^S) the Sasaki structure of TM, with torsion.

Some properties of the Sasaki metric related with its Hermitian structures I, J or $K = IJ$ and quaternionic-Hermitian structure (I, J, K), given by the natural almost complex structure $I = I^S$ and by an almost complex structure J on M pulled-back as $J \oplus J$, were studied in [6]. There we also admitted a metric connection with torsion for the study of I^S. We had in view the quaternionic-Kähler structure on TM, and may be generalized into the present setting too. In the next Theorem we need a formula from [6].

Let

$$\psi = \varphi_2 - \varphi_1, \qquad \overline{\psi} = \varphi_2 + \varphi_1. \tag{29}$$

We then define an endomorphism I^G by $I^G X = \mathrm{e}^{\psi}\theta^t X - \mathrm{e}^{-\psi}\theta X$ for all $X \in TTM$. Also we consider the associated *symplectic* structure ω^G, defined by

$$\omega^G(X,Y) = G(I^G X, Y). \tag{30}$$

Proposition 1.3. *I^G is an almost complex structure compatible with the metric G. The associated symplectic 2-form satisfies*

$$\omega^G = \mathrm{e}^{\overline{\psi}}\omega^S. \tag{31}$$

Proof. Indeed $(I^G)^2 = (\mathrm{e}^{\psi}\theta^t - \mathrm{e}^{-\psi}\theta)(\mathrm{e}^{\psi}\theta^t - \mathrm{e}^{-\psi}\theta) = -\theta^t\theta - \theta\theta^t = -1$. And

$$\begin{aligned} G(I^G X, I^G Y) &= \mathrm{e}^{2\psi} f_1 \langle \theta^t X, \theta^t Y\rangle + \mathrm{e}^{-2\psi} f_2 \langle \theta X, \theta Y\rangle \\ &= \mathrm{e}^{2\varphi_2}\langle X^v, Y^v\rangle + \mathrm{e}^{2\varphi_1}\langle X^h, Y^h\rangle \end{aligned}$$

and this is clearly $G(X,Y)$. Since $f_1 \mathrm{e}^{\psi} = f_2 \mathrm{e}^{-\psi} = \mathrm{e}^{\overline{\psi}}$, we easily get the conformality of ω^G with the Sasaki structure. ∎

Theorem 1.2. *(i) The almost complex structure I^G is integrable if and only if ∇ is flat and*

$$T^\nabla = \mathrm{d}\psi\wedge 1 \tag{32}$$

or equivalently $T^\nabla(X,Y)=X(\psi)Y-Y(\psi)X,\ \forall X,Y\in TM$. It is thus a vectorial torsion type metric connection.

In particular, if ∇ is torsion free, then I^G is integrable if and only if M is Riemannian flat and $f_2/f_1=$constant.

(ii) (TM,ω^G) is a symplectic manifold if and only if

$$T^\nabla = \mathrm{d}\overline{\psi}\wedge 1. \tag{33}$$

In particular, with ∇ the Levi-Civita connection, $\mathrm{d}\omega^G=0$ if and only if $f_2f_1=$constant.

Proof. (i) Let $i=\sqrt{-1}$ and let us denote $I^G=I$. As it is well known, if for all v,w in the $+i$-eigenbundle of I we have ∇^G_wv in the $+i$-eigenbundle, then $[w,v]=\nabla^G_wv-\nabla^G_vw$ will be in the very same space and the I structure will be integrable by the well known Newlander-Niremberg's Theorem. Reciprocally, the integrability of I implies the first condition on the Riemannian connection (the proof in this general setting is simple, cf. [6] or the original reference by S. Salamon [24]).

Recall the Levi-Civita connection for G is $\nabla^G=D^*-\frac{1}{2}\mathcal{R}^\xi+A+B+\tau$, with these tensors given in Theorem 1.1. Let X,Y be any real vector fields on TM. Let $w=X-iIX$ and $v=Y-iIY$. Then

$$\nabla^G_wv \ = \ \nabla^G_{X-iIX}(Y-iIY) \ = \ \nabla^G_XY-\nabla^G_{IX}IY-i(\nabla^G_{IX}Y+\nabla^G_XIY).$$

Now suppose X,Y are horizontal vector fields. Then

$$A(X,Y)=0,\quad B(X,Y)=0,\quad B(X,\theta Y)=0$$

$$B(\theta X,Y)=Y(\varphi_2)\theta X,\quad B(\theta X,\theta Y)=-\mathrm{e}^{2\psi}\langle X,Y\rangle\mathrm{grad}\,\varphi_2$$

and hence

$$\begin{aligned}\nabla^G_wv \ &= \ \nabla^G_XY-\mathrm{e}^{-\psi}\nabla^G_{\theta X}\mathrm{e}^{-\psi}\theta Y+i(\nabla^G_X\mathrm{e}^{-\psi}\theta Y+\mathrm{e}^{-\psi}\nabla^G_{\theta X}Y)\\ &= \ \nabla^{f_1}_XY-\frac{1}{2}\mathcal{R}^\xi(X,Y)+\tau(X,Y)-\mathrm{e}^{-2\psi}(\nabla_{\theta X}\theta Y+B(\theta X,\theta Y))+\\ &\qquad +i\mathrm{e}^{-\psi}\big(-X(\psi)\theta Y+\nabla_X\theta Y+X(\varphi_2)\theta Y+A(X,\theta Y)+\\ &\qquad\qquad \nabla^{f_1}_{\theta X}Y+A(\theta X,Y)+B(\theta X,Y)\big)\\ &= \ \nabla_XY+\pi^*C_1(X,Y)-\frac{1}{2}\mathcal{R}^\xi(X,Y)+\tau(X,Y)-\mathrm{e}^{-2\psi}\theta\nabla_{\theta X}Y+\langle X,Y\rangle\mathrm{grad}\,\varphi_2+\\ &\qquad +i\mathrm{e}^{-\psi}\big(X(\varphi_1)\theta Y+\theta\nabla_XY+A(X,\theta Y)+\nabla_{\theta X}Y+A(\theta X,Y)+Y(\varphi_2)\theta X\big)\end{aligned}$$

because $(\theta X)(\psi)=0$, because θ is ∇-parallel and π^*C_1 only depends on horizontals. Now

$$\begin{aligned}\mathrm{Re}\,I\nabla^G_wv \ &= \ -\mathrm{e}^{-\psi}(\theta\nabla_XY+X(\varphi_1)\theta Y+Y(\varphi_1)\theta X-\langle X,Y\rangle\theta\mathrm{grad}\,\varphi_1)\\ &\qquad -\frac{1}{2}\mathrm{e}^{\psi}\theta^t\mathcal{R}^\xi(X,Y)-\mathrm{e}^{-\psi}\theta\tau(X,Y)-\mathrm{e}^{-\psi}\nabla_{\theta X}Y-\mathrm{e}^{-\psi}\langle X,Y\rangle\theta\mathrm{grad}\,\varphi_2\end{aligned}$$

and

$$\mathrm{Re}\, i\nabla^G_w v = -\mathrm{e}^{-\psi}\big(X(\varphi_1)\theta Y + \theta\nabla_X Y + A(X,\theta Y) + \nabla_{\theta X}Y + A(\theta X, Y) + Y(\varphi_2)\theta X\big)$$

Finally putting in equation, $\mathrm{Re}\,(I - i1)\nabla^G_w v = 0$, it is easy to see the terms appearing with ∇ cancel. So we are left with

$$\begin{aligned} X(\varphi_1)\theta Y + Y(\varphi_1)\theta X + \langle X,Y\rangle\theta\mathrm{grad}\,(\varphi_2-\varphi_1) + \frac{1}{2}\mathrm{e}^{2\psi}\theta^t\mathcal{R}^\xi(X,Y) + \\ +\theta\tau(X,Y) = X(\varphi_1)Y + A(X,\theta Y) + A(\theta X,Y) + Y(\varphi_2)\theta X. \end{aligned}$$

Looking at horizontal and vertical parts,

$$\begin{cases} \frac{1}{2}\mathrm{e}^{2\psi}\theta^t\mathcal{R}^\xi(X,Y) = A(\theta X,Y) + A(X,\theta Y) \\ Y(\varphi_1)X + \langle X,Y\rangle\mathrm{grad}\,(\varphi_2-\varphi_1) + \tau(X,Y) = Y(\varphi_2)X \end{cases}.$$

Let us see the first equation: $\theta(A(\theta X,Y) + A(X,\theta Y)) = \frac{\mathrm{e}^{2\psi}}{2}\mathcal{R}^\xi(X,Y)$. Following from the very definition of A we have

$$\langle\theta A((\theta X,Y) + A(X,\theta Y)),\theta Z\rangle = \frac{\mathrm{e}^{2(\varphi_2-\varphi_1)}}{2}(\langle\mathcal{R}^\xi(Y,Z),\theta X\rangle + \langle\mathcal{R}^\xi(X,Z),\theta Y\rangle).$$

The two equations combine and on the base M it is easy to see they read

$$R(Y,Z)X + R(X,Z)Y = R(X,Y)Z.$$

The symmetries of R imply $R=0$, notice independently of the Bianchi identity. The second equation reads $\tau(X,Y) = Y(\psi) - \langle X,Y\rangle\mathrm{grad}\,\psi$. Since

$$-T(X,Y) = \tau(X,Y) - \tau(Y,X) = Y(\psi)X - X(\psi)Y$$

we find $T^\nabla = \mathrm{d}\psi\wedge 1$. In particular for the Sasaki metric we get the already known result.

The imaginary part of $I\nabla^G_w v = i\nabla_w v$ gives an equivalent condition, since we may use the above and change iw for w. Notice we have used $X,Y\in H$. It is enough, since the projection $X \rightsquigarrow X - iIX = X + i\mathrm{e}^{-\psi}\theta X$ becomes a $\mathbb{C}$-isomorphism between $H\otimes\mathbb{C}$ and the $+i$-eigenbundle of I. This proves the sufficiency of the condition in order to have integrability.

(ii) From (31) we get

$$\mathrm{d}\omega^G = \mathrm{e}^{\overline{\psi}}(\mathrm{d}\overline{\psi}\wedge\omega^S + \mathrm{d}\omega^S).$$

Now we need to choose a basis of g-orthonormal vectors e_i together with their mirror images $e_{i+m} = \theta e_i$, $i = 1,\ldots,m$. From [6] we find the formula:

$$\begin{aligned} \mathrm{d}\omega^S \;=\; & \sum_{i<j<k}^m \big(\langle\mathcal{R}^\xi(e_k,e_i),\theta e_j\rangle + \langle\mathcal{R}^\xi(e_j,e_k),\theta e_i\rangle + \langle\mathcal{R}^\xi(e_i,e_j),\theta e_k\rangle\big)e^{ijk} + \\ & + \sum_{i<j}^m\sum_{k=1}^m(\tau_{ijk} - \tau_{jik})e^{ij,k+m} \end{aligned}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. Since $\tau_{ijk} - \tau_{jik} = -T_{ijk}$ and the curvature components do not involve vertical indices, the equation $\mathrm{d}\omega^G = 0$ is satisfied under the conditions

$$\begin{cases} \underset{ijk}{\circlearrowright\!\!\!\!\!\bigoplus} R_{ijk} = 0 \\ \mathrm{d}\overline{\psi}(e_i)e^{ij,j+m} - T_{ijk}e^{ij,k+m} = 0 \end{cases}.$$

That is, the Bianchi identity and $T = \mathrm{d}\overline{\psi} \wedge 1$. Finally we recall a result stated in [4]. A metric connection with vectorial torsion V satisfies

$$\underset{X,Y,Z}{\circlearrowright\!\!\!\!\!\bigoplus} R(X,Y)Z = \underset{X,Y,Z}{\circlearrowright\!\!\!\!\!\bigoplus} \mathrm{d}V(X,Y)Z.$$

In our case, V is a gradient, hence $\mathrm{d}V = \mathrm{d}\mathrm{d}\overline{\psi} = 0$ and thence Bianchi identity is immediately satisfied. ■

The Theorem above suggests some observation. In the strict Sasaki metric case we had $T^\nabla = 0$ as necessary condition of both integrability of I^S and $\mathrm{d}\omega^S = 0$. In the general case, things are distinguished, as they should, by ψ and $\overline{\psi}$.

Clearly we may draw the following conclusion.

Corollary 1.2. *(TM, G, I^G, ω^G) is Kähler if and only if (M, ∇) is a Riemannian flat manifold ($T^\nabla = 0,\ R^\nabla = 0$) and f_1, f_2 are constants. In this case, TM is flat.*

The last assertion follows easily.

1.7 A natural contact structure

Recall T^*M has a natural symplectic structure: $\mathrm{d}\lambda$ where λ is the Liouville 1-form, i.e. the unique 1-form λ on T^*M such that on a point α we have $\lambda_\alpha = \alpha \circ \pi_*$. Equivalently, such that $\alpha^*\lambda = \alpha$ for any section $\alpha \in \Omega^1_M$. Once we introduce the Riemannian structure, the tangent and co-tangent (sphere) bundles become isomorphic. We easily deduce that μ defined in (9) corresponds by that isomorphism to the Liouville form — so it does not depend on the connection.

By Proposition 1.2 we know the torsion of $\nabla^* \oplus \nabla^*$ for any metric connection on M. It is then easy to deduce as in [7], writing $T = \pi^*T^\nabla$:

$$\mathrm{d}\mu = \omega^S + \mu \circ T. \tag{34}$$

The same is to say ω^S corresponds with the pull-back of the Liouville symplectic form if and only if $T^\nabla = 0$. Notice $T(X,Y)$ vanishes if one direction X or Y is vertical.

Regarding the contact structure on $S_rM \subset TM$, as in classical Y. Tashiro [26], the restriction of the Liouville 1-form defines indeed a contact structure — always, no matter the metric, the radius function or the metric connection. We follow e.g. [17] for the definition.

Theorem 1.3. *For any $r \in \mathrm{C}^\infty_M(\mathbb{R}^+)$, the 1-form μ defines a contact structure on S_rM.*

Proof. Let $n = \dim M - 1$ and let $e_0, \ldots, e_n$ be a local orthonormal basis of TM with $e_0 = u \in TM$ a generic point. We lift the frame and extend with $\theta e_0, \ldots, \theta e_n$ over TM. We may assume locally $\mu = e^0$. We denote $e_{j+n} = \theta e_j$. Then $\omega^S = -\sum_{j=1}^n e^{j,j+n}$ (cf. [7] for these formulas). We may clearly write $\mu \circ T_u = \|u\|^2 \sum_{0\leq i<j\leq n} T_{ij0} e^{ij}$. Let $\iota : S_rM \to TM$ denote the inclusion map. Then

$$\iota^*\mu \wedge (\mathrm{d}\iota^*\mu)^n = \iota^*(\mu \wedge (\mathrm{d}\mu)^n)$$

so we may omit the ι in the following. With a moments thought, we see

$$\mu \wedge (\mathrm{d}\mu)^n = \mu \wedge \big(-\sum_{1\leq j\leq n} e^{j,j+n} + r^2 \sum_{0\leq i<j\leq n} T_{ij0} e^{ij}\big)^n = (-1)^{n-1} n! e^{012\cdots(2n)}.$$

To see that this is $\neq 0$ on S_rM we take a 1-form on TM which has kernel TS_rM: $\Gamma = \xi^\flat - r\mathrm{d}r$. Indeed, differentiating the hypersurface equation $\langle \xi, \xi \rangle - r^2 = 0$ with the aid of ∇^*, we get the 1-form Γ. Finally,

$$\Gamma \wedge \mu \wedge (\mathrm{d}\mu)^n = (-1)^{n-1} n! \xi^\flat \wedge e^{012\cdots(2n)} \neq 0$$

since $\mathrm{d}r$ is a horizontal 1-form. This implies $\mu \wedge (\mathrm{d}\mu)^n \neq 0$ over S_rM. ■

For r constant, a metric associated to μ is recovered as the Tashiro metric contact structure on S_1M if and only if $T^\nabla = 0$, due to (34). Such contact structure is given by

$$\tilde{g} = \frac{1}{4} g^S, \quad \eta = \frac{1}{2r}\mu, \quad \varphi = \theta - \theta^t - \frac{1}{r^2}\xi \otimes \mu, \quad \zeta = \frac{2}{r}\theta^t \xi$$

in order to satisfy standard identities. ζ is the characteristic vector field and φ is the associated (1,1)-tensor such that $\varphi^2 = -1 + \zeta \otimes \eta$ and $\varphi(\zeta) = 0$. Notice $\eta = \zeta \lrcorner \tilde{g}$, $\tilde{g}(\varphi, \varphi) = \tilde{g} - \eta \otimes \eta$ and $\mathrm{d}\eta = 2\tilde{g}(\ ,\varphi\ )$, as expected.

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