Relaxation for some integral functionals in $W^{1,p}_w \times L^q_w$

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Abstract

An integral representation result is obtained for the relaxation of a class of energy functionals depending on two vector fields with different behaviors.

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1 Introduction

Minimization of energies depending on two independent vector fields arise frequently to model several phenomena. This is the case of thermochemical equilibria for multiphase solids (see [4]) or TV (total variation) image decomposition models (see [8], [9]). In both cases, the functionals to minimize have an integral form with densities depending on the total variation of a vector field u and on another independent vector field v. However, the natural space setting is different in each case. For the first one, u is a Sobolev function, whereas in the image decomposition models the natural space for u is BV, the space of bounded variation functions.

In this paper we consider functionals of the form

$$\int_\Omega f(x, u(x), \nabla u(x), v(x))\, dx, \quad \text{for } u \in W^{1,p}(\Omega; \mathbb{R}^d) \text{ and } v \in L^q(\Omega; \mathbb{R}^m) \ (p, q \geq 1).$$

As well known for the case with no x and u dependence, sequential weak lower semicontinuity of the previous functional is related to convexity conditions of the density f (cf. [4], [7]). More precisely, under convenient growth and coercivity conditions, a joint quasiconvexity of f on the gradient variable and convexity on the v variable, here called quasiconvexity-convexity, cf. Definition 1.3 below, is a necessary and sufficient condition for weak lower semicontinuity. In the lack of such condition it is of interest to characterize the relaxed functional of the original one. In this paper we provide a characterization for the relaxed energy of the functional above for the weak convergence in $W^{1,p}$ and the weak convergence in L^q, generalizing a result due to Fonseca, Kinderlehrer and Pedregal cf. [4] where no dependence on x and u was considered. Moreover, this is a first step to deal with the analogous relaxation problem in a $BV \times L^q_w$ setting, as it is the case for image decomposition models.

We introduce next some notation and we establish our main result. Let Ω be an open bounded set of $\mathbb{R}^N$ and let $\mathcal{A}(\Omega)$ denote the family of all open subsets of Ω.

Let $1 \leq p, q < \infty$ and let $F : W^{1,p}\left(\Omega; \mathbb{R}^d\right) \times L^q\left(\Omega; \mathbb{R}^m\right) \times \mathcal{A}(\Omega) \to \mathbb{R}$ be defined by

$$F(u, v, A) := \int_A f\left(x, u(x), \nabla u(x), v(x)\right) dx \tag{1.1}$$

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where $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying

$$0 \le f(x,u,\xi,b) \le C\left(1+|u|^p+|\xi|^p+|b|^q\right) \tag{1.2}$$

for a.e. $x \in \Omega$, for every $(u,\xi,b) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$ and for some $C > 0$.

Consider the relaxed localized energy of (1.1) given by

$$\mathcal{F}(u,v;A) := \inf\left\{\liminf_{n\to\infty}\int_A f(x,u_n(x),\nabla u_n(x),v_n(x))\,dx : u_n \rightharpoonup u \text{ in } W^{1,p}\left(A;\mathbb{R}^d\right),\ v_n \rightharpoonup v \text{ in } L^q(A;\mathbb{R}^m)\right\}.$$

Our goal is to find an integral representation for $\mathcal{F}$. Precisely

Theorem 1.1 *Let $1 \le p,q < \infty$ and assume that $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is Carathéodory satisfying the growth condition* (1.2).

Then for every $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v \in L^q(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}(u,v;A) = \int_A QCf(x,u(x),\nabla u(x),v(x))\,dx,$$

where QCf stands for the quasiconvex-convex envelope of f with respect to the last two variables (see (1.5) *and* (1.6)*).*

Remark 1.2 *Assuming that f satisfies the coercivity condition*

$$\frac{1}{C}\left(|\xi|^p+|b|^q\right) - C \le f(x,u,\xi,b) \tag{1.3}$$

for a.e. $x \in \Omega$, for every $(u,\xi,b) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$ and for some $C > 0$ then $\mathcal{F}$ is $W^{1,p}_w \times L^q_w$ lower semicontinuous.

We also emphasize that Theorem 1.1 constitutes an extension of [1, Theorem 1.3] for the case of the differential operator $\mathcal{A} = (\mathrm{curl}, 0)$, since, in the present result, we impose two different growth assumptions on the $\mathcal{A}$-free fields.

Following [4, 7], see also [5], we recall the definition of quasiconvexity-convexity.

Definition 1.3 *A Borel measurable function $f : \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is said to be quasiconvex-convex if there exists a bounded open set D of $\mathbb{R}^N$ such that*

$$f(\xi,b) \le \frac{1}{|D|}\int_D f(\xi+\nabla\varphi(x), b+\eta(x))\,dx, \tag{1.4}$$

for every $(\xi,b) \in \mathbb{R}^{d\times N}\times\mathbb{R}^m$, for every $\varphi \in W^{1,\infty}_0\left(D;\mathbb{R}^d\right)$ and for every $\eta \in L^\infty(D;\mathbb{R}^m)$, with $\int_D \eta(x)dx = 0$.

If $f : \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is any given Borel measurable function bounded from below, it can be defined the quasiconvex-convex envelope of f, that is the largest quasiconvex-convex function below f:

$$QCf(\xi,b) := \sup\{g(\xi,b) :\ g \le f,\ g \text{ quasiconvex-convex}\}. \tag{1.5}$$

Moreover, by Theorem 4.16 in [7]

$$QCf(\xi,b) = \inf\left\{\frac{1}{|D|}\int_D f(\xi+\nabla\varphi(x), b+\eta(x))\,dx :\ \varphi \in W^{1,\infty}_0(D;\mathbb{R}^d), \eta \in L^\infty(D;\mathbb{R}^m), \int_D \eta(x)\,dx = 0\right\}, \tag{1.6}$$

D being any bounded open set.

Remark 1.4 *It can also be showed that if f satisfies* (1.2) *for some $p,q \ge 1$, then in* (1.4) *and* (1.6) *the spaces $W^{1,\infty}_0$ and L^∞ can be replaced by $W^{1,p}_0$ and L^q, respectively.*

2 Results

All over this section $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying (1.2).

We start by presenting the following auxiliary lemmas.

Lemma 2.1 *Let $1 \leq p, q < \infty$. For any $A \in \mathcal{A}(\Omega)$, $u \in L^p\left(A;\mathbb{R}^d\right)$, $v \in L^q(A;\mathbb{R}^m)$, and any sequences $\{u_n\} \subset L^p\left(A;\mathbb{R}^d\right)$ and $\{v_n\} \subset L^q(A;\mathbb{R}^m)$ such that $u_n \rightharpoonup u$ in $W^{1,p}\left(A;\mathbb{R}^d\right)$ and $v_n \rightharpoonup v$ in $L^q(A;\mathbb{R}^m)$ there exist sequences $\{\overline{u}_n\} \subset W^{1,p}\left(A;\mathbb{R}^d\right)$, $\{\overline{v}_n\} \subset L^q(A;\mathbb{R}^m)$ such that $\overline{u}_n \rightharpoonup u$ in $W^{1,p}\left(A;\mathbb{R}^d\right)$, $\{\nabla \overline{u}_n\}$ is $p-$equi-integrable, $\int_A \nabla \overline{u}_n(x)\,dx = \int_A \nabla u(x)\,dx$, $\overline{v}_n \rightharpoonup v$ in $L^q(A;\mathbb{R}^m)$, $\{\overline{v}_n\}$ is $q-$equi-integrable, $\int_A \overline{v}_n(x)\,dx = \int_A v(x)\,dx$ and*

$$\liminf_{n\to\infty} \int_A f(x, u(x), \nabla \overline{u}_n(x), \overline{v}_n(x))\,dx \leq \liminf_{n\to\infty} \int_A f(x, u_n(x), \nabla u_n(x), v_n(x))\,dx.$$

Proof. If $p,\ q > 1$, the proof is identical to [1, Lemma 3.1], together with the second part of the proof of [1, Theorem 1.3, page 560]. The mentioned lemma gives the construction of the sequences $\{w_n\}$, (curl-free) and $\{\overline{v_n}\}$ while the argument of [1, Theorem 1.3] ensures that the sequence $\{w_n\}$ is in fact a sequence of gradients $\{\nabla \overline{u_n}\}$.

If $p = 1$ ($q = 1$, respectively), then $\overline{u}_n = u_n$ ($\overline{v}_n = v_n$, respectively) because the original sequences are already equi-integrable. ∎

Let $\mathcal{L} : W^{1,p}\left(\Omega;\mathbb{R}^d\right) \times L^q(\Omega;\mathbb{R}^m) \times \mathcal{A}(\Omega) \to \mathbb{R}$ be defined by

$$\begin{aligned}\mathcal{L}(u, v; A) := \inf \Bigg\{ & \liminf_{n\to\infty} \int_A f(x, u(x), \nabla u_n(x), v_n(x))\,dx : \{\nabla u_n\} \subset L^p\left(A;\mathbb{R}^{d\times N}\right), \\ & \nabla u_n \rightharpoonup \nabla u \text{ in } L^p\left(A;\mathbb{R}^{d\times N}\right),\ \{\nabla u_n\} \text{ is } p-\text{equi-integrable}, \\ & \{v_n\} \subset L^q(A;\mathbb{R}^m),\ v_n \rightharpoonup v \text{ in } L^q(A;\mathbb{R}^m),\ \{v_n\} \text{ is } q-\text{equi-integrable} \Bigg\}.\end{aligned}$$

Lemma 2.2 *Let $1 \leq p, q < \infty$. Then $\mathcal{F}(u, v; \cdot)$ and $\mathcal{L}(u, v; \cdot)$ are the traces of Radon measures absolutely continuous with respect to $\mathcal{L}^N \lfloor \Omega$.*

Proof. The proof is standard relying on De Giorgi-Letta criterion (see [3]) and we refer to Lemma 3.4 in [1] for more details. ∎

Lemma 2.3 *Let $1 \leq p, q < \infty$. Then*

$$\mathcal{F}(u, v; A) = \mathcal{L}(u, v; A)$$

for any $u \in W^{1,p}\left(\Omega;\mathbb{R}^d\right)$, $v \in L^q(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

Proof. The result is achieved by double inequality. The proof of $\mathcal{L}(u, v; A) \leq \mathcal{F}(u, v; A)$ relies on Lemma 2.1. Indeed, for $\{u_n\} \subset W^{1,p}\left(A;\mathbb{R}^d\right)$ and $\{v_n\} \subset L^q(A;\mathbb{R}^m)$ such that $u_n \rightharpoonup u$ in $W^{1,p}\left(A;\mathbb{R}^d\right)$ and $v_n \rightharpoonup v$ in $L^q(A;\mathbb{R}^m)$ we may find sequences $\{\overline{u}_n\} \subset W^{1,p}\left(A;\mathbb{R}^d\right)$ with $\{\nabla \overline{u}_n\}$ $p-$ equi-integrable, and $\{\overline{v}_n\} \subset L^q(A;\mathbb{R}^m)$, $q-$ equi-integrable such that

$$\liminf_{n\to\infty} \int_A f(x, u(x), \nabla \overline{u}_n(x), \overline{v}_n(x))\,dx \leq \liminf_{n\to\infty} \int_A f(x, u_n(x), \nabla u_n(x), v_n(x))\,dx.$$

Using the definition of $\mathcal{L}$ and taking the infimum on the sequences $\{u_n\}$ and $\{v_n\}$ we obtain $\mathcal{L}(u, v; A) \leq \mathcal{F}(u, v; A)$.

To obtain the converse inequality, consider admissible sequences for $\mathcal{L}$ and, using Poincaré inequality and the fundamental theorem of Young measures, construct an admissible sequence for $\mathcal{F}$ with the same limit energy. See the proof of Theorem 1.3 in [1] for more details. ∎

We sketch the proof of the main result.

Proof of Theorem 1.1. The characterization of $\mathcal{F}$ is obtained by double inequality.

To prove that

$$\mathcal{F}(u,v;A) \geq \int_A QCf(x,u(x),\nabla u(x),v(x))\,dx, \tag{2.1}$$

for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $v \in L^q(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$, we can exploit blow-up techniques developed by Fonseca and Müller in [6] and argue as in the proof of [2, Theorem 1.2], introducing the dependence on u in the density f. We also observe that the same proof of [2, Theorem 1.2] can still be performed replacing the growth condition therein by (1.2). Moreover, the result still holds for $q = 1$.

To prove the converse inequality of (2.1), we observe that by Lemma 2.3, it is enough to show that

$$\mathcal{L}(u,v;A) \leq \int_A QCf(x,u(x),\nabla u(x),v(x))\,dx \tag{2.2}$$

for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $v \in L^q(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

By Lemma 2.2 $\mathcal{L}(u,v;\cdot)$ is a measure. In the case $p > 1$ and $q > 1$, to prove (2.2) we argue as in the second part of the proof of [2, Theorem 1.2], generalizing [2, Theorem 3.1] to the present growth condition, namely, replacing condition *iii*) therein by

$$|\mathcal{L}(u,v;A)| \leq \int_A a(x) + C\left(|\nabla u(x)|^p + |v(x)|^q\right)\,dx$$

for some $C > 0$ and $a \in L^1_{loc}(\Omega)$.

If $p = 1$ and (or) $q = 1$ the upper bound can be proved arguing as in [1, Theorem 3.6]. ■

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