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Existence, Nonexistence and Multiplicity Results for Some Beam Equations

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Abstract. This paper studies the fourth order nonlinear fully equation

$$u^{(4)}(x) + f\big(x, u(x), u'(x), u''(x), u'''(x)\big) = s\ p(x)$$

for $x \in [a, b]$, $f : [a, b] \times \mathbb{R}^4 \to \mathbb{R}, p : [a, b] \to \mathbb{R}^+$ continuous functions and $s \in \mathbb{R}$, with the boundary conditions

$$u(a) = A\,, \quad u'(a) = B\,,$$
$$k_1\ u''(a) - k_2\ u'''(a) = C\,, \quad k_3\ u''(b) + k_4\ u'''(b) = D$$

for $A, B, C, D, k_1, k_3 \in \mathbb{R}$, $k_2, k_4 \geq 0$ such that $k_1^2 + k_2 > 0$ and $k_3^2 + k_4 > 0$.

This problem models several phenomena, such as, a cantilevered beam with a linear relation between the curvature and the shear force at both endpoints. For some values of the real constants, it will be presented an Ambrosetti–Prodi type discussion on s. The arguments used apply lower and upper solutions technique, *a priori* estimations and topological degree theory.

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1. Introduction

In this paper it is studied the fourth order nonlinear fully equations

$$u^{(iv)}(x) + f\left(x, u(x), u'(x), u''(x), u'''(x)\right) = sp(x)\,, \tag{E$_s$}$$

for $f : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ and $p : [a, b] \to \mathbb{R}^+$ continuous functions and s a real parameter, with several types of two-point boundary conditions.

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If

$$\begin{aligned} u(a) = A\,, \quad u'(a) = B\,, \\ k_1\ u''(a) - k_2\ u'''(a) = C\,, \quad k_3\ u''(b) + k_4\ u'''(b) = D \end{aligned} \tag{1.1}$$

for $A, B, C, D, k_1, k_3 \in \mathbb{R},\ k_2, k_4 \geq 0$ such that $k_1^2 + k_2 > 0$ and $k_3^2 + k_4 > 0$ it will be proved the existence of solutions to problem (E_s)–(1.1) for the values of s such that there are lower and upper solutions.

In Section 3 it is considered, for clearness, a particular case of the above boundary conditions in $[0, 1]$:

$$\begin{aligned} u(0) = 0\,, \quad u'(0) = 0 \\ k_1\ u''(0) - k_2\ u'''(0) = 0\,, \quad k_3\ u''(1) + k_4\ u'''(1) = 0 \end{aligned} \tag{1.2}$$

with $k_1, k_2, k_3, k_4 \geq 0$ such that $k_1 + k_2 > 0$ and $k_3 + k_4 > 0$ and the existence of solution for the problem (E_s)–(1.2) will depend on s.

Taking, in (1.2), $k_2 = k_4 = 0$ and $k_1, k_3 > 0$, the two-point boundary conditions are

$$u(0) = u'(0) = u''(0) = u''(1) = 0 \tag{1.3}$$

and it is obtained in Section 4 an Ambrosetti–Prodi type result, that is, there are $s_0, s_1 \in \mathbb{R}$ such that (E_s)–(1.3) has no solution if $s < s_0$, it has at least one solution if $s = s_0$ and (E_s)–(1.3) has at least two solutions for $s \in]s_0, s_1]$.

As far as we know these Ambrosetti–Prodi results were never applied to fourth order nonlinear fully equations. The arguments used were suggested by several papers namely [2], applied to second order periodic problems, [10], to third order three points boundary value problems, [1] for two-point boundary value problems, and make use of Nagumo-type growth condition, [9], upper and lower solutions technique for higher order boundary value problems, [3, 4, 7], and degree theory, [6].

We point out that the localization of solutions provided by lower and upper solutions method, combined with Ambrosetti–Prodi type results or by itself, can be useful to prove the existence of positive solutions (if the lower function is non-negative) or multiple solutions (if there are solutions in two disjoint branches). In fact, this property is a sharp tool in some applications where bounds on the solution or its derivatives are important, as it is illustrated in last section.

2. Existence and non-existence results

In the following, $C^k([a, b])$ denotes the space of real valued functions with continuous i-derivative in $[a, b]$, for $i = 1, ..., k$, equipped with the usual norms. The nonlinearity of (E_s) must verify some growth conditions, given by next definition, providing also an *a priori* estimate for u''', if some bounds on u, u' and u'' are verified.

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Definition 2.1. *A continuous function* $g : [a,b]\times\mathbb{R}^4 \to \mathbb{R}$ *is said to satisfy Nagumo-type conditions in*

$$E = \big\{(x,y_0,y_1,y_2,y_3) \in [a,b]\times\mathbb{R}^4 : \gamma_i(x) \le y_i \le \Gamma_i(x),\ \ i=0,1,2\big\}\,,$$

with $\gamma_i(x)$ *and* $\Gamma_i(x)$ *continuous functions such that* $\gamma_i(x) \le \Gamma_i(x)$*, for each* i *and every* $x \in [a,b]$*, if there exists a continuous function* $h_E : \mathbb{R}_0^+ \to [k,+\infty]$*, for some fixed* $k > 0$*, such that*

$$|g(x,y_0,y_1,y_2,y_3)| \le h_E(|y_3|)\,, \quad \forall (x,y_0,y_1,y_2,y_3) \in E\,, \tag{2.1}$$

with

$$\int_0^{+\infty} \frac{\xi}{h_E(\xi)}\, d\xi = +\infty\,. \tag{2.2}$$

Lemma 2.2. *Let* $f : [a,b]\times\mathbb{R}^4 \to \mathbb{R}$ *be a continuous function that satisfies Nagumo-type conditions* (2.1) *and* (2.2) *in*

$$E = \big\{(x,y_0,y_1,y_2,y_3) \in [a,b]\times\mathbb{R}^4 : \gamma_i(x) \le y_i \le \Gamma_i(x),\ \ i=0,1,2\big\}\,,$$

where $\gamma_i(x)$ *and* $\Gamma_i(x)$ *are continuous functions. Then there is* $r > 0$ *such that every solution* $u(x)$ *of* (E_s) *verifying* $\gamma_i(x) \le u^{(i)}(x) \le \Gamma_i(x)$*, for* $i = 0,1,2$ *and every* $x \in [a,b]$*, satisfies* $\|u'''\| < r$.

The proof is contained in [4].

To apply upper and lower solutions method it will be considered the following type of functions:

Definition 2.3. *Consider* $A,B,C,D,k_1,k_2,k_3,k_4 \in \mathbb{R}$ *such that* $k_2,k_4 \ge 0$*,* $k_1^2 + k_2 > 0$ *and* $k_3^2 + k_4 > 0$.

A function $\alpha(x) \in C^4(]a,b[) \cap C^3([a,b])$ *is a lower solution of* (E_s)–(1.1) *if*

$$\alpha^{(4)}(x) + f\big(x,\alpha(x),\alpha'(x),\alpha''(x),\alpha'''(x)\big) \ge s\ p(x)\,, \tag{2.3}$$

for $x \in]a,b[$*, and*

$$\alpha(a) \le A\,, \quad \alpha'(a) \le B\,,$$
$$k_1\ \alpha''(a) - k_2\ \alpha'''(a) \le C\,, \quad k_3\ \alpha''(b) + k_4\ \alpha'''(b) \le D\,.$$

A function $\beta(x) \in C^4(]a,b[) \cap C^3([a,b])$ *is an upper solution if the inequalities are reversed.*

For s such that there are upper and lower solutions of (E_s)–(1.1) with the second derivatives “well ordered”, it is obtained not only an existence result but also some information concerning the location of the solution of (E_s)–(1.1) and its derivatives.

Theorem 2.4. *Let* $f : [a,b]\times\mathbb{R}^4 \to \mathbb{R}$ *be a continuous function. Suppose that there are lower and upper solutions of* (E_s)–(1.1)*,* $\alpha(x)$ *and* $\beta(x)$*, respectively, such that,* $\alpha''(x) \le \beta''(x)$*, for* $x \in [a,b]$*, and* f *satisfies Nagumo-type conditions in*

$$E_* = \Big\{(x,y_0,y_1,y_2,y_3) \in [a,b]\times\mathbb{R}^4 : \alpha^{(i)}(x) \le y_i \le \beta^{(i)}(x), i=0,1,2\Big\}\,.$$

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If f verifies

$$f\big(x,\alpha(x),\alpha'(x),y_2,y_3\big) \le f(x,y_0,y_1,y_2,y_3) \le f\big(t,\beta(x),\beta'(x),y_2,y_3\big)\,, \tag{2.4}$$

for fixed $(x,y_2,y_3) \in [a,b]\times\mathbb{R}^2$ and $\alpha(x) \le y_0 \le \beta(x)$, $\alpha'(x) \le y_1 \le \beta'(x)$ then (E_s)–(1.1) *has at least a solution $u(x) \in C^4([a,b])$ satisfying*

$$\alpha(x)\le u(x)\le\beta(x)\,,\quad \alpha'(x)\le u'(x)\le\beta'(x)\,,\quad \alpha''(x)\le u''(x)\le\beta''(x)\,,\quad \forall x\in[a,b]\,.$$

The proof is a particular case of the main result of [4].

A first discussion on s about the existence and nonexistence of a solution will be done, for clearness, in [0,1] with $A = B = C = D = 0$ and $k_1, k_2, k_3, k_4 \ge 0$ with $k_1 + k_2 > 0$, $k_3 + k_4 > 0$, that is, for problem (E_s)–(1.2). Lower and upper solutions definition for this problem are obtained considering these restrictions:

Definition 2.5. *For k_1, k_2, k_3, k_4 nonnegative real numbers such that $k_1 + k_2 > 0$ and $k_3 + k_4 > 0$, a function $\alpha(x) \in C^4(]0,1[) \cap C^3([0,1])$ is a lower solution of* (E_s)*–*(1.2) *if it verifies* (2.3) *in* $]0,1[$ *and*

$$\alpha(0) \le 0\,,\quad \alpha'(0) \le 0\,,$$
$$k_1\ \alpha''(0) - k_2\ \alpha'''(0) \le 0\,,\quad k_3\ \alpha''(1) + k_4\ \alpha'''(1) \le 0\,.$$

A function $\beta(x)$ is an upper solution if it satisfies the reversed inequalities.

Theorem 2.6. *Let $f : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying Nagumo-type condition and such that:*

($\mathbf{H_1}$) *f is nondecreasing on the second and third variables;*
($\mathbf{H_2}$) *f is nonincreasing on the fourth variable;*
($\mathbf{H_3}$) *there are $s_1 \in \mathbb{R}$ and $r > 0$ such that*

$$\frac{f(x,0,0,0,0)}{p(x)} < s_1 < \frac{f(x,y_0,y_1,-r,0)}{p(x)}\,, \tag{2.5}$$

for every $x \in [0,1]$ and every $y_0 \le -r$ and $y_1 \le -r$. Then there is $s_0 < s_1$ (with the possibility that $s_0 = -\infty$) such that: for $s < s_0$, (E_s)–(1.2) *has no solution; for $s_0 < s \le s_1$,* (E_s)–(1.2) *has at least one solution.*

Proof. Defining $s^* = \max\{f(x,0,0,0,0)/p(x), x \in [0,1]\}$, by (2.5), there exists $x^* \in [a,b]$ such that

$$\frac{f(x,0,0,0,0)}{p(x)} \le s^* = \frac{f(x^*,0,0,0,0)}{p(x^*)} < s_1\,,\quad \forall x \in [0,1]\,,$$

and, by the first inequality, $\beta(x) \equiv 0$ is an upper solution of (E_{s^*})–(1.2).

The function $\alpha(x) = -r\ x^2/2$ is a lower solution of (E_{s^*})–(1.2) and, by Theorem 2.4, there is $s^* < s_1$ and a solution of (E_{s^*})–(1.2) with $s^* < s_1$. Suppose that (E_σ)–(1.2) has a solution $u_\sigma(x)$. For s such that $\sigma \le s \le s_1$,

$$u_\sigma^{(4)}(x) \le s\ p(x) - f\big(x,u_\sigma(x),u_\sigma'(x),u_\sigma''(x),u_\sigma'''(x)\big)$$

and so $u_\sigma(x)$ is an upper solution of (E_s)–(1.2) for every s such that $\sigma \le s \le s_1$.

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For $r > 0$ given by (2.5) take $R \geq r$ large enough such that

$$u''_\sigma(0) \geq -R,\ u''_\sigma(1) \geq -R \quad \text{and} \quad \min_{x\in[0,1]} u'_\sigma(x) \geq -R \tag{2.6}$$

the function $\alpha(x) = -Rx^2/2$ is a lower solution of (E_s)–(1.2) for $s \leq s_1$. If there is $x \in [0,1]$ such that $u''_\sigma(x) < -R$, define

$$\min_{x\in[0,1]} u''_\sigma(x) := u''_\sigma(x_0)\ \ (< -R)\,,$$

then, by (2.6), $x_0 \in\ ]0,1[\,,\ u'''_\sigma(x_0) = 0,\ u^{(4)}_\sigma(x_0) \geq 0$. By (H_1), (H_2), (2.6) and (2.5), the following contradiction is obtained

$$\begin{aligned} 0 \leq u^{(4)}_\sigma(x_0) &\leq \sigma\ p(x_0) - f\big(x_0, u_\sigma(x_0), u'_\sigma(x_0), -R, 0\big) \\ &\leq s_1\ p(x_0) - f(x_0, -R, -R, -R, 0) < 0\,. \end{aligned}$$

So $-R \leq u'_\sigma(x)$, for every $x \in [0,1]$, and, by Theorem 2.4, problem (E_s)–(1.2) has at least a solution $u(x)$ for every s such that $\sigma \leq s \leq s_1$. Let $S = \{s \in \mathbb{R} :$ (E_s)–(1.2) has at least a solution$\}$. As $s^* \in S$ then $S \neq \emptyset$. Defining $s_0 = \inf S$, therefore, $s_0 \leq s^* < s_1$ and (E_s)–(1.2) has at least a solution for $s \in]s_0, s_1]$ and it has no solution for $s < s_{0.}$ Observe that if $s_0 = -\infty$ then (E_s)–(1.2) has a solution for every $s \leq s_1$. $\square$

3. Multiplicity results

In the particular case of boundary conditions (1.1) where $k_2 = k_4 = A = B = C = D = 0$ and $k_1, k_3 > 0$ it will be proved the existence of a second solution for problem (E_s)–(1.3) as a consequence of a non null degree for the same operator in two disjoint sets.

The arguments are based on strict lower and upper solutions an some new assumptions on the nonlinearity.

Definition 3.1. *Consider $\alpha, \beta : [0,1] \to \mathbb{R}$ such that $\alpha, \beta \in C^3(]0,1[) \cap C^2([0,1])$. A function $\alpha(x)$ is a strict lower solution of* (E_s)*–*(1.3 *) if*

$$\begin{aligned} &\alpha^{(4)}(x) + f\big(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)\big) > s\ p(x)\,, \qquad \textit{for } x \in]0,1[\,, \\ &\alpha(0) \leq 0\,, \quad \alpha'(0) \leq 0\,, \quad \alpha''(0) < 0\,, \quad \alpha''(1) < 0\,. \end{aligned} \tag{3.1}$$

A function $\beta(x)$ is a strict upper solution of (E_s)*–*(1.3*) if the reversed inequalities hold.*

Define the set $X = \{y \in C^2([0,1]) : y(0) = y'(0) = y''(0) = y''(1) = 0\}$ and the operators $L : dom\ L \to C([a,b])$, with $dom\ L = C^4([0,1]) \cap X$, given by $Lu = u^{(4)}$ and, for $s \in \mathbb{R}$, $N_s : C^3([0,1]) \cap X \to C([0,1])$ given by

$$N_s u = f\big(x, u(x), u'(x), u''(x), u'''(x)\big) - s\ p(x)\,.$$

For an open and bounded set $\Omega \subset X$, the operator $L + N_s$ is L-compact in $\overline{\Omega}$, [6]. Remark that in $dom\ L$ the equation $Lu + N_s u = 0$ is equivalent to problem (E_s)–(1.3).

Next result will be an important tool to evaluate the Leray–Schauder topological degree.

Lemma 3.2. *Consider a continuous function $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ verifying a Nagumo-type condition and (H_1). If there are strict lower and upper solutions of* (E_s)*–*(1.3)*, $\alpha(x)$ and $\beta(x)$, respectively, such that*

$$\alpha''(x) < \beta''(x)\,, \quad \forall x \in [0,1]\,, \tag{3.2}$$

then there is $\rho_3 > 0$ such that $d(L + N_s, \Omega) = \pm 1$ for

$$\Omega = \left\{ y \in domL : \alpha^{(i)}(x) < y^{(i)}(x) < \beta^{(i)}(x), i = 0,1,2, \|y'''\| < \rho_3 \right\}.$$

Remark 3.3. The set Ω can be taken the same, independent of s, as long as α and β are strict lower and upper solutions for (E_s)–(1.3) and s belongs to a bounded set.

Proof. For $i = 0,1,2$ define the truncations

$$\delta_i(x, y_i) = \max\left\{\alpha^{(i)}(x), \min\left\{y_i, \beta^{(i)}(x)\right\}\right\}, \quad \forall (x, y_i) \in [0,1] \times \mathbb{R}$$

consider the modified problem

$$\begin{cases} u^{(4)}(x) + F\big(x, u(x), u'(x), u''(x), u'''(x)\big) = s\ p(x) \\ u(0) = u'(0) = u''(0) = u''(1) = 0 \end{cases} \tag{3.3}$$

with $F : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ a continuous function given by

$$F(x, y_0, y_1, y_2, y_3) = f\big(x, \delta_0(x, y_0), \delta_1(x, y_1), \delta_2(x, y_2), y_3\big) - y_2 + \delta_2(x, y_2)$$

and define the operator $F_s : C^3([0,1]) \cap X \to C([0,1])$ by

$$F_s u = F\big(x, u(x), u'(x), u''(x), u'''(x)\big) - s\ p(x)\,.$$

With these definitions problem (3.3) is equivalent to the equation $Lu + F_s u = 0$ in *dom* L. For $\lambda \in [0,1]$ and $u \in$ *dom* L consider the homotopy

$$\mathcal{H}_\lambda u := Lu - (1-\lambda)\ u'' + \lambda\ F_s u$$

and take $\rho_2 > 0$ large enough such that, for every $x \in [0,1]$,

$$-\rho_2 \le \alpha''(x) < \beta'' \le \rho_2\,,$$

$$s\ p(x) - f(x, \alpha(x), \alpha'(x), \alpha''(x), 0) - \rho_2 - \alpha''(x) < 0$$

and

$$s\ p(x) - f\big(x, \beta(x), \beta'(x), \beta''(x), 0\big) + \rho_2 - \beta''(x) > 0\,.$$

Following similar arguments to the proof of Theorem 2.4, there is $\rho_3 > 0$ such that every solution $u(x)$ of $\mathcal{H}_\lambda u = 0$ satisfies $\|u''\| < \rho_2$ and $\|u'''\| < \rho_3$, independently of $\lambda \in [0,1]$. Defining $\Omega_1 = \{y \in \text{dom } L : \|y''\| < \rho_2, \|y'''\| < \rho_3\}$ then, every solution u of $\mathcal{H}_\lambda u = 0$ belongs to Ω_1 for every $\lambda \in [0,1]$, $u \notin \partial\Omega_1$ and the degree $d(\mathcal{H}_\lambda, \Omega_1)$ is well defined, for every $\lambda \in [0,1]$.

For $\lambda = 0$ the equation $\mathcal{H}_0 u = 0$, that is, the linear problem

$$\begin{cases} u^{(4)}(x) - u''(x) = 0 \\ u(0) = u'(0) = u''(0) = u''(1) = 0 \end{cases}$$

has only the trivial solution and, by degree theory, $d(\mathcal{H}_0, \Omega_1) = \pm 1$. By the invariance under homotopy

$$\pm 1 = d(\mathcal{H}_0, \Omega_1) = d(\mathcal{H}_1, \Omega_1) = d(L + F_s, \Omega_1)\,. \tag{3.4}$$

By (3.4), there is $u_1(x) \in \Omega_1$ solution of $Lu + F_s u = 0$. Assume, by contradiction, that there is $x \in [0,1]$ such that $u_1''(x) \le \alpha''(x)$ and define

$$\min_{x\in[0,1]} \big[u_1''(x) - \alpha''(x)\big] := u_1''(x_1) - \alpha''(x_1)\ (\le 0)\,.$$

From (3.1) $x_1 \in]0,1[$, $u_1'''(x_1) - \alpha'''(x_1) = 0$ and $u_1^{(4)}(x_1) - \alpha^{(4)}(x_1) \ge 0$. By (H_1), the following contradiction is achieved

$$\begin{aligned} u_1^{(4)}(x_1) &= s\ p(x_1) - F\big(x_1, u_1(x_1), u_1'(x_1), u_1''(x_1), u_1'''(x_1)\big) \\ &\le s\ p(x_1) - f\big(x_1, \alpha(x_1), \alpha'(x_1), \alpha''(x_1), \alpha'''(x_1)\big) + u_1''(x_1) - \alpha''(x_1) \\ &\le s\ p(x_1) - f\big(x_1, \alpha(x_1), \alpha'(x_1), \alpha''(x_1), \alpha'''(x_1)\big) < \alpha^{(4)}(x_1)\,. \end{aligned}$$

Therefore $u_1''(x) > \alpha''(x)$, for $x \in [0,1]$. By a similar way it can be proved that $u_1''(x) < \beta''(x)$, for every $x \in [0,1]$. By integration and (1.3), $u_1 \in \Omega$.

As the equations $Lu + F_s u = 0$ and $Lu + N_s u = 0$ are equivalent on Ω then

$$d(L + F_s, \Omega_1) = d(L + F_s, \Omega) = d(L + N_s, \Omega) = \pm 1\,,$$

by (3.4) and the excision property of the degree. □

The main result is attained assuming that f is bounded from below and it satisfies some adequate condition of monotonicity-type which requires different "speeds" of growth.

Theorem 3.4. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function such that the assumptions of Theorem* 2.6 *are fulfilled. Suppose that there is $M \in \mathbb{R}$ such that every solution u of* (E_s)–(1.3)*, with $s \le s_1$, satisfies*

$$u''(x) < M\,, \quad \forall x \in [0,1]\,, \tag{3.5}$$

and there exists $m \in \mathbb{R}$ such that

$$f(x, y_0, y_1, y_2, y_3) \ge m\ p(x)\,, \tag{3.6}$$

for every $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-r, |M|]^2 \times [-r, M] \times \mathbb{R}$, with r given by (2.5)*. Then s_0, provided by Theorem* 2.6*, is finite and: if $s < s_0$,* (E_s)–(1.3) *has no solution; if $s = s_0$,* (E_s)–(1.3) *has at least one solution.*

Moreover, let $M_1 := \max\{r, |M|\}$ and assume that there is $\theta > 0$ such that, for every $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-M_1, M_1]^3 \times \mathbb{R}$ and $0 \le \eta_0, \eta_1 \le 1$,

$$f(x, y_0 + \eta_0\ \theta, y_1 + \eta_1\ \theta, y_2\ \theta, y_3) \le f(x, y_0, y_1, y_2, y_3)\,. \tag{3.7}$$

Then for $s \in]s_0, s_1]$, (E_s)–(1.3) *has at least two solutions.*

Proof. If, by contradiction, there are $s \in]s_0, s_1]$, u solution of (E_s)–(1.3) and $x_2 \in [0,1]$ such that

$$u''(x_2) := \min_{x\in[0,1]} u''(x) \le -r\,.$$

By (1.3), $x_2 \in]0,1[$, $u'''(x_2) = 0$ and $u^{(4)}(x_2) \ge 0$. By (H_2),

$$0 \le u^{(4)}(x_2) \le s_1\; p(x_2) - f\big(x_2, u(x_2), u'(x_2), -r, 0\big)\,.$$

If $u(x_2) < -r$ and $u'(x_2) < -r$, from (2.5) the following contradiction is obtained

$$0 \le s_1\; p(x_2) - f\big(x_2, u(x_2), u'(x_2), -r, 0\big) < 0\,.$$

If $u(x_2) \ge -r$ and $u'(x_2) \le -r$ (the case $u(x_2) \le -r$ and $u'(x_2) \ge -r$ is similar), from (H_1) and (2.5), this contradiction is achieved

$$0 \le s_1\; p(x_2) - f\big(x_2, u(x_2), u'(x_2), -r, 0\big) \le s_1\; p(x_2) - f\big(x_2, u(x_2), -r, -r, 0\big) < 0\,.$$

If $u(x_2) \ge -r$ and $u'(x_2) \ge -r$ then

$$0 \le s_1\; p(x_2) - f\big(x_2, u(x_2), u'(x_2), -r, 0\big) \le s_1\; p(x_2) - f(x_2, -r, -r, -r, 0) < 0\,.$$

Therefore, every solution u of (E_s)–(1.3), with $s_0 < s \le s_1$, verifies $u''(x) > -r$, for $x \in [0,1]$, and, by (3.5), $-r < u''(x) < M$, for every $x \in [0,1]$. Integrating on $[0,x]$, it is obtained $-r \le -rx < u'(x) < M\,x \le |M|\,,\ \forall x \in [0,1]$.

Suppose that $s_0 = -\infty$, that is, by Theorem 2.6, for every $s \le s_1$ problem (E_s)–(1.3) has at least a solution. Define

$$p_1 := \min_{x\in[0,1]} p(x) > 0$$

and take s sufficiently small such that

$$m - s > 0 \quad \text{and} \quad \frac{(m-s)\;p_1}{16} > M\,.$$

If $u(x)$ is a solution of (E_s)–(1.3), then, by (3.6), $u^{(4)}(x) \le (s-m)\;p(x)$ and, by (1.3), there is $x_3 \in]0,1[$ such that $u'''(x_3) = 0$. For $x < x_3$

$$u'''(x) = -\int_x^{x_3} u^{(4)}(\xi)\;d\xi \ge \int_x^{x_3} (m-s)\;p(\xi)\;d\xi \ge (m-s)(x_3 - t)\;p_1\,.$$

For $x \ge x_3$

$$u'''(x) = \int_{x_3}^{x} u^{(4)}(\xi)\;d\xi \le (s-m)(x - x_3)\;p_1\,.$$

Choose $I = \big[0, \frac14\big]$, or $I = \big[\frac34, 1\big]$, such that $|x_3 - t| \ge \frac14$, for every $x \in I$. If $I = \big[0, \frac14\big]$, then $u'''(x) \ge (m-s)p_1/4$, for $x \in I$, and if $I = \big[\frac34, 1\big]$, then $u'''(x) \le (s-m)\;p_1/4$, for $x \in I$. In the first case,

$$\begin{aligned} 0 &= \int_0^{\frac14} u'''(x)\;dx + \int_{\frac14}^{1} u'''(x)\;dx \ge \int_0^{\frac14} \frac{(m-s)\;p_1}{4}\;dx - u''\left(\frac14\right) \\ &= \frac{1}{16}(m-s)\;p_1 - u''\left(\frac14\right) > M - u''\left(\frac14\right), \end{aligned}$$

which is in contradiction with (3.5).

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For $I = \left[\frac{3}{4}, 1\right]$ a similar contradiction is achieved. Therefore, s_0 is finite.

By Theorem 2.6, for $s_{-1} < s_0$, $(E_{s_{-1}})$–(1.3) has no solution. By Lemma 2.2, consider $\rho_3 > 0$ large enough such that the estimate $\|u'''\| < \rho_3$ holds for every u solution of (E_s)–(1.3), with $s \in [s_{-1}, s_1]$.

Define $M_1 := \max\{r, |M|\}$ and the set

$$\Omega_2 = \left\{y \in \mathrm{dom}\ L : \|y''\| < M_1, \|y'''\| < \rho_3\right\}.$$

Then $d(L + N_{s_{-1}}, \Omega_2) = 0$. If u is a solution of (E_s)–(1.3), with $s \in [s_{-1}, s_1]$, then $u \notin \partial\Omega_2$. Defining the convex combination of s_1 and s_{-1} as $H(\lambda) = (1-\lambda)s_{-1} + \lambda s_1$ and considering the corresponding homotopic problems $(E_{H(\lambda)})$–(1.3), the degree $d(L + N_{H(\lambda)}, \Omega_2)$ is well defined for every $\lambda \in [0,1]$ and for every $s \in [s_{-1}, s_1]$. Therefore, by the invariance of the degree

$$0 = d(L + N_{s-1}, \Omega_2) = d(L + N_s, \Omega_2)\,, \tag{3.8}$$

for $s \in [s_{-1}, s_1]$. Let $\sigma \in]s_0, s_1] \subset [s_{-1}, s_1]$ and $u_\sigma(x)$ be a solution of (E_σ)–(1.3), which exists by Theorem 2.6. Take $\varepsilon > 0$ such that

$$|u_\sigma''(x) + \varepsilon| < M_1\,, \quad \forall x \in [0,1]\,. \tag{3.9}$$

Then $\widetilde{u}(x) := u_\sigma(x) + \varepsilon\frac{x^2}{2}$ is a strict upper solution of (E_s)–(1.3), with $\sigma < s \le s_1$. In fact, by (3.7) with $\theta = \varepsilon$, $\eta_0 = \frac{x^2}{2}$ and $\eta_1 = x$ for such σ,

$$\begin{aligned}
\widetilde{u}^{(4)}(x) = u_\sigma^{(4)}(x) &< s\ p(x) - f\big(x, u_\sigma(x), u_\sigma'(x), u_\sigma''(x), \widetilde{u}'''(x)\big) \\
&\le s\ p(x) - f\left(x, u_\sigma(x) + \varepsilon\frac{x^2}{2}, u_\sigma'(x) + \varepsilon x, \widetilde{u}''(x) + \varepsilon, \widetilde{u}'''(x)\right) \\
&= s\ p(x) - f\big(x, \widetilde{u}(x), \widetilde{u}'(x), \widetilde{u}''(x), \widetilde{u}'''(x)\big)\,; \\
\widetilde{u}(0) = 0\,, \quad \widetilde{u}'(0) = 0\,, &\quad \widetilde{u}''(0) = \widetilde{u}''(1) = \varepsilon > 0\,.
\end{aligned}$$

Moreover $\alpha(x) := -r\ \frac{x^2}{2}$ is a strict lower solution of (E_s)–(1.3), for $s \le s_1$. Indeed, by (2.5) and (H_1),

$$\begin{aligned}
&\alpha^{(4)}(x) = 0 > s_1\ p(x) - f(x, -r, -r, -r, 0) \ge s\ p(x) - f\left(x, -r\frac{x^2}{2}, -rx, -r, 0\right); \\
&\alpha(0) = \alpha'(0) = 0\,, \quad \alpha''(0) = \alpha''(1) = -r < 0\,.
\end{aligned}$$

As $-r < u_\sigma''(x)$ for every $x \in [0,1]$ and therefore $-r < u_\sigma''(x) + \varepsilon$, $\forall x \in [0,1]$, that is, $\alpha''(x) < \widetilde{u}''(x)$. Integrating on $[0,x]$, $\alpha'(x) \le \alpha'(x) - \alpha'(0) < \widetilde{u}'(x) - \widetilde{u}'(0) = \widetilde{u}'(x)$, for every $x \in [0,1]$. Then, by (3.9), Lemma 3.2 and Remark 3, there is $\overline{\rho}_3 > 0$, independent of s, such that for

$$\Omega_\varepsilon = \left\{y \in domL : \alpha^{(i)}(x) < y^{(i)}(x) < \widetilde{u}^{(i)}(x), i = 0,1,2, \|y'''\| < \overline{\rho}_3\right\}$$

the degree of $L + N_s$ in Ω_ε satisfies

$$d(L + N_s, \Omega_\varepsilon) = \pm 1\,, \quad \text{for } s \in]\sigma, s_1]\,. \tag{3.10}$$

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Taking ρ_3 in Ω_2 large enough such that $\Omega_\varepsilon \subset \Omega_2$, by (3.8), (3.9) and the additivity of the degree, we obtain

$$d(L + N_s, \Omega_2 - \overline{\Omega_\varepsilon}) = \mp 1\,, \qquad \text{for } s \in]\sigma, s_1]\,. \tag{3.11}$$

So, problem (E_s)–(1.3) has at least two solutions u_1, u_2 such that $u_1 \in \Omega_\varepsilon$ and $u_2 \in \Omega_2 - \overline{\Omega_\varepsilon}$, for $s \in]s_0, s_1]$, since σ is arbitrary in $]s_0, s_1]$.

Consider a sequence (s_m) with $s_m \in]s_0, s_1]$ and $\lim s_m = s_0$. By Theorem 2.6, for each s_m, (E_{s_m})–(1.3) has a solution u_m. Using the estimates of Step 1, $\|u_m^{(i)}\| < M_1,\ i = 0, 1, 2$, independently of m. As there is $\widetilde{\rho}_3 > 0$ large enough such that $\|u_m'''\| < \widetilde{\rho}_3$, independently of m, t hen sequences (u_m), (u_m') and (u_m''), $m \in \mathbb{N}$, are bounded in $C([0,1])$. By the Arzèla-Ascoli theorem, we can take a subsequence of (u_m) that converges in $C^3([0,1])$ to a solution $u_0(x)$ of (E_{s_0})–(1.3). Hence, there is at least one solution for $s = s_0$. $\square$

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