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Rational Map of CP^2 with No Invariant Foliation

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Rational Map of $\mathbb{CP}^2$ with Equal Dynamical Degree and No Invariant Foliation

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Background

Foliations

A *singular holomorphic foliation* on X is a foliation $\mathcal{F}$ on $X \setminus P$, where P is a discrete set of points, called the *singular points*, through which $\mathcal{F}$ does not extend. We use the word "foliation" to mean singular holomorphic foliation.

It is easy to write $\mathcal{F}$ as the integral curves of some holomorphic vector field in a neighborhood of any $x \in X \setminus P$. This can also be done in a neighborhood of any singular point:

Theorem: (Ilyashenko [IL] and [IY, Thm. 2.22]) In a neighborhood of each singular point $p \in P$, $\mathcal{F}$ is generated as the integral curves of some holomorphic vector field.

Thus, any foliation $\mathcal{F}$ on a surface X to be described in two equivalent ways:

- A foliation is an open cover $\{U_i\}$ of X and a system of holomorphic vector fields v_i on U_i having isolated zeros satisfying the compatibility condition that $v_i = g_{ij} v_j$ for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.
- A foliation is an open cover $\{U_i\}$ of X and a system of holomorphic one-forms ω_i on U_i having isolated zeros satisfying the compatibility condition that $\omega_i = g_{ij}\omega_j$ for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.

Within a given U_i the tangent direction to a leaf is described by either v_i or the kernel of ω_i, respectively. The zeros of v_i or ω_i, respectively, correspond to the singular points of $\mathcal{F}$.

Lemma 1: If $\mathcal{F}$ and $\mathcal{G}$ are foliations on a surface X and are equal outside of some analytic curve C, then $\mathcal{F} = \mathcal{G}$.

Pulling Back

Let $\phi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$. The *pullback* $\phi^*\mathcal{F}$ is defined by $\{(\phi^{-1}(U_i), \hat{\omega}_i)\}$ where each form $\hat{\omega}_i$ is obtained by rescaling $\phi^*\omega_i$, i.e., dividing it by a suitable holomorphic function in order to eliminate any non-isolated zeros.

Lemma 2: Let $\phi : X \to Y$ be a dominant holomorphic map between complex surfaces that is finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\phi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\phi^*\mathcal{F}$.

Let p be a point from the indeterminacy set $\mathcal{I}_\phi$. A sequence of point blow-ups $\pi : \tilde{X} \to X$ *resolves the indeterminacy at* p if ϕ lifts to a rational map $\tilde{\phi} : \tilde{X} \dashrightarrow X$ which is holomorphic in a open neighborhood of $\pi^{-1}(p)$ and makes the diagram to the right commute wherever $\tilde{\phi}$ and $\phi \circ \pi$ are both defined.

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow \pi & \overset{\tilde{\phi}}{\searrow} & \\ X & \overset{\phi}{\dashrightarrow} & Y, \end{array}$$

One can do a sequence of point blow-ups $\pi : \tilde{X} \to X$ over $\mathcal{I}_\phi$ so that ϕ lifts to a holomorphic map $\tilde{\phi} : \tilde{X} \to X$ for which all of the indeterminate points in $\mathcal{I}_\phi$ have been resolved. See, for example, [SH, Ch. IV §3.3].

The *pullback of* $\mathcal{F}$ *under* ϕ is defined by $\phi^*\mathcal{F} := \pi_*\tilde{\phi}^*\mathcal{F}$, where $\pi : \tilde{X} \to X$ is a sequence of blow-ups over $\mathcal{I}_\phi$ and $\tilde{\phi} : \tilde{X} \to X$ is a holomorphic map for which all of the indeterminacy of ϕ has been resolved. This coincides with pulling back by $\phi|(X \setminus \mathcal{I}_\phi)$ and then extending through $\mathcal{I}_\phi$.

Useful Lemmas

Lemma 3: Let $\phi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ be any dominant rational maps. For any foliation $\mathcal{F}$ on Z we have $(\psi \circ \phi)^*\mathcal{F} = \phi^*(\psi^*\mathcal{F})$.

Proof: $(\psi \circ \phi)^*\mathcal{F}$ and $\phi^*(\psi^*\mathcal{F})$ agree on $X \setminus (\mathcal{I}_\phi \cup \phi^{-1}(\mathcal{I}_\psi))$. Since $\phi^{-1}(\mathcal{I}_\psi)$ is at most an analytic curve, the result then follows from Lemma 1. ■

Lemma 4: Let $\mathcal{F}$ be a foliation on a surface X and suppose $C \subset X$ is an irreducible algebraic curve. Then, either C is a leaf of $\mathcal{F}$ or C is transverse to $\mathcal{F}$ away from finitely many points.

Proof: Let $\psi : \tilde{C} \to X$ be the normalization of C. (For background on normalization, see [SH, Ch. II §5].) Since C is irreducible, $\tilde{C}$ is a connected Riemann surface. Let $\{(U_i, \omega_i)\}$ be a compatible system of one-forms describing $\mathcal{F}$. Then, the zeros of $\{(\psi^{-1}(U_i), \psi^*\omega_i)\}$ are a well-defined analytic subset of $\tilde{C}$. Any point of $C \setminus \mathrm{Sing}(C)$ where C is parallel to $\mathcal{F}$ corresponds to the image under ψ of such a zero, so the the result follows. ■

Lemma 5: Let $\phi : X \dashrightarrow Y$ be a dominant rational map of surfaces, let $p \in \mathcal{I}_\phi$, and suppose $\pi : \tilde{X} \to X$ and $\tilde{\phi} : \tilde{X} \dashrightarrow Y$ give a resolution of the indeterminacy at p. If $\mathcal{F}$ is a foliation on Y and one of the irreducible components of $\tilde{\phi}(\pi^{-1}(p))$ is generically transverse to $\mathcal{F}$, then $\phi^*\mathcal{F}$ is singular at p.

Proof: Let C be an irreducible component of $\tilde{\phi}(\pi^{-1}(p))$ that is generically transverse to $\mathcal{F}$ and suppose $\mathcal{F}$ is represented by a compatible system of one-forms $\{(U_i, \omega_i)\}$. Then, the tangent spaces to generic points of C are not in the kernels of any of the one-forms based at these points.

Let $E \subset \pi^{-1}(p)$ be an irreducible component that is mapped by $\tilde{\phi}$ onto C. Then, at generic $x \in E$ we have that x and $\tilde{\phi}(x)$ are smooth points of E and C, respectively, and $D\tilde{\phi}$ is an isomorphism between their tangent spaces. This implies that $\{(\tilde{\phi}^{-1}(U_i), \tilde{\phi}^*\omega_i)\}$ are non-vanishing at generic points of E and have kernel transverse to E at these points. Thus, $\tilde{\phi}^*\mathcal{F}$ is generically transverse to E.

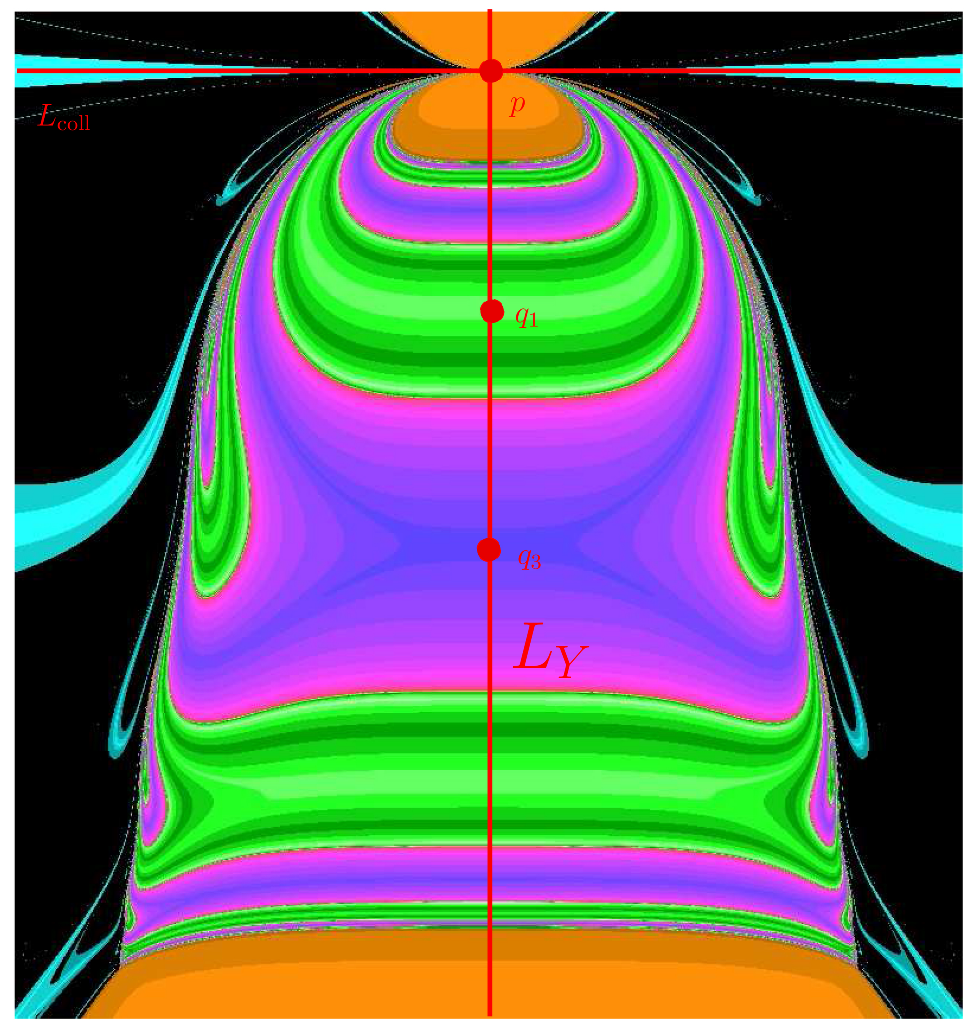
Now we can choose two disjoint holomorphic disks γ_1 and γ_2 within leaves of $\tilde{\phi}^*\mathcal{F}$ that intersect $\pi^{-1}(p)$ transversally at two distinct points of $x_1, x_2 \in E$. Then, $\pi(\gamma_1)$ and $\pi(\gamma_2)$ are distinct integral curves of $\phi^*\mathcal{F} = \pi_*\tilde{\phi}^*\mathcal{F}$ going through p. This implies that p is a singular point for $\phi^*\mathcal{F}$. ■

Main Theorem

Let $\phi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ be given by

$$\phi[X : Y : Z] = [-Y^2 : X(X-Z) : -(X+Z)(X-Z)]$$

is not cohomologically hyperbolic ($\lambda_1(\phi) = \lambda_2(\phi) = 2$), and no iterate of ϕ preserves a singular holomorphic foliation.



Structure of ϕ

Indeterminant and Postcritical Sets

The set of indeterminant points is:

$$\mathcal{I}(\phi) = p := [1 : 0 : 1].$$

It is easy to see there are critical curves:

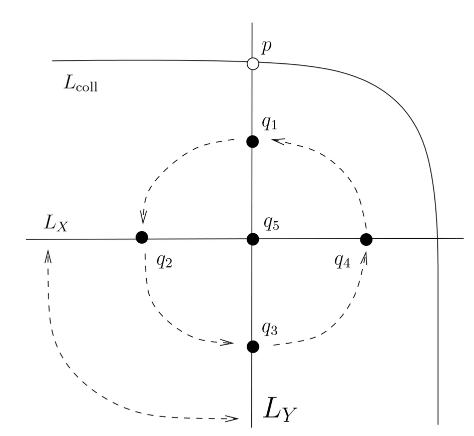
$$L_{\text{coll}} := \{X = Z\}$$
$$L_Y := \{Y = 0\}.$$

Let $L_X := \{X = 0\}$, and note that

$$L_Y \rightleftarrows L_X$$

Let $q_1, \dots, q_4$ be the 4-cycle:

$$L_{\text{coll}} \setminus \{p\} \to [1:0:0] \to [0:1:-1] \to [-1:0:1] \to [0:1:0] \quad ([0:1:0] \to [1:0:0])$$



Resonant Dynamical Deg.

The orbit of the collapsed curve L_{coll} lies in the four cycle $\{q_1, \dots, q_4\}$, which is disjoint from p, so there is no curve V such that $\phi(V) \subset \mathcal{I}_\phi$. Thus, ϕ is algebraically stable by [SI], and $\lambda_1(\phi) = \deg_{alg}(\phi) = 2$.

It can be shown that $[1 : 1 : -1]$ is neither a critical value nor in the image of the indeterminacy, so it is a generic point for ϕ. It has two preimages $[1 : \pm i : 0]$ under ϕ, so that $\lambda_2(\phi) = 2$. Thus,

Lemma: $\lambda_1(\phi) = \lambda_2(\phi) = 2$; ϕ is not cohomologically hyperbolic.

Fatou Set

ϕ^2 fixes L_Y and L_X; each line is transversally superattracting under ϕ^2. Also, $\phi^2 \mid L_Y$ and $\phi^2 \mid L_X$ are each conjugate to $z \mapsto z^2 - 1$ with period two superattracting cycles $q_1 \leftrightarrow q_3$ and $q_2 \leftrightarrow q_4$, respectively . Lastly, ϕ (and hence ϕ^2) has a superattracting fixed point $q_5 = [0 : 0 : 1]$.

Thus, q_i are five superattracting fixed points for ϕ^4. The picture above shows the real slice in local coordinates $(y, z) = (Y/X, Z/X)$, with basins of $q_1, \dots, q_5$ shown in green, black, purple, blue, and orange, respectively.

Resolving the Indeterminacy

Local Coordinates

Let $\widehat{\mathbb{P}^2}$ be the blow up at p with exceptional divisor E_1, and let $\widehat{L}_{\text{coll}}$ denote the proper transform of L_{coll}. Let $\widetilde{\mathbb{P}^2}$ be the blow up of $\widehat{\mathbb{P}^2}$ at the point $E_1 \cap \widehat{L}_{\text{coll}}$, resulting in a new exceptional divisor E_2.

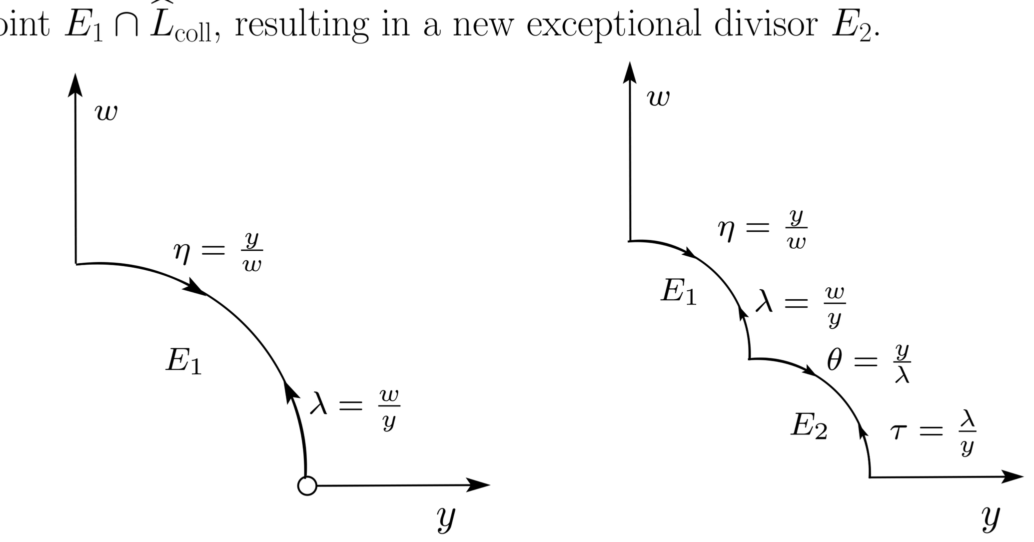


Figure 1: Left: coordinates for first blowup of p. The new indeterminate point $(y, \lambda) = (0, 0)$ for $\widehat{\phi}$ is marked by an open circle. Right: coordinates for second blow-up.

Note ϕ lifts to a rational map $\tilde{\phi} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ that resolves the indeterminacy of ϕ at p, satisfying $\tilde{\phi}(E_1) = [0 : 1 : -2]$ and $\tilde{\phi}(E_2) = \{Z = -2Y\}$.

Consider two systems of affine coordinates on $\mathbb{P}^2$ given by $(y, z) = (Y/X, Z/X)$ and $(x, \zeta) = (X/Y, Z/Y)$. Let $w = z - 1$, so coordinates (y, w) place p locally at $(0, 0)$. Writing $(x', \zeta') = \phi(y, w)$,

$$(x', \zeta') = (y^2/w, -w - 2)\,.$$

More Blowups

The two systems of coordinates in a neighborhood of E_1 within $\widehat{\mathbb{P}^2}$ are (y, λ), where $w = \lambda y$, and (w, η), where $y = \eta w$. Expressed in these coordinates, ϕ lifts to a rational map $\widehat{\phi} : \widehat{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ given by

$$(x', \zeta') = \left(\frac{y}{w}, -wy - 2\right) \text{ and } (x', \zeta') = \left(\eta^2 w, -w - 2\right).$$

The exceptional divisor E_1 is given in the first coordinates by $y = 0$ and in the second coordinates by $w = 0$. Thus, $\widehat{\phi}$ extends holomorphically to all points of $E_1 \setminus \{w = 0\}$, sending all of these points to $(x, \zeta) = (0, -2)$.

In (y, w) coordinates, L_{coll} is given by $w = 0$, and in (y, w) coordinates, the proper transform $\widehat{L}_{\text{coll}}$ is given by $\lambda = 0$. Thus, $E_1 \cap \widehat{L}_{\text{coll}} = (0, 0)$, the indeterminate point for $\widehat{\phi}$ on E_1.

Blow-up $(y, w) = (0, 0)$, using two new systems of coordinates in a neighborhood of the new exceptional divisor E_2: (y, τ), where $w = \tau y$, and (w, θ), where $y = \theta w$. Then $\widehat{\phi}$ lifts to a map $\tilde{\phi}$, expressed locally as

$$(y', z') = \left(\tau, -\tau^2 y^2 - 2\tau\right) \text{ and } (x', \zeta') = \left(\theta, -w^2\theta - 2\right)$$

Thus, $\tilde{\phi} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ is holomorphic in a neighborhood of $E_1 \cup E_2$. Since E_2 is given in these systems of coordinates by $y = 0$ and $w = 0$, respectively, one can see that $\tilde{\phi}(E_2) = \{Z = -2Y\}$.

Notice that the line $C_1 := \{Z = -2Y\}$ passes through no points of

$$\mathcal{I}_{\phi^3} = \{[1 : 0 : 1], [0 : \pm i : 1], [1 : 0 : -1 \pm i]\}.$$

This implies that

$$\tilde{\phi}^4 := \phi^3 \circ \tilde{\phi} : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$$

is holomorphic in a neighborhood of $E_1 \cup E_2$, i.e. that $\tilde{\phi}^4$ resolves the indeterminacy of ϕ^4 at p.

The Fourth Iterate

Proof of the main theorem relies on careful analysis of ϕ^4. The indeterminacy set of ϕ^4 is

$$\mathcal{I}_{\phi^4} = \{p\} \cup \phi^{-1}(\{p\}) \cup \phi^{-2}(\{p\}) \cup \phi^{-3}(\{p\})$$
$$= \{[1:0:1], [0:\pm i:1], [1:0:-1\pm i], [0:i:\pm\sqrt{1\pm i}]\}.$$

We will only need to resolve the indeterminacy of ϕ^4 at p.

Lemma 6: The map $\phi^4 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ lifts to a rational map $\tilde{\phi}^4 : \widetilde{\mathbb{P}^2} \dashrightarrow \mathbb{P}^2$ that resolves the indeterminacy of ϕ^4 at p. Also, $\tilde{\phi}^4(E_1) = [1 : 0 : -9] =: s$ and $\tilde{\phi}^4(E_2)$ is an irreducible algebraic curve C of degree 8 that is singular at s.

Since $\tilde{\phi}(E_1) = [0 : 1 : -2]$, once can check that $\tilde{\phi}^4(E_1) = s = [1 : 0 : -9]$. Meanwhile, $\tilde{\phi}(E_2)$ is the projective line C_1. Thus, $\phi^3(C_1)$ is an algebraic curve $C \equiv C_4$ of degree $8 = \deg(\phi^3)$. Since C_1 does not intersect $\mathcal{I}_{\phi^3}$, C_4 is irreducible.

It remains to show that $C_4 = \phi^3(C_1)$ is singular at s. Notice that ϕ maps a neighborhood of $[0 : 1 : 3]$ biholomorphically to a neighborhood of s. Thus, it is sufficient to show $C_3 := \phi^2(C_1)$ is singular at $[0 : 1 : 3]$.

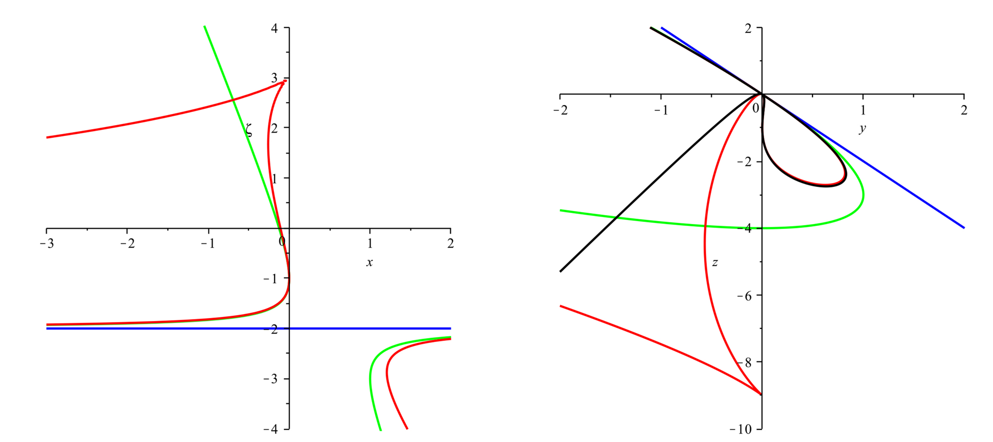


Figure 2: Curves C_1, C_2, and C_3 shown in blue, green, and red (respectively) on the left in the (x, ζ) coordinates and shown on the right in the (y, z) coordinates. The curve C_4 is shown in black on the right. Note the cusps at $(x, \zeta) = (0, 3)$ and $(y, z) = (0, -9)$.

Working in the (x, ζ) coordinates, we have

$$(x', \zeta') = \phi^2(x, \zeta) = \left(\frac{x^2(x-\zeta)^2}{-1 + x^2 - \zeta^2}, -1 - x^2 + \zeta^2\right).$$

Parameterizing C_1 by $t \mapsto (t, -2)$ and then composing with ϕ^2, we obtain the following parameterization of C_3:

$$t \mapsto \left(\frac{t^2(t+2)^2}{-5 + t^2}, 3 - t^2\right).$$

To see that C_3 is singular at $(0, 3)$, check that C_3 intersects any line through $(0, 3)$ with intersection number ≥ 2. Such a generic line is given by $Ax + B(\zeta - 3) = 0$. Substituting the parameterization into this equation, we obtain

$$\frac{At^2(t+2)^2}{-5 + t^2} - Bt^2 = 0,$$

which clearly has a double zero at $t = 0$. Thus, the intersection number of C_3 with the line given by $Ax + B(\zeta - 3) = 0$ is at least two.

Proof of Theorem

Implicitly throughout the proof, assume that $(\phi^m)^*(\phi^n)^*\mathcal{F} = (\phi^{m+n})^*\mathcal{F}$ for $m, n \geq 1$, which follows from Lemma 6. Also, the whole proof takes place in the coordinates $(y, z) = (Y/X, Z/X)$.

The ϕ-preimage of L_{coll} is $S := \{y^2 + z^2 = 1\}$, intersecting L_Y at the indeterminate point $p = (0, 1)$ and at $q_3 = (0, -1)$. Then ϕ^2 is a finite holomorphic map at each point of L_Y other than p and q_3. Thus, ϕ^4 is a finite holomorphic map at every point of L_Y except at $\mathrm{NF} := \{(0, \mathrm{z}) \mid \mathrm{z} = 1, -1, 0, -2, -1 \pm \mathrm{i}\}$ consisting of p, q_3, and their ϕ^2-preimages.

One can check that $r := (0, -1 + \sqrt{2}i) \notin \mathrm{NF}$ is a ϕ^4-preimage of the singular point s of the blow-up curve C. We first show that for any foliation $\mathcal{F}$, $(\phi^4)^*\mathcal{F}$ is singular either at the indeterminate point p or at r. By Lemma 4 , either C is a leaf of $\mathcal{F}$ or generically transverse to $\mathcal{F}$. In the second case, Lemma 5 immediately implies that $(\phi^4)^*\mathcal{F}$ is singular at p. On the other hand, if C is a leaf of $\mathcal{F}$, s must be a singular point for $\mathcal{F}$, since s is a singular point of C by Lemma 6 . Since ϕ^4 is a finite holomorphic map at r and $\phi^4(r) = s$, by Lemma 2 $(\phi^4)^*\mathcal{F}$ is singular at r as well.

Neither p nor r are critical points of $\phi|L_Y$, so they are not exceptional points. One can check that any preorbit of p or r under $\phi^4|L_Y$ is disjoint from NF. Let a_0 be the point (p or r) where $(\phi^4)^*\mathcal{F}$ is singular. If we denote some $\phi^4|L_Y$-preorbit of a_0 by $\{a_{-i}\}_{i=0}^{\infty}$, then for each i, $\left(\phi^{4(i+1)}\right)^*\mathcal{F}$ will be singular at a_{-i}.

Now suppose the foliation $\mathcal{F}$ is preserved by ϕ^ℓ for any positive integer ℓ. Then $\mathcal{F} = \left(\phi^{4\ell k}\right)^*\mathcal{F}$ is singular at $a_{(-\ell k)}$ for all $k \geq 1$. This implies $\mathcal{F}$ has infinitely many singular points, giving a contradiction.

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