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Results and Examples Regarding Bifurcation with a Two-Dimensional Kernal

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RESULTS AND EXAMPLES REGARDING BIFURCATION WITH A TWO-DIMENSIONAL KERNEL

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RESULTS AND EXAMPLES REGARDING BIFURCATION WITH A TWO-DIMENSIONAL KERNEL

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ABSTRACT

Many problems in pure and applied mathematics entail studying the structure of solutions to $F(x, y) = 0$, where F is a nonlinear operator between Banach spaces and y is a real parameter. A parameter value where the structure of solutions of F changes is called a bifurcation point. The particular method of analysis for bifurcation depends on the dimension of the kernel of $D_x F(0, \lambda)$, the linearization of F.

The purpose of our study was to examine some consequences of a recent theorem on bifurcations with 2-dimensional kernels. This resent theorem was compared to previous methods. Also, some specific classes of equations were identified in which the theorem always holds, and an algebraic example was found that illustrates bifurcations with a 2-dimensional kernel.

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CHAPTER I

INTRODUCTION

Many problems in pure and applied mathematics entail studying the structure of solutions to $F(x, y) = 0$, where $F \colon U \times V \to Z$ is a continuous nonlinear map, $U \times V \subseteq X \times Y$ is open, and X, Y, and Z are Banach spaces. Typically, y is a parameter, and Y is one-dimensional. A bifurcation is a change in the structure of the solutions at a particular parameter value. The study of bifurcations arises naturally from the mathematical description of the states of physical systems as these states undergo changes in stability [1]. These descriptions often take the form of nonlinear differential equations whose solutions correspond to the state of the system.

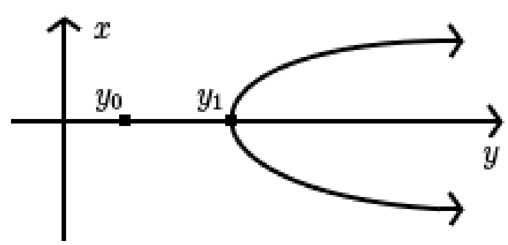

Figure 1.1: Pitchfork bifurcation

The solution to a bifurcation problem is often described graphically by a bifurcation diagram. A simple example is depicted in Figure 1.1. The point labeled y_0 is not a bifurcation point; at y_0 any small change in the parameter will not change

change the structure of the solutions. However, the point labeled y_1 is a bifurcation point. Values less than y_1 yield only one solution, whereas greater values yield three. Bifurcations of the type in Figure 1.1 are appropriately called pitchfork bifurcations. Bifurcations, such as the one at y_1, are often associated with a loss of stability of the solution $x = 0$.

Bifurcation problems are often formulated so that $F(0, y) = 0$, $\forall y \in Y$, so the line $x = 0$ is a branch of solutions, referred to as the trivial branch of solutions. A concern thus becomes identifying the nontrivial branches of solutions, which typically emanate from the trivial branch at bifurcation points. Hence, we must know the bifurcation points. These can be found using $D_xF(x, y)$, the linearization of $F(x, y)$ with respect to x. The Implicit Function Theorem implies that the points where this linearization has no bounded inverse are the candidates for the bifurcation points, and hence, are the candidates for the points where the structure of the solution set changes.

The Implicit Function Theorem provides necessary conditions for bifurcation. However, these conditions are not sufficient; results giving sufficient conditions for bifurcation typically depend on both the dimension of the kernel of $D_xF(x, y)$ and the dimension of the parameter space. This is plausible because the dimension of the kernel of $D_xF(x, y)$ at a bifurcation point is a determining factor in the existence of a bounded inverse. There are several methods for analyzing bifurcations when the dimension of the kernel is one. In particular, the Crandall-Rabinowitz Theorem is an important theorem providing sufficient conditions for bifurcation when the dimension of the kernel of the linearization and the dimension of the parameter space are both one. The proof of this theorem relies on the Implicit Function Theorem and can be found in [2].

Problems with higher dimension kernels and one-dimensional parameter spaces are less readily analyzed. A recently published theorem by Kömer et al. [3] provides a relatively simple method for analyzing bifurcations when the dimension of the kernel of the linearization is two, and the dimension of the parameter space is one. While other methods currently exist for problems with higher dimension kernel, these methods are often difficult to use in practice [2].

The purpose of this thesis is to explore some specific consequences of the new result of Krömer et al. and to construct several examples illustrating applications of Krömer's result. We begin by presenting sufficient background to state the theorem. This background includes a discussion of the reduction method of Lyapunov-Schmidt. Specific examples are given that meet the different sufficiency conditions of each theorem. Next, we discuss several algebraic examples illustrating Krömer's Theorem. To conclude, we analyze a differential operator that meets the sufficiency conditions of the theorem by Krömer et al. and not the other theorems. The final chapter presents a variation on the Crandall-Rabinowitz Theorem that is related to problems with higher dimension kernels.

CHAPTER II

BACKGROUND

In this chapter we present the background material necessary to state the bifurcation theorem of Krömer et al.

2.1 The Implicit Function Theorem and Fredholm Operators

There are several methods for examining the solution sets for nonlinear mappings. Many of these methods rely at some point upon the Implicit Function Theorem. Proof of this theorem can be found in [4].

Theorem 2.1.1 (Implicit Function Theorem)**.** *Suppose* X, Y, Z *are Banach spaces,* $U \subset X$*,* $V \subset Y$ *are open sets,* $F\colon U \times V \to Z$ *is continuously differentiable,* $(x_0, y_0) \in U \times V$*,* $F(x_0, y_0) = 0$*, and* $D_x F(x_0, y_0)$ *has a bounded inverse. Then there is a neighborhood* $U_1 \times V_1 \subset U \times V$ *of* (x_0, y_0) *and a function* $f\colon V_1 \to U_1$*, with* $f(y_0) = x_0$ *such that* $F(x, y) = 0$ *for* $(x, y) \in U_1 \times V_1$ *if and only if* $x = f(y)$*. If* $F \in C^k(U \times V, Z)$*,* $k \geq 1$ *or analytic in a neighborhood of* (x_0, y_0)*, then* $f \in C^k(V_1, X)$ *or is analytic in a neighborhood of* y_0 *[5].*

Bifurcation problems on infinite-dimensional Banach spaces can often be reduced to problems on finite-dimensional spaces. A commonly used method for this is the Reduction Method of Lyapunov-Schmidt, which relies on the Implicit Function Theorem. A sufficient condition to apply the Method of Lyapunov-Schmidt is that F satisfy the following definition:

Definition 2.1.2. *A continuous mapping* $F\colon U \to Z$*, where* X *and* Z *are Banach spaces and* U *is open in* X*, is a nonlinear Fredholm operator if it is Fréchet differentiable on* U *and if* $D_xF(x)$ *has the following properties:*

(i) $dim N(D_xF(x)) < \infty$ *where* $N(D_xF)$ *is the kernel of* D_xF*,*
(ii) $codim R(D_xF(x)) < \infty$ *where* $R(D_xF)$ *is the range of* D_xF*,*
(iii) $R(D_xF(x))$ *is closed in* Z *[2].*

The Lyapunov-Schmidt method exploits the fact that the kernel of the linearization of a Fredholm operator F has finite dimension to split the domain and range of F by projecting the entire Banach spaces X and Z onto finite-dimensional subspaces. To further explain the Lyapunov-Schmidt reduction requires some background information from functional analysis on projection operations in Banach spaces.

2.2 Functional Analysis Background

Definition 2.2.1. *Let* X *be a Banach space, and let* $M, N \subset X$ *be linear manifolds.* M *and* N *are complementary subspaces of* X *if* $M + N = X$ *and* $M \cap N = \{0\}$*. If* M *and* N *are complementary subspaces of* X*, we write* $X = M \oplus N$ *[6].*

Definition 2.2.2. *Let* X *be a Banach space, and let* $M \subset X$ *be a linear manifold.* M *has finite codimension if there is a finite-dimensional linear manifold* N *such that* $X = M \oplus N$*.*

Definition 2.2.3. *Let X be a Banach space, and let $P\colon X \to X$ be a linear transformation. Then P is a projection if $P^2 = P$.*

The next several observations establish some conditions under which a linear manifold M of a Banach space X has a complementary subspace and is the image of a continuous projection on X. These results, which are well-known, are collected here for convenience. Theorems 2.2.4, 2.2.5, and 2.2.6 are exercises in [6].

Theorem 2.2.4. *Let X be a Banach space and let $P\colon X \to X$ be a projection. If P is continuous, then $M = R(P)$ and $N = N(P)$ are closed, complementary subspaces of X [6].*

Proof. First note that since P is a bounded, linear operator, N is a closed, linear subspace of X [6]. Now let $Px_n \subset M$ such that $Px_n \to x \notin M$. Then $(Px_n - x) \to 0$, and since P is continuous, $(P^2x_n - Px) = (Px_n - Px) \to 0$. Hence $x = Px$, which is a contradiction, so $x \in M$ and M is closed.

Next, $P^2x = Px = y \in M$ implies $P^2x = Py$, and $y = Py$. Thus, $M \cap N = \{0\}$. Recall that $M+N = \{m+n \mid m \in M, n \in N\}$. Clearly, since $M, N \subseteq X$, we have $M + N \subseteq X$. Now, let $x \in X$, and note that x can be written as $x = (I - P)x + Px$. Since $P(I - P)x = (P - P^2)x = 0$, we have that $(I - P)x \in N$, so x is the sum of an element in N and an element in M. Hence, $X \subseteq M + N$. □

Also, there are cases when a projection is not needed in order to know of the existence of a complementary space, as seen in the following theorem.

Theorem 2.2.5. *Let X be a Banach space and let $P\colon X \to X$ be a projection. If $R(P)$ and $N(P)$ are closed subspaces of X, then P is continuous.*

Proof. Since P is a projection, we know $(I-P)\colon X \to N(P)$. Let $x \in N(P) \cap R(P)$, so $Px = 0$. Then for $y \in X$ such that $Py = x$, we have $x = Py = P^2y = Px = 0$, which implies that $y = 0$. Hence, $R(P) \cap N(P) = \{0\}$.

Now for $x_n \subset X$ such that $x_n \to x$, suppose that $Px_n \to y$ for some $y \in R(P)$. Then $(I-P)x_n = x_n - Px_n \to x - y \in N(P)$ because $N(P)$ is closed. Also, P maps to $R(P)$, so $0 = P(x-y) = Px - Py = Px - y$, since $y \in R(P)$, and P is a projection. Hence, $Px = y$, so if $x = 0$, then it must also be that $y = 0$, so the graph of P is closed. Then by the Closed Graph Theorem [6], P is continuous. □

Theorem 2.2.6. *Let X be a Banach space and M be a finite-dimensional linear manifold in X. Then there is a continuous projection of X onto M and M has a closed complementary subspace [6].*

Proof. Since M has finite dimension, M is closed [6], and we have M $= span\{x_1, \ldots, x_n\}$. Then for $1 \leq i \leq n$, define a linear functional $f_i\colon M \to \mathbb{R}$ by $f_i(x) = f_i(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_i$. Let $\|x\| = \max_{i=1}^{n}\{|\alpha_1|\}$, and note that since M has finite dimension, all norms are equivalent [6]. It is clear then that each f_i is bounded since $|f_i(x)| \leq \|x\|$. By the Hahn-Banach Theorem, there is a bounded linear functional $F_i\colon X \to \mathbb{R}$ such that $F_i\mid_M = f_i$.

Now define $P\colon X \to M$ by $P(w) = \sum_{i=1}^{n} F_i(w)x_i$. Observe that

$$\|P(w)\| = \left\| \sum_{i=1}^{n} F_i(w)x_i \right\| \leq \sum_{i=1}^{n} \|F_i(w)x_i\| \leq \left(\sum_{i=1}^{n} \|F_i\|\|x_i\| \right) \|w\|.$$

Hence, P is bounded. Since each $F_i(w) \in \mathbb{R}$, we have $\sum_{i=1}^{n} F_i(w)x_i \in M$. Then

$$\begin{aligned} P^2(w) &= P\left(\sum_{i=1}^{n} F_i(w)x_i\right) \\ &= \sum_{i=1}^{n} F_i\left(\sum_{i=1}^{n} F_i(w)x_i\right) x_i \\ &= \sum_{i=1}^{n} f_i\left(\sum_{i=1}^{n} F_i(w)x_i\right) x_i \\ &= \sum_{i=1}^{n} F_i(w)x_i = P(w). \end{aligned}$$

Hence, P is a projection, and then by Theorem 2.2.4, $N(P)$ is complementary to M. □

Theorem 2.2.7. *Let X be a Banach space and M be a closed linear manifold in X with finite codimension. Then there is a continuous projection of X onto M.*

Proof. By Definition 2.1.2, there is a finite-dimensional linear manifold N such that $X = M \oplus N$. For $x \in X$, write $x = m+n$, where $m \in M$ and $n \in N$. Define $Px = m$. Then $P^2 = P$, $R(P) = M$, and $N(P) = N$. Also, note M and N are closed, so by Theorem 2.2.5, P is continuous. □

2.3 Reduction Method of Lyapunov-Schmidt

The information presented in Section 2.2 provides the tools necessary to explain the Reduction Method of Lyapunov-Schmidt. We begin by assuming that F is a map from $U \times V$ into Z, where U is open in X, V is open in Y, and X, Y, and Z are

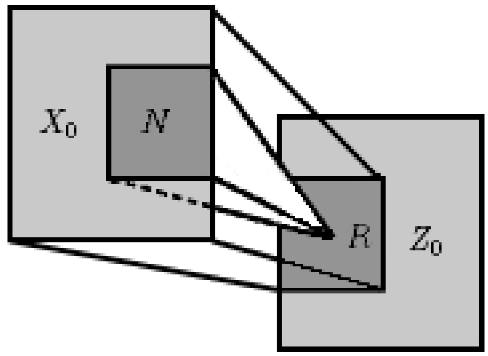

Figure 2.1: Partitioning of the spaces X and Z

Banach spaces. Also, we assume

$$\begin{gathered} F(x_0, y_0) = 0 \text{ for some } (x_0, y_0) \in U \times V, \\ F \in C(U \times V, Z), \\ D_x F \in C(U \times V, L(X, Z)). \end{gathered} \tag{2.1}$$

We assume that $F(\cdot, y_0)$ is a nonlinear Fredholm operator. We define $N = N(D_xF(x_0, y_0))$ and $R = R(D_xF(x_0, y_0))$. Since $D_xF(x_0, y_0)$ is continuous, we know N is closed, and since F is a Fredholm operator, we know $\dim N < \infty$ and that R is closed with finite codimension. Finally, by Theorems 2.2.6 and 2.2.7, we can write $X = N \oplus X_0$ and $Z = R \oplus Z_0$, where X_0 and Z_0 are closed, and we can define continuous projections $P\colon X \to N$ and $Q\colon Z \to Z_0$. We note that Z_0 is finite-dimensional.

Clearly, we have that $x = Px + (I - P)x$. In a similar, though less intuitive fashion, the equation $F(x, y) = 0$ can be written equivalently as the two equations

$$\begin{aligned} QF(x, y) &= 0, \\ (I - Q)F(x, y) &= 0. \end{aligned} \tag{2.2}$$

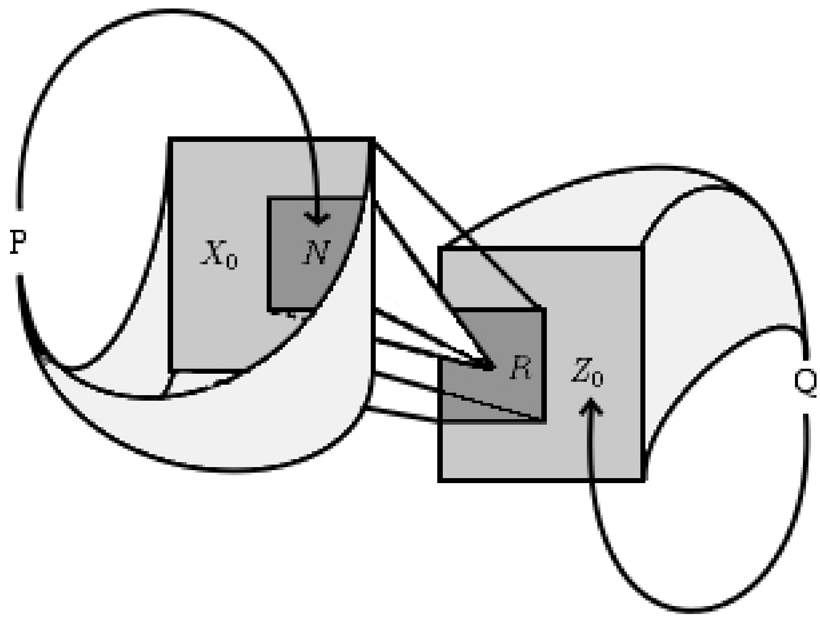

Figure 2.2: Projections P and Q

To see this, begin by assuming $F(x, y) = 0$. Then it is clear that $QF(x, y) = 0$ and $(I-Q)F(x, y) = 0$ are both true. Now, assume $QF(x, y) = 0$ and $(I-Q)F(x, y) = 0$. Then we have $QF(x, y) = (I - Q)F(x, y)$, and because $QF(x, y) \in Z_0$ and $(I - Q)F(x, y) \in R$ are members of two complementary spaces, it must be that $F(x, y) = 0$. It follows that the equation $F(x, y) = 0$ is equivalent to the system of equations

$$QF(Px + (I - P)x, y) = 0 \tag{2.3}$$

$$(I - Q)F(Px + (I - P)x, y) = 0. \tag{2.4}$$

Hence, we have effectively "split" the equation $F(x, y) = 0$ by using the linearization of F to define complementary spaces.

Now define a function $G\colon U_1 \times W_1 \times V_1 \to R$ by

$$G(v, w, y) \quad = \quad (I - Q)F(v + w, y) \tag{2.5}$$

where $v = Px \in U_1 \subset N$ and $w = (I - P)x \in W_1 \subset X_0$. Note for $v_0 = Px_0$ and $w_0 = (I - P)x_0$ that $D_w G(v_0, w_0, y_0) = (I - Q)D_x F(x_0, y_0)$, which as a map from X_0 to R is bijective and bounded [2]. Then by the Implicit Function Theorem, we have a function $\psi\colon U_0 \times V_0 \to X_0$, where $U_0 \times V_0$ is a neighborhood of (v_0, y_0), such that $G(v + \psi(v, y), y) = 0$.

Then using the implicit function ψ in (2.3), we define $\Phi\colon W_0 \times V_0 \to Z_0$ by $\Phi(v, y) = QF(v + \psi(v, y), y)$. Thus we have reduced (2.3) and (2.4) to

$$\Phi(v, y) = 0. \tag{2.6}$$

Equation (2.6) is called the bifurcation function; studying $F(x, y) = 0$ is equivalent to studying (2.6) near the bifurcation point y_0. However, $\Phi\colon W_0 \times V_0 \to Z_0$, so this method has reduced the problem to a finite dimensional one.

CHAPTER III

A THEOREM OF KROMER ET AL.

The Implicit Function theorem provides a necessary but not sufficient condition for the existence of a solution branching from the trivial branch at a bifurcation point. When the kernel of the linearization about the trivial branch is one-dimensional, the Crandall-Rabinowitz theorem is a well-known theorem that provides a sufficient condition for bifurcation. However, the Crandall-Rabinowitz theorem does not directly generalize to higher-dimensional problems depending on a single real parameter. Recently, Kromer et al. [3] offered a method for proving the existence of local continua through a bifurcation point when the kernel of the linearization is two-dimensional. In this section we briefly describe the results of Kromer et al.

3.1 Background for the Theorem

We consider the non-linear problem $F(x, \lambda) = 0$, where $F : U \times V \to Z$, where $U \subset X$ is open, $V \subset \mathbb{R}$ is open, and both X and Z are Banach spaces. We assume that

$$\begin{gathered} F(0, \lambda) = 0 \text{ for all } \lambda \in \mathbb{R}, \\ \exists \lambda_0 \in V \text{ such that } F(\cdot, \lambda_0) \text{ is a Fredholm operator} \\ \dim N(D_x F(0, \lambda_0)) = \operatorname{codim} R(D_x F(0, \lambda_0)) = 2, \\ F \in C^2(U \times V, Z), \text{ where } 0 \in U \subset X \text{ and } \lambda_0 \in V \subset \mathbb{R}. \end{gathered} \tag{3.1}$$

After performing the Method of Lyapunov-Schmidt to $F(x, \lambda) = 0$, we are left with $\Phi(v, \lambda) = 0$, where $\Phi\colon U_0 \times V_0 \to Z_0$, using the notation from Section 2.3. Next we write the Taylor expansion about $(0, \lambda_0)$ of Φ. To do so, introduce the following notation

$$\Phi_{ji}(v) \quad = \quad \frac{1}{j!i!} D_v^i D_\lambda^j \Phi(0, \lambda_0)[\underbrace{v, ..., v}_{i}]. \tag{3.2}$$

Hence, we can write

$$\Phi(v, \lambda) \quad = \quad \sum_{\substack{j=0 \\ i=1}}^{n} \lambda^j \Phi_{ji}(v) + R(v, \lambda). \tag{3.3}$$

We let k be the order of the first non-zero pure v-derivative of Φ at $(0, \lambda_0)$. Then we write

$$\Phi(v, \lambda) = \Phi_{0k}(v) + \lambda \Phi_{11} v + R(v, \lambda). \tag{3.4}$$

It can be shown that the remainder term $R(v, \lambda)$ in (3.4) contains the following terms:

- terms of order 0 in λ and order greater than or equal to $k+1$ in v,
- terms of order 1 in λ and order greater than or equal to 2 in v,
- terms of order greater than or equal to 2 in λ.

See Table 3.1, which represents the terms in the Taylor expansion of Φ. The top row is the order of the λ-derivative, and the left-most column is the order of the v-derivative. Terms left in the remainder are labeled with an R in the table, and terms that vanish are labeled in the table with 0.

Let $\{\hat{v}_1, \hat{v}_2\}$ be a basis for N, so $\forall v \in N$, we can write $v = x_1\hat{v}_1 + x_2\hat{v}_2$, for $x_1, x_2 \in \mathbb{R}$. The following theorem is proved in [3].

v \ λ	0	1	2	3 $\cdots$
0	0	0	0	0 $\cdots$
1	0	$\Phi_{11}v$	R	R $\cdots$
2	0	R	R	R $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$\Phi_{0k}(v)$	R	R	R $\cdots$
k+1	R	R	R	R $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$ $\ddots$

Table 3.1: Taylor expansion of $\Phi(v, \lambda)$.

Theorem 3.1.1. *Let F satisfy the hypotheses (3.1). Let $R_{\pi/2}$ denote the rotation*

$$\begin{aligned} &R_{\pi/2}v = -x_2\hat{v}_1 + x_1\hat{v}_2; \\ &\textit{observe that } \langle v, R_{\pi/2}v\rangle = 0 \textit{ for all } v \in Z_0. \end{aligned} \tag{3.5}$$

Assume that

$$\begin{aligned} &\Phi_{11} = QD^2_{x\lambda}F(0, \lambda_0)\colon N \to Z_0 \\ &\textit{is an isomorphism} \end{aligned} \tag{3.6}$$

and that there exist $\tilde{v}_1, \tilde{v}_2 \in N$ with $\|\tilde{v}_1\| = \|\tilde{v}_2\| = 1$ such that

$$\begin{aligned} &\langle \Phi_{0k}(\tilde{v}_1), R_{\pi/2}\Phi_{11}\tilde{v}_1\rangle < 0, \\ &\langle \Phi_{0k}(\tilde{v}_2), R_{\pi/2}\Phi_{11}\tilde{v}_2\rangle > 0. \end{aligned} \tag{3.7}$$

Then there exists a local continuum $C \subset X \times \mathbb{R}$ of non-trivial solutions of F through $(0, \lambda_0)$, and $C/\{(0, \lambda_0)\}$ consists of at least two components [3].

We see in several examples in the next chapter that the hypotheses of this theorem are straightforward to check. However, the theorem provides little information about the nature of the bifurcation branches.

CHAPTER IV

BIFURCATION IN SPECIFIC CASES

In this chapter, we begin by illustrating a specific example in $\mathbb{R}^3$ of the Lyapunov-Schmidt reduction and an application of Theorem 3.1.1. This is followed by two different generalizations of this example; both establish a set of sufficient conditions to apply Theorem 3.1.1. The chapter concludes with a discussion of the theorem from Krömer et al. as compared to classical methods for studying bifurcation problems with two-dimensional kernels.

4.1 Example in $\mathbb{R}^3$

Consider the map $F\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$F(x) = F(x_1, x_2, x_3, \lambda) = \begin{pmatrix} f_1(x_1, \lambda) \\ f_2(x_2, \lambda) \\ f_3(x_3) \end{pmatrix} = \begin{pmatrix} x_1 \sin(\lambda) + x_1 \sin(x_1) \\ x_2 \sin(\lambda) + x_2 \sin(x_2) \\ \sin(x_3) \end{pmatrix}, \tag{4.1}$$

where $x = (x_1, x_2, x_3)$. Let $\lambda_0 = 0$. One checks that

$$\begin{array}{rrr} f_1(0,\lambda) = 0, & f_2(0,\lambda) = 0, & f_3(0) = 0, \\ D_{x_1} f_1(0,\lambda_0) = 0, & D_{x_2} f_2(0,\lambda_0) = 0, & D_{x_3} f_3(0) = 1, \\ D^2_{x_1\lambda} f_1(0,\lambda_0) = 1, & D^2_{x_2\lambda} f_2(0,\lambda_0) = 1. & \end{array} \tag{4.2}$$

Also,

$$\begin{aligned} D^2_{x_1} f_1(0,\lambda_0) &= \cos(0) + \cos(0) - 0\sin(0) = 2, \\ D^2_{x_2} f_2(0,\lambda_0) &= \cos(0) + \cos(0) - 0\sin(0) = 2. \end{aligned} \tag{4.3}$$

Next, we note that $D_xF(x,\lambda)$ equals

$$\begin{pmatrix} \sin(\lambda)+\sin(x_1)+x_1\cos(x_1) & 0 & 0 \\ 0 & \sin(\lambda)+\sin(x_2)+x_2\cos(x_2) & 0 \\ 0 & 0 & \cos(x_3) \end{pmatrix}, \quad (4.4)$$

and hence

$$D_xF(\mathbf{0},\lambda_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.5)$$

Clearly, (4.5) implies that

$$\dim\left[N\left(D_xF\left(\mathbf{0},\lambda_0\right)\right)\right] = \operatorname{codim}\left[R\left(D_xF\left(\mathbf{0},\lambda_0\right)\right)\right] = 2.$$

Note that F satisfies (2.1) and is a Fredholm operator, so the method of Lyapunov-Schmidt can be applied to this example. Let $N = N\left(D_xF(\mathbf{0},\lambda_0)\right)$ and $R = R\left(D_xF(\mathbf{0},\lambda_0)\right)$. Moreover, let X_0 and Z_0 be their respective complements. From (4.3), it is clear that $N = Z_0 = \{(x_1,x_2,0)\colon x_1,x_2\in\mathbb{R}\}$ and $X_0 = R$ $= \{(0,0,x_3)\colon x_3\in\mathbb{R}\}$. Hence, the projections $P\colon\mathbb{R}^3\to N$ and $Q\colon\mathbb{R}^3\to Z_0$ are both defined by $(x_1,x_2,x_3)\mapsto(x_1,x_2,0)$. Also, the map $G(v,w,\lambda)=(I-Q)F(v+w,\lambda)$ defined by (2.5) is $(v,w,\lambda) = ((x_1,x_2,0),(0,0,x_3),\lambda) \mapsto (0,0,x_3)$. Hence, $D_wG(0,0,\lambda_0)$ is the same as the right hand side of (4.5), which is bijective as a map from X_0 to R. Here, the implicit function ψ is identically 0. See the paragraph containing (2.5). We have

$$\begin{aligned} \Phi(v,\lambda) = QF(v+\psi(v,\lambda),\lambda) &= \begin{pmatrix} f_1(x_1,\lambda) \\ f_2(x_2,\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1\sin(\lambda)+x_1\sin(x_1) \\ x_2\sin(\lambda)+x_2\sin(x_2) \\ 0 \end{pmatrix} = 0. \end{aligned} \quad (4.6)$$

Note that in this situation Φ does not depend on the implicit function ψ.

Now we verify that Φ satisfies the hypothesis of Theorem (5.1). For notational convenience, we drop the third component of Φ, which is identically 0, and we write $v = (x_1, x_2, 0)$ as just (x_1, x_2). We Taylor expand Φ about $(\mathbf{0}, \lambda_0)$,

$$\Phi(v, \lambda) = (\lambda - \lambda_0) D^2_{v\lambda}\Phi(\mathbf{0}, \lambda_0)v + \tfrac{1}{2} D^2_v \Phi(\mathbf{0}, \lambda_0)[v, v] + R(v, \lambda). \tag{4.7}$$

From (4.1), we see that

$$D_v \Phi\left(\mathbf{0}, \lambda\right) v = \begin{pmatrix} \sin(\lambda) & 0 \\ 0 & \sin(\lambda) \end{pmatrix} v.$$

It follows that

$$D^2_{v\lambda} \Phi\left(\mathbf{0}, \lambda_0\right) v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and hence, $D^2_{v\lambda}(\mathbf{0}, \lambda_0)$ is an isomorphism, fulfilling the hypothesis in (3.6).

For the next computation, we have

$$D^2_v \Phi(\mathbf{0}, \lambda_0)[v, v] \;=\; \begin{pmatrix} (x_1, x_2) \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (x_1, x_2) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix} = 2 \begin{pmatrix} {x_1}^2 \\ {x_2}^2 \end{pmatrix}.$$

Then substituting into (4.7), we have

$$\Phi(v, \lambda) \;=\; (\lambda - \lambda_0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} {x_1}^2 \\ {x_2}^2 \end{pmatrix} + R(v, \lambda). \tag{4.8}$$

To satisfy the hypothesis (3.7) of Theorem 3.1.1, it remains to show that there exist $\tilde{c}, \tilde{d} \in \mathbb{R}^2$ with $\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $\tilde{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, and $\|\tilde{c}\| = \|\tilde{d}\| = 1$ such that

$$\begin{aligned} \begin{pmatrix} {c_1}^2 \\ {c_2}^2 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right] &< 0, \\ \begin{pmatrix} {d_1}^2 \\ {d_2}^2 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right] &> 0, \end{aligned} \tag{4.9}$$

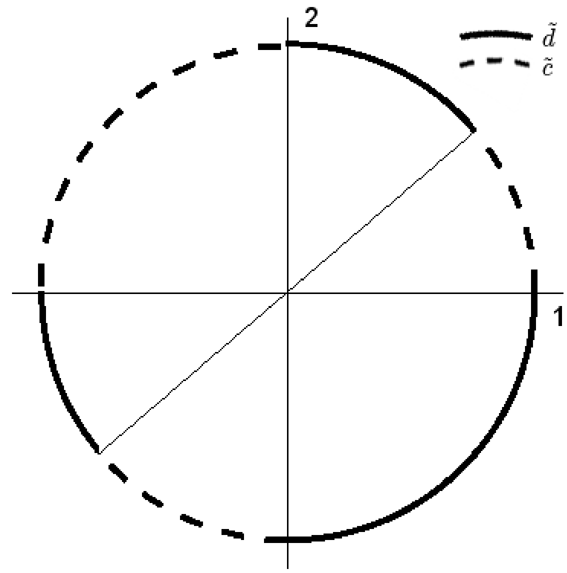

Figure 4.1: Unit vectors $\tilde{c}$ and $\tilde{d}$, which satisfy hypothesis (3.7) of Theorem 3.1.1

or

$$\begin{aligned} c_1c_2(c_2 - c_1) &< 0, \\ d_1d_2(d_2 - d_1) &> 0. \end{aligned} \tag{4.10}$$

The inequalities in (4.10) are true, thus satisfying hypothesis (3.7), for any unit vector $\tilde{c}_1 = (\cos\theta_1, \sin\theta_1)$ where $\theta_1 \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ and any unit vector $\tilde{d}_2 = (\cos\theta_2, \sin\theta_2)$ where $\theta_2 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$, as illustrated in Figure 4.1. For example, we can pick

$$\tilde{c} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \ \tilde{d} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}.$$

At this point, we have satisfied all of the hypotheses of Theorem 3.1.1, so the result hold.

4.2 Generalization of Example in $\mathbb{R}^2$

The example in Section 4.1 suggests the following result, which generalizes the example and a is special case of Theorem 3.1.1.

Theorem 4.2.1. *Let F satisfy (3.1), and let $N, X_0, R,$ and Z_0 be as in Section 2.3. Let $\{\hat{v}_1, \hat{v}_2\}$ be a basis for N, so $\forall v \in N$, we can write $v = x_1\hat{v}_1 + x_2\hat{v}_2$, for $x_1, x_2 \in \mathbb{R}$. Also, let $\{\hat{z}_1, \hat{z}_2\}$ be a basis for Z_0. Suppose that by using the method of Lyapunov-Schmidt $F(x, \lambda) = 0$ is reduced to*

$$\Phi(v, \lambda) = \Phi(x_1\hat{v}_1 + x_2\hat{v}_2, \lambda) = f_1(x_1, \lambda)\hat{z}_1 + f_2(x_2, \lambda)\hat{z}_2 = 0,$$

where $f_i \colon \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$. Suppose there are constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and

$$D^2_{x_1\lambda} f_1(0, \lambda_0) = \alpha_1, \quad D^2_{x_2\lambda} f_2(0, \lambda_0) = \alpha_2. \tag{4.11}$$

Suppose that $D^j_{x_1} f_1(0, \lambda_0) = D^j_{x_2} f_2(0, \lambda_0) = 0$ for $1 \leq j < k$, and further suppose that

$$D^k_{x_1} f_1(0, \lambda_0) = \alpha_3, \quad D^k_{x_2} f_2(0, \lambda_0) = \alpha_4, \tag{4.12}$$

where α_3 and α_4 are not both 0. Then there exists a local continuum $C \subset X \times \mathbb{R}$ of non-trivial solutions of F through $(0, \lambda_0)$, and $C/\{(0, \lambda_0)\}$ consists of at least two components.

Proof. We verify that the hypotheses of Theorem 3.1.1 hold in this situation. Using the bases $\{\hat{v}_1, \hat{v}_2\}$ and $\{\hat{z}_1, \hat{z}_2\}$, we let $x = (x_1, x_2)$ and see that $\Phi(v, \lambda) = 0$ yields

$$\widetilde{\Phi}(x, \lambda) = \widetilde{\Phi}(x_1, x_2, \lambda) = \begin{pmatrix} f_1(x_1, \lambda) \\ f_2(x_2, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.13}$$

First, we check hypothesis (3.6). Let $v = (v_1, v_2) \in \mathbb{R}^2$. Here $QD^2_{x\lambda}F(\mathbf{0}, \lambda_0) = D^2_{x\lambda}\widetilde{\Phi}(\mathbf{0}, \lambda_0)$, and

$$D^2_{x\lambda}\widetilde{\Phi}(\mathbf{0}, \lambda_0)v \quad = \quad D_\lambda \begin{pmatrix} D_{x_1}f_1 & 0 \\ 0 & D_{x_2}f_2 \end{pmatrix} v = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} v,$$

which is an isomorphism because by assumption (4.11), both α_1 and α_2 are non-zero.

Now we check hypothesis (3.7). We claim that for $j \geq 1$,

$$D^j_x\widetilde{\Phi}(\mathbf{0}, \lambda_0)[v, \ldots, v] = \begin{pmatrix} D^j_{x_1}f_1(0, \lambda_0) & 0 \\ 0 & D^j_{x_2}f_2(0, \lambda_0) \end{pmatrix} \begin{pmatrix} {v_1}^j \\ {v_2}^j \end{pmatrix}. \tag{4.14}$$

Note first that for $j = 1$, (4.14) is clear.

Suppose that for $j \geq 1$,

$$D^j_x\widetilde{\Phi}(\mathbf{0}, \lambda_0)[v, \ldots, v] = \begin{pmatrix} D^j_{x_1}f_1(0, \lambda_0) & 0 \\ 0 & D^j_{x_2}f_2(0, \lambda_0) \end{pmatrix} \begin{pmatrix} {v_1}^j \\ {v_2}^j \end{pmatrix}, \tag{4.15}$$

and observe that to find $D^{j+1}_x\widetilde{\Phi}(0, \lambda_0)[v, \ldots, v]$, we first compute

$$\begin{aligned} D_x & \left[\begin{pmatrix} D^j_{x_1}f_1(x_1, \lambda_0) & 0 \\ 0 & D^j_{x_2}f_2(x_2, \lambda_0) \end{pmatrix} \begin{pmatrix} {v_1}^j \\ {v_2}^j \end{pmatrix} \right] \\ & = \begin{pmatrix} D^{j+1}_{x_1}f_1(x_1, \lambda_0){v_1}^j & 0 \\ 0 & D^{j+1}_{x_2}f_2(x_2, \lambda_0){v_2}^j \end{pmatrix}. \end{aligned}$$

Evaluating with $x_1 = x_2 = 0$ and letting this matrix act on v yields

$$D^{j+1}_x\widetilde{\Phi}(\mathbf{0}, \lambda_0)[v, \ldots, v] = \begin{pmatrix} D^{j+1}_{x_1}f_1(0, \lambda_0) & 0 \\ 0 & D^{j+1}_{x_2}f_2(0, \lambda_0) \end{pmatrix} \begin{pmatrix} {v_1}^{j+1} \\ {v_2}^{j+1} \end{pmatrix}. \tag{4.16}$$

Hence, the claim follows by induction. Then, using our assumption (4.12) about the k-derivatives, we have

$$\begin{aligned} \widetilde{\Phi}(v, \lambda) = \quad & (\lambda - \lambda_0) \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} v \\ + \quad & \frac{1}{k} \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_4 \end{pmatrix} \begin{pmatrix} {v_1}^k \\ {v_2}^k \end{pmatrix} + R(v, \lambda). \end{aligned}$$

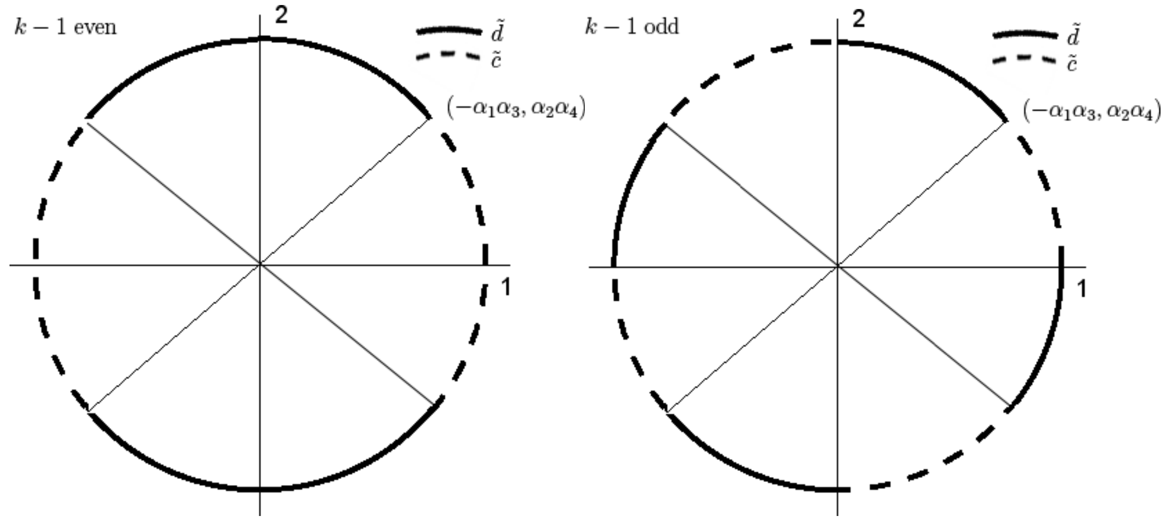

Figure 4.2: Unit vectors $\tilde{c}$ and $\tilde{d}$, which satisfy hypothesis (3.7) of Theorem 3.1.1

To satisfy hypothesis (3.7) of Theorem 3.1.1, it remains to show that there exist $\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and $\tilde{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ with $\|\tilde{c}\| = \|\tilde{d}\| = 1$ such that

$$\begin{aligned}
\left[\begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_4 \end{pmatrix}\begin{pmatrix} {c_1}^k \\ {c_2}^k \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right] &< 0 \\
\left[\begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_4 \end{pmatrix}\begin{pmatrix} {d_1}^k \\ {d_2}^k \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\right] &> 0,
\end{aligned}$$

or

$$\begin{aligned}
(c_1c_2)\left[\left(c_1^{k-1}, c_2^{k-1}\right) \cdot (-\alpha_1\alpha_3, \alpha_2\alpha_4)\right] &< 0, \\
(d_1d_2)\left[\left(d_1^{k-1}, d_2^{k-1}\right) \cdot (-\alpha_1\alpha_3, \alpha_2\alpha_4)\right] &> 0.
\end{aligned} \tag{4.17}$$

There are always vectors $\tilde{c}$ and $\tilde{d}$ to satisfy this provided the inner products are not 0. Since $(-\alpha_1\alpha_3, \alpha_2\alpha_4) \neq 0$, c^{k-1} and d^{k-1} are required to be neither the zero vector nor orthogonal to $(-\alpha_1\alpha_3, \alpha_2\alpha_4)$. The inequalities in (4.17) are thus true, satisfying hypothesis (3.7). See Figure 4.2. $\square$

4.3 General Example in $\mathbb{R}^3$

The example in Section (4.1) also suggests the following result, which is another generalization of the example in Section 4.1 and a special case of Theorem 3.1.1.

Theorem 4.3.1. *Let F satisfy (3.1), and let $N, X_0, R,$ and Z_0 be as in Section 2.3. Let $\{\hat{v}_1, \hat{v}_2\}$ be a basis for N, so $\forall v \in N$, we can write $v = x_1\hat{v}_1 + x_2\hat{v}_2$, for $x_1, x_2 \in \mathbb{R}$. Also, let $\{\hat{z}_1, \hat{z}_2\}$ be a basis for Z_0. Suppose that by using the method of Lyapunov-Schmidt $F(x, \lambda) = 0$ is reduced to*

$$\Phi(v, \lambda) = \Phi(x_1\hat{v}_1 + x_2\hat{v}_2, \lambda) = f_1(x_1, x_2, \lambda)\hat{z}_1 + f_2(x_1, x_2, \lambda)\hat{z}_2 = 0. \tag{4.18}$$

Also, suppose there are constants $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ with $1 \le i \le 4$,

$$\begin{array}{cc} D_x f_1(0,0,\lambda_0) = (0,0), & D_x f_2(0,0,\lambda_0) = (0,0), \\ D^2_{x\lambda} f_1(0,0,\lambda_0) = (\alpha_1, 0), & D^2_{x\lambda} f_2(0,0,\lambda_0) = (0, \alpha_2), \end{array}$$

$$\begin{pmatrix} D^2_{x_1} f_1 & D^2_{x_1x_2} f_1 \\ D_{x_2x_1} f_1 & D^2_{x_2} f_1 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_4 \end{pmatrix}, \quad \begin{pmatrix} D^2_{x_1} f_2 & D^2_{x_1x_2} f_2 \\ D^2_{x_2x_1} f_2 & D^2_{x_2} f_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_4 \end{pmatrix},$$

where the second partial derivatives in the last two equalities are evaluated at $(0, 0, \lambda_0)$. Finally, suppose that $\alpha_1, \alpha_2, \beta_4, \gamma_1 \neq 0$, and the following quantities are not both zero:

$$2\alpha_2\gamma_2 - \alpha_1\beta_1 \quad \text{and} \quad \alpha_2\gamma_4 - 2\alpha_1\beta_2. \tag{4.19}$$

Then there exists a local continuum $C \subset X \times \mathbb{R}$ of non-trivial solutions of F through $(0, \lambda_0)$, and $C/\{(0, \lambda_0)\}$ consists of at least two components.

Proof. As in the previous proof, we verify that the hypotheses of Theorem 3.1.1 hold in this situation. Using the bases $\{\hat{v}_1, \hat{v}_2\}$ and $\{\hat{z}_1, \hat{z}_2\}$, we see that $\Phi(v, \lambda) = 0$ is equivalent to

$$\widetilde{\Phi}(x,\lambda) = \widetilde{\Phi}(x_1, x_2, \lambda) = \begin{pmatrix} f_1(x_1, x_2, \lambda) \\ f_2(x_1, x_2, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.20}$$

where $x = (x_1, x_2)$.

First, we verify that hypothesis (3.6) holds in this situation. Let $v = (v_1, v_2) \in \mathbb{R}^2$. Here

$$QD^2_{x\lambda}F(\mathbf{0}, \lambda_0) = D^2_{x\lambda}\widetilde{\Phi}\,(\mathbf{0}, \lambda_0)\, v = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} v,$$

which is an isomorphism because both α_1 and α_2 are non-zero by assumption.

Now we check hypothesis (3.7). Note that $D^2_v\widetilde{\Phi}(\mathbf{0}, \lambda_0)[v, v]$ equals

$$\begin{pmatrix} (v_1, v_2) \cdot \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ (v_1, v_2) \cdot \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \beta_1 v_1 v_1 + 2\beta_2 v_1 v_2 + \beta_4 v_2 v_2 \\ \gamma_1 v_1 v_1 + 2\gamma_2 v_1 v_2 + \gamma_4 v_2 v_2 \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} \Phi(v, \lambda) &= (\lambda - \lambda_0) \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} v \\ &\quad + \frac{1}{2} \begin{pmatrix} \beta_1 v_1 v_1 + 2\beta_2 v_1 v_2 + \beta_4 v_2 v_2 \\ \gamma_1 v_1 v_1 + 2\gamma_2 v_1 v_2 + \gamma_4 v_2 v_2 \end{pmatrix} + R(v, \lambda). \end{aligned}$$

To satisfy hypothesis (3.7) of Theorem 3.1.1, it remains to show that there exist $\tilde{c}, \tilde{d} \in \mathbb{R}^2$ with $\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and $\tilde{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ with $\|\tilde{c}\| = \|\tilde{d}\| = 1$ such that

$$\begin{aligned} \begin{pmatrix} \beta_1 c_1 c_1 + 2\beta_2 c_1 c_2 + \beta_4 c_2 c_2 \\ \gamma_1 c_1 c_1 + 2\gamma_2 c_1 c_2 + \gamma_4 c_2 c_2 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right] &< 0 \\ \begin{pmatrix} \beta_1 d_1 d_1 + 2\beta_2 d_1 d_2 + \beta_4 d_2 d_2 \\ \gamma_1 d_1 d_1 + 2\gamma_2 d_1 d_2 + \gamma_4 d_2 d_2 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right] &> 0. \end{aligned}$$

Let $m \in \mathbb{R}^2$ with $m = \binom{m_1}{m_2}$ and $\|m\| = 1$, and let θ denote the expression

$$\begin{pmatrix} \beta_1 v_1 v_1 + 2\beta_2 v_1 v_2 + \beta_4 v_2 v_2 \\ \gamma_1 v_1 v_1 + 2\gamma_2 v_1 v_2 + \gamma_4 v_2 m_2 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ m_2 \end{pmatrix} \right].$$

Then θ equals

$$-\alpha_1\beta_1 {m_1}^2 m_2 - 2\alpha_1\beta_2 m_1 {m_2}^2 - \alpha_1\beta_4 {m_2}^3 + \alpha_2\gamma_1 {m_1}^3 + 2\alpha_2\gamma_2 {m_1}^2 m_2 + \alpha_2\gamma_4 m_1 {m_2}^2$$

$$= {m_1}^2 \left[\alpha_2\gamma_1 m_1 + (2\alpha_2\gamma_2 - \alpha_1\beta_1) m_2\right] + {m_2}^2 \left[(\alpha_2\gamma_4 - 2\alpha_1\beta_2) m_1 - \alpha_1\beta_4 m_2\right].$$

Then define $w = (\alpha_2\gamma_1, 2\alpha_2\gamma_2 - \alpha_1\beta_1)$ and $z = (\alpha_2\gamma_4 - 2\alpha_1\beta_2, -\alpha_1\beta_4)$, so we have

$$\theta = ({m_1}^2, {m_2}^2) \cdot (m \cdot w, m \cdot z). \tag{4.21}$$

Existence of vectors $\tilde{c}$ and $\tilde{d}$ is thus dependent on the vectors w and z, and hypothesis (4.19) gives w, z are not both the zero vector. We have

$$\theta = ({m_1}^2, {m_2}^2) \cdot (m \cdot w, m \cdot z) = {m_1}^2(m \cdot w) + {m_2}^2(m \cdot z).$$

If either $m \cdot w$ or $m \cdot z$ is zero, the choice of $\tilde{c}$ and $\tilde{d}$ becomes clear. Also, if $w = z$, then $m \cdot w = m \cdot z$, so choose $\tilde{c}$ to be any vector in the third quadrant and $\tilde{d}$ in the first quadrant. If $w \perp z$, then choosing $\tilde{c}$ and $\tilde{d}$ to be $\pm w$ or $\pm z$ reduces the sum ${m_1}^2(m \cdot w) + {m_2}^2(m \cdot z)$ to one term whose sign is determined by the sign of w or z.

Now suppose $w \neq z$ and $w \cdot z \neq 0$, so without loss of generality, we have the following situation: with regard to the quantity ${m_1}^2(m \cdot w) + {m_2}^2(m \cdot z)$. Note that $m \cdot w$ (and hence, ${m_1}^2(m \cdot w)$) will be approach 0 for m chosen near $w^\perp$. Thus, the sign of ${m_1}^2(m \cdot w) + {m_2}^2(m \cdot z)$ is determined by the sign of m. Should $w^\perp$ happen to be the 0 vector, choose m near $z^\perp$.

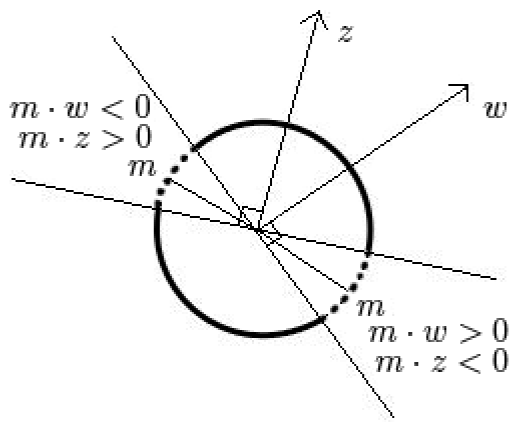

Figure 4.3: Unit vector m, which determines $\tilde{c}$ and $\tilde{d}$, satisfying hypothesis (3.7) of Theorem 3.1.1

Hence, provided w and z are not both 0, there are vectors $\tilde{c}, \tilde{d} \in \mathbb{R}^2$, thus satisfying all hypotheses of Theorem 3.1.1. □

4.4 A Comparison to Other Techniques for Two-Dimensional Kernels

Theorem 3.1.1 provides a method for finding a nontrivial solution curve for single-parameter bifurcation problems with a two dimensional kernel. Classical analytical techniques based on the Implicit Function theorem can also be used to analyze single parameter bifurcation problems with two-dimensional kernels. In this section, we give a class of algebraic examples to which Theorem 3.1.1 applies but to which the classical analytical techniques cannot be applied.

First we describe briefly the classical analytical techniques appropriate for single parameter problems with two-dimensional kernels. We consider $F(x, \lambda) = 0$, where F satisfies (3.1) and is a Fredholm operator. Let $\Phi(v, \lambda) = 0$ be the reduced problem after applying the method of Lyapunov-Schmidt. Also, we call an operator regular if it is invertible.

Theorem 4.4.1 (Kielhofer)**.** *Suppose* Φ *satisfies*

$$\begin{aligned} &\Phi_{02}[v_0, v_0] + \Phi_{11}v_0 = 0 \textit{ for some } v_0 \neq 0, \\ &2\Phi_{02}[v_0, \cdot] + \Phi_{11} \textit{ is regular in } L(N, Z_0). \end{aligned} \tag{4.22}$$

Then there is a nontrivial solution curve with $\tilde{v}(0) = \tilde{v}_0$*,*

$$\left\{ (\lambda\tilde{v}(\lambda), \lambda) \middle| \lambda \in (-\delta, \delta) \right\} \tag{4.23}$$

of $\Phi(v, \lambda) = 0$ *[2].*

Proof of this theorem can be found in [2]. Note that the conclusion here gives information regarding the smoothness of the branching solution. We note that by contrast, Theorem 3.1.1 presents a more relaxed set of sufficient conditions for establishing the existence of non-trivial solutions. These conditions are easier to verify in practice than the conditions in (4.22). On the other hand, Theorem 3.1.1 provides only the existence of a local continuum of non-trivial solutions consisting of at least two components. In particular, there is no further information about the behavior or smoothness of the branching solutions.

The purpose the examples in this section is to explore differences between these two methods. In particular, we provide an example where Theorem 3.1.1 can be applied, but Theorem 4.4.1 cannot. The example in Section 4.3 addresses the same problem as the methods of Theorem 3.1.1 and 4.4.1, but begins in a less general setting. While making use of Theorem 3.1.1, our next claim requires a set of conditions more specific than Theorem 3.1.1. The goal is to eventually establish how much more specific we must be with the conditions of Theorem 3.1.1 in order to gain more information about the non-trivial solution curve.

Theorem 4.4.2. *With regard to a specific instance of the hypotheses of Theorem 4.3.1, the sufficiency conditions of Theorem 3.1.1 are met, but the conditions of Theorem 4.4.1 are not.*

Proof. It was shown in an earlier proof that Theorem 3.1.1 holds with the hypotheses from Theorem 4.3.1. Further suppose that

$$\begin{aligned}
&\beta_2 = \gamma_2 = 0,\\
&\beta_1 = \beta_4 = \gamma_1 = \gamma_4 = \tfrac{1}{2},\\
&\alpha_1 = \alpha_2 = 1.
\end{aligned}$$

If $\Phi_{02}[v_0, v_0] + \Phi_{11}v_0 = 0$ for some $v_0 \neq 0$, then the conditions of Theorem 4.4.1 with these hypotheses yield

$$\begin{pmatrix} \beta_1 v_1 v_1 + 2\beta_2 v_1 v_2 + \beta_4 v_2 v_2 + \alpha_1 v_1 \\ \gamma_1 v_1 v_1 + 2\gamma_2 v_1 v_2 + \gamma_4 v_2 v_2 + \alpha_2 v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}{v_1}^2 + \frac{1}{2}{v_2}^2 + v_1 \\ \frac{1}{2}{v_1}^2 + \frac{1}{2}{v_2}^2 + v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $1 + v_1 + v_2 = 0$. Moreover, we also have

$$\begin{aligned}
2\Phi_{02}[v_0, \cdot] + \Phi_{11} &= \begin{pmatrix} 2\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ 2\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \\
&= \begin{pmatrix} 2\beta_1 v_1 + 2\beta_2 v_2 + \alpha_1 & 2\beta_2 v_1 + 2\beta_4 v_2 \\ 2\gamma_1 v_1 + 2\gamma_2 v_2 & 2\gamma_2 v_1 + 2\gamma_4 v_2 + \alpha_2 \end{pmatrix} \\
&= \begin{pmatrix} v_1 + 1 & v_2 \\ v_1 & v_2 + 1 \end{pmatrix},
\end{aligned}$$

which must be regular. Note however that this is not the case because

$$(v_1 + 1)(v_2 + 1) - v_1 v_2 \quad = \quad (v_1 + v_2 + 1) + v_1 v_2 - v_1 v_2 = 0.$$

Hence, it is not possible under these circumstances to satisfy the sufficiency conditions of Theorem 4.4.1. Recall however, that the sufficiency conditions of Theorem 3.1.1 were met. □

CHAPTER V

AN ALGEBRAIC APPROACH

Theorem 3.1.1 provides sufficient conditions for the existence of a local continuum of nontrivial solutions for a bifurcation problem with a two-dimensional kernel. In this chapter, we construct an example with a specific bifurcation equation, and plot its bifurcation branches. We then use the method outlined in the proof of Theorem 3.1.1 [3] on the same bifurcation equation to generate an example of the results provided by Theorem 3.1.1.

5.1 The Bifurcation Equation Graphically

Let F satisfy (3.1), and let N, X_0, R, and Z_0 be as in Section 2.3. Let $\{\hat{v}_1, \hat{v}_2\}$ be a basis for N, so $\forall v \in N$, we can write $v = x_1\hat{v}_1 + x_2\hat{v}_2$, for $x_1, x_2 \in \mathbb{R}$. Also, let $\{\hat{z}_1, \hat{z}_2\}$ be a basis for Z_0. Suppose that by using the method of Lyapunov-Schmidt $F(x, \lambda) = 0$ is reduced to

$$\Phi(v, \lambda) = \Phi(x_1\hat{v}_1 + x_2\hat{v}_2, \lambda) = f_1(x_1, \lambda)\hat{z}_1 + f_2(x_2, \lambda)\hat{z}_2 = 0,$$

where $f_i \colon \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$. Here, we assume $k = 2$, where k is the order of the first non-zero v-derivative. We also suppose that all but three remainder terms in the Taylor expansion of Φ, about a particular solution $(0, \lambda_0)$, vanish.

	λ 0	1	2	3 $\cdots$
0	0	0	0	0 $\cdots$
1	0	$\Phi_{11}v$	0	0 $\cdots$
2	$\Phi_{02}(v)$	R	0	0 $\cdots$
3	R	R	0	0 $\cdots$
4	0	0	0	0 $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$ $\ddots$

Table 5.1: Taylor expansion of a specific $\Phi(v, \lambda)$

Specifically, we assume that the Taylor expansion of Φ is

$$\begin{aligned} \Phi(v,\lambda) &= \lambda\Phi_{11}v + \Phi_{02}(v) + \Phi_{03}(v) + \lambda\Phi_{12}(v) + \lambda\Phi_{13}(v) \\ &= \lambda D_v D_\lambda \Phi(0,\lambda_0)v + \tfrac{1}{2}D_v^2\Phi(0,\lambda_0)(v) \\ &\quad + \tfrac{1}{6}D_v^3\Phi(0,\lambda_0)(v) + \tfrac{1}{2}\lambda D_v^2 D_\lambda\Phi(0,\lambda_0)(v) + \tfrac{1}{6}\lambda D_v^3 D_\lambda \Phi(0,\lambda_0)(v) \end{aligned} \tag{5.1}$$

Next, we must calculate partial derivatives. We will represent the first v-derivative using the gradient and the second v-derivative with a Hessian. To simplify calculations and maintain a degree of generality, we assign a sequence of variables to the derivative entries of each of the matrices. For example,

$$\begin{aligned} \lambda D_v D_\lambda \Phi(0,\lambda_0)v &= \lambda D_\lambda \begin{pmatrix} D_{x_1}f_1 & D_{x_2}f_1 \\ D_{x_1}f_2 & D_{x_2}f_2 \end{pmatrix} v \\ &= \lambda \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} v = \lambda \begin{pmatrix} \alpha_1 v_1 + \alpha_2 v_2 \\ \alpha_3 v_1 + \alpha_4 v_2 \end{pmatrix}, \end{aligned} \tag{5.2}$$

where we have assumed

$$\begin{pmatrix} D^2_{x_1\lambda}f_1 & D^2_{x_2\lambda}f_2 \\ D^2_{x_1\lambda}f_2 & D^2_{x_2\lambda}f_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

Also,

$$
\begin{aligned}
\tfrac{1}{2}D_v^2\Phi(0,\lambda_0)(v) &= \tfrac{1}{2}D_x(D_x\Phi(0,\lambda_0))[v,v] \\
&= \tfrac{1}{2}D_x\begin{pmatrix} D_{x_1}f_1 & D_{x_2}f_1 \\ D_{x_1}f_2 & D_{x_2}f_2 \end{pmatrix}[v,v] \\
&= \tfrac{1}{2}\begin{pmatrix} [v_1,v_2]\cdot\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ [v_1,v_2]\cdot\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} \beta_1v_1v_1+\beta_2v_1v_2+\beta_3v_2v_1+\beta_4v_2v_2 \\ \gamma_1v_1v_1+\gamma_2v_1v_2+\gamma_3v_2v_1+\gamma_4v_2v_2 \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} \beta_1v_1v_1+\beta_2v_1v_2+\beta_3v_2v_2 \\ \gamma_1v_1v_1+\gamma_2v_1v_2+\gamma_3v_2v_2 \end{pmatrix} \text{ after reindexing.}
\end{aligned} \tag{5.3}
$$

where we have assumed

$$
\begin{pmatrix} D^2_{x_1}f_1 & D^2_{x_1x_2}f_1 \\ D_{x_2x_1}f_1 & D^2_{x_2}f_1 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_4 \end{pmatrix}, \quad \begin{pmatrix} D^2_{x_1}f_2 & D^2_{x_1x_2}f_2 \\ D^2_{x_2x_1}f_2 & D^2_{x_2}f_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_4 \end{pmatrix}.
$$

After similar calculations for the last three terms of Φ, we have

$$
\begin{aligned}
\Phi(v,\lambda) &= \lambda\begin{pmatrix} \alpha_1v_1+\alpha_2v_2 \\ \alpha_3v_1+\alpha_4v_2 \end{pmatrix} \\
&\quad +\tfrac{1}{2}\begin{pmatrix} \beta_1{v_1}^2+\beta_2v_1v_2+\beta_3{v_2}^2 \\ \gamma_1{v_1}^2+\gamma_2v_1v_2+\gamma_3{v_2}^2 \end{pmatrix} \\
&\quad +\tfrac{1}{6}\begin{pmatrix} a_1{v_1}^3+a_2{v_1}^2v_2+a_3v_1{v_2}^2+a_4{v_2}^3 \\ b_1{v_1}^3+b_2{v_1}^2v_2+b_3v_1{v_2}^2+b_4{v_2}^3 \end{pmatrix} \\
&\quad +\tfrac{1}{2}\lambda\begin{pmatrix} c_1{v_1}^2+c_2v_1v_2+c_3{v_2}^2 \\ d_1{v_1}^2+d_2v_1v_2+d_3{v_2}^2 \end{pmatrix} \\
&\quad +\tfrac{1}{6}\lambda\begin{pmatrix} e_1{v_1}^3+e_2{v_1}^2v_2+e_3v_1{v_2}^2+e_4{v_2}^3 \\ f_1{v_1}^3+f_2{v_1}^2v_2+f_3v_1{v_2}^2+f_4{v_2}^3 \end{pmatrix}
\end{aligned} \tag{5.4}
$$

This illustrates how difficulty often arises in dealing with bifurcation equations with higher dimensional kernels. Even while using our very conservative restraints, this example has quickly become cumbersome. For simplicity of calculation,

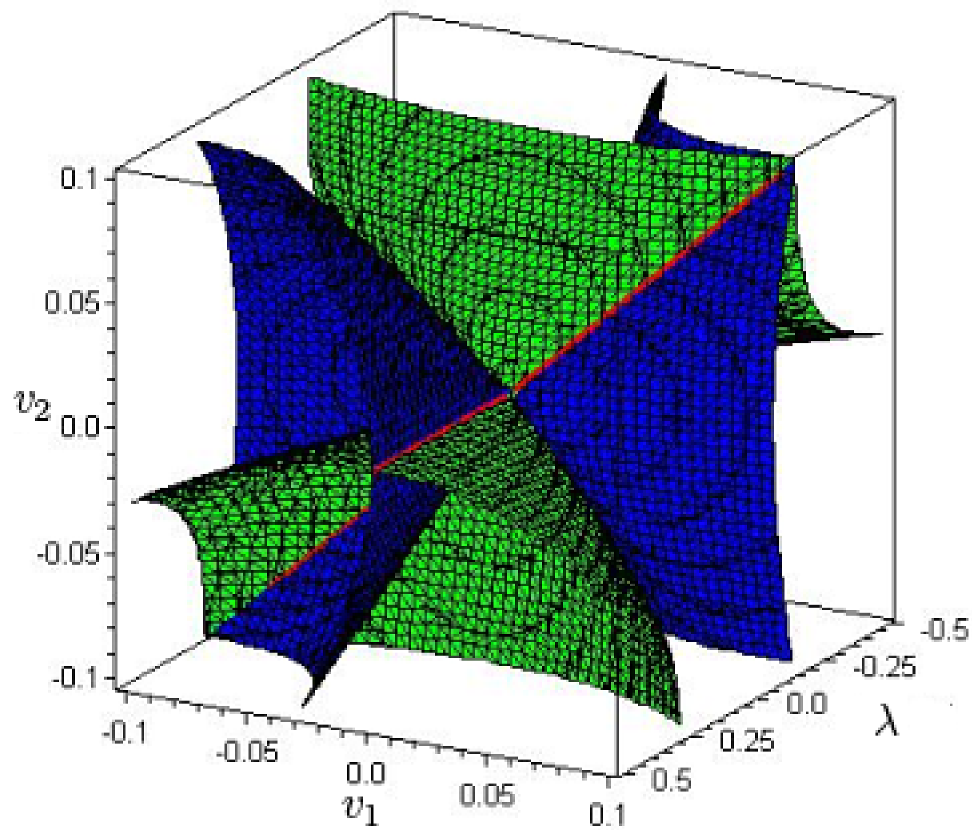

Figure 5.1: The intersection of the zero sets of ϕ_1 and ϕ_2 correspond with the solution set of $\Phi(v, \lambda) = 0$. The non-trivial solutions are indicated by the red lines.

we suppose at this point that $\{\alpha_i\}_{i=1}^4 = \{1, 0, 0, 1\}$. We further suppose that any coefficient of a term containing both v_1 and v_2 to be 0, and all others to be the appropriate constants such that all non-zero coefficients are 1. We then have

$$\begin{aligned}\Phi(v, \lambda) &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} {v_1}^2 + {v_2}^2 \\ {v_1}^2 + {v_2}^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} {v_1}^3 + {v_2}^3 \\ {v_1}^3 + {v_2}^3 \end{pmatrix} + \lambda \begin{pmatrix} {v_1}^2 + {v_2}^2 \\ {v_1}^2 + {v_2}^2 \end{pmatrix} + \lambda \begin{pmatrix} {v_1}^3 + {v_2}^3 \\ {v_1}^3 + {v_2}^3 \end{pmatrix}\end{aligned} \tag{5.5}$$

If we regard Φ as two functions, one in the first component and one in the second, we define

$$\begin{aligned}\phi_1(v_1, v_2, \lambda) &= \lambda v_1 + {v_1}^2 + {v_2}^2 \\ &\quad + {v_1}^3 + {v_2}^3 + \lambda {v_1}^2 + \lambda {v_2}^2 + \lambda {v_1}^3 + \lambda {v_2}^3 \\ \phi_2(v_1, v_2, \lambda) &= \lambda v_2 + {v_1}^2 + {v_2}^2 \\ &\quad + {v_1}^3 + {v_2}^3 + \lambda {v_1}^2 + \lambda {v_2}^2 + \lambda {v_1}^3 + \lambda {v_2}^3,\end{aligned} \tag{5.6}$$

Then the solutions of $\Phi = 0$ correspond with the intersection of the surfaces $\phi_1 = 0$ and $\phi_2 = 0$. These surfaces are plotted together in Figure 5.1. Since $\phi_1 = \phi_2 = 0$ implies

$$\phi_1(v_1, v_2, \lambda) - \phi_2(v_1, v_2, \lambda) = \lambda (v_1 - v_2), \tag{5.7}$$

our solutions are the intersection of the surfaces where $v_1 = v_2$. This is illustrated by the red line in Figure 5.1.

5.2 Krömer's Method

We now provide an example of the non-trivial solution branches as they result from Theorem 3.1.1. To do this, we follow the steps outlined in the proof [3]. Suppose Φ

is as in the previous section. We first make the substitutions $v = s\tilde{v}$, where $\|\tilde{v}\| = 1$, and $\lambda = s\tilde{\lambda}$ yield:

$$\begin{aligned} \Phi(\tilde{v}, \tilde{\lambda}, s) &= s^2\tilde{\lambda}\Phi_{11}\tilde{v} + s^2\Phi_{02}(\tilde{v}) + s^3\Phi_{03}(\tilde{v}) + s^3\tilde{\lambda}\Phi_{12}(\tilde{v}) + s^4\tilde{\lambda}\Phi_{13}(\tilde{v}) \\ &= s^2\left(\tilde{\lambda}\Phi_{11}\tilde{v} + \Phi_{02}(\tilde{v}) + s\Phi_{03}(\tilde{v}) + s\tilde{\lambda}\Phi_{12}(\tilde{v}) + s^2\tilde{\lambda}\Phi_{13}(\tilde{v})\right), \end{aligned} \tag{5.8}$$

so we can define

$$\widetilde{\Phi}(\tilde{v}, \tilde{\lambda}, s) = \tilde{\lambda}\Phi_{11}\tilde{v} + \Phi_{02}(\tilde{v}) + s\Phi_{03}(\tilde{v}) + s\tilde{\lambda}\Phi_{12}(\tilde{v}) + s^2\tilde{\lambda}\Phi_{13}(\tilde{v}). \tag{5.9}$$

We can also make this substitution in ϕ_1 and ϕ_2, and after removing a factor of s^2, define

$$\begin{aligned} \tilde{\phi}_1(v_1, v_2, \tilde{\lambda}, s) &= \tilde{\lambda}v_1 + {v_1}^2 + {v_2}^2 \\ &\quad + s{v_1}^3 + s{v_2}^3 + s\tilde{\lambda}{v_1}^2 + s\tilde{\lambda}{v_2}^2 + s^2\tilde{\lambda}{v_1}^3 + s^2\tilde{\lambda}{v_2}^3, \\ \tilde{\phi}_2(v_1, v_2, \tilde{\lambda}, s) &= \tilde{\lambda}v_2 + {v_1}^2 + {v_2}^2 \\ &\quad + s{v_1}^3 + s{v_2}^3 + s\tilde{\lambda}{v_1}^2 + s\tilde{\lambda}{v_2}^2 + s^2\tilde{\lambda}{v_1}^3 + s^2\tilde{\lambda}{v_2}^3. \end{aligned} \tag{5.10}$$

Also, we have $\widetilde{\Phi}(\tilde{v}, \tilde{\lambda}, s) = 0$ if and only if

$$\tilde{\phi}_1(v_1, v_2, \tilde{\lambda}, s) = \tilde{\phi}_2(v_1, v_2, \tilde{\lambda}, s) = 0. \tag{5.11}$$

We now reformulate the problem as in the proof of Theorem 3.1.1 [3]. Define

$$f_1(\tilde{v}, \tilde{\lambda}, s) = \begin{pmatrix} \phi_1(v_1, v_2, \tilde{\lambda}, s) \\ \phi_2(v_1, v_2, \tilde{\lambda}, s) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{5.12}$$

$$f_2(\tilde{v}, \tilde{\lambda}, s) = \begin{pmatrix} \phi_1(v_1, v_2, \tilde{\lambda}, s) \\ \phi_2(v_1, v_2, \tilde{\lambda}, s) \end{pmatrix} \cdot \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}. \tag{5.13}$$

Note now that for $s \neq 0$, $\tilde{\Phi}(\tilde{v}, \tilde{\lambda}, s) = 0$ if and only of $f_1(\tilde{v}, \tilde{\lambda}, s) = f_2(\tilde{v}, \tilde{\lambda}, s) = 0$. Since we have explicit definitions of f_1 and f_2, we can solve for $\tilde{\lambda}$ in $f_1 = 0$ and substitute the resulting function into $f_2 = 0$. Then we will have a function in $\tilde{v}$ and

s only, whose solutions (for $s \neq 0$) are the nontrivial solution branches of $\Phi(v, \lambda) = 0$ near $(0, \lambda_0)$.

We have that (5.12) yields

$$\begin{aligned}
&\left(\tilde{\lambda} v_1 + {v_1}^2 + {v_2}^2 + s{v_1}^3 + s{v_2}^3 + s\tilde{\lambda}{v_1}^2 + s\tilde{\lambda}{v_2}^2 + s^2\tilde{\lambda}{v_1}^3 + s^2\tilde{\lambda}{v_2}^3\right) v_1 \\
&\quad + \left(\tilde{\lambda} v_2 + {v_1}^2 + {v_2}^2 + s{v_1}^3 + s{v_2}^3 + s\tilde{\lambda}{v_1}^2 + s\tilde{\lambda}{v_2}^2 + s^2\tilde{\lambda}{v_1}^3 + s^2\tilde{\lambda}{v_2}^3\right) v_2 \\
&= \tilde{\lambda}\left({v_1}^2 + s{v_1}^3 + s{v_2}^2 v_1 + s^2 {v_1}^4 + s^2 {v_2}^3 v_1 + {v_2}^2 + s{v_1}^2 v_2 + s{v_2}^3 + s^2 {v_1}^3 v_2 + s^2 {v_2}^4\right) \\
&\quad + {v_1}^3 + {v_2}^2 v_1 + s{v_1}^4 + s{v_2}^3 v_1 + {v_1}^2 v_2 + {v_2}^3 + s{v_1}^3 v_2 + s{v_2}^4 = 0.
\end{aligned} \tag{5.14}$$

Similarly, we have that (5.13) yields

$$\begin{aligned}
&\left(\tilde{\lambda} v_2 + {v_1}^2 + {v_2}^2 + s{v_1}^3 + s{v_2}^3 + s\tilde{\lambda}{v_1}^2 + s\tilde{\lambda}{v_2}^2 + s^2\tilde{\lambda}{v_1}^3 + s^2\tilde{\lambda}{v_2}^3\right) v_2 \\
&\quad - \left(\tilde{\lambda} v_1 + {v_1}^2 + {v_2}^2 + s{v_1}^3 + s{v_2}^3 + s\tilde{\lambda}{v_1}^2 + s\tilde{\lambda}{v_2}^2 + s^2\tilde{\lambda}{v_1}^3 + s^2\tilde{\lambda}{v_2}^3\right) v_1 \\
&= \tilde{\lambda}\left({v_2}^2 + s{v_1}^2 v_2 + s{v_2}^3 + s^2 {v_1}^3 v_2 + s^2 {v_2}^4 - {v_1}^2 - s{v_1}^3 - s{v_2}^2 v_1 - s^2 {v_1}^4 - s^2 {v_2}^3 v_1\right) \\
&\quad + {v_1}^2 v_2 + {v_2}^3 + s{v_1}^3 v_2 + s{v_2}^4 - {v_1}^3 - {v_2}^2 v_1 - s{v_1}^4 - s{v_2}^3 v_1 = 0.
\end{aligned} \tag{5.15}$$

Since v_1 and v_2 are orthogonal unit vectors, we have ${v_1}^2 + {v_2}^2 = 1$, so simplifying yields

$$\begin{aligned}
f_1(\tilde{v}, \tilde{\lambda}, s) \;&=\; \tilde{\lambda}\left(1 + s(v_1 + v_2) + s^2({v_1}^4 + v_1 v_2 + {v_2}^4)\right) \\
&\quad + v_1 + v_2 + s(v_1 + v_2)({v_1}^3 + {v_2}^3) = 0,
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
f_2(\tilde{v}, \tilde{\lambda}, s) \;&=\; \tilde{\lambda}\left({v_2}^2 - {v_1}^2 + s(v_2 - v_1) + s^2(v_2 - v_1)({v_2}^3 + {v_1}^3)\right) \\
&\quad + v_2 - v_1 + s(v_2 - v_1)({v_2}^3 + {v_1}^3) = 0.
\end{aligned} \tag{5.17}$$

Solving (5.16) for $\tilde{\lambda}$ yields

$$\tilde{\lambda}(\tilde{v}, s) = -\frac{v_1 + v_2 + s(v_1 + v_2)({v_1}^3 + {v_2}^3)}{1 + s(v_1 + v_2) + s^2({v_1}^4 + v_1 v_2 + {v_2}^4)}. \tag{5.18}$$

Then we can define $g(v_1, v_2, s) = f_2(\tilde{v}, \tilde{\lambda}(\tilde{v}, s), s) =$

$$\begin{aligned}
&v_2 - v_1 + s(v_2 - v_1)({v_2}^3 + {v_1}^3) \\
&- \left(\frac{v_1 + v_2 + s(v_1 + v_2)({v_1}^3 + {v_2}^3)}{1 + s(v_1 + v_2) + s^2({v_1}^4 + v_1 v_2 + {v_2}^4)}\right)({v_2}^2 - {v_1}^2) \\
&- \left(\frac{v_1 + v_2 + s(v_1 + v_2)({v_1}^3 + {v_2}^3)}{1 + s(v_1 + v_2) + s^2({v_1}^4 + v_1 v_2 + {v_2}^4)}\right)(s(v_2 - v_1))) \\
&- \left(\frac{v_1 + v_2 + s(v_1 + v_2)({v_1}^3 + {v_2}^3)}{1 + s(v_1 + v_2) + s^2({v_1}^4 + v_1 v_2 + {v_2}^4)}\right)(s^2(v_2 - v_1)({v_2}^3 + {v_1}^3))\,.
\end{aligned} \tag{5.19}$$

Define $g(\theta, s)$ by substituting $v_1 = \cos\theta$ and $v_2 = \sin\theta$. After simplifying, we have

$$g(\theta, s) = -\frac{2s\sin\theta\cos^3\theta + \sin\theta - s\sin\theta\cos\theta + s - 2s\cos^2\theta - \cos\theta}{s\sin\theta + s^2\sin\theta\cos\theta + 2s^2\cos^4\theta + s^2 - 2s^2\cos^2\theta + s\cos\theta + 1} \tag{5.20}$$

The graph of $g(\theta, s)$ in Figure 5.2 is a bifurcation diagram, depicting the behavior of the nontrivial solutions of $\Phi = 0$ for s in a neighborhood of 0. Since s is a parameterization relating v and λ, this is equivalent to a neighborhood of the bifurcation point, $(0, \lambda_0)$. Note that in this neighborhood, $\theta = \frac{\pi}{4}$, and this is equivalent in rectangular coordinates to $v_1 = v_2$, as we saw in Section 5.1. We can generate another bifurcation diagram by defining a function $h(v_1, v_2, s)$ by translating $g(\theta, s)$ back into rectangular coordinates, so

$$\begin{aligned}
&h(v_1, v_2, s) = \\
&-\frac{({v_1}^2 + {v_2}^2 + {v_1}^3 + {v_2}^3)({v_2}^3 - {v_1}^3 - v_1{v_2}^2 + v_2{v_1}^2)}{s^2\left({v_1}^2 + {v_1}^3 + v_1{v_2}^2 + {v_1}^4 + v_1{v_2}^3 + {v_2}^2 + v_2{v_1}^2 + {v_2}^3 + v_2{v_1}^3 + {v_2}^4\right)}.
\end{aligned} \tag{5.21}$$

The $\frac{1}{s^2}$ factor is a result of the s^2 that was removed from Φ to define $\tilde{\Phi}$ in (5.9), so

$$\begin{aligned}
&h(v_1, v_2) = \\
&-\frac{({v_1}^2 + {v_2}^2 + {v_1}^3 + {v_2}^3)({v_2}^3 - {v_1}^3 - v_1{v_2}^2 + v_2{v_1}^2)}{\left({v_1}^2 + {v_1}^3 + v_1{v_2}^2 + {v_1}^4 + v_1{v_2}^3 + {v_2}^2 + v_2{v_1}^2 + {v_2}^3 + v_2{v_1}^3 + {v_2}^4\right)}.
\end{aligned} \tag{5.22}$$

This bifurcation diagram can be seen in Figure 5.3.

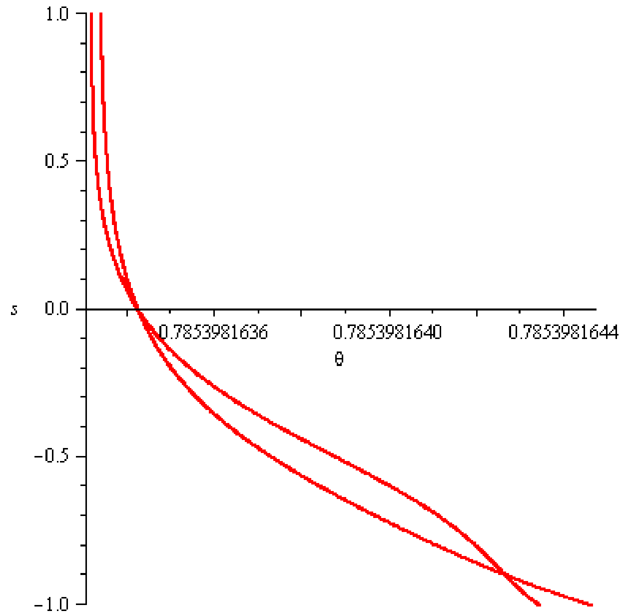

Figure 5.2: Bifurcation diagram $g(s, \theta)$

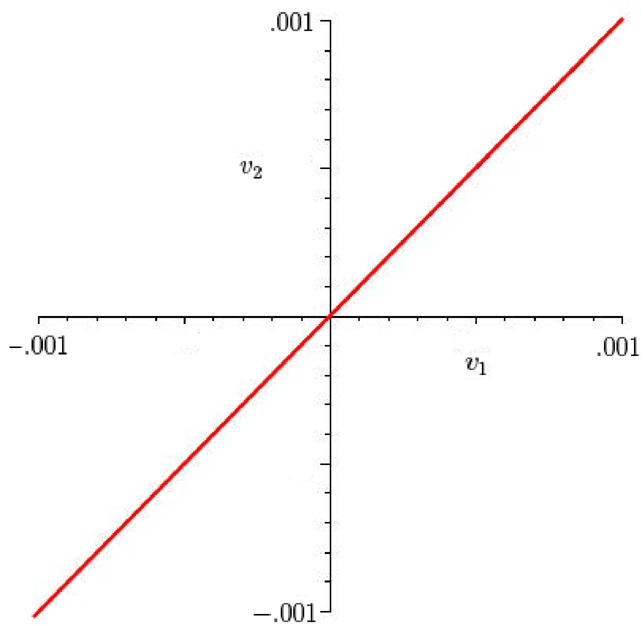

Figure 5.3: Bifurcation diagram $h(v_1, v_2)$

CHAPTER VI

A DIFFERENTIAL OPERATOR EXAMPLE

We now examine the application of Theorem 3.1.1 to an infinite-dimensional problem. We consider a nonlinear boundary-value problem

$$\begin{gathered} u'''' + \lambda u'' + 4u + u^3 = 0, \\ 0 \leq x \leq \pi, \end{gathered} \tag{6.1}$$

$$\begin{gathered} u(0) = u(\pi) = 0, \\ u''(0) = u''(\pi) = 0. \end{gathered} \tag{6.2}$$

Consider (6.1) as an operator

$$\begin{gathered} F\colon X \times \mathbb{R} \to Z \text{ by} \\ F(u, \lambda) = u'''' + \lambda u'' + 4u + u^3, \text{ where} \end{gathered} \tag{6.3}$$

$$\begin{gathered} X = \{u \in C^4([0, \pi], \mathbb{R}) : u \text{ satisfies } (6.2)\}, \\ Z = C([0, \pi], \mathbb{R}). \end{gathered} \tag{6.4}$$

Our bifurcation problem takes the form

$$F(u, \lambda) = 0. \tag{6.5}$$

We must check that hypotheses (3.1), (3.6), and (3.7) of Theorem 3.1.1 are satisfied. Part of checking (3.1) is verifying that $F(\cdot, \lambda)$ is Fréchet differentiable, and part of checking this is showing that $D_u F(0, \lambda)$ is a bounded operator. For differential operators, this entails addressing certain technicalities related to the choice of function spaces. These are not central to our main goal of applying Theorem 3.1.1, so while we do eventually work in $L^2([0,\pi])$, we do not show below that $D_uF(0,\lambda)$ is bounded. See [7] and [8].

We begin by analyzing the linearization of F. Then we perform the reduction method of Lyapunov-Schmidt and verify that the kernel of the linearization and the codimension of the range of the linearization both have dimension two. Finally, we calculate partial derivatives in order to generate the Taylor expansion of the bifurcation equation and verify (3.6) and (3.7). We conclude this chapter with some plots of the graph of the bifurcation equation.

6.1 The Linearization of F

We start by noting that $u_0 = 0$, the zero function, is a solution to (6.5) for any λ. Next we linearize about the trivial branch. To calculate $D_uF(0,\lambda)$, we set $u = u_0 + \epsilon\breve{u}$ in (6.5), and take the partial derivative with respect to ϵ. We get

$$\frac{\partial}{\partial\epsilon}\begin{cases} (u_0+\epsilon\breve{u})'''' + \lambda\,(u_0+\epsilon\breve{u})'' + 4\,(u_0+\epsilon\breve{u}) + (u_0+\epsilon\breve{u})^3 = 0, \\ u_0(0)+\epsilon\breve{u}(0) = 0, \\ u_0(\pi)+\epsilon\breve{u}(\pi) = 0, \\ (u_0+\epsilon\breve{u})''(0) = 0, \\ (u_0+\epsilon\breve{u})''(\pi) = 0, \end{cases} \tag{6.6}$$

which yields

$$\begin{gathered} \breve{u}'''' + \lambda\breve{u}'' + 4\breve{u} + 3\,(u_0+\epsilon\breve{u})^2\,\breve{u} = 0, \\ \breve{u}(0) = 0, \qquad \breve{u}(\pi) = 0, \\ \breve{u}''(0) = 0, \qquad \breve{u}''(\pi) = 0. \end{gathered} \tag{6.7}$$

Then setting $\epsilon = 0$ gives

$$\begin{gathered} \breve{u}'''' + \lambda\breve{u}'' + 4\breve{u} + 3{u_0}^2\breve{u} = 0, \\ \breve{u}(0) = 0, \qquad \breve{u}(\pi) = 0, \\ \breve{u}''(0) = 0, \qquad \breve{u}''(\pi) = 0. \end{gathered} \tag{6.8}$$

Hence, the linearization $D_uF(0,\lambda)\breve{u} = 0$ of (6.5) about the trivial branch corresponds to the linear boundary-value problem

$$\begin{gathered} \breve{u}'''' + \lambda\breve{u}'' + 4\breve{u} = 0, \\ \breve{u}(0) = \breve{u}(\pi) = 0, \\ \breve{u}''(0) = \breve{u}''(\pi) = 0. \end{gathered} \tag{6.9}$$

To find the candidates for bifurcation points of (6.5), we must find the values of λ for which the Implicit Function Theorem fails, or where $D_uF(0,\lambda)$ does not have a bounded inverse. It is sufficient to locate values where $N\left(D_uF(0,\lambda)\right)$ has dimension greater than 0, which correspond to points where (6.9) has non-trivial solutions.

6.2 Analysis of $D_uF(0,\lambda)$

To find the general solution to (6.9), observe that (6.9) yields the characteristic equation $m^4 + \lambda m^2 + 4 = 0$, which has roots

$$m = \pm\sqrt{\tfrac{-\lambda\pm\sqrt{\lambda^2-16}}{2}}. \tag{6.10}$$

We assume $\lambda > 4$, in which case the 4 roots are distinct. Then the general solution is

$$\breve{u} = c_1 e^{s\sqrt{\frac{-\lambda+\sqrt{\lambda^2-16}}{2}}} + c_2 e^{s\sqrt{\frac{-\lambda-\sqrt{\lambda^2-16}}{2}}} + c_3 e^{-s\sqrt{\frac{-\lambda-\sqrt{\lambda^2-16}}{2}}} + c_4 e^{-s\sqrt{\frac{-\lambda-\sqrt{\lambda^2-16}}{2}}}. \tag{6.11}$$

To rewrite (6.11) in a more useful form, we consider how the roots (6.10) depend on λ. Consider the functions f and g defined by

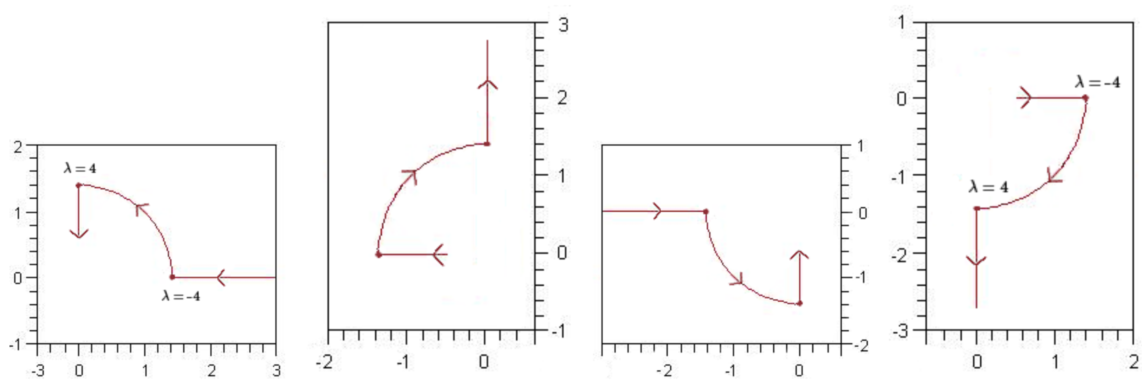

Figure 6.1: Left to right: $i \cdot m_1(\lambda)$, $i \cdot m_2(\lambda)$, $i \cdot m_3(\lambda)$, $i \cdot m_4(\lambda)$

$$f(\lambda) = \tfrac{-\lambda+\sqrt{\lambda^2-16}}{2} \text{ and } g(\lambda) = \tfrac{-\lambda-\sqrt{\lambda^2-16}}{2}.$$

Note that for $\lambda \geq 4$, it is alway the case that $-\lambda \pm \sqrt{\lambda^2 - 16} \leq 0$, and note also that $f(\lambda) \to 0$ and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. Noting that the function $h\colon \mathbb{C} \to \mathbb{C}$ defined by $h(z) = z^{1/2}$ takes a point with argument θ to a point with argument $\frac{\theta}{2}$, we define functions m_i, $i = 1, \ldots, 4$, by

$$\begin{aligned}
i \cdot m_1(\lambda) &= \sqrt{\tfrac{-\lambda+\sqrt{\lambda^2-16}}{2}} &&= i\sqrt{\tfrac{\lambda-\sqrt{\lambda^2-16}}{2}}, \\
i \cdot m_2(\lambda) &= -\sqrt{\tfrac{-\lambda-\sqrt{\lambda^2-16}}{2}} &&= i\sqrt{\tfrac{\lambda+\sqrt{\lambda^2-16}}{2}}, \\
i \cdot m_3(\lambda) &= -\sqrt{\tfrac{-\lambda+\sqrt{\lambda^2-16}}{2}} &&= -i\sqrt{\tfrac{\lambda-\sqrt{\lambda^2-16}}{2}} = -im_1(\lambda), \\
i \cdot m_4(\lambda) &= \sqrt{\tfrac{-\lambda-\sqrt{\lambda^2-16}}{2}} &&= -i\sqrt{\tfrac{\lambda+\sqrt{\lambda^2-16}}{2}} = -im_2(\lambda).
\end{aligned} \tag{6.12}$$

These functions are sketched parametrically in Figure 6.1. For $\lambda > 4$, im_1 and im_3 are conjugates, and im_2 and im_4 are conjugates. Hence, in the usual way, we can rewrite the general solution (6.11) for $\lambda > 4$ as

$$\breve{u} \;=\; c_1 \cos(sm_1(\lambda)) + c_2 \cos(sm_2(\lambda)) + c_3 \sin(sm_1(\lambda)) + c_4 \sin(sm_2(\lambda)). \tag{6.13}$$

To enforce the boundary conditions in (6.9), we note that

$$\begin{aligned}\breve{u}' &= -c_1 m_1(\lambda)\sin(sm_1(\lambda)) - c_2 m_2(\lambda)\sin(sm_2(\lambda)) \\ &\quad + c_3 m_1(\lambda)\cos(sm_1(\lambda)) + c_4 m_2(\lambda)\cos(sm_2(\lambda)) \\ \breve{u}'' &= -c_1(m_1(\lambda))^2\cos(sm_1(\lambda)) - c_2 m_2((\lambda))^2\cos(sm_2(\lambda)) \\ &\quad - c_3(m_1(\lambda))^2\sin(sm_1(\lambda)) - c_4(m_2(\lambda))^2\sin(sm_2(\lambda))\end{aligned} \tag{6.14}$$

Imposing the boundary conditions, we have

$$\begin{aligned}\breve{u}(0) = 0 &= c_1\cos(0) + c_2\cos(0) + c_3\sin(0) + c_4\sin(0) \\ &= c_1 + c_2, \\ \breve{u}(\pi) = 0 &= c_1\cos(\pi m_1(\lambda)) + c_2\cos(\pi m_2(\lambda)) \\ &\quad + c_3\sin(\pi m_1(\lambda)) + c_4\sin(\pi m_2(\lambda)), \\ \breve{u}''(0) = 0 &= -c_1(m_1(\lambda))^2\cos(0) - c_2 m_2((\lambda))^2\cos(0) \\ &\quad - c_3(m_1(\lambda))^2\sin(0) - c_4(m_2(\lambda))^2\sin(0) \\ &= -c_1(m_1(\lambda))^2 - c_2(m_2(\lambda))^2, \\ \breve{u}''(\pi) = 0 &= -c_1(m_1(\lambda))^2\cos(\pi m_1(\lambda)) - c_2(m_2(\lambda))^2\cos(\pi m_2(\lambda)) \\ &\quad - c_3(m_1(\lambda))^2\sin(\pi m_1(\lambda)) - c_4(m_2(\lambda))^2\sin(\pi m_2(\lambda)).\end{aligned} \tag{6.15}$$

If we define A by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ \cos(\pi m_1) & \cos(\pi m_2) & \sin(\pi m_1) & \sin(\pi m_2) \\ -{m_1}^2 & -{m_2}^2 & 0 & 0 \\ -{m_1}^2\cos(\pi m_1) & -{m_2}^2\cos(\pi m_2) & -{m_1}^2\sin(\pi m_1) & -{m_2}^2\sin(\pi m_2) \end{pmatrix}, \tag{6.16}$$

then (6.16) is equivalent to

$$A\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \mathbf{0}. \tag{6.17}$$

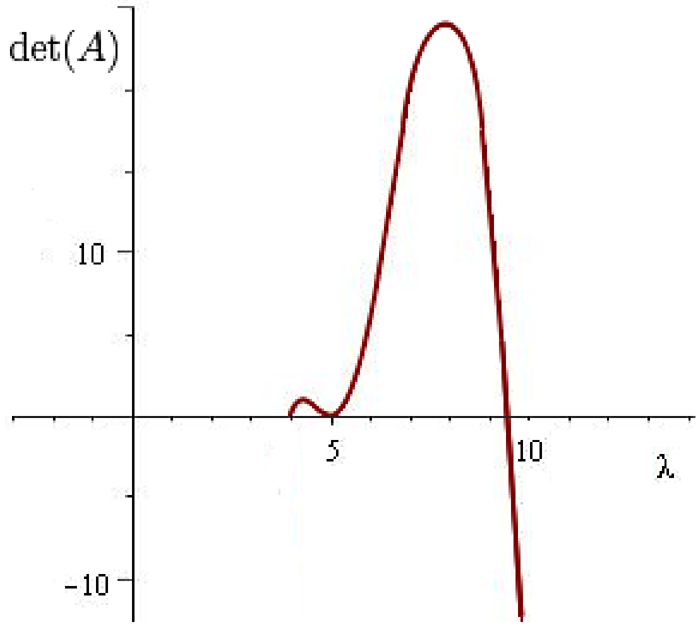

Figure 6.2: $\det(A)$ as a function of λ

Therefore the λ values for which the determinant of A is zero are the values for which (6.9) has non-trivial solutions. A straightforward computation shows that

$$\det(A) = (\lambda^2 - 16)\sin(\pi m_1(\lambda))\sin(\pi m_2(\lambda)). \tag{6.18}$$

Since we assume that $\lambda > 4$, $\det(A) = 0$ exactly when $m_1(\lambda) \in \mathbb{N}$ or $m_2(\lambda) \in \mathbb{N}$, which happens when $\lambda = n^2 + 4n^{-2}$ with $n \in \mathbb{N}$. Figure 6.2 shows a graph of $\det(A)$ as a function of λ.

The first value of λ greater than 4 for which $\det(A) = 0$ is $\lambda = 5$, and $m_1(5) = 1$, $m_2(5) = 2$. Also, for $\lambda = 5$,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 1 & -4 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.19}$$

Note that

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is a basis for the null space of A for $\lambda = 5$. These two vectors have entries that correspond to the four c_i values from the general solution (6.13), so using the general solution, we have $\{\sin(s), \sin(2s)\}$ as a basis for $N(D_uF(0,5))$. Now we seek to establish a bifurcation for (6.5) at $\lambda = 5$ by performing the Lyapunov-Schmidt reduction and then applying Theorem 3.1.1.

As preparation for the Lyapunov-Schmidt reduction, we study the range of $D_uF(0,5)$. We consider $D_uF(0,5)$ as a map from $L^2(0,\pi)$ to $L^2(0,\pi)$ [8] with

$$\text{dom} D_uF(0,5) = \left\{u \colon u^{'''} \text{ is a.c. on } [0,\pi], u^{''''} \in L^2(0\pi), u \text{ satisfies (6.9)}\right\}. \tag{6.20}$$

Also, we have by definition

$$\text{dom}\,[D_uF(0,5)]^* = \{g \in L^2 \mid u \mapsto \langle D_uF(0,5)u, g\rangle \text{ bounded on dom} D_uF(0,5)\},$$

where $[D_uF(0,5)]^*$ is the adjoint of $D_uF(0,5)$. To find the adjoint, we pick $u \in \text{dom}D_uF(0,5)$, and let $g \in L^2(0,\pi)$ such that $g^{'''}$ is a.c. and $g^{''''} \in L^2(0,\pi)$. We then compute

$$\langle D_u F(0,5)u, g\rangle = \left\langle u'''' + 5u'' + 4u, g\right\rangle = \int_0^\pi \left(u'''' + 5u'' + 4u\right) g \tag{6.21}$$

$$= \int_0^\pi u'''' g + 5\int_0^\pi u'' g + 4\int_0^\pi u\overline{g} \tag{6.22}$$

We evaluate the first two integrals in (6.22) using integration by parts. The first integral is

$$\begin{aligned}
\int_0^\pi u'''' g &= u''' g\Big|_0^\pi - \int_0^\pi u''' g' \\
&= u''' g\Big|_0^\pi - u'' g'\Big|_0^\pi + \int_0^\pi u'' g'' \\
&= u''' g\Big|_0^\pi - u'' g'\Big|_0^\pi + u' g''\Big|_0^\pi - \int_0^\pi u' g''' \\
&= u''' g\Big|_0^\pi - u'' g'\Big|_0^\pi + u' g''\Big|_0^\pi + u g'''\Big|_0^\pi - \int_0^\pi u g'''' \\
&= u''' g\Big|_0^\pi + u' g''\Big|_0^\pi + \int_0^\pi u g'''',
\end{aligned} \tag{6.23}$$

where in the last equality in (6.23) we use that u satisfies the boundary conditions (6.9). The second integral in (6.22) is

$$\begin{aligned}
5\int_0^\pi u'' g &= 5u' g\Big|_0^\pi - 5\int_0^\pi u' g' \\
&= 5u' g\Big|_0^\pi - u g'\Big|_0^\pi + 5\int_0^\pi u g'' \\
&= 5u' g\Big|_0^\pi + 5\int_0^\pi u g'',
\end{aligned} \tag{6.24}$$

where again, we have used (6.9). Hence, if we require g to satisfy the same boundary conditions (6.9) as u, then

$$\begin{aligned} \langle D_u F(05)u, g\rangle &= \int_0^\pi ug'''' + 5\int_0^\pi ug'' + 4\int_0^\pi ug \\ &= \int_0^\pi u\left(g'''' + 5g'' + 4g\right) \\ &= \left\langle u, g'''' + 5g'' + 4g\right\rangle . \end{aligned} \tag{6.25}$$

If follows that $D_uF(0,5)$ is self-adjoint. Moreover, since $[R\,(D_uF(0,5))]^\perp = N([D_uF(0,5)]^*)$ [6] and $\{\sin(s), \sin(2s)\}$ is a basis for $N(D_uF(0,5)) = N([D_uF(0,5)]^*)$, we know $\{\sin(s), \sin(2s)\}$ is a basis for $[R\,(D_uF(0,5))]^\perp$.

6.3 Lyapunov-Schmidt Reduction

Now we apply the method of Lyapunov-Schmidt to find the bifurcation equation for the nonlinear problem $F(u,\lambda) = 0$ defined in (6.2) to (6.5). The first step is to define the appropriate projections. We let $N = N(D_uF(0,5))$ and $Z_0 = [R\,(D_uF(0,5))]^\perp$. Noting that $\int_0^\pi \sin(s)\sin(2s)ds = 0$, we normalize the elements of $\{\sin(s), \sin(2s)\}$ to get $\left\{\sqrt{\frac{2}{\pi}}\sin(s), \sqrt{\frac{2}{\pi}}\sin(2s)\right\}$, which, by the result in the previous section, is an orthonormal basis for both N and Z_0. We set $\phi_1 = \sqrt{\frac{2}{\pi}}\sin(s)$ and $\phi_2 = \sqrt{\frac{2}{\pi}}\sin(2s)$. Next we define

$$\begin{aligned} L_1(f) &= \langle f, \phi_1\rangle = \int_0^\pi f\phi_1 ds, \\ L_2(f) &= \langle f, \phi_2\rangle = \int_0^\pi f\phi_2 ds. \end{aligned} \tag{6.26}$$

We then define the projections $P\colon X \to N$ and $Q\colon Z \to Z_0$ by

$$Pf = Qf = L_1(f)\phi_1 + L_2(f)\phi_2. \tag{6.27}$$

Now, just as in Section 2.3, the equation $F(u, \lambda) = 0$ can be rewritten as the pair of equations

$$\begin{aligned} QF(Pu + (I-P)u, \lambda) &= 0, \\ (I-Q)F(Pu + (I-P)u, \lambda) &= 0. \end{aligned} \tag{6.28}$$

Then we define

$$G(x, w, \lambda) = (I-Q)F(x + w, \lambda) = 0, \text{ where } x \in N, w \in N^{\perp}. \tag{6.29}$$

We apply the Implicit Function theorem to (6.29), which yields a function $\psi\colon W_0 \times V_0 \to N^{\perp}$ such that W_0 is a neighborhood of 0 in N and V_0 is a neighborhood of 5 in $\mathbb{R}$, and $G(x+\psi(x, \lambda), \lambda) = 0$ on $W_0 \times V_0$. Hence, we have reduced solving $F(u, \lambda) = 0$ to the finite-dimensional problem

$$QF(x + \psi(x, \lambda), \lambda) = 0, \tag{6.30}$$

where

$$W_0 \times V_0 \ni (x, \lambda) \mapsto QF(x + \psi(x, \lambda), \lambda) \in Z_0 \tag{6.31}$$

and $W_0 \times V_0 \subset N \times \mathbb{R}$ with N and Z_0 both 2 dimensional. Now we introduce coordinates. For any $x \in N$, we can write $x = x_1\phi_1 + x_2\phi_2$ and define

$$\widetilde{F}(x_1, x_2, \lambda) = F(x_1\phi_1 + x_2\phi_2 + \psi(x_1\phi_1 + x_2\phi_2, \lambda), \lambda), \tag{6.32}$$

$$\begin{aligned} h_1(x_1, x_2, \lambda) &= L_1(Q\widetilde{F}(x_1, x_2, \lambda)), \\ h_2(x_1, x_2, \lambda) &= L_2(Q\widetilde{F}(x_1, x_2, \lambda)), \end{aligned} \tag{6.33}$$

$$H(x_1, x_2, \lambda) = \begin{pmatrix} h_1(x_1, x_2, \lambda) \\ h_2(x_1, x_2, \lambda) \end{pmatrix}. \tag{6.34}$$

Then $H\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, and (6.30) is equivalent to $H(x_1, x_2, \lambda) = \mathbf{0}$. We note that for any $\lambda \in V_0$, we have

$$\begin{aligned} h_1(0,0,\lambda) &= L_1(Q\widetilde{F}(0,0,\lambda)) \\ &= L_1(QF(\psi(0,\lambda),\lambda)) = 0 \\ h_2(0,0,\lambda) &= L_2(Q\widetilde{F}(0,0,\lambda)) \\ &= L_2(QF(\psi(0,\lambda),\lambda)) = 0 \end{aligned} \tag{6.35}$$

6.4 Verifying the Hypothesis of Theorem 3.1.1

In this section, we verify the hypotheses of Theorem 3.1.1 for $F(u, \lambda) = 0$ at $\lambda = 5$. Note that here H plays the role of Φ in the statement of Theorem 3.1.1, so the hypotheses we need to check are on H_{11} and H_{0k}. Checking the hypotheses entails generating the Taylor expansion of H around the bifurcation point $(0, 0, 5)$. In the course of taking partial derivatives of H, we must compute various derivatives of F with respect to u. We showed above in (6.6) to (6.8) that $D_uF(u_0, 5)v_1 = v_1^{''''} + 5v_1^{''} + 4v_1 + 3{u_0}^2 v_1$. To compute higher order derivatives, we let $u_0 = u_{00} + \epsilon v_2$. Then

$$\begin{aligned} \tfrac{\partial}{\partial\epsilon} D_uF(u_{00} + \epsilon v_2)v_1 &= \tfrac{\partial}{\partial\epsilon}\left(v_1^{''''} + 5v_1^{''} + 4v_1 + 3(u_{00} + \epsilon v_2)^2 v_1\right), \text{ so} \\ D_{uu}F(u_{00}, 5)[v_2, v_1] &= 6u_{00}v_1v_2. \end{aligned} \tag{6.36}$$

Then since $D_{uu}F(u_{00}, 5)[v_1, v_2] = 6u_{00}v_1v_2$, we let $u_{00} = u_0 + \epsilon v_3$ and calculate

$$\begin{aligned} D_{uuu}F(u_0, 5)[v_3, v_2, v_1] &= \tfrac{\partial}{\partial\epsilon} D_{uu}F(u_0, 5)[v_1, v_2] \\ &= \tfrac{\partial}{\partial\epsilon}\left[6(u_0 + \epsilon v_3)v_1v_2\right] \\ &= 6v_1v_2v_3. \end{aligned} \tag{6.37}$$

Then we have

$$\begin{aligned} D_uF(0, 5)v_1 &= v_1^{''''} + 5v_1^{''} + 4v_1, \\ D_{uu}F(0, 5)[v_2, v_1] &= 0, \\ D_{uuu}F(0, 5)[v_3, v_2, v_1] &= 6v_1v_2v_3. \end{aligned} \tag{6.38}$$

The computation of the partial derivatives of h_1 and h_2 at $(0,0,5)$ are similar. These computations are carried out in Appendix B. The partial derivatives of h_1 and h_2 also require derivatives of the implicit function $\psi(x,\lambda)$, and these can be seen in Appendix A. From these appendices, we have the following results:

$$\frac{\partial h_i}{\partial x_j}(0,0,5) = \frac{\partial^2 h_i}{\partial {x_j}^2}(0,0,5) \quad = \quad 0 \quad \text{for } 1 \le i,j \le 2, \tag{6.39}$$

$$\frac{\partial^3 h_n}{\partial x_i \partial x_j \partial x_k}(0,0,5) \quad = \quad 6 L_n Q \phi_i \phi_j \phi_k \quad \text{for } 1 \le n,i,j,k \le 2, \tag{6.40}$$

$$\frac{\partial^2 h_1}{\partial \lambda \partial x_1}(0,0,5) \quad = \quad L_1 Q {\phi_1}'' = -1, \tag{6.41}$$

$$\frac{\partial^2 h_2}{\partial \lambda \partial x_2}(0,0,5) \quad = \quad L_2 Q {\phi_2}'' = -4, \tag{6.42}$$

$$\frac{\partial^2 h_1}{\partial \lambda \partial x_2}(0,0,5) = \frac{\partial^2 h_2}{\partial \lambda \partial x_1}(0,0,5) \quad = \quad 0. \tag{6.43}$$

From (6.39) it follows that both H_{01} and H_{02} are the zero operator, where we are using the notation introduced in (3.2). Also, letting $x = (x_1, x_2)$, we have that

$$H_{03}(x) = \frac{1}{3!} D_x^3 H(0,0,5)[x,x,x] \quad = \quad \begin{pmatrix} D_x^3 h_1(0,0,5)[x,x,x] \\ D_x^3 h_2(0,0,5)[x,x,x] \end{pmatrix}, \tag{6.44}$$

and we have that

$$
\begin{aligned}
&D^3_x h_1(0,0,5)[x,x,x] \\
&\quad = \begin{bmatrix} (x_1,x_2)\cdot\begin{pmatrix} \partial_{x_1x_1x_1}h_1(0,0,5) & \partial_{x_1x_1x_2}h_1(0,0,5) \\ \partial_{x_1x_2x_1}h_1(0,0,5) & \partial_{x_1x_2x_2}h_1(0,0,5) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (x_1,x_2)\cdot\begin{pmatrix} \partial_{x_2x_1x_1}h_1(0,0,5) & \partial_{x_2x_1x_2}h_1(0,0,5) \\ \partial_{x_2x_2x_1}h_1(0,0,5) & \partial_{x_2x_2x_2}h_1(0,0,5) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{bmatrix}\cdot(x_1,x_2) \\
&\quad = \begin{bmatrix} (x_1,x_2)\cdot\begin{pmatrix} \frac{9}{\pi} & 0 \\ 0 & \frac{6}{\pi} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (x_1,x_2)\cdot\begin{pmatrix} 0 & \frac{6}{\pi} \\ \frac{6}{\pi} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{bmatrix}\cdot(x_1,x_2) \\
&\quad = \tfrac{9}{\pi}{x_1}^3 + \tfrac{18}{\pi}x_1{x_2}^2 \\
&D^3_x h_2(0,0,5)[x,x,x] \\
&\quad = \begin{bmatrix} (x_1,x_2)\cdot\begin{pmatrix} \partial_{x_1x_1x_1}h_2(0,0,5) & \partial_{x_1x_1x_2}h_2(0,0,5) \\ \partial_{x_1x_2x_1}h_2(0,0,5) & \partial_{x_1x_2x_2}h_2(0,0,5) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (x_1,x_2)\cdot\begin{pmatrix} \partial_{x_2x_1x_1}h_2(0,0,5) & \partial_{x_2x_1x_2}h_2(0,0,5) \\ \partial_{x_2x_2x_1}h_2(0,0,5) & \partial_{x_2x_2x_2}h_2(0,0,5) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{bmatrix}\cdot(x_1,x_2) \\
&\quad = \begin{bmatrix} (x_1,x_2)\cdot\begin{pmatrix} 0 & \frac{6}{\pi} \\ \frac{6}{\pi} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (x_1,x_2)\cdot\begin{pmatrix} \frac{6}{\pi} & 0 \\ 0 & \frac{9}{\pi} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{bmatrix}\cdot(x_1,x_2) \\
&\quad = \tfrac{18}{\pi}{x_1}^2x_2 + \tfrac{9}{\pi}{x_2}^3.
\end{aligned} \tag{6.45}
$$

$$
H_{03}(x) = \frac{1}{3!}\begin{pmatrix} \frac{9}{\pi}{x_1}^3 + \frac{18}{\pi}x_1{x_2}^2 \\ \frac{18}{\pi}{x_1}^2x_2 + \frac{9}{\pi}{x_2}^3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2\pi}{x_1}^3 + \frac{3}{\pi}x_1{x_2}^2 \\ \frac{3}{\pi}{x_1}^2x_2 + \frac{3}{2\pi}{x_2}^3 \end{pmatrix}. \tag{6.46}
$$

Note that in (6.46) and in several places below we use $\partial_{x_i} h$ for $\frac{\partial h}{\partial x_i}$, etc., to save space.

Next we compute

$$\begin{aligned} H_{11}x = D_{\lambda x}H(0,0,5)x &= \begin{pmatrix} D_{\lambda x}h_1(0,0,5) \\ D_{\lambda x}h_2(0,0,5) \end{pmatrix} x \\ &= \begin{pmatrix} (\partial \lambda x_1 h_1(0,0,5), \partial_{\lambda x_2} h_1(0,0,5)) \\ (\partial \lambda x_1 h_2(0,0,5), \partial_{\lambda x_2} h_2(0,0,5)) \end{pmatrix} x \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} x \\ &= \begin{pmatrix} -x_1 \\ -4x_2 \end{pmatrix}. \end{aligned} \tag{6.47}$$

Finally, using (6.46) and (6.47), we have the Taylor expansion of H around the bifurcation point $(0,0,5)$:

$$\begin{aligned} H(x_1,x_2,\lambda) = H(x,\lambda) &= (\lambda-5)H_{11}x + H_{03}(x) + R(x,\lambda) \\ &= (\lambda-5)\begin{pmatrix} -x_1 \\ -4x_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{2\pi}{x_1}^3 + \frac{3}{\pi}x_1{x_2}^2 \\ \frac{3}{\pi}{x_1}^2x_2 + \frac{3}{2\pi}{x_2}^3 \end{pmatrix} + R(x,\lambda). \end{aligned} \tag{6.48}$$

H_{11} is clearly an isomorphism from $\mathbb{R}^2 \to \mathbb{R}^2$, so (3.6) is satisfied. To satisfy hypothesis (3.7) of Theorem 3.1.1, it remains to show that there exist $\tilde{c}, \tilde{d} \in \mathbb{R}^2$ with $\tilde{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $\tilde{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, and $\|\tilde{c}\| = \|\tilde{d}\| = 1$ such that

$$\begin{aligned} \begin{pmatrix} \frac{3}{2\pi}{c_1}^3 + \frac{3}{\pi}c_1{c_2}^2 \\ \frac{3}{\pi}{c_1}^2c_2 + \frac{3}{2\pi}{c_2}^3 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right] &< 0, \\ \begin{pmatrix} \frac{3}{2\pi}{d_1}^3 + \frac{3}{\pi}d_1{d_2}^2 \\ \frac{3}{\pi}{d_1}^2d_2 + \frac{3}{2\pi}{d_2}^3 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right] &> 0. \end{aligned} \tag{6.49}$$

This is equivalent to

$$\begin{aligned} \tfrac{3}{2\pi}c_1c_2\left(2{c_1}^2 + 7{c_2}^2\right) &< 0, \\ \tfrac{3}{2\pi}d_1d_2\left(2{d_1}^2 + 7{d_2}^2\right) &> 0. \end{aligned} \tag{6.50}$$

Since $2{c_1}^2 + 7{c_2}^2 > 0$, $\forall c_1 \in \mathbb{R}$, it is clear that this condition is met for any $\tilde{c}$ in the second or fourth quadrant and any $\tilde{d}$ in the first or third quadrant. We then have

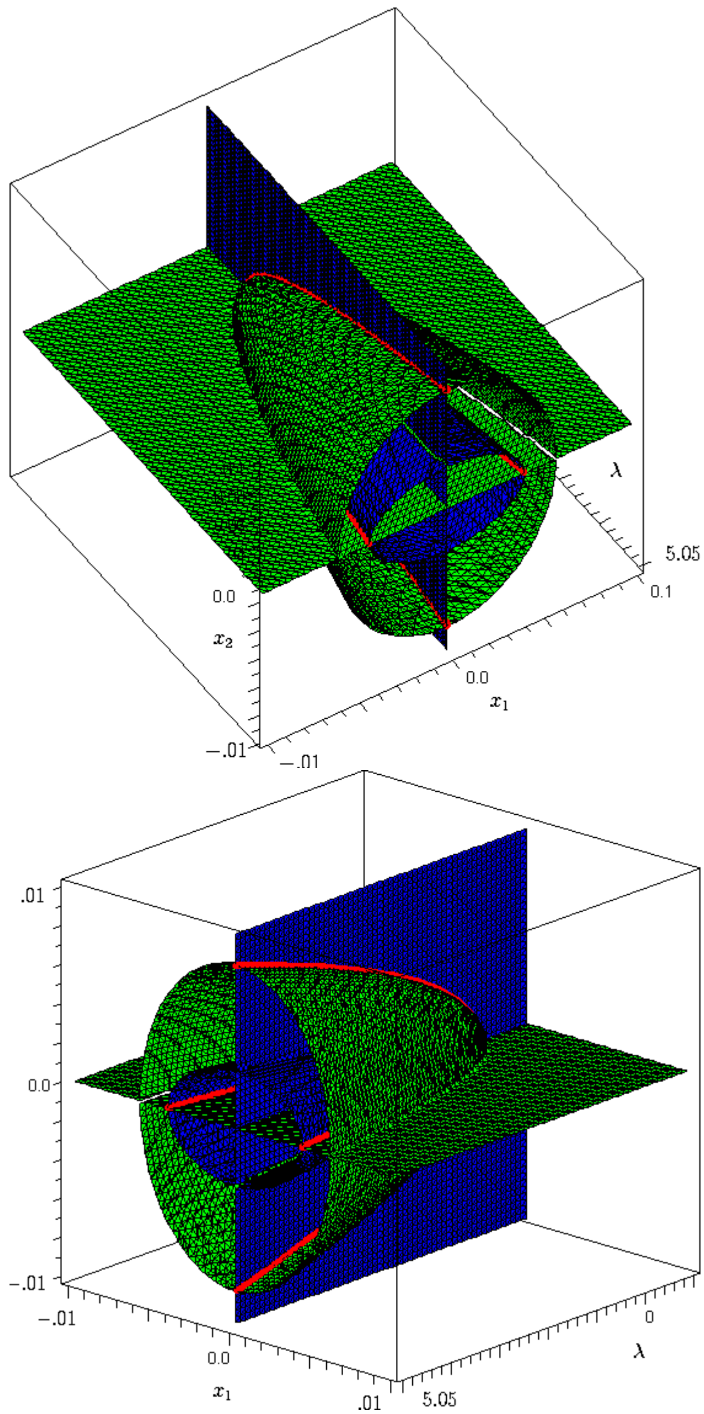

Figure 6.3: These graphs are two different views of the intersection of the zero sets of ϕ_1 and ϕ_2 approximate the solution set of $H(x_1, x_2, \lambda)$ in a neighborhood of $(0, 0, 5)$. The non-trivial solutions are indicated by the red lines.

from Theorem 3.1.1 that there is a local continuum of nontrivial solutions of $F(x, \lambda)$ through $(0, 5)$.

Because we have computed the Taylor expansion for the bifurcation equation in this case, we can, as in Section 5.1, sketch approximately the graph of the nontrivial branches. Recall $H(x_1, x_2, \lambda)$ from (6.48). We define

$$\begin{aligned} \phi_1(x_1, x_2, \lambda) &= (5-\lambda)x_1 + \tfrac{3}{2\pi}{x_1}^3 + \tfrac{3}{\pi}x_1{x_2}^2, \\ \phi_2(x_1, x_2, \lambda) &= (5-\lambda)4x_2 + \tfrac{3}{\pi}{x_1}^2x_2 + \tfrac{3}{2\pi}{x_2}^3. \end{aligned} \tag{6.51}$$

Then plotting the solution sets to ϕ_1 and ϕ_2 yields two surfaces, and as in Section 5.1, the intersection of the two surfaces approximates the solution set to $H(x_1, x_2, \lambda) = 0$ for (x_1, x_2, λ) near $(0, 0, 5)$. This can be seen in Figure 6.3. The non-trivial solutions are indicated by the red lines.

CHAPTER VII

THE CRANDALL-RABINOWITZ THEOREM

7.1 The Crandall-Rabinowitz Theorem

The Crandall-Rabinowitz Theorem is a well-known result that provides a sufficient condition for bifurcation in problems for which the kernel of the linearization is one-dimensional. In this section we note a variation on the Crandall-Rabinowitz Theorem relevant for problems in which the kernel has dimension greater than one.

First we state the standard Crandall-Rabinowitz Theorem. We assume $F\colon X \times \mathbb{R} \to Z$, where X and Z are Banach Spaces, and we also assume that F satisfies

$$\begin{gathered} F(0,\lambda) = 0 \text{ for all } \lambda \in \mathbb{R}, \\ \text{and } \exists \lambda_0 \in \mathbb{R} \text{ such that} \\ \dim N(D_x F(0,\lambda_0)) = \operatorname{codim} R(D_x F(0,\lambda_0)) = 1. \end{gathered} \tag{7.1}$$

Theorem 7.1.1 (Crandall-Rabinowitz Theorem)**.** *Let F satisfy the conditions in (7.1) and suppose*

$$\begin{gathered} F \in C^2(U \times V, Z), \text{ where } F \text{is a Fredholm operator on } U, \\ 0 \in U \subset X,\ U \text{ open in } X, \text{ and } \lambda_0 \in V \subset \mathbb{R},\ V \text{ open in } \mathbb{R}, \\ N(D_x F(0,\lambda_0)) = span[\hat{v}_0],\ \hat{v}_0 \in X,\ \|\hat{v}_0\| = 1, \\ D^2_{x\lambda} F(0,\lambda_0)\hat{v}_0 \notin R(D_x F(0,\lambda_0)). \end{gathered} \tag{7.2}$$

Then there is a nontrivial continuously differentiable curve through $(0,\lambda_0)$*,*

$$\left\{ \big(x(s),\lambda(s)\big) \,\middle|\, s \in (-\delta,\delta), \big(x(0),\lambda(0)\big) = (0,\lambda_0) \right\},$$

such that

$$F(x(s), \lambda(s)) = 0 \text{ for } s \in (-\delta, \delta),$$

and all solution to $F(x,\lambda)$ in a neighborhood of $(0, \lambda_0)$ are on the trivial solution line or on this nontrivial curve [2].

Shortly we present a result whose proof is based on the basic idea in the proof of the Crandall-Rabinowitz Theorem. We illustrate this idea by considering the standard example of the pitchfork bifurcation. Hence, we define $G\colon \mathbb{R}^2 \to \mathbb{R}$ by $G(x, \lambda) = x^3 + \lambda x$. We emphasize that the point of this discussion is to illustrate the basic idea in the proof of the Crandall-Rabinowitz Theorem, not to analyze the equation $G(x, \lambda)$. We know from elementary algebra that $G(x, \lambda) = 0$ has a pitchfork bifurcation at $\lambda_0 = 0$, as seen in Figure 7.1. To illustrate the idea of the proof, we note that

$$\dim N(D_x G(0,0)) = \operatorname{codim} R(D_x G(0,0)) = 1, \tag{7.3}$$

and hence, $\tilde{v}_0 = 1$ is a basis for $N(D_x G(0,0))$. Also,

$$D^2_{x\lambda} G(0,0)\hat{v}_0 = \hat{v}_0 \notin R(D_x G(0,0)) = 0,$$

so the hypotheses of the Crandall-Rabinowitz Theorem are satisfied.

Having noted that $G(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, we have the line $x = 0$ as the trivial branch of solutions. Now we write

$$G(x,\lambda) = \int_0^1 \frac{d}{dt} G(tx,\lambda)dt = \int_0^1 D_x G(tx,\lambda)x dt = x \int_0^1 D_x G(tx,\lambda)dt. \tag{7.4}$$

We define

$$\begin{aligned} \widetilde{G}(x,\lambda) &= \int_0^1 D_x G(tx,\lambda)dt \\ &= (x^2 + \lambda)\,. \end{aligned} \tag{7.5}$$

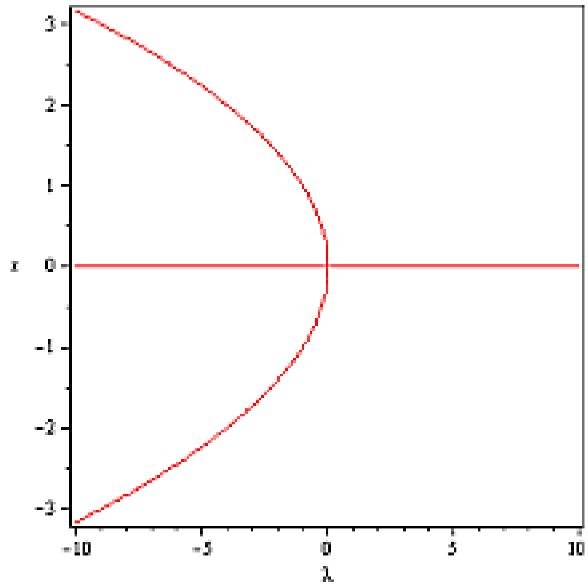

Figure 7.1: $G(x, \lambda) = 0$

After "factoring out" x, the idea now is to apply the Implicit Function Theorem to the equation $\tilde{G}(x, \lambda) = 0$ in order to describe λ in terms of x. Because $G(x, \lambda) = x\tilde{G}(x, \lambda)$, a non-trivial branch of solutions to $\tilde{G}(x, \lambda) = 0$ corresponds to a non-trivial branch of solutions to the original equation. We have that

$$\begin{aligned} D_\lambda \tilde{G}(0,0) &= \int_0^1 D^2_{x\lambda} G(0,0)dt \\ &= D^2_{x\lambda} G(0,0) \neq 0. \end{aligned} \tag{7.6}$$

Hence, the Implicit Function Theorem implies that $\lambda = \hat{\lambda}(x) = -x^2$, and this describes the non-trivial solution curve. We see that, loosely speaking, the basic hypothesis $D_{x\lambda}F(0, \lambda_0)v \notin R(D_xF(0, \lambda_0)$ in (7.2) is used in the proof to show that the Implicit Function Theorem applies to the equation after factoring out the trivial branch.

7.2 Crandall-Rabinowitz in 2 Dimensions

We show in this section that the key assumption in the Crandall-Rabinowitz Theorem, that $D^2_{x\lambda}F(0,\lambda_0)\hat{v}_0 \notin R(D_xF(0,\lambda_0))$, is sufficient to guarantee the existence of non-trivial solutions near $(0,\lambda_0)$ even when $N(D_x(0,\lambda_0))$ has dimension greater than one.

Theorem 7.2.1. *Let X and Z be Banach spaces, and let $F\colon X\times\mathbb{R}\to Z$ such that $F\in C^2(U\times V,Z)$ where $0\in U\subset X$, U is open, and V is open in $\mathbb{R}$. Suppose $F(0,\lambda)=0$, $\forall\lambda\in\mathbb{R}$. Also, suppose $\lambda_0\in V$ such that $F(0,\lambda_0)$ is a Fredholm operator on U and $dimN(D_xF(0,\lambda_0))=2$, and $codimR(D_xF(0,\lambda_0))=1$. Let $\{\hat{v}_1,\hat{v}_2\}$ be a basis for N, and suppose $D^2_{x\lambda}F(0,\lambda_0)\hat{v}_1\notin R(D_xF(0,\lambda_0)$. Under these conditions, there exists a nontrivial continuously differentiable curve through $(0,\lambda_0)$.*

Proof. Applying the Lyapunov-Schmidt reduction gives that solving $F(x,\lambda)=0$ near $(0,\lambda_0)$ is equivalent to solving

$$\begin{gathered}\Phi(v,\lambda)=0,\text{ where}\\ (0,\lambda_0)\in\tilde{U}_1\times V_1\subset N\times\mathbb{R}\text{ and }\Phi\colon\tilde{U}_1\times V_1\to Z_0\text{ with }\dim Z_0=1,\text{ and}\\ \Phi\in C^2(\tilde{U}_1\times V_1,Z_0)\end{gathered}\tag{7.7}$$

We know that $\Phi(0,\lambda)=0$ for all $\lambda\in V_2$. Since N has 2 dimensions, by introducing coordinates, we can write $v\in N$ as $v=(v_1,v_2)$ and write $\Phi(v,\lambda)$ as $\Phi(v_1,v_2,\lambda)$. We have

$$\tfrac{d}{dt}\Phi(t\hat{v}_1,\lambda) = D_v\Phi(t\hat{v}_1,\lambda)\hat{v}_1,\tag{7.8}$$

and hence

$$\begin{aligned}\Phi(\hat{v}_1,\lambda) &= \int_0^1\frac{d}{dt}\Phi(t\hat{v}_1,\lambda)dt\\ &= \int_0^1 D_v\Phi(t\hat{v}_1,\lambda)\hat{v}_1dt.\end{aligned}\tag{7.9}$$

Next we let $s \in (-\delta, \delta)$ and observe that

$$\begin{aligned}\Phi(s\hat{v}_1, \lambda) &= \int_0^1 D_v\Phi(st\hat{v}_1, \lambda)s\hat{v}_1 dt \\ &= s\int_0^1 D_v\Phi(st\hat{v}_1, \lambda)\hat{v}_1 dt.\end{aligned} \tag{7.10}$$

Then we define

$$\widetilde{\Phi}(s, \lambda) = \int_0^1 D_v\Phi(st\hat{v}_1, \lambda)\hat{v}_1 dt. \tag{7.11}$$

Having factored out an s, we now seek to use the Implicit Function Theorem. We calculate $D_\lambda\widetilde{\Phi}(0, \lambda_0)$. Since

$$\begin{aligned}D_\lambda\widetilde{\Phi}(0, \lambda) &= D_\lambda \int_0^1 D_v\Phi(0, \lambda)\hat{v}_1 dt \\ &= D_\lambda \left(D_v\Phi(0, \lambda)\hat{v}_1\right),\end{aligned} \tag{7.12}$$

we must calculate $D_\lambda(D_v\Phi(v, \lambda)\hat{v}_1)$, which, recalling the definition of Φ in the paragraph containing (2.6), entails computing

$$\begin{aligned}&D_\lambda\left\{QD_xF(v + \psi(v, \lambda), \lambda)(\hat{v}_1 + D_v\psi(v, \lambda)\hat{v}_1)\right\} \\ &\quad = D_\lambda[QD_xF(v + \psi(v, \lambda), \lambda)](\hat{v}_1 + D_v\psi(v, \lambda)\hat{v}_1) \\ &\qquad +QD_xF(v + \psi(v + \lambda), \lambda)D_\lambda[(\hat{v}_1 + D_v\psi(v, \lambda)\hat{v}_1)] \\ &\quad = QD^2_{xx}F(v + \psi(v, \lambda), \lambda)[D_\lambda\psi(v, \lambda), \hat{v}_1 + D_v\psi(v, \lambda)\hat{v}_1] \\ &\qquad +[QD^2_{x\lambda}F(v + \psi(v, \lambda), \lambda)](\hat{v}_1 + D_v\psi(v, \lambda)\hat{v}_1) \\ &\qquad +QD_xF(v + \psi(v, \lambda), \lambda)D^2_{\lambda v_1}\psi(v, \lambda)\hat{v}_1.\end{aligned} \tag{7.13}$$

Then substituting $(v, \lambda) = (0, \lambda_0)$, we have

$$\begin{aligned}D_\lambda\widetilde{\Phi}(0, \lambda_0) &= QD^2_{xx}F(0 + \psi(0, \lambda_0), \lambda_0)[D_\lambda\psi(0, \lambda_0), \hat{v}_1 + D_v\psi(0, \lambda_0)\hat{v}_1] \\ &\quad +QD^2_{x\lambda}F(0 + \psi(0, \lambda_0), \lambda_0)(\hat{v}_1 + D_v\psi(0, \lambda_0)\hat{v}_1) \\ &\quad +QD_xF(0 + \psi(0, \lambda_0), \lambda_0)D^2_{\lambda v}\psi(0, \lambda_0)\hat{v}_1 \\ &= QD^2_{xx}F(0, \lambda_0)[\hat{v}_1, 0] + QD^2_{x\lambda}F(0, \lambda_0)\hat{v}_1 \\ &\quad +QD_xF(0, \lambda_0)D^2_{\lambda v}\psi(0, \lambda_0)\hat{v}_1 \\ &= QD^2_{x\lambda}F(0, \lambda_0)\hat{v}_1.\end{aligned} \tag{7.14}$$

Recall that Q projects onto Z_0, which is complementary to $R(D_xF(0, \lambda_0))$, and recall that by hypothesis $D_{x\lambda}F(0, \lambda_0)\hat{v}_1 \notin R(D_xF(0, \lambda_0))$, so the previous calculation shows that $D_\lambda\widetilde{\Phi}(0, \lambda_0) = QD^2_{x\lambda}F(0, \lambda_0)\hat{v}_1 \neq 0$.

Then by the Implicit Function Theorem, there is $\delta > 0$ and a continuously differentiable function $\phi\colon (-\delta, \delta) \to V_2 \subset V_1$, with V_2 open and $\lambda_0 \in V_2$, such that $\phi(0) = \lambda_0$ and such that $\widetilde{\Phi}(s, \lambda) = 0$ for $(s, \lambda) \in (-\delta, \delta) \times V_2$ if and only if $\lambda = \phi(s)$. Hence, $\widetilde{\Phi}(s, \phi(s)) = 0$ for all $s \in (-\delta, \delta)$. Then we have

$$\Phi(s\hat{v}_1, \phi(s)) = s\widetilde{\Phi}(s, \phi(s)) = 0 \text{ for } s \in (-\delta, \delta), \tag{7.15}$$

and hence, we have a non-trivial branch of solutions through $(0, \lambda_0)$. □

Note that in the proof, Φ is a map from $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, and hence, we expect in general that the non-trivial branch is contained in a larger set of solutions forming a surface. It is for this reason that Theorem 7.2.1 does not assert that all non-trivial solutions in a neighborhood of the bifurcation point are on the non-trivial branch.

We illustrate this in an example by defining $F\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ by $F(x_1, x_2, \lambda) = {x_1}^3 + \lambda x_1 + {x_2}^2$. Then the verify the hypotheses of Theorem 7.2.1, we calculate

$$\begin{aligned} D_x F(x_1, x_2, \lambda) &= \begin{pmatrix} 3{x_1}^2 + \lambda \\ 2x_2 \end{pmatrix}, \\ D_{x\lambda} F(x_1, x_2, \lambda) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{7.16}$$

Evaluating these at $(0, 0, 0)$, we have

$$D_x F(0, 0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_{x\lambda} F(0, 0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.17}$$

and it is clearly the case that $dim N(D_x F(0, 0, \lambda_0)) = 2$, and $codim R(D_x F(0, 0, \lambda_0)) = 1$. Moreover, if we have $\{1, 1\}$ as a basis for N, then $D^2_{x\lambda} F(0, 0, \lambda_0)1 = 1$. Note that $1 \notin R(D_x F(0, 0, \lambda_0)$. Thus, the results of Theorem 7.2.1 hold.

We illustrate the results by following the procedure outlined in the proof. Note that

$$\begin{aligned} F(x_1,\lambda) &= \int_0^1 \frac{d}{dt} F(tx_1,\lambda)dt \\ &= \int_0^1 D_x F(tx_1,\lambda) x dt \\ &= x \int_0^1 D_x F(tx_1,\lambda)dt, \end{aligned} \tag{7.18}$$

so define

$$\begin{aligned} \tilde{F}(x_1,\lambda) &= \int_0^1 D_x F(tx_1,\lambda)dt \\ &= {x_1}^2 + \lambda. \end{aligned} \tag{7.19}$$

This yields a nontrivial solution curve , $\lambda = -{x_1}^2$, which can be seen in Figure 7.2. Note, however, that the nontrivial solution is not the only nontrivial solution for F through the bifurcation point, $(0, 0, 0)$. This can be seen in Figure 7.2. The non-trivial solution curve provided by Theorem 7.2.1 is indicated by the red line.

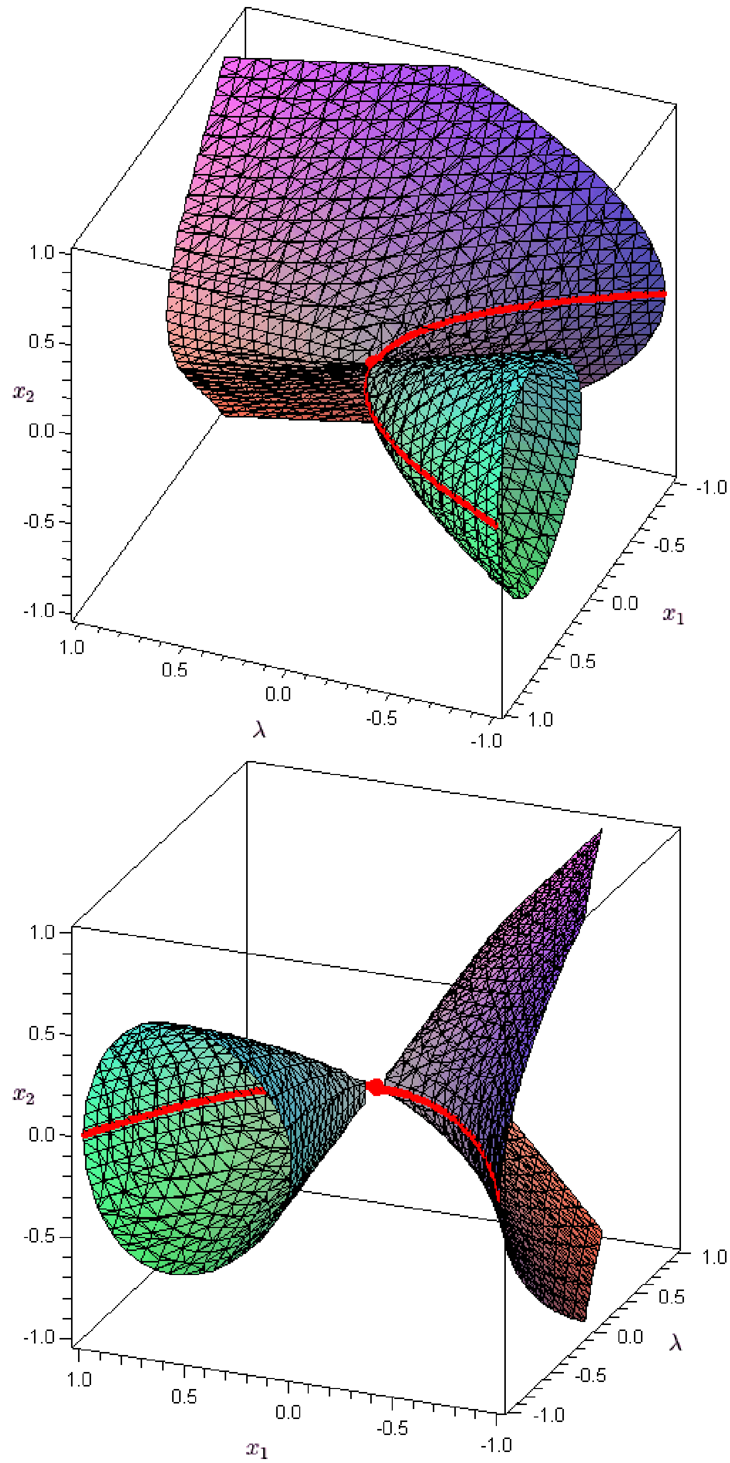

Figure 7.2: These graphs are two different views of the zero set of $F(x_1, x_2, \lambda)$ in a neighborhood of $(0, 0, 0)$. The non-trivial solution curve provided by Theorem 7.2.1 is indicated by the red line.

CHAPTER VIII

CONCLUSION

The main goal of this thesis is to examine several special cases of a recent theorem of Krömer et al. on bifurcation with two dimensional kernel. We also develop a number of examples of applications of the theorem.

We began by developing the background necessary to state the theorem, including an extensive treatment of the reduction method of Lyapunov-Schmidt. Then we presented two special cases of the theorem that identify specific classes of equations in which the theorem always holds, and we used these to construct several examples illustrating applications of Krömer's result. Specific examples are given that meet the different sufficiency conditions of the theorem of Krömer et al. and classical bifurcation theorems. An algebraic example illustrating the results of Krömer's Theorem was presented.

Additionally, we analyzed a non-linear boundary value problem that meets the sufficiency conditions of the theorem by Krömer et al. and not the classical theorems. Finally, we presented a variation on the Crandall-Rabinowitz Theorem that is related to problems with kernels of dimension two.

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APPENDICES

APPENDIX A

DERIVATIVES OF ψ

From the Implicit Function Theorem we know that $(I-Q)F(x+\psi(x,\lambda),\lambda) = 0$ $\forall(x,\lambda)$ in a neighborhood of $(0,5)$, and we will exploint this fact to find ways to define the derivatives of $\psi(x,\lambda)$. We now calculate $D_x(I-Q)F(0,5)$ in order to find $D_x\psi(0,5)$. Since we already have that $(I-Q)F(x+\psi(x,\lambda),\lambda)=0$,

$$\begin{aligned} 0 &= D_x(I-Q)F(x+\psi(x,\lambda),\lambda), \text{ so} \\ 0 &= (I-Q)D_uF(x+\psi(x,\lambda),\lambda)(I_N+D_x\psi(x,\lambda)), \text{ so} \\ 0 &= (I-Q)D_uF(x+\psi(x,\lambda),\lambda)D_x\psi(x,\lambda). \end{aligned} \tag{A.1}$$

Evaluating this at $(0,5)$, we have

$$\begin{aligned} 0 &= (I-Q)D_uF(\psi(0,5),5)D_x\psi(0,5) \\ &= (I-Q)D_uF(0,5)D_x\psi(0,5) \text{ since } \psi(0,5)=0 \text{ by (6.30).} \end{aligned} \tag{A.2}$$

Since $(I-Q)$ maps to $R = \text{ran}D_uF(0,5)$, it must be that $D_x\psi(0,5)=0$. We can use this to calculate $D_{xx}(I-Q)F(0,5)$ in order to find $D_{xx}\psi(0,5)$.

$$\begin{aligned} 0 &= D_x\left[(I-Q)D_uF(x+\psi(x,\lambda),\lambda)(I_N+D_x\psi(x,\lambda))\right], \text{ so} \\ 0 &= (I-Q)D_{uu}F(x+\psi(x,\lambda),\lambda)[I_N+D_x\psi(x,\lambda), I_N+D_x\psi(x,\lambda)] \\ &\quad +(I-Q)D_uF(x+\psi(x,\lambda),\lambda)D_{xx}\psi(x,\lambda), \text{ so} \\ 0 &= (I-Q)D_{uu}F(0,5)[I_N,I_N]+(I-Q)D_uF(0,5)D_{xx}\psi(0,5), \text{ so} \\ 0 &= (I-Q)D_uF(0,5)D_{xx}\psi(0,5) \text{ by (6.38).} \end{aligned} \tag{A.3}$$

Since $(I-Q)D_uF(0,5)\colon N^{\perp} \to Z_0$ is bijective, it must then be that $D_{xx}\psi(0,5) = 0$. Again we can use A.3 to calculate $D_{xxx}(I-Q)F(0,5)$ in order to find $D_{xxx}\psi(0,5)$.

$$0 = D_x\Big[(I-Q)D_{uu}F(x+\psi(x,\lambda),\lambda)[I_N + D_x\psi(x,\lambda), I_N + D_x\psi(x,\lambda)] + (I-Q)D_uF(x+\psi(x,\lambda),\lambda)D_{xx}\psi(x,\lambda)\Big], \text{ so}$$

$$\begin{aligned} 0 = {} & (I-Q)D_{uuu}F(x+\psi(x,\lambda),\lambda)\cdot[I_N + D_x\psi(x,\lambda), I_N + D_x\psi(x,\lambda), I_N + D_x\psi(x,\lambda)] \\ & +(I-Q)D_{uu}F(x+\psi(x,\lambda),\lambda)[D_{xx}\psi(x,\lambda), I_N + D_x\psi(x,\lambda)] \\ & +(I-Q)D_{uu}F(x+\psi(x,\lambda),\lambda)[I_N + D_x\psi(x,\lambda), D_{xx}\psi(x,\lambda)] \\ & +(I-Q)D_{uu}F(x+\psi(x,\lambda),\lambda)[I_N + D_x\psi(x,\lambda), D_{xx}\psi(x,\lambda)] \\ & +(I-Q)D_uF(x+\psi(x,\lambda),\lambda)D_{xxx}\psi(x,\lambda), \text{ so} \end{aligned}$$

$$\begin{aligned} 0 = {} & (I-Q)D_{uuu}F(0,5)[I_N + D_x\psi(0,5), I_N + D_x\psi(0,5), I_N + D_x\psi(0,5)] \\ & +(I-Q)D_{uu}F(0,5)[D_{xx}\psi(0,5), I_N + D_x\psi(0,5)] \\ & +2(I-Q)D_{uu}F(0,5)[D_{xx}\psi(0,5), D_{xx}\psi(0,5)] \\ & +(I-Q)D_uF(x+\psi(x,\lambda),\lambda)D_{xxx}\psi(x,\lambda), \text{ so} \end{aligned}$$

$$0 = (I-Q)D_{uuu}F(0,5)[I_N, I_N, I_N] + (I-Q)D_uF(0,5)D_{xxx}\psi(0,5) \text{ by (6.30).} \tag{A.4}$$

Then since $(I-Q)D_uF(0,5)$ is bijetive, we have

$$D_{xxx}\psi(0,5) = -\big((I-Q)D_uF(0,5)\big)^{-1}\big((I-Q)D_{uuu}F(0,5)[I_N, I_N, I_N]\big) \neq 0. \tag{A.5}$$

APPENDIX B

PARTIAL DERIVATIVES FOR THE TAYLOR EXPANSION OF H

B.1 Partial Derivatives with Respect to x

We must calculate the first three partial derivatives of h_1 with respect to x_1. We begin by calculating $\frac{\partial h_1}{\partial x_1}(0,0,5)$. We have

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\left\{L_1(Q\widetilde{F}(x_1,0,5))\right\}\\
&\quad= L_1\left(Q\tfrac{\partial}{\partial x_1}\widetilde{F}(x_1,0,5)\right)\\
&\quad= L_1Q\tfrac{\partial}{\partial x_1}F(x_1\phi_1+\psi(x_1\phi_1,5),5)\\
&\quad= L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,5),5)\left(\phi_1+D_x\psi(x_1\phi_1,5)\phi_1\right).
\end{aligned}\tag{B.1}$$

Hence, evaluating at $(0,0,5)$, we have

$$\begin{aligned}
\frac{\partial h_1}{\partial x_1}(0,0,5) &= L_1QD_uF(0,5)\left(\phi_1+D_x\psi(0,5)\phi_1\right)\\
&= L_1QD_uF(0,5)\left(\phi_1\right)=0 \text{ by (B.2) and (6.38).}
\end{aligned}\tag{B.2}$$

Next we compute $\frac{\partial^2 h_1}{\partial {x_1}^2}(0,0,5)$. Using (B.1), we compute

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\left\{L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,5),5)\left(\phi_1+D_x\psi(x_1\phi_1,5)\phi_1\right)\right\}\\
&\quad= L_1Q\tfrac{\partial}{\partial x_1}\left[D_uF(x_1\phi_1+\psi(x_1\phi_1,5),5)\left(\phi_1+D_x\psi(x_1\phi_1,5)\phi_1\right)\right]\\
&\quad= L_1Q\tfrac{\partial}{\partial x_1}\left[D_uF(x_1\phi_1+\psi(x_1\phi_1,5),5)\right]\left(\phi_1+D_x\psi(x_1\phi_1,5)\phi_1\right)\\
&\qquad+L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,5),5)\tfrac{\partial}{\partial x_1}\left[\phi_1+D_x\psi(x_1\phi_1,5)\phi_1\right]\\
&\quad= L_1QD_{uu}F(x_1\phi_1+\psi(x_1\phi_1,5),5)\left[\phi_1+D_x\psi(x_1\phi_1,5)\phi_1,\phi_1+D_x\psi(x_1\phi_1,5)\phi_1\right]\\
&\qquad+L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,5),5)D_{xx}\psi(x_1\phi_1,5)\left[\phi_1,\phi_1\right].
\end{aligned}\tag{B.3}$$

Hence, evaluating at $(0,0,5)$, we have

$$\begin{aligned}\frac{\partial^2 h_1}{\partial {x_1}^2}(0,0,5) &= L_1 Q D_{uu} F(0,5)[\phi_1,\phi_1], \text{ since } D_x\psi(0,5)\phi_1 = 0 \text{ by (A.2)} \\ &\quad + L_1 Q D_u F(0,5) D_{xx}\psi(0,5)\,[\phi_1,\phi_1] \\ &= L_1 Q D_{uu} F(0,5)\,[\phi_1,\phi_1] = 0 \text{ by (6.38).}\end{aligned} \tag{B.4}$$

Next we compute $\frac{\partial^3 h_1}{\partial {x_1}^3}(0,0,5)$. Using (B.3), we compute

$$\frac{\partial}{\partial x_1}\Big\{ L_1 Q D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5)\cdot[\phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1] + L_1 Q D_u F(x_1\phi_1 + \psi(x_1\phi_1,5),5) D_{xx}\psi(x_1\phi_1,5)\,[\phi_1,\phi_1]\Big\} \tag{B.5}$$

$$\begin{aligned}&= L_1 Q \tfrac{\partial}{\partial x_1}\Big[D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5)\cdot[\phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1]\Big] \\ &\quad + L_1 Q \tfrac{\partial}{\partial x_1}\Big[D_u F(x_1\phi_1 + \psi(x_1\phi_1,5),5) D_{xx}\psi(x_1\phi_1,5)\,[\phi_1,\phi_1]\Big]\end{aligned} \tag{B.6}$$

The first term of (B.6) equals

$$\begin{aligned}& L_1 Q \tfrac{\partial}{\partial x_1}\Big[D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5)\Big]\cdot[\phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1] \\ &\quad + L_1 Q D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5)\tfrac{\partial}{\partial x_1}\Big[[\phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1]\Big] \\ &= L_1 Q D_{uuu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5) \\ &\qquad \cdot[\phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1] \\ &\quad + L_1 Q D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5) \\ &\qquad \cdot[D_{xx}\psi(x_1\phi_1,5)[\phi_1,\phi_1], \phi_1 + D_x\psi(x_1\phi_1,5)\phi_1] \\ &\quad + L_1 Q D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1,5),5) \\ &\qquad \cdot[\phi_1 + D_x\psi(x_1\phi_1,5)\phi_1, D_{xx}\psi(x_1\phi_1,5)[\phi_1,\phi_1]]\end{aligned} \tag{B.7}$$

The second term of (B.6) equals

$$L_1 Q \frac{\partial}{\partial x_1}\left[D_u F(x_1\phi_1 + \psi(x_1\phi_1, 5), 5)\right] D_{xx}\psi(x_1\phi_1, 5)\,[\phi_1, \phi_1]$$
$$+L_1 Q D_u F(x_1\phi_1 + \psi(x_1\phi_1, 5), 5)\frac{\partial}{\partial x_1}\left[D_{xx}\psi(x_1\phi_1, 5)\,[\phi_1, \phi_1]\right]$$

$$= L_1 Q D_{uu} F(x_1\phi_1 + \psi(x_1\phi_1, 5), 5)\,[\phi_1 + D_x\psi(x_1\phi_1, 5)\phi_1, D_{xx}\psi(x_1\phi_1, 5)\,[\phi_1, \phi_1]]$$
$$+L_1 Q D_u F(x_1\phi_1 + \psi(x_1\phi_1, 5), 5) D_{xxx}\psi(x_1\phi_1, 5)\,[\phi_1, \phi_1, \phi_1] \tag{B.8}$$

Using (B.7) and (B.8), and evaluating at $(0, 0, 5)$, we have

$$\begin{aligned}
\frac{\partial^3 h_1}{\partial {x_1}^3}(0,0,5) &= L_1 Q D_{uuu} F(0,5)[\phi_1, \phi_1, \phi_1] \\
&\quad +L_1 Q D_{uu} F(0,5)\,[D_{xx}\psi(0,5)[\phi_1, \phi_1], \phi_1 + D_x\psi(0,5)\phi_1] \\
&\quad +L_1 Q D_{uu} F(0,5)\,[\phi_1 + D_x\psi(0,5)\phi_1, D_{xx}\psi(0,5)[\phi_1, \phi_1]] \\
&\quad +L_1 Q D_{uu} F(0,5)\,[\phi_1 + D_x\psi(0,5)\phi_1, D_{xx}(0,5)[\phi_1, \phi_1]] \\
&\quad +L_1 Q D_u F(0,5) D_{xxx}\psi(0,5)[\phi_1, \phi_1, \phi_1] \\
&= L_1 Q D_{uuu} F(0,5)[\phi_1, \phi_1, \phi_1] \\
&\quad +L_1 Q D_{uu} F(0,5)\,[0, \phi_1] \\
&\quad +2L_1 Q D_{uu} F(0,5)\,[\phi_1, 0]\,,\ \text{by (6.29)} \\
&\quad +L_1 Q D_u F(0,5) D_{xxx}\psi(0,5)[\phi_1, \phi_1, \phi_1] \\
&= L_1 Q D_{uuu} F(0,5)[\phi_1, \phi_1, \phi_1],\ \text{by (6.38)} \\
&= L_1 Q 6\phi_1\phi_1\phi_1,\ \text{again, by (6.38).}
\end{aligned} \tag{B.9}$$

As the only difference between h_1 and h_2 is the use of L_2 rather than L_1, and derivatives with respect to x_2 rather than x_1 only change ϕ_1 to ϕ_2, we may use these calculations for the other third order partial derivatives with respect to x.

B.2 Mixed Partial Derivatives

We now calculate, $\frac{\partial^2 h_1}{\partial\lambda\partial x_1}(0,0,5)$, the partial derivative of h_1 with respect to x_1, then λ. Using (B.1), we compute

$$\frac{\partial}{\partial\lambda}\left\{L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,\lambda),\lambda)\left(\phi_1+D_x\psi(x_1\phi_1,\lambda)\phi_1\right)\right\} \tag{B.10}$$

$$\begin{aligned} &= \tfrac{\partial}{\partial\lambda}\left[L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,\lambda),\lambda)\right]\left(\phi_1+D_x\psi(x_1\phi_1,\lambda)\phi_1\right) \\ &\qquad +L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,\lambda),\lambda)\tfrac{\partial}{\partial\lambda}\left[\left(\phi_1+D_x\psi(x_1\phi_1,\lambda)\phi_1\right)\right] \\ &= L_1QD^2_{uu}F(x_1\phi_1+\psi(x_1\phi_1,\lambda),\lambda)[D_\lambda\psi(x_1\phi_1,\lambda),\phi_1+D_x\psi(x_1\phi_1,\lambda)\phi_1] \\ &\qquad +QD^2_{u\lambda}F(x_1\phi_1+\psi(x_1\phi_1,\lambda),\lambda)(\phi_1+D_x\psi(x_1\phi_1,\lambda)\phi_1) \\ &\qquad +L_1QD_uF(x_1\phi_1+\psi(x_1\phi_1,\lambda),\lambda)D^2_{\lambda x}\psi(x_1\phi_1,\lambda)\phi_1 \end{aligned} \tag{B.11}$$

Hence, evaluating at $(0,0,5)$, we have

$$\begin{aligned} \frac{\partial^2 h_1}{\partial\lambda\partial x_1}(0,0,5) &= L_1QD^2_{uu}F(0,5),\lambda)[D_\lambda\psi(0,5),\phi_1+D_x\psi(0,5)\phi_1] \\ &+ L_1QD^2_{u\lambda}F(0,5)(\phi_1+D_x\psi(0,5)\phi_1) \\ &+ L_1QD_uF(0,5)D^2_{\lambda x}\psi(0,5)\phi_1 \\ &= L_1QD^2_{u\lambda}F(0,5)\phi_1. \end{aligned} \tag{B.12}$$

Thus, we have

$$\begin{aligned} D^2_{u\lambda}F(u_0,5)[v_1] &= \tfrac{\partial}{\partial\lambda}D_uF(u_0,\lambda)v_1 \\ &= \tfrac{\partial}{\partial\lambda}\left(v_1'''' + \lambda v_1'' + 4v_1 + 3{u_0}^2v_1\right) \\ &= {v_1}'', \text{ so} \\ \frac{\partial^2 h_1}{\partial\lambda\partial x_1}(0,0,5) &= L_1Q{\phi_1}''. \end{aligned} \tag{B.13}$$

These calculations may also be used for the other mixed partial derivatives, $\frac{\partial^2 h_1}{\partial\lambda\partial x_2}(0,0,5)$, $\frac{\partial^2 h_2}{\partial\lambda\partial x_1}(0,0,5)$, and $\frac{\partial^2 h_2}{\partial\lambda\partial x_2}(0,0,5)$.

B.3 Evaluating Derivatives

Note first that for $f \in X$,

$$\begin{aligned}
L_1 Qf &= L_1\left[L_1 f\phi_1 + L_2 f\phi_2\right] \\
&= L_1\left[\int_0^\pi f\phi_1 ds\phi_1 + \int_0^\pi f\phi_2 ds\phi_2\right] \\
&= \int_0^\pi f\phi_1 ds \int_0^\pi {\phi_1}^2 ds + \int_0^\pi f\phi_2 ds \int_0^\pi \phi_1\phi_2 ds \\
&= \int_0^\pi f\phi_1 ds(1) + \int_0^\pi f\phi_2 ds(0) \\
&= \int_0^\pi f\phi_1 \\
&= L_1 f.
\end{aligned} \tag{B.14}$$

Since Q projects f to Z_0, making it a linear combination of $\{\phi_1, \phi_2\}$, the basis of Z_0. L_1 and L_2 use the fact that ϕ_1 and ϕ_2 are orthonormal to produce the coefficients of the basis elements. For a function $f \in X$ projected by Q, since $L_1 f$ and $L_2 f$ are the coefficients of the basis elements, we have that $L_1 Q = L_1$ and similarly, $L_2 Q = L_2$. Now we can calculate the following:

$$\begin{aligned}
\frac{\partial^3 h_1}{\partial {x_1}^3}(0,0,5) &= L_1 Q D_{uuu} F(0,5)[\phi_1,\phi_1,\phi_1] \\
&= L_1 Q 6\phi_1\phi_1\phi_1 \\
&= 6L_1\left(\tfrac{\sqrt{2\pi}}{\pi}\sin(s)\right)^3 \\
&= \tfrac{24}{\pi^2}\int_0^\pi sin^4(s) ds \\
&= \tfrac{24}{\pi^2}\cdot\tfrac{3\pi}{8} \\
&= \tfrac{9}{\pi}.
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\frac{\partial^3 h_2}{\partial {x_1}^3}(0,0,5) &= L_2 Q D_{uuu} F(0,5)[\phi_1,\phi_1,\phi_1] \\
&= L_2 Q 6\phi_1\phi_1\phi_1 \\
&= 6L_2\left(\tfrac{\sqrt{2\pi}}{\pi}\sin(s)\right)^3 \\
&= \tfrac{24}{\pi^2}\int_0^\pi \sin^3(s)\sin(2s) ds \\
&= 0.
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
\frac{\partial^3 h_1}{\partial {x_2}^3}(0,0,5) &= L_1 Q D_{uuu} F(0,5)[\phi_2,\phi_2,\phi_2] \\
&= L_1 Q 6\phi_2\phi_2\phi_2 \\
&= 6L_1 \left(\tfrac{\sqrt{2\pi}}{\pi}\sin(2s)\right)^3 \\
&= \tfrac{24}{\pi^2}\int_0^\pi \sin^3(2s)\sin(s)ds \\
&= \tfrac{24}{\pi^2}\cdot 0 \\
&= 0.
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
\frac{\partial^3 h_2}{\partial {x_2}^3}(0,0,5) &= L_2 Q D_{uuu} F(0,5)[\phi_2,\phi_2,\phi_2] \\
&= L_2 Q 6\phi_2\phi_2\phi_2 \\
&= 6L_2 \left(\tfrac{\sqrt{2\pi}}{\pi}\sin(2s)\right)^3 \\
&= \tfrac{24}{\pi^2}\int_0^\pi \sin^4(2s)ds \\
&= \tfrac{24}{\pi^2}\cdot \tfrac{3\pi}{8} \\
&= \tfrac{9}{\pi}.
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
\frac{\partial^3 h_1}{\partial {x_1}^2\partial x_2}(0,0,5) &= \frac{\partial^3 h_1}{\partial x_2\partial {x_1}^2}(0,0,5) \\
&= L_1 Q D_{uuu} F(0,5)[\phi_2,\phi_1,\phi_1] \\
&= L_1 Q 6\phi_2\phi_1\phi_1 \\
&= 6L_1 \left(\tfrac{\sqrt{2\pi}}{\pi}\right)^3 \sin(2s)\sin^2(s) \\
&= \tfrac{24}{\pi^2}\int_0^\pi \sin^3(s)\sin(2s)ds \\
&= \tfrac{24}{\pi^2}\cdot 0 \\
&= 0.
\end{aligned} \tag{B.19}$$

$$\begin{aligned}
\frac{\partial^3 h_2}{\partial {x_1}^2\partial x_2}(0,0,5) &= \frac{\partial^3 h_2}{\partial x_2\partial {x_1}^2}(0,0,5) \\
&= L_2 Q D_{uuu} F(0,5)[\phi_2,\phi_1,\phi_1] \\
&= L_2 Q 6\phi_2\phi_1\phi_1 \\
&= 6L_2 \left(\tfrac{\sqrt{2\pi}}{\pi}\right)^3 \sin(2s)\sin^2(s) \\
&= \tfrac{24}{\pi^2}\int_0^\pi \sin^4(s)ds \\
&= \tfrac{24}{\pi^2}\cdot \tfrac{\pi}{4} \\
&= \tfrac{6}{\pi}.
\end{aligned} \tag{B.20}$$

$$
\begin{aligned}
\frac{\partial^3 h_1}{\partial x_1 \partial {x_2}^2}(0,0,5) &= \frac{\partial^3 h_1}{\partial {x_2}^2 \partial x_1}(0,0,5) \\
&= L_1 Q D_{uuu} F(0,5)[\phi_2, \phi_2, \phi_1] \\
&= L_1 Q 6 \phi_2 \phi_2 \phi_1 \\
&= 6 L_1 \left(\tfrac{\sqrt{2\pi}}{\pi}\right)^3 \sin(s)\sin^2(2s) \\
&= \tfrac{24}{\pi^2} \int_0^\pi \sin^2(s)\sin^2(2s) ds \\
&= \tfrac{24}{\pi^2} \cdot \tfrac{\pi}{4} \\
&= \tfrac{6}{\pi}.
\end{aligned}
\tag{B.21}
$$

$$
\begin{aligned}
\frac{\partial^3 h_2}{\partial x_1 \partial {x_2}^2}(0,0,5) &= \frac{\partial^3 h_2}{\partial {x_2}^2 \partial x_1}(0,0,5) \\
&= L_2 Q D_{uuu} F(0,5)[\phi_2, \phi_2, \phi_1] \\
&= L_2 Q 6 \phi_2 \phi_2 \phi_1 \\
&= 6 L_2 \left(\tfrac{\sqrt{2\pi}}{\pi}\right)^3 \sin(s)\sin^2(2s) \\
&= \tfrac{24}{\pi^2} \int_0^\pi \sin(s)\sin^3(2s) ds \\
&= \tfrac{24}{\pi^2} \cdot 0 \\
&= 0.
\end{aligned}
\tag{B.22}
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$$
\begin{aligned}
\frac{\partial^2 h_1}{\partial \lambda \partial x_1} &= L_1 Q \phi_1'' \\
&= -\tfrac{\sqrt{2\pi}}{\pi} L_1 \text{sin}(s) \\
&= -\tfrac{2}{\pi} \int_0^\pi \sin^2(s) ds \\
&= -1.
\end{aligned}
\tag{B.23}
$$

$$
\begin{aligned}
\frac{\partial^2 h_1}{\partial \lambda \partial x_2} &= L_1 Q \phi_2'' \\
&= -\tfrac{4\sqrt{2\pi}}{\pi} L_1 \sin(2s) = 0.
\end{aligned}
\tag{B.24}
$$

$$
\begin{aligned}
\frac{\partial^2 h_2}{\partial \lambda \partial x_2} &= L_2 Q \phi_2'' \\
&= -\tfrac{4\sqrt{2\pi}}{\pi} L_2 \text{sin}(2s) \\
&= -\tfrac{8}{\pi} \int_0^\pi \sin^2(2s) ds \\
&= -\tfrac{8}{\pi} \cdot \tfrac{\pi}{2} \\
&= -4.
\end{aligned}
\tag{B.25}
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$$
\begin{aligned}
\frac{\partial^2 h_2}{\partial \lambda \partial x_1} &= L_2 Q \phi_1^{''} \\
&= -\tfrac{\sqrt{2\pi}}{\pi} L_2 \sin(s) = 0.
\end{aligned}
\tag{B.26}
$$