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Geometric Limits of Julia Sets of Maps z^n + exp(2πiθ) as n → ∞

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GEOMETRIC LIMITS OF JULIA SETS WITH PARAMETERS ON THE CIRCLE

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ABSTRACT. We show that the geometric limit as $n \to \infty$ of the Julia sets $J(P_{n,c})$ for the maps $P_{n,c}(z) = z^n + c$ does not exist for almost every c on the unit circle. Furthermore, we show that there is always a subsequence along which the limit does exist and equals the unit circle.

Consider the family of maps

$$P_{n,c}(z) \;=\; z^n + c,$$

where $n \geq 2$ is an integer and $c \in \mathbb{C}$ is a parameter. These maps all share the quality that there is only one free critical point; that is, the critical point at infinity is fixed under iteration, and the iterates of the remaining critical point, $z = 0$, depend on both c and n. Because of this uni-critical property, many dynamical properties of the classical quadratic family $z \mapsto z^2 + c$ are also exhibited by this family of maps. Details of this family are readily available in the literature [6, 8, 5].

In this note, we will consider the filled Julia set $K(P_{n,c})$, the set of points in $\mathbb{C}$ that remain bounded under iteration and its boundary, the Julia set $J(P_{n,c})$. In [2], the structure of the filled Julia set $K(P_{n,c})$ and its boundary $J(P_{n,c})$, the Julia set, as $n \to \infty$ was examined. One of the major results is this work was

Theorem [Boyd-Schulz]. *Let $c \in \mathbb{C}$, and let $CS(\hat{\mathbb{C}})$ denote the collection of all compact subsets of $\hat{\mathbb{C}}$. Then under the Hausdorff metric $d_{\mathcal{H}}$ in $CS(\hat{\mathbb{C}})$,*

(1) *If $c \in \mathbb{C}\backslash\overline{\mathbb{D}}$, then*
$$\lim_{n\to\infty} J(P_{n,c}) = \lim_{n\to\infty} K(P_{n,c}) = S^1.$$
(2) *If $c \in \mathbb{D}$, then*
$$\lim_{n\to\infty} J(P_{n,c}) = S^1 \text{ and } \lim_{n\to\infty} K(P_{n,c}) = \overline{\mathbb{D}}.$$
(3) *If $c \in S^1$, then if $\lim_{n\to\infty} J(P_{n,c})$ and/or $\lim_{n\to\infty} K(P_{n,c})$ (and/or any liminf or limsup) exists, it is contained in $\overline{\mathbb{D}}$.*

The purpose of this note is to improve part (3) of this result. While there may be no limit as $n \to \infty$ for $J(P_{n,c})$ or $K(P_{n,c})$, experimentation suggests given $c \in S^1$, there is almost always a predictable pattern for the filled Julia set for $P_{n,c}$ as $n \to \infty$. This experimentation led to the following result:

Theorem 1. *Let $c = e^{2\pi i\theta} \in S^1$ such that $\theta \neq 0$ and $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then*

$$\lim_{n\to\infty} J(P_{n,c}) \text{ and } \lim_{n\to\infty} K(P_{n,c})$$

do not exist. Moreover, if θ is rational, $\theta \neq 0$, and $\theta \neq \frac{3q\pm1}{3(6p-1)}$, then there exist N and subsequences a_k and b_k partitioning $\{n \in \mathbb{N} \colon n \geq N\}$ such that

$$\lim_{k\to\infty} K(P_{a_k,c}) \;=\; S^1 \qquad \text{and} \qquad \lim_{k\to\infty} K(P_{b_k,c}) \;=\; \overline{\mathbb{D}}.$$

In Section 2, we present the background material and motivation for this result. The proof of Theorem 1 is the focus of Section 3.

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2. Background and Motivation

2.1. **Notation and Terminology.** The main results in this note rely on the convergence of sets in $\hat{\mathbb{C}}$, where the convergence is with respect to the Hausdorff metric. Given two sets A, B in a metric space (X, d), the Hausdorff distance $d_{\mathcal{H}}(A, B)$ between the sets is defined as

$$\begin{aligned} d_{\mathcal{H}}(A,B) &= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \\ &= \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}. \end{aligned}$$

Each point in A has a minimal distance to B, and vice versa. The Hausdorff distance is the maximum of all these distances. For example, a regular hexagon A inscribed in a circle B of radius r has sides of length r. In this case, $d_{\mathcal{A}}(A, B) = r(1 - \sqrt{3}/2)$, the shortest distance from the circle to the midpoint of a side of the hexagon. See Figure 3. Julia sets $J(P_{n,c})$ and filled Julia sets $K(P_{n,c})$ are compact [1] in the compact space $\hat{\mathbb{C}}$. Moreover, with the Hausdorff metric $d_{\mathcal{H}}$, $\hat{\mathbb{C}}$ is complete [3].

Suppose S_n and S are compact subsets of $\mathbb{C}$. If for all $\epsilon > 0$, there is $N > 0$ such that for any $n \geq N$, we have $d_{\mathcal{H}}(S_n, S) < \epsilon$, then we say S_n converges to S and write $\lim_{n\to\infty} S_n = S$.

We adopt the notation from [2]. For an open annulus with radii $0 < r < R$,

$$\mathbb{A}(r, R) := \{z \in \mathbb{C} \colon r < |z| < R\}.$$

Also, the open ball of radius $\epsilon > 0$ centered at z will be denoted $B(z, \epsilon)$.

2.2. **Motivation.** A basic fact from complex dynamics (see [1] or [7]) is that $K(P_{n,c})$ is connected if and only if the orbit of 0 stays bounded; otherwise it is totally disconnected. For each $n \geq 2$, we define the Multibrot sets

$$\mathcal{M}_n := \{c \in \mathbb{C} \colon J(P_{n,c}) \text{ is connected}\}.$$

Since 0 is the only free critical point, $\mathcal{M}_n$ is also the set of parameters c such that the orbit of 0 under iteration by $P_{n,c}$ remains bounded [7]. Since the maps $P_{n,c}$ are uncritical, much of their dynamical behavior mimics the family of complex quadratic polynomials [8].

It was proven in [2] that for sufficiently large N,

(1) $c \in \mathbb{D}$ implies for any $n \geq N$, $0 \in K(P_{n,c})$ (the orbit of 0 is bounded and $c \in \mathcal{M}_n$), and
(2) $c \in \mathbb{C}\backslash\overline{\mathbb{D}}$ implies for any $n \geq N$, $0 \notin K(P_{n,c})$ (the orbit of 0 is not bounded and $c \notin \mathcal{M}_n$).

For parameters $c \in S^1$, $P_{n,c}(0) \in S^1$ for any n, and this obstructs the direct proof that the orbit of 0 remains bounded (or not). However, one finds that in most cases, $P^2_{n,c}(0) \notin S^1$ and should expect that in these situations, determining whether the orbit of zero stays bounded depends heavily on where $P^2_{n,c}(0)$ is relative to the circle. In fact, working with the second iterate of 0 will be sufficient for all of our proofs.

Noting that $P^2_{n,c}(0) = P_{n,c}(c)$, we have the following convenient formula:

Proposition 1. *For $c = e^{2\pi i\theta} \in S^1$ and any positive integer n, $|P_{n,c}(c)| \geq 1$ if and only if*

$$\cos(2\pi\theta(n-1)) \geq -\frac{1}{2},$$

where equality holds if and only if $|P_{n,c}(c)| = 1$.

Proof. Note first that for $c = e^{2\pi i\theta}$, we have $P_{n,c}(c) = (e^{2\pi i\theta})^n + e^{2\pi i\theta}$, so

$$\begin{aligned} P_{n,c}(c) &= \cos(2\pi\theta n) + i\sin(2\pi\theta n) + \cos(2\pi\theta) + i\sin(2\pi\theta) \\ &= \cos(2\pi\theta n) + \cos(2\pi\theta) + i(\sin(2\pi\theta n) + \sin(2\pi\theta)). \end{aligned}$$

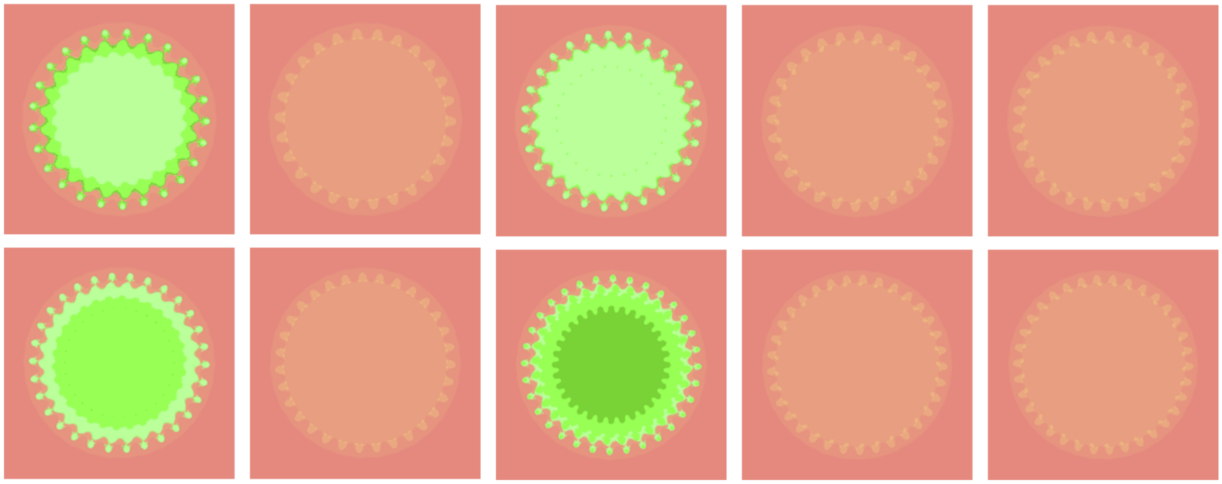

FIGURE 1. $J(P_{n,c})$ for $c = e^{4\pi i/5}$ and $n = 25 \ldots 34$, starting from the upper left to the lower right.

If $P_{n,c}(c) \geq 1$, then

$$\begin{aligned} 1 &\leq (\cos(2\pi\theta n) + \cos(2\pi\theta))^2 + (\sin(2\pi\theta n) + \sin(2\pi\theta))^2 \\ &= 2\cos(2\pi\theta n)\cos(2\pi\theta) + 2\sin(2\pi\theta n)\sin(2\pi\theta) + 2 \\ &= 2\cos(2\pi\theta(n-1)) + 2 \end{aligned}$$

from which the result follows. □

Experimentation indicates that $P_{n,c}(c)$ being inside (or outside) S^1 very consistently dictates that $c \in \mathcal{M}_n$ (or $c \notin \mathcal{M}_n$). See Figure 1. Then the condition on $P_{n,c}(c)$ from Proposition 1 can be used to very consistently predict the structure of $K(P_{n,c})$, which Proposition 1 also suggests is periodic with respect to n. This will be made precise (with quantifiers) in Proposition 2 below.

More efficient experimentation with checking whether the orbit of 0 stays bounded clearly present this periodic (with respect to n) structure for $K(P_{n,c})$ when c is a rational angle on S^1. Figure 2 shows powers $421 \leq n \leq 450$ and $c = e^{\pi i p/q} \in S^1$ where $q = 15$ and p is an integer with $1 \leq p \leq 30$. A star indicates the Julia set $J(P_{n,c})$ is connected. There is, however, an inconsistency when the orbit of 0 remains on S^1. Note that the situation in which $P_{n,c}(c) \in S^1$ corresponds to having $\cos(2\pi\theta(n-1)) = -1/2$. This can be seen in Figure 2 for $n = 426$ and $2\theta = 26/15$ and $2\theta = 28/15$. The program that generated this data can provide a similar table for any equally distributed set of angles and any consecutive set of iterates.

This experimentation yields an intuition that is supported further by another result from [2]:

Theorem [Boyd-Schulz]. *Under the Hausdorff metric $d_{\mathcal{H}}$ in $CS(\hat{\mathbb{C}})$,*

$$\lim_{n\to\infty} M(P_{n,c}) = \overline{\mathbb{D}}.$$

For a fixed $c \in S^1$, as n increases, c will fall into and out of $\mathcal{M}_n$. See Figure 3. Thus, Proposition 1 provides nice visual evidence that this is truly periodic behavior. The Multibrot sets in Figure 3 are in logarithmic coordinates, so the horizontal axis is the real values $-1 \leq \theta \leq 1$, where $c = e^{2\pi i\theta}$. We are using logarithmic coords since we are interested in the angle θ.

It remains an open question what happens for parameters with angles $\theta = \frac{3q\pm1}{3(6p-1)}$ for $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. We prove in Proposition 3 that the parameters corresponding to these angles force $P_{n,c}(c)$ to be a fixed point on S^1. In this case, the critical orbit is clearly bounded, so we know the filled

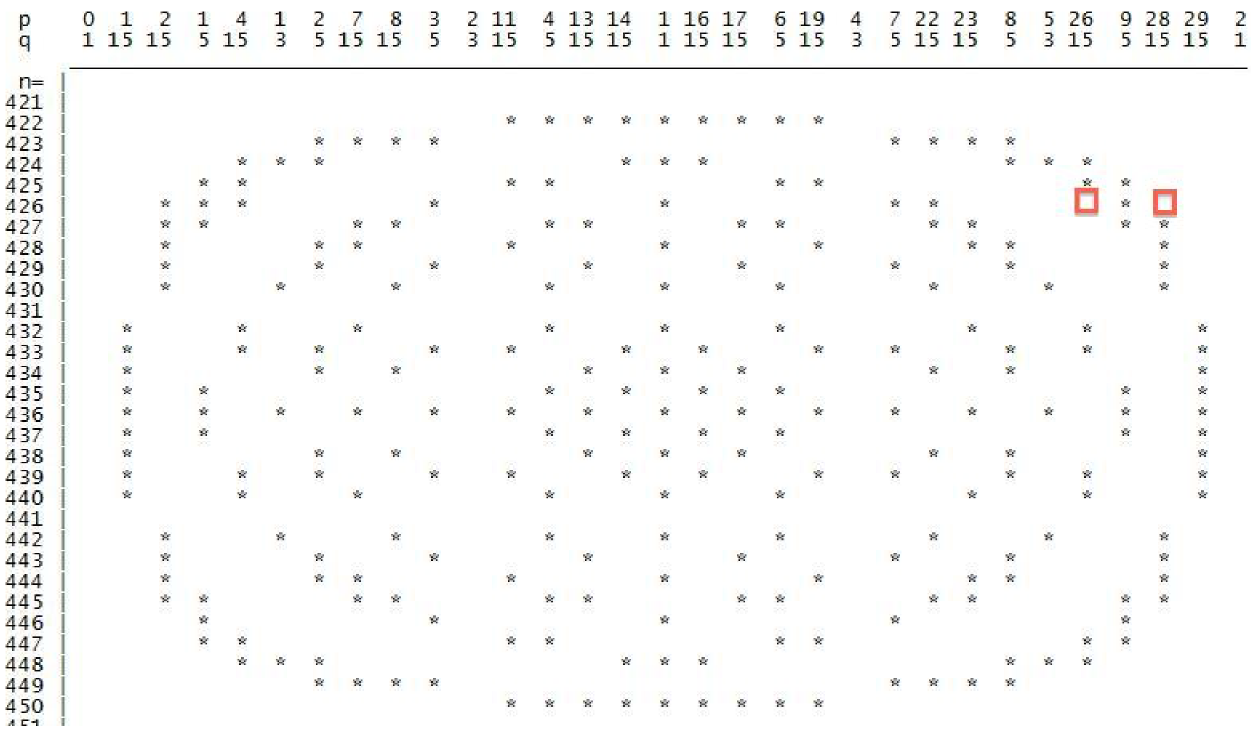

FIGURE 2. A star indicated $J(P_{n,c})$ is connected, where $c = e^{\pi i p/q}$

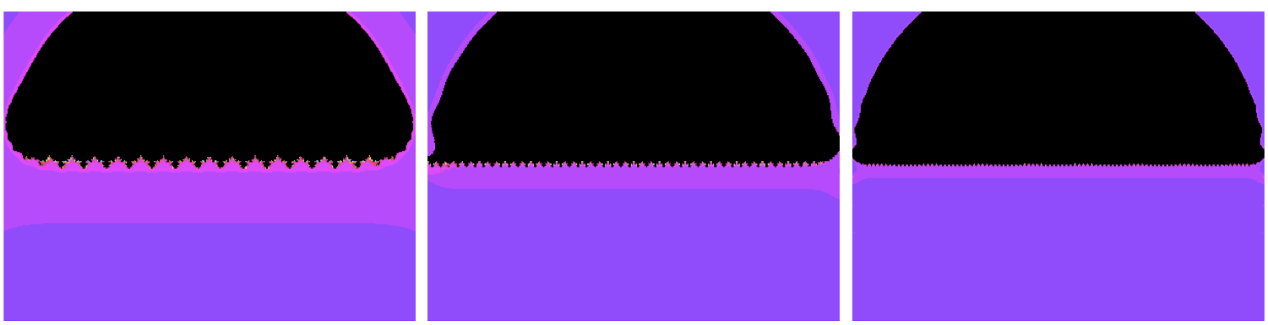

FIGURE 3. $\mathcal{M}_n$, where $c = e^{2\pi i\theta}$, $\theta \in \mathbb{C}$, and $n = 10, 25, 50$. Almost all fixed Reθ, falls into and out of $\mathcal{M}_n$ as n increases.

Julia set $K(P_{n,c})$ must be connected. See Figure 4. However, the behavior of the boundary $J(P_{n,c})$ is extremely complicated, as in the left-most image in Figure 4.

3. Proof of Theorem 1

We now prove that $P_{n,c}(c) \notin S^1$ does allow us to determine whether $c \in \mathcal{M}_n$.

Proposition 2. *Let $c \in S^1$. For any $\epsilon > 0$ there exists $N > 0$ so that for all $n \geq N$ one has:*

1. *if $|P_{n,c}(c)| < 1 - \epsilon$, then $\mathbb{D}_{1-\epsilon} \subset K(P_{n,c})$.*
2. *if $|P_{n,c}(c)| > 1 + \epsilon$, then $\mathbb{D}_{1-\epsilon} \subset \mathbb{C} \setminus K(P_{n,c})$.*

Noting that $0 \in \mathbb{D}_{1-\epsilon}$, it follows immediately from Propositions 1 and 2 that the orbit of 0 is bounded (or not) depending respectively on whether $P_{n,c}(c)$ is inside $\mathbb{D}_{1-\epsilon}$ (or outside $\mathbb{D}_{1+\epsilon}$). That is,

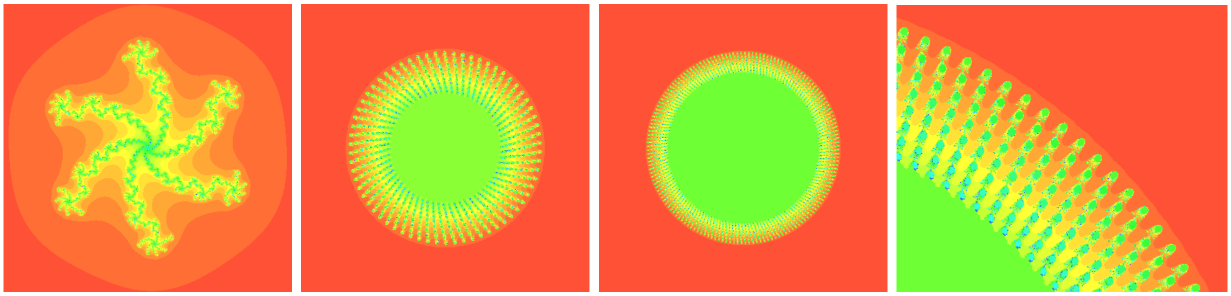

FIGURE 4. From left to right: $K(P_{n,c})$ for $c = e^{2\pi i/15}$ and $n = 6, 66, 156$. The far left image is a closer look at the boundary when $n = 165$

Corollary 1. *For all $\epsilon > 0$, there is an N such that for any $n \geq N$,*

1. *if $\cos(2\pi\theta(n-1)) < -1/2 - \epsilon/2$, then $K(P_{n,c})$ is connected and*
2. *if $\cos(2\pi\theta(n-1)) > -1/2+\epsilon/2$, then $K(P_{n,c})$ is totally disconnected and $K(P_{n,c}) = J(P_{n,c})$.*

Proof of Proposition 2. Fix $c \in S^1$. Let $\epsilon > 0$ and $r_n := |P^2_{n,c}(0)| = |c^n + c|$. Observe

$$\begin{aligned} \left|P^2_{n,c}(z)\right| \;=\; |(z^n+c)^n + c| \;&=\; \left| c^n + c + \sum_{k=1}^{n} \binom{n}{k} (z^n)^k c^{n-k} \right| \\ &\leq\; |c^n + c| + \sum_{k=1}^{n} \binom{n}{k} |z|^{nk} \;=\; r_n + (1+|z|^n)^n - 1. \end{aligned}$$

Then $|P^2_{n,c}(z)| \leq |z|$ when $r_n + (1+|z|^n)^n - 1 < |z|$. That is, for any $\eta \in (0,1)$, if

$$r_n \;\leq\; \eta + 1 - (1+\eta^n)^n, \tag{1}$$

then the disk $\mathbb{D}_\eta$ is forward invariant under $P^2_{n,c}$. Note that $(1+\eta^n)^n > 1$ and for fixed η, $(1+\eta^n)^n \to 1$ as $n \to \infty$. Fix $\eta = 1 - \epsilon/2$, so there is a positive integer N such that for all $n \geq N$,

$$(1+\eta^n)^n - 1 < \frac{\epsilon}{2}.$$

Thus, for any $n \geq N$ such that $r_n < 1 - \epsilon$,

$$r_n < \eta - \frac{\epsilon}{2} < \eta + 1 - (1+\eta^n)^n,$$

so, $\mathbb{D}_{1-\epsilon} \subset \mathbb{D}_\eta$ is forward invariant under $P^2_{n,c}$. This implies that the orbit of any point in $\mathbb{D}_{1-\epsilon}$ must be bounded in a disk of radius $\eta^n + 1$, so we have $\mathbb{D}_{1-\epsilon} \subset K(P_{n,c})$.

On the other hand, note that

$$\left|P^2_{n,c}(z)\right| \;=\; |(z^n+c)^n + c| \;\geq\; \left| |c^n + c| - \sum_{k=1}^{n} \binom{n}{k} |z|^{nk} \right| \;=\; |r_n - (1+|z|^n)^n + 1|.$$

Again, fix $\eta = 1 - \epsilon/2$, so there is an N such that for any $n \geq N$, if $r_n > 1 + \epsilon$ and $|z| < 1 - \epsilon/2$, then

$$(1+|z|^n)^n - 1 < (1+\eta^n)^n - 1 < \frac{\epsilon}{2}.$$

That is, for $n \geq N$ and $z \in \mathbb{D}_\eta$,

$$|P^2_{n,c}(z)| \geq |r_n - (1+|z|^n)^n + 1| \geq 1 + \frac{\epsilon}{2}.$$

By Lemma 1, we can also choose N large enough that $K(P_{n,c}) \subset \mathbb{D}_{1+\epsilon/2}$ as well. Then for any $n > N$ and $z \in \mathbb{D}_\eta$, if $|P_{n,c}(c)| = r_n < 1 + \epsilon$, then $P^2_{n,c}(z) \notin K(P_{n,c})$. It follows that $z \notin K(P_{n,c})$, so $\mathbb{D}_\eta \subset \mathbb{C} \backslash K(P_{n,c})$. $\square$

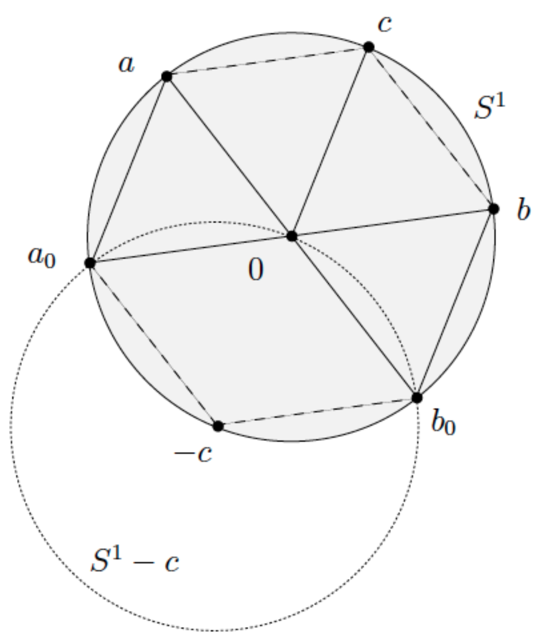

FIGURE 5. $P_{n,c}(c)$ is on the circle if and only if $c^n = a_0$ or $a^n = b_0$.

What remains is to examine $c \in S^1$ such that $P_{n,c}(c) \in S^1$ as well. This case is simpler and occurs less frequently than one might expect.

Proposition 3. *Let $c = e^{2\pi i\theta}$ and $P_{n,c}(z) = z^n + c$. Then $P^2_{n,c}(c) \in S^1$ if and only if $P_{nc}(c)$ is a fixed point, in which case, $(n, \theta) \in N$, where*

$$N := \left\{ (n, \theta) \in \mathbb{N} \times \mathbb{R} \mid n = 6p, \theta = \frac{3q \pm 1}{3(6p-1)}, \text{ where } p \in \mathbb{N} \text{ and } q \in \mathbb{Z} \right\}.$$

Proof. Since $|c| = 1$, note that the set $S^1 - c := \{z - c \mid z \in S^1\}$ is a circle centered at $-c \in S$, so it intersects S^1 in exactly two points, call them a_0 and b_0. By construction, $a_0 + c, b_0 + c \in S^1$, so define

$$\begin{aligned} a &:= a_0 + c \\ b &:= b_0 + c. \end{aligned}$$

Moreover, the points $\{c, a, a_0, -c, b_0, b\}$ form a hexagon inscribed in S^1 whose sides are all length one. Thus, we have

$$\begin{aligned} a &= e^{2\pi i(\theta+1/6)} \\ a_0 &= e^{2\pi i(\theta+1/3)} \\ b_0 &= e^{2\pi i(\theta-1/3)} \\ b &= e^{2\pi i(\theta-1/6)}. \end{aligned}$$

See Figure 3. For any $z \in S^1$, we have that $P_{n,c}(z) = z^n + c$ and $z^n \in S^1$, so $P_{n,c}(z) \in S^1$ if and only if

$$z^n \in (S^1 - c) \cap S^1 = \{a_0, b_0\};$$

that is, $P_{n,c}(z) \in \{a, b\}$. It follows that $|P^k_{n,c}(c)| = 1$ for all $k \geq 0$ if and only if one of the following is true: a is a fixed point, b is a fixed point, or a and b are a two-cycle.

Assume that $P_{n,c}(c) \in S^1$. First observe that $P_{n,c}(c) \in \{a, b\}$, so

$$P_{n,c}(c) = e^{2\pi i(\theta \pm 1/6)}.$$

Since $P_{n,c}(c) = c^n + c = e^{2\pi i\theta n} + e^{2\pi i\theta}$, it follows that

$$e^{2\pi i\theta n} = e^{2\pi i(\theta \pm 1/6)} - e^{2\pi i\theta} = e^{2\pi i(\theta \pm 1/3)}.$$

Thus, $\theta n = \theta \pm 1/3 + q$ for some integer q, so

$$\theta(n-1) = q + \frac{1}{3} \text{ if } P_{n,c}(c) = a \text{ and} \tag{2}$$

$$\theta(n-1) = q - \frac{1}{3} \text{ if } P_{n,c}(c) = b. \tag{3}$$

Proceeding to the next iterate, note that $P^2_{n,c}(c) \in \{a, b\}$ as well, so we need only examine $P_{n,c}(a)$ and $P_{n,c}(b)$. Since $P_{n,c}(a), P_{n,c}(b) \in \{a, b\}$, it must be for some integer p_0,

$$P_{n,c}\left(e^{2\pi i(\theta \pm 1/6)}\right) = e^{2\pi i(\theta \pm 1/6)n} + e^{2\pi i\theta} \in \{a, b\} = \left\{e^{2\pi i(\theta + 1/6 + p_0)}, e^{2\pi i(\theta - 1/6 + p_0)}\right\}.$$

Then it follows that from the definition of a and b that $e^{2\pi i(\theta \pm 1/6 + p_0)} \in \{a_0, b_0\}$, so we have $(\theta \pm 1/6)n = \theta \pm 1/3 + p_0$. In particular,

$$(n-1)\theta = p_0 + \frac{1}{3} - \frac{n}{6}, \quad \text{if } P_{n,c}(a) = a, \tag{4}$$

$$(n-1)\theta = p_0 - \frac{1}{3} - \frac{n}{6}, \quad \text{if } P_{n,c}(a) = b, \tag{5}$$

$$(n-1)\theta = p_0 + \frac{1}{3} + \frac{n}{6}, \quad \text{if } P_{n,c}(b) = a, \text{ and} \tag{6}$$

$$(n-1)\theta = p_0 - \frac{1}{3} + \frac{n}{6}, \quad \text{if } P_{n,c}(b) = b. \tag{7}$$

If a and b are a two cycle, then equations (5) and (6) together imply $q \pm 1/3 = p_0$. This contradicts the fact that q and p_0 are both integers. A similar contradiction arises from the cases when $P_{n,c}(b) = a$ and a is fixed, or when $P_{n,c}(a) = b$ and b is fixed.

The only remaining possibilities are that $P_{n,c}(c) = P_{n.c}(a) = a$ or $P_{n,c}(c) = P_{n.c}(b) = b$. Thus, we have shown that $|P^k_{n,c}(c)| = 1$ for all $k \geq 0$ if and only if for all $k \geq 1$, $P^k_{n,c}(c) = a$ or $P^k_{n,c}(c) = b$.

It remains to show that $(n, \theta) \in N$ is an equivalent statement. Supposing that for all $k \geq 1$, $P^k_{n,c}(c) = a$ or $P^k_{n,c}(c) = b$, we have

$$q \pm \frac{1}{3} = \theta(n-1) = p_0 \pm \frac{1}{3} \mp \frac{n}{6}.$$

From this equation, one can see that $n = 6p$, where $p = q - p_0 \in \mathbb{N}$. Moreover, the equations (2) and (3) derived from the first iterate of c yield

$$\theta(n-1) = q \pm \frac{1}{3},$$

so

$$\theta = \frac{3q \pm 1}{3(n-1)} = \frac{3q \pm 1}{3(6p-1)}.$$

□

The following lemmas are from [2]. The third is a subtle variation, so we include the proof.

Lemma 1 (Boyd-Schulz)**.** *Let $c \in \mathbb{C}$. For any $\epsilon > 0$, there is an N such that for all $n \geq N$,*

$$K(P_{c,n}) \subset \mathbb{D}_{1+\epsilon}.$$

Lemma 2 (Boyd-Schulz)**.** *Let $z \in J(P_{n,c})$. If ω is an n-th root of unity, then $\omega z \in J(P_{n,c})$.*

Lemma 3 (Boyd-Schulz). *Let $\epsilon > 0$ and $c = e^{2\pi i\theta} \in S^1$ such that $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. There is an $N \geq 2$ such that for all $n \geq N$ and for any $e^{i\phi} \in S^1$,*

$$B(e^{i\phi}, \epsilon) \cap J(P_{n,c}) \neq \emptyset.$$

Proof. By Proposition 2, there is an N_1 such that for any $n \geq N_1$, we have $J(P_{n,c}) \subset \mathbb{A}(1-\epsilon/2, 1+\epsilon/2)$. Let $e^{i\phi} \in S^1$ and $\alpha > 0$ be the angle so that

$$U := \{re^{i\tau} \colon r > 0, \phi - \alpha < \tau < \phi + \alpha\} \cap \mathbb{A}(1-\epsilon/2, 1+\epsilon/2)$$

is contained in $B(e^{i\phi}, \epsilon)$. The same α works for each different ϕ.

For any n, let $\omega_n = e^{2\pi i/n}$, and choose $N > N_1$ such that $2\pi/N < \alpha$, noting that N is also independent of ϕ. We have $2\pi/n < \alpha$ for any $n \geq N$.

Since $J(P_{n,c})$ is nonempty for any n [7], choose $z_n \in J(P_{n,c})$ for each $n \geq N$. Then for some integer $1 \leq j_n \leq n-1$, we have

$$\omega_n^{j_n} z_n \in U \subset B(e^{i\phi}, \epsilon).$$

Thus, for all $n \geq N$, $B(e^{i\phi}, \epsilon) \cap J(P_{n,c}) \neq \emptyset$. □

Proof of Theorem 1. Fix $c = e^{2\pi i\theta} \in S^1$ and assume $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then by Proposition 3, $|P_{n,c}(c)| \neq 1$, and by Proposition 1, we have $\cos(2\pi\theta(n-1)) \neq -\frac{1}{2}$. In particular,

(1) $|P_{n,c}(c)| < 1$ when $\cos(2\pi\theta(n-1)) < -\frac{1}{2}$, and
(2) $|P_{n,c}(c)| > 1$ when $\cos(2\pi\theta(n-1)) > -\frac{1}{2}$.

Note that $\cos(2\pi\theta(n-1))$ has period $1/\theta$ as a function of n. If θ is a rational number, then this function takes a finite number of values. In this case, $|P_{n,c}(c)|$ can be bound away from S^1 by a fixed distance for any n. Let $\epsilon > 0$ be smaller than this minimum distance. Then, Proposition 2 gives that that there is $N > 0$ such that for all $n \geq N$, we have either

1. $|P_{n,c}(c)| < 1 - \epsilon$ and $\mathbb{D}_{1-\epsilon} \subset K(P_{n,c})$, or
2. $|P_{n,c}(c)| > 1 + \epsilon$ and $\mathbb{D}_{1-\epsilon} \subset \mathbb{C} \setminus K(P_{n,c})$.

Moreover, if we consider θ as a rational rotation of the circle, the periodic orbit (with respect to n) induces intervals on S^1 that are permuted by this rotation [4]. Since $\cos(2\pi\theta(n-1)) \neq -\frac{1}{2}$, we must have n and m such that $\cos(2\pi\theta(n-1)) \geq -\frac{1}{2}$ and $\cos(2\pi\theta(m-1)) \geq -\frac{1}{2}$. Again, since this rotation is periodic, we can find such n and m for any $N > 0$. Thus, no limit as $n \to \infty$ can exist for $K(P_{n,c})$.

Now suppose θ is irrational. For any sufficiently small $\epsilon > 0$ let $N > 0$ be given by Corollay 1. Since the values $\cos(2\pi(n-1)\theta)$ are equidistributed in $[-1, 1]$ according to $\cos_*(\text{Leb})$ (where Leb is the Lebesgue measure on the circle) [4], there will be arbitrarily large values of $m, n > N$ such that $\cos(2\pi(n-1)\theta) < -1/2 - \epsilon$ and $\cos(2\pi(m-1)\theta) > -1/2 + \epsilon$. In this case $K_{n,c}$ contains the disc $\mathbb{D}_{1-\epsilon}$ while, $\mathbb{D}_{1-\epsilon}$ is contained in the complement of $K_{m,c}$. Thus, no limit as $n \to \infty$ can exist for $K(P_{n,c})$.

Having established the claim in Theorem 1 that no limit exists, we move on to prove the claim that if θ is rational, $\theta \neq 0$, and $\theta \neq \frac{3q\pm1}{3(6p-1)}$, then there are subsequences a_k and b_k partitioning $\{n \in \mathbb{N} \colon n \geq N\}$ such that

$$\lim_{k\to\infty} K(P_{a_k,c}) \;=\; S^1 \qquad \text{and} \qquad \lim_{k\to\infty} K(P_{b_k,c}) \;=\; \overline{\mathbb{D}}.$$

We know from Proposition 3 that $|P_{n,c}(c)| \neq 1$ for any positive integer n. Thus, for any $\epsilon > 0$, we can use Proposition 2 to find an $N \in \mathbb{N}$ and construct subsequences

$$\begin{aligned} A_\epsilon &= \{n \in \mathbb{Z}_+ \colon |P_{n,c}(c)| < 1 - \epsilon\} \text{ and} \\ B_\epsilon &= \{n \in \mathbb{Z}_+ \colon |P_{n,c}(c)| > 1 + \epsilon\} \end{aligned}$$

such that for any $n \geq N$,

(1) if $n \in A_\epsilon$, then $K(P_{n,c})$ is full and connected, and
(2) if $n \in B_\epsilon$, then $K(P_{n,c}) = J(P_{nc})$ is totally disconnected.

Moreover, as $\epsilon \to 0$, these two sets partition $\mathbb{N}$.

With the structure of $K(P_{n,c})$ consistent in each of the sets A_ϵ and B_ϵ, the remainder of the proof very closely follows the proof of Theorem 1.2 in [2].

Let $\epsilon > 0$ and a_k the subsequence of $n \in A_\epsilon$. Then $|P_{a_k,c}(c)| < 1 - \epsilon$, so by Proposition 1, there is an N_1 such that for any $a_k \geq N_1$, we have $\mathbb{D}_{1-\epsilon} \subseteq K(P_{a_k,c})$. By Lemma 1, there is an $N_2 \geq N_1$ such that for any $a_k \geq N_2$, we have $K(P_{a_k,c}) \subseteq \mathbb{D}_{1+\epsilon}$. Thus, for any $z \in K(P_{a_k,c})$,

$$d(z, \overline{\mathbb{D}}) = \inf_{w \in \overline{\mathbb{D}}} |z - w| < \epsilon.$$

Now let $w \in \overline{\mathbb{D}}$. Since $\mathbb{D}_{1-\epsilon} \subseteq K(P_{a_k,c}) \subseteq \mathbb{D}_{1+\epsilon}$, we have

$$d(w, K(P_{a_k,c})) = \inf_{z \in K(P_{a_k,c})} |z - w| < \epsilon.$$

If follows that

$$d_{\mathcal{H}}(K(P_{a_k,c}), \overline{\mathbb{D}}) = \max \left\{ \sup_{z \in K(P_{a_k,c})} d(z, \overline{\mathbb{D}}), \sup_{w \in \overline{\mathbb{D}}} d(w, K(P_{a_k,c})) \right\} < \epsilon.$$

Thus, $\lim_{k\to\infty} K(P_{a_k,c}) = \overline{\mathbb{D}}$.

Now let b_k be the subsequence of $n \in B_\epsilon$. Again, by Proposition 1 and Lemma 1, there is an N_1 such that for any $b_k \geq N_1$, we have $K(P_{n,c}) \subset \mathbb{A}(1 - \epsilon/2, 1 + \epsilon/2)$. Also, note that $0 \notin K(P_{n,c})$, so $K(P_{nc})$ is totally disconnected and $J(P_{n,c}) = K(P_{n,c})$. Then for any $z \in J(P_{b_k,c})$, we have

$$d(z, S^1) = \inf_{s \in S^1} |z - s| < \epsilon.$$

By Lemma 3, there is an $N_2 \geq N_1$ such that for any $b_k \geq N_2$ and for any $s \in S^1$,

$$d(s, J(P_{b_k,c})) = \inf_{z \in J(P_{b_k,c})} |z - s| < \epsilon.$$

Thus, it follows that $d_{\mathcal{H}}(J(P_{b_k,c}), S^1) < \epsilon$ and $\lim_{k\to\infty} J(P_{b_k,c}) = S^1$. □

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