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# The Hamiltonian index of graphs

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**Abstract**

The Hamiltonian index of a graph $G$ is defined as

$$h(G) = \min\{m : L^m(G) \text{ is Hamiltonian}\}.$$

In this paper, using the reduction method of Catlin [P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–44], we constructed a graph $\tilde{H}^{(m)}(G)$ from $G$ and prove that if $h(G) \geq 2$, then

$$h(G) = \min\{m : \tilde{H}^{(m)}(G) \text{ has a spanning Eulerian subgraph}\}.$$

## 1. Introduction

We follow Bondy and Murty [1] for basic terminologies and notations. Let $G$ be a connected graph, which is neither a path nor a cycle, i.e. $\Delta(G) > 2$. A $(u, v)$-path is a path with end-vertices $u$ and $v$. We will also denote a $(u, v)$-path by $P[u, v]$. If $G$ has a cycle containing every vertex of $G$, then $G$ is called *Hamiltonian*. A graph is *Eulerian* if it is connected and every vertex has even degree. An Eulerian subgraph $H$ of $G$ is called a *spanning Eulerian subgraph* if $V(H) = V(G)$. An Eulerian subgraph $H$ of $G$ is called a *D-circuit* if $E(G - V(H)) = \emptyset$. The line graph of $G$ is denoted by $L(G)$ or $L^1(G)$. For a positive integer $m$, define $L^m(G) = L(L^{m-1}(G))$; $L^0(G) = G$. Define

$$h(G) = \min\{m : L^m(G) \text{ is Hamiltonian}\},$$

$h(G)$ is called the Hamiltonian index of $G$. A relationship between a $D$-Circuit and Hamiltonian line graph was given by Harary and Nash-Williams [7].

**Theorem A** (*Harary and Nash-Williams [7]*). *Let $G$ be a connected graph with at least three edges. $L(G)$ is Hamiltonian if and only if $G$ has a D-circuit.*

For a graph $G$ with a connected subgraph $H$, the contraction $G/H$ is the graph obtained from $G$ by replacing $H$ by a vertex $v_H$, such that the number of edges in $G/H$ joining any vertex $v \in V(G-H)$ to $v_H$ in $G/H$ equals to the number of edges joining $v$ in $G$ to $V(H)$.

Using Catlin's reduction method [2], Lai [3] gave several results on the Hamiltonian index of a graph. For this problem and other related topics, see [4–6,8]. In this paper, we will give new results on Hamiltonian index which generalize some of Lai's results. We will discuss some properties on collapsible graphs in the next section first. The main results will be given in Section 3. Several corollaries of the main results will be given in the last section.

## 2. Collapsible graphs

In [2], Catlin introduced the concept of collapsible graphs. A graph $G$ is called collapsible if for every even subset $S \subseteq V(G)$, there is a subgraph $H$ of $G$ (called the *S-subgraph* of $G$) such that $G - E(H)$ is connected and $S = O(H)$, $O(H)$ is the set of vertices of odd degree of $H$.

First we give some equivalent conditions for collapsible graphs.

**Theorem 1.** *The following conditions are equivalent:*

(1) *$G$ is collapsible;*
(2) *for any even set $S \subseteq V(G)$, $G$ has a spanning connected subgraph $G_S$ with $S = O(G_S)$;*
(3) *for any even set $S \subseteq V(G)$, $|S| = 2m$, there are edge-disjoint paths $P_1, P_2, \ldots, P_m$, joining the vertices in $S$ pairwise, such that $G - E(P_1 \bigcup P_2 \bigcup \cdots \bigcup P_m)$ is connected.*

**Proof.** (1) $\Rightarrow$ (2) Let $S \subseteq V(G)$ be an even set. Let $X = S \Delta O(G)$ ($A \Delta B = (A-B) \bigcup (B-A)$ is the symmetric difference of sets $A$ and $B$). Then $X$ is an even subset. Since $G$ is collapsible, $G$ has an $X$-subgraph $H$. Then $G_S = G - E(H)$ is a spanning connected subgraph of $G$ and $S = X \Delta O(G) = O(G_S)$.

(2) $\Rightarrow$ (1) Let $S \subseteq V(G)$ be an even subset. Let $X = S \Delta O(G)$. Then $X$ is an even subset of $V(G)$. By (2), $G$ has a connected spanning subgraph $G_X$ with $X = O(G_X)$. Let $H = G - E(G_X)$. Then $H$ is the $S$-subgraph of $G$, since $G - E(H) = G_X$ is connected and $S = X \Delta O(G) = O(H)$.

(1) $\Rightarrow$ (3) Let $S \subseteq V(G)$ be an even set and let $H$ be an $S$-subgraph of $G$. Since $O(H) = S$, for any vertex $v_1 \in S$, there is a path $P_1$ in $H$ beginning at $v_1$ and ending at another vertex in $S$, say $v_2$. Thus we have paths $P_1, P_2, \ldots, P_m \subseteq H$, joining the vertices in $S$ pairwise.

Assume that the paths $P_1$ (joining $v_1, v_2 \in S$) and $P_2$ (joining $v_3, v_4 \in S$) have common edges. Let $v_1', v_2'$ be the first and the last vertices on $P_1$ which are also on $P_2$. Then the paths $P_1[v_1, v_1']$ and $P_1[v_2', v_2]$ are two sub-paths on $P_1$ which have no internal vertices on $P_2$. The vertices $v_1'$ and $v_2'$ divide $P_2$ into three paths, say $P_2[v_3, v_1']$, $P_2[v_1', v_2']$ and $P_2[v_2', v_4]$. Then, we have a path $P[v_1, v_3]$ formed by $P_1[v_1, v_1']$ and $P_2[v_1', v_3]$ joining $v_1$ and $v_3$, and a path formed by $P_1[v_2, v_2']$ and $P_2[v_2', v_4]$ joining $v_2$ and $v_4$. Obviously, $P[v_1, v_3]$ and $P[v_2, v_4]$ are edge-disjoint paths.

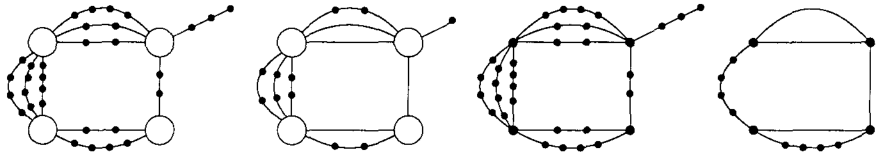
Following the same arguments, we can find edge-disjoint paths $P_1', P_2', \ldots, P_m'$ joining the vertices of $S$ pairwise such that $P_1' \bigcup P_2' \bigcup \cdots \bigcup P_m' \subseteq P_1 \bigcup P_2 \bigcup \cdots \bigcup P_m \subseteq H$. Since $H$ is an $S$-subgraph of $G$, $G - E(H)$ is connected. Since $G - E(H) \subseteq G - E(P_1' \bigcup P_2' \bigcup \cdots \bigcup P_m')$, $G - E(P_1' \bigcup P_2' \bigcup \cdots \bigcup P_m')$ is connected.

(3) $\Rightarrow$ (1) Assume that $P_1, P_2, \ldots, P_m$ are edge-disjoint paths joining the vertices of $S$ pairwise, and $G - E(P_1 \bigcup P_2 \bigcup \cdots \bigcup P_m)$ is connected. Then $H = P_1 \bigcup P_2 \bigcup \cdots \bigcup P_m$ is an $S$-subgraph of $G$. The theorem is proved. $\square$

## 3. Main results

Let $G$ be a connected graph. Let $D_2(G)$ be the subset of $V(G)$ containing all vertices of degree 2. Let $V_0 = V(G) - D_2(G)$. The components of the subgraph $G[V_0]$ are denoted by $C_1, C_2, \ldots, C_k$ (some of them may contain only a single vertex). Let $E_0 = E(G[V_0])$. A lane $\pi$ of $G$ is either a path whose internal vertices are of degree 2 and its end vertices belong to two (or one) components of $G[V_0]$ (we say $\pi$ joins these two components), or a cycle which has a vertex of degree $\geq 3$ in $G$ and other vertices are of degree 2. We also use $\pi(u, v)$ to denote a lane with the end-vertices $u$ and $v$. The length $l(\pi)$ of a lane $\pi$ is defined as the number of the edges of $\pi$. A lane $\pi$ of $G$ is called an end-lane if one of its end-vertices is of degree 1. It is easy to see that $E(G) - E_0$ is the union of all lanes in $G$. Define

$$l(G) = \max\{l(\pi) : \pi \text{ is a lane of } G \text{ but not a cycle}\}.$$



$G(m = 3)$ $L^{(2)}(G)$ $H^{(3)}(G)$ $\tilde{H}^{(3)}(G)$

Fig. 1.

If $G$ has no such lanes, then define $l(G) = 1$.

In the following we assume that $l(G) \geq 2$. For a positive integer $m \geq 2$, let $G(m)$ be the union of lanes of length $< m$. Denote the components of $G[V_0] \bigcup G(m)$ by $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Each $\Gamma_j$ consists of components of $G[V_0]$ connected by lanes of length $< m$.

For a lane $\pi$ of $G$, each edge of $\pi$ corresponds to a vertex in $L(G)$ and each internal vertex of $\pi$ corresponds to an edge in $L(G)$. Thus $\pi$ corresponds to a path in $L(G)$ of length $l(\pi) - 1$, denote by $I(\pi)$. If $l(\pi) > 2$, $I(\pi)$ is a lane in $L(G)$. If $l(\pi) = 2$, $I(\pi)$ is an edge in $L(G)$ joining two complete subgraphs $K_{n_1}$ and $K_{n_2}(n_1, n_2 \neq 2)$, which is called a degenerate lane in $L(G)$.

Consider a component $C_j$ of $G[V_0]$, which is not a single vertex of degree 1 in $G$. Each edge in $C_j$ is adjacent to a vertex of degree at least 3. Let $C'_j$ be the subgraph of $G$ consisting of $C_j$ and the edges with one end in $C_j$. Then each edge of $L(C'_j)$ belongs to a triangle of $L(G)$. Thus $L(C'_j)$ is collapsible.

Thus, $L(G)$ consists of collapsible subgraphs $L(C'_1), L(C'_2), \ldots, L(C'_k)$ and lanes (degenerated lanes) $I(\pi_1), I(\pi_2), \ldots, I(\pi_t)$, with $l(I(\pi_i)) = l(\pi_i) - 1$, $1 \leq i \leq t$. Now consider the graph $L^2(G)$. If $L(C'_{j_1})$ and $L(C'_{j_2})$ are joint by a degenerated lane $\pi$, then $L(L(C'_{j_1}) \bigcup L(C'_{j_2}) \bigcup \pi)$ is collapsible and all its vertices are of degree $\geq 3$.

For $m \geq 2$, let $\pi$ be a lane of $G$. If $l(\pi) \geq m + 1$, then $\pi$ corresponds to a lane in $L^{m-1}(G)$ (denoted by $I^{m-1}(\pi)$) of length $l(\pi) - m + 1$. If $l(\pi) = m$, then $\pi$ corresponds to a degenerated lane of $L^{m-1}(G)$. Also each lane $\omega$ in $L^{m-1}(G)$ can be obtained from a lane $\pi$ in $G$ such that $\omega = I^{m-1}(\pi)$. And each component $\Gamma$ of $G[V_0] \bigcup G(m)$ corresponds to a collapsible subgraph (denoted by $\tilde{L}^{(m-1)}(\Gamma)$) of $L^{m-1}(G)$.

Let $H^{(m)}(G)$ be the graph obtained from $G$ by contracting subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ to distinct vertices. A vertex in $H^{(m)}(G)$ obtained by contracting a $\Gamma \in \{\Gamma_1, \Gamma_2, \ldots, \Gamma_s\}$ is called a contraction image of $\Gamma$.

Now we construct a graph $\tilde{H}^{(m)}(G)$ from $H^{(m)}(G)$ by the following process.

(1) Delete all lanes beginning and ending at the same vertex $\Gamma_j$.

(2) Let $\Gamma_{j_1}$ and $\Gamma_{j_2}$ be two vertices in $H^{(m)}(G)$ corresponding to two components of $G[V_0] \bigcup G(m)$, if they are connected by more than two lanes with length $\geq m$, say $m_1$ lanes with length $> m + 1$ and $n_1$ lanes with length $m$ or $m + 1$ (thus $m_1 + n_1 \geq 3$), then we delete some of them, so that there are $m_2$ lanes with length $> m + 1$ and $n_2$ lanes with length $m$ or $m + 1$, where

$$(m_2, n_2) = \begin{cases} (2, 0) & m_1 \text{ even}, n_1 = 0; \\ (1, 0) & m_1 \text{ odd}, n_1 = 0; \\ (1, 1) & n_1 = 1; \\ (0, 2) & n_1 \geq 2. \end{cases}$$

(3) Delete all end-lanes of length $m$ and replace each lane with length $m$ or $m + 1$ by a single edge (see Fig. 1).

Now we give the following theorem.

**Theorem 2.** *The following are equivalent:*

(1) $L^{m-1}(G)$ *has a D-circuit;*

(2) $H^{(m)}(G)$ *has an Eulerian subgraph obtained by deleting some lanes of length m and* $m + 1$*;*

(3) $\tilde{H}^{(m)}(G)$ *has a spanning Eulerian subgraph.*

**Proof.** (1) $\Rightarrow$ (2) Let $M_0$ be a $D$-circuit of $L^{m-1}(G)$. Let $\theta$ be the contraction homomorphism from $L^{m-1}(G)$ to the graph $\tilde{L}^{(m-1)}(G)$ by contracting $\tilde{L}^{(m-1)}(\Gamma_j)$ to a single vertex denoted by $\tilde{\Gamma}_j (1 \leq j \leq s)$. For $1 \leq j \leq s$, if $\tilde{L}^{(m-1)}(\Gamma_j)$ is not a single vertex, then $M_0$ contains at least one of them, so $\theta(M_0)$ contains $\tilde{\Gamma}_j$. If a vertex $v$ of degree

2 (in $L^{m-1}(G)$) is not in $M_0$, then it must belong to a lane of length 2. Thus $M_0$ contains all lanes of length $\geq 3$ of $L^{m-1}(G)$. Note that $\tilde{L}^{(m-1)}(G) = \theta(L^{m-1}(G))$ is obtained from $H^{(m)}(G)$ by contracting all lanes of length $l(\pi)$ to $l(\pi) - m + 1$ (if $l(\pi) \geq m$). Denote this contraction homomorphism by $\theta_1$. Thus $\theta_1$ induces a bijection between the sets of lanes in $H^{(m)}(G)$ and $\tilde{L}^{(m-1)}(G)$ (if a cycle contains a vertex of degree 2, it must contain the lane containing this vertex). Hence $\theta_1^{-1}(\theta(M_0))$ is an Eulerian subgraph of $H^m(G)$ obtained by deleting some lanes of length $m$ and $m+1$.

(2) $\Rightarrow$ (3) Let $M_1$ be an Eulerian subgraph of $H^{(m)}(G)$ obtained by deleting some lanes of length $m$ and $m+1$. $M_1$ contains all lanes of length $>m+1$. If $M_1$ contains more than two lanes joining two vertices $\Gamma_{j_1}$ and $\Gamma_{j_2}$, then delete even number of them, we obtain a graph $M_1'$ which is also an Eulerian subgraph of $H^{(m)}(G)$. By the same process, we can find an Eulerian subgraph $M_1^*$ of $H^{(m)}(G)$, so that each pair of vertices $\Gamma_{j_1}$ and $\Gamma_{j2}$ are joint by no more than 2 lanes in $M_1^*$. $M_1^*$ passes through all vertices $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Deleting lanes ending at same vertex $\Gamma_j$, replace the lanes of length $m$ and $m+1$ in $M_1^*$ by a single edge, we obtain a spanning Eulerian subgraph of $\tilde{H}^{(m)}(G)$.

(3) $\Rightarrow$ (1) Now assume that $M_2$ is a spanning Eulerian subgraph of $\tilde{H}^{(m)}(G)$. Recover all lanes of length $>m+1$ deleted (when the number of such lanes joining vertices $\Gamma_{j_1}$ and $\Gamma_{j2}$ are odd, we can recover one lane of length $m$ or $m+1$, or delete one lane of length $m$ or $m+1$ in $M_2$, so that the total number of lanes recovered is even), we obtain an Eulerian subgraph $M_2'$, which can be obtained by deleting some lanes of length $m$ or $m+1$ from $H^{(m)}(G)$. Contracting each lane $\pi$ to a lane of length $l(\pi) - m + 1$, we obtain an Eulerian subgraph $M_2''$ of $\tilde{L}^{(m-1)}(G)$, which contains vertices $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ and all lanes with length $\geq 3$. If a lane of length 2 or a degenerated lane is not in $M_2''$ then its vertices are in $M_2''$, thus $M_2''$ is a $D$-circuit of $\tilde{L}^{(m-1)}(G)$. Since $\tilde{L}^{(m-1)}(G)$ is obtained from $L^{m-1}(G)$ by contracting collapsible graphs $\tilde{L}^{(m-1)}(\Gamma_j)$, $1 \leq j \leq s$, thus the conclusion is obtained by the following lemma. □

**Lemma.** *Let $G$ be a graph, $H$ be a collapsible subgraph of $G$. Then $G$ has $D$-circuit if and only if $G/H$ has a $D$-circuit.*

**Proof.** Let $\Gamma$ be a $D$-circuit of $G$, then $\Gamma/H$ is obviously a $D$-circuit of $G/H$.

Assume that $G/H$ has a $D$-circuit $\Gamma'$. Then $E(\Gamma')$ is a subset of $E(G) - E(H)$. Let $G_0 = G[E(\Gamma')]$. Since the number of edges in $E(\Gamma')$ incident to $H$ is even, and each vertex in $V(G_0) - V(H)$ is of even degree (in $G_0$), we have $O(G_0) \subseteq V(H)$. Since $H$ is collapsible, there is a connected spanning subgraph $\Gamma''$ of $H$ with $O(\Gamma'') = O(G_0)$. Let $\Gamma_0 = G_0 \bigcup \Gamma''$, then $\Gamma_0$ is connected and

$$O(\Gamma_0) = O(G_0) \Delta O(\Gamma'') = \emptyset,$$

Again, each edge of $G$ is in $\Gamma_0$ or incident to an edge in $\Gamma_0$. This means $\Gamma_0$ is a $D$-circuit of $G$. □

**Theorem 3.** *If $G$ is connected, $\Delta(G) \geq 3$, $h(G) \geq 2$, then*

$$h(G) = \min\{m : \tilde{H}^{(m)}(G) \text{ has a spanning Eulerian subgraph}\}.$$

**Proof.** By Theorem A, we have, if $h(G) \geq 2$, $L^{m-1}(G)$ has a $D$-circuit if and only if $L^m(G)$ is Hamiltonian. Thus, Theorem 3 follows from Theorem 2. □

## 4. Corollaries

As corollaries of Theorem 3, we give the following results.

**Corollary 1.** *For a connected graph $G$, such that $\Delta(G) \geq 3$, then $h(G) \leq |V(G)| - \Delta(G)$.*

**Proof.** Let $n = |V(G)|$ and $k = |V(G)| - \Delta(G)$. Then $k \geq 1$. If $k = 1$, then $G$ is spanned by a $K_{1,n-1}$, and so by Theorem A, $L(G)$ is Hamiltonian ($h(G) \leq 1$). Therefore, we assume that $k \geq 2$.

If $h(G) \leq 2$, then $h(G) \leq k = |V(G)| - \Delta(G)$. Thus we assume that $h(G) > 2$. Since $G$ has a vertex $v$, which is adjacent to $n-k$ vertices in $G$. Then the lanes of $G$ are either of length $\leq k$, or contained in a cycle of length at most $k+2$. Thus, $G[V_0] \bigcup G(m)$ has only one component (i.e., is connected), so $\tilde{H}^{(k)}(G)$ is a single point. □

**Corollary 2** (*Lai [3]*). $h(G) \leq l(G) + 1$.

**Proof.** Let $m = l(G) + 1$, then $\tilde{H}^{(m)}(G)$ is collapsible, so $h(G) \leq l(G) + 1$. $\square$

Lai [3] also gave the condition such that $h(G) = l(G) + 1$, which is also implied in Theorem 3.

Sarazin [9] gave the following results, which is also implied by Theorem 3. A bridge-lane is a lane containing a bridge.

**Corollary 3** *(Sarazin [9]). If $G$ is not a path and all cyclic blocks of $G$ are Hamiltonian, then*

$$h(G) = \max\{l(P) + 1, l(Q)\}.$$

*Where the maximum is taken over all bridge-lanes $P$ and all end-lanes $Q$.*

**Proof.** Let $m = \max\{l(P)+1, l(Q)\}$. Then if $G$ satisfies the condition, then $\tilde{H}^{(m)}(G)$ has spanning Eulerian subgraph and $\tilde{H}^{(m-1)}(G)$ has not. $\square$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