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Collapsible graphs and reductions of line graphs

Zhi-Hong Chen
Butler University, chen@butler.edu

Peter C.B. Lam

Wai-Chee Shiu

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ABSTRACT

A graph G is collapsible if for every even subset $X \subseteq V(G)$, G has a subgraph Γ such that $G - E(\Gamma)$ is connected and the set of odd-degree vertices of Γ is X. A graph obtained by contracting all the non-trivial collapsible subgraphs of G is called the reduction of G. In this paper, we characterize graphs of diameter two in terms of collapsible subgraphs and investigate the relationship between the line graph of the reduction and the reduction of the line graph. Our results extend former results in [H.-J. Lai, Reduced graph of diameter two, J. Graph Theory 14 (1) (1990) 77–87], and in [P.A. Catlin, Iqblunnisa, T.N. Janakiraman, N. Srinivasan, Hamilton cycles and closed trails in iterated line graphs, J. Graph Theory 14 (1990) 347–364].

1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. Let G be a graph. For a vertex v in G, the neighborhood of v, written $N_G(v)$ or $N(v)$ is $\{u \in V(G) \mid uv \in E(G)\}$. The cardinality of $N(v)$ is denoted by $d_G(v)$ or $d(v)$ and is called the *degree* of v in G. The smallest, respectively largest, degree of any vertex in G is denoted by $\delta(G)$, respectively $\Delta(G)$. A graph is *Eulerian* if it is connected and every vertex has even degree. The *line graph* of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. An Eulerian subgraph H of G is called a *spanning Eulerian subgraph* if $V(H) = V(G)$. If G has a cycle containing every vertex of G, then G is called *Hamiltonian*. A cycle of length t is denoted by C_t. The *girth* of a graph G is the length of any shortest cycle in G. The distance between two vertices u and v of a connected graph is the minimum length of all paths joining u and v, and is denoted by $d(u, v)$. The diameter of G, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices in G, i.e.

$$\mathrm{diam}(G) = \max_{u,v \in V(G)} d(u, v).$$

Consider the set of all regular graphs of degree r and girth g, and a graph from this set of minimal order is called an (r, g)-*cage*. If $g = 2d + 1$, an (r, g)-cage with $n_0(r, g)$ vertices is called an (r, d)-*Moore graph*, where $n_0(r, g) = 1 + r + r(r-1) + \cdots + r(r-1)^{(g-3)/2}$ [2].

For a set $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by contracting the edges of X and deleting all resulting loops. When H is a connected subgraph of G, we use G/H for $G/E(H)$, and let v_H be the new vertex obtained by contracting H in G/H. The vertex v_H is called the *contraction image of H in G/H*.

In this paper, we first study unavoidable subgraphs of non-reduced graphs of diameter two. In Section 4, we characterize graphs of diameter two in terms of collapsible graphs. In Section 5, we introduce a concept, L-collapsible, to study the

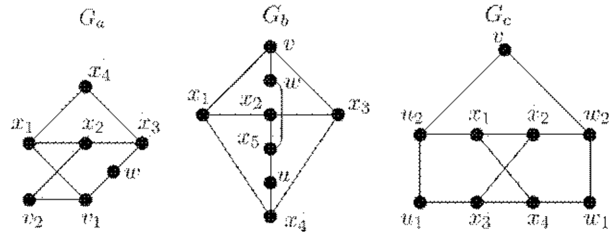

Fig. 1.

reduction of line graphs. Then we will investigate the relationship between the line graph of reduction of a graph and the reduction of the line graph. We will discuss some applications in the last section. In the following, we discuss Catlin's reduction method first.

2. Catlin's reduction method

In [3], Catlin defines the collapsible graphs. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, G has a subgraph Γ such that $G - E(\Gamma)$ is connected and the set of odd-degree vertices of Γ is R. Let R be the set of all odd degree vertices of G. If G is collapsible, then $G - E(\Gamma)$ is a connected Eulerian subgraph of G. Thus, a collapsible graph is a connected and has a spanning Eulerian subgraph. The graph K_1 is regarded as both collapsible and having spanning Eulerian subgraph. In [3], Catlin proved:

Collapsible Partition Theorem (*Catlin, [3]*). Every graph G has a unique collection of vertex disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$ such that $V(G) = V(H_1) \cup V(H_2) \cup \cdots \cup V(H_c)$.

Thus, every vertex of a graph G is in a unique maximal collapsible subgraph of G. Contracting the subgraphs $H_1, H_2, \ldots, H_c$ to distinct vertices, we obtain a new graph from G, denoted by G'. This new graph is called the reduction of G. Let H be a maximal collapsible subgraph of G and let $v \in V(G')$ be the vertex obtained by contracting H. Then H is called the preimage of v and v is called the image of H in G'. If $|V(H)| = 1$, then v is a trivial vertex in G' and H is called a trivial collapsible subgraph of G. A graph is called *reduced* if it contains no non-trivial collapsible subgraphs. It is easy to see that cycles C_3 and C_2 are collapsible, and any C_t with $t \geq 4$ is a reduced graph.

Theorem A (*Catlin [3]*). *Let G be a graph, and let H be a collapsible subgraph of G. Then each of the following holds:*

(a) *G has a spanning Eulerian subgraph if and only if G/H has a spanning Eulerian subgraph.*
(b) *G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.*
(c) *If G is a reduced graph, then $\delta(G) \leq 3$ and G is simple and K_3-free.*

In [4], Catlin introduced a reduction method to handle reduced 4-cycles. Let G be a graph containing a 4-cycle $xyzwx$, and define $E = \{xy, yz, zw, wx\}$. Let G/π be the graph obtained from $G \setminus E$ by identifying x and z to form a vertex v_1, by identifying w and y to form a vertex v_2, and by adding a new edge v_1v_2. The following theorem shows the usefulness of this technique.

Theorem B (*Catlin [4]*). *Let G be a graph and let G/π be a graph defined above. Then each of the following holds:*

(a) *If G/π is collapsible then G is collapsible.*
(b) *If G/π has a spanning Eulerian subgraph, then G has a spanning Eulerian subgraph.*

Examples. The following K_3-free graphs are all collapsible.

(i) $W_1 = K_{3,3} - e$, where e is an edge in $K_{3,3}$.
(ii) G_a a graph obtained from W_1 by subdividing an edge of W_1 that is incident with two vertices of degree three in W_1 (see Fig. 1).
(iii) G_b and G_c are defined above in Fig. 1.

One can easily show that graphs in Fig. 1 are all collapsible. Fig. 2 illustrates the application of Theorem B(a) and Theorem A(b) to graph G_b. Since $\Phi = C_2$, $H = K_3$, and $((G/\pi)/\Phi)/H = K_3$ are all collapsible, by Theorem A(b), G/π is collapsible, and so by Theorem B(a), $G = G_b$ is collapsible.

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G contains none of the subgraphs in $\mathcal{F}$. In this paper, we define

$$\mathcal{Z} = \{K_3, K_{3,3} - e, G_a, G_b, G_c\}.$$

Let m, l be two positive integers. Let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,l}$ be two complete bipartite graphs. Let x, v be two non-adjacent vertices of degree m in H_1, and let u, y be two non-adjacent vertices of degree l in H_2. Let $S_{m,l}$ denote the graph

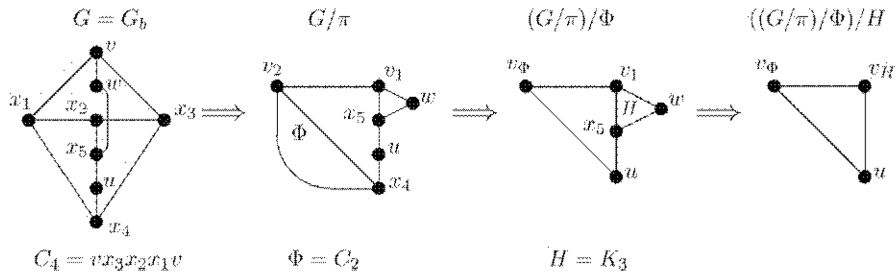

Fig. 2.

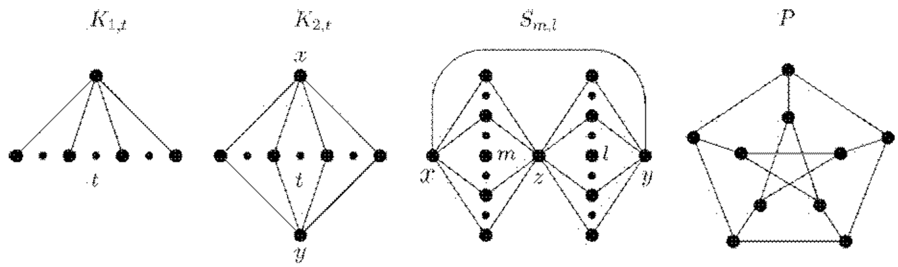

Fig. 3.

obtained from H_1 and H_2 by identifying v and u, and by connecting x and y with a new edge xy (see Fig. 3). Obviously, $S_{1,1} \cong C_5$, the 5-cycle. It is easy to check that the following graphs have no non-trivial collapsible subgraphs, where $t \geq 2$ and P is the Petersen graph.

We should use $\mathcal{L}$ to denote the set of graphs defined in Fig. 3, i.e.,

$$\mathcal{L} = \{K_{1,t}, K_{2,t}, S_{m,l}, P\}.$$

Lai in [10] showed that if G is a reduced graph of diameter two then $G \in \mathcal{L}$.

3. $\mathcal{Z}$-free graphs

Here is our main result in this section.

Theorem 1. *Let G be a graph of diameter two. If G is a $\mathcal{Z}$-free graph, then either $G \in \mathcal{L}$, or G is the Hoffman and Singleton graph (see [9]) or G is a $(57, 2)$-Moore graph (if it exists).*

We shall use the following theorems.

Theorem C (*Singleton [11]*). *Every graph with diameter d and girth $2d + 1$ is regular.*

Theorem D (*Hoffman and Singleton [9]*). *Suppose there is an r-regular graph G of order $n = r^2 + 1$ and diameter 2 (and so girth 5). Then $r = 2, 3, 7$ or 57.*

Remark. It is known that for $r = 2, 3$, and 7, there are unique $(r, 2)$-Moore graphs (r-regular graphs of order $r^2 + 1$ and diameter 2). In particular, for $r = 2$ it is a pentagon C_5 and for $r = 3$ it is the Petersen graph. The graph with $r = 7$, called the Hoffman and Singleton graph, was constructed and proved unique by Hoffman and Singleton (1960) [9] (also see [2], page 189). However, it is not known whether there is a $(57, 2)$-Moore graph.

The following simple results will be needed.

Lemma 1. *Let G be a K_3-free graph and* $\operatorname{diam}(G) = 2$ *and $\delta(G) = r$.*

(a) *If $\delta(G) = r = 1$, then $G \cong K_{1,t}$ for some $t \geq 2$.*
(b) *If G has girth 5, then G is an r-regular graph and $|V(G)| = r^2 + 1$.*

Proof. (a) is obvious. For (b), since G has diameter two, by Theorem C, and $\delta(G) = r$, G is an r-regular graph. Let v be a vertex in G, then $d(v) = r$. Define

$$S_i = \{u \in V(G) \mid d(u, v) = i\}.$$

Since G is r-regular and has girth 5, and $\mathrm{diam}(G) = 2$, we have

$$|S_1| = r, \qquad |S_2| = r(r-1), \quad \text{and} \quad |S_i| = 0 \quad \text{for } i \geq 3.$$

Therefore, $|V(G)| = 1 + r + r(r-1) = r^2 + 1$. Lemma 1 is proved. $\sqcup$

Lemma 2. *If G is a simple and K_3-free graph with $\mathrm{diam}(G) = 2$, then every path L with length 3 in G lies in a 4-cycle or a 5-cycle.*

Proof. Let $L = xuvy$ be a path of length 3. Since $\mathrm{diam}(G) = 2$, $d(x, y) \leq 2$. If $d(x, y) = 1$, then L lies in a 4-cycle. If $d(x, y) = 2$, then there is a (x, y)-path, say xwy, of length 2. Since G is K_3-free, neither $w = u$ nor $w = v$. Hence L lies in a 5-cycle. $\sqcup$

Lemma 3. *Let H be a simple graph with $\mathrm{diam}(H) = 2$ and girth 5, and $\delta(H) \geq 3$. Let v be a vertex not in H. Let G be a graph with $V(G) = V(H) \cup \{v\}$ and $E(H) \subseteq E(G)$. If v is adjacent with only two distinct vertices (of H), then either G has a K_3 subgraph or $\mathrm{diam}(G) = 3$.*

Proof. Assume G is K_3-free. Let x and y be the two distinct vertices of H which are adjacent with v. Since G is K_3-free, $xy \notin E(H)$. Since $\mathrm{diam}(H) = 2$, there is a vertex, say z, in $V(H)$, such that $xz, zy \in E(H)$. Since H has girth 5, $(N(x) \setminus \{z\}) \cap (N(y) \setminus \{z\}) = \varnothing$. Let w_1 be a vertex in $N(x) \setminus \{z\}$. Since $L = w_1xzy$ is path of length 3 and $\mathrm{diam}(H) = 2$, by Lemma 2, L must be in a 5-cycle. Hence, there is a vertex, say w_2, in $N(y) \setminus \{z\}$, such that $w_1w_2 \in E(H)$ and $w_2y \in E(H)$. Since $d(w_1) \geq 3$, there is a vertex, say u, in $N(w_1) \setminus \{x, w_2\}$. Since G is K_3-free, $ux \notin E(G)$. Since H has girth 5, $uy \notin E(H)$, otherwise, uw_1w_2yu is a 4-cycle in H, a contradiction. This shows that $d_G(u, v) = 3$, and so $\mathrm{diam}(G) = 3$. The proof is complete. $\square$

Lemma 4. *Let G be a simple and K_3-free graph with $\mathrm{diam}(G) = 2$ and girth 4. If G does not have a 4-cycle that contains a vertex of degree 2, then G contains a subgraph isomorphic to a subgraph in $\mathcal{Z}$.*

Proof. Since G has girth 4, G has a 4-cycle $C = x_1x_2x_3x_4x_1$. Then by the assumption in the lemma, $d(x_i) \geq 3$ $(1 \leq i \leq 4)$. We will divide the proof into three cases.

Case 1. There is another 4-cycle H with $|E(H) \cap E(C)| = 2$.

Without loss of generality, since G is K_3-free we may assume that $E(C) \cap E(H) = \{x_1x_2, x_2x_3\}$, and let v be the other vertex in H, i.e. $H = x_1x_2x_3vx_1$. Then G has a $K_{2,3}$ subgraph, say Φ, formed by these two 4-cycles. Then $d_\Phi(x_1) = d_\Phi(x_3) = 3$ and $d_\Phi(x_2) = d_\Phi(x_4) = d_\Phi(v) = 2$. Since $d(x_2) \geq 3$, $N(x_2) \setminus \{x_1, x_3\} \neq \varnothing$. Let x_5 be a vertex in $N(x_2) \setminus \{x_1, x_3\}$. Note that $L_1 = x_5x_2x_1x_4$ and $L_2 = x_5x_2x_1v$ are two paths with length 3 in G. By Lemma 2, L_1 and L_2 must be in a 4-cycle or a 5-cycle. If L_1 is in a 4-cycle, then $x_5x_4 \in E(G)$. Therefore, $G[E(H) \cup E(C) \cup \{x_5x_2, x_5x_4\}] \cong K_{3,3} - e$. Similarly, if L_2 is in a 4-cycle, then G contains $K_{3,3} - e$ subgraph. We are done in this case if one of L_i is in a 4-cycle.

Next we assume that both L_i's are not in a 4-cycle, and so they must be in 5-cycles. Let $x_5x_2x_1x_4ux_5$ be a 5-cycle containing L_1, and let $x_5x_2x_1vwx_5$ be a 5-cycle containing L_2. Therefore, $G[E(C) \cup E(H) \cup \{x_5x_2, ux_5, x_4u, x_5w, wv\}] \cong G_b$. We are done in this case.

Case 2. There is another 4-cycle H with $|E(H) \cap E(C)| = 1$.

Let $E(C) \cap E(H) = \{x_1x_2\}$. Let v_1 and v_2 be the other two vertices in H such that $H = v_1x_1x_2v_2v_1$. Note that $L = v_1v_2x_2x_3$ is a path with length 3 in G. By Lemma 2, L must be in a 4-cycle or a 5-cycle. If L is in a 4-cycle, this is the same as Case 1. So we may assume that L is in a 5-cycle. Then there is a vertex, say w such that $v_1w \in E(G)$ and $wx_3 \in E(G)$. Therefore, $G[E(C) \cup E(H) \cup \{v_1w, wx_3\}] \cong G_a$. This shows that the statement holds.

Case 3. There is no 4-cycle in G which shares an edge of C.

By the assumption, no 4-cycle shares an edge with $C = x_1x_2x_3x_4x_1$. Since $d(x_3) \geq 3$, there is a vertex $u_1 \in N(x_3) \setminus \{x_2, x_4\}$. Consider the path $L_1 = x_1x_2x_3u_1$. By Lemma 2 this path lies in a 5-cycle, say $H_1 = x_1x_2x_3u_1u_2x_1$. Similarly, as $d(x_4) \geq 3$, there is a vertex $w_1 \in N(x_4) - \{x_1, x_3\}$ and the path $L_2 = x_2x_3x_4w_1$ lies in a 5-cycle, say $H_2 = x_2x_3x_4w_1w_2x_2$. Since $L = u_2x_1x_2w_2$ is a path of length 3, by Lemma 2 again L must be in a 4-cycle or a 5-cycle. Since x_1x_2 in C cannot be an edge of another 4-cycle, L must be in a 5-cycle. Let v be a vertex such that $vu_2x_1x_2w_2v$ be a 5-cycle containing L. Therefore, $G[E(C) \cup \{x_1u_2, u_2u_1, u_1x_3, x_4w_1, w_1w_2, w_1x_2, vu_2, w_2v\}] \cong G_c$. Case 3 is proved. The proof of Lemma 4 is complete. $\square$

Proof of Theorem 1. If $\delta(G) = 1$ then by Lemma 1, $G \cong K_{1,t}$ for some $t \geq 2$. We are done in this case. In the following we assume that $\delta(G) \geq 2$. By way of contradiction, let G be a counterexample with smallest order. Since G is $\mathcal{Z}$-free, G is K_3-free. Since G has diameter two, G has girth either 4 or 5.

Case 1. G contains 4-cycles.

If none of the 4-cycles in G contains a vertex of degree 2, then by Lemma 4, G has a subgraph isomorphic to a member in $\mathcal{Z}$, a contradiction. Therefore, we may assume that G has a 4-cycle in which one of the vertex, say v, has degree 2 in G. Let $H = G - v$. Then since G is $\mathcal{Z}$-free, H is $\mathcal{Z}$-free and also $\mathrm{diam}(H) = \mathrm{diam}(G) = 2$. Since $|V(H)| = |V(G)| - 1$, and G is a smallest counterexample, the theorem holds for $H = G - v$.

If $G - v \cong K_{1,t}$, then since $\delta(G) = 2$, $t = 2$ and G is a 4-cycle $K_{2,2}$.

If $G - v \cong K_{2,t}$, then since G is $\mathcal{Z}$-free and has $\mathrm{diam}(G) = 2$, $G \cong K_{2,t+1}$.

If $G - v \cong S_{m,l}$, then $G \cong S_{m+1,l}$ or $G \cong S_{m,l+1}$, since $\mathrm{diam}(G) = 2$ and G is $\mathcal{Z}$-free.

If $G - v \cong P$, the Petersen graph, or $G - v \cong$ the Hoffman and Singleton graph, or a $(57, 2)$-Moore graph (if it exists), then since $\delta(G - v) \geq 3$ and $G - v$ has girth 5, by Lemma 3, either G has a K_3 subgraph or $\mathrm{diam}(G) = 3$, a contradiction.

Since in any case, a contradiction arises. This shows that Case 1 is impossible and G contains no 4-cycles.

Case 2. The girth G is 5.

Since $\operatorname{diam}(G) = 2$, by Lemma 1 G is an r-regular graph with order $r^2 + 1$. By Theorem D, $r = 2, 3, 7$, or 57. It follows from the remark after Theorem D, we know that G can not be a counterexample to Theorem 1 in this case. Theorem 1 is proved. □

Corollary 1 *(H.-J. Lai [10]). Let G be a reduced graph with diameter two. Then $G \in \mathcal{L} = \{K_{1,t}, K_{2,t}, S_{m,l}, P\}$.*

Proof. Since G is a reduced graph, G is $\mathbb{Z}$-free. By Theorem A, $\delta(G) \leq 3$ and so G is neither the Hoffman and Singleton graph nor a $(52, 2)$-Moore graph. By Theorem 1, $G \in \mathcal{L}$. □

4. Collapsible graphs with diameter two

Let G be a graph. Let H be a subgraph of G. Let $A(G, H)$ be the set of vertices in H which are adjacent to some vertex not in $V(H)$, i.e.,

$$A(G, H) = \{v \in V(H) \mid N(v) \setminus V(H) \neq \varnothing\}.$$

The set of edges in $E(G) \setminus E(H)$ incident with a vertex in $A(G, H)$ is denoted by

$$E(G, H) = \{uv \in E(G) \setminus E(H) \mid u \in (V(G) \setminus V(H)) \text{ and } v \in A(G, H)\}.$$

Let v_H be the vertex in G/H obtained by contracting H in G. Obviously,

$$d_{G/H}(v_H) = |E(G, H)| \geq |A(G, H)|. \tag{1}$$

Proposition 1. *Suppose G is a graph with diameter 2. Let H be a maximal collapsible subgraph of G. Let $x, y \in V(G) \setminus V(H)$. Suppose $xu, yv \in E(G, H)$ for some $u, v \in V(H)$, then $x \neq y$ and $xy \notin E(G)$.*

Proof. Suppose not, let $H_1 = G[V(H) \cup \{x, y\}]$. Then $H_1/H \cong K_3$ or $H_1/H \cong C_2$. In either case, by Theorem A(b), H_1 is collapsible. It is a contradiction. □

Lemma 5. *Let G be a graph with diameter two. Let H be a maximal collapsible subgraph of G. Then for each vertex v in $A(G, H)$, $N_H(v) = \{u \in V(H) \mid uv \in E(H)\} = V(H) \setminus \{v\}$.*

Proof. Let $x \in V(H) \setminus \{v\}$. Let z be a vertex in $V(G) \setminus V(H)$ adjacent with v. Since $d(x, z) \leq 2$, $x \in V(H)$, either $zx \in E(G)$ or there is a vertex y in G such that $zy, yx \in E(G)$. By Proposition 1, only the last case holds with $y = v$. Thus, $x \in N_H(v)$ and hence $N_H(v) = V(H) \setminus \{v\}$. □

Lemma 6. *Let G be a non-collapsible graph with diameter two. Let H be a maximal collapsible subgraph of G. If $G/H \ncong K_{1,t}$ for some $t \geq 1$, then*

$$|E(G, H)| \geq |A(G, H)| = |V(H)|.$$

Proof. By (1), we only need to show that $|A(G, H)| = |V(H)|$. By way of contradiction, suppose that $|A(G, H)| < |V(H)|$. Then there is a vertex (say x) in $V(H) \setminus A(G, H)$. Since G is not collapsible, $|V(G/H)| > 1$. Since $G/H \ncong K_{1,t}$, by Proposition 1 there is a vertex y in $V(G) \setminus V(H)$ such that y is not adjacent to any vertex in $A(G, H)$. Therefore, $d(x, y) \geq 3$, a contradiction. This shows that $|A(G, H)| = |V(H)|$. The lemma is proved. □

Lemma 7. *Let G be a simple non-collapsible graph with diameter two and $\delta(G) \geq 2$. If H is a maximal non-trivial collapsible subgraph in G, then H is complete.*

Proof. Since $\delta(G) \geq 2$, $G/H \ncong K_{1,t}$. Since G is simple, by Lemmas 5 and 6, H is complete. □

Lemma 8. *Let G be a non-collapsible graph with diameter two. Then G has at most one non-trivial maximal collapsible subgraph.*

Proof. Suppose that G contains two vertex disjointed non-trivial maximal collapsible subgraphs H and K. Then both H and K have order at least two. Since G is connected, there exists a path joining a vertex of H to a vertex of K. Since $\operatorname{diam}(G) = 2$ and $|V(H)| \geq 2$ and $|V(K)| \geq 2$, there are at least two such paths having length at most two. Choose any two such paths, say P_1 and P_2. If either both have length one or exactly one has length two, then $H \cup K \cup P_1 \cup P_2$ is collapsible, a contradiction. Thus, we may assume that no path between H and K has length less than 2, and so P_1 and P_2 have both length two. In all cases, the assumption on the diameter implies that there is an edge $\{f\}$ joining the middle vertices of P_1 and P_2. The graph $G_1 = H \cup K \cup P_1 \cup P_2 \cup \{f\}$ is collapsible. Indeed, $G_1/H/K \cong K_4 - e$. This contradicts the assumption that H and K are two maximal collapsible subgraphs. Lemma 8 is proved. □

Lemma 9. *Let G be a graph of $\mathrm{diam}(G) = 2$. If $\delta(G) = 1$, then either $G \cong K_{1,t}$ for some integer t or G contains a maximal collapsible subgraph H having the following properties:*

(a) *Each edge in H is in a K_3 subgraph,*
(b) *$G/H \cong K_{1,t}$ for some $t \geq 1$, and*
(c) *v_H, the contraction image of H in $K_{1,t}$, has degree t in $K_{1,t}$.*
(d) *If $t \geq 2$, $A(G, H)$ has only one vertex, say v, and all the edges in $E(G) \setminus E(H)$ are incident with v in G.*

Proof. Suppose that $G \ncong K_{1,t}$. Since $\delta(G) = 1$, by Lemma 1(a) G contains a K_3 subgraph. Let H be a maximal collapsible subgraph in G. Since $\delta(G) = 1$, G is not collapsible and so $G \neq H$. Since $\mathrm{diam}(G) = 2$, by Lemma 8 H is the only non-trivial collapsible subgraph of G. Let $G_1 = G/H$. If $\mathrm{diam}(G_1) = 1$, then $G_1 \cong K_{1,1}$, and so Lemma 9 holds in this case. Because, by Lemma 6, if $G_1 \neq K_{1,t}$, then $A(G, H) = V(H)$ is a clique. If $\mathrm{diam}(G_1) = 2$, since G has no other non-trivial collapsible subgraphs, and $\delta(G_1) = \delta(G) = 1$, by Lemma 1(a), $G_1 \cong K_{1,t}$ for some $t \geq 2$. Let v_H be the vertex in $G_1 \cong K_{1,t}$ obtained by contracting H. If $d(v_H) = 1$ in G_1, then since $t \geq 2$ and $|V(H)| > 1$, G will have diameter greater than two, a contradiction. Therefore, $d(v_H) = t$ in $G_1 \cong K_{1,t}$. Suppose that there are two vertices, say x and y, in $V(G_1)$ which are adjacent with two distinct vertices in $V(H)$, then $d_G(x, y) \geq 3$, a contradiction. This shows that in each case $A(G, H)$ can have only one vertex, say v, and all the edges in $E(G) \setminus E(H)$ must be incident with v. By Lemma 5, we know that $N_H(v) = V(H) \setminus \{v\}$. This implies that each edge of H must be in a K_3 since collapsible graphs are 2-edge-connected. The proof is complete. □

Theorem 2. *Let G be a simple graph with diameter two. Then exactly one of the following holds:*

(a) *G is collapsible;*
(b) *G has a maximal collapsible subgraph H in which every edge of H is in a K_3 subgraph and such that $G/H \cong K_{1,t}$ for some $t \geq 1$, and $d_{G/H}(v_H) = \Delta(K_{1,t}) = t$, and G has a vertex v such that $N(v) = V(G) \setminus \{v\}$;*
(c) *G has a complete subgraph H such that $G/H \cong K_{2,t}$ and $t \geq |V(H)|$, and $d_{G/H}(v_H) = \Delta(K_{2,t}) = t$, and each vertex in H is incident with an edge that is incident with v_H in G/H;*
(d) *$G \cong S_{m,l}$;*
(e) *$G \cong P$, the Petersen graph.*

Proof. By Theorem 1, we know that if G is $\mathcal{Z}$-free, then either $G \in \mathcal{L}$, or G is the Hoffman and Singleton graph or G is (57,2)-Moore graph (if it exists).

If G is the Hoffman and Singleton graph or a graph with girth 5 and $\delta(G) = 57$ (if it exists), then by Theorem A(c), G is not reduced. Therefore, G has a maximal collapsible subgraph H. Since in this case G is K_3-free, H is not complete, then by Lemma 7, G is collapsible.

If $G \in \mathcal{L}$, we are done by choosing $H = K_1$.

Next we only need to consider the case that G is not $\mathcal{Z}$-free, and so G has a non-trivial maximal collapsible subgraph H with

$$|V(H)| \geq 3. \tag{2}$$

If G is collapsible then this is case (a), and so we may assume that G is not collapsible.

If $\delta(G) = 1$, then by Lemma 9, Theorem 2(b) is proved.

If $\delta(G) \geq 2$, since G is not collapsible, by Lemma 8 H is the unique non-trivial collapsible and by Lemma 7 H is a complete graph, and $\mathrm{diam}(G/H) = 2$. Let $G_1 = G/H$. Then G_1 is reduced. Therefore, G_1 is a $\mathcal{Z}$-free graph. Since $\mathrm{diam}(G_1) = 2$ and $\delta(G) \geq 2$, we have $\delta(G_1) \geq 2$. Then G_1 is isomorphic to one of the graphs $K_{2,t}$, $S_{m,l}$ and P, where $t \geq 2$. By Lemma 6 and (2), $|E(G, H)| \geq |V(H)| \geq 3$, and so by (1) $d_{G_1}(v_H) \geq 3$.

If $G_1 \cong K_{2,t}$ for some $t \geq 2$, then since $d_{G_1}(v_H) \geq 3$,

$$\Delta(K_{2,t}) = d_{G_1}(v_H) = t \geq 3.$$

By (1) and Lemma 6, we have $t = d_{G_1}(v_H) = |E(G, H)| \geq |V(H)|$. Since G has diameter two, it is easy to check that each vertex in H is incident with an edge that is incident with v_H in G/H. Theorem 2(c) holds.

To complete the proof, we only need to show that if $|V(H)| \geq 3$, it is impossible to have $G_1 = G/H \cong S_{m,l}$ or P.

If $G_1 \cong S_{m,l}$, since $d_{G_1}(v_H) \geq 3$, $v_H \in \{x, y, z\}$. Using the fact that $|E(G, H)| \geq |A(G, H)| = |V(H)| \geq 3$ (Lemma 6), and the structure of $S_{m,l}$, it is easy to check that G has diameter at least three, a contradiction. This case is impossible.

If $G_1 \cong P$, the Petersen graph, then $d_{G_1}(v_H) = 3$, and so $|E(G, H)| = 3$. By Lemma 6, and (2), $H \cong K_3$. It is easy to check that graph G has diameter 3, a contradiction. □

Theorem 3. *Let G be a K_3-free simple graph with diameter two. Then either $G \in \mathcal{L} = \{K_{1,t}, K_{2,t}, S_{m,l}, P\}$ or G is collapsible.*

Proof. It follows from Theorem 2. □

5. Reductions on line graphs and diameters

In this section, we extend Catlin's reduction method to study the reductions of line graphs. For any graphs G and H, we denote $H \preceq G$ if $H \cong G/X$ for some $X \subseteq E(G)$ and denote $D_i(G)$ to be the set of vertices of degree i in G with $i \geq 1$ and $D_3^+(G) = \bigcup_{i \geq 3} D_i(G)$.

Lemma 10. *Let $H \preceq G$. Then each of the following holds,*

(i) $\mathrm{diam}(H) \leq \mathrm{diam}(G)$;
(ii) *if H is reduced, then $H \preceq G'$, where G' is the reduction of G.*

Proof. Lemma 10(i) follows from the definition of diameter. Next we will prove Lemma 10(ii). By the definition of $H \preceq G$, $H \cong G/X$ for some $X \subseteq E(G)$. Let $G_1 = G[X]$ denote the edge-induced subgraph of G with edge set X.

Claim. If $H = G/X$ is reduced, then G_1 contains all non-trivial collapsible subgraphs of G.

Let H_0 be a collapsible subgraph of G with $M = E(H_0)$. If $M \not\subseteq X$, then $M \setminus X \neq \varnothing$. Since collapsible graphs are closed under contraction, $H_0/(M \cap X)$ is a non-trivial collapsible of $H = G/X$, contrary to the assumption that H is reduced. The claim is proved.

By the claim and the assumption that $H = G/X$ is reduced, we may assume that $X = X_0 \cup X_1$, where X_0 is the union of the edge sets of the maximal non-trivial collapsible subgraphs of G and $X_1 = X \setminus X_0$. Then $H = G/X = (G/X_0)/X_1 = G'/X_1$, and so $H \preceq G'$. Lemma 10(ii) is proved. □

For a graph G, let J be a subgraph of G. J is called a *L-collapsible* if $L(J)$ is a maximal collapsible subgraph in $L(G)$. For a graph J, define

$$\mathcal{E}_1(J) = \{uv \in E(J) \mid \text{ either } d_J(u) = 1 \text{ or } d_J(v) = 1\} \quad \text{and} \quad J^- = G[V(J) \setminus D_1(J)].$$

Therefore,

$$V(L(J)) = E(J) = E(J^-) \cup \mathcal{E}_1(J). \tag{3}$$

By Catlin's Collapsible Partition Theorem (Section 2), $L(G)$, the line graph of G, has a unique collection of vertex disjoint maximal collapsible subgraphs, denoted by $\mathcal{CL}(L(G)) = \{L_1, L_2, \ldots, L_c\}$ such that $V(L(G)) = V(L_1) \cup V(L_2) \cup \cdots \cup V(L_c)$. Therefore, since $L(G)$ is the line graph of G, G has a unique collection of vertex disjoint L-collapsible subgraphs, denoted by $\mathcal{CJ}(G) = \{J_1, J_2, \ldots, J_c\}$ such that $L_i = L(J_i)(1 \leq i \leq c)$ and $E(J_i) \cap E(J_j) = \emptyset\ (i \neq j)$. We call J_i the preimage of L_i in G and denoted $J_i = L^{-1}(L_i)$. Therefore, for a graph G, the following collections corresponding to the collection $\mathcal{CJ}(G)$ are unique:

$$\mathcal{J}(G) = \{J_1^-, J_2^-, \ldots, J_c^-\};$$
$$\mathcal{EE}(G) = \{\mathcal{E}_1(J_1), \mathcal{E}_1(J_2), \ldots, \mathcal{E}_1(J_c)\}$$

where $J_i \in \mathcal{CJ}(G)$ and so $J_i^- \cap J_j^- = \emptyset$ and $\mathcal{E}_1(J_i) \cap \mathcal{E}_1(J_j) = \emptyset$ for $i \neq j$.

By contracting subgraphs $J_1^-, J_2^-, \ldots, J_c^-$ in G to distinct vertices, we obtain a new graph from G, denoted by $\tilde{G}$. Let $X = E(J_1^-) \cup E(J_2^-) \cup \cdots \cup E(J_c^-)$. Then

$$\tilde{G} = G/X = (\cdots((G/J_1^-)/J_2^-/)\cdots)/J_c^-.$$

For a subgraph $J^- \in \mathcal{J}(G)$, let v_J be the vertex in $\tilde{G}$ obtained by contracting J^-. Let $\mathcal{E}_1(J)$ be the edge subset in $\mathcal{EE}(G)$ corresponding to J^-. Then $\mathcal{E}_1(J)$ is a vertex subset in the line graph $L(G)$. Since each edge in $\mathcal{E}_1(J)$ is incident with a vertex in $V(J^-)$, each edge in $\mathcal{E}_1(J)$ is incident with v_J after contracting J^- in G. Thus, the vertex subset $\mathcal{E}_1(J)$ in the line graph induces a connected subgraph in $L(\tilde{G})$.

For each $\mathcal{E}_1(J_i) \in \mathcal{EE}(G)$, let Y_i be the subgraph in $L(\tilde{G})$ induced by $\mathcal{E}_1(J_i)$, i.e., $Y_i = L(\tilde{G})[\mathcal{E}_1(J_i)]$. Therefore, $\{Y_1, Y_2, \ldots, Y_c\}$ is a collection of vertex disjoint connected subgraphs in $L(\tilde{G})$. Contracting $Y_1, Y_2, \ldots, Y_c$ into distinct vertices, we obtained a new graph from $L(\tilde{G})$, denoted by $L(\tilde{G})^*$. Figs. 4 and 5 illustrate the relationship among G, $L(G)$, $L(\tilde{G})$, $L(\tilde{G})'$ and $L(\tilde{G})^*$.

From Fig. 4, we can see that $L(G)' \cong K_{1,3}$ and

$$\mathcal{J}(G) = \{J_1^-, J_2^-, J_3^-, J_4^-\} = \{G[x], G[y], G[e_7, e_8, e_9], G[e_{12}, e_{13}, e_{14}]\};$$
$$\mathcal{EE}(G) = \{\mathcal{E}_1(J_1), \mathcal{E}_1(J_2), \mathcal{E}_1(J_3), \mathcal{E}_1(J_4)\} = \{\{e_1, e_2, e_3\}, \{e_4, e_5, e_{10}\}, \{e_6\}, \{e_{11}, e_{15}, e_{16}\}\}.$$

Theorem 4. *Let Γ be a maximal non-trivial collapsible subgraph of $L(G)$ and let $J = L^{-1}(\Gamma)$. Let $J^- = G[V(J) \setminus D_1(J)]$ and let $Y = L(G/J^-)[\mathcal{E}_1(J)]$. Then $L(G)/\Gamma \cong L(G/J^-)/Y$ and so $L(G)' \cong L(\tilde{G})^*$.*

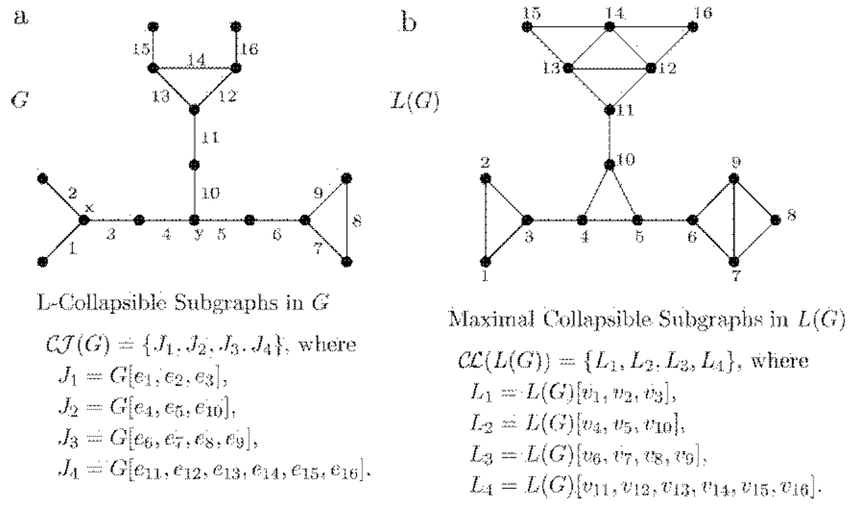

Fig. 4.

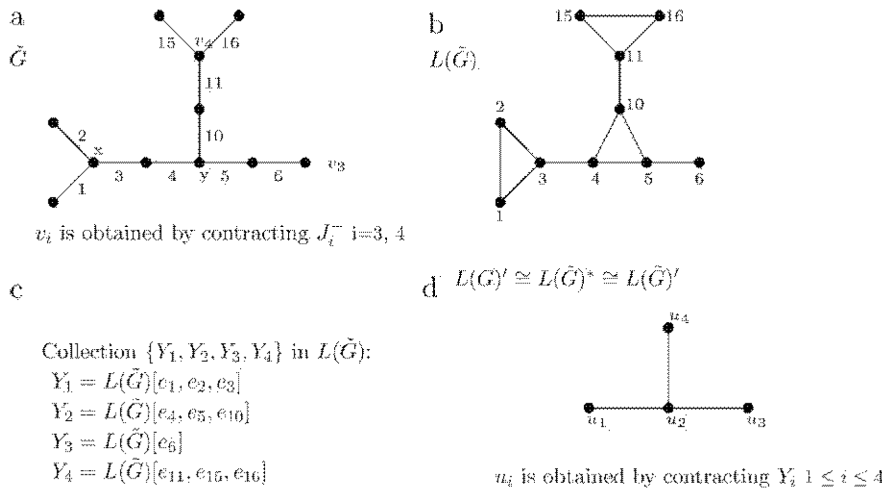

Fig. 5.

Proof. Let v_Γ be the vertex obtained by contracting Γ in $L(G)$. Let v_y be the vertex obtained by contacting Y in $L(G/J^-)$. Since $\Gamma = L(J)$, $V(\Gamma) = E(J)$. By (3), $V(\Gamma) = E(J^-) \cup \mathcal{E}_1(J)$. By the definition of a line graph and the definition of contractions above, we have

$$V(L(G)/\Gamma) = (E(G) \setminus V(\Gamma)) \cup \{v_\Gamma\} \tag{4}$$

and

$$\begin{aligned} V(L(G/J^-)/Y) &= (E(G) \setminus E(J^-)) \cup \{v_y\} \setminus \mathcal{E}_1(J) \\ &= (E(G) \setminus (E(J^-) \cup \mathcal{E}_1(J))) \cup \{v_y\} \\ &= (E(G) \setminus V(\Gamma)) \cup \{v_y\}. \end{aligned} \tag{5}$$

Thus, by (4) and (5), the mapping $\Phi : V(L(G)/\Gamma) \longrightarrow V(L(G/J^-)/Y)$ defined by

$$\Phi(e) = \begin{cases} e & \text{if } e \neq v_\Gamma; \\ v_y & \text{if } e = v_\Gamma \end{cases}$$

is a bijection. This shows that

$$L(G)/\Gamma \cong L(G/J^-)/Y. \tag{6}$$

By the procedures we defined $L(G)'$ and $L(\tilde{G})^*$, and repeatedly applying (6), we have $L(G)' \cong L(\tilde{G})^*$. Theorem 4 is proved. □

Proposition 2. *If $G \neq K_1$ is a collapsible graph, then $L(G)$ is collapsible.*

Proof. Let $L(G)'$ be the reduction of $L(G)$. Suppose that $L(G)$ is not collapsible. Then by Theorem A(c), $\delta(L(G)') \leq 3$ and so there is a vertex of degree at most 3 in $L(G)'$. Therefore, $L(G)$ has an edge cut E of size $|E| = \delta(L(G)') \leq 3$ and no edge in E lies in a 3-cycle of $L(G)$. By the definition of the line graph of graph G, an edge in $L(G)$ that is not in a 3-cycle is obtained from two edges in G that are incident with a common vertex of degree 2. Since E is an edge cut in $L(G)$, those degree 2 vertices in G corresponding to the edges in E forms a vertex cut in G. Thus, G has a vertex cut U with $|U| = |E| = \delta(L(G)') \leq 3$ and every vertex in U has degree 2 in G. If $|U|$ is even, let $S = U$. If $|U|$ is odd, let $S = U \cup \{v\}$ where v is a vertex in $V(G) - U$. Then S is an even subset of $V(G)$. However, it is impossible for graph G to have a subgraph Γ such that $G - E(\Gamma)$ is connected and the set of odd-degree vertices of Γ is S. This is contrary to that G is collapsible. Thus, $L(G)$ is collapsible. $\square$

Theorem 5. $L(G)' \cong L(\tilde{G})^* \preceq L(G')$.

Proof. Let $H \subseteq G$ be a maximal collapsible subgraph with $E(H_1) \neq \varnothing$. Then by Proposition 2, $L(H)$ is a collapsible subgraph in $L(G)$. Thus, there is a maximal collapsible subgraph Γ in $L(G)$ containing $L(H)$ as a subgraph. Let $J = L^{-1}(\Gamma)$. Then J is a L-collapsible subgraph in G such that $H \subseteq J$. Since collapsible graphs are 2-edge-connected, H is 2-edge-connected. Hence, $H \subseteq J^-$. Let $Y = L(G/J^-)[\mathcal{E}_1(J)]$. Therefore, by Theorem 4, $L(G)/\Gamma = L(G/J^-)/Y$. Let $F = E(J^-) - E(H)$. Let $X = L(G/H)[F \cup \mathcal{E}_1(J)]$. Therefore, $L(G/J^-)/Y \cong L(G/H)/X$. Thus, $L(G/J^-)/Y \preceq L(G/H)$.

Since the maximal collapsible subgraphs of a graph are vertex disjoint, repeatedly applying the argument above, we have $L(\tilde{G})^* \preceq L(G')$. The proof is completed. $\square$

By Theorem 5 and by Lemma 10, we have the following

Corollary 2. $L(G)' \preceq L(G')'$ *and* $\mathrm{diam}(L(G)') \leq \mathrm{diam}(L(G')')$.

In the following an (e_x, e_y)-path is a path whose first edge is e_x and the last edge is e_y. If an edge $e \in E(G)$ is incident with a vertex in $v \in D_3^+$, then e is in a non-trivial complete subgraph of $L(G)$. Therefore, e is a contractible vertex in $L(G)$, and so $e \notin V(L(G)')$. A vertex $e \in L(G)'$ is called a *trivial vertex* if e is not a vertex obtained from a non-trivial collapsible subgraph in $L(G)$.

Theorem 6. $\mathrm{diam}(L(G)') \leq \mathrm{diam}(G) - 1$, *unless* $G = C_n$.

Proof. Suppose that G is not a cycle. Let e'_x and e'_y be two vertices in $L(G)'$. It suffices to show that $L(G)'$ has an (e'_x, e'_y)-path with length at most $m - 1$ where $m = \mathrm{diam}(G)$.

Let Γ_x and Γ_y be the two preimages of e'_x and e'_y in $L(G)$, respectively. Let $H_x = L^{-1}(\Gamma_x)$ and $H_y = L^{-1}(\Gamma_y)$. Then H_x and H_y are two L-collapsible subgraphs in G. Then $E(H_x) \neq \varnothing$ and $E(H_y) \neq \varnothing$. Let $e_x = uv \in E(H_x)$ and $e_y = zw \in E(H_y)$. The following simple fact will be needed.

Proposition 3. *If G has an (e_x, e_y)-path P_e of length at most m, then $L(P_e)$ is a path of length at most $m - 1$ in $L(G)$. Hence $L(G)'$ has an (e'_x, e'_y)-path with length at most $m - 1$.*

Let P_1 be a shortest (v, w)-path in G. We let P_e be the (e_x, e_y)-path formed by the path P_1 and $\{e_x, e_y\}$.

If $|E(P_1)| \leq m-2$, then P_e is an (e_x, e_y)-path of length at most m in G. By Proposition 3, we are done in this case. Therefore, $m - 1 \leq |E(P_1)| \leq m$.

Case 1. The ends of e_x and e_y are in $D_1(G) \cup D_2(G)$, i.e., $u, v, z, w \in D_1(G) \cup D_2(G)$.

Subcase 1(A). $|E(P_1)| = m - 1$.

Then $L(P_e)$ is an (e_x, e_y) path with length at most m in $L(G)$. If there is an internal vertex of P_1 which is in $D_3^+(G)$, then at least two edges in P_1 that are incident with the vertex in $D_3^+(G)$ are in a non-trivial collapsible subgraph of $L(G)$. Thus, at least one edge in $L(P_e)$ will be contracted in $L(G)$. Hence, $L(G)'$ has an (e_x, e_y)-path of length at most $m - 1$. We are done in this case. In the following we assume that

$$V(P_1) \subseteq D_1(G) \cup D_2(G).$$

If at least one among e_x or e_y is in $E(P_1)$, then the path P_e in G is an (e_x, e_y)-path with length at most m. Therefore, $L(P_e)$ is an (e_x, e_y)-path in $L(G)$ with length at most $m - 1$. We are done in this case.

If e_x and e_y are not in $E(P_1)$, then $V(P_1) \subseteq D_2(G)$. Let P_2 be a shortest (u, z)-path in G. Since $u, z \notin V(P_1)$, $P_1 \neq P_2$. By the same argument above, we have $m - 1 \leq |E(P_2)| \leq m$. Then $C = E[P_1 \cup P_2]$ is a cycle of length $2m$ or $2m + 1$. Since G is not a cycle, there is a vertex $a \in V(P_2) \cap D_3^+(G)$. Suppose there is only one vertex in P_2 of degree at least 3. Let b be a vertex not in C adjacent with a. Then there is a vertex in C having distance $m + 1$ from b, a contradiction. Thus, P_2 contains two vertices of degree at least 3. So at least two edges in $L(P_2)$ will be contracted in $L(G)'$. Hence, $L(G)'$ has an (e_x, e_y)-path of length at most $m - 1$. We are done in this case.

Subcase 1(B). $|E(P_1)| = m$.

If e_x and e_y are both in $E(P_1)$, then the path $L(P_e)$ in $L(G)$ is an (e_x, e_y)-path with length at most $m - 1$. Then we are done in this case. Next we consider the case that at least one of the edges in $\{e_x, e_y\}$, say e_y, is not in $E(P_1)$.

Since P_1 is a shortest (v, w)-path with length m in G and $e_y \notin E(P_1)$, $z \notin V(P_1)$, and so $P_e = P_1 \cup \{e_y\}$ is a (v, z)-path with length $m+1$. Since $m = \text{diam}(G)$, there is a shortest (v, z)-path P_2 with length at most m in G. If the length of P_2 is less than $m-2$, then P_2 with edge e_y is a (v, w)-path with length at most $m-1$, contrary to the fact that P_1 is a shortest (v, w)-path with length m. Thus, $m - 1 \le |E(P_2)| \le m$. Similar to the argument above, $V(P_2) \subseteq D_2(G)$. Therefore, $G \in \{C_{2m}, C_{2m+1}\}$, a contradiction. The proof of Case 1 is complete.

Case 2. One of the vertices of $\{u, v, z, w\}$ is in $D_3^+(G)$, (say $u \in D_3^+(G)$).

By using the fact that $\text{diam}(G) \le m$, and any edge incident with a vertex in $D_3^+(G)$ is in collapsible subgraph of $L(G)$, one can always construct an (e_x, e_y)-path P_e such that after contraction, $L(P_e)'$ is an (e'_x, e'_y)-path with length at most $m-1$ in $L(G)'$. The details of the proof is similar to Case 1, and hence is omitted. □

Corollary 3. $\text{diam}(L(G)') \le \text{diam}(G') - 1$ *unless* $G' = C_n$.

Proof. Apply Theorem 6 to G', we have $\text{diam}(L(G')') \le \text{diam}(G') - 1$ unless $G' = C_n$. By Corollary 2, we have $\text{diam}(L(G)') \le \text{diam}(L(G')')$. Therefore, $\text{diam}(L(G)') \le \text{diam}(L(G')') \le \text{diam}(G') - 1$ unless G' is a cycle. □

For integer $m > 0$, define $L^m(G) = L(L^{m-1}(G))$ with $L^0(G) = G$.

Corollary 4. $\text{diam}(L^r(G)') \le \text{diam}(G') - r$ *unless* $L^i(G)'$ *is a cycle for some* i *with* $0 \le i < r$.

Proof. When $r = 1$, by Corollary 3, the statement holds. Assume that G' is not a cycle. Then we have the induction assumption that

$$\text{diam}(L^{r-1}(G)') \le \text{diam}(G') - (r-1). \tag{7}$$

Let $\Gamma = L^{r-1}(G)$. If Γ is a cycle, then we are done. Otherwise, by (7)

$$\text{diam}(\Gamma') \le \text{diam}(G') - (r-1).$$

Then by Corollary 3,

$$\text{diam}(L^r(G)') = \text{diam}(L(\Gamma)') \le \text{diam}(\Gamma') - 1 \le \text{diam}(G') - r.$$

The proof is completed. □

An (x, y)-path in $L^k(G)'$ is called *non-trivial path* if x and y are non-trivial vertices in $L^k(G)'$ and all the internal vertices are trivial.

Proposition 4. *Let* $k > 0$ *and* $e \in E(L^k(G)')$. *If the both ends of* e *are non-trivial, then each of the following holds:*

(a) *There is a path* $P = v_0v_1 \cdots v_{k+1}$ *in* G *such that* $d(v_0, v_{k+1}) = k+1$ *and each internal vertex of* P *has degree* 2 *and* v_0 *and* v_{k+1} *have degree at least* 3 *in* G *and* $L^k(P) = e$ *in* $L^k(G)$.
(b) $\text{diam}(G) \ge k+3$.

Proof. (a) Let $e = v_0^{(k)}v_{k+1}^{(k)}$ be an edge in $L^k(G)'$. Then there is a path $P_{k-1} = v_0^{(k-1)}v_1^{(k-1)}v_{k+1}^{(k-1)}$ in $L^{k-1}(G)$ such that $L(P_{k-1}) = e$, and P_{k-1} is also a path in $L^{k-1}(G)'$. Since the both ends of e are non-trivial, $v_0^{(k-1)}$ and $v_{k+1}^{(k-1)}$ are incident with two non-trivial collapsible subgraphs, respectively, and so $v_0^{(k-1)}$ and $v_{k+1}^{(k-1)}$ have degree at least 3 in $L^{k-1}(G)$. Obviously, $v_1^{(k-1)}$ has degree 2 in $L^{k-1}(G)$. Otherwise, e will be in a non-trivial collapsible subgraph of $L^k(G)$, a contradiction. Following the same argument, we know that for each $0 < i \le k$, there is a path $P_{k-i} = v_0^{(k-i)}v_1^{(k-i)} \cdots v_i^{(k-i)}v_{k+1}^{(k-i)}$ in $L^{k-i}(G)'$ such that each internal vertex has degree 2 and $v_0^{(k-i)}$ and $v_{k+1}^{(k-i)}$ have degree at least 3 in $L^{k-i}(G)$. Proposition 4(a) is proved.

(b) By way of contradiction, suppose that for any two vertices u and v in G, $d(u, v) \le k+2$. Let $P = v_0v_1 \cdots v_kv_{k+1}$ be a path in G as stated in part (a). Let $N^-(v_0) = N(v_0) \setminus \{v_1\}$ and let $N^-(v_{k+1}) = N(v_{k+1}) \setminus \{v_k\}$. Since v_0 and v_{k+1} have degree at least 3 in G, $|N^-(v_0)| \ge 2$ and $|N^-(v_{k+1})| \ge 2$.

Let x be any vertex in $N^-(v_0)$. Suppose $d(x, v_{k+1}) \le k+1$. Let P_x be a shortest (x, v_{k+1})-path. Then v_0P_x is a (v_0, v_{k+1})-path with length at most $k+2$. Then $L^k(v_0P_x)$ is a path with length at most 2 in $L^k(G)$ with both ends are incident with non-trivial collapsible subgraphs. Then e is an edge in a C_2 or K_3 in $L^k(G)$, and so e will be contracted in $L^k(G)'$, a contradiction. Hence $d(x, v_{k+1}) = k+2$. Similarly, $d(v_0, y) = k+2$ for any vertex $y \in N^-(v_{k+1})$. Let P_{xy} be a shortest (x, y)-path in G. Since $d(x, v_{k+1}) = k+2$ and $d(v_0, y) = k+2$, the length of P_{xy} must be $k+1$ or $k+2$. Note that since each internal vertex of P is of degree 2, P_{xy} and P are disjoint. Suppose all vertex of P_{xy} are of degree 2 in G. Since $|N^-(v_0)| \ge 2$, there is a vertex $x_1 \in N^-(v_0) \setminus \{x\}$. Note that $d(x_1, v_{k+1}) = k+2$. Since all vertices of P_{xy} and all internal vertices of P are of degree 2, $d(x_1, y) = k+3$, a contradiction. Thus, P_{xy} contains a vertex of degree 3 in G.

Now we let $x \in N^-(v_0)$ and $y \in N^-(v_{k+1})$ such that $d(x, y)$ is as small as possible.

Case 1. Suppose the length of P_{xy} is $k+1$ and suppose that there is a vertex of P_{xy} of degree 3 in G. Then $L^k(v_0P_{xy}v_{k+1})$ is a path of length at most 3 in $L^k(G)$ and at least one edge is in a collapsible subgraph of $L^k(G)$. Then e will be contracted in $L^k(G)'$, a contradiction.

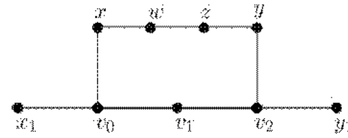

Fig. 6.

Case 2. Suppose the length of P_{xy} is $k+2$. Note that, in this case the distance between each vertex in $N^-(v_0)$ with each vertex in $N^-(v_{k+1})$ is $k+2$. If there are two vertices of P_{xy} of degree 3 in G then by a similar argument of Case 1, we will obtain a contradiction. Thus, P_{xy} contains exactly one vertex of degree 3 in G.

Suppose $k \geq 2$. Then $L^k(v_0 P_{xy} v_{k+1})$ is a path of length at most 4 in $L^k(G)$ and at least two edges are in some collapsible subgraphs of $L^k(G)$. Then e will be contracted in $L^k(G)'$, a contradiction. Suppose $k = 1$. We have a subgraph of G described in Fig. 6. Without loss of generality, we may assume $d_G(x) \geq 3$ or $d_G(w) \geq 3$ but not both. Suppose $d_G(x) \geq 3$. Since the vertices w, z, y and v_1 are of degree 2 and $d(x_1, y_1) = 3$, $d(x_1, y) = 4$, a contradiction. Suppose $d_G(w) \geq 3$. Since $d(x_1, y) = 3$ and $d(x_1, y_1) = 3$, $x_1 w \in E(G)$. Then e will be contracted in $L(G)'$, a contradiction.

The proof of Proposition 4(b) is complete. □

6. Applications

A graph is an *even* graph if it has no odd degree vertices. For a graph G, a connected even subgraph H is called a *dominating Eulerian* subgraph if every edge of G is incident with a vertex in H. A *double cycle cover of* a graph G is a collection of even subgraphs $H_1, H_2, \ldots, H_m$ of G, such that each edge of G occurs in exactly two of the H_i's. If $m = 3$, then we say G admits a double cycle cover with three even subgraphs. For example, consider $K_{2,t}$ for $t \geq 2$. If t is odd, then we can choose three even subgraphs $H_1 \cong H_2 \cong K_{2,t-1}$ and $H_3 \cong K_{2,2}$. If t is even, then we can choose $H_1 \cong H_2 \cong K_{2,t-2}$ and $H_3 \cong K_{2,4}$. Thus, $K_{2,t}$ admits a double cycle cover with three even subgraphs.

Theorem E. *Let G be a connected simple graph with at least three edges.*

(a) *(Catlin [5]). Let G be a graph and let H be a subgraph of G. If H is collapsible or H is a 4-cycle, then G admits a double cycle cover with three even subgraphs if and only if G/H admits a double cycle cover with three even subgraphs.*
(b) *(Catlin [6]). If G has a spanning Eulerian subgraph, then G admits a double cycle cover with three even subgraphs.*
(c) *(Harary and Nash-Williams [8]). L(G) is Hamiltonian if and only if G has a dominating Eulerian subgraph.*
(d) *(Catlin [3]). G has a dominating Eulerian subgraph if and only if G′, the reduction of G, has a dominating Eulerian subgraph containing all non-trivial vertices of G′.*

It is known that the Petersen graph cannot have a double cycle cover with three even subgraphs. By Theorem E(a), one can see that $S_{m,l}$ admits a double cycle cover with three even subgraphs. We also know that if $G \in \mathcal{L}$, then G has a dominating Eulerian subgraph. By Theorem 2 and Theorem E, we have the following corollary:

Corollary 5. *Let G be a connected graph with diameter at most* 2.

(a) *(Veldman [12]). If G has at least three edges, then L(G) is Hamiltonian.*
(b) *(H.-J. Lai [10]). If G is 2-edge-connected, then either G admits a double cycle cover with three even subgraphs, or $G \cong P$, the Petersen graph.*

The smallest m such that $L^m(G)$ is Hamiltonian is called the *Hamiltonian index* of G and denoted by $h(G)$. The following theorem generalize Corollary 5(a), and improves a result in [7] stating that $h(G) \leq \mathrm{diam}(G)$ unless G is a path or a C_2.

Theorem 11. *Let G be connected simple graph. Then $h(G) \leq \mathrm{diam}(G) - 1$ unless G is a path.*

Proof. If $\mathrm{diam}(G) \leq 2$, then by Corollary 5(a) the theorem holds. In the following we assume that G is not Hamiltonian and $\mathrm{diam}(G) \geq 3$. Let r be the largest non-negative integer such that $\mathrm{diam}(L^r(G)') \geq 3$ and $L^r(G)$ is not Hamiltonian. Then either $\mathrm{diam}(L^{r+1}(G)') \leq 2$ or $L^{r+1}(G)$ is Hamiltonian. (Note that if no such integer r exists, this implies that $\mathrm{diam}(L(G)') \leq 2$. Our proof is still valid for this case.) By Corollary 4,

$$r \leq \mathrm{diam}(G) - \mathrm{diam}(L^r(G)') \leq \mathrm{diam}(G) - 3. \tag{8}$$

If $L^{r+1}(G)$ is Hamiltonian, then we are done. If $\mathrm{diam}(L^{r+1}(G)') = 0$, then $L^{r+1}(G)$ is collapsible. By Theorem A(a) and Theorem E(c), $L^{r+2}(G)$ is Hamiltonian. By (8) $h(G) \leq r+2 \leq \mathrm{diam}(G) - 1$. We are done in this case. In the following we will consider the case that $1 \leq \mathrm{diam}(L^{r+1}(G)') \leq 2$.

Note that since $1 \le \mathrm{diam}(L^{r+1}(G)') \le 2$, $L^{r+1}(G)' \in \{K_2, K_{1,t}, K_{2,s}, S_{m,l}, P\}$. If $L^{r+1}(G)'$ has a dominating Eulerian subgraph containing all non-trivial vertices of $L^{r+1}(G)'$, then by Theorem E(d) $L^{r+1}(G)$ has a dominating Eulerian subgraph. Therefore, by Theorem E(c) $L^{r+2}(G)$ is Hamiltonian. By (8), $h(G) \le r + 2 \le \mathrm{diam}(G) - 1$.

Next we assume that $L^{r+1}(G)'$ has no dominating Eulerian subgraph containing all non-trivial vertices of $L^{r+1}(G)'$. For each possible case of $L^{r+1}(G)' \in \{K_2, K_{1,t}, K_{2,s}, S_{m,l}, P\}$, $L^{r+1}(G)'$ has at least an edge $e = xy$ such that x and y are non-trivial in $L^{r+1}(G)'$. Then by Proposition 4 with $k = r + 1$ in this case,

$$\mathrm{diam}(G) \ge k + 3 = r + 4. \tag{9}$$

Since each vertex of degree at least 3 in $L^{r+1}(G)' \in \{K_2, K_{1,t}, K_{2,s}, S_{m,l}, P\}$ is a non-trivial vertex, and the fact that $L^{r+1}(G)'$ has at least two non-trivial vertices, one can check that $L^{r+2}(G)$ is collapsible. Therefore, by Theorem E(c), $L^{r+3}(G)$ is Hamiltonian. Hence, by (9), $h(G) \le r + 3 \le \mathrm{diam}(G) - 1$. The proof is complete. $\square$

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