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WACKY WORDS AND STEINER SYSTEMS

I suspect that many readers will have noticed that all the vowels (including Y) occur at odd positions in the alphabet (let's call them odd letters). This led me to think of the set of words using only odd letters.

In choosing a name for such words, I decided, on the precedent of "Miami Words", to make it **autological**, i.e. applicable to itself. Most of the definitions I will be making in this article are autological in this sense. The antonym of **autological** is **heterological** – and don't ask whether this word is autological or heterological because either way you'll get a contradiction (this is known to logicians as the **Grelling-Nelson Paradox**, and it's closely related to Russell's Paradox the discovery of which was one of the milestones leading to the development of modern logic).

I had originally intended to call such words "magic", but decided to use this word in another sense (see below). So I propose to use the term **wacky words**. Not only is this autological, but "wacky" suggests "odd", and also the phrase is what I call **attractively alliterative** (as are "Word Ways" and several other phrases I use in this article), by which I mean that not only do the words begin with the same letter but they have the same length.

One thing one can do with wacky words is to construct a **perfect pangram** – i.e. sentence containing every odd letter exactly once. Because of the imbalance between consonants and vowels, this is not easy with the full alphabet, and nobody has come up with a sentence that doesn't need explanation.

Not that it's all that easy to come up with a natural sounding wacky perfect pangram. I think my best is **O QUICK, MY WAGES!** I also like **SWIM QUAY, GECKO!**

A group of devotees of Indian mysticism put on a party at which they played an animal imitation game. A report of this party: **YOGIS QUACK, MEW**. Or perhaps the party was Jewish, with a token gentile: **I QUACK, GOY MEWS**.

The words "quack" and "news" both have alternative meanings, so how about saying to a knowitall doctor: **GO, MY WISE QUACK**; or directing someone to a London exercise group which meets in a back street and is just about to start: **YOGA? QUICK, MEWS!**

Now let's take a different subset of the alphabet – all the letters whose position is a multiple of 3. Let's call them **colourful** words – an autological definition that works just as well with the American spelling. For a perfect pangram, one might reasonably say to one's hairdresser: **O FIX CURL!**

While I can't find any other colourful perfect pangrams that I like, this set of letters does have a very unusual property: it can be split into a number-name – **FOUR** – and letters representing Roman numerals, which can form the numbers **CXLI = 141**, **CLIX = 159**, or **CLXI = 161**. Or one can make the number 604 out of all of them as **FOUR × CLI**.

Next let's divide the alphabet into letters which in upper case are made entirely out of straight lines (AEFHILMNTVWXYZ) and those which have curves (BCDGOPQRSU) – let's call them **lineal** and **proud** respectively. (Can any reader come up with a proud word which suggests curviness?) In neither case is the consonant-vowel balance conducive to perfect pangrams, but we can do other things such as look for the longest lineal or proud word. For the former the chemical **HEXAMETHYLETHANE** has 16 letters, for the latter **SUBGROUPS** has 9. Can anyone beat these?

Other subsets of the alphabet, which I think are too small to warrant names, include every 4th letter from A, i.e. AEIMQUY, which seems to have too many vowels to make up a good perfect pangram! Or one can take the letters whose positions are Fibonacci numbers (ABCEHMU, making up the perfect pangram **BE A CHUM**, or perhaps **HUM "EABC"**, the opening notes of the last movement of Beethoven's Pathétique Sonata transposed into the key of A minor). The squares make up the perfect pangram **I'D PAY**, but despite a good consonant-vowel balance it is difficult to get anything out of the triangular numbers, for which the relevant letters are ACFJOU.

My final subset of the alphabet consists of those letters whose position-numbers have remainder 1 or 3 when divided by 6. I call words consisting of such letters **magic**. A couple of magic perfect pangrams: **MOSAIC GUY** and **YOGA MUSIC** (music while you work out?).

Numbers which have a remainder of 1 or 3 when divided by 6 are of interest to devotees of combinatorial problems, including some kinds of wordplay, because they are the orders of so-called **Steiner Systems**, or Steiner Triple Systems to be precise. These are named after the 19th century Swiss geometer Steiner, though they had in fact been discovered earlier by the English mathematician Kirkman. If one considers a set of n

objects – let’s call them points – a Steiner Triple System of order n is a set of triples of points – let’s call them lines – which has the property that every pair of points occurs in exactly one line. The orders that occur are, as stated above, exactly the positive integers which have remainder 1 or 3 when divided by 6.

For order 1 there are no pairs, so the empty set of lines satisfies this definition. Almost as trivial is order 3 when a single line makes up a Steiner System. There are unique solutions (up to permutation of the points) when the order is 7 or 9: for the former, ABD, BCE, CDF, DEG, EFA, FGB, GAC (note that all of these can be obtained from one another by cycling the letters); and for the latter, ABC, DEF, GHI, ADG, BEH, CFI, AEI, BFG, CDH, AFH, BDI, CEG, i.e. the rows, columns, diagonals and broken diagonals of the square

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A B C
D E F
G H I.
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A natural problem is to find a Steiner System on a set of 7 or 9 letters such that each line can be rearranged to form a word. This is relatively easy for the case of 7 letters: replace ABCDEFG by AYTEROU and form the words AYE, TRY, TOE, RUE, OAR, YOU, TAU for example. Because of the consonant-vowel problem it’s more difficult for the case of 9 letters, but one solution I’ve found is to replace ABCDEFGHI by YOUETGAMR and form the words YOU, GET, RAM, AYE, TOM, RUG, TRY, AGO, EMU, GYM, ORE, TAU. The word “tom” isn’t a proper name when it’s used as short for “tomcat” – or one might instead choose the word “mot”, which, though French, occurs in several naturalized English phrases.

In keeping with the spirit of this article, one may then try to find a Steiner System of words out of the very letters that express the possible orders of Steiner Systems, i.e. the letters that form magic words. I managed to create such a system but only by using abbreviations and acronyms: replace ABCDEFGHI by YOMCAUIGS forming the “words” YOM (as in Yom Kippur War), AUC (for Anno Urbis Conditae, referring to the Roman dating of years from the founding of Rome), GIs (plural of GI), ICY, AGO, SUM, SAY, IOU (a well known abbreviation), GMC (General Medical Council), GUY, COS (cosine), and AIM.

If one writes the above letters as a 3×3 square as above, then not only are all the words magic but the word MAGIC can be spelt out by king’s moves. There are other ways of arranging the letters to give what one might call a **magic square**.

Of course the phrase “magic square” is more usually used in a quite different sense, to indicate a square of distinct numbers in which all the rows and columns, and the two main diagonals, add up to the same number. Can one construct a 3×3 square of letters which is magic if the letters are replaced by the numbers representing their positions in the alphabet, and in which the word MAGIC can be spelt out by king’s moves? The fact that all the letters in MAGIC have odd positions, i.e. that MAGIC is wacky, should in theory help here. But unfortunately the answer is no. However using a computer I did find such a 4×4 square using the first 16 letters of the alphabet, which was unique up to rotations and reflections. Can any reader find and prove the uniqueness of the answer without a computer?

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L H A M
N B G K
C I P F
E O J D
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The rest of this article deals with the construction of Steiner Systems, and can be skipped by those allergic to mathematics. Our methods cover systems of every permissible order up to 25 (i.e. all orders which can be represented by letters of the alphabet). This will help any readers who may wish to construct systems in which all lines rearrange to words or abbreviations.

It is sometimes convenient to add what might be called “degenerate lines” in which a point can occur more than once. Two ways of doing this are to add a line AAA for any point A (let’s call this method 1), or add a new point, say 0, and a line 0AA for any point A (including 0), which we call method 2. In either case every pair of points, equal or not, will complete to a unique line.

We can then take the **direct product** of two systems – if ABC and DEF are points making up a line in the first and second system respectively, then AD, BE and CF will be points making up a line in the direct product. The order of the direct product will be the product of the two orders.

If we take systems of order m and n and extend them by method 1, then after removing the degenerate lines we get a system of order mn . If we use method 2 we get a system of order $mn + m + n$.

If we apply method 2 repeatedly to the system of order 1 we get systems of order 3, 7, 15, 31 and so on. These have a high degree of symmetry, as do the systems of order 9, 27 and so on obtained by applying method 1 to the system of order 3. We can also apply methods 1 and 2 to get systems of order 21 (out of orders 3 and 7) and order 19 (out of orders 1 and 9).

There is also a different type of construction that can be applied to any number of the form $m^2 + mn + n^2$ not divisible by 3, which includes all the numbers up to 25 (indeed up to 49) which leave remainder 1 when divided by 6.

Consider a triangular lattice. One point can be adjacent to another in 6 ways: let's call them left, left up, right up, right, right down and left down. (In the first array below the letters bearing these relations to any of the A's are G, E, F, B, D and C respectively.) It is possible to label all the points of our triangular lattice with our $m^2 + mn + n^2$ points in such a way that the labels repeat if one goes m steps right and n steps right up, or m steps left down and n steps right down, or (which is a consequence) m steps left up and n steps left. Below we show parts of the arrays that result in the cases $m = 2$ or 3 and $n = 1$.

A B C D E F G A	A B C D E F G H I J K L M A
C D E F G A B C	D E F G H I J K L M A B C D
F G A B C D E F	H I J K L M A B C D E F G H
A B C D E F G A	K L M A B C D E F G H I J K
D E F G A B C D	B C D E F G H I J K L M A B

If the spacing is chosen correctly then the basic triangles of the lattice will be equilateral. Now consider all triples of different letters that occur as vertices of equilateral triangles in the lattice. For the first case these are ABD, BCE, CDF, DEG, EFA, FGB, GAC, ABF, BCG, CDA, DEB, EFC, FGD, GAE. It will be seen that some of these (e.g. ABD) "point downwards" while others (e.g. ABF) "point upwards". Take the ones that point downwards and these will form the Steiner System given earlier.

For larger orders there will be more than one type of triple – to be precise for order $6k + 1$ there will be k types. Each type can point in two directions. Choose a direction for each type and they will make up a Steiner System. For example in the second case there are small triangles (e.g. ABE pointing downwards or ABK pointing upwards) and triangles of twice the size (e.g. ACI pointing downwards or ACH pointing upwards). If we choose all the triangles pointing downwards, which are the cyclic permutations of ABE and ACI, the 26 lines form a Steiner System.

This method gives a unique Steiner System for orders 7, 13 or 25, but two distinct systems of order 19.

We now show how to get from one Steiner System to another of the same order. Let A and B be points on a line ABC. If D is another point, then let A and D determine a line ADE, B and E determine a line BFE, A and F determine a line AFG, and so on till we get back to D. If this happens before we've exhausted all the letters then in general we can get a different Steiner System by interchanging A and B in all the lines we've encountered.

The simplest case is when we get back to D immediately after G, i.e. there are lines ADE, BFE, AFG and BDG. In this set of 4 lines every pair intersects in a point, forming a **complete quadrilateral** (CQ), and such a CQ has a symmetry between three pairs (in this case AB, DF and EG), so we may call it (AB|DF|EG). Then we get the other Steiner System by replacing the above 4 lines by those in the CQ (BA|FD|GE).

As stated earlier, for orders up to 9 there is a unique Steiner System. For orders 15 upwards the number increases very rapidly. Order 13 is intermediate with two distinct systems. One is the one described earlier, the other is obtained from it by applying the above transformation to the CQ (AF|IB|CE) or any of its cyclic permutations.

Incidentally it is easy to find which type a Steiner System of order 13 belongs to. Find a pair AF which extends to a CQ. Complete AF to a line (AFH for either of our systems). If both AH and FH extend to CQs, then the Steiner System belongs to the original type; otherwise it belongs to the new type.

I hope that some reader will be able to find a Steiner System using the 13 odd letters whose 26 lines can all be rearranged into wacky words (including abbreviations etc.).