The Arithmetic of Word Ladders

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The relationship between word games and mathematical recreations is well-known. Martin Gardner has often described them in his "Mathematical Games" column in the *Scientific American*. The editor of WORD WAYS, in his recent book BEYOND LANGUAGE, has derived logological structures from mathematical ones.

This article describes certain maximized sets of word ladders derived from related sets of words. For four-letter word ladders there are 384 ladders derived from a basic set of 16 words. The four-letter case is just difficult enough to be interesting. The one-, two-, and three-letter cases are too easy, and the five-letter case appears impossible to solve. Strangely, however, as we shall see later, one-, two-, three-, and five-letter solutions display a type of symmetry that is impossible, mathematically, in the four-letter case.

WORD LADDERS

By changing the letters of a word one at a time, forming a new word each time, until we arrive at some desired word (in most instances, one semantically related to the starting word), we develop a series called a "word ladder." For four-letter words, this requires a minimum of four steps, as in this example:

(1)	(2)	(3)	(4	Position Changed
C	O	L	D	
C	O	R	D	3
W	O	R	D	1
W	A	R	D	2
W	A	R	M	4

FIGURE 1

Note that, in Figure 1, we have changed the 3rd, 1st, 2nd, and 4th letters, in that order, to convert COLD into WARM.

PERMUTATIONS AND ARRANGEMENTS

The sequence 3124 is called a *permutation,* or more correctly, an *arrangement* of the symbols 1234. Strictly speaking, a *permutation* is the *operation* of changing one *arrangement* into another. The change from 1234 to 3124 may be written in permutational notation as: (132) (4).

For "n" symbols there are "n!" such arrangements, and for n = 4 there are $4! = 4 \times 3 \times 2 \times 1 = 24$ arrangements.

THE ARITHMETIC OF WORD LADDERS

Is it possible to form 24 different word ladders changing word "A" to word "B"? The answer is in the affirmative; indeed, we shall see that, by finding a suitable set of 16 words, we can form from them a collection of $2^4 \times 4! = 16 \times 24 = 384$ distinct word ladders from these 16 basic words only.

The key is the set of 16 basic words, and the game or puzzle is to find more such sets. We show one below:

(1)	(2)	(3)	(4)
S	I	N	S
F	A	T	E

FIGURE 2

By selecting one of the two letters, in order, from the four columns, we form 16 words:

SINS, SINE, SITS, SITE, SANS, SANE, SATS, SATE
FATE, FATS, FANE, FANS, FITE, FITS, FINE, FINS

FIGURE 3

A FANE is a temple or church; SATS are eternal and immutable existences in Hinduism; and FITE is a Scottish way of spelling "white." All of these less common words are in Webster's Third Edition.

Starting from any word in Fig. 3 and ending with the word above it or below it, as the case may be, we can form 16 sets of 24 word ladders each. For instance, there are 24 ways of going from SINS to FATE. Two of these ways are shown in Fig. 4:

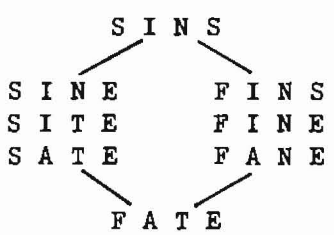

FIGURE 4

There is a total of 384 distinct word ladders inherent in the 16 basic words. Not all pairs of starting and ending words are semantically related, of course. Sets found by readers could be rated for quality in proportion to the fraction of the 16 that are semantically related in one or more subsets.

GENERAL CASE AND BINOMIAL PATTERNS

The number of words in the basic set of n-letter words is 2^n. Powers of 2—$(1 + 1)^n$—may be expanded binomially to form a "triangle of Pascal" (see any college algebra text for an explanation). The binomial expansions for powers from the first to the fifth inclusive form the triangle in Fig. 5.

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        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
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FIGURE 5

It is clear that the numbers in each horizontal row add up to the number of words in the basic set for 1-, 2-, 3-, 4-, and 5-letter words. We note that in each row but the fourth, each number divides, or is divisible by, the number to the right or to the left of it. This divisibility property is absent, also, in the case of the sixth and all subsequent rows. A proof of this is left to the mathematically-minded reader to construct.

What has this binomial divisibility property to do with the patterns of the word ladders? Figs. 6 through 9, below, supply the answer. The reader will note that there is no configuration included for four-letter ladders, simply because six is not a factor of ten. Consequently, the pattern would have to have both single and double connecting lines in one row, thereby destroying its symmetry. Perhaps there is a lesson for us in this. The redundancy structure of English (discussed by the Editor, Dmitri A. Borgmann, in the work cited above) determines that four-letter word ladders are the most interesting ones, but another mathematical theory, the binomial theorem, denies them a certain symmetry pattern. We cannot have perfection!

CONCLUSION

There should be more 4-letter sets of 16 words having the property described.

The number of word ladders at each letter length is shown in Fig. 10. Perhaps some reader can do the "impossible" and find a 32-word, 5-letter set!

FIGURES

FIGURE 6

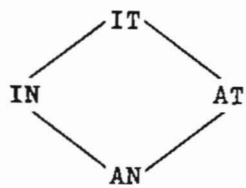

FIGURE 7

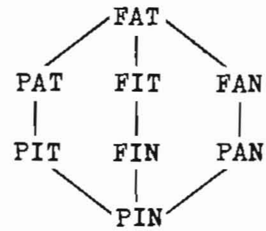

FIGURE 8

NOTE: Figures 6 through 9, inclusive, are called "partially ordered series" (see *A Survey of Modern Algebra*, by Birkhoff and Lane, section on "partial orderings").

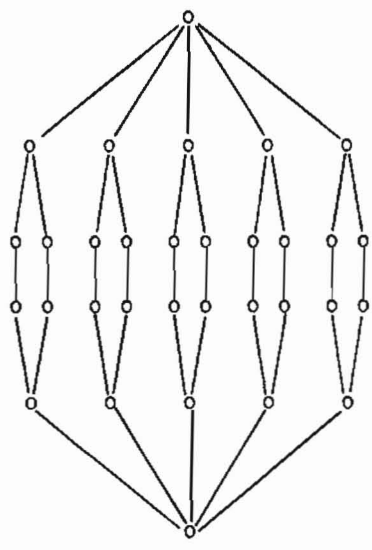

FIGURE 9

Word Length	Words in Basic Set	Ladders
1	2	2
2	4	8
3	8	48
4	16	384
5	32	3840
6	64	46080

FIGURE 10

EPILEGOMENA

Word ladder construction has mathematical analogues in (1) the Gray code, used to program binary digital computers, and in (2) the Icosian Game.

If we substitute a zero (0) for the letters of the initial word and a one (1) for the letters of the final word, the problem of running once through the basic list becomes identical with the problem of constructing the Gray code for a computer. Figure 11 illustrates the Gray code and the analogous word list, which is closed. It is left to the reader to enumerate all analogues to codes of the "Gray" type.

If we put the sixteen words on the vertices of a suitable polyhedron, the problem of running through the words once, as in Fig. 11, is known as "The Icosian Game," widely discussed in the literature of mathematical recreations.

From this binary property of the 4-letter word ladder puzzle, the reader is invited to explore further relationships, as with the Game of Nim, for example.

Decimal	Gray Code	Icosian Word
0	0000	SANS
1	0001	SANE
2	0011	SATE
3	0010	SATS
4	0110	SITS
5	0111	SITE
6	0101	SINE
7	0100	SINS
8	1100	FINS
9	1101	FINE
10	1111	FITE
11	1110	FITS
12	1010	FATS
13	1011	FATE
14	1001	FANE
15	1000	FANS

FIGURE 11

* * *

SPOTS BEFORE YOUR EYES?

JIUJITSU, a Japanese art of self-defense, and UJIJI, a town in Tanzania (formerly Tanganyika), have this interesting feature in common: written or printed in lower-case style, four out of five consecutive letters in each term are dotted. An alternative name for "sesame seed," JINJILI (a name of Hindustani derivation), shares this distinction and adds the further one of having five out of seven consecutive dotted letters.

Can any reader top the examples just given?