A BOUQUET FOR GARDNER

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Our tribute bouquet starts with several PELARGONIUMS: all but two of which are red, all but two of which are yellow, and all but two of which are green. The reader should be able to determine from this information just how many flowers are in our main bouquet. The answer to this riddle which is an adaptation of one of Martin Gardner's charming Snarkteasers (#52, in [G73]) will be given later.

Meanwhile, there is some mathematics to be explained about our PELARGONIUMS and the hybrids we obtain from them. We start with one of Gardner's earliest articles "The Five Platonic Solids" (Chapter 1 in [G61]). There he proves that the tetrahedron, the hexahedron (a cube), the octahedron, the dodecahedron, and the icosahedron are the only possible regular convex solids (Fig. 1).

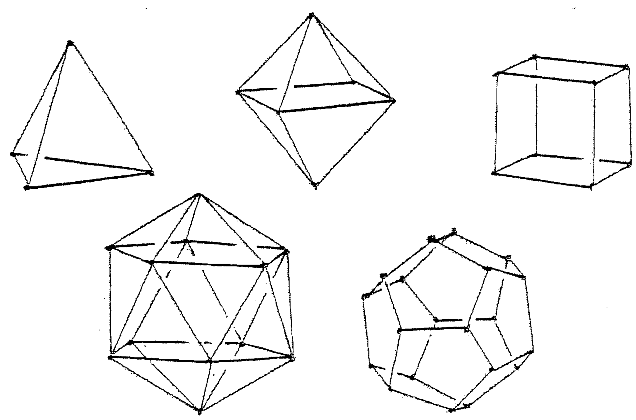


Figure 1. The five regular convex Platonic solids.

Most of our flowers will grow from these five Platonic seeds. In the article Gardner points out that "the cube and octahedron are 'duals' in the sense that if the centers of all pairs of adjacent faces on one are connected by straight lines, the lines form the edges of the other. The dodecahedron and icosahedron are dually related in the same way. The tetrahedron is its own dual". We will combine planar graphs of the solids with their duals in a special way to obtain our PELARGONIUMS. Such planar graphs are called Schlegel diagrams. This is a model that is, as Gardner explains (p. 23 in [G83]) ". is .simply the distorted diagram of the solid, with its back face stretched to become the figure's outside border." In Figure 2 we draw with solid, curved lines the Schlegel diagrams of a tetrahedron, an octagon and an icosahedron. Superimposed with dashed lines are the Schlegel diagrams of the respective duals, the tetrahedron, hexahedron, and the dodecahedron. The three outer dashed lines in each diagram are to be regarded as meeting in the back face.

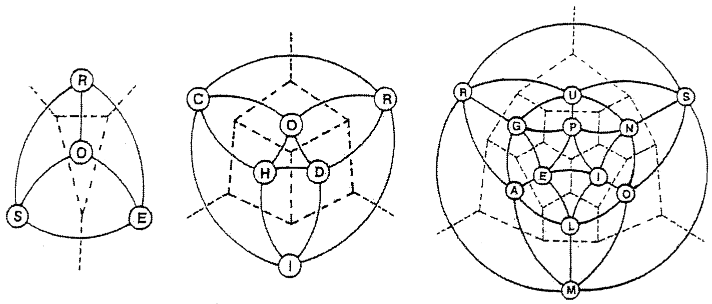


Figure 2. Schlegel graphs and their duals.

After labeling the nodes as ROSE, ORCHID, and PELARGONIUMS, the drawings are extended to the completed flowers where the new nodes become the intersections of the former dashed and solid arcs and inherit the two-letter labels of the endpoints of the former nodes. A study of Figure 3 will make this clear. Thus three flowers answers our opening riddle.

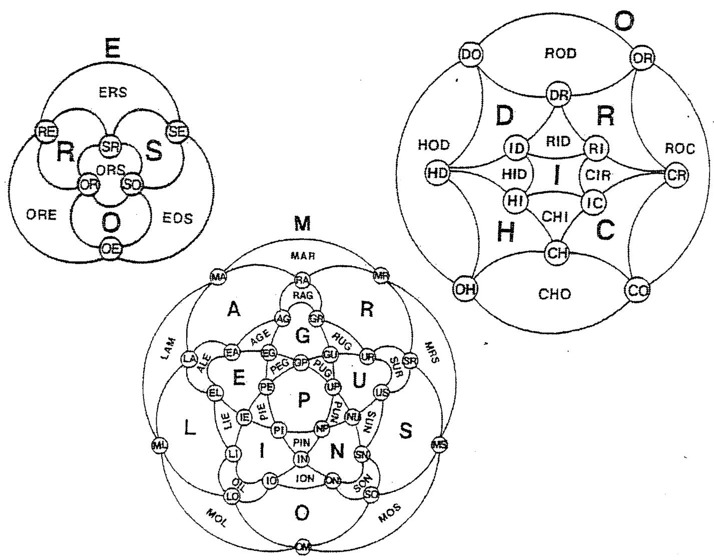
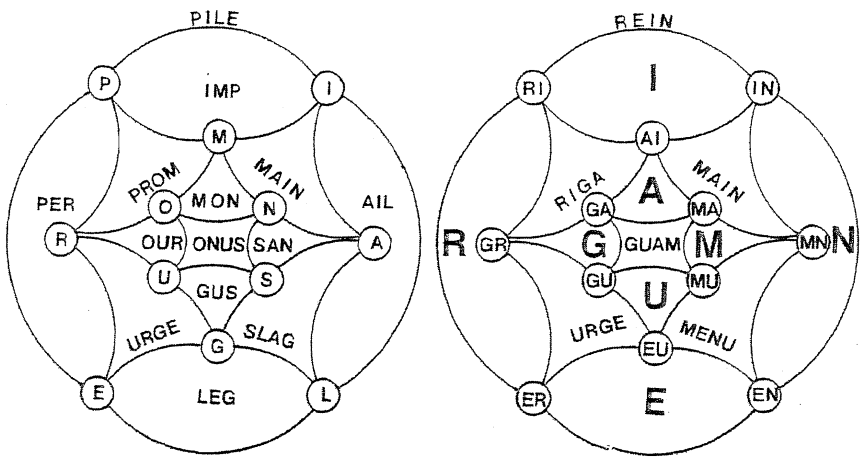


Figure 3. The three main flowers: ROSE, ORCHID, and PELARGONIUMS.

It is possible to form hybrids by judicious relabeling of the flowers. For instance, Figure 4 illustrates the new flowers GERANIUM and PELARGONIUMS formed on ORCHID. It may be possible to obtain other hybrids using as labels WILD ROSE, VIOLET, MARIGOLD, MYRTLE, etc. Remember any labeling should yield bona fide dictionary entries on the new names of the parts.

Figure 4. The Two ORCHID Hybrids.



In addition to the nodes our flowers contain 3-, 4- and 5-cycle petals corresponding to the former regular faces of the Platonic solids. All of the petals inherit labels from their boundaries and all labels are words which are main entries in most unabridged dictionaries or atlases. (We used the Merriam-Webster New International, 3rd ed.) We will be able to use these labels for certain puzzles and games we have in mind on the flower graphs. The number of flower parts are:

	Nodes	3-cycles	4-cycles	5-cycles
ROSE	6	8	0	0
ORCHID	12	8	6	0
PELARGONIUMS	30	20	0	12

Recall that the outside of each flower is also a cycle.

There is a famous puzzle invented in the 1850s by the Irish mathematician Sir William Rowan Hamilton that was originally played on a solid dodecahedron that can be played on our green PELARGONIUMS grid. Gardner first described this puzzle in "The Icosian Game and the Tower of Hanoi" (Chapter 6 in [G59]). ". . .the basic puzzle is as follows. Start at any corner of the solid (Hamilton labeled each corner with the name of a large city), then by traveling along the edges make a complete 'trip around the world', visiting each vertex once and only once, and return to the starting corner." Today this is called finding a Hamiltonian circuit. The 20 3-cycles of PELARGONIUMS are the "vertices" that must be visited in a circuit on our graph. They are each joined by a two-letter node on the flower. It is also possible to write the 20 three-letter words on tiles and try to arrange them in a chain so that abutting tiles have two letters in common. If the chain closes, you have solved the puzzle.

Gardner writes "On a dodecahedron with unmarked vertices there are only two Hamiltonian circuits that are different in form, one a mirror image of the other. But if the corners are labeled, and we consider each route 'different' if it passes through the 20 vertices in a different order, there are 30 separate circuits, not counting reverse runs of these same sequences. Similar Hamiltonian paths can be found on the other four Platonic solids". One of the 30 solutions is given at the end of this article. Other informative Gardner articles about Hamiltonian circuits include "Graph Theory: (Chapter 10 in [G71]), "Knights of the Square Table" (Chapter 14 in [G77], and "Uncrossed Knight's Tours (p. 186 in [G79]). John H. Conway's very interesting "A Dodecahedron – Quintomino Puzzle" (p. 23 in [G83]) can be adapted to our flower.

Gardner has often written about magic squares in his Scientific American columns and we were especially interested in his report about Room squares reprinted in "The Császár Polyhedron" (Chapter 11 in [G88]). "A Room square is an arrangement of an even number of objects, n+1, in a square array of side n. Each cell is either empty or holds exactly two different objects. In addition, each object appears exactly once in every row and column, and each (unordered) pair of objects must occur in exactly one cell."

The Australian mathematician Thomas G. Room had called this concept "A New type of magic square" in 1955 but it was later discovered that they had been in use before 1900 in scheduling bridge tournaments. We have discovered in our flowers a generalization of these squares. For instance using the yellow ORCHID we can form this 4x4 square from the 12 nodes.

4	RI		HD	CO
3	CH	ID	OR	
2		OH	IC	DR
1	DO	CR		HI
	a	b	c	d

This square is magic on the rows and columns in the sense that each set of three entries transposes into the word ORCHID, the magic constant. It is not a Room square since the taboo pairs IO, HR, and CD never occur together. It is instructive to locate these pairs on the ORCHID graph.

A pleasant little puzzle is possible by preparing 12 tiles with the two-letter words on them and trying to reconstruct one of the 1152 solutions to the puzzle that look different to the eye. Two persons can play the puzzle as a game by drawing a tile in turn and placing it on the grid so that no common letter occurs in any row or column. The last player to be able to play wins. To play expertly one must heed complementary pairs of words consisting of taboo mates. That is, RI-OH, ID-CO, IC-DO, HI-OR, CR-HD, and CH-DR. If any of these occur in the same row or column it will be impossible to complete the trio of words in that row or column. It can also be proved that where the blank squares in the grid intersect, we must insert complementary words. For example, the two blanks at a2 and b4 intersect at a4 and b2 where the complements RI and OH occur.

This last property can be used in a magic trick. Let the subject find one of the possible solutions to the puzzle and then turn all the tiles face down. The subject turns over and exposes any tile and the magician can then call out another (the complement) and is able to locate it in the grid. This is repeated until all tiles are exposed. If the grid is on a board it may be carefully rotated before any tile is exposed and of course the trick will still work. Interchanging pairs of rows or columns – including the blanks – can make the trick even more mysterious.

The 12 words are the edges of a 3-dimensional cube and similar square grids are possible using the edges of an n-dimensional cube. This generalization, as far as we know, has never been explored.

The 20 three-letter words in the green PELARGONIUMS flower yield another magic square. One solution with magic constant PELARGONIUMS is:

MAR	OIL		SUN	PEG
PUG	MRS	ALE		ION
LIE	PUN	MOS	RAG	
	AGE	PIN	MOL	SUR
SON		RUG	PIE	LAM

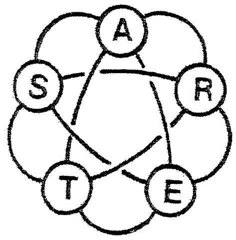
The taboo pairs are MP, AN, LU, RI, GO and ES which lead to the complementary pairs MAR-PIN, MRS-PIE, MOS-PEG, MOL-PUG, LAM-PUN, ALE-SUN, RAG-ION, AGE-SON, SUR-LIE and RUG-OIL.

There are 28,800 solutions to this puzzle that will look different to the eye. (Purists would not regard all of these as truly different because of certain symmetries of the grid.) Each of the remarks about the 4x4 ORCHID grid hold as well for this 5x5 PELARGONIUMS grid.

These magic squares may have applications in tournament schedules. For instance suppose we have three two-person teams that are to play one-on-one games of four types, a, b, c, and d, over four days. If the team members have initials IO, HR, and CD respectively, our 4x4 square gives the pairings on each day.

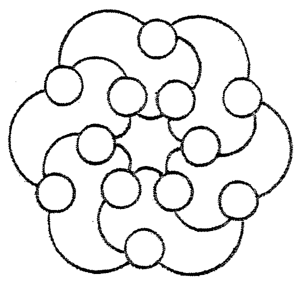
It is possible to extend our flower garden into higher dimensions. For example, there are six regular polytopes in four-dimensions that are the analogues of the Platonic Solids. One is the hypercube (see Chapter 13 [G01]) but perhaps the simplest is the polytope ASTER whose Schlegel diagram appears in Figure 5. See also Gardner's "Tetrahedrons" (Chapter 19, [G71]).

Figure 5. The ASTER



ASTER, or the "regular simplex" as geometers call it, has 5 nodes, 10 lines, 10 3-cycles and 5 4-cycles. Specifically, the parts are A-REST, S-TEAR, T-SEAR, E-STAR, R-SEAT, AS-RET, AT-ERS, EA-STR, RA-SET, ST-ARE, SE-RAT, SR-TEA, TE-RAS, RT-SEA, and RE-SAT. This flower is self-dual, like the tetrahedron it is similar to, and so a dual ASTER, interchanging nodes with 4-cycles and lines with 3-cycles could be superimposed on the graph. When reproduced in two dimensions the result would be a very "busy" flower!

There are of course other solids of interest that are not regular. One such infinite class is the prisms, solids with polygonal bases and tops with quadrilateral sides. As an example consider the following flower.



This flower is the Schelgel diagram of a hexagonal prism on which we ask the reader as a puzzle to place the 12 letters of PELARGONIUMS in the nodes so that each of the six 4-cycles as well as the two 6-cycles forming the base and top of the prism transpose into words. Our solution will be given later.

The basic flower diagrams can be used for a variety of board games. One game that can be played on any of the three flowers starts by placing tokens on all the nodes. Two players alternately remove one or more tokens from any one of the cycles on the board. The player that removes the last token wins the game. This game is a nim type game that superficially resembles David Gale's CHOMP. See Gardner's accounts "Nim and Tac Tix" (Chapter 15 in [G59]) and "Sim, Chomp and Race Track" (Chapter 9 in [G86]) for details. On our flower boards, however, these games are second person wins. Perhaps the reader can discover the strategy before we give a hint on how to play.

A more challenging game, and one we can not predict the winner of, is Ten Men's Morris played on the green PELARGONIUMS flower. Each of two players has ten distinctive tokens which they alternately place on the nodes. When a player obtains a cycle of tokens, that player has formed a Mill and may remove one of his or her opponent's tokens. A token that is part of a Mill cannot be removed. After all their tokens have been placed, the players alternately move their tokens to empty, adjacent nodes trying to form Mills. The game continues until one player loses by being reduced to two tokens.

All of our games and puzzles may be played strictly as word games without using the board at all. This usually makes them immensely harder to play. In "Jam, Hot and Other Games" (Chapter 16 in [G75]) Gardner recounts a word version of ticktacktoe by Canadian mathematician Leo Moser, who called it "Hot". Without the symmetry of the board or grid as a guide, the games take on new life. Even our Ten Men's Morris game can be played as a word game. The players have a word list composed from the 3- and 5-cycles of PELARGONIUMS and begin play by drawing, in turn, ten tiles from a face-up bone pile of 30 tiles. The tiles contain the two-letter words of the nodes. When someone is able to form a word Mill with their tiles they take a tile from their opponent's ten and place it back, face up, in the bone pile. After drawing their ten tiles the players continue by exchanging one of the tiles in front of them with one from the bone pile that has a letter in common, still hoping to form Mills. When a player is reduced to just two tiles, they have lost the game.

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The co-authors, long-time collaborators, are avid collectors of all things Gardner: books, articles, reviews, magic, puzzles, games, et.al.