SCRABBLE AND RETROGRADE ANALYSIS

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Reading Mike Keith's article "Scrabble-Tile Double Word Squares and Rectangles" (WW Feb 2011) led me to ask the following question: which of the word grids he gave are actually legal positions in a game of Scrabble? Chess problemists use the term "retrograde analysis" for the study of how chess positions might have arisen in a legal game; this concept is of course equally applicable to other games, including Scrabble.

I should say, for the benefit of readers who may wish to pursue any of the questions I ask, that when writing this article I have not attempted to consult any online word lists; all the words used here are either words I know or words used in Mike Keith's article.

The rules of Scrabble are easy to axiomatise. (The following includes references to fixed numbers – 7, 15, 100 – which are correct for the standard game of Scrabble, but which can in principle be changed.)

1. The **alphabet** is a set of symbols called **letters.** A **word** is a sequence of at least 2 letters. (Note that 1-letter words play no role in Scrabble.) A **subword** of a word is a sequence of at least 2 letters obtainable by removing (possibly zero) letters from the beginning and end; it is **proper** if at least one letter is removed. The **lexicon** is a set of words (those which are allowed in Scrabble). The **ensemble** is the collection of letters, each with a given multiplicity, which can be used in a Scrabble position. Apart from a letter for each lettered tile, the ensemble includes "wild cards" – the equivalent of the blank tiles which can represent any letter – and its total size is 100.

2. A **polyomino** is a geometrical figure, made up of unit squares, where any two squares are connected by a sequence in such a way that adjacent squares in the sequence share a common (full length) edge. The **board** is a particular polyomino, a square of side 15, whose central square plays a distinguished role. A polyomino is **possible** if it has between 2 and 100 squares and can be placed within the board so as to include the central square. In a polyomino a **chain** of unit squares consists of an $n \times 1$ or $1 \times n$ rectangle (with $n \geq 2$) entirely within the polyomino, and such that if extended (in either of the two possible directions) to an $n+1 \times 1$ or $1 \times n+1$ rectangle then it is no longer entirely within the polyomino.

3. A **crossword** is a polyomino, with left-right or up-down directions defined on every chain in a **compatible way, in which each unit square is labelled with a letter.** It is **possible** if the polyomino is possible, and if every chain, read left to right or top to bottom, forms a word within the lexicon. A word is **possible** if its letters fit within the ensemble (i.e. excluding words like RAZZMATAZZ), and if it fits within the board (i.e. has at most 15 letters).

4. An **initial move** consists of the setting up of a chain, of length at most 7, which is labelled and oriented to form a word within the lexicon and positioned on the board in such a way as to include the central square. A **move** is either an initial move, or an operation that takes a crossword A to a crossword B, both possible and with the same orientation, by inserting between 1 and 7 squares, labelled with letters, in such a way that there is a single word of B which contains all the inserted letters. An **unmove** is the reverse of a move.

5. A crossword is **legal** if it is the result of an initial move followed by a (possibly empty) sequence of moves. A polyomino is **legal** if it can be labelled with letters so as to form a legal crossword. A word is **legal** if there is a legal crossword in which it appears as a word. A polyomino is **true** if any crossword obtained by labelling it is legal if it is possible, and if this remains so with an arbitrary lexicon. (That means essentially that it can be built up without using any proper subwords of the words that appear in it.) A **grid** is a rectangular polyomino in which each dimension is at least 2. Without loss of generality we can assume that its length – the distance from left to right – is at least as great as its height – the distance from top to bottom. (This was the case with all the grids depicted in Mike Keith's article.)

Note the following:

(a) I am assuming that all 100 tiles can be placed on the board. This is not true of a real game with at least 2 players, as a game ends as soon as any player has no tiles left. In general, in an n-player game, the maximum number of tiles than can be placed before the game ends is $101 - n$. But in a solitaire game all 100 tiles can be placed (as the above formula implies).

Q1: Is there any other way in which a crossword can be legal in a solitaire game but not in a game with at least 2 players? To show how this might happen, imagine a 2-player game in which 86 tiles had been placed. This would leave each player with 7 tiles. It would therefore be impossible to extend this game by three 4-letter moves. Can one find a crossword which cannot be reached by a different sequence of moves?

(b) The above does not involve itself with matters of scoring, or the possibility of "pass moves" in which a player exchanges letters. Neither is relevant to the question of which positions can be the result of a legal game.

To start the ball rolling (and illustrate the definition of trueness) we prove that any grid of size at most 8×3 is true. In fact, we can state a stronger result: a crossword based on such a grid can be positioned with any letter being at the central square of the board, and however this is done the crossword will be legal. The proof is simple: choose as the initial move the column (of length 2 or 3) containing the central square; then complete the top row; then complete the bottom row; then, if the height of the grid is 3, complete the middle row. It can easily be seen that the only words created are those that appear in the full grid.

This means that the smallest possible grid which might be possible but not legal is 4×4. It is easy to see that such a grid is not true, because any possible unmove must create at least one new word (in fact more than one).

Q2: Is there a 4×4 grid which is possible but not legal? If not, what is the smallest grid of this type?

Mike Keith's article included the "heterogrammatic" 4×4 grid shown below (Table 1, centre). This grid *is* legal: one can start by removing the B, then the S, E and W, then the rest is easy.

There are polyominoes smaller than the 4×4 grid which are not true; this can happen when there are words too long to be playable in a single move, but it can also happen for the same reason as with a 4×4 grid. I am pretty sure that the smallest case of the latter is a 4×4 square with its corners removed.

Q3: Is there a crossword based on this polyomino which is possible but not legal? If this is to happen, none of the proper subwords of its 4-letter words may appear in the lexicon. If there is no crossword of this type, what is the smallest crossword consisting of more than one word that is possible but not legal?

Mike Keith said that there was a unique 7×7 grid which was possible (and where no two words were the same). This grid is reproduced below (Table 1, left). I was delighted to find that it *is* legal: it can be created by the following sequence of moves:

A: Start with the word SNEEZER.

B: In any order, lay the words SNIP, EIDOLIC, ZONATED and RESECTS in the first, third, fifth and bottom rows.

C: In any order, lay the words NOISOME, PHORATE, MI and RECEDES in the second, fourth, sixth and last columns.

D: Complete the words ESERINE, EMINENT, POLITIC and INDENES in that order.

However, I was *unable* to find a legal game leading to his "high scoring" 7×5 grid (Table 1, right)

Let me conclude with some further questions for readers:

Q4: Find a polyomino that is possible but not legal. Mike Keith's article would have proved that this was true of a grid of size larger than those he constructed but for his requirement that all words within any of his grids be distinct. However I am pretty sure that such a polyomino exists.

Q5: Is every word legal if it is possible? Note that even a 15-letter word can be built up independently of whether appropriate subwords are in the lexicon; the relevant unmove is to remove the letters in the 7 even positions and leave the letters in the 8 odd positions. However it is not obvious, though it seems likely, that a crossword can always be chosen in such a way that this unmove is to a possible position and can be followed by further unmoves which prove the crossword legal.

S	N	I	P	P	E	R
N	O	N	H	O	M	E
E	I	D	O	L	I	C
E	S	E	R	I	N	E
Z	O	N	A	T	I	D
E	M	E	S	E	N	E
R	E	S	E	C	T	S

B	R	I	G
A	U	T	O
S	L	E	W
H	Y	M	N

Q	U	I	V	E	R	S
A	L	L	O	X	A	N
I	T	E	M	I	Z	E
D	R	A	I	N	E	D
S	A	L	T	E	R	S

A PUZZLE

across

1 Spring
4 Winning ___
8 Won a tennis serve
12 Adj: suff.
13 *King* ___
14 Tinkers, Evers and Chance
15 Boxing
18 Tie score
19 Thus: Lat.
20 Button or Fleming
23 Croquet hoop
26 Sailor's aid
27 Sheepish remark
28 Adam's rib
29 Bows
30 Spar
31 Solar disc
32 ___ jongg
33 America's ___
34 Abounding in: suff.
35 Belonging to 1964, 1500 m Gold Medalist
37 Puzzle theme
38 King: Fr.
39 100 m race, e.g.
40 Brit. Grand National entrant
46 Back talk
47 Dynamics or drome
48 Basketball court zone
49 ___ in the towel
50 Handicap: sl.
51 Island: Fr.

down

1 Small drink
2 Golfer's goal
3 15A name
4 Football team
5 Postpone
6 Ten yard ___
7 Energy unit
8 Loft
9 British game player
10 The same: Ger.
11 ___ leg
16 Gr. letters
17 Chin. dynasty
20 Bridge coups
21 Islamic book
22 Crossbow expert: arch.
23 43D need
24 Happening
25 Hikers' gear
27 Hit: sl.
30 Team vehicle
31 Hula Bowl welcome
33 Football penalty
34 Conclusion
36 Soil deposit
37 Bone: pref.
39 ___-sea fishing
40 Concorde
41 The way: Ch.
42 ___ Vegas
43 Lodge type
44 Moray
45 Whiskey