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Interpolating with outer functions

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Abstract

The classical theorems of Mittag-Leffler and Weierstrass show that when $(\lambda_n)_{n\geq 1}$ is a sequence of distinct points in the open unit disk $\mathbb{D}$, with no accumulation points in $\mathbb{D}$, and $(w_n)_{n\geq 1}$ is any sequence of complex numbers, there is an analytic function φ on $\mathbb{D}$ for which $\varphi(\lambda_n) = w_n$. A celebrated theorem of Carleson [2] characterizes when, for a bounded sequence $(w_n)_{n\geq 1}$, this interpolating problem can be solved with a bounded analytic function. A theorem of Earl [5] goes further and shows that when Carleson's condition is satisfied, the interpolating function φ can be a constant multiple of a Blaschke product. Results from [4] determine when the interpolating function φ can be taken to be zero free. In this paper we explore when φ can be an outer function.

Keywords Interpolating sequences · Hardy spaces · Outer functions

Mathematics Subject Classification 30H10 · 47B35 · 30E05 · 41A05

Dedicated to Harold S. Shapiro.

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1 Interpolation

Interpolation problems for analytic functions have been a mainstay in complex and harmonic analysis since its conception in the late 19th century. The general idea is that we have a certain class $\mathcal{X}$ of analytic functions on the open unit disk $\mathbb{D}$ (e.g., all analytic functions, bounded analytic functions, analytic self maps of $\mathbb{D}$, Blaschke products, zero-free functions, outer functions). Then, for a sequence $(\lambda_n)_{n\geq 1}$ of distinct points in $\mathbb{D}$ and sequence $(w_n)_{n\geq 1}$ of complex numbers, when can we find an $f \in \mathcal{X}$ such that

$$f(\lambda_n) = w_n \quad \text{for all } n \geq 1? \tag{1.1}$$

If we are not able to solve this problem for all $(\lambda_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$, what restrictions must we have?

Suppose $\mathcal{X}$ is the class of *all* analytic functions on $\mathbb{D}$. For a sequence $(\lambda_n)_{n\geq 1}$ of distinct points in $\mathbb{D}$ (with no limit point in $\mathbb{D}$) and any sequence $(w_n)_{n\geq 1}$, an application of the classical Mittag–Leffler theorem and the Weierstrass factorization theorem produces an $f \in \mathcal{X}$ that satisfies (1.1). In other words, for the class $\mathcal{X}$ of all analytic functions on $\mathbb{D}$, besides the obvious restriction that $(\lambda_n)_{n\geq 1}$ has no limit points in $\mathbb{D}$, there is no other restriction on $(\lambda_n)_{n\geq 1}$ to be able to interpolate any sequence $(w_n)_{n\geq 1}$ with an analytic function.

Of course, there are the finite interpolation problems. For example, a well known result of Lagrange (from 1795) says that given distinct $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$ and arbitrary $w_1, \ldots, w_n$ in $\mathbb{C}$ there is a polynomial p of degree $n-1$ such that $p(\lambda_j) = w_j$ for all $1 \leq j \leq n$. There is also the often-quoted result of Nevanlinna and Pick (from 1916) which says that given distinct $\lambda_1, \ldots, \lambda_n$ in $\mathbb{D}$ and arbitrary $w_1, \ldots, w_n$ in $\mathbb{D}$, there is an analytic $f : \mathbb{D} \to \mathbb{D}$ for which $f(\lambda_j) = w_j$, $1 \leq j \leq n$, if and only if the Nevanlinna-Pick matrix

$$\left[\frac{1-\overline{w_i}w_j}{1-\overline{\lambda_i}\lambda_j}\right]_{1\leq i,j\leq n}$$

is positive semidefinite [1,7].

When $\mathcal{X}$ is the class of bounded analytic functions on $\mathbb{D}$, denoted in the literature by H^∞, a well-known theorem of Carleson [2] (see also [7]) says the following: for a sequence $(\lambda_n)_{n\geq 1} \subseteq \mathbb{D}$ the following conditions are equivalent:

(i) given any bounded sequence $(w_n)_{n\geq 1}$ there is a $\varphi \in H^\infty$ satisfying (1.1);

(ii)

$$\inf_{n\geq 1} \prod_{k=1, k\neq n}^{\infty} \left|\frac{\lambda_k - \lambda_n}{1-\overline{\lambda_k}\lambda_n}\right| > 0. \tag{1.2}$$

Such a $(\lambda_n)_{n\geq 1}$ is called an *interpolating sequence* since the map

$$\Lambda : H^\infty \to \ell^\infty(\mathbb{N}), \quad \Lambda(\varphi) = (\varphi(\lambda_n))_{n\geq 1}$$

is surjective.

In this paper we investigate the type of functions $\varphi \in H^\infty$ that can perform the interpolating. For example, a result of Earl [5] says that when (1.2) holds, one can always take the interpolating function φ to be a constant multiple of a Blaschke product. Blaschke products have zeros in $\mathbb{D}$ which leads us to ask: can one choose φ to be zero free? Can one choose φ to be outer? Under certain circumstances, the answer to the first question is yes and was explored in [4]. The answer to the second question is not as well understood and is the focus of this paper.

2 Some notation

Let us set our notation and review some well-known facts about the classes of analytic functions that appear in this paper. The books [3,7,9] are thorough references for the details and proofs. In this paper, $\mathbb{D}$ is the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}$ the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and $dm = d\theta/2\pi$ is normalized Lebesgue measure on $\mathbb{T}$. For $0 < p < \infty$, we use $L^p(m)$ to denote the space of (Lebesgue) integrable functions and $L^\infty(m)$ to denote the essentially bounded measurable functions.

For $0 < p < \infty$, the Hardy space H^p is the set of analytic functions f on $\mathbb{D}$ for which

$$\|f\|_p := \Big(\sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^p dm(\xi) \Big)^{\frac{1}{p}} < \infty.$$

Standard results say that every $f \in H^p$ has a radial limit

$$f(\xi) := \lim_{r \to 1^-} f(r\xi)$$

for almost every $\xi \in \mathbb{T}$ and $\|f\|_{L^p(m)} = \|f\|_p$. As is the usual practice in Hardy spaces, we use the symbol f to denote the boundary function on $\mathbb{T}$ as well as the analytic function on $\mathbb{D}$.

Let H^∞ denote the bounded analytic functions on $\mathbb{D}$ and observe that $H^\infty \subseteq H^p$ for all p and thus every $f \in H^\infty$ also has a radial boundary function. In fact,

$$\sup_{z \in \mathbb{D}} |f(z)| = \|f\|_{L^\infty(m)}.$$

If $f \in H^p \backslash \{0\}$ then

$$\int_{\mathbb{T}} \log |f| dm > -\infty$$

and thus the boundary function for f does not vanish on any set of positive measure.

3 Known facts about outer functions

For $W \in L^1(m)$ and real valued,

$$\varphi(z) = \exp\Big(\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} W(\xi) dm(\xi)\Big) \tag{3.1}$$

is analytic on $\mathbb{D}$ and is called an *outer function*. The class of outer functions will be denoted by $\mathcal{O}$. For all $z \in \mathbb{D}$, observe that

$$\log|\varphi(z)| = \int_{\mathbb{T}} \Re\Big(\frac{\xi + z}{\xi - z}\Big) W(\xi)\, dm(\xi) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} W(\xi)\, dm(\xi),$$

which is the Poisson integral of W. By some harmonic analysis [7, p. 15],

$$\lim_{r \to 1^-} \log|\varphi(r\zeta)| = \log|\varphi(\zeta)| = W(\zeta) \tag{3.2}$$

for almost every $\zeta \in \mathbb{T}$. Moreover, $\varphi \in H^p$ precisely when $e^W \in L^p(m)$ and $\varphi \in H^\infty$ precisely when $e^W \in L^\infty(m)$.

The outer functions belong to the *Smirnov class*

$$N^+ = \big\{ f/g : f \in H^\infty, g \in H^\infty \cap \mathcal{O} \big\}$$

and every $F \in N^+$ can be factored as $F = I_F O_F$, where I_F is inner ($I_F \in H^\infty$ with unimodular boundary values almost everywhere on $\mathbb{T}$) and O_F is outer. There are also the (proper) inclusions $H^\infty \subsetneq H^p \subsetneq N^+$.

The following are well known classes of outer functions.

Proposition 3.1 *If f is analytic on $\mathbb{D}$, then any of the following conditions implies that $f \in \mathcal{O}$.*

(a) $\Re f > 0$ *on* $\mathbb{D}$.
(b) $f \in H^p$ *for some* $0 < p < \infty$ *and* $1/f \in H^r$ *for some* $0 < r < \infty$.
(c) f *is a rational function with no zeros or poles in* $\mathbb{D}$.

Important to this paper will be outer functions which take the form $f = \varphi^\psi$, where $\varphi, \psi \in \mathcal{O}$. Indeed this is something that needs checking since if $\varphi, \psi \in \mathcal{O}$, then $f = \varphi^\psi$, though analytic on $\mathbb{D}$ and zero-free, need not be outer. In fact with $\varphi = e$ (a constant outer function) and

$$\psi(z) = -\frac{1 + z}{1 - z},$$

which is outer by Proposition 3.1, then

$$f(z) = \varphi(z)^{\psi(z)} = \exp\Big(-\frac{1 + z}{1 - z}\Big)$$

is inner! Note that the only functions that are both inner and outer are the unimodular constants.

If $u \in L^1(m)\backslash\{0\}$ and $u \geq 0$ on $\mathbb{T}$, the *Herglotz integral*

$$H_u(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} u(\xi) dm(\xi) \tag{3.3}$$

is analytic on $\mathbb{D}$ and

$$\Re H_u(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} u(\xi) dm(\xi) > 0, \quad z \in \mathbb{D}.$$

By Proposition 3.1, H_u is outer. In fact $H_u \in H^p$ for all $0 < p < 1$.

Lemma 3.2 *For $f \in H^1$ there are $G_j \in N^+$ with $\Re G_j > 0$ on $\mathbb{D}$ for $j = 1, 2$ such that $f = G_1 - G_2$.*

Proof Functions in H^1 have radial boundary values almost everywhere on $\mathbb{T}$ and so let u_+ and u_- be defined for almost every $\xi \in \mathbb{T}$ by

$$u_+(\xi) = \max(\Re f(\xi), 0), \quad u_-(\xi) = \max(-\Re f(\xi), 0).$$

Since $|\Re f(\xi)| \leq |f(\xi)|$ and $|f|$ is integrable on $\mathbb{T}$, we see that u_+, u_- are nonnegative integrable functions. Furthermore, by the discussion above, H_{u_+} and H_{u_-} belong to N^+ and have positive real parts on $\mathbb{D}$. Finally, $H_{u_+} - H_{u_-}$ belongs to N^+ and has the same real part as f on $\mathbb{T}$. Thus, by the uniqueness of the harmonic conjugate, $f = H_{u_+} - H_{u_-} + ic$ for some $c \in \mathbb{R}$. This completes the proof. □

Lemma 3.3 *If $f \in H^1$, then e^f is outer.*

Proof By the previous lemma, $f = H_{u_+} - H_{u_-} + ic$ and so

$$e^f = e^{ic} \frac{e^{-H_{u_-}}}{e^{-H_{u_+}}}.$$

From the formula for the Herglotz integral in (3.3) and the definition of outer from (3.1), the functions $e^{-H_{u_+}}$ and $e^{-H_{u_-}}$ are outer. Thus, e^f is also outer. □

Proposition 3.4 *Let $\varphi \in \mathcal{O}$ and $\psi \in \mathcal{O} \cap H^\infty$.*

(a) If $\arg \varphi \in L^1(m)$, then $f = \varphi^\psi \in \mathcal{O}$.
(b) If $\arg \varphi \in L^\infty(m)$ and $\Re \psi > 0$ on $\mathbb{D}$, then $f = \varphi^\psi \in \mathcal{O} \cap H^\infty$.

Proof On $\mathbb{T}$ we have

$$\begin{aligned} |\log \varphi| &\leq \big|\log |\varphi|\big| + |\arg \varphi| \\ &= \big|\log(|\varphi|/\|\varphi\|_\infty) + \log \|\varphi\|_\infty\big| + |\arg \varphi| \\ &\leq \big|\log(|\varphi|/\|\varphi\|_\infty)\big| + \big|\log \|\varphi\|_\infty\big| + |\arg \varphi| \end{aligned}$$

$$= -\log(|\varphi|/\|\varphi\|_\infty) + \big|\log\|\varphi\|_\infty\big| + |\arg\varphi|$$
$$= -\log|\varphi| + 2\log^+\|\varphi\|_\infty + |\arg\varphi|.$$

Since $\log|\varphi| \in L^1(m)$ and $|\arg\varphi| \in L^1(m)$, it follows that $|\log\varphi| \in L^1(m)$. Since φ is outer, $\log\varphi \in N^+$. A standard result [3, p. 28] of Smirnov implies $\log\varphi \in H^1$. Therefore, $\psi\log\varphi \in H^1$. By the previous lemma, $f = \exp(\psi\log\varphi)$ is outer. This proves (a).

If we assume that

$$|\Im\log\varphi| = |\arg\varphi| \le M \quad \text{and} \quad \Re\psi \ge 0$$

on $\mathbb{T}$, we have

$$\begin{aligned} |f| &= \exp(\Re\psi\ \log|\varphi| - \Im\psi\ \arg\varphi) \\ &\le \exp(\|\psi\|_\infty \log(1+\|\varphi\|_\infty) + M\|\psi\|_\infty). \end{aligned}$$

Thus, the function f is bounded and outer. Note that

$$\Re\psi\ \log|\varphi| \le \|\psi\|_\infty \log(1+\|\varphi\|_\infty)$$

follows from the fact that $\Re\psi \ge 0$ on $\mathbb{T}$. This proves (b). □

4 Positive results

We start off with examples of bounded $(w_n)_{n\ge1}$ which can be interpolated by outer functions and explore the ones which can not in the next section.

Theorem 4.1 *Suppose $(\lambda_n)_{n\ge1} \subseteq \mathbb{D}$ is interpolating. If $(w_n)_{n\ge1}$ is bounded such that*

$$\inf_{n\ge1}|w_n| > 0,$$

there is a $\varphi \in \mathcal{O}\cap H^\infty$ such that $\varphi(\lambda_n) = w_n$ for all n.

Proof For a suitable branch of the logarithm, the sequence $\log w_n$ is bounded and thus there is an $f \in H^\infty$ such that $f(\lambda_n) = \log w_n$ (Carleson's theorem). By Lemma 3.3, $\varphi = e^f$ is bounded and outer with $\varphi(\lambda_n) = w_n$. □

This next result says that for outer interpolation, we can always assume, for example, that the targets w_n are positive.

Proposition 4.2 *Suppose $(\lambda_n)_{n\ge1}$ is interpolating and $(w_n)_{n\ge1}$ and $(w_n')_{n\ge1}$ are bounded with*

$$0 < m \le \left|\frac{w_n'}{w_n}\right| \le M < \infty, \quad n \ge 1.$$

Then $(w_n)_{n\geq 1}$ *can be interpolated by an outer (bounded outer) function if and only if* $(w'_n)_{n\geq 1}$ *can be interpolated by an outer (bounded outer) function.*

Proof By Theorem 4.1 there is a bounded outer ψ such that $\psi(\lambda_n) = w'_n/w_n$ for all $n \geq 1$. If there is an outer (bounded outer) φ such that $\varphi(\lambda_n) = w_n$ for all n, then the outer (bounded outer) function $\varphi\psi$ (note that the class of outer functions is closed under multiplication) performs the desired interpolation for $(w'_n)_{n=1}^{\infty}$. □

Remark 4.3 If φ is outer (bounded outer) then so is φ^c for any $c > 0$. Thus $(w_n^c)_{n\geq 1}$ can be interpolated by an outer (bounded outer) function whenever $(w_n)_{n\geq 1}$ can.

5 Negative results–existence of an inner factor

If $(\lambda_n)_{n\geq 1}$ is interpolating, we know that given any bounded $(w_n)_{n\geq 1}$ there is a $\varphi \in H^\infty$ such that $\varphi(\lambda_n) = w_n$. This next result says that under certain circumstances, any N^+-interpolating function for $(w_n)_{n\geq 1}$ must have an inner factor.

Theorem 5.1 *If* $(\lambda_n)_{n\geq 1}$ *is interpolating and* $(w_n)_{n\geq 1} \subseteq \mathbb{C}\setminus\{0\}$ *satisfies*

$$\lim_{n\to\infty} (1 - |\lambda_n|) \log |w_n| \neq 0,$$

then any $\varphi \in N^+$ *satisfying* $\varphi(\lambda_n) = w_n$ *for all n must have a non-trivial inner factor.*

Proof The proof of this theorem follows from the following fact from [10]: If $\varphi \in \mathcal{O}$, then

$$\lim_{|z|\to 1^-} (1 - |z|) \log |\varphi(z)| = 0.$$

We include a proof for the sake of completeness.

Let $a > 1$ and $E_a = \{\xi \in \mathbb{T} : |\varphi(\xi)| > a\}$. Then $\log |\varphi| > 0$ on E_a and an application of

$$\int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2}\, dm(\xi) = 1 \quad \text{for all } z \in \mathbb{D}, \tag{5.1}$$

and

$$\frac{1 - |z|^2}{|\xi - z|^2} \leq \frac{2}{1 - |z|} \quad \text{for all } z \in \mathbb{D} \text{ and } \xi \in \mathbb{T}, \tag{5.2}$$

give us

$$\begin{aligned} \log |\varphi(z)| &= \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \xi|^2} \log |\varphi(\xi)| dm(\xi) \\ &= \int_{E_a} \frac{1 - |z|^2}{|z - \xi|^2} \log |\varphi(\xi)| dm(\xi) + \int_{\mathbb{T}\setminus E_a} \frac{1 - |z|^2}{|z - \xi|^2} \log |\varphi(\xi)| dm(\xi) \\ &\leq \frac{2}{1 - |z|} \int_{E_a} \log |\varphi(\xi)| dm(\xi) + \log a \int_{\mathbb{T}\setminus E_a} \frac{1 - |z|^2}{|z - \xi|^2} dm(\xi) \end{aligned}$$

$$\leq \frac{2}{1-|z|}\int_{E_a}\log|\varphi(\xi)|dm(\xi)+\log a.$$

Hence for all $z\in\mathbb{D}$,

$$(1-|z|)\log|\varphi(z)|\leq 2\int_{E_a}\log|\varphi(\xi)|dm(\xi)+(1-|z|)\log a,$$

which implies

$$\overline{\lim_{|z|\to 1^-}}(1-|z|)\log|\varphi(z)|\leq 2\int_{E_a}\log|\varphi(\xi)|dm(\xi).$$

Now let $a\to+\infty$ and use the fact that $\log|\varphi|\in L^1(\mathbb{T})$ to deduce

$$\overline{\lim_{|z|\to 1^-}}(1-|z|)\log|\varphi(z)|\leq 0. \tag{5.3}$$

Since $1/\varphi$ is also outer, the above argument also implies

$$\overline{\lim_{|z|\to 1^-}}(1-|z|)\log|1/\varphi(z)|\leq 0,$$

or equivalently

$$\underline{\lim}_{|z|\to 1^-}(1-|z|)\log|\varphi(z)|\geq 0. \tag{5.4}$$

The result now follows by comparing (5.3) and (5.4). □

Example 5.2 If

$$w_n := \exp\Big(-\frac{1}{1-|\lambda_n|}\Big),\quad n\geq 1,$$

then any interpolating $\varphi\in N^+$ for $(w_n)_{n=1}^{\infty}$ is not outer.

Remark 5.3 One notices from the proof of Theorem 5.1 that if

$$\lim_{n\to\infty}(1-|\lambda_n|)\log|w_n|\neq 0,$$

then any $\varphi\in N^+$ satisfying $|\varphi(\lambda_n)|\asymp|w_n|$ for all n must have a non-trivial inner factor.

Let us comment here that when the hypothesis of Theorem 5.1 is satisfied, the inner factor that appears in the interpolating function φ plays a significant role in the decay of φ.

Corollary 5.4 *Suppose* $(\lambda_n)_{n\geq 1}$ *is interpolating and* $(w_n)_{n\geq 1} \subseteq \mathbb{C}\backslash\{0\}$ *is bounded and satisfies*

$$\lim_{n\to\infty}(1-|\lambda_n|)\log|w_n| \neq 0.$$

If I_φ *is the inner factor for a* $\varphi \in N^+$ *for which* $\varphi(\lambda_n) = w_n$ *for all* n*, then*

$$\varliminf_{n\to\infty}|I_\varphi(\lambda_n)| = 0.$$

Proof Let $\varphi = F_\varphi I_\varphi$, where F_φ is outer and I_φ is inner. If $|I_\varphi(\lambda_n)| \geq \delta > 0$ for all n, then $I_\varphi(\lambda_n) = w_n/F_\varphi(\lambda_n)$ satisfies the hypothesis of Theorem 4.1 and so there is a $\psi \in \mathcal{O} \cap H^\infty$ with $\psi(\lambda_n) = I_\varphi(\lambda_n)$ and hence $F_\varphi\psi$ is outer and interpolates w_n. This says that $(w_n)_{n\geq 1}$ can be interpolated by an outer function – which it can not. □

Remark 5.5 The above says that a subsequence of $(\lambda_n)_{n=1}^\infty$ must approach

$$\Big\{\xi \in \mathbb{T} : \varliminf_{z\to\xi}|I_\varphi(z)| = 0\Big\},$$

the boundary spectrum of the inner factor I_φ. This set will consist of the accumulation of the zeros of the Blaschke factor of I_φ as well as the support of the singular measure associated with the singular inner inner factor of I_φ [6, p. 152].

6 Negative results–existence of a Blaschke factor

This next result says that under the right circumstances, any N^+-interpolating function must have a Blaschke factor.

Theorem 6.1 *Suppose* $(\lambda_n)_{n\geq 1}$ *is interpolating and* $(w_n)_{n\geq 1} \subseteq \mathbb{C}\backslash\{0\}$ *is bounded and satisfies*

$$\varlimsup_{n\to\infty}(1-|\lambda_n|)|\log|w_n|| = \infty.$$

Then any $\varphi \in N^+$ *for which* $\varphi(\lambda_n) = w_n$ *for all* $n \geq 1$ *must have a Blaschke factor.*

Proof Any zero-free $\varphi \in N^+$ can be written as

$$\varphi(z) = \exp\Big(\int_{\mathbb{T}}\frac{\xi+z}{\xi-z}W(\xi)dm(\xi)\Big)\exp\Big(-\int_{\mathbb{T}}\frac{\xi+z}{\xi-z}d\mu(\xi)\Big), \tag{6.1}$$

where W is a real-valued integrable function and μ is a positive measure that is singular with respect to Lebesgue measure m. The theorem will follow from the following fact: if $\varphi \in N^+$ and zero free, then

$$\varlimsup_{|z|\to 1^-}(1-|z|)\Big|\log|\varphi(z)|\Big| < \infty.$$

From (6.1) we have

$$\log|\varphi(z)| = \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\xi|^2} W(\xi)dm(\xi) - \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\xi|^2} d\mu(\xi).$$

The proof of Theorem 5.1 shows that

$$\lim_{|z|\to 1^-} (1-|z|) \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\xi|^2} W(\xi)dm(\xi) = 0.$$

From (5.2) we have

$$0 \leq (1-|z|) \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\xi|^2} d\mu(\xi) \leq 2 \int_{\mathbb{T}} d\mu = 2\mu(\mathbb{T}).$$

Combine these two facts to prove the result. □

This says, for example, that for an interpolating sequence $(\lambda_n)_{n=1}^{\infty}$, any $\varphi \in H^\infty$ for which

$$\varphi(\lambda_n) = \exp\Big(-\frac{1}{(1-|\lambda_n|)^2}\Big)$$

(and such φ exist by Carleson's theorem) must have a Blaschke factor.

Remark 6.2 From the proof of Theorem 6.1 if

$$\overline{\lim_{n\to\infty}} (1-|\lambda_n|)|\log|w_n|| = \infty.$$

Then any $\varphi \in N^+$ for which $|\varphi(\lambda_n)| \asymp |w_n|$ for all $n \geq 1$ must have a Blaschke factor.

7 Growth rates

We know from Corollary 5.4 that if $(\lambda_n)_{n\geq 1}$ is interpolating and if $(w_n)_{n\geq 1}$ can be interpolated by an outer function, then

$$\lim_{n\to\infty} (1-\lambda_n)\log|w_n| = 0.$$

What is the decay rate of $(1-\lambda_n)\log|w_n|$? To make our results less wordy, we focus our attention on the case when $(\lambda_n)_{n\geq 1} \subseteq (0,1)$. Though it does not play a role in our results, it is known [3, p. 156] that $(\lambda_n)_{n\geq 1} \subseteq (0,1)$ is interpolating if and only if there is a $0 < c < 1$ such that

$$(1-\lambda_{n+1}) \leq c(1-\lambda_n), \quad n \geq 1.$$

Such sequences are called *exponential sequences*. Naively speaking, the following discussion from [8] says that the decay rate of $(1-\lambda_n)\log|w_n|$ is controlled by an absolutely continuous function and this can not be of any desired decay. The sharpness of this observation will be studied in Theorem 7.2.

Theorem 7.1 *Suppose $(\lambda_n)_{n\geq 1}\subseteq(0,1)$ is interpolating and $(w_n)_{n\geq 1}\subseteq\mathbb{C}\backslash\{0\}$ satisfies*

$$M := \sup_{n\geq 1}|w_n| < \infty.$$

Suppose there is a $\varphi\in\mathcal{O}$ for which $\varphi(\lambda_n)=w_n$ for all n. Then there is an $h\in L^1[0,1]$ that is positive and decreasing such that

$$-(1-\lambda_n)\log\left|\frac{w_n}{M}\right| \leq \int_0^{1-\lambda_n} h(t)dt, \quad n\geq 1.$$

Next we improve Theorem 7.1 with this sharpness result.

Theorem 7.2 *Suppose $(\lambda_n)_{n\geq 1}\subseteq(0,1)$ is interpolating and $h\in L^1[0,1]$ is positive and decreasing. If $(w_n)_{n\geq 1}\subseteq\mathbb{C}\backslash\{0\}$ is bounded and satisfies*

$$-(1-\lambda_n)\log|w_n| \asymp \int_0^{1-\lambda_n} h(t)dt,$$

then there is a $\psi\in\mathcal{O}\cap H^\infty$ such that

$$-(1-\lambda_n)\log\psi(\lambda_n) \asymp \int_0^{1-\lambda_n} h(t)dt.$$

In the above we use the notation $A_n \asymp B_n$ to mean there is a $c>0$ such that $\frac{1}{c}A_n \leq B_n \leq cA_n$ for all n.

Without getting into the fine details, which are carefully done in [8], let us mention the ideas that go into proving these two results. Let $h:[0,\pi]\to[0,\infty)$ belong to $L^1[0,1]$ be decreasing, right-continuous, and is zero on $(1,\pi]$. If

$$P_r(t) = \frac{1-r^2}{1-2r\cos t+r^2}, \quad 0\leq r<1, \quad -\pi\leq t\leq\pi,$$

is the standard Poisson kernel, consider the function

$$A_h(r) = (1-r)\int_{-\pi}^{\pi} P_r(t)h(|t|)\frac{dt}{2\pi}.$$

Then,

$$A_h(r) \asymp \int_0^{1-r} h(t)dt. \tag{7.1}$$

If $\varphi \in \mathcal{O} \cap H^\infty$ and, without loss of generality, $|\varphi| \leq 1$ on $\mathbb{D}$, then

$$-(1-r)\log|\varphi(r)| = -(1-r)\int_{-\pi}^{\pi} P_r(t)\log|\varphi(e^{it})|\frac{dt}{2\pi}$$
$$\leq (1-r)\int_{-\pi}^{\pi} P_r(t)k(t)\frac{dt}{2\pi},$$

where $k = -\min(0, \log|\psi|)$. If k^* is the symmetric decreasing rearrangement of k, a classical theorem of Hardy and Littlewood shows that

$$(1-r)\int_{-\pi}^{\pi} P_r(r)k(t)\frac{dt}{2\pi} \leq (1-r)\int_{-\pi}^{\pi} P_r(t)k^*(t)\frac{dt}{2\pi}.$$

Thus,

$$-(1-r)\log|\varphi(r)| \leq A_{k^*}(r). \tag{7.2}$$

To prove Theorem 7.2, use (3.2) to define the outer function φ by

$$|\varphi(e^{it})| = \exp(-h(|t|)).$$

Now use (7.1) to obtain the desired estimate. The proof of Theorem 7.1 follows from (7.2).

8 More delicate interpolation

Given h as in Theorem 7.2, there is a $\varphi \in \mathcal{O} \cap H^\infty$ such that

$$\log \varphi(\lambda_n) \asymp \frac{1}{1-\lambda_n}\int_0^{1-\lambda_n} h(t)\,dt \quad \text{for all } n \geq 1.$$

Can we replace $\asymp$ with $=$ in the above? Equivalently, can we find an outer (bounded outer) φ such that

$$\varphi(\lambda_n) = \exp\Big(-\frac{1}{1-\lambda_n}\int_0^{1-\lambda_n} h(t)dt\Big) \quad \text{for all } n \geq 1?$$

We certainly can find $0 < \alpha \leq d_n \leq \beta < \infty$ such that

$$\varphi(\lambda_n)^{d_n} = \exp\Big(-\frac{1}{1-\lambda_n}\int_0^{1-\lambda_n} h(t)dt\Big).$$

By Theorem 4.1 there is a $\psi \in \mathcal{O} \cap H^\infty$ with $\Re\psi > 0$ with $\psi(\lambda_n) = 1/d_n$ for all n. The function $f = \varphi^\psi$ is analytic on $\mathbb{D}$ with

$$f(\lambda_n) = \exp\Big(-\frac{1}{1-\lambda_n}\int_0^{1-\lambda_n} h(t)dt\Big)$$

and thus performs the interpolation. But of course we need to check that f is outer (bounded outer).

Let us use the results above to refine Theorem 7.2.

Theorem 8.1 *Suppose $h \in L^1[0,\pi]$ is positive and decreasing. Let $(\lambda_n)_{n\geq 1} \subseteq (0,1)$ be interpolating and $(w_n)_{n\geq 1} \subseteq \mathbb{C}\backslash\{0\}$ be bounded with*

$$-(1-\lambda_n)\log|w_n| \asymp \int_0^{1-\lambda_n} h(t)dt, \quad n \geq 1.$$

(a) If $h(|t|)\log^+ h(|t|) \in L^1[-\pi,\pi]$ then there is an $f \in \mathcal{O}$ such that $f(\lambda_n) = w_n$ for all n.
(b) If

$$\mathrm{PV}\int_{-\pi}^{\pi} \cot\Big(\frac{\theta - t}{2}\Big) h(|t|)\frac{dt}{2\pi}$$

is bounded on $[-\pi,\pi]$ then there is an $f \in \mathcal{O} \cap H^\infty$ such that $f(\lambda_n) = w_n$ for all n.

Proof From the discussion at the very beginning of this section, we can find $\varphi, \psi \in \mathcal{O} \cap H^\infty$ such that $f = \varphi^\psi$ satisfies $f(\lambda_n) = w_n$ for all n. We just need to check that f is outer (bounded outer).

By the proof of Theorem 7.2, $\log|\varphi(e^{it})| = -h(|t|)$ and

$$\begin{aligned}\varphi(z) &= \exp\Big(\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\log|\varphi(\xi)|dm\Big)\\ &= \exp\Big(\int_{\mathbb{T}} \Re\Big(\frac{\xi+z}{\xi-z}\Big)\log|\varphi(\xi)|dm + i\int_{\mathbb{T}} \Im\Big(\frac{\xi+z}{\xi-z}\Big)\log|\varphi(\xi)|dm\Big).\end{aligned}$$

From

$$\arg\varphi(z) = i\int_{\mathbb{T}} \Im\Big(\frac{\xi+z}{\xi-z}\Big)\log|\varphi(\xi)|dm \quad \text{for all } z \in \mathbb{D},$$

and standard theory involving the Hilbert transform on the circle, we have

$$\arg\varphi(e^{i\theta}) = -\,\mathrm{PV}\int_{-\pi}^{\pi} \cot\Big(\frac{\theta-t}{2}\Big)h(|t|)\frac{dt}{2\pi}.$$

A classical result of Zygmund [11, Vol I, p. 254] says that if

$$h(|t|)\log^+ h(|t|) \in L^1[-\pi,\pi],$$

then $\arg\varphi \in L^1(m)$. An application of Proposition 3.4 yields $f = \varphi^\psi \in \mathcal{O}$.

If the above Hilbert transform is bounded, another application of Proposition 3.4, along with the fact that we can always choose ψ so that $\Re\psi > 0$, yields $f = \varphi^\psi \in \mathcal{O} \cap H^\infty$. ◻

Example 8.2 If $(\lambda_n)_{n\geq 1} \subseteq (0, 1)$ is interpolating, Theorem 5.1 says that any $\varphi \in N^+$ with

$$\varphi(\lambda_n) = \exp\Big(-\frac{2}{1-\lambda_n}\Big) \quad \text{for all } n \geq 1,$$

must have an inner factor. In fact, the obvious guess at an analytic function that interpolates this sequence is

$$\varphi(z) = \exp\Big(-\frac{2}{1-z}\Big)$$

which turns out to be a constant multiple of an inner function. Indeed, the singular inner function

$$\exp\Big(-\frac{1+z}{1-z}\Big)$$

can be written as

$$\exp\Big(-\frac{1+z}{1-z}\Big) = \exp\Big(-\frac{2-(1-z)}{1-z}\Big) = \exp\Big(-\frac{2}{1-z}\Big)e.$$

Thus φ is a constant multiple of a singular inner function.

Example 8.3 Let $(\lambda_n)_{n\geq 1} \subseteq (0, 1)$ be interpolating and

$$w_n = \exp\Big(-\frac{1}{1-\lambda_n}\frac{1}{(\log\frac{100}{1-\lambda_n})^2}\Big).$$

The funciton

$$h(t) = \frac{2}{t(\log(\frac{100}{t}))^3} \quad \text{for } 0 < t < 1,$$

is positive and decreasing on $[0, 1]$ and $h(|t|)\log^+ h(|t|)$ belongs to $L^1[-1, 1]$. Thus $(w_n)_{n\geq 1}$ can be interpolated with an outer function.

Example 8.4 Let

$$w_n = \exp\Big(-\frac{1}{(1-\lambda_n)^\alpha}\Big),$$

where $0 < \alpha < 1$. In this case,

$$h(t) = \frac{1-\alpha}{t^\alpha}$$

is positive, decreasing, and

$$h(|t|)\log^+ h(|t|) \in L^1[-\pi,\pi].$$

Thus, by the previous theorem, $(w_n)_{n\geq 1}$ can be interpolated by an outer function. In fact, one can take $\varphi \in \mathcal{O} \cap H^\infty$. To see this, observe that $(1-z)^{-\alpha} \in H^1$ and so

$$\varphi(z) = \exp\Big(-\frac{1}{(1-z)^\alpha}\Big)$$

is outer (Lemma 3.3). Furthermore,

$$\frac{1}{1-e^{i\theta}} = \frac{e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}} = \frac{e^{-i\theta/2}}{-2i\sin(\theta/2)} = \frac{1}{2\sin(\theta/2)}e^{i\frac{\pi-\theta}{2}}.$$

Thus,

$$|\varphi(e^{i\theta})| \leq e^{-2^{-\alpha}\cos(\pi\alpha/2)} \quad \text{for all } \theta \in [-\pi,\pi],$$

and so $\varphi \in \mathcal{O}$ and is bounded on $\mathbb{T}$. A result of Smirnov [3, p. 28] says that $\varphi \in H^\infty$. If $0 < m \leq d_n \leq M < \infty$, one can also interpolate

$$w_n = \exp\Big(-d_n\frac{1}{(1-\lambda_n)^\alpha}\Big)$$

with an outer function.

Example 8.5 If $(\lambda_n)_{n\geq 1} \subseteq (0,1)$ is interpolating and $(d_n)_{n\geq 1}$ satisfies $0 < m \leq d_n \leq M < \infty$ for all $n \geq 1$, one can appeal to Proposition 3.4 directly to interpolate $w_n = (1-\lambda_n)^{d_n}$ with a bounded outer function. Here $f = \varphi^\psi$, where $\varphi(z) = 1-z$ (which clearly has bounded argument) and ψ is the bounded outer function with $\Re\psi > 0$ and $\psi(\lambda_n) = d_n$ for all $n \geq 1$.

9 A Harnack restriction

As it turns out, a characterization of when one can interpolate with an outer function seems to depend on the regularity of the targets $(w_n)_{n=1}^\infty$ and not merely the decay rate in (5.1). We outline the following example, a variation of one from [4], to better explain what we mean here.

Consider the interpolating sequence $\lambda_n = 1-2^{-n}$ and targets $w_n = 2^{-n}$ for $n \geq 1$. If $\varphi(z) = 1-z$, then $\varphi \in \mathcal{O}$ (Proposition 3.1) and $\varphi(\lambda_n) = w_n$. Furthermore,

$$-(1-\lambda_n)\log w_n \lesssim n2^{-n} \lesssim \int_0^{1-\lambda_n} \log\Big(\frac{1}{t}\Big)dt.$$

On the other hand, if one considers the targets

$$\widetilde{w_n} = \begin{cases} 1 & n \text{ is odd,} \\ 2^{-n} & n \text{ is even,} \end{cases}$$

then Carleson's theorem says there is a $\widetilde{\varphi} \in H^\infty$ for which $\widetilde{\varphi}(\lambda_n) = \widetilde{w_n}$. Observe that

$$-(1-\lambda_n)\log \widetilde{w_n} \lesssim n2^{-n} \lesssim \int_0^{1-\lambda_n} \log\Big(\frac{1}{t}\Big)dt.$$

In other words,

$$-(1-\lambda_n)\log w_n \quad \text{and} \quad -(1-\lambda_n)\log \widetilde{w_n}$$

have the same upper bound.

However, $\widetilde{\varphi} \notin \mathcal{O}$. Otherwise,

$$u(z) = \log\Big(\frac{\|\widetilde{\varphi}\|_\infty}{|\widetilde{\varphi}(z)|}\Big)$$

would be a positive harmonic function on $\mathbb{D}$. The invariant form of Harnack's inequality says that

$$\frac{1-\rho(z_1,z_2)}{1+\rho(z_1,z_2)} \le \frac{u(z_1)}{u(z_2)} \le \frac{1+\rho(z_1,z_2)}{1-\rho(z_1,z_2)} \quad \text{for all } z_1, z_2 \in \mathbb{D},$$

where

$$\rho(z_1,z_2) = \frac{|z_1-z_2|}{|1-\overline{z_1}z_2|}$$

is the pseudohyperbolic metric. Applying this to

$$z_1 = \lambda_{2n} \quad \text{and} \quad z_2 = \lambda_{2n+1}$$

gives us

$$\frac{1-\rho(\lambda_{2n},\lambda_{2n+1})}{1+\rho(\lambda_{2n},\lambda_{2n+1})} \le \frac{u(\lambda_{2n})}{u(\lambda_{2n+1})} \le \frac{1+\rho(\lambda_{2n},\lambda_{2n+1})}{1-\rho(\lambda_{2n},\lambda_{2n+1})},$$

which works out to be

$$\frac{1}{2+(-1+2\cdot 2^{2n})^{-1}} \le \frac{\log(\|\widetilde{\varphi}\|_\infty/2^{-n})}{\log \|\widetilde{\varphi}\|_\infty} \le 2+(-1+2\cdot 2^{2n})^{-1}.$$

These inequalities do not hold for arbitrarily large n. Thus, $\widetilde{\varphi} \notin \mathcal{O}$. That being said, Theorem 8.1, with $h(t) = \log(1/t)$ on $(0, 1]$, supplies $\psi \in \mathcal{O}$ for which

$$\psi(\lambda_n) \asymp \int_0^{1-\lambda_n} \log(1/t)\,dt.$$

Declarations

Data availability This manuscript has no associated data.

References

1. Agler, J., McCarthy, J.E.: Pick Interpolation and Hilbert Function Spaces. Graduate Studies in Mathematics, vol. 44. American Mathematical Society, Providence (2002)
2. Carleson, L.: An interpolation problem for bounded analytic functions. Am. J. Math. **80**, 921–930 (1958)
3. Duren, P.L.: Theory of H^p Spaces. Academic Press, New York (1970)
4. Dyakonov, K., Nicolau, A.: Free interpolation by nonvanishing analytic functions. Trans. Am. Math. Soc. **359**(9), 4449–4465 (2007)
5. Earl, J.P.: On the interpolation of bounded sequences by bounded functions. J. London Math. Soc. **2**(2), 544–548 (1970)
6. Garcia, S.R., Mashreghi, J., Ross, W.T.: Introduction to Model Spaces and their Operators. Cambridge Studies in Advanced Mathematics, vol. 148. Cambridge University Press, Cambridge (2016)
7. Garnett, J.: Bounded Analytic Functions. Graduate Texts in Mathematics, vol. 236, 1st edn. Springer, New York (2007)
8. Mashreghi, J., Ptak, M., Ross, W.T.: A common range problem for model spaces. to appear, Forum Mathematicum
9. Mashreghi, J.: Representation Theorems in Hardy Spaces. London Mathematical Society Student Texts, vol. 74. Cambridge University Press, Cambridge (2009)
10. Yanagihara, N.: Mean growth and Taylor coefficients of some classes of functions. Ann. Polon. Math. **30**, 37–48 (1974)
11. Zygmund, A.: Trigonometric Series. Cambridge Mathematical Library, vol. I, II. Cambridge University Press, Cambridge (2002)