Consensus Building With Individual Consistency Control in Group Decision Making

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***Abstract*—The individual consistency and the consensus degree are two basic measures to conduct group decision making with reciprocal preference relations. The existing frameworks to manage individual consistency and consensus degree have been investigated intensively and follow a common resolution scheme composed by the two phases: the consistency improving process, and the consensus reaching process. But in these frameworks, the individual consistency will often be destroyed in the consensus reaching process, leading to repeat the consistency improving process, which is time consuming. In order to avoid repeating the consistency improving process, a consensus reaching process with individual consistency control is proposed in this paper. This novel consensus approach is based on the design of an optimization-based consensus rule, which can be used to determine the adjustment range of each preference value guaranteeing the individual consistency across the process. Finally, theoretical and numerical analysis are both used to justify the validity of our proposal.**

***Index Terms*—Consensus, consistency, group decision making (GDM), optimization, preference relations.**

I. Introduction

THE reciprocal preference relation (RPR) is one of the most widely used preference representation structures in decision problems. Various types of RPRs have been proposed, such as additive preference relations (also called reciprocal fuzzy preference relations) [1]–[4], multiplicative preference relations [5]–[9], and linguistic preference relations [10], [11]. In group decision making (GDM) problems with RPRs, there are the following two measures that have been considered before obtaining a final solution [12].

1) Individual consistency. The individual consistency is applied to ensure that decision maker is being neither random nor illogical in his/her pairwise comparisons.
2) Consensus. Consensus means that the group of decision makers agreed to their preferences to some extent.

In order to effectively manage individual consistency and consensus in the GDM with RPRs, Chiclana *et al.* [12] and Herrera *et al.* [13] initiated the consensus framework for integrating an individual consistency measure in GDM with additive RPRs. This consensus framework deals with a two-step procedure: consistency improving process and consensus reaching process. Dong *et al.* [14] proposed a new automatic consensus framework to address the GDM with multiplicative RPRs by incorporating consistency and consensus measures into one phase. Based on the framework presented in [14], Wu and Xu [15] presented a consistency consensus based model for GDM with additive RPRs. In recent years, these frameworks were extended to interval-value preference relations [16], intuitionistic preference relations [17], [18], hesitant preference relations [19]–[21], and linguistic preference relations [22]–[26] to manage individual consistency and consensus in GDM.

Although the frameworks to manage individual consistency and consensus have been investigated intensively, and they are very useful in GDM, there still exist gaps that must be filled because of the following reasons.

1) In GDM problems with RPRs, the individual consistency improving process is applied before the consensus reaching process [12], [13] and it could happen that the adjusted preferences in the consensus process are not consistent. Hence, the inconsistency of the individual preferences leads to repeat the consistency improving process and the consensus process again, which is time consuming.
2) The consensus frameworks that can simultaneously manage the individual consistency and consensus in one phase without repeating consistency improving processes [14], [15] are based on an automatic consensus process that does not consider decision makers' opinions during the consensus reaching process.

In order to overcome previous shortcomings, this paper proposes a novel consensus approach to manage individual consistency and consensus in GDM with RPRs. This proposal is carried out by the following points.

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1) An optimization-based consensus rule (OCR) is defined, which is the core idea of this novel consensus approach. The OCR can be used to determine the adjustment ranges of each preference value and guarantee the individual consistency of the RPRs adjusted in the adjustment ranges. In order to obtain the optimum solution of the proposed consensus rule, an approximate algorithm (Algorithm 1) with an adjustment parameter is proposed. It provides a good approximate performance.
2) We develop a consensus reaching process with individual consistency control and provide an algorithm (Algorithm 2) to describe this process. In this process, the decision makers' opinions are considered and the consensus is achieved with the decision makers' participation. Besides, the theoretical analysis shows that the adjusted RPRs are of acceptably consistency and acceptable consensus in the consensus reaching process. Moreover, based on Algorithm 2, we present Algorithm 3 to automatically simulate the proposed consensus reaching process.

Finally, some experimental simulations regarding the consensus convergence with the individual consistency are provided. They show that the consensus improves for each iteration and the individual consistency is guaranteed without repeating the consistency improving process, which justifies the feasibility and viability of the proposed consensus approach.

In order to develop our proposal, we consider the consistency improving method proposed in [27] and the identification and direction rules from the consensus approach presented in [28].

The remainder of this paper is organized as follows: Section II introduces the background regarding the consistency and consensus methods in GDM with RPRs. Section III presents the motivation of this study: how to deal with both the consistency and the consensus in GDM. Section IV develops a consensus rule with individual consistency control via an optimization-based model in GDM with RPRs and also presents the consensus reaching process based on the proposed consensus rule. Section V explores the use of the consensus reaching process with individual consistency control by means of simulation experiments. Finally, concluding remarks are included in Section VI.

II. Preliminaries

This section introduces some basic concepts about GDM based on RPRs, measuring and improving individual consistency and group consensus in GDM with RPRs.

A. GDM Based on RPRs

Let $X = \{x_1, x_2, \ldots, x_n\}(n \geq 2)$ be a finite set of alternatives. When a decision maker makes pairwise comparisons using the preference values in $[0, 1]$, he/she can construct a fuzzy reciprocal preference relation (also called additive preference relation) $F = (f_{ij})_{n\times n}$, whose elements $f_{ij} \in [0, 1]$ represent the preference degree of alternative x_i over x_j. Fuzzy reciprocal preference relations can be formally defined as follows.

Definition 1 ([4], [29]): A fuzzy reciprocal preference relation F on a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ is a fuzzy set on the product set $X \times X$, characterized by a membership function $\mu_F : X \times X \longrightarrow [0, 1]$ fulfilling $\mu_F(x_i, x_j) + \mu_F(x_j, x_i) = 1$.

The preference relation may be conveniently represented by the $n \times n$ matrix $F = (f_{ij})_{n\times n}$ with $f_{ij} = \mu_F(x_i, x_j) \in [0, 1]$ and $f_{ij} + f_{ji} = 1$ $(\forall i, j \in \{1, 2, \ldots, n\})$, where f_{ij} is interpreted as the preference degree or intensity of the alternative x_i over x_j. $f_{ij} = 1$ indicates that the maximum degree of preference of x_i over x_j and each value of f_{ij} in the open interval $(0.5, 1)$ indicates a definite preference of x_i to x_j with the intensity of preference corresponding to the value of f_{ij} (a higher value means a stronger intensity).

There are transformation functions between fuzzy reciprocal preference relations and multiplicative preference relations [5], [30]. In this paper, we focus our study on fuzzy reciprocal preference relations, but the proposed results can be similarly applied to multiplicative preference relations and linguistic preference relations. In order to pursue generality, in this paper, fuzzy reciprocal preference relations will be denoted as RPRs.

Let $D = \{d_1, d_2, \ldots, d_m\}(m \geq 2)$ be the set of decision makers involved in the GDM problem and let $F^k = (f^k_{ij})_{n\times n}$ be the RPR over the alternative set $X = \{x_1, x_2, \ldots, x_n\}(n \geq 2)$, provided by a decision maker $d_k(k = 1, 2, \ldots, m)$. Traditionally, a selection process is applied to a GDM problem to obtain a ranking of alternatives and select the best one. It is divided into two different phases: aggregation phase and exploitation phase [31], [32].

1) In the aggregation phase, the individual RPRs are aggregated into a collective RPR $F^c = (f^c_{ij})_{n\times n}$ by using an aggregation operator. There are different aggregation operators that can be applied, such as the weighted average (WA) [33], the ordered weighted average (OWA) [34], etc. In this paper, without loss of generality, the WA operator is used.

Definition 2: Let $\{f^1_{ij}, f^2_{ij}, \ldots, f^m_{ij}\}$ be the individual fuzzy preference degrees of alternative x_i over x_j, $i, j = 1, 2, \ldots, n$, and $W = \{w_1, w_2, \ldots, w_m\}$ be their associated weights. Then, based on the WA operator, the collective fuzzy preference degree f^c_{ij} is obtained as follows:

$$f^c_{ij} = \mathrm{WA}_W(f^1_{ij}, f^2_{ij}, \ldots, f^m_{ij}) = \sum_{k=1}^{m} w_k \cdot f^k_{ij}, for\, i, \quad j = 1, \ldots, n. \tag{1}$$

2) The exploitation phase obtains the ranking of alternatives. There are various approaches in the literature to rank alternatives, such as the approaches based on dominance and nondominance degrees of alternatives [29], [35].

B. Measuring and Improving Individual Consistency in GDM With RPRs

In this section, we review the consistency index (CI) of RPRs [29] and the optimization-based method to improve the consistency of RPRs [27], which will be used as a basis of our proposal.

Herrera-Viedma *et al.* [29] proposed the CI based on the additive transitivity to evaluate the individual consistency of an RPR F as follows:

$$\mathrm{CI}(F) = 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,k=1}^{n} |f_{ij} + f_{jk} - f_{ik} - 0.5|. \quad (2)$$

The larger the value of $\mathrm{CI}(F)$, the more consistent F. If $\mathrm{CI}(F) = 1$, then F is a consistent RPR.

In general, in GDM problems, decision makers may establish a consistency threshold $\overline{\mathrm{CI}}$ for the consistency index of an RPR F. When $\mathrm{CI}(F) \geq \overline{\mathrm{CI}}$, F is considered of acceptable consistency; otherwise, F is considered of unacceptable consistency.

Zhang *et al.* [27] presented an optimization-based consistency improving model. In order to obtain the adjusted RPR $F' = (f'_{ij})_{n\times n}$, which is not only of acceptable consistency but also preserves as much information as possible in the original RPR F, the optimization-based model [27] minimizes the Manhattan distance between F and F', i.e., $\min \sum_{i,j=1}^{n} \frac{|f_{ij} - f'_{ij}|}{n^2}$.

Simultaneously, to guarantee that the adjusted RPR F' of acceptable consistency, we have

$\mathrm{CI}(F') \geq \overline{\mathrm{CI}}$.

Moreover, based on the definition of an RPR (Definition 1), we have

$f'_{ij} + f'_{ji} = 1$.

As a result, an optimization model to deal with inconsistency in F can be constructed as follows:

$$\begin{cases} \min \sum_{i,j=1}^{n} \frac{|f_{ij} - f'_{ij}|}{n^2} \\ \text{s.t. } f'_{ij} + f'_{ji} = 1 \\ \quad \mathrm{CI}(F') \geq \overline{\mathrm{CI}}. \end{cases} \quad (3)$$

Note 1: In this paper, we use model (3) to automatically deal with the inconsistency of RPRs. If we use other consistency improving methods (e.g., [3] and [12]), our proposal, in this paper, will be still valid.

C. Consensus Reaching Process for GDM With RPRs

The consensus reaching process is defined as a dynamic and iterative group discussion. By computing the consensus degree, we can find out the actual level of consensus among the decision makers. If the consensus level (CL) is not acceptable, a feedback process should be applied to improve the consensus. Otherwise, the CL is acceptable and the selection process should be applied to obtain the final consensus solution to the GDM problem. A general consensus process is shown in Fig. 1.

Different consensus models have been proposed during recent decades [36]–[45]. Generally, the computation of consensus measure for GDM is often done by measuring the difference between individual opinions and group opinion. The CL associated with the decision maker d_k is given by [12], i.e.,

$$\mathrm{CL}_k = 1 - \sum_{i,j=1;i\neq j}^{n} \frac{|f_{ij}^k - f_{ij}^c|}{n(n-1)}. \quad (4)$$

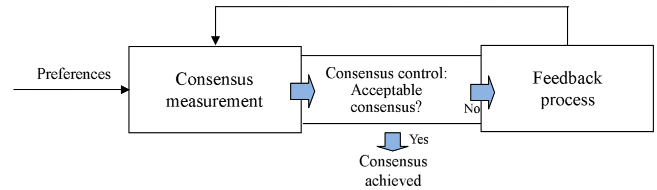

Fig. 1. General consensus reaching process scheme [36].

The CL among the RPRs $F^k = (f_{ij}^k)_{n\times n}(k = 1, 2, \ldots, m)$, associated with all decision makers $\{d_1, d_2, \ldots, d_m\}$, is given as follows:

$$\mathrm{CL}(F^1, F^2, \ldots, F^m) = \frac{1}{m}\sum_{k=1}^{m} \mathrm{CL}_k. \quad (5)$$

Clearly, $\mathrm{CL}_k, \mathrm{CL} \in [0, 1]$. A larger CL value indicates a higher consensus degree among all decision makers $\{d_1, d_2, \ldots, d_m\}$. If $\mathrm{CL}(F^1, F^2, \ldots, F^m) = 1$, then all decision makers $\{d_1, d_2, \ldots, d_m\}$ are of fully consensus.

In practice, a consensus threshold $\overline{\mathrm{CL}}$ is established for defining the necessary consensus level CL. When $\mathrm{CL}(F^1, F^2, \ldots, F^m) \geq \overline{\mathrm{CL}}$, the consensus has been achieved; otherwise, another discussion round starts with the feedback process to make RPRs closer to each other.

In the feedback process, some advice is generated to achieve a solution with a higher degree of consensus. To do so, it is necessary to identify which decision makers are farther from the collective RPR and how they should change their preferences. Therefore, the following two consensus rules are introduced [28].

1) Identification rule. The identification rule identifies the decision makers contributing less to reach a high degree of consensus. Generally, the decision maker d_r, where $\mathrm{CL}_r = \min\{\mathrm{CL}_1, \ldots, \mathrm{CL}_m\}$, needs to change his/her preferences.
2) Direction rule. The direction rule finds out the direction to change the preferences of decision makers. To do this, the following two direction rules are defined.
   a) If f_{ij}^r is smaller than f_{ij}^c, then the decision maker d_r should increase the evaluation associated with the pairwise (x_i, x_j).
   b) If f_{ij}^r is larger than f_{ij}^c, then the decision maker d_r should decrease the evaluation associated with the pairwise (x_i, x_j).

III. Consistency Versus Consensus: How to Deal With Both Concepts in GDM

As mentioned previously in Section I, in the existing studies, there are two main problems:

1) Losing consistency during the consensus process [12], [13].
2) Ignoring decision makers during the consensus reaching process [14], [15].

According to the two problems, one question has been raised: How to deal with both the consistency and the consensus in GDM? In Sections III-A and III-B, we wish to emphasize the

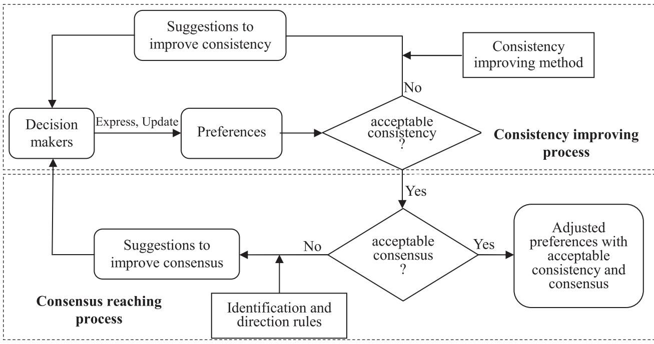

Fig. 2. Consensus framework with individual consistency [12], [13].

motivations of this paper and illustrate the above two problems in details.

A. Individual Consistency and Consensus Reaching Process: Separate Processes

In [12] and [13], the implementation of the consensus building with individual consistency in the GDM deals with a two-step procedure (see Fig. 2).

1) Consistency improving process. This process is used to help the decision makers obtain the preferences with acceptable consistency. There are a number of approaches to improve the consistency of RPRs [3], [12], [27], [29]. Without loss of generality, in this paper, an optimization-based model is used to improve individual consistency [27].
2) Consensus reaching process. Once all RPRs are of acceptable individual consistency, the consensus reaching process is applied to reach an acceptable consensus among all decision makers involved in the GDM problem. Identification rule and direction rule provided in Section II-C are widely used in consensus models to generate the suggestions to improve the CL.

Finally, the selection process is applied to rank alternatives based on the adjusted RPRs with acceptably consistency and acceptable consensus.

In the consensus framework shown in Fig. 2, the individual consistency will often be destroyed in the consensus reaching process, which leads to repeat the consistency improving process until the adjusted RPRs with the acceptably consistency and acceptable consensus are obtained simultaneously. Example 1 shows this issue.

Example 1: Suppose a GDM problem in which four decision makers are involved and they express their preferences over a set of four alternatives. The four RPRs are taken from [12] as follows:

$$F^1 = \begin{pmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & 0.9 & 0.7 \\ 0.4 & 0.1 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.7 & 0.5 \end{pmatrix}$$

$$F^2 = \begin{pmatrix} 0.5 & 0.7 & 0.9 & 0.5 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.5 & 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$$F^3 = \begin{pmatrix} 0.5 & 0.3 & 0.5 & 0.7 \\ 0.7 & 0.5 & 0.1 & 0.3 \\ 0.5 & 0.9 & 0.5 & 0.25 \\ 0.3 & 0.7 & 0.75 & 0.5 \end{pmatrix}$$

$$F^4 = \begin{pmatrix} 0.5 & 0.25 & 0.15 & 0.65 \\ 0.73 & 0.5 & 0.6 & 0.8 \\ 0.85 & 0.4 & 0.5 & 0.5 \\ 0.35 & 0.2 & 0.5 & 0.5 \end{pmatrix}.$$

Based on (2), the CIs of these RPRs are the following ones: $\mathrm{CI}(F^1) = 1, \mathrm{CI}(F^2) = 0.7667, \mathrm{CI}(F^3) = 0.65$, and $\mathrm{CI}(F^4) = 0.8333$.

Let $\overline{\mathrm{CI}} = 0.9$ be the consistency threshold, it can be seen that the decision makers d_2, d_3, and d_4 need to change their preferences. Using the consistency improving method [i.e., (3)] to deal with the inconsistency in F^2, F^3, and F^4, the adjusted RPRs of F^1, F^2, F^3, and F^4 are $F^{1,0}, F^{2,0}, F^{3,0}$, and $F^{4,0}$ as follows:

$$F^{1,0} = F^1$$

$$F^{2,0} = \begin{pmatrix} 0.5 & 0.7 & 0.8 & 0.778 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.778 \\ 0.222 & 0.3 & 0.222 & 0.5 \end{pmatrix}$$

$$F^{3,0} = \begin{pmatrix} 0.5 & 0.685 & 0.5 & 0.491 \\ 0.315 & 0.5 & 0.254 & 0.3 \\ 0.5 & 0.746 & 0.5 & 0.252 \\ 0.509 & 0.7 & 0.748 & 0.5 \end{pmatrix}$$

$$F^{4,0} = \begin{pmatrix} 0.5 & 0.25 & 0.15 & 0.65 \\ 0.75 & 0.5 & 0.6 & 0.8 \\ 0.85 & 0.4 & 0.5 & 0.7 \\ 0.35 & 0.2 & 0.3 & 0.5 \end{pmatrix}$$

where $\mathrm{CI}(F^{1,0}) = 1, \mathrm{CI}(F^{2,0}) = 0.9, \mathrm{CI}(F^{3,0}) = 0.9$, and $\mathrm{CI}(F^{4,0}) = 0.9$.

Based on (1), let $W = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$ be the weighting vector used to aggregate the individual RPRs and obtain the collective RPR $F^{c,0}$

$$F^{c,0} = \begin{pmatrix} 0.5 & 0.459 & 0.513 & 0.58 \\ 0.541 & 0.5 & 0.588 & 0.625 \\ 0.487 & 0.412 & 0.5 & 0.508 \\ 0.42 & 0.375 & 0.492 & 0.5 \end{pmatrix}.$$

The CLs associated with the decision makers $d_k (k = 1, 2, 3, 4)$ are computed by using (4), i.e., $\mathrm{CL}_{1,0} = 0.813$, $\mathrm{CL}_{2,0} = 0.82$, $\mathrm{CL}_{3,0} = 0.793$, and $\mathrm{CL}_{4,0} = 0.83$. Then, according to (5), the CL among $\{d_1, d_2, d_3, d_4\}$ is $\frac{CL_{1,0}+CL_{2,0}+CL_{3,0}+CL_{4,0}}{4} = 0.814$.

Let $\overline{\mathrm{CL}} = 0.84$ be the consensus threshold, since $\mathrm{CL}_3 = \min_k \mathrm{CL}_k = 0.793 < 0.84$, based on the identification rule, we can see that the decision maker d_3 needs to change his/her preferences. The new preference relation $F^{3,1}$ is obtained by following the direction rule

$$F^{3,1} = \begin{pmatrix} 0.5 & 0.68 & 0.51 & 0.56 \\ 0.32 & 0.5 & 0.3 & 0.62 \\ 0.49 & 0.7 & 0.5 & 0.3 \\ 0.44 & 0.38 & 0.7 & 0.5 \end{pmatrix}.$$

Because $F^{1,0}$, $F^{2,0}$, and $F^{4,0}$ are not modified in the first round, we have $F^{1,1} = F^{1,0}$, $F^{2,1} = F^{2,0}$, and $F^{4,1} = F^{4,0}$ in the new round. Based on RPRs $F^{k,1} (k = 1, 2, 3, 4)$, we obtain the CL associated with d_k by using (4), i.e., $\mathrm{CL}_{1,1} = 0.828$, $\mathrm{CL}_{2,1} = 0.838$, $\mathrm{CL}_{3,1} = 0.855$, and $\mathrm{CL}_{4,1} = 0.85$. According to (5), the CL among $\{d_1, d_2, d_3, d_4\}$ is $\frac{\mathrm{CL}_{1,1}+\mathrm{CL}_{2,1}+\mathrm{CL}_{3,1}+\mathrm{CL}_{4,1}}{4} = 0.843$.

It is clear that the CL among the decision makers $\{d_1, d_2, d_3, d_4\}$ has improved in this round of consensus reaching.

However, based on (2), we obtain the consistency index $\mathrm{CI}(F^{3,1}) = 0.827 < \overline{\mathrm{CI}} = 0.9$, which means that the individual consistency in the RPR $F^{3,1}$ has been damaged in the consensus reaching process. As a result, a repeating of the consistency improving process must be used to deal with the unacceptable consistency in $F^{3,1}$.

Naturally, the repeating consistency improving process leads to start with the consensus process again and is very time consuming. The aim of this paper is to develop the consensus reaching process with individual consistency control to avoid the repetition of the consistency improving process.

B. Automatic Consensus Reaching Process With Individual Consistency: Ignoring Decision Makers

In [14] and [15], the automatic consensus process is provided to assist the decision makers reach consensus. Let $F^k = (f_{ij}^k)_{n\times n}$ $(k = 1, 2, \ldots, m)$ be the individual RPRs and $F^c = (f_{ij}^c)_{n\times n}$ be the collective RPR. The main step for the automatic consensus process is to construct a new RPR $\overline{F^k}$ according to F^k. When establishing the new preference relation, the following strategy is adopted:

$$\overline{f_{ij}^k} = \lambda f_{ij}^k + (1-\lambda) f_{ij}^c \quad \text{where } 0 < \lambda < 1. \tag{6}$$

This strategy is adopted until all RPRs reach an acceptable consensus or the maximum number of iterations is obtained.

In the automatic consensus process, the modified RPR has an acceptable individual CI. However, the decision makers are ignored during the consensus reaching process, they cannot change their preferences freely. In other words, all elements in the RPR are changed based on (6) without the participation of decision maker. Example 2 shows this issue.

Example 2: Let $F^{k,0}$ $(k = 1, 2, 3, 4)$ and $F^{c,0}$ be as in Example 1. According to Example 1, $\mathrm{CL}_3 = \min_k \mathrm{CL}_k = 0.793 < \overline{\mathrm{CL}} = 0.84$, so the RPR $F^{3,0}$ needs to be modified for reaching an established consensus level $\overline{\mathrm{CL}}$. Based on (6), set $\lambda = 0.4$, then $F_{ij}^{3,1} = 0.4 \times F_{ij}^{3,0} + 0.6 \times F_{ij}^{c,0}$, for $i, j = 1, 2, \ldots, 4$.

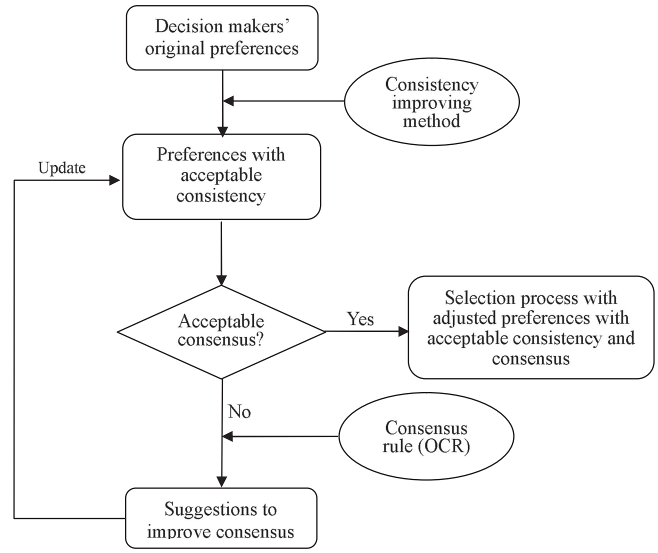

Fig. 3. Consensus framework with individual consistency control.

The new RPR $F^{3,1}$ is obtained as follows:

$$F^{3,1} = \begin{pmatrix} 0.5 & 0.549 & 0.508 & 0.544 \\ 0.451 & 0.5 & 0.454 & 0.495 \\ 0.492 & 0.546 & 0.5 & 0.406 \\ 0.456 & 0.505 & 0.594 & 0.5 \end{pmatrix}$$

where $\mathrm{CI}(F^{3,1}) = 0.955$.

Let $F^{k,1} = F^{k,0}$ for $k = 1, 2, 4$. The new collective RPR $F^{c,1}$ is obtained as follows:

$$F^{c,1} = \begin{pmatrix} 0.5 & 0.425 & 0.515 & 0.593 \\ 0.575 & 0.5 & 0.639 & 0.674 \\ 0.486 & 0.361 & 0.5 & 0.546 \\ 0.407 & 0.326 & 0.454 & 0.5 \end{pmatrix}.$$

According to (4), we obtain the CLs associated with d_k $(k = 1, 2, 3, 4)$ are 0.827, 0.826, 0.886, 0.848, respectively. Based on (5), the CL among decision makers is obtained, $\mathrm{CL}(F^{1,1}, F^{2,1}, F^{3,1}, F^{4,1}) = 0.847$.

In Section IV, we will propose a consensus reaching process to overcome the previous two problems. In the proposed process, we will develop a consensus rule via an optimization model to guarantee the individual consistency and to support the participation of the decision makers.

IV. Consensus Building With Individual Consistency Control

In order to overcome the shortcomings of the consensus process provided in Section III, we propose a consensus rule with individual consistency control, and following the consensus scheme shown in Fig. 2, a novel approach of consensus building with individual consistency control is provided (see Fig. 3).

The proposed consensus framework consists of two processes: consistency improving process and consensus reaching process. The main difference from the consensus framework shown in Fig. 2 is that the OCR is used in the consensus reaching process. The proposed consensus rule provides a consensus adjustable range for each preference and avoids repeating the consistency improving process in the consensus reaching process. Besides, in the consensus process based on the consensus rule, the decision makers can adjust their opinions freely within the adjustable range.

Section IV-A presents the consensus rule with individual consistency control via an optimization model, and Section IV-B introduces an algorithm to obtain its approximate optimal solution. Section IV-C provides the consensus reaching process based on the consensus rule. Finally, an algorithm to automatically revise the preference values is proposed in Section IV-D.

A. Optimization-Based Consensus Rule

In this section, we model the consensus rule with individual consistency control. Let $F^k = (f_{ij}^k)_{n\times n} (k = 1, 2, \ldots, m)$ be the individual RPR of the decision maker d_k and $F^c = (f_{ij}^c)_{n\times n}$ be the collective RPR.

The basic idea of the proposed consensus rule with individual consistency control is to get the adjustable range $[l_{ij}^k, u_{ij}^k]$ for decision maker d_k and pairwise (x_i, x_j), based on F^k and F^c. When the preference value f_{ij}^k is revised within the adjustable range $[l_{ij}^k, u_{ij}^k]$ in the consensus reaching process, the adjusted RPRs satisfy the following two conditions.

1) The CL among all the decision makers $\{d_1, d_2, \ldots, d_m\}$ is improved.
2) The adjusted RPRs are of acceptable consistency.

According to the direction rule to improve the CL among all the decision makers $\{d_1, d_2, \ldots, d_m\}$, if f_{ij}^k is smaller than f_{ij}^c, then the decision maker d_k is suggested to increase the evaluation associated with the pairwise (x_i, x_j); if f_{ij}^k is larger than f_{ij}^c, then the decision maker d_k is suggested to decrease the evaluation associated with the pairwise (x_i, x_j). In other words, in order to achieve a consensus, the revised preferences, associated with decision maker d_k and pairwise (x_i, x_j), are suggested to be within the interval $[\min\{f_{ij}^k, f_{ij}^c\}, \max\{f_{ij}^k, f_{ij}^c\}]$.

Let $[l_{ij}^k, u_{ij}^k]$ be the adjustable range, in order to guarantee the improvement of the CL among the decision makers $\{d_1, d_2, \ldots, d_m\}$, the adjustable range $[l_{ij}^k, u_{ij}^k]$ should be contained in the interval $[\min\{f_{ij}^k, f_{ij}^c\}, \max\{f_{ij}^k, f_{ij}^c\}]$, i.e.,

$$[l_{ij}^k, u_{ij}^k] \subseteq [f_{ij}^k, f_{ij}^c] \quad \text{if } f_{ij}^k \le f_{ij}^c \tag{7}$$

and

$$[l_{ij}^k, u_{ij}^k] \subseteq [f_{ij}^c, f_{ij}^k] \quad \text{if } f_{ij}^k > f_{ij}^c. \tag{8}$$

Let $L^k = (l_{ij}^k)_{n\times n}$ and $U^k = (u_{ij}^k)_{n\times n}$. Let $\varphi^k = \{A^k = (a_{ij}^k)_{n\times n} | a_{ij}^k \in [l_{ij}^k, u_{ij}^k], a_{ij}^k + a_{ji}^k = 1, i, j = 1, 2, \ldots, n, k = 1, 2, \ldots, m\}$ be the set of RPRs based on L^k and U^k. Example 3 illustrates the set φ^k.

Example 3: Let

$$L^k = \begin{pmatrix} 0.5 & 0.2 & 0.45 & 0.35 \\ 0.7 & 0.5 & 0.7 & 0.55 \\ 0.5 & 0.2 & 0.5 & 0.3 \\ 0.45 & 0.4 & 0.5 & 0.5 \end{pmatrix}$$

and

$$U^k = \begin{pmatrix} 0.5 & 0.3 & 0.5 & 0.55 \\ 0.8 & 0.5 & 0.8 & 0.6 \\ 0.55 & 0.3 & 0.5 & 0.5 \\ 0.65 & 0.45 & 0.7 & 0.5 \end{pmatrix}.$$

Then, for any RPR A^k that satisfies the condition $a_{ij}^k \in [l_{ij}^k, u_{ij}^k]$ and $a_{ij}^k + a_{ji}^k = 1$ for $i, j = 1, 2, 3, 4,$ we have $A^k \in \varphi^k$, such as

$$A^k = \begin{pmatrix} 0.5 & 0.25 & 0.47 & 0.45 \\ 0.75 & 0.5 & 0.75 & 0.57 \\ 0.53 & 0.25 & 0.5 & 0.4 \\ 0.55 & 0.43 & 0.6 & 0.5 \end{pmatrix}.$$

In order to guarantee the consistency of the adjusted RPRs based on the $[l_{ij}^k, u_{ij}^k]$, it is required that for any $A^k \in \varphi^k$, it should be of acceptable consistency, that is,

$$\min_{A^k \in \varphi^k} \text{CI}(A^k) \ge \overline{\text{CI}}. \tag{9}$$

Equation (9) guarantees that the consistency of any RPR A^k in the set φ^k is no less than the consistency threshold $\overline{\text{CI}}$.

Finally, the decision makers should have the maximum degree of freedom to revise their preferences, i.e., the width of $[l_{ij}^k, u_{ij}^k]$ is maximal, namely

$$\max \sum_{i=1}^{n} \sum_{j=i+1}^{n} (u_{ij}^k - l_{ij}^k). \tag{10}$$

Based on (7)–(10), an optimization-based model to obtain the adjustable range $[l_{ij}^k, u_{ij}^k]$ can be constructed as follows:

$$\begin{cases} \max \sum_{i=1}^{n} \sum_{j=i+1}^{n} (u_{ij}^k - l_{ij}^k) \\ \text{s.t.} \\ [l_{ij}^k, u_{ij}^k] \subseteq [f_{ij}^k, f_{ij}^c] \quad \text{if } f_{ij}^k \le f_{ij}^c, i, j = 1, 2, \ldots, n \\ [l_{ij}^k, u_{ij}^k] \subseteq [f_{ij}^c, f_{ij}^k] \quad \text{if } f_{ij}^k > f_{ij}^c, i, j = 1, 2, \ldots, n \\ \min_{A^k \in \varphi^k} \text{CI}(A^k) \ge \overline{\text{CI}} \end{cases} \tag{11}$$

where $l_{ij}^k, u_{ij}^k \ (i, j = 1, 2, \ldots, n)$ are decision variables in model (11).

Because $\min_{A^k \in \varphi^k} \text{CI}(A^k)$ in model (11) can be equivalently transformed into the following linear programming model:

$$\begin{cases} \min_{a_{ij}^k, a_{jz}^k, a_{iz}^k} \left(1 - \sum_{i<j<z} \frac{4}{n(n-1)(n-2)} |a_{ij}^k + a_{jz}^k - a_{iz}^k - 0.5|\right) \\ \text{s.t.} \\ a_{ij}^k \in [l_{ij}^k, u_{ij}^k] \quad i, j = 1, 2, \ldots, n \\ a_{ij}^k + a_{ji}^k = 1 \quad i, j = 1, 2, \ldots, n \end{cases} \tag{12}$$

model (11) can be reorganized as follows:

$$\begin{cases} \max \sum_{i=1}^{n} \sum_{j=i+1}^{n} (u_{ij}^k - l_{ij}^k) \\ \text{s.t.} \\ [l_{ij}^k, u_{ij}^k] \subseteq [f_{ij}^k, f_{ij}^c] \quad if\ f_{ij}^k \le f_{ij}^c, i, j = 1, 2, \ldots, n \\ [l_{ij}^k, u_{ij}^k] \subseteq [f_{ij}^c, f_{ij}^k] \quad if\ f_{ij}^k > f_{ij}^c, i, j = 1, 2, \ldots, n \\ \min_{a_{ij}^k, a_{jz}^k, a_{iz}^k} \left(1 - \sum_{i<j<z} \frac{4}{n(n-1)(n-2)} |a_{ij}^k + a_{jz}^k - a_{iz}^k - 0.5|\right) \\ \quad \ge \overline{\text{CI}} \\ a_{ij}^k \in [l_{ij}^k, u_{ij}^k] \quad i, j = 1, 2, \ldots, n \\ a_{ij}^k + a_{ji}^k = 1 \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{13}$$

We call the method to obtain the adjustable range via the optimization-based model (13) as the OCR and $l_{ij}^k, u_{ij}^k, a_{ij}^k$ $(i, j = 1, 2, \ldots, n)$ are the decision variables in the OCR. In the OCR, there exists a minimization model, and we will demonstrate this problem in detail in the following section.

Solving the OCR yields the adjustable range $[l_{ij}^k, u_{ij}^k]$. When the decision maker d_k revises their preferences in the adjustable range $[l_{ij}^k, u_{ij}^k]$, the constraint conditions (7) and (8) guarantee that the CL of the decision maker d_k can be improved. The constraint condition (9) guarantees that the individual consistency of the adjusted RPR associated with d_k will not be damaged, for the RPR A^k associated to $[l_{ij}^k, u_{ij}^k]$, which has the worst consistency, is of acceptable consistency.

B. Algorithm to the OCR

When implementing the OCR, a core problem is how to obtain the optimal solution to the OCR. Particularly, in the constraint conditions of the OCR [model (13)], a linear programming model [model (12)] is involved. The inclusion of model (12) in the OCR leads to the difficulty to obtain the optimum solution to the OCR. Therefore, we need to design an algorithm that obtains an approximate optimal solution to the OCR.

In essence, model (12) is used to find out an RPR $A^k = (a_{ij}^k)_{n \times n}$, whose CI is the smallest under the condition that $a_{ij}^k \in [l_{ij}^k, u_{ij}^k]$ and $a_{ij}^k + a_{ji}^k = 1$. When model (12) is used, we have the following observation: if a_{ij}^k $(i, j = 1, 2, \ldots, n)$ are the optimum solutions to model (12), then we find that in almost all of the cases, there exist $a_{ij}^k = l_{ij}^k$ or $a_{ij}^k = u_{ij}^k$. In other words, some elements a_{ij}^k in the RPR A^k will be on the boundary of the adjustable range $[l_{ij}^k, u_{ij}^k]$. Thus, we set $\psi^k = \{(i, j) | a_{ij}^k = l_{ij}^k \ or\ a_{ij}^k = u_{ij}^k\}$ and $\psi^k \neq \emptyset$. Next, we provide Example 4 to illustrate the set ψ^k.

Example 4: Let the matrices L^k and U^k be as in Example 3. Based on model (12), we find out the RPR

$$A^k = \begin{pmatrix} 0.5 & 0.2 & 0.49 & 0.4 \\ 0.8 & 0.5 & 0.8 & 0.58 \\ 0.51 & 0.2 & 0.5 & 0.45 \\ 0.6 & 0.42 & 0.55 & 0.5 \end{pmatrix}$$

whose CI is the smallest under the condition that $a_{ij}^k \in [l_{ij}^k, u_{ij}^k]$ and $a_{ij}^k + a_{ji}^k = 1$ for $i, j = 1, 2, 3, 4$.

Then, it is clear that $a_{12}^k = l_{12}^k$, $a_{21}^k = u_{21}^k$, $a_{23}^k = u_{23}^k$ and $a_{32}^k = l_{32}^k$. So, we have $\psi^k = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$.

Suppose $(\alpha, \beta) \in \psi^k$ and $u_{\alpha\beta}^k - l_{\alpha\beta}^k = \max_{(i,j) \in \psi^k} (u_{ij}^k - l_{ij}^k)$. We may update the adjustable range $[l_{ij}^k, u_{ij}^k]$ using the following equations:

$$[\overline{l_{ij}^k}, \overline{u_{ij}^k}] = [l_{ij}^k, u_{ij}^k] \quad \text{for}\ i, j \neq \alpha, \beta \tag{14}$$

$$\begin{cases} \overline{l_{\alpha\beta}^k} = l_{\alpha\beta}^k + \theta(u_{\alpha\beta}^k - l_{\alpha\beta}^k) \\ \overline{u_{\alpha\beta}^k} = u_{\alpha\beta}^k \end{cases} \quad \text{if}\ a_{\alpha\beta}^k = l_{\alpha\beta}^k \tag{15}$$

and

$$\begin{cases} \overline{l_{\alpha\beta}^k} = l_{\alpha\beta}^k \\ \overline{u_{\alpha\beta}^k} = u_{\alpha\beta}^k - \theta(u_{\alpha\beta}^k - l_{\alpha\beta}^k) \end{cases} \quad \text{if}\ a_{\alpha\beta}^k = u_{\alpha\beta}^k \tag{16}$$

where $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ is the updated adjustable range, associated with $[l_{ij}^k, u_{ij}^k]$, $\theta \in \{0, 1\}$ is the adjustment parameter, and the larger θ value implies the more adjustment amount.

Let $[l_{ij}^k, u_{ij}^k]$ be the adjustable range for decision maker d_k with pairwise (x_i, x_j) and let $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ be the updated adjustable range based on (14)–(16). Then, we have a desired property for (14)–(16).

Property 1: Let $A^k = (a_{ij}^k)_{n \times n}$ be the RPR with the smallest CI under the condition $a_{ij}^k \in [l_{ij}^k, u_{ij}^k]$ and let $\overline{A^k} = (\overline{a_{ij}^k})_{n \times n}$ be the RPR with the smallest CI under the condition $\overline{a_{ij}^k} \in [\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ and $\overline{a_{ij}^k} + \overline{a_{ji}^k} = 1$. Then, the consistency of $\overline{A^k}$ is greater than or equal to the consistency of A^k, i.e., $\text{CI}(\overline{A^k}) \ge CI(A^k)$.

Proof: According to model (12), $\text{CI}(A^k) = \min(1 - \sum_{i<j<z} \frac{4}{n(n-1)(n-2)} |a_{ij}^k + a_{jz}^k - a_{iz}^k - 0.5|)$, where $a_{ij}^k \in [l_{ij}^k, u_{ij}^k]$.

Since $[\overline{l_{ij}^k}, \overline{u_{ij}^k}] \subseteq [l_{ij}^k, u_{ij}^k]$ and $\overline{a_{ij}^k} \in [\overline{l_{ij}^k}, \overline{u_{ij}^k}]$, we have

$$\text{CI}(\overline{A^k}) \ge \min\left(1 - \sum_{i<j<z} \frac{4}{n(n-1)(n-2)} |a_{ij}^k + a_{jz}^k - a_{iz}^k - 0.5|\right).$$

Thus, $\text{CI}(\overline{A^k}) \ge CI(A^k)$.

This completes the proof of Property 1.

Property 1 shows that the worst consistency degree of the RPRs can be improved by updating the adjustable range based on (14)–(16).

Based on Property 1, we design an algorithm to obtain the approximate optimal solution to the OCR. Initially, we set $[l_{ij}^k, u_{ij}^k] = [\min\{f_{ij}^k, f_{ij}^c\}, \max\{f_{ij}^k, f_{ij}^c\}]$, and then update the adjustable range based on (14)–(16). Follow this procedure until $\text{CI}\overline{A^k}) \ge \overline{\text{CI}}$ and the obtained approximate optimal solution to the OCR is $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$. The detailed algorithm noted as Algorithm 1 is provided below.

Note 2: For the optimum solutions a_{ij}^k $(i, j = 1, 2, \ldots, n)$ to model (12), we find that in almost all of the cases, there

Algorithm 1 The Approximate Algorithm to the OCR.

Input. The established consistency threshold $\overline{CI}$, the individual RPR $F^k = (f_{ij}^k)_{n\times n}$ with the acceptable consistency (i.e., $CI(F^k) \geq \overline{CI}$), the collective RPR $F^c = (f_{ij}^c)_{n\times n}$, the adjustment parameter θ and the established maximum number of iterations M.

Output. The adjusted range $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ and the iteration number t.

Step 1: Let $t = 0$, and $[l_{ij}^{k,t}, u_{ij}^{k,t}] = [l_{ij}^{k,0}, u_{ij}^{k,0}] = [\min\{f_{ij}^k, f_{ij}^c\}, \max\{f_{ij}^k, f_{ij}^c\}]$.

Step 2: Based on model (12), we find out a RPR $A^{k,t} = (a_{ij}^{k,t})_{n\times n}$ whose consistency index is the smallest under the condition that $a_{ij}^{k,t} \in [l_{ij}^{k,t}, u_{ij}^{k,t}]$ and $a_{ij}^{k,t} + a_{ji}^{k,t} = 1$.

Step 3: If $CI(A^{k,t}) < \overline{CI}$ and $t < M$, go to Step 4; Otherwise, go to Step 5.

Step 4: Get $[l_{ij}^{k,t+1}, u_{ij}^{k,t+1}]$. Suppose $\psi^{k,t} = \{(i,j)|a_{ij}^{k,t} = l_{ij}^{k,t} \ or a_{ij}^{k,t} = u_{ij}^{k,t}\}$ and consider the following cases:

Case A: $\psi^{k,t} \neq \emptyset$.

Let $(\alpha,\beta) \in \psi^{k,t}$ and $u_{\alpha\beta}^{k,t} - l_{\alpha\beta}^{k,t} = \max_{(i,j)\in\psi^{k,t}} (u_{ij}^{k,t} - l_{ij}^{k,t})$.

Based on (14)–(16), we structure $[l_{ij}^{k,t+1}, u_{ij}^{k,t+1}]$ as follows:

i) $[l_{ij}^{k,t+1}, u_{ij}^{k,t+1}] = [l_{ij}^{k,t}, u_{ij}^{k,t}]$ for $i,j \neq \alpha,\beta$

ii) $\begin{cases} l_{\alpha\beta}^{k,t+1} = l_{\alpha\beta}^{k,t} + \theta(u_{\alpha\beta}^{k,t} - l_{\alpha\beta}^{k,t}) \\ u_{\alpha\beta}^{k,t+1} = u_{\alpha\beta}^{k,t} \end{cases}$ if $a_{\alpha\beta}^{k,t} = l_{\alpha\beta}^{k,t}$

iii) $\begin{cases} l_{\alpha\beta}^{k,t+1} = l_{\alpha\beta}^{k,t} \\ u_{\alpha\beta}^{k,t+1} = u_{\alpha\beta}^{k,t} - \theta(u_{\alpha\beta}^{k,t} - l_{\alpha\beta}^{k,t}) \end{cases}$ if $a_{\alpha\beta}^{k,t} = u_{\alpha\beta}^{k,t}$

Case B: $\psi^{k,t} = \emptyset$.

Let $u_{\alpha\beta}^{k,t} - l_{\alpha\beta}^{k,t} = \max_{i,j\in\{1,2,\ldots,n\}} (u_{ij}^{k,t} - l_{ij}^{k,t})$. Then, we structure $[l_{ij}^{k,t+1}, u_{ij}^{k,t+1}]$ as follows:

i) $[l_{ij}^{k,t+1}, u_{ij}^{k,t+1}] = [l_{ij}^{k,t}, u_{ij}^{k,t}]$ for $i,j \neq \alpha,\beta$

ii) $\begin{cases} l_{\alpha\beta}^{k,t+1} = l_{\alpha\beta}^{k,t} \\ u_{\alpha\beta}^{k,t+1} = u_{\alpha\beta}^{k,t} - \theta(u_{\alpha\beta}^{k,t} - l_{\alpha\beta}^{k,t}) \end{cases}$ for $i,j = \alpha,\beta$

Let $t = t + 1$ and go to Step 2.

Step 5: Let $[\overline{l_{ij}^k}, \overline{u_{ij}^k}] = [l_{ij}^{k,t}, u_{ij}^{k,t}]$. Output the adjustable range $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ as the approximate optimal solution to the OCR, and output the iteration number t.

exist $a_{ij}^k = l_{ij}^k$ or $a_{ij}^k = u_{ij}^k$. But, it is hard to prove this point analytically. So, in Step 4 of Algorithm 1, we also consider Case B, which will not change the essence of Algorithm 1.

An important problem of the approximate algorithm is its approximate performance. The approximate performance reflects the degree of approximation between the approximate solution and the optimal solution. Let $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ be the approximate optimal solution to the OCR. Let $[l_{ij}^{k*}, u_{ij}^{k*}]$ be the optimum solution to the OCR with Algorithm 1. According to the theory of approximate algorithms [46], if $\frac{u_{ij}^{k*} - l_{ij}^{k*}}{\overline{u_{ij}^k} - \overline{l_{ij}^k}} \leq \rho$, then, Algorithm 1 is considered to be a ρ-approximation algorithm. Because $u_{ij}^{k*} - l_{ij}^{k*} \leq |f_{ij}^k - f_{ij}^c|$, $\frac{u_{ij}^{k*} - l_{ij}^{k*}}{\overline{u_{ij}^k} - \overline{l_{ij}^k}} \leq \frac{|f_{ij}^k - f_{ij}^c|}{\overline{u_{ij}^k} - \overline{l_{ij}^k}}$, so in this paper, we set

$$\rho = \frac{|f_{ij}^k - f_{ij}^c|}{\overline{u_{ij}^k} - \overline{l_{ij}^k}} \tag{17}$$

to measure the approximate performance of Algorithm 1. Clearly, $\rho \geq 1$. The smaller ρ value indicates a better performance. When $\rho = 1$, $[\overline{l_{ij}^k}, \overline{u_{ij}^k}]$ is the optimum solution to the OCR. Section V will show the approximate performance of Algorithm 1.

Example 5 illustrates the process of updating the adjustable range $[l_{ij}^k, u_{ij}^k]$ based on Algorithm 1.

Example 5: Suppose the consistency threshold $\overline{\text{CI}} = 0.85$ and the adjustment parameter $\theta = 0.3$. Consider the individual RPR $F^{4,0}$ and the collective RPR $F^{c,0}$ provided in Example 1.

1) In the first iteration, let $l_{ij}^{4,0} = \min\{f_{ij}^{4,0}, f_{ij}^{c,0}\}$ and $u_{ij}^{4,0} = \max\{f_{ij}^{4,0}, f_{ij}^{c,0}\}$. Solving model (12) obtains the RPR $A^{4,0}$

$$A^{4,0} = \begin{pmatrix} 0.5 & 0.459 & 0.15 & 0.58 \\ 0.541 & 0.5 & 0.588 & 0.8 \\ 0.85 & 0.412 & 0.5 & 0.508 \\ 0.42 & 0.2 & 0.492 & 0.5 \end{pmatrix}$$

where $\text{CI}(A^{4,0}) = 0.79$.

Then, we have $\psi^{4,0} = \{(i,j)|i,j = 1,2,3,4\}$. It is clear that $u_{13}^{4,0} - l_{13}^{4,0} = \max_{(i,j)\in\psi^{4,0}} (u_{ij}^{4,0} - l_{ij}^{4,0})$. Since $a_{13}^{4,0} = l_{13}^{4,0}$, the range $[l_{ij}^{4,1}, u_{ij}^{4,1}]$ is updated as follows:

$$[l_{ij}^{4,1}, u_{ij}^{4,1}] = [l_{ij}^{4,0}, u_{ij}^{4,0} \text{ for } i,j \neq 1,3$$

and

$$\begin{cases} l_{13}^{4,1} = l_{13}^{4,0} + 0.3(u_{13}^{4,0} - l_{13}^{4,0}) \\ u_{13}^{4,1} = u_{13}^{4,0}. \end{cases}$$

We have $l_{13}^{4,1} = 0.2589$ and $u_{13}^{4,1} = 0.513$. Based on the new range $[l_{ij}^{4,1}, u_{ij}^{4,1}]$, solving model (12) obtains $\text{CI}(A^{4,1}) = 0.836$, where

$$A^{4,1} = \begin{pmatrix} 0.5 & 0.459 & 0.2589 & 0.58 \\ 0.541 & 0.5 & 0.588 & 0.8 \\ 0.7411 & 0.412 & 0.5 & 0.508 \\ 0.42 & 0.2 & 0.492 & 0.5 \end{pmatrix}.$$

2) In the second iteration, let $l_{ij}^{4,1} = \min\{f_{ij}^{4,1}, f_{ij}^{c,1}\}$ and $u_{ij}^{4,1} = \max\{f_{ij}^{4,1}, f_{ij}^{c,1}\}$.

We have $\psi^{4,1} = \{(i,j)|i,j = 1,2,3,4\}$. It is clear that $u_{13}^{4,1} - l_{13}^{4,1} = \max_{(i,j)\in\psi^{4,1}} (u_{ij}^{4,1} - l_{ij}^{4,1}) = 0.2541$. Since $a_{13}^{4,1} = l_{13}^{4,1}$, we update the range $[l_{ij}^{4,2}, u_{ij}^{4,2}]$ as follows:

$[l_{ij}^{4,2}, u_{ij}^{4,2}] = [l_{ij}^{4,1}, u_{ij}^{4,1}]$ for $i,j \neq 1,3$ and

$$\begin{cases} l_{13}^{4,2} = l_{13}^{4,1} + 0.3(u_{13}^{4,1} - l_{13}^{4,1}) \\ u_{13}^{4,2} = u_{13}^{4,1}. \end{cases}$$

We have $l_{13}^{4,2} = 0.3351$ and $u_{13}^{4,2} = 0.513$. Based on the new range $[l_{ij}^{4,2}, u_{ij}^{4,2}]$, solving model (12) obtains $\text{CI}(A^{4,2}) = 0.861$, where

$$A^{4,2} = \begin{pmatrix} 0.5 & 0.459 & 0.3351 & 0.58 \\ 0.541 & 0.5 & 0.588 & 0.8 \\ 0.6649 & 0.412 & 0.5 & 0.508 \\ 0.42 & 0.2 & 0.492 & 0.5 \end{pmatrix}.$$

C. Consensus Reaching Process Based on the OCR

In this section, a consensus reaching process with individual consistency control is developed. The core of this approach is based on the use of the OCR, which provides a powerful tool to guide the decision makers to build the acceptably consistency and acceptable consensus.

The proposed consensus reaching process includes the following two steps.

Step 1: We analyze the consistency degree of individual RPRs. If the RPRs are not consistent enough, then the consistency improving method is applied to generate the RPRs with acceptable consistency.

Step 2: We analyze the CL among the decision makers $\{d_1, d_2, \ldots, d_m\}$. If the decision makers are of unacceptable consensus, then solving the OCR yields the adjustable range $[l_{ij}^k, u_{ij}^k]$, associated with the decision maker d_k and the pairwise (x_i, x_j). Following this, decision makers d_k $(k = 1, 2, \ldots, m)$ are suggested to revise their preferences under the adjustable range $[l_{ij}^k, u_{ij}^k]$ $(i, j = 1, 2, \ldots, n)$ to improve the CL. In other words, the modified preferences regarding d_k and (x_i, x_j) should belong to the interval $[l_{ij}^k, u_{ij}^k]$. Follow this procedure until the consensus is reached.

Algorithm 2 formally describes the consensus reaching process with individual consistency control.

Note 3: In Step 4 of Algorithm 2, there exists the case in which all decision makers do not adjust their preference relations, i.e., $F^{k,h+1} = F^{k,h}$ $(k = 1, 2, \ldots, m)$. In order to avoid this issue, it is required that the pairwise preference values with the biggest difference must be adjusted, i.e., if $\left|f_{s\tau}^{\kappa_1,h} - f_{s\tau}^{\kappa_2,h}\right| = \max_{\substack{k_1,k_2 \in \{1,2,\ldots,m\} \\ i,j \in \{1,2,\ldots,n\}}} \left|f_{ij}^{k_1,h} - f_{ij}^{k_2,h}\right|$, then $f_{s\tau}^{\kappa_1,h+1} \in (l_{s\tau}^{\kappa_1,h}, u_{s\tau}^{\kappa_1,h})$ and $f_{s\tau}^{\kappa_2,h+1} \in (l_{s\tau}^{\kappa_2,h}, u_{s\tau}^{\kappa_2,h})$.

Some desired properties of the consensus reaching process with individual consistency control are explained in Theorems 1 and 2.

Theorem 1: Let $\overline{\text{CI}}$ be the consistency threshold in Algorithm 2. Let $F^{k,h} = (f_{ij}^{k,h})_{n \times n}$ be the RPRs generated by Algorithm 2 and $\text{CI}(F^{k,h})$ be the CI of $F^{k,h}$. Then, $\text{CI}(F^{k,h}) \geq \overline{\text{CI}}$ for $k = 1, 2, \ldots, m; h = 0, 1, 2, \ldots, M$.

Proof: In Algorithm 2, $F^{k,0}$ is the RPR with acceptable consistency, i.e.,

$$\text{CI}(F^{k,0}) \geq \overline{CI} \ \ for\ k = 1, 2, \ldots, m. \tag{18}$$

Because $f_{ij}^{k,h+1} \in [l_{ij}^{k,h}, u_{ij}^{k,h}]$, where $[l_{ij}^{k,h}, u_{ij}^{k,h}]$ is the adjustable range obtained from the OCR, associated with the decision maker d_k and the pairwise (x_i, x_j). The constraint condition (9) guarantees that the individual consistency of the adjusted

Algorithm 2 The Consensus Reaching Process with Individual Consistency Control.

Input. The individual RPRs $F^k = (f_{ij}^k)_{n \times n} (k = 1, 2, \ldots, m)$, the weighting vector of the decision makers $W = \{w_1, w_2, \ldots, w_m\}$, the established consistency threshold $\overline{CI}$, the established consensus threshold $\overline{CL}$ and the established maximum number of iterations M.
Output. Adjusted RPRs $\overline{F^k} = (\overline{f_{ij}^k})_{n \times n} (k = 1, 2, \ldots, m)$ and the iteration number h.
Step 1: Calculate $CI(F^k)$. If $CI(F^k) \geq \overline{CI}$ for $k = 1, 2, \ldots, m$, go to Step 2; otherwise, apply the consistency improving method (see Section II-B) to obtain the adjusted RPRs with acceptable consistency (for the sake of simplicity, they are still denoted as F^k), and then go to Step 2.
Step 2: Let $h = 0$ and $F^{k,0} = F^k (k = 1, 2, \ldots, m)$.
Step 3: If $CL(F^{1,h}, F^{2,h}, \ldots, F^{m,h}) \geq \overline{CL}$ or $h > M$, go to Step 5; otherwise, go to Step 4.
Step 4: Aggregate the RPRs $\{F^{1,h}, F^{2,h}, ..., F^{m,h}\}$ to obtain the collective $F^{c,h}$. Then, based on $F^{k,h}$ and $F^{c,h}$, we use the Algorithm 1 to approximate the OCR to obtain the adjustable range $[l_{ij}^{k,h}, u_{ij}^{k,h}]$, associated with the decision maker d_k $(k = 1, 2, \ldots, m)$ and the pairwise (x_i, x_j) $(i, j = 1, 2, \ldots, n)$. Next, for any decision maker d_k $(k = 1, 2, \ldots, m)$, she/he can give the adjusted RPR $F^{k,h+1} = (f_{ij}^{k,h+1})_{n \times n}$, where $f_{ij}^{k,h+1} \in [l_{ij}^{k,h}, u_{ij}^{k,h}]$. Go to Step 3.
Step 5: Let $\overline{F^k} = F^{k,h} (k = 1, 2, \ldots, m)$. Output the adjusted RPRs $\overline{F^k} = (\overline{f_{ij}^k})_{n \times n} (k = 1, 2, \ldots, m)$ and the iteration number h.

RPR associated with d_k will not be damaged, i.e., when $h > 0$

$$\text{CI}(F^{k,h}) \geq \overline{\text{CI}} \text{ for } k = 1, 2, \ldots, m. \tag{19}$$

Thus, $\text{CI}(F^{k,h}) \geq \overline{\text{CI}}$ for $k = 1, 2, \ldots, m; h = 0, 1, 2, \ldots, M$.

This completes the proof of Theorem 1.

From Algorithm 2, the final adjusted RPRs $\overline{F^k}$ are obtained. By Theorem 1, we have the following corollary.

Corollary 1: The consistency of the adjusted RPRs $\overline{F^k}$ is greater than or equal to the consistency threshold $\overline{\text{CI}}$, i.e., $\text{CI}(\overline{F^k}) \geq \overline{\text{CI}}$ for $k = 1, 2, \ldots, m$.

Note 4: According to Theorem 1 and Corollary 1, the consistency of the adjusted RPRs is higher than the consistency threshold $\overline{\text{CI}}$. To guarantee that the consistency of the adjusted RPRs is not lower than the consistency of the original individual RPRs, it should set $\overline{\text{CI}}$ as the largest consistency value of the original RPRs.

Theorem 2: Let $\overline{\text{CL}}$ be the consensus threshold in Algorithm 2. Let $\{F^{k,h} = (f_{ij}^{k,h})_{n \times n} | k = 1, 2, \ldots, m\}$ be the adjusted RPRs sequence in Algorithm 2 and $\text{CL}(F^{1,h}, F^{2,h}, \ldots, F^{m,h})$ be the CL among $\{F^{1,h}, F^{2,h}, \ldots, F^{m,h}\}$. Then, when setting $\overline{\text{CL}} = 1$ and $h \to \infty$, we have $\lim_{h \to \infty} (CL(F^{1,h}, F^{2,h}, \ldots, F^{m,h})) = 1$.

Proof: Let $f_{ij}^{+,h} = \max_{k \in \{1,2,\ldots,m\}} f_{ij}^{k,h}$ and $f_{ij}^{-,h} = \min_{k \in \{1,2,\ldots,m\}} f_{ij}^{k,h}$. Then, we have $f_{ij}^{k,h+1} \in [l_{ij}^{k,h}, \mu_{ij}^{k,h}] \subseteq$

$[f_{ij}^{-,h}, f_{ij}^{+,h}]$, for any $i,j \in \{1,2,\ldots,n\}$ and $k \in \{1,2,\ldots,m\}$. So

$$\left[f_{ij}^{-,h+1}, f_{ij}^{+,h+1}\right] \subseteq \left[f_{ij}^{-,h}, f_{ij}^{+,h}\right]. \tag{20}$$

Meanwhile, if $f_{s\tau}^{+,h} - f_{s\tau}^{-,h} = \max_{\substack{k_1,k_2\in\{1,2,\ldots,m\}\\ i,j\in\{1,2,\ldots,n\}}} |f_{ij}^{k_1,h} - f_{ij}^{k_2,h}|$ and $f_{s\tau}^{+,h} - f_{s\tau}^{-,h} = \left|f_{s\tau}^{\kappa_1,h} - f_{s\tau}^{\kappa_2,h}\right|$, based on

$$L^{1,0} = \begin{pmatrix} 0.5 & 0.22 & 0.513 & 0.4 \\ 0.587 & 0.5 & 0.74 & 0.625 \\ 0.4 & 0.2 & 0.5 & 0.3 \\ 0.42 & 0.3 & 0.493 & 0.5 \end{pmatrix}$$

$$U^{1,0} = \begin{pmatrix} 0.5 & 0.413 & 0.6 & 0.58 \\ 0.78 & 0.5 & 0.8 & 0.7 \\ 0.487 & 0.26 & 0.5 & 0.507 \\ 0.6 & 0.375 & 0.7 & 0.5 \end{pmatrix}$$

$$L^{2,0} = \begin{pmatrix} 0.5 & 0.459 & 0.5232 & 0.6079 \\ 0.3719 & 0.5 & 0.588 & 0.625 \\ 0.2948 & 0.4 & 0.5 & 0.508 \\ 0.222 & 0.3 & 0.3208 & 0.5 \end{pmatrix}$$

$$U^{2,0} = \begin{pmatrix} 0.5 & 0.6281 & 0.7052 & 0.778 \\ 0.541 & 0.5 & 0.6 & 0.7 \\ 0.4768 & 0.412 & 0.5 & 0.6792 \\ 0.3921 & 0.375 & 0.492 & 0.5 \end{pmatrix}$$

$$L^{3,0} = \begin{pmatrix} 0.5 & 0.5125 & 0.5 & 0.491 \\ 0.337 & 0.5 & 0.4058 & 0.3404 \\ 0.487 & 0.4411 & 0.5 & 0.3592 \\ 0.42 & 0.5106 & 0.492 & 0.5 \end{pmatrix}$$

$$U^{3,0} = \begin{pmatrix} 0.5 & 0.663 & 0.513 & 0.58 \\ 0.4875 & 0.5 & 0.5589 & 0.4894 \\ 0.5 & 0.5942 & 0.5 & 0.508 \\ 0.509 & 0.6596 & 0.6408 & 0.5 \end{pmatrix}$$

$$L^{4,0} = \begin{pmatrix} 0.5 & 0.25 & 0.3431 & 0.58 \\ 0.602 & 0.5 & 0.588 & 0.625 \\ 0.4089 & 0.4 & 0.5 & 0.5327 \\ 0.35 & 0.2226 & 0.3112 & 0.5 \end{pmatrix}$$

$$U^{4,0} = \begin{pmatrix} 0.5 & 0.398 & 0.4911 & 0.65 \\ 0.75 & 0.5 & 0.6 & 0.7774 \\ 0.6569 & 0.412 & 0.5 & 0.6888 \\ 0.42 & 0.375 & 0.4673 & 0.5 \end{pmatrix}$$

Note 3, we have $f_{s\tau}^{\kappa_1,h+1} \in \left(l_{s\tau}^{\kappa_1,h}, u_{s\tau}^{\kappa_1,h}\right) \subseteq \left(f_{s\tau}^{-,h}, f_{s\tau}^{+,h}\right)$ and $f_{s\tau}^{\kappa_2,h+1} \in \left(l_{s\tau}^{\kappa_2,h}, u_{s\tau}^{\kappa_2,h}\right) \subseteq \left(f_{s\tau}^{-,h}, f_{s\tau}^{+,h}\right)$. So

$$\left[f_{s\tau}^{-,h+1}, f_{s\tau}^{+,h+1}\right] \subseteq \left(f_{s\tau}^{-,h}, f_{s\tau}^{+,h}\right). \tag{21}$$

Let $Z^h = \sum_{i=1}^n \sum_{j=1;j\neq i}^n (f_{ij}^{+,h} - f_{ij}^{-,h})$. Based on (20) and (21), we have

$$Z^{h+1} < Z^h. \tag{22}$$

For any h, we have $Z^h \geq 0$. Thus, the sequence $\{Z^h | h = 1,2,\ldots\}$ is monotone decreasing and has lower bounds. Then, we have $\lim_{h\to\infty} Z^h = \inf\{Z^h | h = 1,2,\ldots\}$.

Suppose that $\inf\{Z^h | h = 1,2,\ldots\} \neq 0$. Because $\text{CL}(F^{1,h}, F^{2,h}, \ldots, F^{m,h}) < \overline{\text{CL}} = 1$, Algorithm 2 will continue and the Z^h values will continue to decrease with the increase of h. This contradicts the definition of infimum $\inf\{Z^h | h = 1,2,\ldots\}$. So, when setting $\overline{\text{CL}} = 1$, $\lim_{h\to\infty} Z^h = 0$.

Because $1 \geq \text{CL}(F^{1,h}, F^{2,h}, \ldots, F^{m,h}) =$

$$\frac{1}{m}\sum_{k=1}^m \left(1 - \sum_{i,j=1;i\neq j}^n \frac{|f_{ij}^{k,h} - f_{ij}^{c,h}|}{n(n-1)}\right) \geq 1 - \frac{Z^h}{n}(n-1),$$

therefore, $\lim_{h\to\infty}(\text{CL}(F^{1,h}, F^{2,h}, \ldots, F^{m,h})) = 1$.

This completes the proof of Theorem 2.

Theorems 1 and 2 guarantee that the adjusted RPRs, obtained by the proposed consensus reaching process, are of acceptable consistency and consensus. We should point out that the consistency improving method is only used in the first round because of the use of the OCR. As shown in Theorem 1, the proposed consensus reaching process can avoid repeating the consistency improving process.

Example 6 illustrates the consensus reaching process with individual consistency control.

Example 6: (Example 1 continuation): Same to Example 1 in Section III, let $\overline{\text{CL}} = 0.84$ be the consensus threshold, let $\overline{\text{CI}} = 0.9$ be the consistency threshold and let $F^{1,0}$, $F^{2,0}$, $F^{3,0}$, and $F^{4,0}$ be RPRs with acceptable consistency as Example 1.

As shown in Example 1, $\text{CL}(F^{1,0}, F^{2,0}, F^{3,0}, F^{4,0}) = 0.814 < \overline{\text{CL}}$, so we use the consensus reaching process with individual consistency control (Algorithm 2) to improve the consensus. Let $[l_{ij}^{k,0}, u_{ij}^{k,0}]$ be the adjustable range obtained by using Algorithm 1. Suppose $L^{k,0} = (l_{ij}^{k,0})_{4\times 4}$ and $U^{k,0} = (u_{ij}^{k,0})_{4\times 4}$, $k = 1,2,3,4$.

Based on $L^{k,0}$ and $U^{k,0}$ $(k = 1,2,3,4)$, the decision makers construct new RPRs $F^{k,1} = (f_{ij}^{k,1})_{4\times 4}$, where $f_{ij}^{k,1} \in [l_{ij}^{k,0}, u_{ij}^{k,0}]$, as follows:

$$F^{1,1} = \begin{pmatrix} 0.5 & 0.3 & 0.55 & 0.45 \\ 0.7 & 0.5 & 0.8 & 0.65 \\ 0.45 & 0.2 & 0.5 & 0.4 \\ 0.55 & 0.35 & 0.6 & 0.5 \end{pmatrix}$$

$$F^{2,1} = \begin{pmatrix} 0.5 & 0.6 & 0.7 & 0.61 \\ 0.4 & 0.5 & 0.6 & 0.65 \\ 0.3 & 0.4 & 0.5 & 0.55 \\ 0.39 & 0.35 & 0.45 & 0.5 \end{pmatrix}$$

$$F^{3,1} = \begin{pmatrix} 0.5 & 0.55 & 0.51 & 0.54 \\ 0.45 & 0.5 & 0.41 & 0.4 \\ 0.49 & 0.59 & 0.5 & 0.5 \\ 0.46 & 0.6 & 0.5 & 0.5 \end{pmatrix}$$

$$F^{4,1} = \begin{pmatrix} 0.5 & 0.39 & 0.49 & 0.62 \\ 0.61 & 0.5 & 0.59 & 0.71 \\ 0.51 & 0.41 & 0.5 & 0.55 \\ 0.38 & 0.29 & 0.45 & 0.5 \end{pmatrix}.$$

Based on (2), the consistency degrees of $F^{k,1}(k = 1, 2, 3, 4)$ are obtained, i.e., $\text{CI}(F^{1,1}) = 0.983$, $\text{CI}(F^{2,1}) = 0.953$, $\text{CI}(F^{3,1}) = 0.97$, and $\text{CI}(F^{4,1}) = 0.97$.

According to (1), and taking the weighting vector $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$, the collective RPR $F^{c,1}$ is obtained as follows:

$$F^{c,1} = \begin{pmatrix} 0.5 & 0.46 & 0.5625 & 0.555 \\ 0.54 & 0.5 & 0.6 & 0.6025 \\ 0.4375 & 0.4 & 0.5 & 0.5 \\ 0.445 & 0.3975 & 0.5 & 0.5 \end{pmatrix}.$$

Based on (4) and (5), the CLs associated with d_k are obtained, i.e., $\text{CL}_{1,1} = 0.896, \text{CL}_{2,1} = 0.928, \text{CL}_{3,1} = 0.908$, and $\text{CL}_{4,1} = 0.938$. The CL of $\{d_1, d_2, d_3, d_4\}$ is $\text{CL}(F^{1,1}, F^{2,1}, F^{3,1}, F^{4,1}) = 0.918$.

Finally, let $\overline{F^k} = F^{k,1}(k = 1, 2, 3, 4)$ be the adjusted RPRs with acceptably consistency and acceptable consensus. Compared with the existing consensus model (see Example 1), our proposal not only improves the CL among the decision makers greatly, but also maintains the individual consistency in the adjusted RPRs, which avoid the repeating of the consistency improving process.

D. Algorithm to Automatically Revise the Preference Values

In Section IV-C, we proposed Algorithm 2 to describe the consensus reaching process with individual consistency control. In this section, based on Algorithm 2, we propose a new algorithm (Algorithm 3) to automatically revise the decision makers' preference values in the consensus reaching process.

Algorithm 3 will not change the essence of Algorithm 2, and the aim of presenting Algorithm 3 is to explore the use of the consensus reaching process with individual consistency control by means of simulation experiments in Section V. In Algorithm 3, we replace Step 4 from Algorithm 2 with Step 4' as follows.

Step 4': Aggregate the RPRs $\{F^{1,h}, F^{2,h}, \ldots, F^{m,h}\}$ to obtain the collective $F^{c,h}$. If $\text{CL}_{\kappa,h} = \min_{k\in\{1,2,\ldots,m\}} \text{CL}_{k,h}$, then using the OCR and Algorithm 1 obtains the adjustable range $[l_{ij}^{\kappa,h}, u_{ij}^{\kappa,h}]$, associated with the decision maker d_κ and the pairwise (x_i, x_j). Then, let $F^{\kappa,h+1} = (f_{ij}^{\kappa,h+1})_{n\times n}$, where $f_{ij}^{\kappa,h+1}(i < j)$ is uniformly and randomly selected from $[l_{ij}^{\kappa,h}, u_{ij}^{\kappa,h}]$ and $f_{ji}^{\kappa,h+1} = 1 - f_{ij}^{\kappa,h+1}$, and let $F^{k,h+1} = F^{k,h}$ for $k \neq \kappa$. Go to Step 3.

In Algorithm 2, the decision makers participate in the consensus reaching process and modify their preferences according to the adjustable range, whereas in Algorithm 3, the consensus reaching process is automatic and the adjusted value is uniformly and randomly selected from the adjustable range. Algorithm 3 will not change the essence of Algorithm 2 and it automatically simulates the consensus reaching process with individual consistency control.

V. Simulation Experiments

In this section, we explore the use of the consensus reaching process with individual consistency control by means of simulation experiments based on Algorithm 3 from three aspects: the CI, the CL, and the approximate performance.

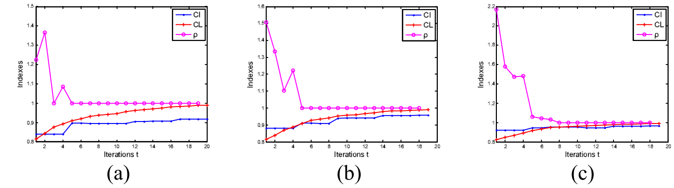

Fig. 4. Process to improve the consensus based on Algorithm 3 using data source 1. (a) $\overline{\text{CI}} = 0.84$, (b) $\overline{\text{CI}} = 0.88$, (c) $\overline{\text{CI}} = 0.92$.

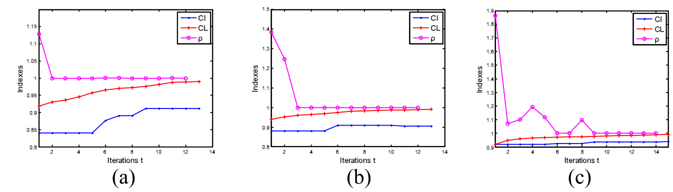

Fig. 5. Process to improve the consensus based on Algorithm 3 using data source 2. (a) $\overline{\text{CI}} = 0.84$, (b) $\overline{\text{CI}} = 0.88$, (c) $\overline{\text{CI}} = 0.92$.

In the simulation experiments, we use five case studies from the existing literatures.

Case 1: Taken from Example 3, [12, Sec 3.3]. In this example, four decision makers provide their RPRs over four alternatives. The worst consistency and the best consistency among the four RPRs are 0.65 and 1, respectively. The CL among decision makers is 0.82.

Case 2: Taken from Example 4, [47, Sec. 3.2]. In this example, four decision makers provide their RPRs over four alternatives. The worst consistency and the best consistency among the four RPRs are 0.7 and 0.97, respectively. The CL among decision makers is 0.9.

Case 3: Taken from [48, Sec. 6.1]. In this case, four decision makers provide their RPRs over four alternatives. The worst consistency and the best consistency among the four RPRs are 0.87 and 0.97, respectively. The CL among decision makers is 0.86.

Case 4: Taken from Example 2, [49, Sec. 4]. In this example, four decision makers provide their RPRs over four alternatives. The worst consistency and the best consistency among the four RPRs are 0.67 and 0.96, respectively. The CL among decision makers is 0.81.

Case 5: Taken from [50, Example 1]. In this example, six decision makers provide their RPRs over four alternatives. The worst consistency and the best consistency among the six RPRs are 0.67 and 0.97, respectively. The CL among decision makers is 0.81.

Following this, we set the consensus threshold $\overline{\text{CL}} = 0.99$ (close to 1) and investigate the variation trends of the CI, the CL, and the approximate performance under different consistency thresholds $\overline{\text{CI}}$ (0.84, 0.88, and 0.92). They are illustrated in Figs. 4–8, respectively.

From Figs. 4–8, the following observations can be drawn.

1) We can find that the CL can be improved rapidly by using the consensus reaching process with individual consis-

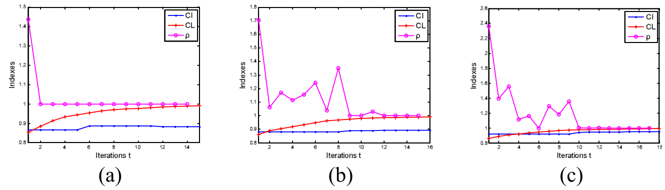

Fig. 6. Process to improve the consensus based on Algorithm 3 using data source 3. (a) $\overline{\text{CI}} = 0.84$, (b) $\overline{\text{CI}} = 0.88$, (c) $\overline{\text{CI}} = 0.92$.

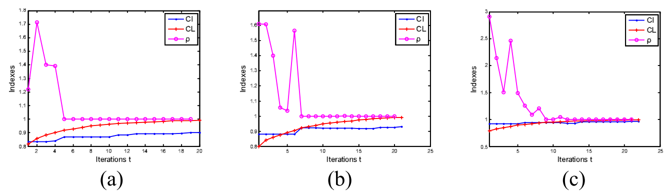

Fig. 7. Process to improve the consensus based on Algorithm 3 using data source 4. (a) $\overline{\text{CI}} = 0.84$, (b) $\overline{\text{CI}} = 0.88$, (c) $\overline{\text{CI}} = 0.92$.

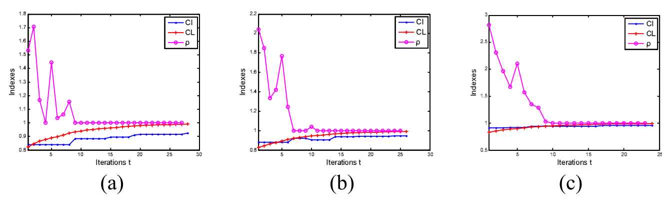

Fig. 8. Process to improve the consensus based on Algorithm 3 using data source 5. (a) $\overline{\text{CI}} = 0.84$, (b) $\overline{\text{CI}} = 0.88$, (c) $\overline{\text{CI}} = 0.92$.

tency control, and the number of the iterations depends on the established consensus thresholds. For example, the CL can reach 0.85 in about 2 iterations, reach 0.9 in about 4 iterations, and reach fully consensus (close to 1) in about 15 iterations. This shows that our proposal provides an effective way to build consensus in the GDM with RPRs.

2) The CI in each iteration is not less than the consistency threshold. Even the CI increases with the number of iterations. This means that our proposal can avoid repeating the consistency improving processes.
3) The largest value of the approximate performances ρ is about 2 and rapidly decreases with the number of iterations. The value of ρ is close to 1, not more than 8 iterations. This shows that Algorithm 3 can yield the solution with a good approximate performance, although it is difficult to obtain the optimum solution to the OCR.

Through the simulation experiments of the five case studies, the above-mentioned observations show that by applying Algorithm 3, the CL is improved and also the consistency of each decision maker is guaranteed without repeating the consistency improving process.

VI. Conclusion

In this paper, the consensus reaching process with individual consistency control is proposed in the GDM with RPRs, and its core part is based on the design of an optimization-based consensus rule to determine the adjustment range of each preference value to guarantee the individual consistency in building consensus. Compared with some existing studies, the consensus reaching process with individual consistency control can not only provide a new way to assist decision makers to reach a consensus in the GDM with RPRs, but also avoid repeating the consistency improving process, which is very time consuming.

In the future, we plan to work on the potential use of the consensus reaching process with individual consistency control for large-scale decision making [51]–[58] to handle large groups with different preference representation structures.

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