Research Article

Factorial Harmonious Graph of a Group

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Abstract
Consider the commutative group G. The Factorial Harmonious graph of G is the undirected graph with vertex set G and two different vertices a and b are adjacent if $\frac{[f(a)+f(b)]!}{[f(a)]![f(b)]!}$ + {$f(a)$ + $f(b)$} (mod m) in G is isomorphism. The results of a study of the Factorial Harmonious graph and its generalizations on Group are presented in this work.

Key Word: Commutative group, Factorial Harmonious graph, Complete bipartite graph, Degree divisor.

1. INTRODUCTION

A graph's vertex labeling G is a planning *f* made up of G's vertices to each edge ab has a label that depends on the vertices a and b and their label *f*(a) and *f*(b). Graph labeling methods began with A. Rosa [9] in 1967. The concept of the Harmonious labeling graph was first introduced by R. L. Graham and N. J. A Sloane [5] in 1980 and the concept of Factorial labeling graph were introduced by A. Edward Samuel and S. Kalaivani [4] in 2018.

In section 2, we drive Some Results on Order not Prime in $Fl_H(G)$ and in section 3, we drive Some Results on Degree Divisor $Fl_H(G)$ on Group.

KNOWN RESULT'S AND DEFINITION

Definition 1.1: [3]

Consider the graph G, which has m edges. If $f : V \rightarrow \{0,1,2,\dots,m-1\}$ is injective and the induced function $f^* : E \rightarrow \{1,2,\dots,m\}$ is bijective, the function $f^*(e = ab) = \big(f(a) + f(b)\big)(mod\ m)$ is called Harmonious labeling of graph G. Harmonious graph is a graph that allows for Harmonious labeling.

Definition 1.2: [4]

A factorial labeling of a connected graph G is a bijection $f : V \rightarrow \{0,1,2,\dots,m\}$ such that the induced function $f^* : E \rightarrow \{1,2,\dots,m\}$ defined as $f^*(e = ab) = \frac{[f(a)+f(b)]!}{[f(a)]![f(b)]!}$ then the edges labels are distinct. Any graph which admits a factorial labeling is called a factorial graph.

Definition 1.3: [6]

An Euler tour of a graph G is a tour that passes around each of the graph G's edge exactly once.

Definition 1.4: [6]

If a graph G has an Euler tour, it is termed an Euler graph or Eulerian.

Theorem 1.5: [6]

If and only if the degree of each vertex is even, a connected graph is Euler.

Definition 1.6: [6]

If there is a cycle that contains every vertex of G exactly once, the connected graph G is termed Hamiltonian Graph.

Theorem 1.7: [7]

The order of H divides the order of G if G is a finite group and H is a subgroup of G.

Definition 1.8 [2]

Let G be a graph and v be one of its vertex. The maximum distance between v and any other vertex is the eccentricity of the vertex v.

In other words, e(v) = max {d (v, w): w in v(G)}

Definition 1.9 [2]

The largest eccentricity among G's vertices equals the diameter of G. As a result, diameter (G) = max { e(v): $v \in$ G}

Definition 1.10 [2]

The length of the shortest cycle in G is the girth of G.

2. Some Results on Order not Prime in $Fl_H(G)$

Definition 2.1

Consider the graph G, which has m edges. If $f : V \rightarrow \{0, 1,2, \ldots, 2m - 1\}$ is injective and the induced function $f^* : E \rightarrow \{0,1,2, \ldots, m\}$ defined as $f^*(e = ab) = \frac{[f(a)+ f(b)]!}{[f(a)]![f(b)]!}$ + {$f(a)$ + $f(b)$} (mod m) is isomorphism. A Factorial Harmonious graph is indicated by the symbol $Fl_H(G)$ and it admits Factorial Harmonious labeling.

Definition 2.2

Consider the commutative group G. The Factorial Harmonious graph has the vertex set G when two different vertices a and b are adjacent in $Fl_H(G)$ such that $f : V \rightarrow \{0, 1,2, \ldots, 2m - 1\}$is injective with order either prime or not prime and $f^*(e = ab) = \frac{[f(a)+ f(b)]!}{[f(a)]![f(b)]!}$ + {$f(a)$ + $f(b)$} (mod m) is isomorphism.

Theorem 2.3

The Factorial Harmonious graph is a commutative group then whose order is not prime.

Proof:

Suppose $Fl_H(G)$ is a complete bipartite graph. So, every pair of vertices are adjacent.

Therefore $o(x) = o\left(x^i\right)$ for some $i \in \{1, 2, \ldots, n - 1\}$.

Then $o(x) \mid o(x^i)$ or $o\left(x^i\right) \mid o\,(x)$

This implies gcd (i, n) ≠ 1 for some $i \in \{1, 2, \ldots, n - 1\}$ and also order of a group element is not prime.

Hence n is not prime.

Remark 2.4

A Factorial Harmonious graph is a group whose order is not a prime number p then G is not a cyclic group.

Theorem 2.5

If G is a commutative group then every connected Factorial Harmonious graph is an Euler cycle.

Proof:

Given G is a commutative group.

If we take $K_{2,n}$ graph that admits Factorial harmonious graph and satisfy commutative group.

By Theorem 1.5, If and only if the degree of each vertex is even, a connected graph is Euler.

Our graph has even degree for every vertex; Hence G is an Euler cycle.

Corollary 2.6

Suppose that G is commutative group and Factorial Harmonious graph is complete bipartite graph then G is Hamiltonian cycle when n = 2, 3.

Theorem 2.7

The order of x divides the order of G if G is a Factorial Harmonious graph with finite group and x is an element of G

Proof:

Assume G is a $K_{2,n}$ graph.

As a result, G accepts both the Factorial harmonious labeling graph and the group.
Assume *x* is an element of G.
By definition, the order of *x* is the order of the subgroup created by *x*.
As a result of Theorem 1.7, the order of G is divisible by the order of *x*.
Hence proved.

Theorem 2.8

A graph $Fl_H(G)$ has a commutative group if and only if $Fl_H(G)$ is a group.

Proof:

Assume G = $Fl_H(G)$ is a $K_{2\,,n}$ graph with an order of 6.
To put it another way, G = {0,1,2,3,4,5} is an element of the $K_{2\,,3}$ graph.
Let's pretend that G is a commutative group.
To prove: G is a group
By default, G is a group.
Conversely,

Assume that G is a group
To prove: G is a commutative group
That is to prove: a + b = b + a where a, $b \in$ G
Let a = 2 and b = 3 then 2 + 3 = 4 ∈ G,
Also, 3 + 2 = 4 ∈ G
Therefore G is a commutative group.
In general, G is also commutative group.
Hence, A graph $Fl_H(G)$ has a commutative group if and only if $Fl_H(G)$ is a group.

3. Degree Divisor $Fl_H(G)$ on Group

Definition: 3.1

Let G be a finite group. Then $Fl_{H_{DD}}(G)$ denotes the degree divisor Factorial harmonious graph whose vertex set is G such that two distinct vertices a and b having same degree are adjacent provided that $f^*(e = ab) = \frac{[f(a)+f(b)]!}{[f(a)]![f(b)]!} + \{f(a) + f(b)\}$ (mod m) is isomorphism then d(a) | $d(b)$ or d (b) | $d(a)$.

Theorem: 3.2

The degree divisor graph $Fl_{H_{DD}}(G)$ is a $K_{2,n}$ graph if and only if every element of the group G has prime degree.

Proof:

Assumed, if every element of G has prime degree, then $Fl_{H_{DD}}(G)$ is a $K_{2,n}$ graph.
Conversely,
Assume $Fl_{H_{DD}}(G)$ is a $K_{2,n}$ graph.
Obviously, graph structure shows each vertex has 2 degree.
Hence each n is prime degree.

Theorem 3.3

If $Fl_{H_{DD}}(G)$ is a finite group whose non-identity vertex degree is a prime number p,then G is a cyclic group. Further $Fl_{H_{DD}}(G)$ is a sequential join $(G_1 \diamond G_2 \diamond G_3) \diamond k_2$.
i.e., Degree sequence of $Fl_{H_{DD}}(G)$ = $(G_1 + G_2 + G_3)$ + K_1 + K_2 always even.

Proof:

Let p be a prime and G be a group, such that deg(G) = p be the result.
Then G is made up of many elements.
Let a ∈ G such that a ≠ e.
Then < a > contains more than one element.
Since, < a > ≤ G

deg(< a >)divides p.

Since deg(< a >) >1 and deg(< a >) divides a prime, deg(< a >)= p = G.
Hence < g > = G
Hence G is cyclic group.
Also note that all vertices in G are independent.
Hence deg(G) = $(G_1 + G_2 + G_3) + K_1 + K_2$and add all prime degree must be even.
Therefore, Degree sequence of $P_{H_{DD}}(G)$ = $(G_1 + G_2 + G_3) + K_1 + K_2$ always even.

Theorem 3.4

If $Fl_{H_{DD}}(G)$ is connected for abelian group G then diam($Fl_{H_{DD}}(G)$) = 2.

Proof:

Let a and b be two distinct vertices of $Fl_{H_{DD}}(G)$. If $(|a|,|b|) = 1$, then a is adjacent to b and hence d (a , b) = 1.
In this manner, we may expect that a and b are non-identity elements of G $(|a|,|b|) \neq 1$.
Note that $(|a|,|e|) = 1$ and $(|b|,|e|) = 1$, then the vertex e is neighboring both a and b and we get d(a, b) =2 .
This implies that $Fl_H(G)$ is connected and diam($Fl_{H_{DD}}(G)$) = 2.

Theorem 3.5

Let G be a group. If $Fl_{H_{DD}}(G)$ contains a cycle, then $g(Fl_{H_{DD}}(G))$ = 4.

Proof:

Permit us to accept $Fl_{H_{DD}}(G)$ contains a cycle. We ensure that the length of most short cycle present in $Fl_{H_{DD}}(G)$ is 4. In this view, if there is an example of length 4, by then outcome follows itself.

In this case, it contains a cycle $a_1 - e - a_2 - \cdots - a_n - a_1$ for n ≥ 2.
Now, for all i, a_i should be same degree.
Subsequently, $a_1 - e - a_2 - e - a_3 - e - a_4 - e - a_1$ is a cycle of length 4 in $Fl_{H_{DD}}(G)$

Hence $g(Fl_{H_{DD}}(G))$ = 4.

Conclusion:

We may deduce that if a Factorial Harmonious Graph is a commutative group with an order that is not prime, it is not a cyclic group, but an Eulerian graph. A Group's order is divided by the order of its elements.

Degree Divisor Factorial Harmonious graph is a commutative group with degree prime, it is also a cyclic group with a diameter of two and a girth of four.

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