Departamento de Física de Partículas

RESURGENCE IN TOPOLOGICAL STRING THEORY

Ricardo Couso Santamaría

TESE DE DOUTORAMENTO

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Santiago de Compostela, setembro de 2014.

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Tese presentada para optar ao grao de Doutor en Física por:

Ricardo Couso Santamaría

Santiago de Compostela, setembro de 2014

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Departamento de Física de Partículas

José D. Edelstein Glaubach, Profesor Titular de Física Teórica da Universidade de Santiago de Compostela,

CERTIFICO: que a memoria titulada *Resurgence in topological string theory* foi realizada, baixo a miña dirección, por Ricardo Couso Santamaría, no departamento de Física de Partículas desta Universidade e constitúe o traballo de Tese que presenta para optar ao grao de Doutor en Física.

Asinado:

José D. Edelstein Glaubach
Santiago de Compostela, setembro de 2014

A mi padre

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Publications related to this work

R. Couso-Santamaría, J. D. Edelstein, R. Schiappa and M. Vonk, "Resurgent Transseries and the Holomorphic Anomaly," `arXiv:1308.1695 [hep-th]`.

R. Couso-Santamaría, J. D. Edelstein, R. Schiappa and M. Vonk, "Resurgent Transseries and the Holomorphic Anomaly: Nonperturbative Closed Strings in Local $\mathbb{CP}^2$," `arXiv:1407.4821 [hep-th]`.

Other publications

R. Couso-Santamaría, J.D. Edelstein, A. Garbarz, G.E. Giribet, "On the Addition of Torsion to Chiral Gravity," *Phys. Rev.* **D83** (2011) 124032, `arXiv:1102.4649 [hep-th]`.

R. Couso-Santamaría, M. Mariño and P. Putrov, "Unquenched Flavor and Tropical Geometry in Strongly Coupled Chern-Simons-Matter Theories," *JHEP* **1110** (2011) 139, `arXiv:1011.6281 [hep-th]`.

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Introduction

In theoretical physics it is natural to distinguish between weakly and strongly coupled systems. In the first case the coupling constant that measures the strength of the interaction is small, and the natural approach is perturbation around the free theory with vanishing coupling. In the second situation the degrees of freedom of the system are strongly correlated with each other and a perturbative analysis fails to describe even the most basic features of the system. We talk about nonperturbative effects. Many physically interesting situations lie in the nonperturbative regime of the theory and it is the physicist's job to develop the appropriate tools for the computation of observables.

A famous example is the WKB approximation, which in quantum mechanics allows to compute the decay time by tunnel effect of a particle in potentials unbounded from below, among other things. The coupling constant here is $\hbar$, and the result is to leading order proportional to $\hbar^{-1}$, so the effect has no interpretation in the classical limit. Powerful nonperturbative methods have been invented in many areas of physics to approach a variety of different problems. See for example [1] for a non-exhaustive survey.

In mathematics the study of nonperturbative problems is typically incarnated in the study of functions, usually complex, and their various representations as integrals, series, etc. More precisely, one is interested in the singular behavior of the functions at different points in the complex plane and their asymptotic limit there. The classification of these functions is eased by the study of the differential equations they satisfy and the properties these have in terms of regular or irregular points. A more general approach is that of the theory of resurgence [2], which deals with a very general class of functions whose singularity structure has interesting algebraic properties. In some sense, it provides a unifying framework in which to accommodate previous separate results and tools to approach more difficult and general problems. In physics, the relevant functions, usually describing observables of the theory, depend on the coupling constant.

For physical applications the coupling has real values but, as usual, much more can be achieved by regarding it as a complex variable. It has been found in the past decades that many observables of physical systems have representations as power series that are not convergent for any nontrivial value of the coupling constant—usually zero or infinity depending on the convention. That is, they have zero radius of convergence. The origin of the divergence is the singular behavior of the functions and the underlying resurgent structure. In practice, working with divergent series is not completely hopeless because in many cases, before the series starts to diverge, the partial sums can approximate the actual value of the observable quite well. This is what allows quantum field theories to produce predictions of high precision when a few loops are included in the calculation and the effective coupling constant is not very large (the divergent nature of the series continues to hold after renormalization). What at first sight would be regarded as a problem, that is, the divergence of perturbation theory for any value of the coupling constant, turns out to provide a very useful connection between perturbative and nonperturbative quantities, as was understood by the theory of resurgence.

In high energy physics a good part of the interesting dynamics occurs in the nonperturbative regime of the relevant coupling constant. Several techniques have been developed during the years to translate difficult problems into others that are amenable to more standard approaches. This process very often brings with it a new perspective on the theory under consideration because new languages are invented to describe it, or new relations to pre-existing theories are discovered. Among various examples we can highlight integrability, where it is found that the theory, or some of its sectors, can be solved exactly for any value of the coupling. The existence of several descriptions of integrability increases the chances of finding relations with seemingly different theories; for an example with links to resurgence see [3]. Another example applies to some quantum field theories including gauge theories and topological string theory is localization, see for example [4, 5], where a symmetry, usually in the form of a supersymmetry, can simplify an infinite dimensional path integral to a finite dimensional one. The latter description is usually easier to approach. For gauge theories it can be identified as a matrix model, for which nonperturbative techniques are available. The most powerful and far-reaching nonperturbative technique high energy physics is the gauge/gravity correspondence [6] resting on the general idea on large N duality [7] and holography [8]. The correspondence compares superstring theory on an asymptotic AdS space and a supersymmetric gauge theory. The theories depend on two parameters, the string coupling and string tension on one side, and the rank of the gauge group and the 't Hooft parameter on the other. In its weakest formulation super Yang–Mills at strong coupling and large N would be equivalent to classical supergravity, which is a weak coupling limit of string theory. There is a large amount of evidence in favor of the conjecture. It has been used to explore the strong coupling regime of several gauge theories. Proving the conjecture would require a nonperturbative understanding of string theory that is still beyond reach. However, nonperturbative effects have been calculated in terms of D-branes, dynamical objects that extend in several dimensions, and on which open strings can end. The possible perfect equivalence between string and gauge theory would in principle provide a nonperturbative definition of the former in terms of the latter. This realization is particularly suggestive in the case of topological string theories and their matrix model duals.

Matrix models, also known as random matrix theory, has multiple connections with problems in mathematics and physics—see for example [9–12]. In physics they can be regarded as toy models, zero-dimensional versions of gauge theories, and as we said before, they appear naturally from localization. They depend on a gauge group of rank N that can be used as a expansion parameter, around infinite N. The large N expansion of the free energy, computed as the logarithm of the matrix integral, can be matched by the perturbative free energy of a topological string theory [13, 14]. This correspondence has an illuminating description in the case of matrix models with polynomial potentials. The matrix model is determined at leading order in $1/N^2$ (planar limit) by a spectral curve [15]. From this spectral curve alone one can compute recursively the complete perturbative large N expansion [16–19]. This is the so-called topological recursion whose application extends beyond matrix models. Topological string theory has less degrees of freedom than the physical string theory but it retains many of its features, as well as being completely integrable in many cases and realizing interesting mathematical ideas like mirror symmetry, see [20]. Like the original one it can describe open and closed strings. The target space the strings probe is a Calabi–Yau threefold, a complex three-dimensional variety with special geometrical properties. This geometry and the spectral curve are related to each other. Moreover, to all orders in perturbation theory, open topological string theory on a particular geometry is equivalent to a Hermitian matrix model whose polynomial potential determines the spectral curve. The string coupling g_s, from the string side, and the rank N, from the gauge side, are related through the 't Hooft parameter $t = g_s N$, which can be calculated on both sides of the duality as an integral over cycles of the geometry. The details of the correspondence can be understood in terms of a geometric transition [21] between the deformed geometry associated to the spectral curve and its resolution as a Calabi–Yau manifold. The identification of both theories beyond the planar limit was done [22] by showing the equivalence between an extension of the topological recursion and the holomorphic anomaly equations [23, 24] that the topological string free energies satisfy. To go beyond perturbation theory in g_s is an even more challenging problem, first of all because topological string theory lacks a nonperturbative definition. The problem of finding a nonperturbative completion of this and the full string theory is a longstanding one, and there have been several proposals inspired by large N duality to address the problem [25–29].

Any nonperturbative completion of string theory must incorporate all the knowledge obtained from gauge/gravity dualities, and in fact, they have been the main inspiration. This thesis explores a different set of tools, those of resurgent transseries, to gain nonperturbative information about topological string theory. Nevertheless this work presented in this thesis relies heavily on previous research involving the duality with matrix models and the relations to resurgence [30–42]. Often, the starting point of resurgence is an asymptotic series, usually divergent. In string theory that series in the free energy, the asymptotic parameter is the string coupling constant. The coefficients are the perturbative free energies. They grow factorially with the order [43, 44] and they are well-defined functions of certain moduli ('t Hooft parameter, complex or Kähler moduli) [45]. This factorial growth implies nonperturbative effects in the string coupling constant of order e^{-1/g_s} [44, 46, 47] that can be identified with D-branes.

At the formal mathematical level the theory of resurgence focuses on complex functions, their asymptotic series representations, the properties of their Borel transforms, and the underlying algebraic structure. This theory provides a convenient and powerful language is which to express questions and answers for nonperturbative problems. An example is the concept of transseries which extends the perturbative series expansion to include nonperturbative corrections. It has to be thought of as a formal object on the coupling constant because it has a zero radius of convergence. Nevertheless, it is a representation of the most general solution to the problem determining a particular observable. As such it has the potential to become the final physical solution. The step going from a formal transseries to an actual function of the coupling constant is called resummation. The problem of resumming divergent perturbative expansions is an old one and many techniques have been developed to extract sensible numbers out of them, see for example [48, 49]. However, due to the singular nature of the underlying functions, resummation is sometimes burdened with the so-called nonperturbative ambiguity that leaves the final result underdetermined. This issue affects quantum mechanical systems and also quantum field theories, and its resolution involves the cancellation of ambiguities coming not only from perturbation theory, but from all nonperturbative sectors. This phenomenon has been observed in many different settings [50–53] and it is a necessary condition for the consistency any physical theory.

One of the key results in resurgence theory is the existence of a set of equations relating coefficients of perturbative expansions around trivial and nontrivial saddle points of the theory. The are called large-order relations because they describe the asymptotic growth of the coefficients in a particular series expansion. The first example of a large-order relation in a physical setting was done in the asymptotic analysis of the energy levels of the anharmonic oscillator [54, 55], and it was later extended to apply in field theories of different types. In well-defined mathematical problems the analysis of the large-order relations could be done in great detail. An interesting realization of this ideas in the context of integral representation of functions was developed in [56], where the resurgent relations appear explicitly from the existence of different saddle points for the integral. One of the fundamental problems in resurgence as applied to quantum field theories is to understand what is the analogous situation for path integrals. See [57] for a recent discussion and references.

In recent years the framework and techniques of resurgence have found application in an increasing number of areas, in mathematics, theoretical physics, and the vague region between the two. Hopefully in the near future the ideas of resurgence, as applied to nonperturbative problems in physics, will be naturally incorporated as useful tools in combination with others. There are many ideas yet to be explored and generalizations to be made that will find their application in problems of today and the near future.

Goal of the thesis

The purpose of this thesis is the study of the resurgent structure of the topological closed string theory. The motivation to do so is twofold.

The first motivation centers on the longstanding problem of providing a nonperturbative definition for string theories in general, and topological string theories in particular. Large N dualities and the AdS/CFT correspondence in its various incarnations have become the preferred tool to learn about nonperturbative aspects of string theories. From a bold point of view, gauge theories present themselves as candidates for nonperturbative definitions of their dual string theories. A more standard approach takes duality as inspiration and guidance to propose purely string theoretic formulations. This thesis sees perturbative string theory as a collection of results that can be further exploited using the adequate tools without necessarily having to cross the bridge to gauge theory. These tools are resurgent transseries and large-order analysis. The asymptotic nature of string perturbation theory hides, from the point of view of resurgence theory, a wealth of information about the nonperturbative formulation of the theory. This information is conveniently stored in the transseries, a formal object beyond perturbation theory. The resummation of the transseries into an actual function of the string coupling constant provides the final step in the nonperturbative formulation of string theory. This last part may well be a difficult practical problem. Finally, a physical interpretation of the nonperturbative elements of the transseries must be provided.

The second motivation has to do with putting the capabilities of the resurgent framework to the test. Resurgence theory has a solid mathematical formulation and has found natural applications in the study of solutions to differential and difference equations. In these areas, rigorous theorems can be proved and numerical resurgence checks can be done to high precision. The application of resurgence to physical problems has been more timid, especially in field theories, partly due to the technical obstacle of being able to compute perturbation theory to high order and/or higher instanton corrections. In matrix models (especially with polynomial potentials) the situation has been more favorable due to the development of powerful computational techniques, such as the use of orthogonal polynomials. On the string theory side, the holomorphic anomaly equations have been used very efficiently to generate perturbative data. It is this possibility of computing perturbation theory to high order, in a very interesting and nontrivial physical problem, that opens up the possibility of applying the techniques of resurgence. At the same time, the holomorphic anomaly equations can be naturally written in a way that admits a transseries solution. Thus, topological string theory satisfies two important properties: computability to high order in perturbation theory and capability to generate transseries solutions. Both features come together through large-order analysis.

Structure of the thesis

This thesis is divided in two parts.

In the first one we introduce the main theoretical aspects of both resurgence and topological string theories. Chapter 1 on resurgence, reviews the definitions of asymptotic series, Borel resummation, and related concepts. Next we discuss the concept of resurgent transseries and their physical interpretation, and end with an overview of alien calculus and large-order relations, as they will be used in chapters 3 and 4. Chapter 2 reviews the construction of topological string theory and the definition of the free energies. We discuss their computation with the holomorphic anomaly equations on the B model. We conclude with the example of the mirror of local $\mathbb{CP}^2$, and the calculation the perturbative free energies to high genus.

The second and main part of the thesis studies the resurgent properties of topological string theory. In chapter 3 we show the extension of the holomorphic anomaly equations and study the structure of different transseries solutions. We discuss the case of resonance and stress the consequences of large-order growth of the perturbative free energies. We finish by analyzing the problem of fixing the holomorphic ambiguities. Chapter 4 applies the techniques described in previous chapters to the case of local $\mathbb{CP}^2$. We study the large-order growth of perturbation theory and discover a number of instanton actions controlling it. A deeper analysis uncovers the existence of one and two-instanton sectors. These are computed from the holomorphic anomaly equations as explained in chapter 3 and found to agree, to high precision, with numerical tests. Resonance is found explicitly in the large-order growth of the one-instanton sector. New transseries sectors computed from the holomorphic anomaly equations control its growth.

Chapters 5 and 6 cover a summary and conclusions of the thesis. Appendices A, B, and C show extra material that can be consulted in parallel with chapters 3 and 4. A list of bibliographic references concludes the thesis.

Chapter 1

Aspects of resurgence

1.1 Introduction

In the study of differential equations it is common to find formal solutions, expressed as powers series in the relevant variable x, that have zero radius of convergence. Often, the cause for this is a factorial growth in the coefficients of the asymptotic series, rather than a milder exponential one. This behavior occurs already in very simple differential equations, or it can be found in the series expansion of functions expressed in integral form, for example. In physical problems such divergent series can appear after performing a perturbative calculation around the weakly coupled regime of the theory. This is the case in quantum mechanics and quantum field theory. In the latter, the fast growth is sometimes due to the factorial number of Feynman diagrams contributing to a given order in perturbation theory, while other in other cases the diagrams themselves grow factorially and one speaks of renormalons. A very generic argument due to Dyson [58] shows that the radius of convergence of, say the ground state energy as a series in the coupling, must be zero. The reason lies in the fact that if a series converges for $0 < x \leq R$, then it will also converge for $-x$—or any other complex value of x with $|x| \leq R$—, but the quantum mechanical system with the opposite coupling may well be unstable, and hence, very different from the original. Even if this argument is not a proof (see [59] for counterexamples), it shows that we should not expect observables to be represented by convergent series if the physics of the problem changes drastically as the coupling x moves in the complex plane.

Beyond the perturbative asymptotic series there usually exists a larger family of formal solutions, not only in x, but also in the object $\mathrm{e}^{-1/x}$. This exponential is nonanalytic at $x = 0$, and from the physical point of view it is a genuine nonperturbative object, in the sense that it cannot be seen in a perturbative computation around $x = 0$. In physical problems one can find such nonanalytic functions of the coupling when looking, for instance, at instanton configurations. These are related to saddle points other than the minimum. Such a formal double power series is the simplest example of transseries—see [60] for a mathematical review—, and it constitutes a very convenient way to store nonperturbative information of physical systems [32]. A very important problem is the passage from the formal transseries to a well-defined function in some region of the complex coupling plane. This is one the main concerns of the theory of resurgence [2]

Resurgence deals with asymptotic series and their resummation, uncovering very interesting and powerful algebraic structures that impose relations between the different sectors and coefficients of the transseries. These resurgence relations, of which we will make extensive use, dictate precisely how the factorial growth is realized, not only for the perturbative solution, but also for any of the other sectors. It turns out, remarkably, that this large-order factorial growth is controlled by other sectors of the transseries and the relation can be made absolutely explicit. This happens in such a way that knowing one of sectors would, in principle, allow us to reconstruct all the others. This determination of one sector in terms of the rest is not really a surprise if we think that the transseries may come from a differential equation, for example. All the information is stored in the equation so we could expect it to be also stored in any of the sectors. However, the actual resurgence relations have a very transparent and hierarchical form that makes them useful. The first thread from which to pull the nonperturbative sectors of the transseries is the factorial growth of the perturbative coefficients.

The asymptotic analysis of perturbation theory in physical systems started with the example of the quantum anharmonic oscillator [54, 55]. From a semiclassical point of view, instanton configurations were found to be deeply related to the asymptotics [50, 51, 61–63]. At the same time, similar asymptotic studies were being carried out for field theories in various fronts [64–68], see [69] for a thorough compilation. After the work of Écalle [2] and the birth of resurgence, approaches became more rigorous and formal [70]. Since then there has been an extensive application of resurgence ideas to the solutions of differential equations, see for example [71]. In high energy physics the usefulness of the resurgent transseries started to appear within the context of matrix models, their double scaling limits and dual topological string theories [31–33, 36, 38]. Especially fruitful was the analysis of the Painlevé I equation [38] for which a two-parameter transseries was needed and resonance was present, that is there are two instanton actions of opposite signs whose physical interpretations remains to be explained. Further examples suggest that this is a common phenomenon [40, 41] in cases where a topological expansion, of powers of x^2, defines the perturbative sector. The concept of resonance that will be explained in this chapter, see section 1.5.3, is indeed present in the topological string theory on local $\mathbb{CP}^2$, as it will be shown in chapter 4. For theories in which a differential—like Painlevé I—or difference equation—like the string equation in polynomial matrix models [16]—is present, it is possible to compute the full transseries in a systematic way and check many relations that resurgence imposes on the coefficients of different sectors. However, in general in physical systems, and especially in quantum field theories, computing the perturbative sector already becomes a challenge. Nevertheless, there has been significant progress in gauge theories with realistic properties such as asymptotic freedom and the presence of renormalons [52, 53, 57, 72]. Some of this progress is based on the understanding of resurgence on quantum mechanical systems on which progress is still ongoing, see for example [73–75]. Exact quantization conditions born in the area of quantum mechanics are also being used in the context of spectral problems in topological string theory and its refined version [76, 77].

In this chapter we are going to review the most important and relevant concepts of asymptotic series, resurgent transseries, alien calculus and large-order relations. The results of the last section will be especially useful when we deal with an actual topological string theory. Because this is an early application of resurgence theory to topological string theory on its own, some of the advanced concepts explained in this chapter will not find a realization in the example of local $\mathbb{CP}^2$ presented in chapter 4. Useful reviews on asymptotics and resurgent transseries can be found in [40, 78–83]. See also [84] for a recent overview of resurgence in quantum physics.

1.2 Asymptotic series

Even if asymptotic series are not necessary divergent this adjective is most often used to suggest a zero radius of convergence. We will follow this common abuse of language since it should not lead to confusion.

1.2.1 Definition and properties

The difference between an asymptotic series and a Taylor expansion can be understood by looking at the sequence of partial sums

$$\sum_{g=0}^{N} a_g\, x^g. \tag{1.1}$$

For a Taylor series expansion, if we fix a value of x and take $N \to \infty$ the partial sum approximates better and better the original function it represents. The series converges with a certain radius in the complex plane. For an asymptotic series, on the other hand, one has to fix the upper bound N and take $x \to 0$ to increase the agreement. Asymptotic series are not expected to converge for any nonzero value of x, so for practical purposes they can be thought of as formal power series around $x = 0$. More precisely, given a formal series,

$$\varphi(x) = \sum_{g=0}^{\infty} a_g\, x^g, \tag{1.2}$$

we say that a function $\Phi(x)$ is asymptotic to it in the sense of Poincaré if for every N

$$\lim_{x\to 0} \left(\Phi(x) - \sum_{g=0}^{N} a_g\, x^g \right) x^{-N} = 0. \tag{1.3}$$

This means that for a fixed and small value of x, we can find a value of N for which the partial sum approximates well the value of the function $\Phi(x)$. Different functions can have the same asymptotic expansion. For example, the functions $\Phi(x)$ and $\Phi(x) + 4\,\mathrm{e}^{-3/x}$ share have the same Poincaré asymptotics as $x \to 0$ (from the right). The reason is that the exponential term is nonanalytic at zero and its naive Taylor series has every coefficient equal to zero as a result. We speak of an exponentially suppressed term that in the language of physics is labelled nonperturbative. Asymptotic series may be convergent or not, but one usually finds divergent examples of a particular kind called Gevrey 1.

A power series $\varphi(x)$ like (1.2) is of class Gevrey 1 if there exist two positive numbers c and A, such that

$$|a_g| \leq c\, g!\, A^{-g}, \tag{1.4}$$

for all g. Series of this class are asymptotic and they are found in the perturbative calculation of many systems. For them we can estimate which partial sum gives the best approximation to a function asymptotic to it. That is, we are looking for the optimal truncation of the series before the factorial growth takes over and makes the result divergent. We need to find the smallest term in the series, the value of g for which $|a_g\, x^g|$ is minimal. That value of g will be our optimal truncation value for N. If x is small enough, optimal N will be large, so we can approximate

$$\left|a_g\, x^N\right| \simeq c\, N!\, A^{-N}\, |x|^N \simeq c\sqrt{2\pi N}\left(\frac{N}{\mathrm{e}}\right)^N \left(\frac{|x|}{A}\right)^N, \tag{1.5}$$

using Stirling's approximation to the factorial. This term is minimal for a value of N approximately equal to

$$N \simeq \frac{A}{|x|}. \tag{1.6}$$

The error in this optimal truncation can be estimated by the order of the next term in the asymptotic series

$$\text{OT ERROR} \simeq \left|a_{N+1}\, x^{N+1}\right| \simeq \tilde{c}\, \mathrm{e}^{-A/|x|}, \tag{1.7}$$

where we have used Stirling's approximation again. Note that this error is nonperturbative in nature. A word of caution is in order here. The resummation of the asymptotic series (1.2) by optimal truncation, or by more powerful methods that will be covered later, does not need to coincide with the value of the full nonperturbative function $\Phi(x)$. Indeed, there may well be a transseries on top of the perturbative asymptotic expansion $\varphi(x)$ that needs to be taken into account in order to match against the value of $\Phi(x)$. We can compare orders of magnitude between subleading sectors (which are exponentially suppressed) and errors of resummation of the perturbative one. Sometimes this error will be far greater than the transseries contributions. So for numerical purposes it will make no sense to include those, because they will be shadowed by the resummation error. Other times, the subleading terms of the transseries have a non-negligible part to play. The underlying reason is that different functions can have the same asymptotic expansion, as we mentioned before.

1.2.2 Borel transform and resummation

A step beyond optimal truncation is Borel resummation. This process comes in two parts. The first deals with the construction of the Borel transform: a new series in a new variable, ξ, that lives in the so-called Borel plane. Given a Gevrey 1 series like (1.2), we define its Borel transform as

$$\mathcal{B}[\varphi](\xi) := \sum_{g=1}^{\infty} \frac{a_g}{(g-1)!}\xi^{g-1}. \tag{1.8}$$

This expression is also sometimes called the minor of φ, and it does not include the constant term a_0. The definition of the Borel transform can be slightly modified to accommodate different factorial growths of the form $(\beta g+\delta)!$, for some values of β and δ. The precise details are not relevant for us and definitions can be adapted accordingly. In any case, it is a theorem—see proposition 3 of [81], for example—that the Borel transform defines an analytic function in some disk around $\xi=0$. The radius of the disk will be equal to the distance from the origin to the closest singularity. The singularities of the Borel transform carry abundant information and they are of primary importance in the theory of resurgence. The second ingredient of Borel resummation involves the formal inverse of the Borel transform: the Laplace transform. Note that for $\mathrm{Re}(1/x)>0$,

$$\int_0^\infty d\xi\, \mathrm{e}^{-\xi/x}\, \xi^{g-1} = x^g\, \Gamma(g) = x^g\, (g-1)!, \tag{1.9}$$

which puts back the factorial that was removed in the Borel transform. If $\mathcal{B}[\varphi](\xi)$ has an analytic continuation in some region around the real axis, we define Borel resummation of the asymptotic series $\varphi(x)$ as

$$(\mathcal{S}\varphi)(x) := a_0 + \int_0^\infty d\xi\, \mathrm{e}^{-\xi/x}\, \mathcal{B}[\varphi](\xi). \tag{1.10}$$

Technically, we need $\mathcal{B}[\varphi](\xi)$ to grow like $\mathrm{e}^{\tau|\xi|}$ at most, for some value τ, when ξ goes to infinity. Then $(\mathcal{S}\varphi)(x)$ defines an analytic function on $\{x \,|\, \mathrm{Re}(1/x)>\tau\}$. The resummed function, $(\mathcal{S}\varphi)(x)$, is still asymptotic to φ in the appropriate domain. See [78] or [81] for details. Borel resummation can be generalized so that the integration is taken along some other line in the Borel plane around which the Borel transform has an analytic continuation,

$$(\mathcal{S}_\theta\varphi)(x) := a_0 + \int_0^{\infty\, \mathrm{e}^{i\theta}} d\xi\, \mathrm{e}^{-\xi/x}\, \mathcal{B}[\varphi](\xi). \tag{1.11}$$

An important obstruction appears along directions where the Borel transform has singularities. The avoidance of singularities from above or below introduces the definition of lateral resummations $(\mathcal{S}_{\theta_\pm}\varphi)(x)$, where $\theta_\pm$ are directions slightly above or below the direction marked by the angle θ where the singularities lie, respectively. The two lateral resummations define different functions. The difference between the two is given by

$$(\mathcal{S}_{\theta_+}\varphi)(x) - (\mathcal{S}_{\theta_-}\varphi)(x) = \int_{\mathcal{C}} d\xi\, \mathrm{e}^{-\xi/x}\, \mathcal{B}[\varphi](\xi), \tag{1.12}$$

where $\mathcal{C}$ is a contour in the Borel plane around the singularities. We can see that the integral on the right-hand-side depends on the possible singularities of the Borel transform along the θ-direction. This quantity is actually nonperturbative in nature. Let us consider the simple example in which $\mathcal{B}[\varphi](\xi)$ only has one pole. The location of the singularity can be determined by the growth of the asymptotic series coefficients, say $a_g \sim c\,(g-1)!\, A^{-(g-1)}$ as $g\to\infty$. To see this we can perform the rough calculation

$$\mathcal{B}[\varphi](\xi) = \sum_{g=1}^{\infty} \frac{a_g}{(g-1)!}\xi^{g-1}$$

$$\simeq (\text{finite piece}) + \sum_{g\gg 1} c\,\frac{(g-1)!}{(g-1)!}\left(\frac{\xi}{A}\right)^{g-1}$$
$$\simeq (\text{finite piece}) + \frac{c}{1-\xi/A}, \tag{1.13}$$

where we have separated the first terms of the series into a finite piece that is not important. The pole is located at $\xi = A$. The contour $\mathcal{C}$ that comes from infinite at angle θ_-, goes around the singularity at $A = |A|\,\mathrm{e}^{i\theta}$, and leaves off to infinity at angle θ_+, can be deformed as to only enclose the pole, because in this case there are no other poles or branch cuts. The residue theorem tells us that, for this very simple example,

$$(\mathcal{S}_{\theta_+}\varphi)(x) - (\mathcal{S}_{\theta_-}\varphi)(x) = -2\pi i\, c\,\mathrm{e}^{-A/x}. \tag{1.14}$$

The connection between the singularities of the Borel plane and the large-order growth of the asymptotic coefficients is not accidental, and it can be studied systematically with the help of the alien calculus.

1.3 Resurgent transseries

1.3.1 Definition and notation

A transseries is a formal power expansion that goes beyond a series in x alone. In the simplest case, the expansion is both in x and $\mathrm{e}^{-A/x}$, for some constant $A \in \mathbb{C}$,

$$\varphi(x) = \sum_{n\in\mathbb{N}} \mathrm{e}^{-nA/x} \sum_{g=0}^{\infty} a_g^{(n)}\, x^g. \tag{1.15}$$

This is, in principle, a formal object, but an ultimate goal is to make sense of the right-hand-side of (1.15) as a proper function of x after some resummation procedure is applied. We call n the instanton number even if the problem does not admit such a physical interpretation. We will speak of the n-instanton sector to refer to quantities associated to this number. An example is the series

$$\sum_{g=0}^{\infty} a_g^{(n)}\, x^g \tag{1.16}$$

which is expected to be asymptotic. Also, we will refer to A as the instanton action regardless of its actual origin. The customary choice of the letter A is not a coincidence in the light of the previous section; the instanton action is involved in the large-order growth of the coefficients $a_g^{(0)}$. This is the first and most simple sign of resurgence.

It should be clear from the form of the transseries that it combines perturbative and nonperturbative information in x. In this way, the transseries proves to be the formal object that, after proper resummation, should become the full nonperturbative solution to the physical problem under study.

Let us spend some time generalizing the transseries in (1.15) and fixing notation that will be used extensively later. It is customary and useful to introduce a transseries parameter σ, which keeps track of the instanton sector, n. Also, we should allow each asymptotic series to start at a generic power of x, $b^{(n)}$. Thus,

$$\varphi(\sigma, x) = \sum_{n \in \mathbb{N}} \sigma^n \, \mathrm{e}^{-nA/x} \sum_{g=0}^{\infty} a_g^{(n)} \, x^{g+b^{(n)}}. \tag{1.17}$$

The presence of σ is natural in the context of ordinary differential equations. There, it has the interpretation of an integration constant which must eventually be fixed by a boundary condition, for example. A natural and common generalization of (1.17) involves allowing for several instanton actions collected into a vector $\boldsymbol{A} = (A_1, A_2, \ldots, A_p)$. The new transseries will be

$$\varphi(\boldsymbol{\sigma}, x) = \sum_{\boldsymbol{n} \in \mathbb{N}^p} \boldsymbol{\sigma}^{\boldsymbol{n}} \, \mathrm{e}^{-\boldsymbol{n} \cdot \boldsymbol{A}/x} \sum_{g=0}^{\infty} a_g^{(\boldsymbol{n})} \, x^{g+b^{(\boldsymbol{n})}}, \tag{1.18}$$

where $\boldsymbol{\sigma}^{\boldsymbol{n}} = \prod_{\alpha=1}^{p} \sigma_\alpha^{n_\alpha}$, $\boldsymbol{n} \cdot \boldsymbol{A} = \sum_{\alpha=1}^{p} n_\alpha \, A_\alpha$. (1.18) is called a multiparameter transseries, where p is the number of parameters. $\boldsymbol{n} \cdot \boldsymbol{A}$, also denoted by $\boldsymbol{A}^{(\boldsymbol{n})}$, is called the total instanton action for the n-instanton sector. The next step in the generalization of the transseries ansatz requires the introduction of another monomial: $\log x$. The presence of logarithms is not uncommon at all and has to be kept in mind when considering a transseries ansatz. In the context of differential equations it appears naturally to lift the degeneracy of solutions when some eigenvalues are equal. We will not consider these cases in as much detail as the other type of transseries, but we can write down the expression

$$\varphi(\boldsymbol{\sigma}, x) = \sum_{\boldsymbol{n} \in \mathbb{N}^p} \boldsymbol{\sigma}^{\boldsymbol{n}} \, \mathrm{e}^{-\boldsymbol{n} \cdot \boldsymbol{A}/x} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k(x) \, \varphi^{(\boldsymbol{n})[k]}(x), \tag{1.19}$$

where

$$\varphi^{(\boldsymbol{n})[k]}(x) = \sum_{g=0}^{\infty} a_g^{(\boldsymbol{n})[k]} \, x^{g+b^{(\boldsymbol{n})[k]}}. \tag{1.20}$$

If there are no logarithms present, the relevant index, k, drops and we just have $\varphi^{(\boldsymbol{n})}(x)$. We should also say that more general transseries may include other nonanalytic functions at $x = 0$, besides $\mathrm{e}^{-1/x}$ or $\log x$, see for example [60]. For each particular problem the resurgent structure should tell us what the general form of the transseries is.

1.3.2 Physical transseries

If φ is to represent a physical observable that includes nonperturbative corrections, we must deal with two questions. One is the resummation of the transseries. The other is the selection of the transseries sectors that will enter the resummation. It is clear that if the instanton sectors of the transseries, with $\boldsymbol{n} \neq \boldsymbol{0}$, have to be exponentially suppressed with respect to the perturbative sector, $\boldsymbol{n} = \boldsymbol{0}$, one must have

$$\mathrm{Re}(A_\alpha / x) > 0. \tag{1.21}$$

We have not imposed any restrictions on the complex numbers A_α, but if we take $x \in \mathbb{R}^+$ (*e.g.*, x is a physical coupling constant) then only sectors with $\mathrm{Re}(A_\alpha) > 0$ must become part of the transseries to be resummed. All the others must have $\sigma_\beta = 0$. This does not mean that the sectors set to zero are not important or relevant—they are very much relevant. On the one hand, they are involved in the large-order growth of the perturbative coefficients $a_g^{(\mathbf{0})}$ and of other higher instanton coefficients. On the other hand, the vanishing of σ_β for these sectors is subordinated to a particular region in the complex x-plane. Moving to different regions involves crossing the so-called Stokes lines, and such Stokes transitions can turn on transseries parameters that were zero before. This effect is called Stokes phenomenon, see for example [85, 86].

The application of the Borel resummation procedure to a full transseries goes by the name of Borel-Écalle resummation. One of the main results of Écalle's work is the proof that Borel-Écalle resummation exists uniquely whenever the singularities of the Borel transforms of the different sectors have isolated singularities (technically one requires that they can be analytically continued along any path in the Borel plane that avoids the singularities). A very important property of Borel-Écalle resummation is the resolution of the nonperturbative ambiguity that is present in Borel resummations along singular directions. In section 1.2 we saw that we must define lateral resummations, either above or below the singular direction. This ambiguity can render perturbation theory inconsistent and ill-defined. Borel resummation of higher instanton series $\varphi^{(\boldsymbol{n})}$ also produces analogous ambiguities. The remarkable result is that, when considered all together, as part of the transseries, the ambiguities cancel each other out, leaving a well-defined object. This cancellation was observed to leading order in quantum mechanics in [50, 51] and has the name of Bogomolny–Zinn-Justin mechanism. Recently this mechanism was shown to work in particular examples of quantum field theories in four and two dimension [52, 53]. A generic study of the cancellation of the nonperturbative ambiguity and reality conditions was carried out in [42]. This procedure also goes by the name of median resummation.

1.4 Alien calculus

1.4.1 Definition and properties

In section 1.2 we saw that the difference between lateral Borel resummations produces a nonperturbative result that depends on the singularities of the Borel transform. The systematic study of these singularities, for the different sectors in the transseries, leads to a complicated algebraic structure that goes by the name of alien calculus. In Écalle's work it was shown that, besides the usual derivations with respect to x, there exist a number of derivatives, labelled by points ω in the Borel plane, which are sensitive to the singularities of the Borel transform. These are the alien derivatives, Δ_ω. In part of this section it will be convenient to work with the inverse variable $z = 1/x$, to keep with standard notation.

The definition of the alien derivative is based on the concept of resurgent function that we now, finally, explain. Given a Gevrey 1 series, φ, we say that it is resurgent if its Borel transform has analytic continuations along any path avoiding the discrete set of singularities of $\mathcal{B}[\varphi](\xi)$. This is a very wide definition. Usually, and because this is the case in many examples, one restricts to the set of resurgent functions whose Borel transform has only simple poles and logarithmic branch cuts at each singularity. More precisely, if $\Omega = \{\omega \in \mathbb{C} \,|\, \omega \text{ singularity of } \mathcal{B}[\varphi](\xi)\}$ is the set of singular points of $\mathcal{B}[\varphi]$, we say that φ is a simple resurgent function if, for every $\omega \in \Omega$,

$$\mathcal{B}[\varphi](\xi) = \frac{a_\omega}{2\pi i(\xi-\omega)} + \mathcal{B}[\psi_\omega](\xi-\omega)\,\frac{\log(\xi-\omega)}{2\pi i} + \text{hol. func.} \tag{1.22}$$

where $a_\omega \in \mathbb{C}$ and $\mathcal{B}[\psi_\omega]$ is holomorphic near the singularity ω.[1] Simple resurgent functions form an algebraically closed set under multiplication. Note that (1.22) is a generalization of the very simple case that we considered in (1.13). When faced with particular problems, which admit a transseries solution, one finds that the function ψ_ω is actually one of the higher instanton sectors, $\varphi^{(\boldsymbol{n})}$, or some combination of them. In this sense, the function φ resurges in different guises when looking near the singularities of the Borel transforms. We will focus on this topic later on.

Now we are in position to define the alien derivative acting on a simple resurgent function. If $\omega \notin \Omega$ then $\Delta_\omega \varphi := 0$. If $\omega \in \Omega$ then

$$(\Delta_\omega \varphi)(z) := a_\omega + \psi_\omega(z). \tag{1.23}$$

Note that the alien derivative takes an asymptotic (formal) series and gives back another formal series.[2] It can be shown that Δ_ω is a derivation, that is, it is linear and it satisfies Leibniz rule for differentiation. A very important property of the alien derivative lies in its commutation relation with the usual derivative

$$[\partial_z, \Delta_\omega] = \omega\, \Delta_\omega. \tag{1.24}$$

Since the computation of alien derivatives is not usually an easy task, (1.24) can help translate alien operations into regular ones.

1.4.2 Stokes automorphism and bridge equation

The principal role of the alien derivative is to provide a systematic way of computing the difference between lateral resummations

$$(\mathcal{S}_{\theta_+}\varphi)(z) - (\mathcal{S}_{\theta_-}\varphi)(z). \tag{1.25}$$

We saw earlier that this difference is determined precisely by the singularity behavior of the Borel transform, captured in (1.23). Let us define the Stokes automorphism along the direction θ, $\underline{\mathfrak{S}}_\theta$, by the relation

$$\mathcal{S}_{\theta_+} := \mathcal{S}_{\theta_-} \circ \underline{\mathfrak{S}}_\theta. \tag{1.26}$$

[1] Strictly speaking, on the left-hand-side of (1.22) we must work with a particular analytic continuation of the Borel transform along a particular path that avoids the singularities up until ω.

[2] Technically, the definition of $\Delta_\omega \varphi$ involves a weighted sum over inequivalent continued paths from the origin to ω. We can think of (1.23) as a schematic version of the actual definition.

This is an automorphism from the space of simple resurgent functions into itself. Note that

$$\mathcal{S}_{\theta_+} - \mathcal{S}_{\theta_-} = -\mathcal{S}_{\theta_-} \circ \mathrm{Disc}_\theta, \tag{1.27}$$

where $\underline{\mathfrak{S}}_\theta = 1 - \mathrm{Disc}_\theta$. This discontinuity operator and the Stokes automorphism are functions of the alien derivatives. In general, the relation between derivations and automorphisms is given by exponentiation. A familiar example is the translation automorphism on a function, $f(z) \to f(z+1)$, and the usual derivative ∂_z. The precise relation is

$$\underline{\mathfrak{S}}_\theta = \exp\left(\sum_{\{\omega_\theta\}} \mathrm{e}^{-\omega_\theta\, z}\, \Delta_{\omega_\theta}\right), \tag{1.28}$$

where $\{\omega_\theta\}$ is the set of singularities in Ω that lie along the θ-direction. A justification for this formula requires a careful examination of the contours involved in (1.25) and the definition of the alien derivative (for a rigorous derivation see [79]). Usually poles along a given direction θ can be written as multiples of an instanton action A with $\arg(A) = \theta$, that is, $\omega = \ell A$, with $\ell \geq 1$. In this case,

$$\underline{\mathfrak{S}}_\theta = 1 + \mathrm{e}^{-Az}\Delta_A + \mathrm{e}^{-2Az}\left(\Delta_{2A} + \frac{1}{2}\Delta_A^2\right) + \cdots \tag{1.29}$$

See [40] for general formulae describing the action of the Stokes automorphism on one and two-parameter transseries. Thus, we see explicitly that knowledge about the alien derivatives on the asymptotic series at each relevant singularity will give us a precise form of the discontinuity of lateral resummations. However, as we mentioned before, the computation of the alien derivatives may be difficult, due to lack of knowledge of the Borel transform. In some situations a huge step forward is provided by the so-called bridge equation. The first step to obtain the bridge equation is to define the pointed alien derivative

$$\dot{\Delta}_\omega = \mathrm{e}^{-\omega\, z}\, \Delta_\omega, \tag{1.30}$$

appearing in (1.28). We can easily see from (1.24) that $\dot{\Delta}_\omega$ commutes with ∂_z. The derivative with respect to the transseries parameter, σ, also trivially commutes with $\dot{\Delta}_\omega$, because the latter does not act on σ. Consider now the situation in which the transseries of interest, φ, satisfies an ordinary differential equation in z, that could be nonlinear. If we act on this equation with either $\dot{\Delta}_\omega$ or ∂_σ we will arrive at a *linear* differential equation for $\dot{\Delta}_\omega\varphi$ or $\partial_\sigma\varphi$, respectively, due to the commutativity with ∂_z. The equation will be the same in both cases. Note that φ itself may be part of this equation, but our focus is on $\dot{\Delta}_\omega\varphi$ and $\partial_\sigma\varphi$, and for them the equation is indeed linear. For the sake of simplicity, we are considering a first order differential equation and hence a one-parameter transseries. This means that any solutions of the linearized equation should be proportional to each other. This means

$$\dot{\Delta}_\omega\varphi(\sigma, z) = S_\omega(\sigma)\, \partial_\sigma\varphi(\sigma, z). \tag{1.31}$$

Here, $S_\omega(\sigma)$ is the proportionality factor, a function of the transseries parameter, independent of z and regular at $\sigma = 0$. (1.31) is an example of bridge equation between alien derivations on the left-hand-side and usual ones on the right. If we start with a higher order differential equation, admitting a multiparameter transseries, the right-hand-side of (1.31) will change to a linear combination with partial derivatives with respect to all the σ_α. If we now plug the transseries ansatz for φ and expand $S_\omega(\sigma)$ in series, we can collect similar powers of σ. This will lead us to explicit expressions of the alien derivatives, $\Delta_\omega \varphi^{(n)}$, in terms of other sectors $\varphi^{(m)}$ of the transseries. The only unknown in this procedure are the coefficients of the series expansion of $S_\omega(\sigma)$, called the Stokes constants. See [40] for explicit formulae and examples.

1.4.3 Dispersion relation

The main use we will make of the Stokes automorphism is the derivation of the large-order growth of the coefficients of various sectors of the transseries, $a_g^{(n)}$, starting with the perturbative ones. We go back to the use of x for the remainder of this section. Consider a function $\varphi(x)$ and write its value as a contour integral using Cauchy's theorem,

$$\varphi(x) = \frac{1}{2\pi i} \oint_{(x)} dy \, \frac{\varphi(y)}{y - x}, \tag{1.32}$$

where (x) denotes a small contour around x. Let us deform this contour, expanding it out to infinity carefully avoiding possible singularities and branch cuts of $\varphi(x)$, until we obtain two contributions. The first is the integral around infinity, which in many cases can be shown to vanish, so we forget about it for simplicity. The other is given by a sum of discontinuities along certain singular rays. We can therefore write

$$\varphi(x) = \sum_{\{\theta\}} \frac{1}{2\pi i} \int_0^{e^{i\theta}\infty} dy \, \frac{\mathrm{Disc}_\theta \varphi(y)}{y - x} \tag{1.33}$$

for some set of singular directions $\{\theta\}$. The discontinuity operator is given in terms of the Stokes automorphism. If we know the alien derivatives we can calculate the right-hand-side of (1.33), and this will give us an explicit expression for the large-order growth of the coefficients of the asymptotic expansion of $\varphi(x)$. For illustration purposes, let us consider a one-parameter transseries with instanton action A, and assume that the bridge equation holds. Then we obtain the set of resurgent equations

$$\Delta_{\ell A} \varphi^{(n)} = S_\ell \, (n + \ell) \, \varphi^{(n+\ell)}, \tag{1.34}$$

where the Stokes function ends up having the form $S_{\ell A}(\sigma) = S_\ell \, \sigma^{1-\ell}$, with S_ℓ the Stokes constant. Since the Stokes function is analytic at $\sigma = 0$ we must have $\ell \leq 1$. All the other alien derivatives are zero. The origin, $\ell = 0$, is trivial as well, so the relevant singular set $\Omega = \{\ell A \mid \ell = 1, -1, -2, \ldots\}$. Let us focus on the perturbative sector, $n = 0$. The Stokes automorphism, (1.28), is only nontrivial for $\theta = 0$, and because in that direction $\Delta_{\ell A} \neq 0$ only for $\ell = 1$, we have

$$\underline{\mathfrak{S}}_{\theta=0} \varphi^{(0)} = \sum_{k=0}^{\infty} \frac{1}{k!} e^{-kAz} \Delta_A^k \varphi^{(0)}$$

$$= \sum_{k=0}^{\infty} S_1^k \mathrm{e}^{-kAz} \varphi^{(k)}. \tag{1.35}$$

while $\underline{\mathfrak{S}}_{\theta \neq 0} \varphi^{(0)} = \varphi^{(0)}$. From this, the discontinuity operator reads

$$\mathrm{Disc}_0 \varphi^{(0)} = -\sum_{\ell=1}^{\infty} (S_1)^\ell \, \mathrm{e}^{-\ell A/x} \, \varphi^{(\ell)}, \tag{1.36}$$

$$\mathrm{Disc}_\pi \varphi^{(0)} = 0. \tag{1.37}$$

See [40] for more details and general cases. Because the direction other than $\theta = 0$ is trivial in this example, we only have to worry about integration along the positive real line. We look at (1.33) for $\varphi = \varphi^{(0)}$, and expand everything as a formal series, to find

$$\begin{aligned} \sum_{g=0} a_g^{(0)} \, x^{g+b^{(0)}} &= -\frac{1}{2\pi i} \int_0^\infty dy \sum_{\ell=1}^{\infty} \sum_{h=0}^{\infty} (S_1)^\ell \frac{\mathrm{e}^{-\ell A/y} \, y^{h+b^{(\ell)}}}{y-x} \, a_h^{(\ell)} \\ &= -\sum_{\ell=1}^{\infty} \frac{(S_1)^\ell}{2\pi i} \sum_{h=0}^{\infty} \sum_{g=0}^{\infty} x^g \, a_h^{(\ell)} \int_0^\infty dy \, \mathrm{e}^{-\ell A/y} \, y^{h+b^{(\ell)}-g-1} \\ &= \sum_{g=0}^{\infty} x^g \sum_{\ell=1}^{\infty} \frac{(S_1)^\ell}{2\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(g-b^{(\ell)}-h)}{(\ell A)^{g-b^{(\ell)}-h}} \, a_h^{(\ell)}, \end{aligned} \tag{1.38}$$

where we have transformed the integral into a Gamma function. After a shift in the index g we find

$$a_g^{(0)} \sim \sum_{\ell=1}^{\infty} \frac{(S_1)^\ell}{2\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(g+b^{(0)}-b^{(\ell)}-h)}{(\ell A)^{g+b^{(0)}-b^{(\ell)}-h}} \, a_h^{(\ell)}. \tag{1.39}$$

This expression determines explicitly how the coefficients $a_g^{(0)}$ grow with the order g. We will examine this relation and some generalizations in the next section.

1.5 Large order relations

1.5.1 Structure and one-instanton contribution

We finished last section with the simplest example of a resurgence relation. Equation (1.39) shows how the growth of the perturbative coefficients $a_g^{(0)}$ is determined by all the other higher instanton coefficients, $a_h^{(l)}$. By looking at other sectors one can find analogous resurgence relations. This network shows how the information captured in one sector is also shared by all the others. More importantly, the information is spread in many layers, in such a way that complete knowledge of the perturbative coefficients can determine all the other higher instanton coefficients. This is a critical point for this thesis because our starting point in topological string theory is a perturbative computation and we need to calculate nonperturbative data. Knowledge of the perturbative sector to all orders is not always possible so one has to rely on numerical methods to make up for this. We will discuss this issues in section 1.5.2.

Let us explore the simplest resurgence relation (1.39). We can expand it out as

$$a_g^{(0)} \sim \frac{\Gamma(g+b^{(0,1)})}{A^{g+b^{(0,1)}}} \frac{S_1}{2\pi i} \left(a_0^{(1)} + \frac{\Gamma(g+b^{(0,1)}-1)}{\Gamma(g+b^{(0,1)})} A\, a_1^{(1)} + \frac{\Gamma(g+b^{(0,1)}-2)}{\Gamma(g+b^{(0,1)})} A^2\, a_2^{(1)} + \dots \right)$$
$$+ \frac{\Gamma(g+b^{(0,2)})}{(2A)^{g+b^{(0,2)}}} \frac{(S_1)^2}{2\pi i} \left(a_0^{(2)} + \frac{\Gamma(g+b^{(0,2)}-1)}{\Gamma(g+b^{(0,2)})} 2A\, a_1^{(2)} + \frac{\Gamma(g+b^{(0,2)}-2)}{\Gamma(g+b^{(0,2)})} (2A)^2\, a_2^{(2)} + \dots \right)$$
$$+ \cdots \tag{1.40}$$

where $b^{(0,\ell)} := b^{(0)} - b^{(\ell)}$. Assuming that $b^{(1)} \leq b^{(2)} \leq \dots$ we can see that the second line in (1.40) is subleading with respect to the first line because it is exponentially suppressed by a factor 2^{-g}. In the same way, the contribution of the ℓ-th instanton sector is suppressed by a factor ℓ^{-g}. Therefore, the leading and most important contribution to the growth of the perturbative coefficients is the one-instanton sector. This fact is very explicitly understood in the context of saddle point methods for integral expressions. Let us focus on the first line of (1.40). Besides the factorial growth and the presence of the instanton action, confirming the Gevrey 1 behavior, we have the one-loop one-instanton action coefficient $a_0^{(1)}$. The ratio of Gamma functions in the first line simplifies to the product

$$\frac{\Gamma(g+b^{(0)}-b^{(1)}-1)}{\Gamma(g+b^{(0)}-b^{(1)})} = \prod_{k=1}^{h} \frac{1}{g+b^{(0)}-b^{(k)}-k} = \mathcal{O}(g^{-h}), \tag{1.41}$$

so the effect of the coefficient $a_h^{(1)}$ is suppressed by a factor g^{-h}, as $g \to \infty$. An analogous description can be made for the other instanton sectors in (1.40). In summary, the coefficient of the transseries $a_h^{(\ell)}$ contributes to the large-order of $a_g^{(0)}$ at order $\ell^{-g}\, g^{-h}$.

One-instanton contribution

The fact that the different contributions are hierarchical is very useful because, when taking the reverse point of view, we can think of extracting the coefficients $a_h^{(\ell)}$ out of the set $\{a_g^{(0)}\}$ by looking at the contributions at the appropriate order. To exemplify this let us take one step back and extract first the instanton action out of the perturbative coefficients. If we consider the combination

$$g\, \frac{a_g^{(0)}}{a_{g+1}^{(0)}} \tag{1.42}$$

and expand around $g = \infty$ we can see that, at leading order, it is given by

$$A\left(1 + \mathcal{O}(g^{-1}) + \mathcal{O}(2^{-g})\right). \tag{1.43}$$

This is simply a consequence of the Gevrey 1 condition. We can therefore write

$$A = \lim_{g\to\infty} g\, \frac{a_g^{(0)}}{a_{g+1}^{(0)}}. \tag{1.44}$$

This means that even if we knew nothing about the transseries we could still find out the instanton action. The next piece of information we can find out is the number $b^{(0)} - b^{(1)}$, with the help of the limit

$$-\left(b^{(0)} - b^{(1)}\right) = \lim_{g\to\infty} g \left(1 - \frac{A}{g} \frac{a^{(0)}_{g+1}}{a^{(0)}_g}\right). \tag{1.45}$$

Note that we need to know A in order to compute the right-hand-side. Now we are in position to calculate the one-instanton coefficients $a^{(1)}_h$. We start with the first one,

$$\frac{S_1}{2\pi i} a^{(1)}_0 = \lim_{g\to\infty} \frac{A^{g+b^{(0)}-b^{(1)}}}{\Gamma(g + b^{(0)} - b^{(1)})} a^{(0)}_g, \tag{1.46}$$

and then, recursively for $h \geq 1$,

$$\frac{S_1}{2\pi i} a^{(1)}_h = \lim_{g\to\infty} \frac{A^{g+b^{(0)}-b^{(1)}}}{\Gamma(g + b^{(0)} - b^{(1)})} \left(a^{(0)}_g - \sum_{k=0}^{h-1} \frac{\Gamma(g + b^{(0)} - b^{(1)} - k)}{A^{g+b^{(0)}-b^{(1)}-k}} \frac{S_1}{2\pi i} a^{(1)}_k\right). \tag{1.47}$$

Without further information we cannot disentangle the Stokes constant from the one-instanton coefficients. This is not necessarily a problem since at the end of the day the Stokes constant may be absorbed into the transseries parameter σ, which still remains to be fixed. In the solution of differential equations $a^{(1)}_0$ is the only coefficient that is not determined by the equations. The freedom to choose its value is transferred to σ and $a^{(1)}_0$ is set to 1 by convention. In this situation the Stokes constant S_1 can be calculated from (1.46). The limit (1.47) is becomes more subtle and fine as we h is higher because the information that is extracted lies deeper in the asymptotic expansion. In this sense the limit becomes increasingly difficult to do in practice.

Richardson extrapolation

In the common situation where data is scarce one cannot take limits all the way to infinite and we have to conform ourselves with numerical approximations. Let

$$q = \lim_{g\to\infty} Q_g \tag{1.48}$$

for some quantities Q_g, q. One can approximate q by $Q_{g_{\max}}$, where $g_{\max}$ is the largest value of g for which data is available. This approximation is almost never sufficiently good. A very efficient way to accelerate the convergence of (1.48) is to use the Richardson transform (or extrapolation) of the sequence Q_g [87]. To be able to apply a Richardson transform we must have

$$Q_g = q_0 + \frac{q_1}{g} + \frac{q_2}{g^2} + \dots \tag{1.49}$$

where we treat Q_g as a function of g. Note that q_0 is the value we want to approximate, that is, q. All the limits we discussed above adapt to this form. The Richardson transform modifies the sequence $\{Q_g\}$ into a new sequence $\{Q^{[1]}_g\}$ defined by

$$Q^{[1]}_g := (g + 1)\, Q_{g+1} - g\, Q_g. \tag{1.50}$$

The key point is that this combination produces a cancellation that removes the $\mathcal{O}(g^{-1})$-term,

$$Q_g^{[1]} = q_0 - \frac{q_2}{g^2} + \frac{q_2 - 2q_3}{g^3} + \dots \tag{1.51}$$

This accelerates the numerical convergence of the sequence. A generalization of (1.50) is given by

$$\begin{aligned} Q_g^{[n]} &:= \sum_{h=0}^{n} (-1)^{n-h} \frac{(g+h)^n}{h!(n-h)!} Q_{g+h} \\ &= q_0 + \frac{q_{n+1}}{g^{n+1}} + \mathcal{O}\left(\frac{1}{g^{n+2}}\right). \end{aligned} \tag{1.52}$$

We say that we have taken n Richardson transforms on the limit (1.48). Note that with each simple Richardson transform we make use of a number in the sequence whose information cannot be used in the next one. So there is a limit to the number of Richardson transforms that can be taken with a finite amount of data.

We will make extensive use of this technique, for different values of n, and it will allow us to compute limits with great precision.

1.5.2 Practical resummation and two-instanton contribution

The extraction of the two-instanton sector coefficients, $a_h^{(2)}$, is not as straightforward. We first need to do a resummation of the one-instanton contribution as a series in $\frac{1}{g}$. It is not a surprise that this series turns out to be asymptotic in most cases. We saw in section 1.2 that an approximate resummation can be done with the technique of optimal truncation. The error of this method is of order $\mathrm{e}^{-\tilde{A}/g}$, where $\tilde{A}$ is the instanton action controlling the large-order of the asymptotic series. Note that $\tilde{A}$ will not coincide with A because the series to resum is not $\varphi^{(1)}$ but one in which the coefficients $a_h^{(1)}$ and A are mixed together. Explicitly, we have to resum

$$I(g) := \sum_{h=0}^{\infty} \frac{\Gamma(g + b^{(0)} - b^{(1)} - h)}{\Gamma(g + b^{(0)} - b^{(1)})} A^h \, a_h^{(1)}. \tag{1.53}$$

The ratio of Gamma functions can be expanded in power series around $\frac{1}{g} = 0$,

$$\frac{\Gamma(g + b^{(0)} - b^{(1)} - h)}{\Gamma(g + b^{(0)} - b^{(1)})} = g^{-h} \left(1 + \mathcal{O}(g^{-1})\right), \tag{1.54}$$

from which we arrive at

$$I(g) = \sum_{h=0}^{\infty} \frac{\tilde{a}_h}{g^h}. \tag{1.55}$$

In order to see if optimal truncation is good enough we must make sure that the optimal truncation error is smaller than the order of magnitude of the two-instanton contribution to $a_g^{(0)}$, for a given value of g [88]. If this is not the case we must resort to more powerful resummation methods. The obvious one that we have discussed is Borel resummation.

This method presents, however, a practical problem which makes it unusable except in very favorable situations. In most cases we only have a finite number of coefficients to work with and the Borel transform cannot be computed. Optimal truncation does not care about this limitation but the Borel transform requires all the terms. This is so because the result must be a function with finite radius of convergence, that can later be extended analytically in some region of the Borel plane. If we truncate the Borel transform series, the resummation will give back the original result. A remedy for this uses Padé approximants for the truncated Borel transform [48]. The Padé approximant takes a polynomial of degree d and approximates it by a rational function whose series expansion matches the original polynomial up to order d. The Padé approximant gives a good approximation to the actual Borel transform because it incorporates some of its singularities. This procedure goes by the name of Borel-Padé resummation, and it is expected to give a better approximation that optimal truncation. We define

$$\mathrm{BP}[I](g) := g \int_0^\infty d\xi\, \mathrm{e}^{-\xi/g}\, \text{Padé}\left(\sum_{h=0}^{h_{\max}} \frac{\tilde{a}_h}{h!}\,\xi^h\right), \tag{1.56}$$

where $h_{\max}$ is the maximum index for which there is available data. Just as with Borel resummation we may find some singularities (poles) along the integration contour. Several options have been considered to tackle this issue, see for example [89–91]: deformation of the contour either above or below the axis, or principal value prescription. The latter has the advantage that no nonperturbative ambiguity is introduced, which may affect later a clean extraction of the coefficients $a_h^{(2)}$.

Once a sufficiently accurate resummation is performed we can consider the asymptotic series

$$X_g^{(1)} := \frac{A^{g+b^{(0)}-b^{(1)}}}{\Gamma(g+b^{(0)}-b^{(1)})}\, a_g^{(0)} - \frac{S_1}{2\pi i}\,\mathrm{Resum}\left[I(g)\right] \tag{1.57}$$

$$\sim \frac{A^{b^{(2)}-b^{(1)}}}{2^{g+b^{(0)}-b^{(2)}}}\,\frac{(S_1)^2}{2\pi i}\sum_{h=0}^{\infty}\frac{\Gamma(g+b^{(0)}-b^{(2)}-h)}{\Gamma(g+b^{(0)}-b^{(1)})}\,(2A)^h\, a_h^{(2)} + \ldots \tag{1.58}$$

By taking suitable ratios and combinations we can extract $b^{(0)}-b^{(2)}$ and the coefficients $a_h^{(2)}$. We could then go on to the third, fourth, etc instanton sectors, if enough data and precision is available.

1.5.3 Generalizations

Generalizations of (1.40) exist when the transseries has more parameters. There are also similar formulae describing the large-order of other sectors. For systems that satisfy a bridge equation like (1.31) one can be very systematic and derive general resurgence relations for any sector [40]. For other examples, we should expect some generalization of (1.40). In any case, the resurgence relations give us a list of relations, with a very combinatorial flavor, in which the transseries coefficients play specific roles. In physical examples that admit a large N description [7] we find that the perturbative sector forms an asymptotic series in x^2 rather than in x. One talks about a topological expansion because the asymptotic series can be interpreted as a sum over Riemann surfaces of all genera. The rest of the transseries sectors are still proper formal series in x. Along with this it has been found in several examples, all related to each other, that the transseries describing them have two parameters with opposite instanton actions, A and $-A$. For a two-parameter resonant transseries with instanton actions A and $-A$, the bridge equation (1.31) generalizes to

$$\dot{\Delta}_\omega \varphi(\boldsymbol{\sigma}, z) = S_\omega(\boldsymbol{\sigma})\, \partial_{\sigma_1} \varphi(\sigma, z) + \widetilde{S}_\omega(\boldsymbol{\sigma})\, \partial_{\sigma_2} \varphi(\boldsymbol{\sigma}, z), \tag{1.59}$$

where the singularities are restricted some integer multiples of A. The set of resurgent equations for the different sectors is more complicated than in the one-parameter case. For the perturbative sector we still have $\Delta_{\ell A}\varphi^{(0|0)} = 0$ for $\ell \geq 2$, but now $\Delta_{\ell A}\varphi^{(0|0)} = \widetilde{S}_\ell\, \varphi^{(\ell+1|1)} \neq 0$ for $\ell = -1$. This means that the Stokes automorphism along $\theta = \pi$ is not trivial anymore and it contributes to the perturbative large-order,

$$\begin{aligned} a_g^{(0|0)} \sim & \sum_{\ell=1}^{\infty} \frac{(S_1)^\ell}{2\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(g + b^{(0|0)} - b^{(\ell|0)} - h)}{(\ell A)^{g+b^{(0|0)}-b^{(\ell|0)}-h}}\, a_h^{(\ell|0)} \\ & + \sum_{\ell=1}^{\infty} \frac{(\widetilde{S}_{-1})^\ell}{2\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(g + b^{(0|0)} - b^{(0|\ell)} - h)}{(-\ell A)^{g+b^{(0|0)}-b^{(0|\ell)}-h}}\, a_h^{(0|\ell)}, \end{aligned} \tag{1.60}$$

where $\widetilde{S}_{-1}$ is another Stokes constant. Because the perturbative series is topological, every other coefficient on the left-hand-side must be zero, which means that there must be a cancellation to all instanton orders in the right-hand-side to make this happen. If $a_{\text{odd}}^{(0|0)} = 0$ we must have

$$b^{(\ell|0)} = b^{(0|\ell)} \quad \text{and} \quad (S_1)^\ell\, a_h^{(\ell|0)} = (-1)^{-b^{(0)}+b^{(0|\ell)}+h}\, (\widetilde{S}_{-1})^\ell\, a_h^{(0|\ell)}, \tag{1.61}$$

for all values of ℓ and h. This symmetry is verified by direct computation in the examples mentioned above and will also be true for topological string theory, see section 3.5.2. If we use the relations (1.61) back into (1.60) we find

$$a_{2g}^{(0|0)} \sim \sum_{\ell=1}^{\infty} \frac{(S_1)^\ell}{\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(2g + b^{(0|0)} - b^{(\ell|0)} - h)}{(\ell A)^{2g+b^{(0|0)}-b^{(\ell|0)}-h}}\, a_h^{(\ell|0)}. \tag{1.62}$$

Because the odd coefficients are all zero it is customary to do a change of notation

$$F_g^{(0|0)} := a_{2g}^{(0|0)} \quad \text{and} \quad F_h^{(\ell|0)} := a_h^{(\ell|0)}, \tag{1.63}$$

where we have already adopted the letter F, for free energy, that will be used in the rest of the thesis.

The large-order growth of the one-instanton coefficients, $F_g^{(1|0)}$, shows resonance explicitly. From the bridge equation (1.59) one can see after a small calculation that the nontrivial alien derivatives on this sector are $\Delta_A\varphi^{(1|0)} = 2\, S_1\, \varphi^{(2|0)}$ and $\Delta_{\ell A}\varphi^{(1|0)} = \tilde{S}_\ell\, \varphi^{(\ell+2|1)}$ for $\ell \leq -1$. The growth is controlled to leading order by the singularities that are closest to the origin in the Borel plane, A and $-A$. Using the dispersion relation (1.33) one finds,

$$F_g^{(1|0)} \sim \frac{\Gamma(g + b^{(1|0)} - b^{(2|0)})}{A^{g+b^{(1|0)}-b^{(2|0)}}} \frac{S_1}{\pi i} F_0^{(2|0)} + \frac{\Gamma(g + b^{(1|0)} - b^{(1|1)})}{(-A)^{g+b^{(1|0)}-b^{(1|1)}}} \frac{1}{2} \frac{\widetilde{S}_{-1}}{\pi i} F_0^{(1|1)} \tag{1.64}$$

Note the explicit presence of $-A$ and the mixed sector $F_0^{(1|1)}$, which was absent in the large-order of $F_g^{(0|0)}$.

Some final remarks to conclude. In some situations, the presence of two instanton sectors of the same absolute value, hence contributing to the same order, can produce a growth of the form

$$F_g^{(0|0)} \sim \frac{\Gamma(2g+b)}{|A|^{2g+b}} \cos(\theta\,(2g+b)+\psi)\,|\mu|\,, \tag{1.65}$$

where $A = |A|\, \mathrm{e}^{i\theta}$ and ψ is a phase coming from the one-instanton sector represented by μ. The oscillations produced by the trigonometric function prevent us from using the Richardson transform in order to extract A, b, and μ. No approach around this obstacle has been found yet, to my knowledge. In multiparameter transseries with different instanton actions A_1, A_2, $\dots$, A_p it is crucial to determine the ordering of the quantities $|\ell A_i|$ from smallest to largest for $i = 1, 2, \dots, p$ and $\ell = 1, 2, \dots$ The smallest this quantity is, the largest contribution the corresponding sector will have in the resurgence relation, due to the dependence $(\ell A_i)^{-g}$. If the problem at hand has moduli or external parameters, as is the case for topological strings, the ordering may change as we move through the parameter space.

1.5.4 An example: Riccati equation

We can illustrate the concept of a transseries and make use of the large-order relations in a particularly simple example, the Riccati equation

$$\varphi'(z) = \varphi(z) - \frac{1}{z}\varphi(z)^2 - \frac{1}{z^2}. \tag{1.66}$$

This is a first order nonlinear equation, so we expect a formal solution around $z = \infty$ (or $x = z^{-1} = 0$) in the form of a one-parameter transseries,

$$\varphi(z) = \sum_{n=0}^{\infty} \sigma^n \mathrm{e}^{-nAz} \sum_{g=0}^{\infty} a_g^{(n)} z^{-g-b^{(n)}}. \tag{1.67}$$

Since we are dealing with a differential equation the argument leading to the bridge equation (1.31) applies, and so do the resurgence relations (1.40). Plugging the transseries ansatz into (1.66) we first find the value of the instanton action $A = -1$, and

$$\begin{aligned}\varphi(z) = {} & z^{-1}\left(1 - \frac{1}{z} + \frac{3}{z^2} - \frac{11}{z^3} + \frac{51}{z^4} - \frac{283}{z^5} + \frac{1831}{z^6} + \dots\right) \\ & + \sigma \mathrm{e}^{z}\left(1 + \frac{2}{z} + \frac{1}{z^2} + \frac{4}{3z^3} - \frac{7}{3z^4} + \frac{34}{3z^5} - \frac{523}{9z^6} + \dots\right) \\ & + \sigma^2 \mathrm{e}^{2z} z^{-1}\left(-1 - \frac{5}{z} - \frac{14}{z^2} - \frac{122}{3z^3} - \frac{421}{3z^4} - \frac{1969}{3z^5} + \dots\right) \\ & + \dots \end{aligned} \tag{1.68}$$

The starting powers are $b^{(0)} = 1$, $b^{(n)} = n-1$ for $n \geq 1$. The first coefficient of the one-instanton sector, $a_0^{(1)}$, is not determined by the equations and is set to one by convention. The integration constant is σ, the transseries parameter, that will have to be fixed by boundary conditions as $|z| \to \infty$ in some direction. For example, we must impose $\sigma = 0$ if we require a nontrivial as $z \to +\infty$; but $\sigma \neq 0$ if we need a solution as $z \to -\infty$. The coefficients of the perturbative sector alternate signs, which indicates Borel summability along the positive x-axis because the pole lies on the negative side. Indeed,

$$a_g^{(0)} \sim \frac{\Gamma(g+1)}{A^{g+1}} \frac{S_1}{2\pi i} a_0^{(1)} = \Gamma(g+1)(-1)^{g+1} \frac{S_1}{2\pi i}. \tag{1.69}$$

The instanton action can be extracted from the limit (1.44) and one finds a numerical value compatible $A = -1$, see first plot in figure 1.1. To be more precise we can show the difference between the exact value of -1 and various Richardson transforms performed on the limit (1.44),

# RT	EXACT − LIMIT
0	$-1.7 \cdot 10^{-2}$
1	$-2.8 \cdot 10^{-4}$
2	$+4.6 \cdot 10^{-6}$
3	$-2.8 \cdot 10^{-7}$
8	$-7.6 \cdot 10^{-10}$

Here the maximum value of g is 60. After around ten Richardson transforms there is no further improvement.

Equation (1.45) shows that $b^{(0)} - b^{(1)} = 1$, as we see from the comparison with the general case (1.40). The next term we can look at is (1.46), for which $a_0^{(1)} = 1$, so we are actually calculating the Stokes constant. One finds, see figure 1.1,

$$\frac{S_1}{2\pi i} = -3.676\,077\,910\,37\ldots = -\frac{\sinh(\pi)}{\pi}. \tag{1.70}$$

The exact value can be calculated by other, analytical, means. See [81, 92] for a thorough resurgent analysis of this and more general Riccati equations. Once the Stokes constant is identified one can check the rest of the one-instanton coefficients, $a_h^{(1)}$, $h = 1, 2, \ldots$, using equation (1.47). Each limit is accelerated with several Richardson transforms. In this simple example it is very easy to compute many perturbative coefficients so one can go very far in the asymptotics. See figure 1.1 for examples of these limits. Plots like these will appear profusely in chapter 4.

To check the two-instanton coefficients of the transseries out of large-order we need to use the resummation procedure explained in section 1.5.2, and the quantities $X_g^{(1)}$ in equation (1.57). To know whether we can use optimal truncation for the resummation procedure or we have to do Borel-Padé resummation we do the following comparison. Let

$$\text{EXACT} = \frac{A^{g+1}}{\Gamma(g+1)} a_g^{(0)}, \tag{1.71}$$

for a particular value of g, say $g = 40$. Denote by OT the optimal truncation of the one-instanton contribution, that is the second term in (1.57). We find

$$|\text{EXACT} - \text{OT}| \simeq 2 \cdot 10^{-13}, \quad \text{OT ERROR} \simeq 6 \cdot 10^{-14} \tag{1.72}$$

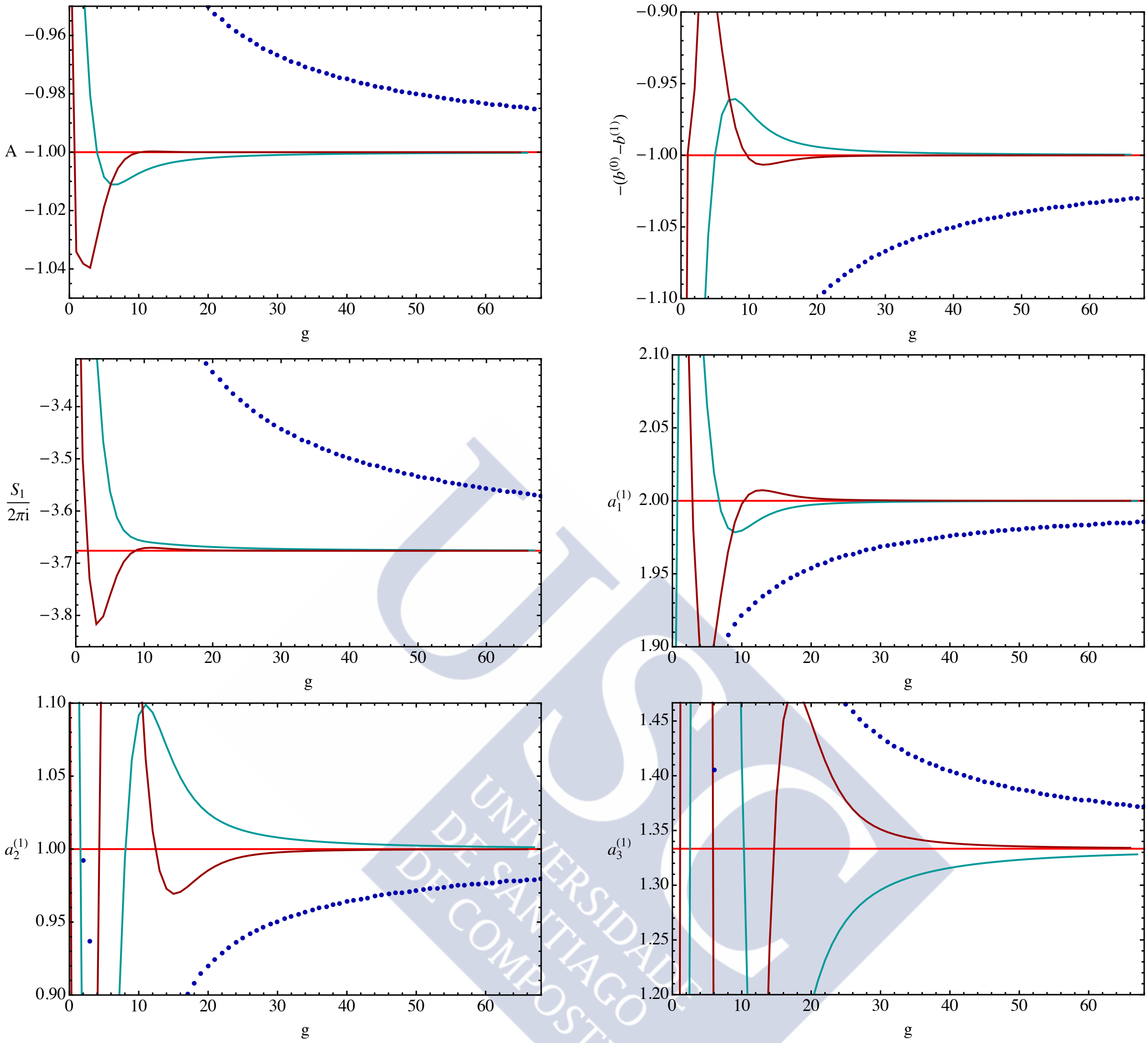


Figure 1.1: In these plots we compare the numerical large-order analysis of the perturbative coefficients of the transseries solution to the Riccati equation against resurgent predictions. We present the sequence of points that converge towards the corresponding quantity, for example $A = -1$ in the first plot coming from the limit (1.44). The first two Richardson transforms are displayed as continuous lines, as well as the predicted limit, in red.

And we need to compare this error against the order of magnitude of the leading contribution to $X^{(1)}_{g=58}$ in (1.58), which is

$$\left| \frac{A}{2^g} \frac{S_1^2}{2\pi i} \frac{1}{g} \right| \simeq 2 \cdot 10^{-12}. \tag{1.73}$$

It turns out that two orders of magnitude between the optimal truncation error and the two-instanton contribution is not big enough for optimal truncation to be sufficiently precise. We need to use the Borel-Padé resummation. The Padé approximants have poles along the

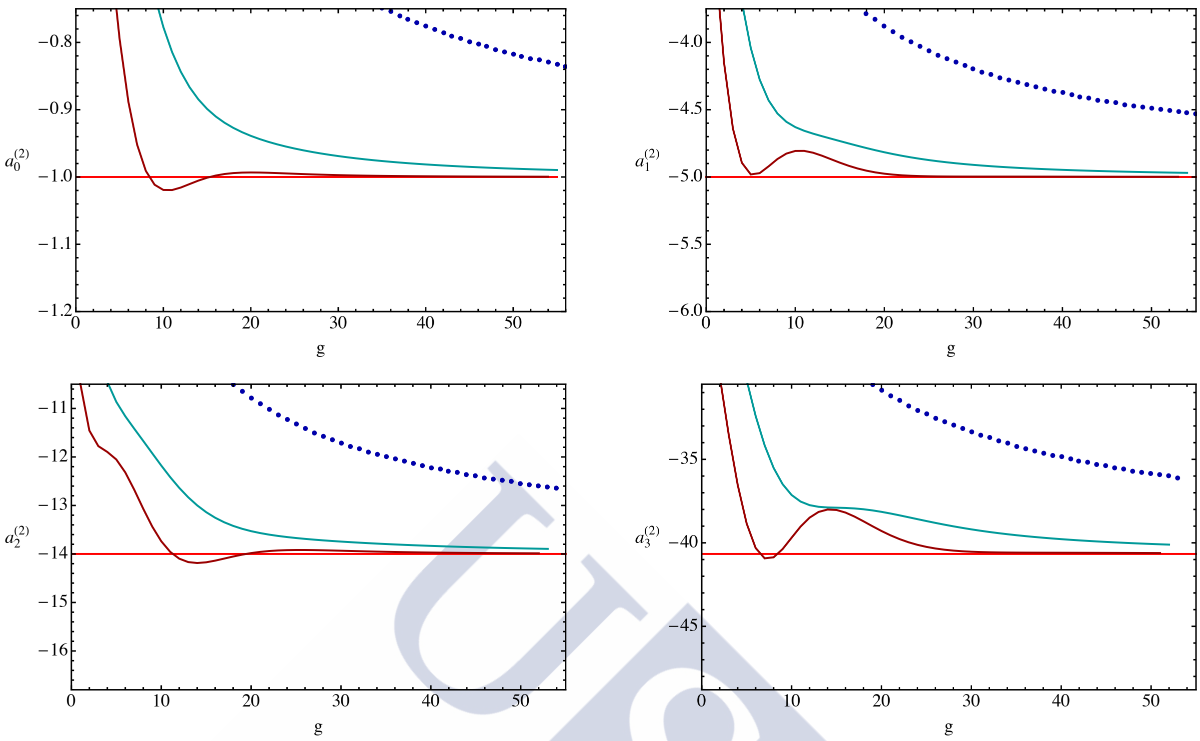


Figure 1.2: Checks for the two-instanton coefficients of the Riccati transseries solution, $a_0^{(2)} = -1$, $a_1^{(2)} = -5$, $a_2^{(2)} = -14$ and $a_3^{(2)} = -\frac{122}{3}$. They are obtained from the large-order growth of $X_g^{(1)}$, (1.58), where the resummation is done with Borel-Padé, (1.56).

positive line of integration so the contour has to be deformed to avoid them. This introduces a nonperturbative ambiguity of subleading order that will be cancelled by ambiguities at higher orders. In order to check the coefficients $a_h^{(2)}$ from the large-order of $X_g^{(1)}$ we can simply omit the ambiguity. We show the checks for the first few coefficients in figure 1.2.

The bridge equation predicts that the large-order growth of the one-instanton sector is controlled by the two-instanton sector. The one-parameter version of (1.64) is just

$$a_g^{(1)} \sim \sum_{h=0}^{\infty} \frac{\Gamma(g + b^{(1)} - b^{(2)} - h)}{A^{g+b^{(1)}-b^{(2)}-h}} \frac{S_1}{\pi i} a_h^{(2)}, \tag{1.74}$$

where we have only included the leading contribution: the two-instanton sector. From this large-order growth we can extract A and $b^{(1)} - b^{(2)}$ whose values we already know from the calculations above. The limits that extract the coefficients $a_h^{(2)}$ produce plots that look very much like those in figure 1.2. We show the examples of $-(b^{(1)} - b^{(2)})$ and the high-loop coefficient $a_5^{(2)}$ in figure 1.3. The appearance of the same two-instanton coefficients in both the subleading contribution to perturbative theory and in the leading contribution to the one-instanton sector is a consequence of resurgence and the precise form of the bridge equation (1.31) and the resurgence equations (1.34) it generates. The resurgent structure that we will uncover for the topological string theory on the mirror of local $\mathbb{CP}^2$ shares many properties with this example, but this feature of the two-instanton sector is not realized

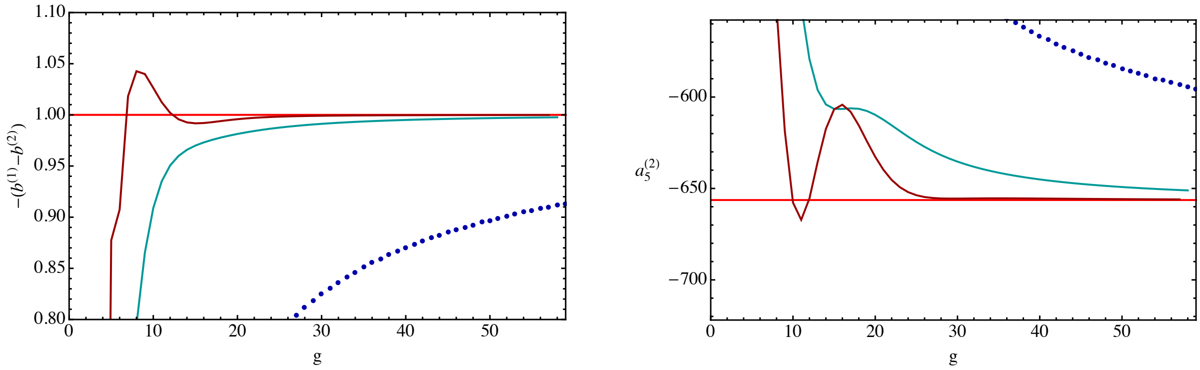


Figure 1.3: Checks for the difference of starting powers and a high loop two-instanton coefficient, $a_5^{(2)} = -\frac{1969}{3}$, from the large-order of the one-instanton sector, (1.74).

there. The Riccati equation does not have a complicate resurgent structure because it is a first order differential equation and the transseries has only one parameter. Resonance between sectors can only appear for higher order equations. We will discuss this in chapters 3 and 4.

Chapter 2

Aspects of topological string theory

2.1 Introduction

String theory can be seen as a conformal field theory in two dimensions coupled to gravity. The two-dimensional manifold, a Riemann surface, is the worldsheet the string creates as it propagates in a ten-dimensional target space, assuming supersymmetry is present. It can further be required that the observables do not depend on the metric of the Riemann surface, hence the adjective topological. This extra feature makes it possible to solve the topological string theory to any perturbative order in many cases, so it is an excellent arena in which to explore string related properties. Topological string theory can compute objects of interest in the full string theory, it realizes the mathematical statement of mirror symmetry, and it also enjoys large N dualities of its own, which in some cases can be explored in great depth.

Topological string theory is built from a topological conformal field theory coupled to gravity on the Riemann surface. This theory is an $\mathcal{N} = (2, 2)$ supersymmetric sigma model whose bosonic field represents a map from a worldsheet of genus g to a six-dimensional target space, which should be thought of as the compactified manifold in the splitting of ten-dimensional string theory. Supersymmetry requires this manifold be complex, and moreover, Kähler. To make the theory topological a modification of the symmetry algebra called twisting is required. It comes in two flavors, A and B. The A-model depends on the Kähler structure of the target space, while the B-model depends on the complex structure. Most observables (correlation functions) for these theories are trivial, and the essential reason is that the metric of the Riemann surface is fixed. There are no holomorphic functions on Riemann surfaces of genus $g \geq 2$. One needs to integrate over all possible metrics—which, in a conformal field theory, reduces to integration over complex structures—in order to overcome the selection rule that dictates which correlation functions vanish. This coupling to gravity leads to the definition of the topological string theory, once the target space has been further constrained to be a Calabi-Yau threefold. In this final setting, the A and B models are not quite holomorphic in their dependence of the Kähler or complex structures, respectively. This failure of holomorphicity can be made quantitatively precise through the holomorphic anomaly equations of Bershadsky, Cecotti, Ooguri and Vafa [23, 24]. These equations involve the perturbative free energies of the theory, $F_g^{(0)}$. They are of central importance for this thesis.

The A and B topological string theories are related by mirror symmetry, a geometrical statement that identifies the moduli space and observables of an A-model on a Calabi-Yau X with the ones of a B-model on the mirror manifold $\widetilde{X}$. This means that computations done on one side, involving topological and geometrical information of one of the geometries, can be translated into the mirror symmetry model and geometry. The most important application of this is the calculation of Gromov–Witten invariants of X—essentially counting holomorphic maps from a Riemann surface of genus g to X with fixed degree—from computations on the B-model, usually performed with the holomorphic anomaly equations.

Topological string theory can also compute relevant quantities for compactifications of string theories on Calabi-Yau threefolds. The resulting effective theory is $\mathcal{N} = 2$ supergravity in four dimensions. The effective couplings of this theory turn out to be determined by the free energies obtained in type A or B topological string theory, depending whether we started with a type IIA or type IIB string theory. In [93], it was shown how the perturbative free energy of the type-A topological string theory,

$$F^{(0)}(t, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2}\, F_g^{(0)}(t), \tag{2.1}$$

(here g_s is the string coupling constant and t represents Kähler structure moduli) can be expressed directly as an index counting BPS states of the supergravity theory. The explicit computation of this index gives a general expression for the type-A perturbative (in g_s) free energy, and it involves integer numbers, the so-called Gopakumar–Vafa invariants, that can be directly related to Gromov–Witten invariants—see [94] for a review.

Topological strings, like physical strings, can be open or closed. Open topological string theory includes topological D-branes that impose boundary conditions on the strings and can also wrap nontrivial cycles of the geometry. There is an open/closed string duality, in the spirit of AdS/CFT correspondence, that is realized through geometric transitions [95] (see [96] for a mathematical review). The first studied example was the transition from a deformed conifold, $T^*\mathbb{S}^3$, with D-branes wrapping around the $\mathbb{S}^3$ and making it shrink into a singularity. The singular geometry is then resolved and the D-branes disappear. The resulting theory, after the transition, is one of closed strings. The proposal for this result was motivated after studying the equivalence between Chern–Simons theory—a gauge theory—on $\mathbb{S}^3$ and an open topological string theory on the deformed conifold. This equivalence can be checked by comparing the free energy of the gauge theory, which can be computed exactly, and the Gopakumar–Vafa expression for the type-A free energy on this particular geometry. This duality between a gauge theory and a topological string theory is a further, more tractable, example of large N duality. In [13, 14] it was shown how geometric transitions can be applied to obtain dualities between type-B topological string theories and matrix models. These matrix models depend on a potential which is determined by the Calabi–Yau geometry. However, these geometries do not have mirrors.

An important class of Calabi–Yau geometries which do have mirrors is that of toric Calabi–Yau manifolds, which are actually non-compact varieties. The mirror geometry can be systematically constructed [97] and turns out to be essentially determined by a Riemann surface, which is the counterpart of the spectral curve in matrix models. The mirror of local $\mathbb{CP}^2$ is the main example we will discuss in this thesis—see section 2.4.

The rest of this chapter will be divided in three sections. The first will include a brief overview of the main elements of topological theories and topological strings. The second will focus on the computation of free energies in the B-model using the holomorphic anomaly equations. The last one will apply this theory to the example of local $\mathbb{CP}^2$, that will be used later in chapter 4. There are several reviews and books on topological string theory that cover in more detail the topics presented here, [11, 20, 98–101]. For questions on complex and algebraic geometry we refer to [102–104].

2.2 Topological theories and topological strings

2.2.1 Topological theories

Topological field theories depend on a particular manifold but not on its metric, $g_{\mu\nu}$. This can be so either because the theory does not require a metric to be formulated—Schwarz type —like Chern-Simons theory, or because one is able to show that physical observables do not actually depend on the metric—Witten or cohomological type . Topological strings are based on the second type.

Topological field theories of the Witten type include a symmetry, realized in an operator $\mathcal{Q}$, that squares to a bosonic symmetry of the theory, $\mathcal{Q}^2 = \mathcal{L}_B$. In some cases $\mathcal{Q}^2 = 0$. The topological property of the theory is guaranteed if the energy-momentum tensor is exact with respect to $\mathcal{Q}$,

$$T_{\mu\nu} := \frac{\delta S}{\delta g^{\mu\nu}} = \{\mathcal{Q}, G_{\mu\nu}\} \tag{2.2}$$

for some $G_{\mu\nu}$. Indeed, if we consider a correlator involving operators that respect the symmetry, $\mathcal{Q}\mathcal{O} = 0$, we find,

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle &= \frac{\delta}{\delta g^{\mu\nu}} \int \mathcal{D}\phi \, \mathcal{O}_1 \dots \mathcal{O}_n \, \mathrm{e}^{-S[\phi]/\hbar} = -\frac{1}{\hbar} \int \mathcal{D}\phi \, \mathcal{O}_1 \dots \mathcal{O}_n \, T_{\mu\nu} \\ &= -\frac{1}{\hbar} \langle \mathcal{O}_1 \dots \mathcal{O}_n \, \{\mathcal{Q}, G_{\mu\nu}\} \rangle = 0, \end{aligned} \tag{2.3}$$

because we can write the operator in the correlator as a $\mathcal{Q}$-exact term. (Here we can assume that $\delta\mathcal{O}_i/\delta g^{\mu\nu}$ is $\mathcal{Q}$-exact instead of zero to arrive at the same result). $\mathcal{Q}$-exactness of the energy-momentum tensor implies its closeness, which is always the case because $\mathcal{Q}$ represents a symmetry. A way to enforce this is by requiring the action itself to be $\mathcal{Q}$-exact,

$$S = \{\mathcal{Q}, V\}. \tag{2.4}$$

If this is the case one can go further and show that the correlators are actually independent of $\hbar$—just repeat (2.3) but take the derivative with respect to $\hbar^{-1}$ instead of $g_{\mu\nu}$. This means that the semiclassical limit of the theory ($\hbar \to 0$) is actually exact. This reduces drastically the complexity of the path integral to be evaluated in a process called localization. Equivariant cohomology is the theory that formalizes these arguments, see [4] for example. We will see how topological string theory incorporates this property.

2.2.2 Superconformal algebra and twisting

As we said in the introduction 2.1, we need to start with an $\mathcal{N} = (2,2)$ superconformal sigma model in two dimensions describing maps from a Riemann surface of genus g, Σ_g, to a target space, X, which must be at least a complex Kähler manifold. The presence of supersymmetry and R-symmetry will be the crucial ingredients needed to construct the topological symmetry $\mathcal{Q}$. $\mathcal{Q}$ will be different for the A and B models. The supersymmetry algebra is given by

$$\{Q_{\alpha+}, Q_{\beta-}\} = \gamma^{\mu}_{\alpha\beta} P_{\mu}, \qquad \{Q_{\alpha\pm}, Q_{\beta\pm}\} = 0, \tag{2.5}$$

$$\{J, Q_{\pm a}\} = 0, \tag{2.6}$$

$$\{F_V, Q_{\pm +}\} = +\frac{1}{2} Q_{\pm +}, \qquad \{F_A, Q_{\pm +}\} = \pm\frac{1}{2} Q_{\pm +}, \tag{2.7}$$

$$\{F_V, Q_{\pm -}\} = -\frac{1}{2} Q_{\pm +}, \qquad \{F_A, Q_{\pm +}\} = \mp\frac{1}{2} Q_{\pm +}. \tag{2.8}$$

Here, $Q_{\alpha a}$ are the supersymmetry generators, where α, β are spin indices and a, b are R-charge indices—in the twisting of the algebra both indices will mix. J generates the (Euclideanized) Lorentz symmetry and P_μ generates translations. F_V and F_A generate the R-symmetries, vectorial and axial, respectively. See [20, 99] for more details.

We now define a set of chiral and antichiral multiplets, Φ^i and $\Phi^{\bar{i}}$, where i and $\bar{i}$ run from 1 to the complex dimension of the target space, X. The action of this theory is given in terms of a Kähler potential $K(\Phi, \bar{\Phi})$, integrated over superspace.

The twisting of the algebra produces an odd but scalar symmetry $\mathcal{Q}$ that makes the theory topological. It is obtained by modifying the Lorentz generator that assigns spin values. There are two options, the A-twist or the B-twist,

$$J_A := J + F_V \quad \text{or} \quad \mathrm{J}_B := J + F_A \tag{2.9}$$

(the choice of relative signs is conventional). With respect to this new spin generator the supercharges have now either spin zero or spin one, instead of one half. In the case of the A-twist, Q_{+-} and Q_{-+} are scalars, whereas in the B-twist the scalars are Q_{+-} and Q_{--}. This allows us to define topological charges

$$\mathcal{Q}_A := Q_{+-} + Q_{-+}, \qquad \mathcal{Q}_B := Q_{+-} + Q_{--}. \tag{2.10}$$

It can be checked that $\mathcal{Q}_A^2 = \mathcal{Q}_B^2 = 0$, but we still need to show that the energy momentum tensor is $\mathcal{Q}$-exact. This is done separately for the two twists, showing that the metric dependent parts of the actions are actually exact.

An important question to address is whether the vector and axial symmetries, generated by F_V and F_A, are anomalous or not. One can see by looking at the actions (see [20, 99]) that only the axial symmetry is anomalous, and the obstruction is given by the index of the Dirac operator acting on the fermionic sector. This can be calculated to be

$$\int_{\Sigma_g} x^* \left(c_1(X)\right). \tag{2.11}$$

$c_1(X)$ is the first Chern class of the target manifold X, which measures the nontriviality of the canonical bundle on X. Since the A-twist is done by using F_V we have no constraints to impose, but the B-model will remain ill-defined unless $c_1(X) = 0$. This is the Calabi–Yau condition for the Kähler manifold X. For topological strings we will find that this condition has to be imposed also on the A-model.

2.2.3 A-model

Let us turn our attention now to the observables of the theory, starting with the A-model. One can find a perfect one-to-one correspondence between $\mathcal{Q}_A$ and the de Rham differential on X, so the operator ring can be described by the cohomology group $H^*(X)$. Since the axial symmetry is anomalous, any correlation function must absorb a number of zero modes of the twisted fermionic fields, as can be seen in the path integral formulation. For $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$, the selection rule

$$\sum_{k=1}^{n} \deg(\mathcal{O}_k) = 2d(1-g) + 2\int_{\Sigma_g} x^*\left(c_1(X)\right) \tag{2.12}$$

must be satisfied. Here $d = \dim_{\mathbb{C}} X$, g is the genus of Σ_g, and $\deg(\mathcal{O}_k)$ is the degree of the differential form to which $\mathcal{O}_k$ corresponds. For Calabi–Yau threefolds, the right-hand-side of (2.12) is $6(1-g)$. Note that for $g = 1$ only the partition function (no insertions) is nonvanishing, while for $g \geq 2$ all correlators must necessarily be trivial because the right-hand-side is negative and the left can never be. As we mentioned before, this is due to the fact that we are considering a fixed metric on the Riemann surface Σ_g and for $g \geq 2$ there are no holomorphic maps. We couple the theory to gravity in order to have a richer theory.

If we focus on the $g = 0$ correlators, we must impose the insertion of three operators. The localization of the path integral due to the exactness of the action imposes that only holomorphic maps $x : \Sigma_g \to X$ contribute to the observable. These maps turn out to be classified by the homology group of two-cycles, $H_2(X)$. The correlation function depends on

$$\int_{x_*[(\Sigma_g)]} \omega, \tag{2.13}$$

where ω is the (complexified) Kähler form on X. $x_*[(\Sigma_g)]$ is the homology class of the image of Σ_g by x and it can be expanded in a basis of $H_2(X)$, call it $[S_i]$. The general form of the three point function turns out to be

$$\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3 \rangle = (D_1 \cap D_2 \cap D_3) + \sum_{\beta \neq 0} I_{0,3,\beta}(\phi_1, \phi_2, \phi_3)\, Q^\beta. \tag{2.14}$$

Here $Q^\beta := \prod_{i=1}^{b_2(X)} \mathrm{e}^{-n_i\, t_i}$, where $b_2(X) = \dim H_2(X)$, $\beta = \sum_i n_i\, [S_i] \in H_2(X)$ and $t_i = \int_{[S_i]} \omega$ are the Kähler parameters. The first term in (2.14) is the contribution from the trivial class, $\beta = 0$, and can be written in terms of the intersection number of divisors D_k dual to the operators $\mathcal{O}_k$. The second term includes (worldsheet) instanton corrections to the classical result. The coefficients $I_{0,3,\beta}$ are related to the Gromov–Witten invariants, $N_{0,\beta}$, through the formula

$$I_{0,3,\beta}(\phi_1, \phi_2, \phi_3) = N_{0,\beta} \int_\beta \phi_1 \int_\beta \phi_2 \int_\beta \phi_3, \tag{2.15}$$

where ϕ_k is the differential form associated to $\mathcal{O}_k$. These invariants can be collected into a very important function called the prepotential,

$$F_0(t) = \sum_{\beta} N_{0,\beta}\, Q^{\beta} \tag{2.16}$$

Note that it is associated to genus $g = 0$ and that it depends on the Kähler structure moduli of X.

2.2.4 B-model

In the B-model the observables are in one-to-one correspondence with elements of the Dolbeault cohomology group $H^*_{\bar{\partial}}(X, \wedge^* TX)$. We still have a selection rule to fulfill. Let $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$ an n-point function with $\mathcal{O}_k \in H^{p_k}_{\bar{\partial}}(X, \wedge^{q_k} TX)$. Then,

$$\sum_{k=1}^{n} p_k = \sum_{k=1}^{n} q_k = d(1-g), \tag{2.17}$$

where we have already used that the target space must be Calabi–Yau. In this model the localized configurations are simply the constant maps $x : \Sigma_g \to X$. This implies that the path integrals reduce to regular integrals over the target space only. The correlation function for $g = 0$, which includes three operators, makes use of the nowhere vanishing $(3,0)$-form Ω that exists on every Calabi–Yau manifold. Ω allows to write the combination of differential forms associated to $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ as one differential form of top degree (d,d) which can be integrated over X. Again, see [20, 99] for details. The correlation function turns out to depend only on the complex structure of X, which can be parametrized by integrals of Ω over non-trivial three-cycles of X. More precisely, take a basis of $H_3(X)$, $\{(A_a, B^a)\}_{a=0}^{h^{2,1}}$. Here we use that $\dim H_3(X) = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} = 2(1 + h^{2,1})$. The moduli space of complex structures of X has precisely dimension $h^{2,1}$. If we define the periods of Ω as

$$z_a = \int_{A_a} \Omega, \quad \mathcal{F}^a = \int_{B^a} \Omega, \tag{2.18}$$

we can take coordinates z_a for the moduli space and deduce that $\mathcal{F}^a$ must be a function of them. Actually the coordinates are projective, so we can construct inhomogeneous ones by $t_a = \frac{z_a}{z_0}$, which we call the complex structure parameters. $\mathcal{F}^a$ can be computed from a function $\mathcal{F}(z)$ by taking a z_a-derivative. Dehomogeneizing, we can define the prepotential

$$F_0(t) = \frac{1}{z_0} \mathcal{F}(z). \tag{2.19}$$

In terms of the prepotential, the periods (2.18) will be

$$1, \quad t_a, \quad 2F_0 - \sum_{i=1}^{h^{2,1}} t_i \frac{\partial F_0}{\partial t_i}, \quad \frac{\partial F_0}{\partial t_i}. \tag{2.20}$$

The theory that studies deformations and dependence on the complex structure goes by the name of special geometry [105, 106]. A final result expresses the three point function as a third derivative of the prepotential

$$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \frac{\partial F_0}{\partial t_i \partial t_j \partial t_k}. \tag{2.21}$$

Here $\mathcal{O}_i$ is, in a precise sense, related to the coordinate t_i, see [20]. For a general three point function one would have to introduce covariant derivations. We call t_i a flat coordinate. The fact that we denote the prepotential, F_0, by the same symbol in the A and B models is of course no coincidence. Once the relation between Kähler and complex structure moduli is found for two mirror geometries—this is called the mirror map—, the prepotential on the A and B sides are equal. This equivalence is completed to all genera once we define the appropriate free energies in topological string theory.

2.2.5 Topological strings

We have seen that the selection rule, originating in the axial anomaly, severely restricts the nontrivial observables of the topological theories. The coupling to two-dimensional gravity removes this obstruction because the degrees of freedom associated to the integration over the space of metrics on the Riemann surfaces are enough to absorb all the necessary zero modes. However this counting only works for Calabi–Yau manifolds in three dimensions, since the Riemann surface moduli space has precisely dimension 3(g-1) for $g \geq 2$. The Calabi–Yau condition and the particular complex dimension of three is so relevant because of this matching. In more detail, we can define a nonzero, genus g, free energy by the appropriate integration over the moduli space of Riemann surfaces of genus g, $\bar{M}_g$,

$$F_g := \int_{\bar{M}_g} \langle \prod_{k=1}^{3g-3} (G, \mu_k)(\bar{G}, \bar{\mu}_k) \rangle. \tag{2.22}$$

The field $G = G_{\mu\nu}$ already appeared in the $\mathcal{Q}$-exactness relation for the energy-momentum tensor (2.2), and μ_k are the so-called Beltrami differentials. For $g = 1$, the correlator $\langle \ldots \rangle$ has one insertion. For $g = 0$ the free energy is given by the prepotential, and no integration over the moduli space is needed. The correlator can be computed on the A or B model, giving rise to A or B topological string free energies. The respective dependence on the Kähler and complex structure moduli of X remains. Our main interest will be on the B side but let us mention that the Gromov–Witten invariants of (2.16) generalize to

$$F_g(t) = \sum_\beta N_{g,\beta}\, Q^\beta. \tag{2.23}$$

Recall that these count, in a particular sense, holomorphic maps from a genus g Riemann surface to the Calabi–Yau X with fixed degree given by the class β.

Let us now focus on the B model and consider the question of whether topological string theory is still topological, now that we have coupled the original topological theory to gravity.

It turns out that, due to insertions of $G_{\mu\nu}$ in (2.22), the usual argument does not give a zero answer, which would prove the topological property, but it produces a quantity that depends on the boundary of the moduli space $\bar{M}_g$, instead. This is so because the calculation of $\langle \mathcal{Q}(\ldots) \rangle$ involves $\mathcal{Q}$ actions on $G_{\mu\nu}$ that give back the energy-momentum tensor, which in turn can be written as a variation of the action with respect to the metric. This finally gives a total derivative in the integral over $\bar{M}_g$ that translates into a contribution from the boundary $\partial\bar{M}_g$.

An important consequence of this is that the dependence of the free energies on the complex structure—or the Kähler structure for the A-model—is not entirely holomorphic. Before coupling to gravity, dependence on $\bar{t}$—the complex conjugate of the moduli t—was washed away by $\mathcal{Q}$-exactness. However, the integration over $\bar{M}_g$ obstructs this argument producing the so-called *holomorphic anomaly*. Let us notice that there is no anomaly for F_0, essentially because the moduli space for this genus is zero-dimensional under some technical conditions. The precise way in which this failure of holomorphicity occurs is encoded in the holomorphic anomaly equations of [23, 24], which we write here but explain in the next section,

$$\bar{\partial}_{\bar{i}}\partial_i F_1^{(0)} = \frac{1}{2} C_{ijk}\, C_{\bar{i}}^{jk} - \left(\frac{\chi}{24} - 1\right) G_{i\bar{j}}, \tag{2.24}$$

$$\bar{\partial}_{\bar{i}} F_g = \frac{1}{2} C_{\bar{i}}^{jk} \left(D_j D_k F_{g-1} + \sum_{h=1}^{g-1} D_j F_h\, D_k F_{g-h} \right), \tag{2.25}$$

for $g \geq 2$. These equations can compute the topological string free energies very efficiently. All of them together can be packed into the perturbative closed topological string free energy

$$F^{(0)}(g_s; t, \bar{t}) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{(0)}(t, \bar{t}), \tag{2.26}$$

where g_s is the string coupling constant, and we have added a superscript indicating the perturbative nature, in g_s of the free energies.

2.3 Perturbative free energies

2.3.1 Genus zero

We finished last section with the holomorphic anomaly equations. They provide a way to compute the perturbative free energies in a very efficient way due to the recursive nature of the equations and their integrability, once the appropriate variables have been introduced. Besides the holomorphic anomaly equations, there are other techniques to compute free energies. On the A-model there is Kontsevich's localization method on toric Calabi–Yau manifolds [107], or the topological vertex [108, 109] can provide complete computations of the free energies. Other approaches aiming at finding Gromov–Witten invariants include the use of relative Gromov–Witten invariants for compact geometries or attempts at rigorous formulations of the Gopakumar–Vafa invariants. One can also use type IIA string theory and heterotic duality in order to extract invariants. A very general but less efficient approach relies on the use of the topological recursion [18, 110]. Many of these techniques have a firm mathematical basis.

In this section we will focus on the holomorphic anomaly equations of [24] for the B-model. We will start with the genus-0 free energy and its relation to the periods of the geometry. After briefly reviewing the origin and derivation of the equations we will focus on their integration. We will follow the standard technique of introducing the propagators as nonholomorphic integrating variables. Finally, the holomorphic ambiguity as subproduct of the integration will be addressed.

The genus-0 free energy, $F_0^{(0)}$, has no holomorphic anomaly, it is holomorphic, as we mentioned in the last section. Equation (2.20) hints that we should be able to compute $F_0^{(0)}$ by integrating one of the periods of Ω with respect to the complex structure moduli, t_i. An explicit computation of the periods as integrals on particular three-cyles is usually a complicated task. However, all these periods satisfy a set of linear differential equations in the complex structure moduli that go by the name of Picard-Fuchs equations. These equations can be obtained in several ways, and we will give an example in the next section when we focus on local $\mathbb{CP}^2$. Once a basis of solutions is computed one has to determine which combination reproduces each period. Having explicit expressions for t_i and $\partial_{t_i} F_0^{(0)}$ we can compute the free energy. Recall that from the prepotential we can calculate the three point functions (2.21). The are known as Yukawa couplings and are denoted by C_{ijk}. They will be relevant for the integration of higher genus free energies. We leave the details for the example of section 2.4.

2.3.2 Holomorphic anomaly equations

The holomorphic anomaly equations allow us to compute $F_g^{(0)}$, $g \geq 1$, recursively up to a holomorphic function called the holomorphic ambiguity. As we saw in the previous section, the failure of holomorphicity appears as a consequence of boundary terms in the moduli space of Riemann surfaces of genus g. Riemann surfaces at the boundary can be obtained as limits of surfaces by pinching one of the cycles. This can be done in two ways. In the first the shrinking of the cycle produces two disconnected surfaces whose genera, h and $g-h$, add up to the initial genus g. The second the contraction only removes one of the holes, thereby leaving a surface of genus $g-1$. Using the explicit form of the free energies (2.22) and carrying out a careful analysis (see [24]) the holomorphic anomaly equations are produced

$$\bar{\partial}_{\bar{i}} F_g^{(0)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \left(D_j D_k F_{g-1}^{(0)} + \sum_{h=1}^{g-1} D_j F_h^{(0)} \, D_k F_{g-h}^{(0)} \right), \tag{2.27}$$

for $g \geq 2$. For genus 1 the equation has a different aspect,

$$\bar{\partial}_{\bar{i}} \partial_i F_1^{(0)} = \frac{1}{2} C_{ijk} \, \bar{C}_{\bar{i}}^{jk} - \left(\frac{\chi}{24} - 1 \right) G_{i\bar{j}}. \tag{2.28}$$

and has a geometric interpretation as a Ray-Singer torsion [20]. Let us explain briefly what the various ingredients in (2.27) and (2.28) are. $G_{i\bar{j}}$ is the (Weil–Peterssen) metric on the complex structure moduli space. It is derived from the Kähler potential K, related to the nowhere vanishing form Ω by the formula

$$\mathrm{e}^{-K} = \int_X \bar{\Omega} \wedge \Omega. \tag{2.29}$$

χ is the Euler characteristic of the moduli space. D_i is the covariant derivative in this space, and it includes the Christoffel symbol, Γ^i_{jk}, associated to the metric $G_{i\bar{j}}$, and $\partial_i K$, associated to a line bundle whose origin is the multiplicative ambiguity in the definition of the Calabi–Yau three-form Ω. $\bar{\partial}_{\bar{i}}$ is the antiholomorphic derivative. C_{ijk} are the Yukawa couplings. Finally, $C^{jk}_{\bar{i}}$ is given by

$$C^{jk}_{\bar{i}} = \bar{C}_{\bar{i}\bar{j}\bar{k}}\, G^{\bar{j}i}\, G^{\bar{k}k}\, \mathrm{e}^{2K}, \tag{2.30}$$

with $\bar{C}_{\bar{i}\bar{j}\bar{k}} = \overline{C_{ijk}}$, the complex conjugate of the Yukawa couplings.

The genus-1 free energy, $F^{(0)}_1$, can be integrated up to the holomorphic ambiguity [23],

$$F^{(0)}_1 = \log\left(\mathrm{e}^{\frac{K}{2}\left(3+h^{2,1}-\frac{\chi}{12}\right)} \det(G)^{-1/2}\,|f|^2\right), \tag{2.31}$$

where f is the holomorphic ambiguity. We will review how to fix it later on.

2.3.3 Propagators

In order to compute the free energies for higher genera it is convenient and efficient to introduce the propagators [24, 111, 112]. To motivate their use we can note that the function $C^{jk}_{\bar{i}}$ in front of (2.27) complicates the integration in the antiholomorphic complex structure coordinates. This is because $C^{jk}_{\bar{i}}$ is not holomorphic and difficult to calculate in practice. However, this can be overcome with the definition of the propagators, S^{ij},

$$\bar{\partial}_{\bar{i}} S^{ij} = \bar{C}^{ij}_{\bar{i}}. \tag{2.32}$$

Note that S^{ij} is only defined up to a holomorphic function. This freedom has to be fixed for every particular example. In order to a complete picture one needs to define further propagators, S^j, S, by

$$\bar{\partial}_{\bar{i}} S^j = G_{i\bar{i}}\, S^{ij}, \quad \bar{\partial}_{\bar{i}} S = G_{i\bar{i}}\, S^i. \tag{2.33}$$

A very important property of these variables is that, while containing the nonholomorphic dependence, they form a closed set under the action of the covariant derivative, as long as we also include $K_i := \partial_i K$. One can show that this is indeed the case [112] by starting by with an identity from the theory of special geometry

$$\bar{\partial}_{\bar{i}} \Gamma^i_{jk} = \delta^i_j\, G_{k\bar{i}} + \delta^i_k\, G_{j\bar{i}} - C_{jkl}\, \bar{C}^{li}_{\bar{i}}. \tag{2.34}$$

The idea is to write the expression for $D_i S^{jk}$ involving the Christoffel symbols, Γ^i_{jk}, for the metric and the connection for the line bundle, K_i, and then apply a $\bar{\partial}_{\bar{i}}$-derivative. After some manipulations one is able to write

$$\bar{\partial}_{\bar{i}}(D_i S^{jk}) = \bar{\partial}_{\bar{i}}\left(\delta^j_i\, S^k + \delta^k_i\, S^j - C_{ilm}\, S^{lj}\, S^{lk}\right), \tag{2.35}$$

from which we determine $D_i S^{jk}$ in terms of the propagators and K_i, only,

$$D_i S^{jk} = \delta_i^j\, S^k + \delta_i^k\, S^j - C_{ilm}\, S^{lj}\, S^{lk} + f_i^{jk}. \tag{2.36}$$

Here, f_i^{jk} is a holomorphic function to be fixed by some particular choice, and related to the ambiguous definition of the propagator. A similar exercise can be done for S^i, S and K_i. See [112–115]. The next step is to apply the chain rule to express $\{\bar{t}_i\}$ in terms of $\{S^{ij}, S^i, S, K_i\}$, and use this in (2.27),

$$\frac{\partial F_g^{(0)}}{\partial S^{ij}} = \frac{1}{2}\left(D_j D_k F_{g-1}^{(0)} + \sum_{h=1}^{g-1} D_j F_h^{(0)}\, D_k F_{g-h}^{(0)} \right), \tag{2.37}$$

$$0 = S^{ij}\, \frac{\partial F_g^{(0)}}{\partial S^j} + S^i\, \frac{\partial F_g^{(0)}}{\partial S} + \frac{\partial F_g^{(0)}}{\partial K_i}. \tag{2.38}$$

The nonholomorphic dependence of the free energies is stored in all the propagators. In the holomorphic limit, $\bar{t}_i \mapsto -i\infty$, the propagators acquire a holomorphic value. It turns out that in the case of local, that is, noncompact, Calabi–Yau manifolds those values can be chosen to vanish for S^i and S—recall the ambiguity in the propagator definitions. For local geometries K goes to a constant in the holomorphic limit so $\partial_i K$ vanishes. S^{ij} does not vanish in this limit. In many situations one is only interested in the holomorphic limit of the perturbative free energies, mainly because they produce Gromov–Witten invariants after the mirror map is applied. So it is customary to turn off all propagators except for S^{ij} and proceed with the integration. It was shown in [112] that the dependence of the free energies on the propagators is polynomial. This means that, at the formal level we can treat the propagators as external parameters. In this thesis we study resurgent properties of the nonperturbative free energy, and this implies an analysis of the large-order behavior of different sectors. This is a formal operation in which S^i, S and K_i can be considered as parameters. This is indeed the view we set on S^{ij} for this thesis. Therefore, in order to simplify the analysis we choose to turn off the other propagators from the start, knowing that some nonperturbative dependence is stored in S^{ij} and that a particular value of S^{ij} reproduces the holomorphic limit. Further resurgent analyses aiming at resummation for the full nonholomorphic free energy should approach the complete problem. This being a first analysis, we choose a simplified, yet relevant, setting. So we will focus on the holomorphic anomaly equations

$$\frac{\partial F_g^{(0)}}{\partial S^{ij}} = \frac{1}{2}\left(D_j D_k F_{g-1} + \sum_{h=1}^{g-1} D_j F_h\, D_k F_{g-h} \right), \tag{2.39}$$

with $D_i F_g^{(0)} = \partial_i F_g^{(0)}$ and $D_i D_j F_g^{(0)} = \partial_i \partial_j F_g^{(0)} - \Gamma_{ij}^k\, \partial_k F_g^{(0)}$. For later use we mention that from (2.34) the Christoffel symbols can be written in terms of the propagator as

$$\Gamma_{jk}^i = -C_{jkl}\, S^{li} + \tilde{f}_{jk}^i, \tag{2.40}$$

where $\tilde{f}_{jk}^i$ is a holomorphic function not unrelated to f_k^{ij}. Also, one can show using (2.40) that

$$D_i F_1^{(0)} = \frac{1}{2} C_{ijk}\, S^{jk} + \alpha_i, \tag{2.41}$$

where $\alpha_i = -\frac{1}{2}\tilde{f}^{j}_{ij} + \partial_i \log f$. Since $D_i F_1^{(0)}$ is the starting point of the holomorphic anomaly equations (2.39), and S^{ij} reproduces itself under the action of D_i, the higher genus free energies, $F_g^{(0)}$, will have a polynomial dependence in the propagator.

2.3.4 Integration and ambiguities

The holomorphic ambiguities appear in the process of integration and must be determined at each genus. One way to do it is by using the mirror map and comparing against previously computed Gromov–Witten invariants. This can only be done for a small number of geometries and for low genus in general. Actually, the usual goal is to proceed the other way around, obtaining invariants efficiently from the B-model. As was argued in [116], in the case of local Calabi–Yau manifolds, there is a systematic way to fix the holomorphic ambiguity based on a well-understood behavior of the free energies for certain values of the complex structure. We need to separate the cases $g = 1$ and $g \geq 2$.

For genus one, the explicit expression for $F_1^{(0)}$ as a correlation function can be computed, in the holomorphic limit, near special points in the complex structure moduli space: the large-radius point and the conifold point[1]. If t_i denote flat coordinates (periods) around a given special point, the holomorphic limit of the free energy satisfies [24]

$$\lim_{\bar{t}_i \mapsto -i\infty} F_1^{(0)} = -\frac{1}{24} \sum_{i=1}^{h^{2,1}} t_i \int_X c_2 \wedge J_i, \tag{2.42}$$

where c_2 is the second Chern class and J_i is the i-th complex structure (1,1)-form (see [117] for examples). We will review the application of this formula to the example of local $\mathbb{CP}^2$ in the next section.

For $g \geq 2$, we must again look at the free energy $\mathcal{F}_g^{(0)}$ near the conifold locus and the large-radius point. There, the free energy has a well known behavior obtained from the analysis of a supergravity sector in type IIA string theory. This boundary condition, along with regularity of the free energies elsewhere in moduli space, is enough to fix the holomorphic ambiguity completely. We will be more precise and explicit with the concrete case of local $\mathbb{CP}^2$ in the following section. For more comments and generalities of the procedure to fix the holomorphic ambiguity see [116].

2.3.5 Antiholomorphic structure

Part of chapter 3 studies the antiholomorphic dependence of the different nonperturbative sectors of the transseries free energy. In this subsection we review the analysis of the propagator dependence for the perturbative free energies, $F_g^{(0)}$, as done in [112]. Their elegant argument based on the introduction of a natural notion of degree cannot be carried over easily to the nonperturbative case, so we will repeat the argument in a more hands-on, explicit way, that will work in chapter 3.

The efficient integration of the holomorphic anomaly equations started with [111] for the case of the quintic in $\mathbb{P}^4$. There it was noticed that the perturbative free energies, for each

[1] In general the conifold locus has dimension greater than zero. In those cases a specific point is chosen.

genus, could be expressed in terms a set of generators which carried the full antiholomorphic dependence. Moreover, they found that the free energies are polynomials of a specific degree in these generators. For $F_g^{(0)}$ the degree is $3g-3$, for $g \geq 2$. This construction was later extended to any compact Calabi–Yau geometry in [112]. The results also covered the case of several complex structure moduli and was formulated in terms of the propagator variables we reviewed before. It was shown that S^{ij}, S^i, S, and K_i form a differential ring with respect to the covariant derivative D_i, meaning that, apart from holomorphic functions under control, no other antiholomorphic functions can appear in the process of integration of the holomorphic anomaly equations. To show that the dependence of the free energies in the set of propagators is polynomial of degree $3g-3$, a notion of degree compatible with this one was introduced. It assigns degree $+1$ to S^{ij} and K_i, degree $+2$ to S^i and degree $+3$ to S. Because we are only going to work with the propagator S^{ij} we omit all the other ones (see [112] or [113] for the general case). The argument is as follows. Let us assume that the perturbative free energies have a well-defined degree, $F_g^{(0)} \mapsto d(g)$. Because of the relation between the covariant derivative and the propagator (2.36), one must assign degree $+1$ to D_i. Holomorphic quantities have degree zero. The proof that $d(g) = 3g-3$ is done by induction on g using the holomorphic anomaly equations (2.39). The base case is $g = 2$ which we do explicitly for the case of local $\mathbb{CP}^2$ in section 2.4. One finds $d(2) = 3$, see (2.81). If we assume that for $h < g$, $d(h) = 3h-3$, the right-hand-side of the holomorphic anomaly equations (2.39) for $F_g^{(0)}$, have degree $3(g-1)+1+1 = 3g-4$. Since the left-hand-side, by assumption, also has degree $3g-3-1 = 3g-4$, this concludes the proof.

As we mentioned, we will not be able to implement this type of proof for higher instanton sectors. The essential reason is that the dependence on the propagators is not only polynomial, but exponential, as well. Therefore, a more direct approach to studying the antiholomorphic dependence will be needed. Let us exemplify on the perturbative free energies. To show that they are polynomials in the propagators we use induction again but we write explicitly

$$F_g^{(0)} = \mathrm{Pol}(S^{ij}; 3g-3) =: \mathrm{Pol}(3g-3), \quad g \geq 2. \tag{2.43}$$

Here $\mathrm{Pol}(x_1, \ldots, x_n; d)$ represents a polynomial of total degree d in the variables $x_1, \ldots, x_n$. If it is clear what the arguments are we drop them. The base case of the induction is done as before. Assuming that $F_h^{(0)} = \mathrm{Pol}(3h-3)$ for $h < g$, we can see that the right-hand-side of (2.39) has the schematic form

$$\mathrm{Pol}(3(g-1)-3+2) + \sum_{h=1}^{g-1} \mathrm{Pol}(3(g-h)-3+1) \times \mathrm{Pol}(3h-3+1) = \mathrm{Pol}(3g-4), \tag{2.44}$$

where we have used that $D_i F_h^{(0)} = \mathrm{Pol}(3h-3-1)$ and $D_i D_j F_h^{(0)} = \mathrm{Pol}(3h-3-2)$. Integrating the right-hand-side of the holomorphic anomaly equations with respect to the propagator increases the degree of the polynomial by one, arriving at the familiar result.

This fairly easy exercise will become a little bit more complicated in the case of higher sectors of the free energy transseries. Nevertheless, general results will be proved following this procedure.

2.4 Local $\mathbb{CP}^2$

The last section of this chapter is focused entirely on the example of local $\mathbb{CP}^2$. We will analyze this model in the framework of resurgence in chapter 4. Some of the points that were left unspecified with respect to propagator definitions, ambiguities or holomorphic limits will be addressed here in this concrete example. We follow closely [116], but useful references are also [118, 119]. Our main objective is to describe how the perturbative free energies of this topological string theories can be computed systematically to high genus. Since our analysis in chapter 4 is a large-order one, we are interested in the behavior of the free energies, $F_g^{(0)}$, for very high g. We will also spend some time describing the periods of the geometry, not only because they they determine the genus-zero free energy, the Yukawa coupling and the holomorphic limit of the propagator, but because they will be relevant for the study of the instanton actions that control the large-order.

2.4.1 Geometry and genus-zero free energy

We want to compute the perturbative free energies of the B-type topological string theory on the mirror of the local $\mathbb{CP}^2$. Local $\mathbb{CP}^2$ is a noncompact Calabi–Yau three fold of toric type. There is a standard method to construct the mirror manifold of a toric Calabi–Yau [97, 120, 121], so we will follow this procedure. As a manifold local $\mathbb{CP}^2$ is a total bundle of $\mathbb{CP}^2$, explicitly, $\mathcal{O}(-3) \to \mathbb{CP}^2$. The property of being a Calabi–Yau comes from the cancellation between the Chern class associated to $\mathbb{CP}^2$ and the one associated to the line bundle. As a toric variety local $\mathbb{CP}^2$ can be described in terms of a quotient of $\mathbb{C}^n$, with some points removed, by a particular group action G,

$$(\mathbb{C}^4 - \{x_1 = x_2 = x_3 = 0\})/G. \tag{2.45}$$

Here $x_0, \ldots, x_3$ are standard coordinates on $\mathbb{C}^4$. The group $G = \{(t^{-3}, t, t, t) \in (\mathbb{C}^*)^4\}$ acts by component wise multiplication on $\mathbb{C}^4$, and is isomorphic to $\mathbb{C}^*$ (the complex plane minus the origin). In this description the embedded $\mathbb{CP}^2$ can be seen by projection of $\mathbb{C}^4$ onto the last three components. The first component, with the particular exponent of the action group, -3, has the information about the fiber. The construction of the mirror manifold of a toric Calabi–Yau depends on combinatorial data encoded in the group action. For our case it is simply the vector $Q = (-3, 1, 1, 1)$. The sum of the components being zero is equivalent to the Calabi–Yau condition of the variety. An alternative description of the variety can be given in which the presence of the Kähler parameter is present, and it is connected to the Higgs branch of a supersymmetric two-dimensional linear sigma model [122]. There is only one Kähler modulus in the A-geometry, which means that there is only one complex structure modulus in the mirror geometry. This simplifies the computations, which is important if we need to go to high genus.

The mirror construction produces a new Calabi–Yau manifold which can be described very explicitly once we have introduced the appropriate coordinates,

$$w^+ w^- = \sum_{i=0}^{3} X_i, \tag{2.46}$$

$$z = \prod_{i=0}^{3} X_i^{Q_i}. \tag{2.47}$$

Here $w^+, w^- \in \mathbb{C}$. $X_0, \ldots, X_3 \in \mathbb{C}^\star$ are projective coordinates, $X_i \sim \lambda X_i$, $i = 0, \ldots, 3$, $\forall \lambda \in \mathbb{C}^\star$. Note the presence of the combinatorial numbers Q_i, and the complex structure modulus z. In principle $z \in \mathbb{CP}^1$ although for some special values the geometry becomes singular. These special points are very important for the computations ahead. In order to make the B-geometry more explicit we can fix the homogeneity invariance by defining $x_i := X_i/X_0$, for $i = 0, 2, 3$. Equation (2.47) allows us to solve for x_2. We can then substitute the value back into (2.46) to find

$$w^+w^- = 1 + x + z\,\frac{x^3}{y} + y \equiv H, \tag{2.48}$$

where we have renamed $x := x_0$ and $y := x_3$, and denoted by H the expression on the right. In the local description (2.48) of the B-geometry we can see the presence of a Riemann surface, $H = 0$, embedded in it. This lower dimensional geometry determines much of the geometrical properties we need to find. The reason is that when computing periods of the geometry, the cycles over which we integrate reduce to cycles of the Riemann surface. Recall from section 2.2.4 that periods are integrals over cycles of the nonvanishing $(3,0)$-form Ω. The explicit form of Ω in this kind of geometries the mirror of local $\mathbb{CP}^2$ belongs to, is given by

$$\Omega = \frac{\mathrm{d}H \wedge \mathrm{d}x \wedge \mathrm{d}y}{Hxy}, \tag{2.49}$$

in local coordinates. Part of the integration, over the pair of lines $w^+w^- = 0$ can be performed and the remaining form of the periods is

$$\int \lambda, \quad \text{where} \quad \lambda = \log y\,\frac{dx}{x}. \tag{2.50}$$

There are two independent cycles A and B on the Riemann surface, plus a third integral which essentially picks up a residue and gives a constant value. This is the C-period. Let us stress that the periods are functions of the complex structure modulus z. The A and B periods provide the redundant projective coordinates we described in section 2.2.4. From the combination of the two the genus-zero free energy can be computed.

Before going into the computation of the periods we can anticipate that some values of the complex structure modulus z the periods can have a singular behavior. This is because there, the geometry shows some kind of singularity. The simplest example is the case of the conifold point, $z = -1/27$, which is selected as special when looking at the j-function of the torus $H = 0$. For that value, the function has a singularity. Another special point is $z = 0$, the large-radius point. It can be shown that the mirror geometry acquires infinite volume when approaching the corresponding value of the Kähler parameter. Finally, at $z = 0$ the geometry becomes an orbifold, $\mathbb{C}^3/\mathbb{Z}_3$ and we talk about the orbifold point. Around these special points, the nonconstant periods have non trivial monodromies. The monodromy group gives an indication of the modular symmetry that is present in the problem—see [115, 119, 123] for example.

The computation of periods as explicit integrals over cycles is a difficult task. Fortunately, it is possible to derive a so-called Picard–Fuchs equations that the periods, as functions of the complex moduli, must satisfy (see for example [124]). The first step takes (2.46) and (2.47) and rescales $X_i \to a_i X_i$ by some numbers a_i such that $z = \prod a_i^{Q_i} = a_0^{-3} a_1 a_2 a_3$. Then we notice that the operator

$$\partial_{a_0}^3 - \partial_{a_1}\partial_{a_2}\partial_{a_3} = \left(X_0^3 - X_1 X_2 X_3\right) \partial_P^3, \tag{2.51}$$

where $P := w^+ w^- - \sum_i a_i X_i$, is zero because the term on the right vanishes on the manifold—it is (2.47) in the rescaled variables. Using the chain rule the left side can be written, up to a factor, in terms of z as

$$\mathcal{D}_z = \theta_z^3 + 3z\theta_z \left(3\theta_z + 1\right)\left(3\theta_z + 2\right), \tag{2.52}$$

where we have used the logarithmic derivative $\theta_z := z\partial_z$. So any function on the complex structure moduli space has to be annihilated by this operator. A standard analysis of this equation shows that there are two regular singular points in the z-plane at finite distance. One is the conifold point $z = -1/27$ and the other is the large-radius point at $z = 0$. Using an inverse variable z^{-1} one can see that $z^{-1} = 0$; the orbifold point is also a regular singular point. The existence of this regular singularities implies that if we try to look for power series solutions, using Frobenius method, their radius of convergence will be equal to the distance to the closest singularity. This means that the equations must be solved in different patches around each special point and then analytically continued. Note that there is a global constant solution of $\mathcal{D}_z f(z) = 0$, which is associated to the C-cycle we mentioned before and is always present for noncompact manifolds. Since the equation is of order 3 we expect two other, nontrivial, solutions associated to A and B cycles. From the discussion in section 2.2.4, one of them will provide the flat coordinate around a special point (mirror map), and the other the derivative of the prepotential, the genus-zero free energy, with respect to this flat coordinate.

The classification of regular singular points depends on the coordinate we choose in the differential equation. There is a natural coordinate, ψ, that will become relevant in chapter 4, for which the orbifold point is not regular singular anymore. Instead we have three different conifold points, along with the remaining large-radius point. If we define ψ by the relation

$$\psi^{-3} = -27z, \tag{2.53}$$

then the Picard–Fuchs equation becomes

$$f'''(\psi) - \frac{3\psi^2}{1-\psi^3} f''(\psi) - \frac{\psi}{1-\psi^3} f'(\psi) = 0. \tag{2.54}$$

In the ψ-plane the only finite-distance singularities are at the cubic roots of 1, the three conifold points. The point at infinity, $\psi^{-1} = 0$, is the large-radius point. We are going to work with both coordinates. z will predominate in the perturbative calculation, whereas ψ will be necessary to understand the large-order behavior.

Let us turn now to an account of the solutions of the Picard–Fuchs equation. See [125] for a thorough analysis. Around the large-radius point $z = 0$ one can see, studying the equation, that besides the constant solution there are two power series with a $\log z$ term and a $\log^2 z$ term, respectively. A basis is given by

$$X^{(1)} = \frac{1}{2\pi i} \left(\log z + \sigma_1(z) \right), \tag{2.55}$$

$$X^{(1,1)} = \frac{1}{(2\pi i)^2} \left(\log^2 z + 2\sigma_1(z) \, \log z + \sigma_2(z) \right), \tag{2.56}$$

where

$$\sigma_1(z) = \sum_{n=1}^{\infty} 3 \, \frac{(3n-1)!}{(n!)^3} (-z)^n = -6z + 45z^2 - 560z^3 + \cdots, \tag{2.57}$$

$$\sigma_2(z) = \sum_{n=1}^{\infty} 18 \left(\sum_{k=n+1}^{3n-1} \frac{1}{k} \right) \frac{(3n-1)!}{(n!)^3} (-z)^n = -18z + \frac{423}{2} z^2 - 2972z^3 + \cdots. \tag{2.58}$$

$X^{(1)}(z)$ with that normalization provides the mirror map that relates the moduli spaces of the mirror geometries around the large-radius point. A more complicated combination produces the derivative of the prepotential.

$$T = X^{(1)}, \tag{2.59}$$

$$-9 \, \partial_T F_0^{[\mathrm{LR}](0)} = T_D = \frac{3}{2} X^{(1,1)} - \frac{3}{2} X^{(1)} + \frac{3}{4}. \tag{2.60}$$

See [116, 118] for more details. An important piece of information we can calculate from these expressions is the Yukawa coupling, which appears explicitly in expressions necessary to integrate the holomorphic anomaly equations, (2.36), (2.40), (2.41). We find

$$C_{TTT} = (2\pi i)^3 \frac{\partial^3 F_0^{[\mathrm{LR}](0)}}{\partial T^3} = -\frac{1}{3} + \mathcal{O}(z), \tag{2.61}$$

$$C_{zzz} = \left(\frac{\partial T}{\partial z} \right)^3 C_{TTT} = -\frac{1}{3z^3(1+27z)}. \tag{2.62}$$

The Yukawa coupling C_{zzz} is a holomorphic rational globally defined function in moduli space. The pole at the conifold point is a general feature.

Around the conifold point $z = -1/27$ we can do an analogous exercise. It is convenient to use a coordinate centered at the special point, $\Delta = 1 + 27z$. In this case we find a regular solution at $\Delta = 0$ and a logarithmic one. They are analytic continuations of the solutions (2.59) and (2.60) in the following way

$$t_c = \Delta + \frac{11}{18}\Delta^2 + \frac{109}{243}\Delta^3 + \cdots = -\frac{2\pi}{\sqrt{3}} T_D, \tag{2.63}$$

$$t_{cD} = \partial_{t_c} F_0^{[\mathrm{c}](0)} = -T_D \log \Delta + \mathcal{O}(\Delta^0) = \frac{4\pi^2 i}{3\sqrt{3}} T. \tag{2.64}$$

The flat coordinate around the conifold point t_c will be very important when we describe the large-order of perturbative free energies in chapter 4.

The last special point, the orbifold $z^{-1} = 0$, is interesting because the power series solutions can be immediately identified as hypergeometric functions. Around this point it is useful to use the coordinate ψ given in equation (2.53). The flat coordinate and the derivative of the prepotential are given by

$$\sigma = 3\alpha\psi \; {}_3F_2\left(\frac{1}{3},\frac{1}{3},\frac{1}{3};\frac{2}{3},\frac{4}{3}\middle|\,\psi^3\right), \tag{2.65}$$

$$\partial_\sigma F_0^{[\mathrm{orb}](0)} = \frac{1}{6}\left(3\alpha\psi\right)^2 \; {}_3F_2\left(\frac{2}{3},\frac{2}{3},\frac{2}{3};\frac{4}{3},\frac{5}{3}\middle|\,\psi^3\right), \tag{2.66}$$

with $\alpha = (-1)^{1/3}$. The periods computed around the special points are all related to each other by analytic continuation. Since the expressions above are valid in all of the ψ plane we can always write closed expressions for any of the other periods in terms of σ and $\partial_\sigma F_0^{[\mathrm{orb}](0)}$. For future use we show the closed expression for the large-radius and conifold flat coordinates,

$$T = -\frac{1}{2\pi i}\frac{\sqrt{3}}{2\pi} G_{33}^{22}\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} & 1\\ 0 & 0 & 0\end{matrix}\middle|\,-\frac{1}{\psi^3}\right), \tag{2.67}$$

$$t_c = \frac{2\pi}{\sqrt{3}}\left(\frac{3\psi}{\Gamma\left(\frac{2}{3}\right)^3}\;{}_3F_2\left(\frac{1}{3},\frac{1}{3},\frac{1}{3};\frac{2}{3},\frac{4}{3}\middle|\,\psi^3\right) - \frac{\frac{9}{2}\psi^2}{\Gamma\left(\frac{1}{3}\right)^3}\;{}_3F_2\left(\frac{2}{3},\frac{2}{3},\frac{2}{3};\frac{4}{3},\frac{5}{3}\middle|\,\psi^3\right) - 1\right), \tag{2.68}$$

where G_{33}^{22} is a Meijer function. In the next subsection we are going to make explicit use of the periods in order to derive an expression for the holomorphic limit of the genus-one free energy. This will provide an expression for the holomorphic limit of the propagator.

2.4.2 Genus-one free energy and holomorphic propagator

In sections 2.3.2 and 2.3.3 we saw the general expression for the genus-one free energy, $F_1^{(0)}$, and its relation to the propagator. For the computation of higher-genus free energies we do not need to know the explicit expression of $F_1^{(0)}$, but its holomorphic limit is going to determine the holomorphic limit of the propagator. We will use that value to fix the holomorphic ambiguities in the integration process and later in chapter 4. In section 2.3.4 we mentioned briefly that to fix the holomorphic ambiguity $f(z)$ appearing in the general expression for $F_1^{(0)}$, (2.31), we need to use the known behavior of the holomorphic limit of the free energy near the special points of the geometry. Near the large-radius point, the following condition must be satisfied [23],

$$\lim_{z\to 0}\mathcal{F}_1^{[\mathrm{LR}](0)} = \lim_{z\to 0} -\frac{1}{24}\,2\pi i T\int_M c_2 J, \tag{2.69}$$

where, for local $\mathbb{CP}^2$, $\int_M c_2 J = -2$. A useful ansatz for $f(z)$ is $\Delta^r\, z^b$. Universal behavior at the conifold point dictates that $r = 1/12$, always. This means, using the expression (2.59), that $b = \frac{7}{12}$, so

$$f(z) = (1+27z)^{1/12}\, z^{7/12}. \tag{2.70}$$

Finally, using the fact that the metric $G_{z\bar{z}}$ is proportional to $\partial_z T$ in the holomorphic limit [20], we find

$$\mathcal{F}_1^{[\mathrm{LR}](0)} = -\frac{1}{2}\log\frac{\partial T}{\partial z} - \frac{1}{12}\log z^7\left(1+27z\right), \tag{2.71}$$

where we have omitted any additive constants. From this expression, the closed form of T (2.67), and (2.62) we can write the holomorphic limit of the propagator. For that we need to take equation (2.41) in the holomorphic limit and notice that indices only have one value since the moduli space is one dimensional. We use the notation $i=j=k=z$. At this point we need to completely fix the holomorphic function α_z associated to the ambiguity in the definition of the propagator. Following [116] we set α_z to zero. All in all, we find

$$S^{zz}_{[\mathrm{LR}],\mathrm{hol}} = \frac{2}{C_{zzz}}\left(\frac{1}{12z\left(1+27z\right)} - \frac{{}_2F_1\left(\frac{2}{3},\frac{4}{3},1\,\middle|\,-27z\right)}{6z\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},1\,\middle|\,-27z\right)}\right). \tag{2.72}$$

In the expression (2.71) we have labelled the holomorphic free energy with a reference to the special point near which the limit is taken. This label is necessary due to the nature of the free energies on and off the holomorphic limit. We briefly mentioned the modular symmetry present in the example of local $\mathbb{CP}^2$. The nonholomorphic perturbative free energies are modular functions and can be written as appropriate combinations of modular forms so that the total weight is zero [119, 123]. In this language the propagator is represented by a nonholomorphic modular form $\widehat{E}_2(\tau,\bar{\tau})$, which is the nonholomorphic extension of a quasimodular form called the second Eisenstein series, $E_2(\tau)$. The modular symmetry of the free energies is only present in the full nonholomorphic regime and is spoiled when taking the holomorphic limit. Taking the holomorphic limit is actually associated to the notion of frame, and modular transformations are equivalent to a change of frame. Because holomorphicity breaks modularity to quasimodularity, taking a holomorphic limit carries a frame label. Preferred frames can be associated to the special points in the geometry: the large-radius, conifold, and orbifold points. We use the corresponding labels [LR], [c], and [orb].

The same calculation that we did for the holomorphic limit of the genus-one free energy in the large-radius frame can be carried over to the other frames. This results in expressions analogous to (2.71) but with the flat coordinate T replace by either t_c or σ. For example,

$$\mathcal{F}_1^{[\mathrm{c}](0)} = -\frac{1}{2}\log\frac{\partial t_c}{\partial z} - \frac{1}{12}\log z^7\left(1+27z\right). \tag{2.73}$$

From $\mathcal{F}_1^{[\mathrm{c}](0)}$ and $\mathcal{F}_1^{[\mathrm{orb}](0)}$ we can calculate the corresponding holomorphic values of the propagator,

$$S^{zz}_{[\mathrm{c}],\mathrm{hol}} = \frac{z^2}{2}\left(-1-54z+2\frac{\pi\,P_{2/3}(1+54z)+2\sqrt{3}\,Q_{2/3}(1+54z)}{\pi\,P_{-1/3}(1+54z)+2\sqrt{3}\,Q_{-1/3}(1+54z)}\right), \tag{2.74}$$

$$S^{zz}_{[\mathrm{orb}],\mathrm{hol}} = \frac{z}{54}\left(-27z+\left(1+27z\right)\frac{{}_2F_1\left(\frac{4}{3},\frac{4}{3},\frac{5}{3}\,\middle|\,-\frac{1}{27z}\right)}{{}_2F_1\left(\frac{1}{3},\frac{1}{3},\frac{2}{3}\,\middle|\,-\frac{1}{27z}\right)}\right). \tag{2.75}$$

Here $P_\nu(x)$ and $Q_\nu(x)$ are Legendre functions, see [126], for example.

2.4.3 Higher genus integration

Now that we have calculated the Yukawa coupling (2.62) and know how to take holomorphic limits, we are almost ready to discuss the integration of the holomorphic anomaly equations (2.39). First we need to compute the holomorphic functions f_z^{zz} and $\tilde{f}_{zz}^z$ in

$$D_z S^{zz} = -C_{zzz}\,(S^{zz})^2 + f_z^{zz}, \tag{2.76}$$
$$\Gamma_{zz}^z = -C_{zzz}\,S^{zz} + \tilde{f}_{zz}^z. \tag{2.77}$$

Using the relation $\alpha_z = -\frac{1}{2}\tilde{f}_{zz}^z + \partial_z \log f$, and the values $\alpha_z = 0$ and $f = (1+27z)^{1/12}\,z^{7/12}$, we find

$$\tilde{f}_{zz}^z(z) = -\frac{7+216z}{6z(1+27z)}. \tag{2.78}$$

The function f_z^{zz} is computed by taking the holomorphic limit of (2.77) and using that $(\Gamma_{zz}^z)_{[\mathrm{LR}],\mathrm{hol}} = (\partial_z T)^{-1}\partial_z^2 T$,

$$f_z^{zz}(z) = -\frac{z}{12(1+27z)}. \tag{2.79}$$

Now any covariant derivative on a holomorphic tensor or on the propagator is completely defined.

Let us exemplify the integration of the holomorphic anomaly equations and fixing of the ambiguity with the first $g=2$ perturbative free energy. Its equations is

$$\partial_{S^{zz}} F_2^{(0)} = \frac{1}{2}\left(D_z^2 F_1^{(0)} + \left(\partial_z F_1^{(0)}\right)^2\right), \tag{2.80}$$

which can be integrating in the propagator as soon as the right-hand-side is spelled out as a polynomial in S^{zz}. We need to use (2.76), (2.77) and (2.41). We obtain

$$F_2^{(0)} = C_{zzz}^2\left(\frac{5}{24}\,(S^{zz})^3 - \frac{3z^2}{16}\,(S^{zz})^2 + \frac{z^4}{16}\,S^{zz}\right) + f_2^{(0)}(z). \tag{2.81}$$

The function $f_2^{(0)}(z)$ is the holomorphic ambiguity. The rest of the expression (2.81) is completely determined and polynomial in the propagator. To fix the ambiguity we need to use the values of the free energy near the large-radius and conifold points, in the holomorphic limit, and the meromorphic properties as a function on the complex structure moduli space. Due to the presence of the Yukawa coupling in (2.81) and the regularity of $S^{zz}_{[\mathrm{c}],\mathrm{hol}}$ at the conifold point, $\mathcal{F}_2^{(0)}$ has a pole at $z=-1/27$. This is actually the only pole of $\mathcal{F}_2^{(0)}$ as a function in moduli space, no matter the frame. However the details of the singularity do not match the universal behavior described by the gap condition [24, 93, 127–131],

$$\mathcal{F}_g^{[\mathrm{c}](0)} = \frac{\mathfrak{c}^{g-1}\,B_{2g}}{2g\,(2g-2)\,t_{\mathrm{c}}^{2g-2}} + \mathcal{O}(t_{\mathrm{c}}^0), \qquad g \geq 2. \tag{2.82}$$

B_{2g} are the Bernoulli numbers, and with our normalizations $\mathfrak{c}=3$. Equation (2.82) describes a pole of order 2 for $g=2$ with a given residue and then a regular tail. Recall that t_c given by (2.68) vanishes at the conifold point. To agree with this behavior near the conifold, and imposing that no other point is singular, leads to the ansatz

$$f_g^{(0)}(z) = \frac{p_{g,0}^{(0)}(z)}{\Delta^{2g-2}}, \tag{2.83}$$

where $\Delta = 1 + 27z$. $p_{g,0}^{(0)}$ has to be a polynomial in z of degree $2g-2$ to have regularity at the orbifold point. All of the coefficients of $p_{g,0}^{(0)}$ are fixed by the gap condition. The remaining one is computed by comparing with the behavior of the free energy at the large-radius point. It is given by the constant map contribution [24, 93, 129, 132, 133],

$$\mathcal{F}_g^{[\mathrm{LR}](0)} = \frac{(-1)^{g-1}\chi\, B_{2g-2}B_{2g}}{4g\,(2g-2)\,(2g-2)!} + \mathcal{O}(z), \qquad g \geq 2, \tag{2.84}$$

where $\chi = 3$ for local $\mathbb{CP}^2$. Putting everything together one can finally calculate the value of the holomorphic ambiguity for $g = 2$,

$$f_2^{(0)}(z) = C_{zzz}^2\, z^6\, \frac{729z^2 + 162z - 11}{1920}. \tag{2.85}$$

One can proceed recursively and compute higher-genus free energies, fixing the holomorphic ambiguities in the same way. In general [112], the free energy $F_g^{(0)}$ is a polynomial in the propagator S^{zz} of degree $3g-3$ whose coefficients are rational functions of z.

The procedure we have reviewed above can be automatized and put on a computer. Knowing the general structure of the solution can accelerate the computation significantly. Fixing the ambiguity requires knowing the expansion of t_c as a powers series in z and inverting this relation, $z = z(t_c)$. This can be very time consuming. In order to thoroughly analyze the large-order growth of the perturbative sector of the topological string free energy, we need to be have data for very high genus. For the case of local $\mathbb{CP}^2$ we computed free energies, $F_g^{(0)}$, up to $g = 114$. For some large-order calculations we do not need to use such high genus because numerical convergence with good precision is obtained long before genus 114. However, for other quantities, especially when analyzing the contribution of higher instanton sectors, having such a number of free energies is helpful.

Chapter 3

Nonperturbative holomorphic anomaly equations

3.1 Introduction

The holomorphic anomaly equations [24] constitute a tower of differential equations in the complex structure moduli and the propagators. As such they have no reference to the string coupling constant, g_s. It is only when we construct the perturbative free energy as a generating function that g_s appears,

$$F^{(0)}(g_s; z^i, S^{ij}) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{(0)}(z^i, S^{ij}). \tag{3.1}$$

The result is an asymptotic series in the small parameter, g_s. The goal in this chapter is to move from this perturbative series to a resurgent transseries, always at the formal level. For this we need to find the generalizations of the holomorphic anomaly equations for the higher instanton free energies. The idea is to write an equation for the perturbative series involving not only the moduli and propagators but also g_s. Such master equation was already considered back in [24, 134]. The next step is to assume that the same equations will generate a full transseries solution. To justify this important step we can use an analogy with an ordinary differential equation. If we try a power series solution on this example equation we will typically find a recursive relation for the coefficients that determines the perturbative solution. The recursive relation is analogous to the holomorphic anomaly equations. On the other hand, given that same relation it is not too difficult to reconstruct the original equation. But this equation may admit a family of solutions in the form of a transseries ready to be discovered with a more general ansatz. This is what we do in this chapter. The analogy is not quite perfect because the holomorphic anomaly equations are not equations in the transseries variable g_s. This is a consequence of the ample domain of validity of the equations but results in a restriction to their computational power. There are holomorphic ambiguities at the perturbative level and there will be analogous ones in the transseries.

In this chapter we construct various transseries solutions and describe their structure. We study the general consequences of the asymptotic nature of the perturbative sector and make use of it to deal with the problem of the holomorphic ambiguities. This chapter is based on the results presented in [135].

3.2 Rewriting the holomorphic anomaly equations

The holomorphic anomaly equations of [24] are a tower of infinite recursive equations for the perturbative free energies in topological string theory. Since we want to go past perturbation series into a transseries we can ask the question of whether there is information in the perturbative equations that knows about nonperturbative sectors. From the point of view of resurgence this should not be a surprise. A recurrent motive is that each sector of a resurgent system knows about all others, although in a codified form. Our goal is to write a single equation which we know is valid at the perturbative level, because it reproduces the holomorphic anomaly equations (2.39), but such that it admits a full transseries ansatz.

The existence of such equation was already proposed originally in [23] under the name of master equation. This would be a single equation for the perturbative free energy, or rather, the partition function $Z = \exp F^{(0)}$. An explicit master equation was put forward in [134], in the form

$$\left(\partial_{\bar{i}} - \frac{1}{2} g_s^2 \, \bar{C}_{\bar{i}}^{\,jk} D_j D_k\right) Z = 0. \tag{3.2}$$

As it stands, equation (3.2) does not quite reproduce all the details of the holomorphic anomaly equations. The resulting quadratic part of the equations for the free energies do not have the appropriate lower and upper limits for the sum for h ($h = 1$ to $h = g - 1$, see (2.39)), necessary for a proper recursion. The explicit master equation considered in [24] focuses not on the full free energy but on $\widehat{F} := \sum_{g=1}^{\infty} g_s^{2g-2} \, F_g^{(0)}$. Note that $F_0^{(0)}$ is not included. The equation

$$\left(\partial_{\bar{i}} - \partial_{\bar{i}} F_1^{(0)}\right) \mathrm{e}^{\widehat{F}} = \frac{1}{2} g_s^2 \, \bar{C}_{\bar{i}}^{jk} D_j D_k \mathrm{e}^{\widehat{F}}, \tag{3.3}$$

reproduces the holomorphic anomaly equations for $g \geq 2$. Leaving out the genus-zero free energy $F_0^{(0)}$ can be justified noting that its role in the equation is achieved only through the Yukawa couplings. In the context of matrix models and their relations to topological string theories, [22] considered the equation

$$\frac{1}{\widetilde{Z}} \, \partial_{\bar{i}} \widetilde{Z} = \frac{1}{2} g_s^2 \, \frac{1}{\widehat{Z}} \, \bar{C}_{\bar{i}}^{jk} D_j D_k \widehat{Z}, \tag{3.4}$$

where the role of g_s was identified with N^{-1} in the language of matrix models, up to the 't Hooft coupling, and

$$Z = \exp \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{(0)}, \qquad \widetilde{Z} = \mathrm{e}^{-\frac{1}{g_s^2} F_0^{(0)} - F_1^{(0)}} \, Z, \qquad \widehat{Z} = \mathrm{e}^{-\frac{1}{g_s^2} F_0^{(0)}} \, Z, \tag{3.5}$$

Note how it is necessary here to separate $F_0^{(0)}$ and also $F_1^{(0)}$. The genus-one free energy $F_1^{(0)}$ enters the holomorphic anomaly equations through its derivative, and this can be written in terms of the propagator (2.41). From this point of view, both $F_0^{(0)}$ and $F_1^{(0)}$ are in some sense geometrical initial data for the recursive holomorphic anomaly equations.

(3.4) is already the equation we are looking for. It is constructed out of the tower of perturbative holomorphic anomaly equations but it may admit a transseries ansatz. To be able to use such ansatz directly we need to rewrite equation (3.4) in terms of Z alone. This means that $\widehat{Z}$ and $\widetilde{Z}$ will disappear at the cost of having $F_0^{(0)}$ and $F_1^{(0)}$ explicitly in the equations. As we mentioned, we are going to think of these perturbative free energies as initial data that set the recursion in motion, even at the transseries level, and are present in the form of the Yukawa coupling and the propagator, respectively[1].

Since we are going to work entirely in the language of propagators, S^{ij}, we use these antiholomorphic variables to write the master equation,

$$\left(\frac{\partial}{\partial S^{ij}} + \frac{1}{2}\left(U_i D_j + U_j D_i\right) - \frac{1}{2} g_s^2\, D_i D_j\right) Z = \left(\frac{1}{g_s^2}\, W_{ij} + V_{ij}\right) Z. \tag{3.6}$$

Equation (3.6) has the usual terms involving an antiholomorphic derivative and two covariant derivatives. Together they resemble a generalized heat equation for the partition function. See [134, 136, 137] for an exploration of the possibility of Z being a theta function. The extra terms in (3.6), involving U_i, V_{ij}, and W_{ij}, are related directly to $F_0^{(0)}$ and $F_1^{(0)}$. Their presence ensures that plugging $Z = \exp \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{(0)}$ reproduces the holomorphic anomaly equations and nothing more, see below. Using the relation $F^{(0)} = \log Z$ we can write the master equation for the perturbative free energy

$$\frac{\partial F^{(0)}}{\partial S^{ij}} + \frac{1}{2}\left(U_i D_j F^{(0)} + U_j D_i F^{(0)}\right) - \frac{1}{2} g_s^2 \left(D_i D_j F^{(0)} + D_i F^{(0)} D_j F^{(0)}\right) = \frac{1}{g_s^2}\, W_{ij} + V_{ij}. \tag{3.7}$$

The extra step now requires assuming that this equation will give us relevant information of the nonperturbative sector if $F^{(0)}$ is promoted from a perturbative series in g_s, to a full transseries, like the ones we saw in chapter 1. We will explore the consequences of this in the rest of the chapter.

Let us conclude this section by computing the values of U_i, V_{ij}, and W_{ij} that reproduce the holomorphic anomaly equations when a perturbative ansatz for F is used,

$$F^{(0)} = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{(0)}. \tag{3.8}$$

Let us stress that U_i, V_{ij}, and W_{ij} are functions of the complex structure moduli and the propagator, but not of g_s. The quadratic part of equation (3.7) gives

$$D_i F^{(0)} D_j F^{(0)} = \sum_{g=0}^{\infty} g_s^{2g-4} \sum_{h=0}^{g} D_i F_{g-h}^{(0)} D_j F_h^{(0)}. \tag{3.9}$$

[1] We will see that the instanton actions are given in terms of periods, so one can think of the genus-zero free energy as being present through them as well.

Note that the sum in h does not have the appropriate upper and lower limits. The function U_i will make sure that the extra two terms $h = 0$ and $h = g$ are removed. If we collect similar powers of g_s in the equation we find

$$\begin{aligned}
&\frac{1}{g_s^2}\left(\frac{\partial F_0^{(0)}}{\partial S^{ij}} + \frac{1}{2}U_i D_j F_0^{(0)} + \frac{1}{2}U_j D_i F_0^{(0)} - \frac{1}{2}D_i F_0^{(0)} D_j F_0^{(0)}\right)\\
&+ g_s^0\left(\frac{\partial F_1^{(0)}}{\partial S^{ij}} + \frac{1}{2}U_i D_j F_1^{(0)} + \frac{1}{2}U_j D_i F_1^{(0)} - \frac{1}{2}D_i D_j F_0^{(0)} - \frac{1}{2}D_i F_0^{(0)} D_j F_1^{(0)} - \frac{1}{2}D_i F_1^{(0)} D_j F_0^{(0)}\right)\\
&+ \sum_{g=2}^{\infty} g_s^{2g-2}\left(\frac{\partial F_g^{(0)}}{\partial S^{ij}} + \frac{1}{2}U_i D_j F_g^{(0)} + \frac{1}{2}U_j D_i F_g^{(0)} - \frac{1}{2}D_i D_j F_{g-1}^{(0)} - \frac{1}{2}\sum_{h=0}^{g} D_i F_h^{(0)} D_j F_{g-h}^{(0)}\right)\\
&= \frac{1}{g_s^2}W_{ij} + g_s^0 V_{ij}.
\end{aligned} \tag{3.10}$$

In the third line, choosing the right value of U_i will reproduce exactly the holomorphic anomaly equations (2.39). On the other hand, the first two lines must be taken care of by W_{ij} and V_{ij}, so that no other equations remain. The correct values for the functions are

$$U_i = D_i F_0^{(0)}, \tag{3.11}$$

$$V_{ij} = \frac{\partial F_1^{(0)}}{\partial S^{ij}} - \frac{1}{2}D_i D_j F_0^{(0)}, \tag{3.12}$$

$$W_{ij} = \frac{\partial F_0^{(0)}}{\partial S^{ij}} + \frac{1}{2}D_i F_0^{(0)} D_j F_0^{(0)}. \tag{3.13}$$

The genus-zero free energy is holomorphic so the first term in (3.13) is actually zero.

Equation (3.7) must be promoted to an equation for the full nonperturbative free energy F,

$$\frac{\partial F}{\partial S^{ij}} + \frac{1}{2}\left(U_i D_j F + U_j D_i F\right) - \frac{1}{2}g_s^2\left(D_i D_j F + D_i F D_j F\right) = \frac{1}{g_s^2}W_{ij} + V_{ij}. \tag{3.14}$$

Note that this is not an equation in g_s, but an equation in the complex structure moduli and the propagators. This is a major difference with respect to the usual differential equations studied in the framework of resurgence, in which the independent variable of the equation coincides with the small parameter of the transseries. Such small parameter is g_s here, but the equation is not in g_s. This has the important consequence that the equation will not be able to fix all the ingredients of the transseries. Just as in the perturbative situation, there will be holomorphic ambiguities, in the sense of quantities that are not computed by the holomorphic anomaly equations and must be fixed in another way. This is not really a surprise since the holomorphic anomaly equations must be valid for all geometries, and although there is geometrical input from the start, we should not expect it to be enough. For the nonperturbative situation, the resurgent properties of F and the analysis of the large-order of the perturbative and other sectors, will be determinant to provide a way to fix the holomorphic ambiguities.

Finally, since our working example, local $\mathbb{CP}^2$, has a one-dimensional moduli space, and a single propagator S^{zz}, the master equation (3.14) simplifies to

$$\partial_{S^{zz}} F + U\, D_z F - \frac{1}{2} g_s^2 \left(D_z D_z F + (D_z F)^2 \right) = \frac{1}{g_s^2} W + V. \tag{3.15}$$

At this point in the discussion one may object against the importance that we have put on the particular form of (3.14), and wonder if there are other possibilities for the equation that F should satisfy. The first reason for the use of this particular equation is that its solutions are able to reproduce every number obtained from a large-order analysis in the example of local $\mathbb{CP}^2$ presented in chapter 4. This means that any possible extensions must be compatible with what is obtained there. A possible class of extensions the inclusion of exponentially suppressed terms in g_s. This is, however, unnatural because the instanton actions of those terms should be precisely tuned to match the instanton actions in each particular model. Another variation would try to reproduce the perturbative holomorphic anomaly equations not only for $g \geq 2$, but also $g = 1$ or even $g = 0$. Due to the different nature of the free energies for $g < 2$ and $g \geq 2$ this approach is more difficult to implement, and we were not able to construct such an equation. At the end of the day it must be resurgence the one who decides what master equation for F is the correct one.

3.3 One-parameter transseries

3.3.1 Tower of equations and instanton action

In this section we work out the example of a transseries ansatz with one parameter, that is, depending on a single instanton action A. Although this not a very realistic example, because one expects several instanton actions and resonance to play a role, it is useful to explore as a first case. Some sectors of a multiparameter transseries are effectively one-parameter transseries, and many properties of the solutions can be generalized for the second case. Indeed, in a multiparameter transseries with instanton sectors labelled by $(n_1|\cdots|n_\alpha|\cdots|n_p)$, the sectors of the form $(0|\cdots|n_\alpha|\cdots|0)$ satisfy equations involving only kA_α with $k < n_\alpha$ and no other sectors. To simplify the analysis we will assume the complex structure moduli space to be one-dimensional. This is the case for local $\mathbb{CP}^2$. See appendix A for the general case.

The equation we need to solve is (3.15), and the ansatz for the free energy is

$$F(\sigma, g_s) = \sum_{n=0}^{\infty} \sigma^n \, \mathrm{e}^{-A^{(n)}/g_s} \, F^{(n)}(g_s). \tag{3.16}$$

This is a transseries like (1.17) where g_s plays the role of x, the small parameter. σ is a constant (independent of g_s, z and S^{zz}) that will help us keep track of the instanton sector, n. The sector n has total instanton action $A^{(n)} := n\, A$. The instanton action A is, in principle, a function $A = A(z, S^{zz})$, although an important result is that it is actually independent of the propagator, hence holomorphic. Around each instanton sector there is a perturbative expansion

$$F^{(n)}(g_s) = \sum_{g=0}^{\infty} g_s^{g+b^{(n)}} F_g^{(n)}(z,S). \tag{3.17}$$

As we mentioned in section 1.3.1, $b^{(n)}$ is an starting power that must be allowed for full generality. Plugging the ansatz (3.16) into (3.15) requires the computation of

$$D_z F = \sum_n \sigma^n \, \mathrm{e}^{-A^{(n)}/g_s} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n)} \right) F^{(n)}, \tag{3.18}$$

$$D_z^2 F = \sum_n \sigma^n \, \mathrm{e}^{-A^{(n)}/g_s} \left(D_z - \frac{1}{g_s} \partial_z A^{(n)} \right) \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n)} \right) F^{(n)}, \tag{3.19}$$

$$(D_z F)^2 = \sum_n \sigma^n \, \mathrm{e}^{-A^{(n)}/g_s} \sum_{m=0}^{n} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(m)} \right) F^{(m)} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n-m)} \right) F^{(n-m)}, \tag{3.20}$$

where we have used that $A^{(m)} + A^{(n-m)} = A^{(n)}$. Collecting similar powers of σ we find

$$\begin{aligned} \sum_n \sigma^n \, \mathrm{e}^{-A^{(n)}/g_s} \Bigg\{ & \left(\partial_{S^{zz}} - \frac{1}{g_s} \partial_{S^{zz}} A^{(n)} \right) F^{(n)} + U \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n)} \right) F^{(n)} \\ & - \frac{1}{2} g_s^2 \left(D_z - \frac{1}{g_s} \partial_z A^{(n)} \right) \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n)} \right) F^{(n)} \qquad (3.21) \\ & - \frac{1}{2} g_s^2 \sum_{m=0}^{n} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(m)} \right) F^{(m)} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n-m)} \right) F^{(n-m)} \\ & - \delta_{n,0} \left(\frac{1}{g_s^2} W + V \right) \Bigg\} = 0. \qquad (3.22) \end{aligned}$$

For $n = 0$, the perturbative sector, we recover the holomorphic anomaly equations once the values (3.11), (3.12) and (3.13) are used. Let us focus on the nonperturbative sectors, $n > 0$. If we collect together all the terms in involving $F^{(n)}$ we find

$$\begin{aligned} & \left(\partial_{S^{zz}} - \frac{1}{g_s} \partial_{S^{zz}} A^{(n)} \right) F^{(n)} - \frac{1}{2} g_s^2 \left(D_z - \frac{1}{g_s} \partial_z A^{(n)} + 2 \, \partial_z \widehat{F}^{(0)} \right) \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n)} \right) F^{(n)} \\ & = \frac{1}{2} g_s^2 \sum_{m=1}^{n-1} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(m)} \right) F^{(m)} \left(\partial_z - \frac{1}{g_s} \partial_z A^{(n-m)} \right) F^{(n-m)}. \end{aligned} \tag{3.23}$$

Here we have defined $\partial_z \widehat{F}^{(0)}(g_s) := \partial_z F^{(0)}(g_s) - \frac{1}{g_s} U$. Notice that the right-hand-side of (3.23) depends exclusively on lower instanton sectors and comes from the quadratic term in (3.15). Because of the instanton action, the covariant derivatives and antiholomorphic derivatives pick up an extra term due to

$$\partial_z^k \left(\mathrm{e}^{-A^{(n)}/g_s} F^{(n)} \right) = \mathrm{e}^{-A^{(n)}/g_s} \left(\partial_z - \frac{1}{g_s} \, \partial_z A^{(n)} \right)^k F^{(n)}, \qquad k = 1, 2, \ldots. \tag{3.24}$$

The left-hand-side of (3.23) can be compactly written in terms of the g_s-dependent operator

$$\mathcal{D}^{(n)}(g_s) := \partial_{S^{zz}} - \frac{1}{g_s}\partial_{S^{zz}}A^{(n)} - \frac{1}{2}g_s^2\left(D_z - \frac{1}{g_s}\partial_z A^{(n)} + 2\,\partial_z\widehat{F}^{(0)}\right)\left(\partial_z - \frac{1}{g_s}\partial_z A^{(n)}\right). \quad (3.25)$$

The presence of $\widehat{F}^{(0)}(g_s)$ makes the operator into a formal power series

$$\mathcal{D}^{(n)}(g_s) = \sum_{g=-1}^{\infty} g_s^g\,\mathcal{D}_g^{(n)}, \quad (3.26)$$

where $\mathcal{D}_{-1}^{(n)} = -\partial_{S^{zz}}A^{(n)}$, and

$$\mathcal{D}_0^{(n)} = \partial_{S^{zz}} - \frac{1}{2}\left(\partial_z A^{(n)}\right)^2, \quad (3.27)$$

$$\mathcal{D}_1^{(n)} = \frac{1}{2}\,D_z^2 A^{(n)} + \partial_z A^{(n)}\left(\partial_z + \partial_z F_1^{(0)}\right), \quad (3.28)$$

$$\mathcal{D}_2^{(n)} = -\frac{1}{2}\,D_z^2 - \partial_z F_1^{(0)}\,D_z, \quad (3.29)$$

$$\mathcal{D}_{2h-1}^{(n)} = \partial_z A^{(n)}\,\partial_z F_h^{(0)}, \qquad h = 2,3,\ldots, \quad (3.30)$$

$$\mathcal{D}_{2h}^{(n)} = -\partial_z F_h^{(0)}\,\partial_z, \qquad h = 2,3,\ldots. \quad (3.31)$$

Notice the presence of the perturbative free energies. The power series $\mathcal{D}^{(n)}(g_s)$ is formally multiplied by the g_s-expansion of $F^{(n)}$ (3.17),

$$\sum_{g=-1}^{\infty} g_s^g\left\{-\partial_{S^{zz}}A^{(n)}F_{g+1}^{(n)} + \sum_{h=0}^{g}\mathcal{D}_h^{(n)}F_{g-h}^{(n)}\right\} = \quad (3.32)$$
$$= \sum_{m=1}^{n-1}\sum_{g=0}^{\infty} g_s^{g+B(n,m)}\,\frac{1}{2}\sum_{h=0}^{g}\left(\partial_z F_{h-1}^{(m)} - \partial_z A^{(m)}F_h^{(m)}\right)\left(\partial_z F_{g-1-h}^{(n-m)} - \partial_z A^{(n-m)}F_{g-h}^{(n-m)}\right).$$

Having collected similar powers of g_s leads to the definition of $B(n,m) := b^{(m)} + b^{(n-m)} - b^{(n)}$. At the end of last section we mentioned that due to the fact that the equations are not in g_s but in the moduli, we would find quantities in the transseries that the equations cannot fix, that is, holomorphic ambiguities. An example of this are the starting powers, $b^{(n)}$, or more precisely, their combination into $B(n,m)$. Even if these values are not fixed, we can see from the equations (3.32) that a very negative value of $B(n,m)$ will imply equations in which no $F_g^{(n)}$ or propagator derivative are present. This is because the left-hand-side will be zero but not the right-hand-side. This is not necessarily an inconsistency because such equations could simply realize symmetry constraints of the particular geometry and be satisfied by themselves. Notice that the right-hand-side depends on lower instanton sector, which in the recursive process of integration should be already computed. Therefore, there is no problem with integrability properties of the equations. From now on we will assume that $B(n,m) \geq 0$ for all n and m, and we will sometimes use the example $B(n,m) = 1$ to give an explicit example.

If we collect all the terms with the same power of g_s we find the tower of holomorphic anomaly equations for the higher sectors of the transseries

$$\begin{aligned}&\partial_{S^{zz}} A^{(n)} F^{(n)}_{g+1} - \sum_{h=0}^{g} \mathcal{D}^{(n)}_h F^{(n)}_{g-h} + \\ &\qquad + \frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=0}^{g-B} \left(\partial_z F^{(m)}_{h-1} - \partial_z A^{(m)} F^{(m)}_h \right) \left(\partial_z F^{(n-m)}_{g-1-B-h} - \partial_z A^{(n-m)} F^{(n-m)}_{g-B-h} \right) = 0,\end{aligned} \tag{3.33}$$

for $g = -1, 0, 1, 2, \ldots$ In the second sum over h we use the convention that $F^{(m)}_h$ is zero if $h < 0$. The first equation in (3.33), for $g = -1$ and any sector n, is simply

$$\partial_{S^{zz}} A^{(n)} F^{(n)}_0 = 0. \tag{3.34}$$

Since by definition $F^{(n)}_0 \neq 0$ (otherwise we change $b^{(n)}$ by one), A must be independent of the propagator,

$$\partial_{S^{zz}} A = 0. \tag{3.35}$$

This means that the instanton action $A = A(z)$ is holomorphic. This is a very important result because it tells us that we can maintain the geometrical interpretation of the instanton action as an object computed from the geometry, and dependent only on the moduli space in a holomorphic way. In matrix models, which are closely related to topological strings, some instanton actions can be interpreted a semiclassical way as the values of the action for configurations that take eigenvalues from one saddle point to another [44, 138, 139]. The expressions for these instanton actions can then be written as integrals over cycles of the spectral curve describing the matrix model in the large N limit of the theory [30, 31, 36, 140–143]. The spectral curve can be identified with the Riemann surface embedded in the geometry of toric Calabi–Yau manifolds [97]. The cycle integrals are nothing but the periods of the geometry. One can move beyond the context of matrix models, but maintaining the spectral curve interpretation, and conjecture that instanton action in topological string theories must be combinations of periods of the Calabi–Yau geometry, which are holomorphic quantities [143]. This interpretation of instanton actions as periods is crucial to obtain an explicit expression for A, because from the point of view of the holomorphic anomaly equations the instanton action is a holomorphic ambiguity. It is only fixed up to a holomorphic quantity, that is the whole function itself.

Using the holomorphicity of A back into (3.33) we write the final set of equations

$$\begin{aligned}&\left(\partial_{S^{zz}} - \frac{1}{2} \left(\partial_z A^{(n)} \right)^2 \right) F^{(n)}_g = \\ &= - \sum_{h=1}^{g} \mathcal{D}^{(n)}_h F^{(n)}_{g-h} + \frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=0}^{g-B} \left(\partial_z F^{(m)}_{h-1} - \partial_z A^{(m)} F^{(m)}_h \right) \left(\partial_z F^{(n-m)}_{g-1-B-h} - \partial_z A^{(n-m)} F^{(n-m)}_{g-B-h} \right),\end{aligned} \tag{3.36}$$

where we have explicitly written the form of the operator $\mathcal{D}^{(n)}_0$ on the left-hand-side. These equations are valid for $n > 1$ and $g \geq 0$, and generalize the perturbative holomorphic anomaly equations of [24]. They are recursive: in order to compute $F^{(n)}_g$ we need to know all the previous free energies in the same instanton sector n, a finite number of lower instanton free energies including the perturbative sector. The equations are second order in z, due to $\mathcal{D}_2^{(n)}$ (for $g \geq 2$) and first order in the propagator due to the left-hand-side of (3.36).

3.3.2 Structure of the transseries solution

Now that the holomorphic anomaly equations for higher instanton free energies have been computed (3.36) we can describe the integration procedure and the antiholomorphic dependence of the solutions. In the same way we reviewed the structure of the perturbative free energies in section 2.3.5, here we are going to show, with some more work, what is the propagator dependence of the higher sectors of the transseries. During the integration of the equations we will find the analog of the holomorphic ambiguities, $f_g^{(n)}$.

Let us denote the right-hand-side of equation (3.36) by $G_g^{(n)}(z, S^{zz})$. By induction, this expression has already been computed at instanton level n and coefficient g. The equation for $F_g^{(n)}$ is very simple, and its general solution is

$$F_g^{(n)}(z, S) = \mathrm{e}^{\frac{1}{2}\left(\partial_z A^{(n)}\right)^2 S} \left(f_g^{(n)}(z) + \int^{S^{zz}} \mathrm{d}\tilde{S}\, \mathrm{e}^{-\frac{1}{2}\left(\partial_z A^{(n)}\right)^2 \tilde{S}}\, G_g^{(n)}(z, \tilde{S}) \right). \tag{3.37}$$

The function $f_g^{(n)}(z)$ is the holomorphic ambiguity. We will attack the problem of fixing it in section 3.7. Due to the generalization to depend on the total instanton action we find explicit exponentials that generalize the simple polynomial dependence on the propagators. Because the equations are recursive, this exponential (and others) will be present generically[2] in all free energies.

To have a better idea of what these free energies look like in general, let us focus first on the one-instanton sector, $n = 1$. For this sector, the quadratic term in (3.36) is zero because the sum over m is empty, so

$$G_g^{(1)} = -\sum_{h=1}^{g} \mathcal{D}_h^{(1)} F_{g-h}^{(1)}. \tag{3.38}$$

This means that the one-instanton sector is independent of the values of $B(n, m)$. For $g = 0$, $G_0^{(1)} = 0$ so $F_0^{(1)}$ satisfies

$$\left(\partial_{S^{zz}} - \frac{1}{2} (\partial_z A)^2 \right) F_0^{(1)} = 0, \tag{3.39}$$

and integrates to

$$F_0^{(1)} = \mathrm{e}^{\frac{1}{2}(\partial_z A)^2 S^{zz}}\, f_0^{(1)}(z). \tag{3.40}$$

Notice the exponential dependence in the propagator and the holomorphic ambiguity, $f_0^{(1)}(z)$, which by itself forms a polynomial of degree 0. The next equation is for $g = 1$. Now $G_1^{(1)}$ is not zero but

$$G_1^{(1)} = -\mathcal{D}_1^{(1)} F_0^{(1)}, \tag{3.41}$$

[2] In the multiparameter case some sectors may have total instanton action zero due to resonance. We investigate that case in section 3.5.2.

where $\mathcal{D}_1^{(1)}$ is given by (3.28) with $n = 1$. When putting the explicit expression (3.40) for $F_0^{(1)}$ in (3.41) we obtain the product of the same exponential and a polynomial of degree 2 in S^{zz}. This polynomial appears when the derivative $\mathcal{D}_1^{(1)}$ acts on the propagator sitting in the exponential, and this produces the square of a propagator according to (2.36). If we use $G_1^{(1)}$ in (3.37), the exponentials in the integral cancel and the remaining polynomial has its degree increased by one. So the final form of the $g = 1$, $n = 1$ free energy is

$$F_1^{(1)} = \mathrm{e}^{\frac{1}{2}(\partial_z A)^2 S^{zz}} \mathrm{Pol}(S^{zz}; 3). \tag{3.42}$$

where $\mathrm{Pol}(S^{zz}; d)$ represents a polynomial of degree d in the propagator S^{zz}. We could continue with the calculation of $F_2^{(0)}$ and find that the structure is the same except that the degree for this case is 6. Trying a few more cases we can convince ourselves that the general structure for $F_g^{(1)}$ should be

$$F_g^{(1)} = \mathrm{e}^{\frac{1}{2}(\partial_z A)^2 S^{zz}} \, \mathrm{Pol}\left(S^{zz}; 3g\right). \tag{3.43}$$

To prove this we need to proceed by induction on g as explained in section 2.3.5. However, it is just as easy to extend the proof to the instanton number n on top of g. To apply the proof by induction we need to guess the structure of any free energy $F_g^{(n)}$.

The equations for the two-instanton sector, $n = 2$, depend explicitly on the value of $B(2,1)$. In order to have concrete expression let us set $b^{(1)} = b^{(2)} = 1$ which gives $B(2,1) = 1$. At the end of this section we will provide a general proof for the antiholomorphic structure of the free energies for any values of $B(n,m)$. The free energies $F_g^{(2)}$ comprise now two different exponentials. The first one, $\exp 4\frac{1}{2}(\partial_z A)^2 S^{zz}$, comes from the exponential term in front of (3.37) for $n = 2$. The second one appears due to the quadratic term

$$\sum_{h=0}^{g-1} \left(\partial_z F_{h-1}^{(1)} - \partial_z A^{(1)} F_h^{(1)}\right) \left(\partial_z F_{g-2-h}^{(1)} - \partial_z A^{(1)} F_{g-1-h}^{(n-m)}\right). \tag{3.44}$$

This is the product of two different one-instanton sector each proportional to an exponential $\exp \frac{1}{2}(\partial_z A)^2 S^{zz}$. Together they produce $\exp 2\frac{1}{2}(\partial_z A)^2 S^{zz}$. Accompanying each exponential there are polynomials of a certain degree. After studying the first few cases, one can conjecture

$$F_g^{(2)} = \mathrm{e}^{\frac{1}{2}\,2(\partial_z A)^2 S^{zz}} \, \mathrm{Pol}\left(S; d_1\right) + \mathrm{e}^{\frac{1}{2}\,4(\partial_z A)^2 S} \, \mathrm{Pol}\left(S^{zz}; d_2\right), \tag{3.45}$$

where $d_1 = 3(g + 1 - 2)$ and $d_2 = 3(g + 1 - 1)$. The $+1$ in these degrees is given in the general case by $b^{(2)}$, while 2 and 1 are combinatorial numbers to be described later (they also depend on $b^{(2)}$ and $b^{(1)}$ so the final answer depends on $B(2,1)$ only). We use the convention that a polynomial of negative degree is identically zero.

To guess the structure of the free energies in n we can calculate the first few cases and see that the number of different exponentials will grow. For $n = 3$ and $b^{(3)} = 1$ we find exponentials $\exp a\frac{1}{2}(\partial_z A)^2 S^{zz}$ with $a \in \{3, 5, 9\}$ and polynomials of degrees $d = 3(g+1-\lambda)$, for $\lambda \in \{3, 2, 1\}$, respectively. For general instanton number n, we guess the ansatz

$$F_g^{(n)} = \sum_{\{\gamma_n\}} \mathrm{e}^{\frac{1}{2}a(n;\gamma_n)(\partial_z A)^n S^{zz}} \, \mathrm{Pol}\left(S^{zz}; 3(g + 1 - \lambda(n; \gamma_n))\right). \tag{3.46}$$

Here γ_n is a set of indices the number of which varies depending on n. For $n = 1$ there is only one, for $n = 2$ there are 2, for $n = 7$ there are 13. For each value of n and γ_n there are two combinatorial numbers $a(n;\gamma_n)$ and $\lambda(n;\gamma_n)$ that generalize the examples we showed above. Actually, this is the notation for the specific example of $b^{(n)} = 1$ for all n. In the more general situation we conjecture that

$$F_g^{(n)} = \sum_{\{\gamma_n\}} \mathrm{e}^{\frac{1}{2}\,a(n;\gamma_n)(\partial_z A)^2 S^{zz}}\ \mathrm{Pol}\left(S; 3\left(g + b^{(n)} - \lambda_b\left(n;\gamma_n\right)\right)\right). \tag{3.47}$$

The a-numbers remain the same but the λ ones have to be modified. This expression for the free energy contains the holomorphic ambiguity $f_g^{(n)}$. It sits in the polynomial accompanying biggest exponential, $\exp n^2 \frac{1}{2}(\partial_z A)^2 S^{zz}$, for which $a = n^2$.

The combinatorial coefficients a and λ are determined the structure of the holomorphic anomaly equations (3.36) only. They can be computed, as we prove in theorem 1, from the following generating function,

$$\Phi_b = \prod_{m=1}^{\infty} \frac{1}{1 - \varphi^{b^{(m)}}\, E^{m^2}\, \rho^m}. \tag{3.48}$$

This generating function has three formal variables: φ, E, and ρ. We can expand each term in the product of (3.48) formally using $(1-x)^{-1} = 1 + x + x^2 + \cdots$ and obtain

$$\Phi_b = \prod_{m=1}^{\infty} \sum_{r=0}^{\infty} \varphi^{r b^{(m)}}\, E^{r m^2}\, \rho^{r m}. \tag{3.49}$$

Expanding the infinite product gives, taking care of relabeling $r \to r_m$,

$$\Phi_b = \sum_{r_m} \varphi^{\sum_{m=0}^{\infty} r_m b^{(m)}}\, E^{\sum_{m=0}^{\infty} r_m m^2}\, \rho^{\sum_{m=0}^{\infty} r_m m}. \tag{3.50}$$

Similar powers of ρ are collected by considering the values of $\{r_m\}$ for which $\sum_{m=0}^{\infty} r_m m = n$. This is actually the definition of an integer partition of n. Thus,

$$\Phi_b = \sum_{n=0}^{\infty} \rho^n \sum_{\substack{\{r_m\}:\\ \sum_m r_m m = n}} \varphi^{\sum_{m=0}^{\infty} r_m b^{(m)}}\, E^{\sum_{m=0}^{\infty} r_m m^2}. \tag{3.51}$$

Now we separate the values $\{r_m\}$ within a given partition of n with respect to the values of $\sum_{m=0}^{\infty} r_m m^2$. For each different value

$$a(n;\gamma_n) := \sum_{m=0}^{\infty} r_m m^2 \tag{3.52}$$

we introduce a label γ_n. So for each n, we have the partitions of n, $\{r_m\}$ such that $\sum_m r_m m = n$. Those partitions are grouped in classes with the same value of a. The label γ_n distinguishes between all the possible values of a. In partition language, a is the sum of the squares of the elements of the partition specified by $\{r_m\}$. Now we can write

$$\Phi_b = \sum_{n=0}^{\infty} \rho^n \sum_{\{\gamma_n\}} E^{a(n;\gamma_n)} \sum_{\{r_m\}\in\gamma_n} \varphi^{\sum_{m=0}^{\infty} r_m b^{(m)}}. \tag{3.53}$$

Finally, we focus on the smallest power of φ for each class γ_n and give it a name

$$\lambda_b(n;\gamma_n) := \min_{\{r_m\}\in\gamma_n} \left\{ \sum_{m=0}^{\infty} r_m b^{(m)} \right\}. \tag{3.54}$$

So the generating function can be written as

$$\Phi_b = \sum_{n=0}^{\infty} \rho^n \sum_{\{\gamma_n\}} E^{a(n;\gamma_n)} \varphi^{\lambda_b(n;\gamma_n)}\, \mathcal{O}(\varphi^0). \tag{3.55}$$

The first terms of the series are

$$\begin{aligned}\Phi_b = {} & 1 + \rho\, E\, \varphi^{b^{(1)}} + \rho^2 \left(E^2\, \varphi^{2b^{(1)}} + E^4\, \varphi^{b^{(2)}} \right) + \rho^3 \left(E^3\, \varphi^{3b^{(1)}} + E^5\, \varphi^{b^{(1)}+b^{(2)}} + E^9\, \varphi^{b^{(3)}} \right) \\ & + \rho^4 \left(E^4\, \varphi^{4b^{(1)}} + E^8\, \varphi^{2b^{(2)}} + E^6\, \varphi^{2b^{(1)}+b^{(2)}} + E^{10}\, \varphi^{b^{(1)}+b^{(3)}} + E^{16}\, \varphi^{b^{(4)}} \right) + \cdots \end{aligned} \tag{3.56}$$

The numbers a and λ_b and the label γ_n that we have defined after expanding the generating function are the same as the ones appearing in (3.47). We prove this in theorem 1. So in order to know the propagator structure of a given instanton sector we just need to expand the generating function to the appropriate order, n, and read from the powers of E the coefficients in the exponentials, $a(n;\gamma_n)$, and from the powers of φ the coefficients $\lambda_b(n;\gamma_n)$ that determine the degree of the corresponding polynomial. Since the holomorphic anomaly equations only depend on $b^{(n)}$ through the combination $B(n,m)$ the description should only depend on this function. See (3.75) in the footnote in page 65.

From the general discussion of the generating function and its interpretation in terms of integer partitions we can see that for each n there is a special class $\widehat{\gamma}_n$ whose only element is given by $r_m = \delta_{m,n}$. Equivalently $a(n;\widehat{\gamma}_n) = n^2$. Also, for this class $\lambda_b(n;\widehat{\gamma}_n) = b^{(n)}$. This is the class to which the holomorphic ambiguity, $f_g^{(0)}$, belongs. It will have a special role in the proof of theorem 1.

Theorem 1. *For any $n \geq 1$ and $g \geq 0$, the structure of the nonperturbative free energies has the form*

$$F_g^{(n)} = \sum_{\{\gamma_n\}} \mathrm{e}^{\frac{1}{2}\, a(n;\gamma_n)(\partial_z A)^2 S^{zz}}\, Pol\left(S; 3\left(g + b^{(n)} - \lambda_b\left(n;\gamma_n\right)\right)\right), \tag{3.57}$$

where the set of numbers $\{a(n;\gamma_n)\}$ and $\{\lambda_b(n;\gamma_n)\}$ are read from the generating function

$$\Phi_b = \prod_{m=1}^{\infty} \frac{1}{1 - \varphi^{b^{(m)}}\, E^{m^2}\, \rho^m} = \sum_{n=0}^{\infty} \rho^n \sum_{\{\gamma_n\}} E^{a(n;\gamma_n)} \varphi^{\lambda_b(n;\gamma_n)}\, \mathcal{O}(\varphi^0). \tag{3.58}$$

Here, Pol(S^{zz}; d) stands for a polynomial of degree d in the variable S^{zz} and whose coefficients have a holomorphic dependence on z. Whenever $d < 0$, the polynomial is identically zero. We assume that $b^{(m)} + b^{(n-m)} - b^{(n)} \geq 0$.

The proof of this theorem relies on a lemma that we state here and prove, in more generality, in appendix B.

Lemma 1. *The set of numbers $\{a(n;\gamma_n)\}$ and $\{\lambda_b(n;\gamma_n)\}$, and the range of labels $\{\gamma_n\}$, appearing in (3.57) are determined by the recursions*

$$\{a(n;\gamma_n)\}_{\gamma_n\neq\widehat{\gamma}_n} = \bigcup_{m,\gamma_m,\gamma_{n-m}}{}' \{a(m;\gamma_m) + a(n-m;\gamma_{n-m})\} \tag{3.59}$$

and

$$\lambda_b(n;\gamma_n) = \min\{\lambda_b(m;\gamma_m) + \lambda_b(n-m;\gamma_{n-m})\}, \quad \forall \gamma_n \neq \widehat{\gamma}_n, \tag{3.60}$$

where min *ranges over $m \in \{0,\cdots,n\}'$, and γ_m and γ_{n-m} are such that*

$$a(m;\gamma_m) + a(n-m;\gamma_{n-m}) = a(n;\gamma_n). \tag{3.61}$$

The prime means that $m = 0$ and $m = n$ are excluded from the range. Further, the following initial data must be specified

$$a(n;\widehat{\gamma}_n) = n^2, \quad \forall n, \tag{3.62}$$

$$\lambda_b(n;\widehat{\gamma}_n) = b^{(n)}, \quad \forall n. \tag{3.63}$$

Proof of theorem 1. The proof of the theorem is by induction both in n and in g. We assume that the structure of the free energies $F_h^{(m)}$ is given by (3.57) for $m \leq n$ and $h < g$. Then we analyze the propagator structure of $G_g^{(n)}$, the right-hand-side of the holomorphic anomaly equation for $F_g^{(n)}$. We use the fact that a z-derivative on the propagator produces a propagator squared. We also need to use that $\partial_z F_h^{(0)}$ is a polynomial of degree $3h-2$ in the propagators for $h \geq 1$. To simplify notation we are going to write E instead of $\exp \frac{1}{2}(\partial_z A)^2 S^{zz}$, and drop S^{zz} in Pol(S^{zz}; d). The dependence of the polynomial in the propagator should always be understood.

The right-hand-side of (3.36), denoted by $G_g^{(n)}$, has a linear and a quadratic term. We analyze both terms separately. The linear term gives

$$\sum_{h=1}^{g} \mathcal{D}_h^{(n)} F_{g-h}^{(n)} = \sum_{\{\gamma_n\}} E^{a(\gamma_n;\gamma_n)}\, \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\gamma_n)\right) - 1\right). \tag{3.64}$$

To obtain the right-hand-side of this expression we focus on the terms will polynomial of highest degree, as long as they accompany the same exponential. The leading term comes from $\mathcal{D}_1^{(n)} F_{h-1}^{(n)}$.

The quadratic term in $G_g^{(n)}$ is a little bit more difficult to analyze. It involves a product and requires using lemma 1. We calculate explicitly,

$$\sum_{m=1}^{n-1}\sum_{h=0}^{g-B} \left(\partial_z F_{h-1}^{(m)} - \partial_z A^{(m)} F_h^{(m)}\right)\left(\partial_z F_{g-1-B-h}^{(n-m)} - \partial_z A^{(n-m)} F_{g-B-h}^{(n-m)}\right) =$$

$$
\begin{aligned}
&= \sum_{m=1}^{n-1} \sum_{h=0}^{g-B} \left\{ \sum_{\{\gamma_m\}} E^{a(m;\gamma_m)} \mathrm{Pol}\left(3\left(h + b^{(m)} - \lambda_b(m;\gamma_m)\right)\right) \right\} \times \\
&\times \left\{ \sum_{\{\gamma_{n-m}\}} E^{a(n-m;\gamma_{n-m})} \mathrm{Pol}\left(3\left(g - h - B(n,m) + b^{(n-m)} - \lambda_b(n-m;\gamma_{n-m})\right)\right) \right\} \\
&= \sum_{m=0}^{n}{}' \sum_{\{\gamma_m, \gamma_{n-m}\}} E^{a(m;\gamma_m)+a(n-m;\gamma_{n-m})} \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(m;\gamma_m) - \lambda_b(n-m;\gamma_{n-m})\right)\right).
\end{aligned}
\tag{3.65}
$$

In the last step we have made use of the definition of $B(n,m)$ and dropped the sum over h. We can simplify the different appearances of γ, a and λ_b using lemma 1.

$$
(3.65) = \sum_{\{\gamma_n\}}{}' E^{a(n;\gamma_n)} \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\gamma_n)\right)\right). \tag{3.66}
$$

The prime on the sum means that the special class $\widehat{\gamma}_n$ is not included. If we put (3.64) and (3.66) together and separating the contribution from the special class $\widehat{g}_n$, we find

$$
\begin{aligned}
G_g^{(n)} &= E^{a(n;\widehat{\gamma}_n)} \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\widehat{\gamma}_n)\right) - 1\right) + \\
&+ \sum_{\{\gamma_n\}}{}' E^{a(n;\gamma_n)} \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\gamma_n)\right)\right).
\end{aligned}
\tag{3.67}
$$

We make this separation because

$$
E^{a(n;\widehat{\gamma}_n)} = \mathrm{e}^{a(n;\widehat{\gamma}_n)\frac{1}{2}(\partial_z A)^2 S^{zz}} = \mathrm{e}^{\frac{1}{2}(\partial_z A^{(n)})^2 S^{zz}} \tag{3.68}
$$

cancels precisely the exponential inside the integral in (3.37). The integral in (3.37) can be of two types

$$
\int^{S^{zz}} d\tilde{S}\, \mathrm{Pol}(\tilde{S}; d) = \mathrm{Pol}(S^{zz}; d+1), \tag{3.69}
$$

$$
\int^{S^{zz}} d\tilde{S}\, \mathrm{e}^{c\tilde{S}} \mathrm{Pol}(\tilde{S}; d) = \mathrm{e}^{cS^{zz}} \mathrm{Pol}(S^{zz}; d), \quad c \neq 0. \tag{3.70}
$$

Using this expression we calculate the free energy $F_g^{(n)}$

$$
\begin{aligned}
F_g^{(n)} &= \mathrm{e}^{\frac{1}{2}\left(\partial_z A^{(n)}\right)^2 S} \left(f_g^{(n)}(z) + \int^{S^{zz}} \mathrm{d}\tilde{S}\, \mathrm{e}^{-\frac{1}{2}\left(\partial_z A^{(n)}\right)^2 \tilde{S}} G_g^{(n)}(z, \tilde{S}) \right) \\
&= E^{a(n;\widehat{\gamma}_n)} \Bigg\{ f_g^{(n)} + \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\widehat{\gamma}_n)\right) - 1 + 1\right) + \\
&\qquad + \sum_{\{\gamma_n\}}{}' E^{a(n;\gamma_n) - a(n;\widehat{\gamma}_n)} \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\gamma_n)\right)\right) \Bigg\}
\end{aligned}
$$

$$= \sum_{\{\gamma_n\}} E^{a(n;\gamma_n)} \mathrm{Pol}\left(3\left(g + b^{(n)} - \lambda_b(n;\gamma_n)\right)\right), \tag{3.71}$$

which is exactly what we wanted to show.

The proof will be complete once we check the base case of the induction. The base case is given by $g = 0$ for any n, and in order to prove it we need to use induction on n. The general equation for $g = 0$ and any n is

$$\left(\partial_{S^{zz}} - \frac{1}{2}\left(\partial_z A^{(n)}\right)^2\right) F_0^{(n)} = \frac{1}{2} \sideset{}{'}\sum_{\substack{m=\mathbf{0},\\ B(n,m)=0}}^{n} \partial_z A^{(m)}\, \partial_z A^{(n-m)}\, F_0^{(m)}\, F_0^{(n-m)}. \tag{3.72}$$

For $n = 1$, the base case of this induction, the right-hand-side is zero and integration gives

$$F_0^{(1)} = \mathrm{e}^{\frac{1}{2}(\partial_z A)^2 S^{zz}} = E^{a(1;\gamma_1)} \mathrm{Pol}(S^{zz};0), \tag{3.73}$$

This expression agrees with the statement of the theorem we want to prove because for $n = 1$ there is only one class γ_1 that is actually the special class $\widehat{\gamma}_1$, for which (3.62) says $a = 1$. Using this and (3.63) in (3.57) we prove the base case. Now we need to show that

$$F_0^{(n)} = \sum_{\substack{\{\gamma_n\}\\ \lambda_b(n;\gamma_n)=b^{(n)}}} E^{a(n;\gamma_n)} \mathrm{Pol}(S^{zz};0), \tag{3.74}$$

which is (3.57) evaluated for $g = 0$ taking into account that polynomials are zero if their degree is negative. Since[3] $\lambda_b \geq b^{(n)}$,

$$F_0^{(m)} = \sum_{\{\gamma_m\}} E^{a(m;\gamma_m)} \mathrm{Pol}\left(b^{(m)} - \lambda_b(m;\gamma_m)\right). \tag{3.76}$$

This implies that the right-hand-side of (3.72) is equal to

$$\sideset{}{'}\sum_{\substack{m=0,\\ B(n,m)=0}}^{n} \sum_{\{\gamma_m,\gamma_{n-m}\}} E^{a(m;\gamma_m)+a(n-m;\gamma_{n-m})} \mathrm{Pol}\left(b^{(n)} + b^{(m)} - \lambda_b(m;\gamma_m) - \lambda_b(n-m;\gamma_{n-m})\right) =$$

$$= \sideset{}{'}\sum_{\{\gamma_n\}} E^{a(n;\gamma_n)} \mathrm{Pol}\left(b^{(n)} - \lambda_b(n;\gamma_n)\right) = \sideset{}{'}\sum_{\substack{\{\gamma_n\},\\ \lambda_b(n;\gamma_n)=b^{(n)}}} E^{a(n;\gamma_n)} \mathrm{Pol}(0), \tag{3.77}$$

where we have used that $b^{(m)} + b^{(n-m)} = b^{(n)}$ since $B(n,m) = 0$, and the recursions in lemma 1. The integration of (3.72) concludes the proof of the base case and the proof of the theorem. □

[3] If we define $\lambda_B\,(n;\gamma_n) := \lambda_b\,(n;\gamma_n) - b^{(n)}$, (3.60) and (3.63) become

$$\lambda_B\,(n;\gamma_n) = \min\left\{\lambda_B\,(m;\gamma_m) + \lambda_B\,(n-m;\gamma_{n-m}) + B\,(n,m)\right\}, \tag{3.75}$$

with $\lambda_B\,(n;\widehat{\gamma}_n) = 0$. Because we assume $B \geq 0$, $\forall n, m$, by induction on n, $\lambda_B\,(n;\gamma_n) \geq 0$.

The generating function Φ_b was found by guessing the pattern of the first few numbers a and λ_b for $b = 1$ and it was later generalized for general values of $b^{(n)}$. The multiparameter case, that will follow next, enjoys a similar generating function that was easily found from the one-parameter case. Since we are only interested in the numbers a and λ_b that appear as powers of the formal variables E and φ, respectively, it could be possible that the generating function is not unique. This is actually the case. It can be checked that for a generic power series $f(x)$, the composition $f \circ R$, where $R(\rho) := \sum_{m=1}^{\infty} \varphi^{b(m)} E^{m^2} \rho^m$, provides the same combinatorial numbers needed to describe the propagator dependence of the free energies—the particular coefficients will differ. The general formula for the composition of two formal power series is given by the Faá di Bruno formula. It can be written in terms of Bell polynomials that can also be described in terms of partitions (of sets). For the proof of lemma 1 provided in appendix B the particular form of the generating function we have been working with is useful. This function only describes the propagator dependence but we could ask how difficult it is to generalize it to describe the holomorphic dependence. This is a much messier problem and it is even difficult to give a heuristic description for the first few instanton sectors—see appendix C for an attempt in the case of local $\mathbb{CP}^2$. However, the problem of determining completely the structure of the free energies may prove to be a necessary step in the resummation of the transseries in closed form. This is of course a long term goal, far beyond the scope of this work.

3.4 Multiparameter transseries

In the previous section we analyzed the holomorphic anomaly equations for a one-parameter transseries and study the structure of the solution with respect to the propagator. We saw that the free energies $F_g^{(n)}$ for $n > 1$ generalize the dependence on S^{zz} from just polynomial for perturbative $F_g^{(0)}$ to combinations of polynomials and exponentials. Theorem 1 describes precisely what the functional dependence on the antiholomorphic variable is. However, in most situations a one-parameter transseries will not be general enough to describe the complete solution of a problem. It may be the case that the transseries involves a number of different instanton actions with no relation with each other. In a more common scenario [38, 40, 41], some of the instanton actions satisfy $\mathbb{Z}$-linear relations. This is the case of resonance and its simplest incarnation is by having instanton actions come in pairs of opposite signs. We mentioned this situation briefly in section 1.5.3. In this section we will derive the holomorphic anomaly equations for the free energies of a multiparameter transseries and analyze the structure of the solutions in the generic case. We leave the case of resonance for the next section.

3.4.1 Multiparameter transseries

Most of this section is an almost direct generalization of the topics discussed in section 3.3. Notation there was already slightly adapted to this more general situation. We start by assuming the existence of p different instanton actions A_α, $\alpha = 1, \ldots, p$. The multiparameter transseries has the form

$$F(\boldsymbol{\sigma}, g_s) = \sum_{\boldsymbol{n}\in\mathbb{N}^p} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-A^{(\boldsymbol{n})}/g_s}\, F^{(\boldsymbol{n})}(g_s). \tag{3.78}$$

Vector notation is the same as in section 1.3.1. The total instanton action is $A^{(\boldsymbol{n})} = \sum_{\alpha=1}^{p} n_\alpha A_\alpha$. Each instanton action is in principle assumed to be a function of both z and S^{zz}. We saw in section 3.3.1 that for a one-parameter transseries the instanton action is holomorphic. This result generalizes. Each sector $F^{(\boldsymbol{n})}(g_s)$ is expected to be an asymptotic series in g_s of the form

$$F^{(\boldsymbol{n})} = \sum_{g=0}^{\infty} g_s^{g+b^{(\boldsymbol{n})}} F_g^{(\boldsymbol{n})}, \tag{3.79}$$

where $F_g^{(\boldsymbol{n})}$ depends on z and S^{zz} and $b^{(\boldsymbol{n})}$ is the starting power.

Just as in the one-parameter case we plug in the transseries ansatz into the equation (3.15) and collect similar powers of $\boldsymbol{\sigma}$. For $\boldsymbol{n} = \boldsymbol{0}$ we recover the perturbative holomorphic anomaly equations as it must be, and for $\boldsymbol{n} \neq \boldsymbol{0}$ we have a set of recursive equations that generalizes (3.23),

$$\begin{aligned}&\left(\partial_{S^{zz}} - \frac{1}{g_s}\partial_{S^{zz}} A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})} - \frac{1}{2} g_s^2 \left(D_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{n})} + 2\,\partial_z \widehat{F}^{(\boldsymbol{0})}\right)\left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})} = \\ &= \frac{1}{2} g_s^2 \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})} \left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{n}-\boldsymbol{m})}\right) F^{(\boldsymbol{n}-\boldsymbol{m})}.\end{aligned} \tag{3.80}$$

Here we just need to note that the sum over lower instanton sectors $\boldsymbol{m}$ with a prime on it means

$$\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' := \sum_{\substack{m_1,\dots,m_p=0\\ \boldsymbol{m}\neq\boldsymbol{0},\, \boldsymbol{m}\neq\boldsymbol{n}}}^{n_1,\dots,n_p}. \tag{3.81}$$

As in the one-parameter case, the two terms involving $F^{(\boldsymbol{n})}$ were moved to the left-hand-side of (3.80). Next, we expand in g_s and collect similar powers

$$\begin{aligned}&\partial_{S^{zz}} A^{(\boldsymbol{n})} F_{g+1}^{(\boldsymbol{n})} - \sum_{h=0}^{g} \mathcal{D}_h^{(\boldsymbol{n})} F_{g-h}^{(\boldsymbol{n})} + \\ &+ \frac{1}{2} \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{h=0}^{g-B(\boldsymbol{n},\boldsymbol{m})} \left(\partial_z F_{h-1}^{(\boldsymbol{m})} - \partial_z A^{(\boldsymbol{m})} F_h^{(\boldsymbol{m})}\right)\left(\partial_z F_{g-1-B(\boldsymbol{n},\boldsymbol{m})-h}^{(\boldsymbol{n}-\boldsymbol{m})} - \partial_z A^{(\boldsymbol{n}-\boldsymbol{m})} F_{g-B(\boldsymbol{n},\boldsymbol{m})-h}^{(\boldsymbol{n}-\boldsymbol{m})}\right) = 0,\end{aligned} \tag{3.82}$$

for $g = -1, 0, 1, \dots$. Here $B(\boldsymbol{n}, \boldsymbol{m}) = b^{(\boldsymbol{m})} + b^{(\boldsymbol{n}-\boldsymbol{m})} - b^{(\boldsymbol{n})}$. As we did in the previous section we are going to assume general $b^{(\boldsymbol{n})}$ as long as $B(\boldsymbol{n}, \boldsymbol{m}) \geq 0$ to avoid the case of extra equations coming form (3.82) with the first two terms vanishing. The operators $\mathcal{D}_h^{(\boldsymbol{n})}$ are straightforward generalizations of the ones in (3.27)–(3.31).

The first equation in (3.82), for $g = -1$, comprises only the first term. Since by definition $F_0^{(\boldsymbol{n})} \neq$ for every $\boldsymbol{n}$ we find

$$\partial_{S^{zz}} A^{(\boldsymbol{n})} = 0. \tag{3.83}$$

If we focus on the one-instanton sectors, that is, those of the form $(\boldsymbol{n}) = (0|\cdots|0|1|0|\cdots|0)$ we prove the holomorphicity of every instanton action in the transseries,

$$\partial_{S^{zz}} A_\alpha = 0, \quad \alpha = 1, \ldots, p. \tag{3.84}$$

In section 3.6 we show the holomorphicity of the instanton actions directly through a large-order argument involving only perturbation theory.

Using this information back into (3.82) and using the explicit expression for $\mathcal{D}_0^{(\boldsymbol{n})}$ we find

$$\left(\partial_{S^{zz}} - \frac{1}{2}\left(\partial_z A^{(\boldsymbol{n})}\right)^2\right) F_g^{(\boldsymbol{n})} = -\sum_{h=1}^{g} \mathcal{D}_h^{(\boldsymbol{n})} F_{g-h}^{(\boldsymbol{n})} + \tag{3.85}$$
$$+\frac{1}{2}\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{h=0}^{g-B(\boldsymbol{n},\boldsymbol{m})} \left(\partial_z F_{h-1}^{(\boldsymbol{m})} - \partial_z A^{(\boldsymbol{m})} F_h^{(\boldsymbol{m})}\right)\left(\partial_z F_{g-1-B(\boldsymbol{n},\boldsymbol{m})-h}^{(\boldsymbol{n}-\boldsymbol{m})} - \partial_z A^{(\boldsymbol{n}-\boldsymbol{m})} F_{g-B(\boldsymbol{n},\boldsymbol{m})-h}^{(\boldsymbol{n}-\boldsymbol{m})}\right).$$

This is a equation for $F_g^{(\boldsymbol{n})}$, for each $\boldsymbol{n} \neq \boldsymbol{0}$ and $g \geq 0$ which can be solved recursively because the right-hand-side only depends on previously computed data. We study the structure of the solutions next.

3.4.2 Structure of the transseries solution

Even though the transseries can become very complicated as we increase the number of parameters, the antiholomorphic dependence can be systematically described in the generic case. We follow the same line of argument presented in section 3.3.2 generalized to accommodate the indices α, β, ... labeling the instanton actions. If we study the solutions of the holomorphic anomaly equations (3.85) for low instanton sector $\boldsymbol{n}$ and small g we find the following structure,

$$F_g^{(\boldsymbol{n})} = \sum_{\{\gamma_{\boldsymbol{n}}\}} \exp\left(\frac{1}{2}\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}\left(\boldsymbol{n};\gamma_{\boldsymbol{n}}\right) \partial_z A_\alpha\, \partial_z A_\beta\, S^{zz}\right) \mathrm{Pol}\left(S^{zz}; d_b\left(\boldsymbol{n}; g; \gamma_{\boldsymbol{n}}\right)\right). \tag{3.86}$$

Here $\gamma_{\boldsymbol{n}}$ is an index running in a certain finite set, $a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})$ are nonnegative integers independent of g, and $d_b(\boldsymbol{n};g;\gamma_{\boldsymbol{n}})$ is the degree of the polynomial which can be parametrized as

$$d_b(\boldsymbol{n};g;\gamma_{\boldsymbol{n}}) = 3\left(g + b^{(\boldsymbol{n})} - \lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})\right). \tag{3.87}$$

The numbers $\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})$ are also independent of g and do not carry any multiparameter labels, unlike $a_{\alpha\beta}$. Equation (3.86) generalizes the one-parameter case (3.47). The higher instanton free energies $F_g^{(\boldsymbol{n})}$ are given in terms of exponentials and polynomials of certain degrees in the propagator.

All the numbers $a_{\alpha\beta}$, λ_b and the labels $\gamma_{\boldsymbol{n}}$ can be computed from the following generating function

$$\Phi_b \equiv \Phi_b\left(\varphi, E, \rho\right) := \prod_{\boldsymbol{m}=\boldsymbol{0}}^{\infty}{}' \frac{1}{1 - \varphi^{b(\boldsymbol{m})} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{m_\alpha m_\beta} \prod_{\alpha=1}^{p} \rho_\alpha^{m_\alpha}}, \tag{3.88}$$

which can be expanded into

$$\begin{aligned}\Phi_b &= \sum_{\{r_{\boldsymbol{m}}\}} \varphi^{\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}}\, b^{(\boldsymbol{m})}} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}}\, m_\alpha m_\beta} \prod_{\alpha=1}^{p} \rho_\alpha^{\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}}\, m_\alpha}, \\ &= \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty} \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\{\gamma_{\boldsymbol{n}}\}} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\, \varphi^{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\, \mathcal{O}(\varphi^0). \end{aligned} \tag{3.89}$$

Here the indices $r_{\boldsymbol{m}}$ run from 0 to infinity. From the first to the second line we have introduced a multi-index $\boldsymbol{n}$ such that $\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}} \boldsymbol{m} = \boldsymbol{n}$. This reduces to the characterization of an integer partition in the one-parameter transseries case. Further, we have collected similar powers of ρ_α, $E_{\alpha\beta}$ and φ, and defined

$$a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}}) := \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}} m_\alpha m_\beta, \quad \text{for } \{r_{\boldsymbol{m}}\} \text{ such that } \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}} \boldsymbol{m} = \boldsymbol{n}. \tag{3.90}$$

There is a different value of the label $\gamma_{\boldsymbol{n}}$ for each different value of $a_{\alpha\beta}$. Also,

$$\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}}) := \min_{\{r_{\boldsymbol{n}}\}\in\gamma_{\boldsymbol{n}}} \left\{ \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\infty} r_{\boldsymbol{m}}\, b^{(\boldsymbol{m})} \right\}. \tag{3.91}$$

The relation between the free energies $F_g^{(\boldsymbol{n})}$ in (3.86) and the generating function (3.88) is justified in the following theorem

Theorem 2. *For any $\boldsymbol{n} \neq \boldsymbol{0}$ and $g \geq 0$, the structure of the nonperturbative free energies has the form*

$$F_g^{(\boldsymbol{n})} = \sum_{\{\gamma_{\boldsymbol{n}}\}} \mathrm{e}^{\frac{1}{2}\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\partial_z A_\alpha\, \partial_z A_\beta\, S^{zz}}\, \mathrm{Pol}\left(S^{zz}; 3\left(g + b^{(\boldsymbol{n})} - \lambda\left(\boldsymbol{n};\gamma_{\boldsymbol{n}}\right)\right)\right), \tag{3.92}$$

where the set of numbers $\{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\}$ and $\{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})\}$ are read from the generating function

$$\begin{aligned}\Phi_b &= {\prod_{\boldsymbol{m}=\boldsymbol{0}}^{\infty}}' \frac{1}{1-\varphi^{b^{(\boldsymbol{m})}} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{m_\alpha m_\beta} \prod_{\alpha=1}^{p} \rho_\alpha^{m_\alpha}} = \\ &= \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty} \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\{\gamma_{\boldsymbol{n}}\}} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\, \varphi^{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\, \mathcal{O}(\varphi^0). \end{aligned} \tag{3.93}$$

Here $\mathrm{Pol}(S^{zz};d)$ stands for a polynomial of degree d in the variable S^{zz} (and whose coefficients have a dependence in z). Whenever $d<0$, the polynomial is taken to be identically zero. We assume that $b^{(\boldsymbol{m})} + b^{(\boldsymbol{n}-\boldsymbol{m})} - b^{(\boldsymbol{n})} \geq 0$.

The proof of this theorem follows exactly the same inductive argument used in theorem 1. In several occasions it mades use of the generalization of lemma 1, stated and proved in appendix B. The proof concludes with the verification of the base case, which in this case involves all the one-instanton sectors $(0|\cdots|0|1|0|\cdots|0)$.

In order to illustrate the theorem let us focus on a two-parameter transseries, and determine the structure of the $\boldsymbol{n} = (2|1)$ free energies for every g. For definiteness we choose $b^{(\boldsymbol{n})} = 1$. We need to take the generating function $\Phi_{b=1}$, expand it as explained earlier, and collect the terms with powers $\rho_1^2 \rho_2^1$,

$$E_{11}^4\, E_{12}^{2\cdot 2}\, E_{22}\,\varphi + E_{11}^4\, E_{22}\,\varphi^2 + E_{11}^2\, E_{12}^{2\cdot 1}\, E_{22}\,\varphi^2 + E_{11}^2\, E_{22}\,\varphi^3. \tag{3.94}$$

By comparing with the general expansion (3.93) we can see that the $\gamma_{(2|1)}$ runs over four values. Out of these four classes, the special one $\widehat{\gamma}_{(2|1)}$ has $a_{1,1} = 4$, $a_{1,2} = a_{2,1} = 2$, and $a_{2,2} = 1$, along with $\lambda_{b=1} = 1 = b^{(2|1)}$, as expected. The last three terms in (3.94) produce the remaining classes. Collecting the corresponding values of $a_{\alpha\beta}$ and λ_b we can write

$$\begin{aligned} F_g^{(2|1)} = {}& \mathrm{e}^{[2(\partial_z A_1)^2 + 2\,\partial_z A_1\,\partial_z A_2 + \frac{1}{2}(\partial_z A_2)^2] S^{zz}}\, \mathrm{Pol}(3g) + \mathrm{e}^{[2(\partial_z A_1)^2 + \frac{1}{2}(\partial_z A_2)^2] S^{zz}}\, \mathrm{Pol}(3g-3) + \\ & + \mathrm{e}^{[(\partial_z A_1)^2 + 2\,\partial_z A_1\,\partial_z A_2 + \frac{1}{2}(\partial_z A_2)^2] S^{zz}}\, \mathrm{Pol}(3g-3) + \mathrm{e}^{[(\partial_z A_1)^2 + \frac{1}{2}(\partial_z A_2)^2] S^{zz}}\, \mathrm{Pol}(3g-6). \end{aligned} \tag{3.95}$$

This is an expression valid for every g if we follow the convention that negative degree polynomials are zero. Therefore, for $g = 0$ only the first term is present. For $g = 1$ the second and third are turned on and for $g \geq 2$ all of them contribute to the free energy.

3.5 Resonance

The concept of resonance in a resurgent transseries appears when two different instanton sectors, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ have the same total instanton action, $A^{(\boldsymbol{n}_1)} = A^{(\boldsymbol{n}_2)}$. In the realm of differential equations, it is usually possible to write the differential equation in the so-called prepared form, see [71], for example. In such a form there is a matrix whose eigenvalues give precisely the instanton actions. Resonance will present when, for example, two of the eigenvalues have opposite signs, because the sector $(1|1)$ in a two-parameter transseries will have total instanton action equal to zero, the same as the perturbative sector. This is what was found in the case of the Painlevé I equation [38]. In that case it was also discovered that the transseries describing the solution to that differential equation should include logarithmic sectors, $\log g_s$, as in (1.19). Resonance of this sort was also studied in the quartic matrix model, related to Painlevé I through a double scaling limit, in [40]. In that context it was found that the logarithmic sectors $\varphi^{(\boldsymbol{n})[k]}$, as functions, are not independent of $\varphi^{(\boldsymbol{n})[0]}$. In this sense they do not carry any new information. It was also found that the asymptotic series of sectors of the form $(n|n)$ are genus expansions, in the sense of being series in g_s^2, rather than in g_s. Large-order relations are modified by the presence of logarithms but only at subleading order and for higher instanton sectors. The physical interpretation of this sectors is still unclear. As discussed in [39], from the interpretation of the instanton action kA coming from k D-brane one could associate anti-D-branes to $-kA$ [144], the incarnation of ghost D-branes in topological string theory [145], because they are related by a change of sign in g_s that could be tied to a change of sign in the instanton action instead. This interpretation would imply that the sector $(n_1|n_2)$ should be equivalent to $(n_1 - n_2|0)$ because a stack of D and anti-D-branes can be described in terms of only D or only anti-D-branes [13, 144]. However, $F^{(n_1|n_2)}$ and $F^{(n_1-n_2|0)}$ are not equal. So this interpretation is at least incomplete. See also the discussion in [40].

In this section we study how a logarithmic transseries can be used as an ansatz for the holomorphic anomaly equations, and how the results obtained in section 3.4 generalize. We also prove that for sectors with vanishing total instanton action, the asymptotic series is an expansion in g_s^2 under certain conditions.

3.5.1 Structure of the transseries solution

The logarithmic transseries has the generic form

$$F(\boldsymbol{\sigma}, g_s) = \sum_{\boldsymbol{n}\in\mathbb{N}^p} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-A^{(\boldsymbol{n})}/g_s} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k(g_s)\; F^{(\boldsymbol{n})[k]}(g_s), \tag{3.96}$$

where

$$F^{(\boldsymbol{n})[k]}(g_s) = \sum_{g=0}^{\infty} g_s^{g+b^{(\boldsymbol{n})[k]}} F_g^{(\boldsymbol{n})[k]}. \tag{3.97}$$

Note that there is an extra index k in square brackets that runs from 0 to some number $k_{\max}^{(\boldsymbol{n})}$. This upper limit is not fixed by the equations, but logarithms at a given sector can only come from logarithms of lower sectors, so there must be some relations between the different $k_{\max}^{(\boldsymbol{n})}$. The exact structure should be studied in each specific example. Here we are going to use a generic value of $k_{\max}^{(\boldsymbol{n})}$.

The process of plugging the transseries ansatz into the holomorphic anomaly equation (3.15) is analogous to the previous cases. On top of collecting similar powers of $\boldsymbol{\sigma}$ and g_s, we also have to collect powers of $\log g_s$. As before, the first equations just tell us that the instanton actions are holomorphic, even in this more general case. This has to be the case because it is a direct consequence of the factorial growth of the perturbative free energies and the holomorphic anomaly equations they satisfy; see section 3.6.

The rest of the equations are

$$\left(\partial_{S^{zz}} - \frac{1}{2}(\partial_z A^{(\boldsymbol{n})})^2\right) F_g^{(\boldsymbol{n})[k]} + \sum_{h=1}^{g} \mathcal{D}_h^{(\boldsymbol{n})} F_{g-h}^{(\boldsymbol{n})[k]} =$$
$$= \frac{1}{2} {\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}}' \sum_{\ell} \sum_{h=0}^{g-B} \left(\partial_z F_{h-1}^{(\boldsymbol{m})[\ell]} - \partial_z A^{(\boldsymbol{m})}\, F_h^{(\boldsymbol{m})[\ell]}\right) \left(\partial_z F_{g-1-B-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - \partial_z A^{(\boldsymbol{n}-\boldsymbol{m})}\, F_{g-B-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}\right). \tag{3.98}$$

Here, ℓ runs from $\max\left(0, k - k_{\max}^{(\boldsymbol{n}-\boldsymbol{m})}\right)$ to $\min\left(k, k_{\max}^{(\boldsymbol{m})}\right)$, and

$$B = B(\boldsymbol{n}, \boldsymbol{m})[k, \ell] := b^{(\boldsymbol{m})[\ell]} + b^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - b^{(\boldsymbol{n})[k]} \tag{3.99}$$

generalizes the definitions of B we have been using until now.

The structure of the solutions to equation (3.98) can be analyzed in the same way we have done with the one-parameter and multiparameter transseries. The presence of logarithmic sectors and the new index k can be incorporated in a extension of theorem 2. We find that the free energies have the following antiholomorphic dependence

$$F_g^{(\boldsymbol{n})[k]} = \sum_{\{\gamma_{\boldsymbol{n}}\}} \mathrm{e}^{\frac{1}{2}\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\partial_z A_\alpha\, \partial_z A_\beta\, S}\, \mathrm{Pol}\left(S; 3\left(g + b^{(\boldsymbol{n})[k]} - \lambda_{b,k_{\max}}^{[k]}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\right)\right). \tag{3.100}$$

The labels $\gamma_{\boldsymbol{n}}$ and the numbers $a_{\alpha\beta}$ are the same as in lemma 4—they only depend on the quadratic structure of the equations—, while the numbers $\lambda^{[k]}_{b,k_{\max}}$ have to be generalized slightly to incorporate the logarithmic index. All of them can be read from the generating function

$$\Phi_{b,k_{\max}} = \prod_{\boldsymbol{m}=\boldsymbol{0}}^{\infty}{}' \prod_{\ell=0}^{k^{(\boldsymbol{m})}_{\max}} \frac{1}{1-\varphi^{b(\boldsymbol{m})[\ell]}\,\psi^{\ell}\,\prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{m_\alpha m_\beta}\,\prod_{\alpha=1}^{p}\rho_\alpha^{m_\alpha}} = \\ = \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty} \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{k=0}^{k^{(\boldsymbol{n})}_{\max}} \psi^{k} \sum_{\{\gamma_{\boldsymbol{n}}\}} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\,\varphi^{\lambda^{[k]}_{b,k_{\max}}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\,\mathcal{O}(\varphi^0), \tag{3.101}$$

where ψ is the new formal variable associated to logarithmic sectors.

The analysis of the holomorphic anomaly equations with logarithmic sectors can be done in the exact same way if logarithms are replaced by some other nonanalytic functions of g_s at zero. The equations do not determine that the monomial needs to be $\log g_s$ unlike what is found in Painlevé I equation, for example. The way to know the correct transseries is to study the resurgence relations and deduce form them which monomial is the right one.

3.5.2 Genus expansions within transseries

When we talk about a genus expansion we refer to a power series in g_s^2. The perturbative sector satisfies such expansion, $\sum_{g=0}^{\infty} F_g^{(\boldsymbol{0})}\, g_s^{2g-2}$, and $2g-2$ has a direct interpretation as the Euler characteristic of a Riemann surface of genus. In [38, 40, 41] it was found that the free energies $F^{(n|n)}(g_s)$ for certain polynomial matrix models and Painlevé equations are also power series in g_s. The topological interpretation is not known for these sectors, but we will still use this language. The origin of this topological expansion seems to be the existence of pairs of instanton actions A and $-A$. The relation $A+(-A)=0$ implies a symmetry $g_s \to -g_s$ which is manifest in the free energies of sectors $(n|n)$, and implies the topological expansion. The holomorphic anomaly equations are supposed to be valid for any closed topological string theory. Such generality is a disadvantage when trying to compute every detail of the transseries solution, but it becomes useful when proving general statements. One example is the existence of topological expansions within transseries when resonance is present. This is what we show now, first for a two-parameter transseries and then in more generality. Even if we are dealing with the case of resonance we will turn off the logarithmic sectors to make the argument easier.

We are going to work with the transseries

$$F(\boldsymbol{\sigma}, g_s) = \sum_{\boldsymbol{n}\in\mathbb{N}^p} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-A^{(\boldsymbol{n})}/g_s}\, F^{(\boldsymbol{n})}(g_s), \tag{3.102}$$

and focus on $F^{(\boldsymbol{n})}(g_s)$ as formal objects in g_s (recall that these are asymptotic series so to talk about functions requires a process of resummation). The holomorphic anomaly equations applied to this transseries produce the equations

$$\mathcal{D}^{(\boldsymbol{n})}(g_s)\, F^{(\boldsymbol{n})}(g_s) = T^{(\boldsymbol{n})}(g_s). \tag{3.103}$$

We have already seen the operator on the left-hand-side of (3.103),

$$\mathcal{D}^{(\boldsymbol{n})}(g_s) = \partial_{S^{zz}} - \frac{1}{2}g_s^2\left(D_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{n})} + 2\,\partial_z \widehat{F}^{(\boldsymbol{0})}(g_s)\right)\left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{n})}\right), \tag{3.104}$$

where $\widehat{F}^{(\boldsymbol{0})}(g_s) \equiv F^{(\boldsymbol{0})}(g_s) - \frac{1}{g_s^2}F_0^{(\boldsymbol{0})}(g_s) = \sum_{g=1}^{\infty} g_s^{2g-2}F_g^{(\boldsymbol{0})}$ is an even series in g_s. The right-hand-side of (3.103) depends on lower sectors and is quadratic,

$$\begin{aligned} T^{(\boldsymbol{n})}(g_s) &:= \frac{1}{2}g_s^2 \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' H^{(\boldsymbol{m})}(g_s)\, H^{(\boldsymbol{n}-\boldsymbol{m})}(g_s) \\ &:= \frac{1}{2}g_s^2 \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})} \left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{n}-\boldsymbol{m})}\right) F^{(\boldsymbol{n}-\boldsymbol{m})}. \end{aligned} \tag{3.105}$$

We are going to study the parity properties of $T^{(\boldsymbol{n})}(g_s)$ in g_s and using $H^{(\boldsymbol{m})}(g_s)$ as an intermediate step will be convenient. The final goal will be to show that for any sector of the transseries, $\boldsymbol{r}$, with vanishing total instanton action, $A^{(\boldsymbol{r})} = 0$,

$$F^{(\boldsymbol{r})}(-g_s) = F^{(\boldsymbol{r})}(g_s). \tag{3.106}$$

Let us illustrate the strategy of the proof for the case of a two-parameter transseries with opposite instanton actions,

$$A_1 = -A_2. \tag{3.107}$$

We say that the transseries has a $\mathbb{Z}_2$-symmetry. Let us define the following operation at the level instanton sectors,

$$\boldsymbol{n}^\star = (n_1|n_2)^\star := (n_2|n_1). \tag{3.108}$$

Because of the $\mathbb{Z}_2$ symmetry,

$$A^{(\boldsymbol{n})} + A^{(\boldsymbol{n}^\star)} = (n_1A_1 + n_2A_2) + (n_2A_1 + n_1A_2) = (n_1+n_2)(A_1+A_2) = 0. \tag{3.109}$$

From this we can see that $A^{(\boldsymbol{r})} = 0$ only if $\boldsymbol{r}^\star = \boldsymbol{r} = (r|r)$. We want to study how the ingredients of equation (3.103) behave under the action of *. The key point is that this action is equivalent to a change of sign in g_s. For example, it is easy to see that

$$\mathcal{D}^{(\boldsymbol{n})}(-g_s) = \mathcal{D}^{(\boldsymbol{n}^\star)}(g_s). \tag{3.110}$$

Let us suppose for a moment that the right-hand-side of (3.103) behaves similarly,

$$T^{(\boldsymbol{n})}(-g_s) = \varepsilon^{(\boldsymbol{n})}T^{(\boldsymbol{n}^\star)}(g_s), \tag{3.111}$$

where $\varepsilon^{(\boldsymbol{n})}$ is a number equal to $+1$ when $\boldsymbol{n}^\star = \boldsymbol{n}$. With this assumption we can calculate that

$$\begin{aligned} \mathcal{D}^{(\boldsymbol{n})}(-g_s)\left(F^{(\boldsymbol{n})}(-g_s) - \varepsilon^{(\boldsymbol{n})}\,F^{(\boldsymbol{n}^\star)}(g_s)\right) &= \mathcal{D}^{(\boldsymbol{n})}(-g_s)F^{(\boldsymbol{n})}(-g_s) - \varepsilon^{(\boldsymbol{n})}\,\mathcal{D}^{(\boldsymbol{n}^\star)}(g_s)F^{(\boldsymbol{n}^\star)}(g_s) = \\ &= T^{(\boldsymbol{n})}(-g_s) - \varepsilon(\boldsymbol{n})\,T^{(\boldsymbol{n}^\star)}(g_s) = 0. \end{aligned} \tag{3.112}$$

From this equation we want to derive that

$$F^{(\boldsymbol{n})}(-g_s) = \varepsilon^{(\boldsymbol{n})} F^{(\boldsymbol{n}^\star)}(g_s), \tag{3.113}$$

because if we choose the sector to be $\boldsymbol{n} = \boldsymbol{r} = (r|r) = \boldsymbol{r}^\star$ we conclude,

$$F^{(r|r)}(-g_s) = F^{(r|r)}(g_s). \tag{3.114}$$

That is, the free energy $F^{(r|r)}(g_s)$ has a genus expansion in g_s.

The last assumption regarding the symmetry properties of the free energies in (3.113) is the object of the next lemma.

Lemma 2. *The solution, $y(g_s; z, S^{zz})$, of the differential equation*

$$\mathcal{D}^{(\boldsymbol{n})}(g_s)\, y(g_s; z, S^{zz}) = 0, \tag{3.115}$$

is the trivial solution if the corresponding holomorphic ambiguity $y(g_s; z, 0)$, obtained by setting the propagator to zero, vanishes.

Proof. We expand $\mathcal{D}^{(\boldsymbol{n})}(g_s)$ and $y(g_s)$ as a formal series in g_s, and obtain a tower of differential equations,

$$\sum_{h=0}^{g} \mathcal{D}_h^{(\boldsymbol{n})}\, y_{g-h} = 0, \qquad g = 0, 1, 2, \ldots. \tag{3.116}$$

We proceed by induction on g. The first equation, for $g = 0$, can be solved explicitly

$$\mathcal{D}_0^{(\boldsymbol{n})}\, y_0 = 0 \qquad \Rightarrow \qquad y_0(z, S) = \mathrm{e}^{\frac{1}{2}\left(\partial_z A^{(\boldsymbol{n})}\right)^2 S}\, v_0(z), \tag{3.117}$$

where $v_0(z)$ is the corresponding holomorphic ambiguity. If $v_0 = 0$ then $y_0 = 0$, and the base case of the induction is proved. Assume that $y_h = 0$, $\forall\, h < g$. Then

$$\mathcal{D}_0^{(\boldsymbol{n})}\, y_g = 0 \qquad \Rightarrow \qquad y_g(z, S) = \mathrm{e}^{\frac{1}{2}\left(\partial_z A^{(\boldsymbol{n})}\right)^2 S}\, v_g(z). \tag{3.118}$$

The assumptions of the lemma imply that $y_g = 0$, and the proof is complete. □

So we have shown that if (3.111) is true, the diagonal sectors $(r|r)$ have a genus expansion. We show this assumption in the proof of the following lemma, but first let us note that all this discussion can be extended without problems to multiparameter transseries in which all the instanton actions come in opposite pairs,

$$A_2 = -A_1, \quad A_4 = -A_3, \quad \ldots, \quad A_{2q} = -A_{2q-1}. \tag{3.119}$$

We say that we are in the $(\mathbb{Z}_2)^q$-symmetric case. On top of this symmetry there could be other resonances involving instanton actions in different pairs, for example, $A_1 + A_2 + A_3 = 0$. We exclude this cases from the discussion so that whenever we have $A^{(\boldsymbol{n})} = 0$, it means that $n_1 = n_2$, $n_3 = n_4$, $\ldots$, $n_{2q-1} = n_{2q}$. Equivalently, that $A^{(\boldsymbol{n})} = 0$ if and only if $\boldsymbol{n}^\star = \boldsymbol{n}$, where we have extended the action of $\star$ in the natural way

$$\boldsymbol{n}^\star \equiv (n_1|n_2\|n_3|n_4\|\cdots\|n_{2q-1}|n_{2q}) := (n_2|n_1\|n_4|n_3\|\cdots\|n_{2q}|n_{2q-1}). \tag{3.120}$$

Lemma 3. *A 2q-parameter transseries with* $(\mathbb{Z}_2)^q$*-symmetry satisfies*

$$F^{(\boldsymbol{n})}(-g_s) = \varepsilon^{(\boldsymbol{n})}\, F^{(\boldsymbol{n}^\star)}(g_s), \tag{3.121}$$

for any sector $\boldsymbol{n}$ *provided that the associated holomorphic ambiguity satisfies the corresponding relation*

$$f^{(\boldsymbol{n})}(-g_s) = \varepsilon^{(\boldsymbol{n})}\, f^{(\boldsymbol{n}^\star)}(g_s). \tag{3.122}$$

The numbers ε *satisfy* $\varepsilon^{(\boldsymbol{m})}\varepsilon^{(\boldsymbol{n}-\boldsymbol{m})} = \varepsilon^{(\boldsymbol{n})}$*, for any* $\boldsymbol{m} < \boldsymbol{n}$*.*

Proof. The proof is by induction on the instanton sector We show the base case at the end. Assume that the statement of the lemma is true for every $\boldsymbol{m} < \boldsymbol{n}$, then

$$\begin{aligned} H^{(\boldsymbol{m})}(-g_s) &= \left(\partial_z - \frac{1}{(-g_s)}\partial_z A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})}(-g_s) = \left(\partial_z + \frac{1}{g_s}\partial_z A^{(\boldsymbol{m})}\right) \varepsilon^{(\boldsymbol{m})}\, F^{(\boldsymbol{m}^\star)}(g_s) = \\ &= \varepsilon^{(\boldsymbol{m})} \left(\partial_z - \frac{1}{g_s}\partial_z A^{(\boldsymbol{m}^\star)}\right) F^{(\boldsymbol{m}^\star)}(g_s) = \varepsilon^{(\boldsymbol{m})}\, H^{(\boldsymbol{m}^\star)}(g_s). \end{aligned} \tag{3.123}$$

From this result, the same formula is valid for $T^{(\boldsymbol{m})}$,

$$\begin{aligned} T^{(\boldsymbol{n})}(-g_s) &= \frac{1}{2}(-g_s)^2 \sideset{}{'}\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}} H^{(\boldsymbol{m})}(-g_s)\, H^{(\boldsymbol{n}-\boldsymbol{m})}(-g_s) = \\ &= \frac{1}{2}g_s^2 \sideset{}{'}\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}} \varepsilon^{(\boldsymbol{m})}\, \varepsilon^{(\boldsymbol{n}-\boldsymbol{m})}\, H^{(\boldsymbol{m}^\star)}(g_s)\, H^{((\boldsymbol{n}-\boldsymbol{m})^\star)}(g_s) = \\ &= \varepsilon^{(\boldsymbol{n})}\, \frac{1}{2}g_s^2 \sideset{}{'}\sum_{\boldsymbol{m}^\star=\boldsymbol{0}}^{\boldsymbol{n}^\star} H^{(\boldsymbol{m}^\star)}(g_s)\, H^{(\boldsymbol{n}^\star-\boldsymbol{m}^\star)}(g_s) = \varepsilon^{(\boldsymbol{n})}\, T^{(\boldsymbol{n}^\star)}(g_s). \end{aligned} \tag{3.124}$$

In the third line we have used that $\varepsilon^{(\boldsymbol{m})}\varepsilon^{(\boldsymbol{n}-\boldsymbol{m})} = \varepsilon^{(\boldsymbol{n})}$, that $\star$ is a linear operation, and that

$$\sideset{}{'}\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}} = \sideset{}{'}\sum_{\boldsymbol{m}^\star=\boldsymbol{0}}^{\boldsymbol{n}^\star}. \tag{3.125}$$

The differential operator $\mathcal{D}^{(\boldsymbol{n})}(g_s)$ given by (3.104) satisfies

$$\mathcal{D}^{(\boldsymbol{n})}(-g_s) = \mathcal{D}^{(\boldsymbol{n}^\star)}(g_s). \tag{3.126}$$

Together with (3.124) it implies,

$$\begin{aligned} \mathcal{D}^{(\boldsymbol{n})}(-g_s)\left(F^{(\boldsymbol{n})}(-g_s) - \varepsilon^{(\boldsymbol{n})}\, F^{(\boldsymbol{n}^\star)}(g_s)\right) &= \mathcal{D}^{(\boldsymbol{n})}(-g_s)F^{(\boldsymbol{n})}(-g_s) - \varepsilon^{(\boldsymbol{n})}\, \mathcal{D}^{(\boldsymbol{n}^\star)}(g_s)F^{(\boldsymbol{n}^\star)}(g_s) = \\ &= T^{(\boldsymbol{n})}(-g_s) - \varepsilon(\boldsymbol{n})\, T^{(\boldsymbol{n}^\star)}(g_s) = 0. \end{aligned} \tag{3.127}$$

From lemma 2, if $f^{(\boldsymbol{n})}(-g_s) - \varepsilon^{(\boldsymbol{n})}\, f^{(\boldsymbol{n}^\star)}(g_s) = 0$ then

$$F^{(\boldsymbol{n})}(-g_s) = \varepsilon^{(\boldsymbol{n})}\, F^{(\boldsymbol{n}^\star)}(g_s). \tag{3.128}$$

The base case of the induction involves the one-instanton sectors (those with $\|\boldsymbol{n}\| = 1$). For them $T^{(\boldsymbol{n})}$ is zero, so we can directly apply lemma 2. □

The precise statement for the existence of genus expansions within transseries in the case of resonance is given in the following

Theorem 3. *Consider a $2q$–parameters transseries with $(\mathbb{Z}_2)^q$-symmetry. If a sector $\boldsymbol{r}$ has a vanishing total instanton action, $A^{(\boldsymbol{r})} = 0$, then $F^{(\boldsymbol{r})}(g_s)$ has a topological genus expansion in the string coupling g_s (an asymptotic expansion in powers of g_s^2), provided the holomorphic ambiguities respect the symmetry (as in (3.122) with $\varepsilon^{(\boldsymbol{r})} = +1$).*

Proof. If $A^{(\boldsymbol{r})} = 0$, then $\boldsymbol{r}^\star = \boldsymbol{r}$. Using lemma 3 we conclude that

$$F^{(\boldsymbol{r})}(-g_s) = +F^{(\boldsymbol{r})}(g_s), \tag{3.129}$$

where we have used $\varepsilon^{(\boldsymbol{r})} = +1$. □

The argumentation above can be extended to include transseries with logarithmic sectors, which is expected in the case of resurgence. We only need to require that $k_{\max}^{(\boldsymbol{n}^\star)} = k_{\max}^{(\boldsymbol{n})}$. Also, to simplify the notation we have worked in the case of one-dimensional moduli spaces, but everything carries through in the general case.

The $\mathbb{Z}_2$-symmetry that acts on the instanton actions as $A^{(\boldsymbol{n}^\star)} = -A^{(\boldsymbol{n})}$ is directly related to changing the sign of the string coupling constant, $g_s \to -g_s$. From the point of view of resurgence and large-order relations this seems to be the case. We need to have A and $-A$ sectors in order to have a g_s^2 expansion for sectors with vanishing total instanton action, like the perturbative sector. At the level of the Borel plane, one can see that for each pole there is a symmetrical one with opposite sign. But what happens when there is more resonance, or symmetry, than just $(\mathbb{Z}_2)^q$? In those cases the argument we used in the proofs above do not go through anymore. One should define a new $\star$ operation that accommodates the new resonant symmetry, but in general it will not be represented by a change of sign in g_s, like in (3.110).

There is a strong assumption in theorem 3 involving the symmetry properties of the ambiguities. From the holomorphic anomaly equations we cannot say anything about the ambiguities but we should expect the $(\mathbb{Z}_2)^q$-symmetry to validate this assumption. We will see in chapter 4, regarding the case of local $\mathbb{CP}^2$, that the assumptions are satisfied and that the $(1|1\|0|0\|\cdots)$-sector has a genus expansion, in agreement with the general argument presented in this section.

The antiholomorphic structure of the free energies in the resonant case deviates slightly from the generic situation covered in theorem 2. Let us focus on the case with $\mathbb{Z}_2$-symmetry. The results we present here heuristic and we have no proof for them. For free energies, $F_g^{(n_1|n_2)}$ with $n_1 \neq n_2$, the description of the propagator structure in theorem 2 is still valid, if we also impose $E_{11} = E_{22}$ and $E_{12} = E_{11}^{-1}$. These conditions come directly from the interpretation of the formal variables $E_{\alpha\beta}$ as exponentials $\exp \frac{1}{2}\partial_z A_\alpha \partial_z A_\beta S^{zz}$, and the resonant condition $A_1 = -A_2$. For sectors with vanishing total instanton action, those of the form $(r|r)$, the description is more involved. Consider for example the specific case in which $b^{(\boldsymbol{n})} = 1$. Then $F_{2h}^{(r|r)}$ as expected from theorem 3. The odd free energies $F_{2h+1}^{(r|r)}$ are combinations of exponentials times polynomials, as in the general case. The coefficients $a_{\alpha\beta}$ in the exponentials are still given by the generating function (3.93). However for the special class, $\widehat{\gamma}_{(r|r)}$, the exponential becomes one because it depends on $(\partial_z A^{(r|r)})^2 = 0$. So the term associated to the special class is a just polynomial. The degrees of the polynomials can still be described in terms of λ_b, coming from the generating function. We find, analyzing a few examples, that if $\lambda_{b=1}\left((r|r);\gamma_{(r|r)}\right) = 3,4,5,\ldots$ then the degree of the polynomial is still $3(g+1-\lambda_{b=1})$ with $g=2h+1$. There seems to be an exception for $\lambda=4$ and the corresponding value of a is twice the sum of two squares[4]. In this case the degree is $5(h-1)$. For $\lambda=2$ the degree of the polynomials is $5h$. The case $\lambda=1$ only appears for $a=0$ that corresponds to no exponential. We see that it is difficult to find a common rule for all the cases, and every example should be treated on its own. The problem of discovering the antiholomorphic structure of the free energies is important for two reasons. The first is that knowing the form of the solution beforehand can accelerate the process of computation. The second is that the problem of resummation of the transseries can only be approached when the structure of the transseries is well-understood. From [19, 34, 146] we should expect a theta function structure for the resummation and the restitution of modularity in the holomorphic limit. These are difficult but natural problems in the context of resurgent transseries in topological string theories. See [147] for a recent approach to the resummation of the topological string perturbation series.

3.6 Holomorphicity from large-order

In the previous sections we have stressed the first result that is obtained from the integration of the holomorphic anomaly equations (3.14) for any of the transseries ansätze we have discussed: the instanton actions are holomorphic, independent of the propagator. This result keeps intact the geometrical interpretation of the instanton action as integrals over cycles of the Calabi–Yau geometry, that is, periods [143]. As we discussed in chapter 1, the instanton actions appearing in the exponential monomials of the transseries are, loosely speaking, identified as poles in the Borel plane. Using the alien derivative and the Stokes automorphism one can derive large-order expressions describing the precise factorial growth of the different sectors in the transseries, see (1.40) for example. Manipulating the large-order relations we can extract nonperturbative information. We showed several examples of this in section 1.5.1. In particular, we saw in equation (1.44) how the instanton action can be expressed as a ratio of consecutive perturbative free energies in the large g limit. In the language of this chapter

$$A^2\left(z^i,S^{ij}\right) = \lim_{g\to\infty} \frac{\Gamma\left(2g-b+2\right)}{\Gamma\left(2g-b\right)} \frac{F_g^{(0)}\left(z^i,S^{ij}\right)}{F_{g+1}^{(0)}\left(z^i,S^{ij}\right)}. \tag{3.131}$$

At the beginning of this chapter we argued that the tower of holomorphic anomaly equations could be written into a single equation for the perturbative free energy, and then promoted to be valid for any transseries. This step was justified under the principle of resurgence

[4]Moreover, the value of $a/2$ can be neither in the list

$$\{m_1^2 - m_1 m_2 + m_2^2\}_{1\le m_1<m_2} = \{3,7,12,13,19,21,27,\ldots\}, \tag{3.130}$$

which corresponds to terms with $\lambda=3$, nor can it be a perfect square, which corresponds to $\lambda=2$.

that says that all sectors of the transseries know about each other, and in particular, the perturbative sector knows about the nonperturbative ones. Information of all the system is stored in all the ingredients. From this vague idea we should expect to be able to reproduce nonperturbative results of the previous sections just from perturbation theory and large-order relations. It is difficult to see how far this program can be taken, but the first consequence of the extended holomorphic anomaly equations, the holomorphicity of A, can be done. The idea is to take (3.131), apply a propagator derivative to both terms, and show that the limit is zero. To get to this result we should only be able to use the perturbative holomorphic anomaly equations (2.39) and the large-order growth of the perturbative free energies

$$F_g^{(0)} = c(z_i, S^{ij})\, A(z^i, S^{ij})^{-(2g-b)}\, \Gamma\left(2g-b\right) \left(1+\mathcal{O}(g^{-1})\right), \tag{3.132}$$

from which (3.131) is derived. $c(z^i, S^{ij})$ is related to the one-instanton sector but we do not need to use that information here.

If we take a derivative with respect to S^{ij} on both sides of (3.131), and interchange derivative and limit on the right-hand-side (we will show that the limit exists and is zero so this step is justified), we find

$$\begin{aligned}\partial_{S^{ij}} A^2 &= \lim_{g\to\infty} \frac{\Gamma\left(2g-b+2\right)}{\Gamma\left(2g-b\right)} \left(\frac{\partial_{S^{ij}} F_g^{(0)}}{F_{g+1}^{(0)}} - \frac{\partial_{S^{ij}} F_{g+1}^{(0)}\, F_g^{(0)}}{(F_{g+1}^{(0)})^2} \right) = \\ &= \lim_{g\to\infty} \partial_{S^{ij}} F_g^{(0)} \left(\frac{\Gamma\left(2g-b+2\right)}{\Gamma\left(2g-b\right)} \frac{1}{F_{g+1}^{(0)}} - \frac{\Gamma\left(2g-b\right)}{\Gamma\left(2g-b-2\right)} \frac{F_{g-1}^{(0)}}{(F_g^{(0)})^2} \right). \end{aligned} \tag{3.133}$$

The expression in brackets in (3.133) is, for large g,

$$\frac{A^{2g-b}}{\Gamma\left(2g-b\right) c} \left(1+\mathcal{O}(g^{-1})\right) - \frac{A^{2g-b}}{\Gamma\left(2g-b\right) c} \left(1+\mathcal{O}(g^{-1})\right) = \frac{A^{2g-b}}{\Gamma\left(2g-b\right)} \mathcal{O}(g^{-1}), \tag{3.134}$$

because the leading terms cancel each other. The prefactor in (3.133) can be written in terms of lower genera through the holomorphic anomaly equations (2.39). We compute the large-order growth of the derivatives of the free energies from (3.132),

$$\partial_i F_h^{(0)} \sim \partial_i c\, \frac{\Gamma\left(2h-b\right)}{A^{2h-b}} - c\, \partial_i A\, \frac{\Gamma\left(2h-b+1\right)}{A^{2h-b+1}}, \tag{3.135}$$

$$\begin{aligned}\partial_i \partial_j F_h^{(0)} &\sim \partial_i \partial_j c\, \frac{\Gamma\left(2h-b\right)}{A^{2h-b}} - \left(\partial_i c\, \partial_j A + \partial_j c\, \partial_i A + c\, \partial_i \partial_j A\right) \frac{\Gamma\left(2h-b+1\right)}{A^{2h-b+1}} + \\ &+ c\, \partial_i A\, \partial_j A\, \frac{\Gamma\left(2h-b+2\right)}{A^{2h-b+2}}. \end{aligned} \tag{3.136}$$

The second order derivative term $D_i \partial_j F_{g-1}^{(0)}$ has two terms,

$$\begin{aligned}\partial_i \partial_j F_{g-1}^{(0)} \sim \frac{\Gamma\left(2g-b\right)}{A^{2g-b}} \Big\{ & c\, \partial_i A\, \partial_j A - \left(\partial_i c\, \partial_j A + \partial_j c\, \partial_i A + c\, \partial_i \partial_j A\right) A\, \frac{\Gamma\left(2g-b-1\right)}{\Gamma\left(2g-b\right)} + \\ & + \partial_i \partial_j c\, A^2\, \frac{\Gamma\left(2g-b-2\right)}{\Gamma\left(2g-b\right)} \Big\} = \end{aligned}$$

$$= \frac{\Gamma\left(2g-b\right)}{A^{2g-b}}\,\left\{\mathcal{O}(g^{0})+\mathcal{O}(g^{-1})+\mathcal{O}(g^{-1})\right\}, \tag{3.137}$$

$$\begin{aligned}\Gamma_{ij}^{k}\,\partial_{k}F_{g-1}^{(0)} &\sim \frac{\Gamma\left(2g-b\right)}{A^{2g-b}}\,\Gamma_{ij}^{k}\left\{-c\,A\,\partial_{k}A\,\frac{\Gamma\left(2g-b-1\right)}{\Gamma\left(2g-b\right)}+\partial_{k}c\,A^{2}\,\frac{\Gamma\left(2g-b-2\right)}{\Gamma\left(2g-b\right)}\right\}= \\ &= \frac{\Gamma\left(2g-b\right)}{A^{2g-b}}\,\left\{\mathcal{O}(g^{-1})+\mathcal{O}(g^{-2})\right\}.\end{aligned} \tag{3.138}$$

The second term on the right-hand-side of the holomorphic anomaly equations is $\sum_{h=1}^{g-1}\partial_i F_h^{(0)}\partial_j F_{g-h}^{(0)}$, and it involves free energies with large and small h. To deal with it we use the inequality (1.4) that asymptotic series of this type satisfy (there is no rigorous proof that the perturbative series is of Gevrey 1 type but we assume it is),

$$\left|F_{g}^{(0)}\right|\leq\tilde{c}\,|A|^{-(2g-b)}\;\Gamma\left(2g-b\right). \tag{3.139}$$

Then,

$$\begin{aligned}\left|\sum_{h=1}^{g-1}\partial_{i}F_{g-h}^{(0)}\,\partial_{j}F_{h}^{(0)}\right| &\leq \sum_{h=1}^{g-1}\left|\partial_{i}F_{g-h}^{(0)}\right|\left|\partial_{j}F_{h}^{(0)}\right|\leq \\ &\leq \sum_{h=1}^{g-1}\left(\partial_{i}\tilde{c}\,\frac{\Gamma\left(2g-2h-b\right)}{|A|^{2g-2h-b}}+\tilde{c}\,\partial_{i}|A|\,\frac{\Gamma\left(2g-2h-b+1\right)}{|A|^{2g-2h-b+1}}\right)\times \\ &\quad\times\left(\partial_{j}\tilde{c}\,\frac{\Gamma\left(2h-b\right)}{|A|^{2h-b}}+\tilde{c}\,\partial_{j}|A|\,\frac{\Gamma\left(2h-b+1\right)}{|A|^{2h-b+1}}\right).\end{aligned} \tag{3.140}$$

Using the following inequalities involving products of gamma functions

$$\Gamma\left(2g-2h+p\right)\Gamma\left(2h+p\right)\leq\Gamma\left(2g+p-2\right)\Gamma\left(p+2\right), \tag{3.141}$$

$$\Gamma\left(2g-2h+p\right)\Gamma\left(2h+p+1\right)\leq\Gamma\left(2h+p-1\right)\Gamma\left(p+2\right), \tag{3.142}$$

for $1\leq h\leq g-1$, we conclude

$$\left|\sum_{h=1}^{g-1}\partial_{i}F_{g-h}^{(0)}\,\partial_{j}F_{h}^{(0)}\right|\leq\frac{\Gamma\left(2g-b\right)}{|A|^{2g-b}}\,\left\{\mathcal{O}(g^{-1})+\mathcal{O}(g^{0})+\mathcal{O}(g^{0})\right\}. \tag{3.143}$$

Note that there is a factor of order g coming from the sum in h that has to be taken into accoutn. We put all the ingredients, (3.137), (3.138) and (3.143), into (3.133) and find the bound

$$\left|\partial_{S^{ij}}A^{2}\right|\leq\mathcal{O}\left(g^{-1}\right). \tag{3.144}$$

In the large g limit the right-hand-side goes to zero and this proves the holomorphicity of A.

Strictly speaking, this argument is only valid for the dominant instanton action, that is, the smallest in absolute value, or closest to the origin in the Borel plane. Since the instanton actions depend on the moduli it should be expected that different instanton actions take dominance as we move in moduli space. This is what we will see in local $\mathbb{CP}^2$. There may be instanton actions which are never dominant and for those this argument does not apply. In principle we can see subdominant instanton actions from large-order by doing resummation (we do this in chapter 4), but an argument analogous to the one presented here is completely impractical. Nevertheless, numerical calculations can give evidence to the holomorphicity of these instanton actions.

The type of argument we have used above could in principle provide information about higher instanton sectors of the transseries. The first piece of information we could demand is, like for the instanton actions, the antiholomorphic dependence of the free energies. We have seen that the propagator structure of the transseries is essentially independent of the model we are considering. Since the perturbative holomorphic anomaly equations tells us about the dependence on the propagator in terms of other objects, this type of questions should be the easiest to approach. But let us stress that the large-order relations are powerful enough, not only to determine the antiholomorphic dependence, but the complete free energies, see (1.47) for instance. We will put this principle to work in the next section in the region of moduli space near a conifold point. The relevant value of the propagator there is the holomorphic one because the perturbative free energies are then given by the gap condition (2.82). Another interesting value of the propagator is zero. Due to the polynomial nature of the perturbative this value selects the holomorphic ambiguity. Something similar happens for higher instanton sectors. So we can see the resurgent relations as a family of constraints depending on the value of the propagator and is up to us to find out which member of the family is the most appropriate to extract interesting information. For a vanishing value of the propagator we obtain a net of relations involving essentially the holomorphic ambiguities. We will not follow this route because the most relevant members of the family of relations are the ones selected by a holomorphic limit.

3.7 Fixing the holomorphic ambiguities

The holomorphic ambiguities are holomorphic functions in the complex moduli that are left undetermined by the holomorphic anomaly equations, and can be thought of as integration constants that appear when solving the differential equations with respect to the propagator. The computation of the holomorphic ambiguities at each order in perturbation theory requires extra information not included in the equations. This is the price one pays for the extreme generality of the holomorphic anomaly equations and the fact that it is not a set of equations in g_s but in the moduli. Other techniques, like the topological recursion [18, 142] or the use of a string equation in polynomial matrix model [16], do not suffer this problem, or at least not so severely. In the latter example, the determination of the transseries solution is obtained from a string equation whose exact form depends on the polynomial potential. This implies that the solution is completely fixed up to set of constants that can be fixed by comparing to the double-scaled solution [32, 40].

In topological string theory the determination of the holomorphic ambiguities has different success depending on the nature of the Calabi–Yau geometry. For compact manifolds, the holomorphic ambiguities can only be determined up to finite genus [131, 148, 149]. After that there is not enough information to fix them completely. The situation is different for local, that is noncompact, geometries. In those cases the behavior of the free energies close to special points in moduli space along with regularity properties is enough to determine the ambiguities to all orders [113, 116]. This has been checked in several examples, including local $\mathbb{CP}^2$, and the results compared against known Gromov–Witten invariants.

In section 2.4.3 we reviewed the fixing of the holomorphic ambiguities for local $\mathbb{CP}^2$. The perturbative free energies in the holomorphic limit have only a singularity at the conifold locus, and are regular elsewhere. This restricts the form of the ambiguity to be a rational function of the modulus. The singularity at the conifold point is a pole order $2g-2$ for genus g as dictated by the gap condition

$$\mathcal{F}_g^{[c](0)} = \frac{\mathfrak{c}^{g-1}\, B_{2g}}{2g\,(2g-2)\, t_c^{2g-2}} + \mathcal{O}(t_c^0), \qquad g \geq 2. \tag{3.145}$$

$\mathfrak{c}=3$ for local $\mathbb{CP}^2$, and t_c is the flat coordinate around the conifold point, a period of the geometry given by (2.68). This condition alone determines all the coefficients of the rational function except for one, which must be tuned to match the constant map contribution at the large-radius point (2.84).

We have seen in this chapter that the extension of the holomorphic anomaly equations to admit a transseries ansatz retains the problem of having to fix a holomorphic ambiguity, $f_g^{(\boldsymbol{n})}$, coming from integration, for each instanton sector $\boldsymbol{n}$ and order g. In contrast with the perturbative sector, we have no a priori knowledge of the regularity properties of the nonperturbative free energies near special points in moduli space. However, we expect the full nonperturbative free energy to be represented by a resurgent transseries, and the different sectors to be related by large-order relations. These relations determine some sectors in terms of others, but extracting analytical information can be difficult. In section 3.6 we were able to conclude the holomorphicity of the instanton actions from large-order but not much more. It may seem unlikely that we are able to extract enough information out of resurgence, in an analytical way, to fix the holomorphic ambiguities for higher instanton sectors. However, for some sectors of the transseries this is actually possible, as we will explain below. The reason is the universality property of the conifold point as a phase transition point [24, 127, 150]. The class of theories that present a gap condition behavior as in (3.145) near the conifold point, undergo a phase transition at this point and are described by the $c=1$ string at self-dual radius [127]. The singular behavior of the perturbative free energies near the conifold locus is enough to provide conditions that can fix the holomorphic ambiguity of nonperturbative sectors associated to the conifold point, in a sense we explain below.

3.7.1 Fixing holomorphic ambiguities at the conifold

The computation we present here has been checked at the numerical level in the example of local $\mathbb{CP}^2$, see section 4.2, but it is expected to be valid for other geometries lying in the same universality class. We we will comment on generalizations of this method at the end of this section.

Our goal is to use the large-order relations to extract as much information as we can from the perturbative free energies near the conifold point, and use that information to fix the ambiguity of some sectors of the transseries. The gap condition can be written as

$$\mathcal{F}_g^{[\mathrm{c}](0)} = \frac{\mathfrak{c}^{g-1}\, B_{2g}}{2g\,(2g-2)\, t_{\mathrm{c}}^{2g-2}} + a_{0,g} + a_{1,g} t_{\mathrm{c}} + \cdots\,, \qquad g \geq 2. \tag{3.146}$$

where $a_{0,g}$, $a_{1,g}$, ... are numbers that depend on the particular geometry. We call the series $a_{0,g} + a_{1,g} t_{\mathrm{c}} + \cdots$ the tail of the perturbative free energy. The Bernoulli numbers, B_{2g}, grow like

$$B_{2g} \sim (-1)^{g-1} \frac{2(2g)!}{(2\pi)^{2g}} \left(1 + 2^{-2g} + 3^{-2g} + \cdots\right). \tag{3.147}$$

This means that the free energies near the conifold point grow like

$$\mathcal{F}_g^{[\mathrm{c}](0)} = \frac{1}{2\pi^2} \frac{(2g-1)(2g-3)!}{\left(\frac{2\pi i}{\sqrt{\mathfrak{c}}} t_{\mathrm{c}}\right)^{2g-2}} \left(1 + 2^{-2g} + 3^{-2g} + \cdots\right) + a_{0,g} + a_{1,g} t_{\mathrm{c}} + \cdots \tag{3.148}$$

The dependence of the coefficients $a_{k,g}$ on g is not known but we argue that it is irrelevant when we consider also the limit $t_c \to 0$. Let us be more precise on this by computing the dominant instanton action around the conifold point. It can be obtained from the limit

$$\begin{aligned} A_{\mathrm{c}}^2 &= \lim_{g\to\infty} 4g^2 \frac{\mathcal{F}_g^{[\mathrm{c}](0)}}{\mathcal{F}_{g+1}^{[\mathrm{c}](0)}} \\ &= \lim_{g\to\infty} 4g^2 \frac{(2g+1)(2g-1)!}{(2g-1)(2g-3)!} \left(\frac{2\pi i}{\sqrt{\mathfrak{c}}} t_{\mathrm{c}}\right)^2 + \mathcal{O}\left(t_{\mathrm{c}}^{2g+2}\right) \\ &= \left(\frac{2\pi i}{\sqrt{\mathfrak{c}}} t_{\mathrm{c}}\right)^2. \end{aligned} \tag{3.149}$$

In the last step we notice that the tail $\mathcal{O}(t_{\mathrm{c}}^{2g+2})$ involving the coefficients $a_{k,g}$ goes away in the large g limit. That is, when taking the limit we regard the series in t_{c} as a formal one and as g grows the tail contributes less and less. Also, for this calculation we do not need to use the exponentially suppressed terms 2^{-2g}, 3^{-3g}, etc of the asymptotic expansion of the Bernoulli numbers. Those correction will be necessary when we extract information about higher instanton sectors. The conclusion is that the instanton action is proportional to a period

$$A_{\mathrm{c}} = \frac{2\pi i}{\sqrt{\mathfrak{c}}} t_{\mathrm{c}}. \tag{3.150}$$

Along with A_{c} there is also $-A_{\mathrm{c}}$. We call A_{c} the conifold instanton action.

From the point of view of resurgence explained in chapter 1 we expect the large-order of the perturbative sector to be controlled by different sectors of the transseries. When the resurgent system answers to a bridge equation like (1.59), we can write the large-order relations explicitly from the start, up to the Stokes constants, as in (1.62). In general we should expect a similar structure. Analyzing expression (3.146) we can see that its large-order has the form

$$\mathcal{F}_g^{[\mathrm{c}](0)} \sim \sum_{k=1}^{\infty} \sum_{h=0}^{\infty} \frac{\Gamma(2g - c^{(k)} - h)}{(kA_{\mathrm{c}})^{2g-c^{(k)}-h}} \frac{(S_1)^k}{\pi i} \widetilde{\mathcal{F}}_h^{(k)} + \cdots \tag{3.151}$$

This expression is valid very close to the conifold point, so that any other contributions from the transseries are subleading due to the corresponding instanton actions being larger in absolute value. Recall that $A_{\rm c}$ vanishes at the conifold point, and so does $kA_{\rm c}$ for all k. The objects $\widetilde{\mathcal{F}}_h^{(k)}$ are expected to be free energies, or combinations of free energies, associated to the instanton sector k and order h. Until we can determine their nature we use the notation with a tilde. When the bridge equation determines the resurgence relations, these objects are found to be the higher instanton free energies in the transseries themselves, but we will see in the example of local $\mathbb{CP}^2$ that the situation is slightly more complicated, starting at the two-instanton sector. Let us compute now what the ingredients of (3.151) are.

The instanton action has already been computed above. We focus next on the one-instanton sector. The number $c^{(1)}$, which must be related to the perturbative and one-instanton starting powers, can be calculated as in (1.45),

$$\begin{aligned} 2c^{(1)} - 1 &= \lim_{g\to\infty} 2g \left(1 - \frac{A_{\rm c}^2}{4g^2} \frac{\mathcal{F}_{g+1}^{[{\rm c}](0)}}{\mathcal{F}_g^{[{\rm c}](0)}} \right) \\ &= +1. \end{aligned} \tag{3.152}$$

Here we have used the value of $A_{\rm c}$ in (3.150), (3.146), and the fact that the tail dependence goes away for large g. This value of $c^{(1)} = 1$ can be read directly from (3.148). Now we can start extracting the quantities $\frac{S_1}{\pi i}\widetilde{\mathcal{F}}_h^{(1)}$, for $h = 0, 1, 2, \ldots$, in analogy to (1.46) and (1.47). We find

$$\begin{aligned} \frac{S_1}{\pi i}\,\widetilde{\mathcal{F}}_0^{(1)} &= \lim_{g\to\infty} \frac{A_{\rm c}^{2g-c^{(1)}}}{\Gamma\left(2g-c^{(1)}\right)}\,\mathcal{F}_g^{(0)} = \\ &= \lim_{g\to\infty} t_{\rm c}\,\frac{i\,(2g-1)}{2\pi\sqrt{\mathfrak{c}}\,(g-1)} = \frac{i}{\pi\sqrt{\mathfrak{c}}}\,t_{\rm c} = \frac{1}{2\pi^2}\,A_{\rm c}, \end{aligned} \tag{3.153}$$

and

$$\begin{aligned} \frac{S_1}{\pi i}\,\widetilde{\mathcal{F}}_1^{(1)} &= \lim_{g\to\infty} \frac{A_{\rm c}^{2g-2}}{\Gamma\left(2g-2\right)} \left(\mathcal{F}_g^{(0)} - \frac{\Gamma\left(2g-1\right)}{A_{\rm c}^{2g-1}}\,\frac{S_1}{\pi i}\,\widetilde{\mathcal{F}}_0^{(1)} \right) = \\ &= \lim_{g\to\infty} \left(\frac{2g-1}{2\pi^2} - \frac{2g-2}{2\pi^2} \right) = \frac{1}{2\pi^2}. \end{aligned} \tag{3.154}$$

In both cases the dependence on the coefficients $a_{k,g}$ is washed away by the large g limit.

The higher-loop one-instanton free energies are zero as can be shown by induction. We illustrate the base case

$$\begin{aligned} \frac{S_1}{\pi i}\,\widetilde{\mathcal{F}}_2^{(1)} &= \lim_{g\to\infty} \frac{\pi A_{\rm c}^{2g-3}}{\Gamma\left(2g-3\right)} \left(\mathcal{F}_g^{(0)} - \frac{\Gamma\left(2g-1\right)}{\pi A_{\rm c}^{2g-1}}\,\frac{S_1}{2\pi i}\,\widetilde{\mathcal{F}}_0^{(1)} - \frac{\Gamma\left(2g-2\right)}{\pi A_{\rm c}^{2g-2}}\,\frac{S_1}{2\pi i}\,\widetilde{\mathcal{F}}_1^{(1)} \right) = \\ &= \lim_{g\to\infty} \frac{\sqrt{\mathfrak{c}}}{4\pi^3 i\,t_{\rm c}}\,\frac{1}{\Gamma\left(2g-3\right)} \left(\frac{\Gamma\left(2g\right)}{2\left(g-1\right)} - \Gamma\left(2g-1\right) - \Gamma\left(2g-2\right) \right) = 0. \end{aligned} \tag{3.155}$$

The fact that the large-order relation (3.151) at the one-instanton level truncates allows us to go immediately to the two-instanton sector. For this we just need to move the two free energies, $\widetilde{\mathcal{F}}_h^{(1)}$, $h = 0, 1$ in (3.151) to the left-hand-side and repeat the same computation we did above, but for the new large-order formula, that is

$$\begin{aligned}
\mathcal{F}_g^{[c](0)} &- \frac{\Gamma(2g-1)}{A_c^{2g-1}}\frac{S_1}{\pi i}\widetilde{\mathcal{F}}_0^{(1)} - \frac{\Gamma(2g-2)}{A_c^{2g-2}}\frac{S_1}{\pi i}\widetilde{\mathcal{F}}_1^{(1)} \\
&= \mathcal{F}_g^{[c](0)} - \frac{(2g-1)(2g-3)!}{2\pi^2 A_c^{2g-2}} \\
&\sim \frac{1}{2\pi^2}\frac{(2g-1)(2g-3)!}{A_c^{2g-2}}\left(1 + 2^{-2g} + 3^{-2g} + \cdots\right) + a_{0,g} + \cdots - \frac{(2g-1)(2g-3)!}{2\pi^2 A_c^{2g-2}} \\
&= \frac{1}{2\pi^2}\frac{(2g-1)(2g-3)!}{(2A_c)^{2g-2}}\frac{1}{2^2}\left(1 + \left(\frac{3}{2}\right)^{-2g} + \left(\frac{4}{2}\right)^{-2g} + \cdots\right) + a_{0,g} + \cdots \qquad (3.156)
\end{aligned}$$

Notice that the leading contribution to the asymptotics of the Bernoulli numbers is cancelled because that information has already been used. Also, the large-order growth of the new object $\mathcal{H}_g^{(1)}$ is very similar to that of $\mathcal{F}_g^{(0)}$ so the results will be the same modulo factors of two. The functions $\widetilde{\mathcal{F}}_h^{(2)}$ truncate after $h = 1$, as they did for the one-instanton sector, giving

$$c^{(2)} = 1, \quad \frac{(S_1)^2}{\pi i}\widetilde{\mathcal{F}}_0^{(2)} = \frac{1}{2}\frac{A_c}{2\pi^2}, \quad \frac{(S_1)^2}{\pi i}\widetilde{\mathcal{F}}_0^{(2)} = \frac{1}{2^2}\frac{1}{2\pi^2}. \qquad (3.157)$$

One can go on to higher and higher sectors and obtain the general result for the large-order growth of the perturbative free energies in the holomorphic limit of the conifold point (3.151) with

$$c^{(n)} = 1, \qquad (3.158)$$

$$\frac{(S_1)^n}{\pi i}\widetilde{\mathcal{F}}_0^{(n)} = \frac{1}{n}\frac{A_c}{2\pi^2}, \qquad (3.159)$$

$$\frac{(S_1)^n}{\pi i}\widetilde{\mathcal{F}}_1^{(n)} = \frac{1}{n^2}\frac{1}{2\pi^2}, \qquad (3.160)$$

$$\widetilde{\mathcal{F}}_{g\geq 2}^{(n)} = 0. \qquad (3.161)$$

These expressions for higher instanton free energies appear when studying the discontinuity on the string coupling plane of the $c = 1$ string at self-dual radius, which is a first step to a resurgent analysis of the theory [36]. The information about the specific geometry of local $\mathbb{CP}^2$ has been removed expcept for the particular dependence of the instanton action on the complex modulus. As we said before, (3.151) is not the whole story because other sectors exist and are relevant to the large-order growth for points in moduli space away form the conifold.

Let us summarize the computation we have done. By focusing on the holomorphic limit of the perturbative free energies near the conifold point we have been able to compute, in closed form, the leading large-order growth of these free energies. Beyond this leading order there will be other sectors of the transseries, but near the conifold point they are subleading because nA_c can be made arbitrarily small near $z = -1/27$. The importance of this calculation is that it provides the necessary condition to fix the holomorphic ambiguities of, at least, the one and two-instanton free energies associated to the conifold point. We will see this in chapter 4 for the case of local $\mathbb{CP}^2$, where compare the holomorphic limit of free energies computed from the holomorphic anomaly equations will be compared against the analytic functions computed above, and we will solve for the ambiguities.

3.7.2 Fixing holomorphic ambiguities in other cases

The universal character of the conifold point, and the fact that it is a phase transition point for the perturbative free energies, is what has allowed us to perform a computation to all instanton numbers and all orders. Models with this gap condition behavior are said to lie in the universality class of the $c = 1$ string at self-dual radius. There are other topological string theories which lie in other universality classes. The phase transition of the free energies at the corresponding singular point is of a different form and is controlled by a characteristic exponent γ. If τ denotes a flat coordinate around the singular point, then

$$\mathcal{F}_g^{(0)} \sim H_g \, \tau^{(1-g)(2-\gamma)}, \quad \text{as } \tau \to 0, \tag{3.162}$$

for $g \geq 2$. Here H_g is a genus dependent number, expected to grow like $(2g)!$. For topological string theory on local $\mathbb{CP}^2$ $\gamma = 2$, $\tau = t_c$, and H_g is proportional to the Bernoulli numbers, so that (3.146) is recovered. Another important class is $\gamma = \frac{1}{2}$, the class of $c = 0$ string theory or pure 2d gravity, see [150] for worked out geometries in this class. For general γ there is a correspondence with so-called minimal models [151], conformal field theories with central charge characterized by two integers p and q,

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \quad \text{and} \quad \gamma_{p,q} = -\frac{2}{p+q-1}, \tag{3.163}$$

see [9] for a computation in terms of matrix models. Even though topological string theories belonging to these general classes have not been studied very much, the singular behavior of the free energies could be enough to derive a set of conditions like (3.158)-(3.161), which can be used to fix higher instanton holomorphic ambiguities. A precise computation would require knowledge of the factorially growing coefficients H_g, and of the nature of the subleading tail in(3.162). For the class $c = 1$, the coefficients are read from (3.146), and can be identified as those appearing in the Gaussian matrix model and in the double-scaling limit of other models. In general classes, the double-scaling limits turn out to be classified in hierarchies like the KdV one [152]. In principle it should be possible to compute the coefficients H_g from there but it is a complicated problem.

With respect to other special points in moduli space that have transseries sectors naturally associated to them, there is not a systematic way to approach the large-order behavior of holomorphic perturbation theory. For starters there is no general classification of Calabi–Yau threefold singularities and it may be difficult to construct one [153]. This prevents an algorithmic approach to the complete integration of the extended holomorphic anomaly equations. The alternative is, as usual, to turn to resurgence and large-order analysis of the sectors that we can integrate and have their ambiguities fixed. Codified in their factorial growth is the fixing condition for every sector, even if it hides very deep in the asymptotics. Further understanding of these ideas in a rigorous geometric formulation could shed some light on the nature of the higher instanton free energies near special points in moduli space.

Chapter 4

Resurgence in local $\mathbb{CP}^2$

In this chapter we develop the resurgent study of topological string theory on the mirror of local $\mathbb{CP}^2$. It is based on the results presented in [154]. The geometry and integration of the perturbative sector was explained in section 2.4. There we saw that the complex structure moduli space is one-dimensional, parametrized by z or ψ. The antiholomorphic dependence is captured by a single propagator S^{zz}. The various holomorphic limits, depending on the frame, are given in section 2.4.2, although some refinement will be needed. The large-order analysis in the holomorphic limit will be important in order to understand how to fix the holomorphic ambiguity. We have already explored this point in section 3.7.

The first section of this chapter focuses on the question of what different instanton actions appear in the transseries for local $\mathbb{CP}^2$. Since the instanton actions are, for all purposes, holomorphic ambiguities, we must rely on a large-order analysis to approach the problem. We show the holomorphicity of the dominant instanton actions, as expected form the extended holomorphic anomaly equations and proved in section 3.6. We check that instanton actions are periods of the geometry and compare dominance between them. This last exercise uncovers a more complicated structure for the Borel plane than it was expected.

A deeper analysis of the perturbative large-order involves one-instanton and two-instanton sectors. Focusing on a region in moduli space near around the conifold we are able to reproduce the factorial growth to leading and subleading order in terms of free energies computed from the holomorphic anomaly equations. At this point we can already see that the large-order relations found on matrix models and string related differential equations have to be slightly modified, which implies that a simple bridge equation like (1.31) is not the right one for this problem.

Next we analyze the large-order growth of the one-instanton sector. There the presence of resonance between $+A$ and $-A$ is explicit, as was the case in the examples mentioned above. The free energies controlling the growth can be computed form the holomorphic anomaly equations, and the holomorphic and nonholomorphic dependence is checked against numerical calculations.

The large-order numerical analysis requires some specific techniques of convergence acceleration for limits and series resummation that were explained in chapter 1. We will make reference to them as they appear.

We finish this chapter with a general discussion on the structure of the large-order relations, the transseries, and the resurgence structure that is behind local $\mathbb{CP}^2$.

4.1 Instanton actions from large-order

As we saw in chapter 1, instanton actions can be seen from three complementary points of view. As the coefficients sitting in the exponential monomials of the transseries, e^{-A/g_s}; as poles in the Borel plane of a given asymptotic sector; or controlling the large-order factorial growth of the coefficients of an asymptotic sector. We will touch on all these interpretations during this section.

In the description of the transseries solutions that we explained in chapter 3 we emphasized that the instanton actions are holomorphic. They are expected to keep their geometrical interpretations as linear combinations of periods. A general large-order analysis near the conifold point showed that the dominating instanton action there is actually proportional to the conifold flat coordinate, t_c. However, such general analyses are seldom possible to perform and one has to rely on numerics. Even then large-order growth is mostly sensitive to dominant instanton actions, that is smallest in absolute value. We cannot be sure if there are other larger instanton actions unless we go very deep into the large-order growth. But this also means that the corresponding sectors will be greatly suppressed in the transseries (assuming positive large real part) and will have small effect in the final resummation. Another important point on this topic is that the instanton actions are not all linearly independent. We are assuming that they are combinations of periods, that is, solutions of the Picard–Fuchs equation (2.52), and there are only three independent solutions. It may well be that there are in fact $\mathbb{Z}$-linear relations between instanton actions. We could interpret this as resonance between all the corresponding sectors, and that should be reflected in the various large-order relations. But we could also interpret this integer combination of instanton actions as a multi-instanton sector that is not independent of the rest. The two interpretations require to different transseries. In order to know which one is correct one must study the large-order of different sectors carefully, looking for inconsistencies on one or the other interpretation. The holomorphic anomaly equations are not powerful enough to tell them apart. A g_s-equation for the local $\mathbb{CP}^2$ free energy could provide that information. In this chapter we are going to focus on the numerical results and provide analytical formulae, computed from the holomorphic anomaly equations, to match them. We will rely on some interpretations but it will take a more thorough and deeper analysis and a larger interpretative framework, to put each part of the puzzle in its proper place.

4.1.1 Dominant instanton actions

The perturbative free energies, $F_g^{(0)}$, grow factorially with the genus g. This can be expected from the behavior of the holomorphic limits, $\mathcal{F}_g^{(0)}$, near the conifold and large-radius points, (2.82) and (2.84). We see an explicit $(2g)!$ growth due to the Bernoulli numbers. This Gevrey-1 growth is found for all present for all values of the complex modulus and propagator,

$$F_g^{(0)} \sim \frac{\Gamma(2g-b)}{A_{\text{dom}}^{2g-b}} c(z, S^{zz}). \tag{4.1}$$

This formula expressing the Gevrey-1 property is the leading term of a general large-order formula that involves infinitely many other sectors of the transseries. We will worry about those later in this chapter. This section is devoted only to the dominant instanton actions, $A_{\rm dom}$.

As we move in moduli space $A_{\rm dom}$ can change abruptly. This is because another instanton actions that was larger in absolute value has become smaller, and thus, it is less exponentially suppressed as g goes to infinity. Therefore, we can write

$$A_{\rm dom} = \min_{\alpha}\{A_\alpha\}, \tag{4.2}$$

where α runs over the different sectors in the transseries, and the minimum is understood with respect to the absolute value—the instanton actions can be complex in general but their relevance to the perturbative large-order is determined by the absolute value. Before analyzing in more detail $A_{\rm dom}$ as a function on moduli space, we must provide numerical evidence of the claims made in chapter 3 about the holomorphicity of the instanton actions.

At the numerical level we can extract the dominant instanton action as explained in chapter 1 or equation (3.131),

$$A^2_{\rm dom} = \lim_{g\to\infty} 4g^2 \frac{F^{(0)_g}}{F^{(0)}_{g+1}}. \tag{4.3}$$

This limit is independent of the constant b because it appears in a subleading term that goes to zero in the limit. Due to the $(2g)!$ growth we can only extract the square of $A_{\rm dom}$. This is related to the fact that there are instanton actions of opposite signs contributing and implying the $(2g)!$ instead of a simple $g!$ growth. The numerical limit in (4.3) is obtained using Richardson transforms on the right-hand-side sequence as explained in section 1.5.2. To show holomorphic independence of the instanton actions we focus on a point in moduli space, that is, a fixed value of ψ (or z), and study how $A^2_{\rm dom}$ changes with S^{zz}. There are some practical limitations to this approach due to the restricted number of perturbative free energies to work with. The sequence of values $\{4g^2F^{(0)}_g/F^{(0)}_{g+1}\}^{g=113}_{g=2}$ starts converging to $A^2_{\rm dom}$ at different critical values of g depending on S^{zz}. This means that if we depart too much from the holomorphic value of S^{zz}—for which convergence is good—we will fail to see convergence because it happens outside the working domain, $g \leq 114$. We can see an example of this in figure 4.1. This practical inconvenience will be present throughout this chapter, and it will force us to restrict ourselves to values of the propagator for which we can show convergence and Richardson extrapolation can be used effectively.

Having commented on the practical domain of work, we can present evidence of holomorphicity of the dominant instanton action near the conifold point. It is shown in figure 4.2. Each point in the 3d plot is obtained as a Richardson extrapolation for fixed value of ψ and S^{zz}. If the sequence of numerical points is stable we can take a big number of Richardson transforms (large n in equation (1.52)) and achieve great precision. We see that the value of $A^2_{\rm dom}$ remains stable as we change the value of the propagator until we move too far away from the holomorphic limit and the effected illustrated in figure 4.1 takes over. A similar exercise can be done in other points in moduli space.

Let us now explore the moduli space dependence of $A_{\rm dom}$ and study the different regions of dominance. Advancing results that we will explain later, we are going to parametrize the

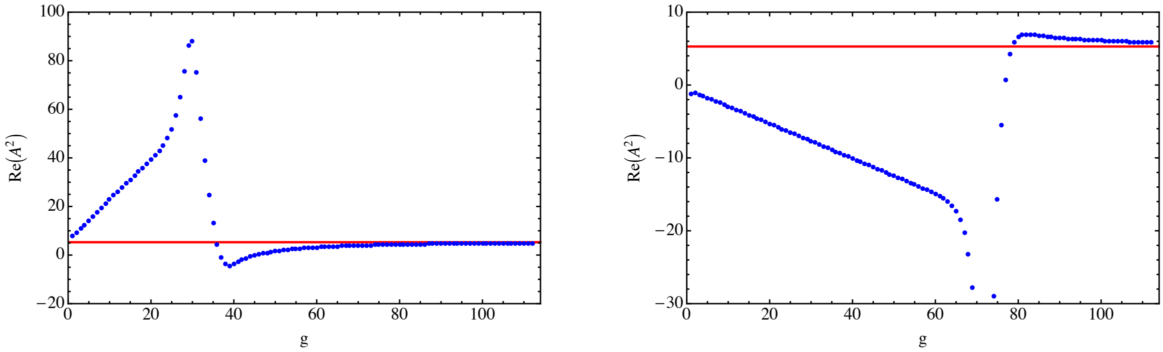


Figure 4.1: Convergence of the limit in the right-hand-side of (4.3) for $\psi = 2e^{i\pi/6}$ and values of the propagator far from the conifold holomorphic limit, $S^{zz} = (4+4i)S^{zz}_{[c],\text{hol}}$ (left) and $S^{zz} = (5+6.5i)S^{zz}_{[c],\text{hol}}$ (right). The critical value of the genus for which convergence begins increases as S^{zz} increases. $S^{zz}_{[c],\text{hol}}$ can be taken as the relevant scale for the value of the propagator.

complex structure moduli space with ψ, which is related to the original z by $z = (-3\psi)^{-3}$. The ψ-plane is a triple cover of the z-plane. Thus we divide the former in three wedges as shown in figure 4.3. We are going to work on wedge 1, but the other wedges will become relevant soon. As we explained in section 2.4, the special points in the ψ-plane are the large-radius point at infinity, and the three conifold points at the cubic roots of unity. There is a $\mathbb{Z}_3$-symmetry relating all the conifold points that can be associated to the orbifold point at $\psi = 0$.

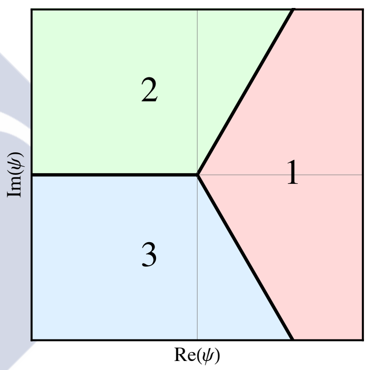


Figure 4.3: ψ-plane split in three wedges, each identified with a complete z-plane.

If we consider the line in moduli space passing through the conifold point, $\psi = 1$, and $\psi = 0$, we find that A^2_{dom} is negative, or equivalently, that A_{dom} is purely imaginary. Taking the limit (4.3) for several values of ψ on this line we obtain the result in figure 4.4. We see two different instanton actions building up to A_{dom}. One of them, on the right part of the plot, has the value $4\pi^2 i$. This is the instanton action we would find using the holomorphic value of the free energies near the large-radius point,

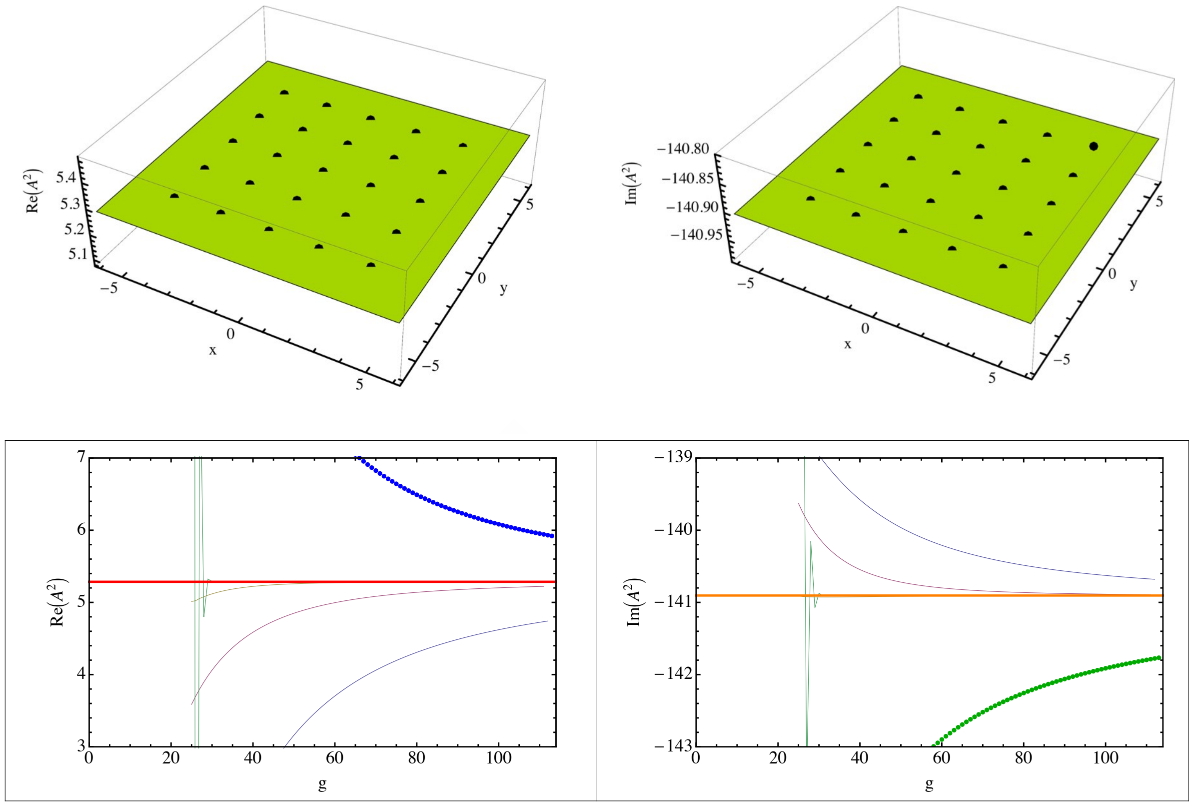
$$\mathcal{F}_g^{(0)[LR]} = \frac{(-1)^{g-1} 3 B_{2g-2} B_{2g}}{4g(2g-2)(2g-2)!} + \mathcal{O}(z). \tag{4.4}$$

Putting this expression in (4.3) and taking the large g limit we find

$$A^2_{\text{dom}} = (4\pi^2 i)^2 + \mathcal{O}(z). \tag{4.5}$$

The $4\pi^2 i$ term comes directly from the constant map contribution in (4.4) and was addressed in detail in [36]. The extra term $\mathcal{O}(z)$ can be shown to vanish if we assume that A_{dom} has to be a combination of periods [143]. Indeed, if that assumption is true then we can write

$$A_{\text{dom}} = a\,T + b\,\partial_T F_0^{(0)} + c, \tag{4.6}$$



$$A^2 = 5.285\,620\,821\,373 - 140.904\,851\,596\,926\, i$$

14 Richardson Transforms $= 5.285\,620\,821\,381 - 140.904\,851\,596\,903\, i$.

Figure 4.2: On top, a three dimensional representation of the holomorphicity of the instanton action at $\psi = 2\mathrm{e}^{i\pi/6}$. We display the real and imaginary parts of (4.3) for different values of the propagator, $S^{zz} = S^{zz}_{[c],\text{hol}} \cdot (1 + x - i\, y)$. The horizontal plane represents the theoretical value given by (4.8). Below, an example of one of the points, $(x, y) = (4, -4)$. If S^{zz} is not too large the numerical convergence can be accelerated greatly. Otherwise, like the point $(x, y) = (4, 4)$, convergence starts at too high genus, see figure 4.1

with T and $\partial_T F_0^{(0)}$ given by (2.59) and (2.60), respectively, for some constants a, b, c. But T and $\partial_T F_0^{(0)}$have a logarithmic dependence on z of different degree, so no combination of them can ever amount to $\mathcal{O}(z)$. Thus, $a = b = 0$ and $c = 4\pi^2 i$. This constant instanton action is always going to be present because the constant map contribution is universal. Once understood, it is best to remove it from the perturbative free energies,

$$F_g^{(0)} \longrightarrow F_g^{(0)} - \frac{(-1)^{g-1}\, 3\, B_{2g-2} B_{2g}}{4g\,(2g-2)\,(2g-2)!}. \tag{4.7}$$

The other instanton action in figure 4.4, which dominates around the conifold point, was

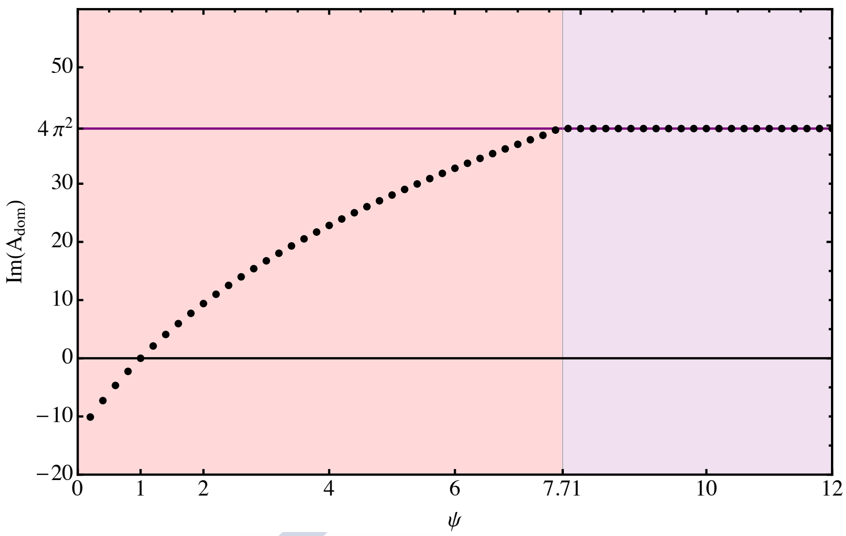


Figure 4.4: Dominant instanton action around $\psi = 1$ as a function of $\psi \in \mathbb{R}$. Two different instanton actions can be distinguished between the conifold and large-radius point. The constant map contribution is included and produces an instanton action $4\pi^2 i$.

already computed in section 3.7 based on the gap condition (2.82). It is

$$A_1(\psi) = \frac{2\pi i}{\sqrt{3}} t_c(\psi), \tag{4.8}$$

where $t_c(\psi)$ is the flat coordinate around the conifold point $\psi = 1$, (2.68),

$$t_c = \frac{2\pi}{\sqrt{3}} \left(\frac{3\psi}{\Gamma\left(\frac{2}{3}\right)^3} \, {}_3F_2 \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{2}{3}, \frac{4}{3} \middle| \psi^3 \right) - \frac{\frac{9}{2}\psi^2}{\Gamma\left(\frac{1}{3}\right)^3} \, {}_3F_2 \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{4}{3}, \frac{5}{3} \middle| \psi^3 \right) - 1 \right). \tag{4.9}$$

If we move around in moduli space we are going to find a picture similar to figure 4.4, but with slightly different takeover point. A very important exception occurs when $\arg(\psi) = \pm\pi/3$. These are the boundaries of wedge 1 in figure 4.3, or the real positive z-line. More on this separate case below.

If we remove the constant map contribution from the free energies, as in (4.7), the instanton action $4\pi^2 i$ disappears in figure 4.4. A_1 takes its place until a smaller instanton action takes over, see figure 4.5. This instanton action is associated to the large-radius point, and just like for the conifold, it is proportional to the corresponding mirror map period,

$$A_K = 4\pi^2 i T, \tag{4.10}$$

where the Kähler parameter T is represented in closed form by a Meijer function,

$$T(\psi) = -\frac{1}{2\pi i} \frac{\sqrt{3}}{2\pi} G^{22}_{33} \left(\begin{matrix} \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 0 \end{matrix} \middle| -\frac{1}{\psi^3} \right). \tag{4.11}$$

The numerical analysis near the large-radius point is not as stable and precise as near the conifold point. This can be seen for instance in figure 4.6 that compares the numerical limit

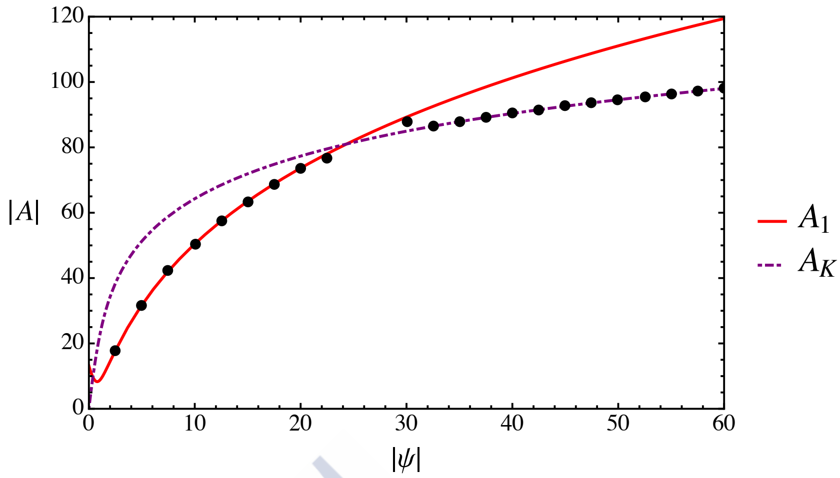
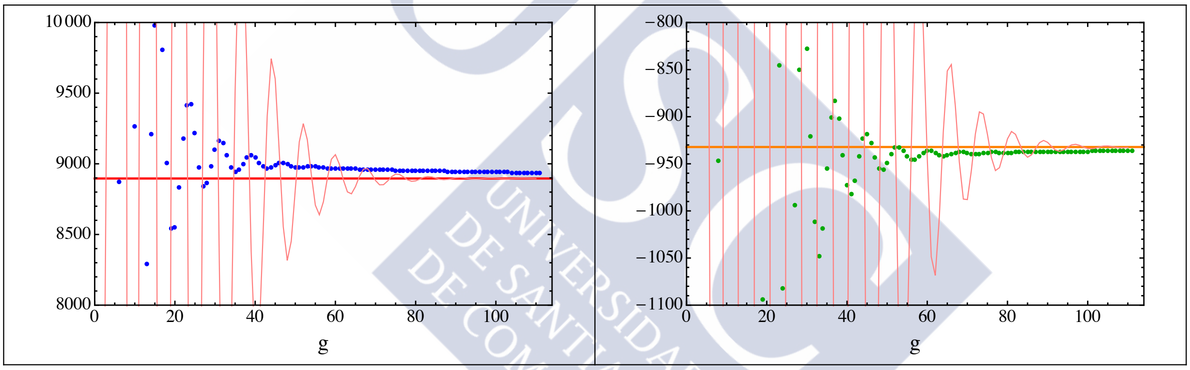


Figure 4.5: Transition between dominant instanton actions controlling the perturbative the large-order growth *without* the constant map contribution. For sufficiently large $|\psi|$ A_1 is taken over by $A_{\rm K}$ as dominant. Here $\arg(\psi) = \pi/4$.



$A_{\rm K}^2 = 8\,896.1 - 932.2\,i$

1 Richardson Transform $= 8\,888.8 - 929.8\,i$

Figure 4.6: Real and imaginary parts of the dominant instanton action squared (4.3) for $\psi = 50\,{\rm e}^{i\pi/4}$, and with one Richardson transform. For values ψ near the large-radius point the limit converges slowly and with oscillations, which makes limits the effectiveness of Richardson extrapolation. Agreement with the analytic value of $A_{\rm K}^2$, (4.10), is around 0.1%.

(4.3) and one Richardson transform with the analytic value of the instanton action. We see how the convergence is oscillatory and the accelerating technique is not well suited for it. Nevertheless, there is agreement between both calculations to less than one percent. This numerical instability is also reflected in the poor resolution of the transition between $|A_1|$ and $|A_{\rm K}|$ in figure 4.5, where two of the points converge outside of the picture.

Borel plane and A_K

So we see that after removing the constant map contribution form the free energies, there is an instanton action related to the Kähler parameter, T, that becomes dominant near the large-radius point. Its explicit expression is given in (4.11). Around the conifold point, and all the way down to $\psi = 0$, we expect $|A_\mathrm{K}|$ to be larger than $|A_1|$ because we have seen that A_1 is the dominant instanton action in that area. However if we plot $|A_\mathrm{K}|$ in that range we find that it approaches zero as $\psi \to 0$, meaning that at some small value of ψ, A_K is the least instanton action. How can A_K have the smallest absolute value and still not be controlling the leading large-order growth? To answer this question we must go back to the origins of the large-order relation (4.1) that we reviewed in chapter 1. The factorial growth of the perturbative sector is determined by the poles of the Borel transform of $F^{(0)}(g_s)$,

$$\mathcal{B}[F^{(0)}](\xi) = \sum_{g=2}^{\infty} \frac{F_g^{(0)}}{(2g-2)!} \xi^{2g-2}. \tag{4.12}$$

In the general theory, the singularity structure of the Borel plane is codified in the action of the alien derivatives and the Stokes automorphism. There can be poles and logarithmic branch cuts endowing the Borel plane with a potentially complicated Riemann sheet structure. Up to leading order, the large-order growth of the perturbative coefficients is determined by the closest singularity to the origin. However, as we vary the complex structure modulus, ψ, this picture changes accordingly. This change does not beed to be continuous. Indeed, due to the branch cuts present, some of the poles may disappear as the modulus is varied. A known explanation for this is that the pole has moved to another Riemann sheet of the Borel complex surface and it is not visible anymore. In order to see if the pole corresponding to A_K actually disappears for some value of ψ as it approaches zero, we need to have a look at the Borel plane. Since we do not know all the perturbative free energies to compute the exact Borel transform—and even then resummation could be challenging—we have to rely on Padé approximants of a truncation (partial sum) of the Borel transform. Padé approximants have the form of a rational function and the poles tend to mimic the singularities in the actual Borel plane. If we do this numerical exercise we can see that a pole corresponding to A_K disappears for some value of ψ around 1, a region where A_K is still very subleading. We show this in figure 4.7. This result points to the fact that the string free energy has a complicated multibranched Borel structure in which the so-called higher Stokes phenomenon [155] is present and has a relevant role. This phenomenon deserves more investigation both at the numerical and, if possible, analytical level.

Instanton action at the large-radius point

We finish this section at the large-radius point, $\psi^{-1} = 0$. We have seen that for ψ near 0 the conifold instanton action A_1 is dominant. As we increase the value of ψ inside wedge 1 of figure 4.3 there is a change of dominance to the Kähler instanton action A_K. This instanton action dominates all the way to the large-radius point, increasing as ψ reaches infinity. At $\psi^{-1} = 0$, A_K diverges logarithmically. Since all the other instanton actions are subdominant, that is, they have larger absolute values, we must conclude that the Borel plane has to be

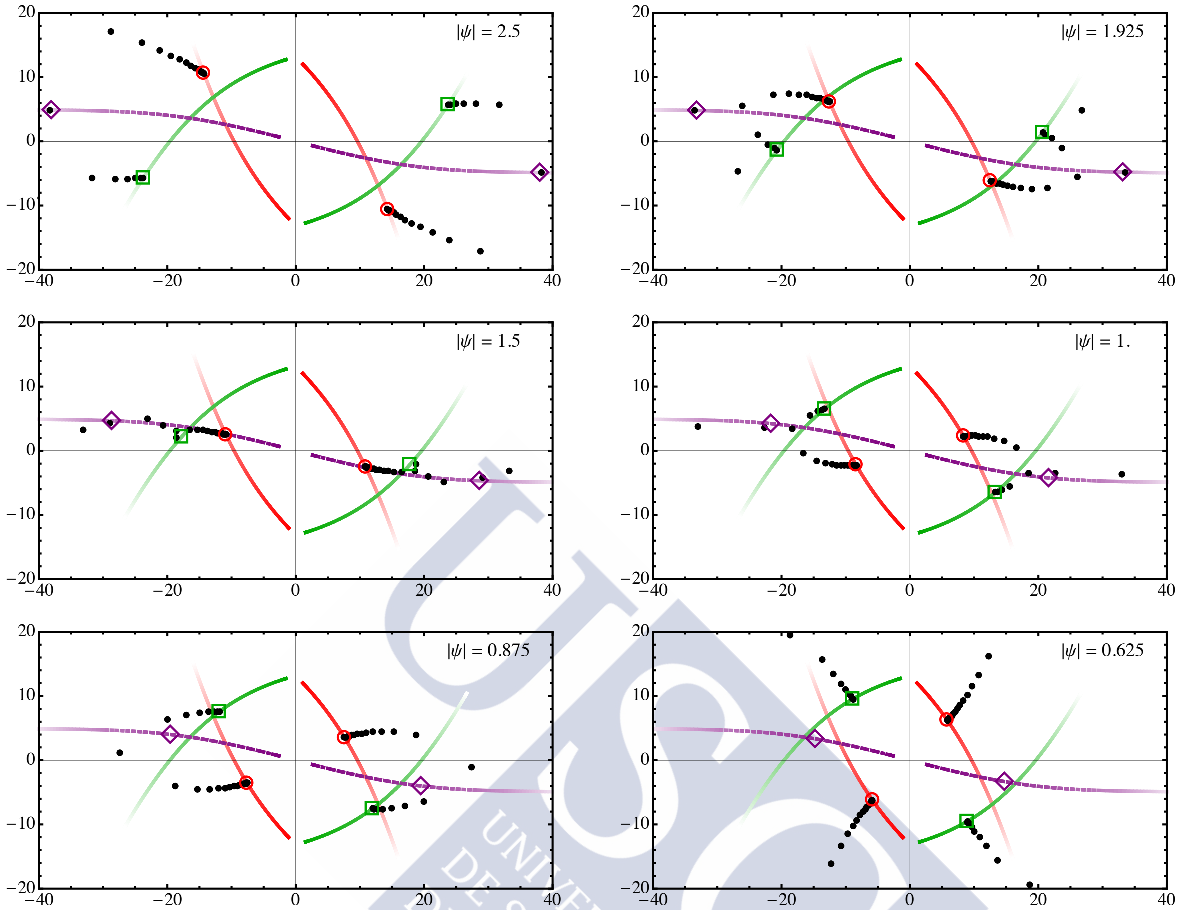


Figure 4.7: Padé analysis of the Borel singularity structure. We illustrate snapshots of the Borel plane for different absolute values of the modulus ψ. The red circle shows the analytical value of the conifold action A_1, and the red line the trajectory it follows as the modulus is varied. Similarly for the green square and trajectory, associated to the conifold action A_2, see section 4.1.2; and for the purple rhombus and trajectory, associated to the large-radius action $A_{\rm K}$. The black dots are the Padé poles of the Borel transform of the perturbative free energy, and their accumulation signals a branch cut. Around $|\psi| \simeq 1$ the Padé pole associated to the large-radius action disappears from the principal Riemann sheet of the perturbative sector.

free of singularities exactly at the large-radius point. This means that the perturbative free energies cannot be of Gevrey-1 type and the $F^{(0)}(g_s)$ series must be convergent. However, this is not what we find. First of all, let us recall that we are working with a version of the free energies where the constant map contribution has been removed. Therefore, in the holomorphic limit, the free energies are exactly zero at the large-radius point. The zero series is trivially convergent so let us explore the antiholomorphic dependence. If we fix the value of the propagator and take the limit $z \to 0$ (equivalent to $\psi^{-1} \to 0$) we get a divergence for the perturbative free energies at each genus. To prevent this we must rescale the propagator

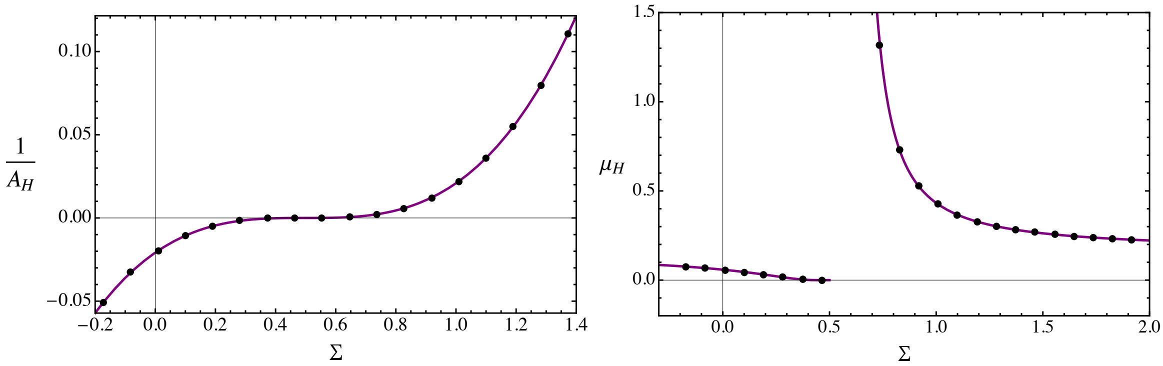


Figure 4.8: On the left, comparison of the analytical expression for the instanton action at the large-radius point, (4.16), against numerical calculation from large order with three Richardson transforms. We represent the inverse of the instanton action versus Σ. A_H diverges for $\Sigma = \frac{1}{2}$. On the right plot we check formula (4.17). μ_H is nonanalytic at $\Sigma = \frac{1}{2}$.

accordingly, in the same way that the holomorphic limit does,

$$S^{zz}_{[LR],\text{hol}} = z^2 \left(\frac{1}{2} + 9z + \cdots \right). \tag{4.13}$$

So we define a new antiholomorphic variable Σ by $S^{zz} := z^2 \Sigma$, and the rescaled free energies at the large-radius point,

$$H_g^{(0)}(\Sigma) := \lim_{z \to 0} F_g^{(0)}(z, z^2\, \Sigma). \tag{4.14}$$

If we numerically analyze the large-order of $H_g^{(0)}$ we find

$$H_g^{(0)} \sim \frac{\Gamma(g-1)}{A_H(\Sigma)^{g-1}}\, \mu_H(\Sigma). \tag{4.15}$$

$A_H(\Sigma)$ is an instanton action whose functional dependence can be guessed easily from the numerics,

$$A_H(\Sigma) = \frac{6}{\left(\Sigma - \frac{1}{2}\right)^3}. \tag{4.16}$$

See the left plot in figure 4.8. The function $\mu_H(\Sigma)$ should be related to the one-instanton sector associated to this instanton action. After some guess work we find,

$$\mu_H(\Sigma) = \frac{1}{2\pi} \exp \frac{\frac{1}{2}}{\Sigma - \frac{1}{2}}. \tag{4.17}$$

There are two important things to notice from the large-order behavior in (4.15). The first is that the instanton action is nonholomorphic. From the results derived in chapter 3 and the numerics in this section we would have expected no instanton action, or in any case, a holomorphic one. Since we are looking at a particular point in moduli space this means a constant instanton action. But we find a simple dependence in Σ. Notice that the instanton action A_H blows up at $\Sigma = \frac{1}{2}$. This is exactly the holomorphic limit of the original $F_g^{(0)}$ free energies which are vanishing. More precisely, $H_g^{(0)} = \left(\Sigma - \frac{1}{2}\right)^{2g-3} \mathrm{Pol}(\Sigma; g)$. The factor $\mu(\Sigma)$ also blows up at $\Sigma = \frac{1}{2}$. The second important feature of this behavior is that the factorial growth is not $(2g)!$ anymore, but simply $g!$ (or actually $(g-2)!$). Let us stress that the new series $H^{(0)}(g_s)$ defined from $F_g^{(0)}$ by (4.15) is still a series in g_s^2. This may be a simple curiosity without further implications but investigating it further may help understand how to take simultaneous limits of the propagator and the complex structure modulus. It would also be nice to see in more detail how this transmutation from a $(2g)!$ to a $g!$ growth happens exactly.

4.1.2 Other instanton actions

Beyond the dominant instanton actions we have analyzed above there may be, and indeed there are, other instanton actions that cannot be directly detected at leading order. In this subsection we show that there are at least two other instanton actions besides A_1 and A_{K} that we denote by A_2 and A_3. Just like A_1 is related to the first conifold point at $\psi = 1$, A_2 and A_3 are related to the remaining conifold points in the ψ-plane at $\psi = \mathrm{e}^{+2\pi i/3}$ and $\psi = \mathrm{e}^{-2\pi i/3}$, respectively. The effect of these instanton actions is quite indirect to leading order and one has to go to subleading terms of the perturbative large-order relation to see them explicitly. We will do this later in section 4.2.2. Nevertheless, this is a good point to introduce them and lay out a picture of what the transseries for local $\mathbb{CP}^2$ looks like.

Let us focus on a region of moduli space around the conifold point $\psi = 1$. We have mentioned before that for almost all the points in wedge 1, the dominant instanton action is A_1. However, when $\arg(\psi) = \pm\pi/3$, the behavior of the perturbative free energies is oscillatory in the genus g, and a dominant instanton action cannot be extracted by a large g limit. For real values of the propagator we find

$$F_g^{(0)} \sim \frac{\Gamma(2g-1)}{|A_1|^{2g-1}} |c| \cos\left(\theta_A(2g-1) - \theta_c\right) \tag{4.18}$$

where θ_A is the argument of A_1 and θ_c is the argument of c, recall (1.65). We show the oscillations in g in figure 4.9. A full explanation for this plot, including frequency, phase, and amplitude of oscillation will have to wait until section 4.2.1. Without going into details, we can say that because the perturbative free energies are real, the right-hand-side of their large-order growth must somehow be real as well. The instanton action A_1 is complex for $\arg(\psi) = \pm\pi/3$ so a complex conjugate contribution must appear for everything to be consistent. This contribution is either A_2 or A_3 depending on the sign of $\arg(\psi)$. These are the instanton actions associated to the conifold points at $\psi = \mathrm{e}^{+2\pi i/3}$ and $\psi = \mathrm{e}^{-2\pi i/3}$ found across the border of wedge 1. The three conifold points at the cubic roots of unity are related by a $\mathbb{Z}_3$-symmetry whose origin is the orbifold symmetry at $\psi = 0$. This symmetry is naturally translated to the instanton actions by rotation of a third of a full turn in the ψ-plane. In this way we define,

$$A_i(\psi) = \frac{2\pi i}{\sqrt{3}}\, t_{\mathrm{c},i}(\psi), \qquad i = 1, 2, 3, \tag{4.19}$$

where

$$t_{\mathrm{c},1}(\psi) = t_{\mathrm{c}}(\psi), \tag{4.20}$$

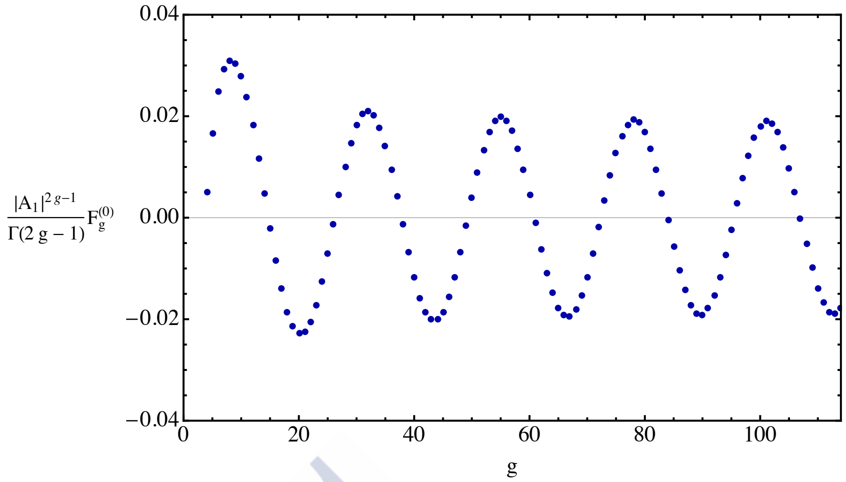


Figure 4.9: Oscillatory behavior of the perturbative sector, due to complex conjugate contributions of A_1 and A_2, at boundary of wedge 1, $\arg(\psi) = \pi/3$, and $|\psi| = 1.25$ and $S^{zz} = 10^{-5} \simeq 0.15 \left| S^{zz}_{[1],\text{hol}} \right|$.

$$t_{c,2}(\psi) = t_c(e^{-2\pi i/3}\,\psi), \tag{4.21}$$

$$t_{c,3}(\psi) = t_c(e^{+2\pi i/3}\,\psi). \tag{4.22}$$

This shows the importance of working on the covering ψ-plane. The wedges 1 to 3 in figure 4.3 are all equivalent to each other by rotational symmetry, so we only need to work with one of them. We choose wedge 1. Notice that the perturbative free energies incorporate the $\mathbb{Z}_3$-symmetry in their holomorphic dependence because they are functions of z, or ψ^3. But higher instanton free energies will break this symmetry, depending on the transseries sector, as we will see later.

From the explicit form of A_2 and A_3 we can see that they are always subleading with respect to A_1 except at the boundaries of wedge 1, as shown in figure 4.10. At those two

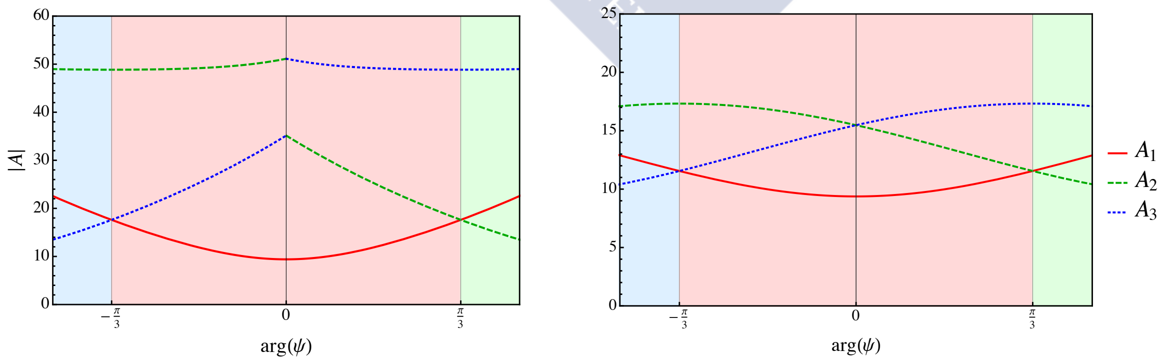


Figure 4.10: Absolute value of the conifold points for $|\psi| = 2$ (left) and $|\psi| = \frac{1}{4}$ (right), as $\arg(\psi)$ is varied. A_2 and A_3 are always subdominant in the interior of wedge 1, and of equal absolute value at the boundaries.

boundary lines in moduli space, A_1 and A_2, or A_1 and A_3, respectively, become complex conjugate of each other (up to a sign which is compensated by another sign in c in (4.18)). In the interior of the sector, A_2 and A_3 are subdominant. We also see in figure 4.10 that the instanton actions can have discontinuities on the ψ-plane. If we have a look at the functional form of A_i, $i = 1, 2, 3$, expressed in terms of two hypergeometric functions, we see that there are three potential branch cuts starting at the cubic roots of unity and going off to infinity. However, only two of the three are present. For A_1 there is no branch cut in wedge 1, and similarly for A_2 and A_3. This situation is represented in figure 4.11. In the numerical large-

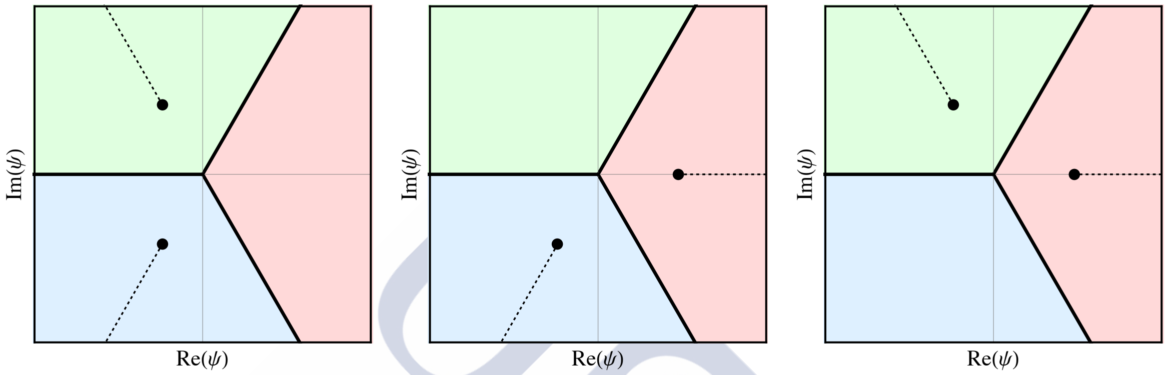


Figure 4.11: Branch points and cuts of the conifold instanton actions A_1 (left), A_2 (center) and A_3 (right), in the complex ψ plane. The branch points correspond to the conifold points $\psi = 1$, $\psi = \mathrm{e}^{2\pi i/3}$, $\psi = \mathrm{e}^{-2\pi i/3}$.

order analysis we perform in this section and the rest of this chapter the discontinuities of the instanton actions do not lead to any inconsistencies, although dominance and subdominance regions do not allow for a thorough exploration. These discontinuities may be relevant when discussing the resummation of the transseries. We show the discontinuities for a fixed value of $|\psi|$ (greater than 1) and varying argument in 4.12 and a three-dimensional representation in figure 4.13.

As we mentioned in the introduction to this section, there can only be three linearly independent instanton actions because they are solutions of the Picard–Fuchs equation. Since the conifold instanton actions are not constant, some combination of them must be. From their explicit dependence in ψ and $\mathbb{Z}_3$-symmetry it is immediate to check that

$$A_1 + A_2 + A_3 = -4\pi^2 i. \tag{4.23}$$

Notice that $4\pi^2 i$ is the instanton action associated to the constant map contribution, not a mere complex number. Moreover, the relation between all these instanton actions is not only $\mathbb{C}$-linear but $\mathbb{Z}$-linear. This leads to resonance between all the corresponding sectors. Removing the constant map contribution should be done carefully because there may be a full transseries sector associated with it. We will comment more on this later. Besides this linear relation there is another one expressing the Kähler instanton action in terms of the conifold ones. Due to brach cuts we have $A_{\mathrm{K}} = A_1 - A_2$ for $\arg(\psi) > 0$ and $A_{\mathrm{K}} = A_3 - A_1$ for $\arg(\psi) < 0$. Finally, there is resonance between each instanton action and its negative, $+A + (-A) = 0$. This must be so in order to have a g_s^2-expansion for the perturbative sector

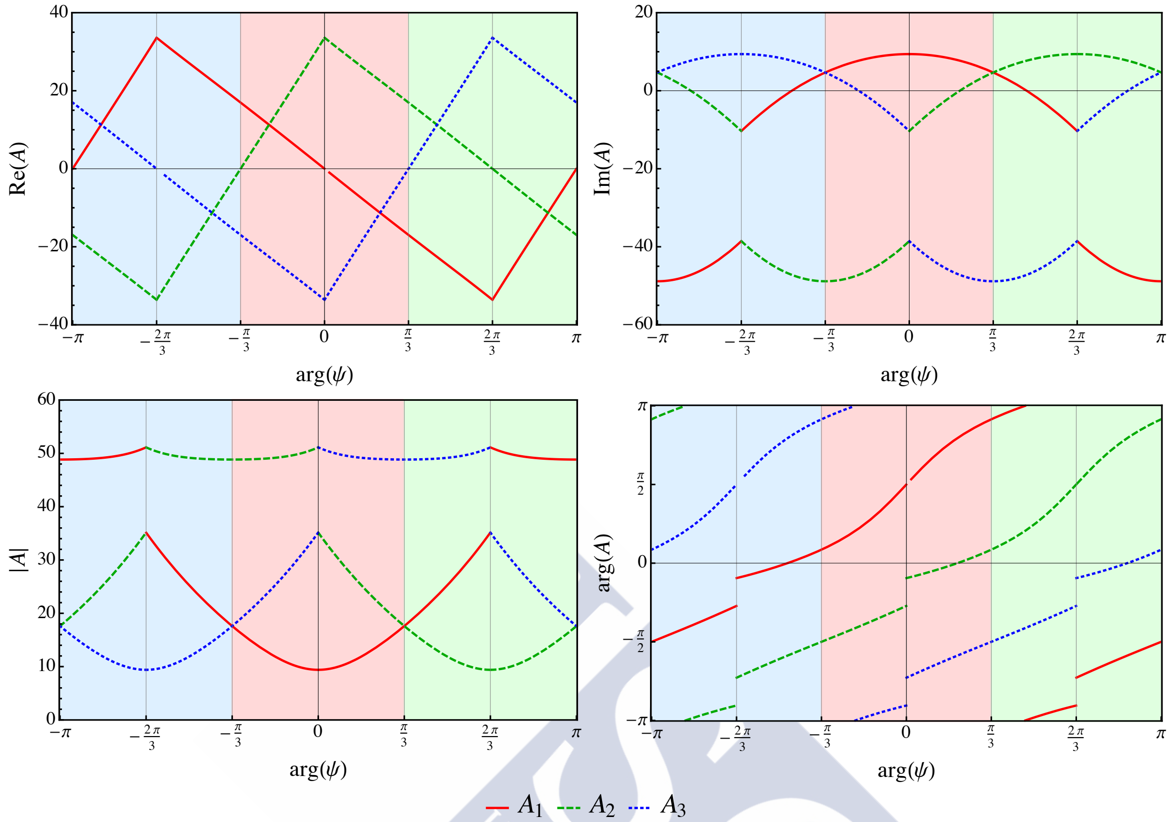


Figure 4.12: Real parts, imaginary parts, absolute values, and arguments of the three instanton actions for fixed absolute value $|\psi| = 2$ and varying $\arg(\psi)$. We did not include the negative instanton actions, $-A_i$, $i = 1, 2, 3$.

and will be checked explicitly in the large-order of the one-instanton sector associated to A_1 in section 4.3. Also in the Borel plane in figure 4.7 poles of both signs appear. All in all the transseries for local $\mathbb{CP}^2$ can be very complicated with a large number of parameters and multiple resonances.

Phase diagrams

We end this section with some comments on the phase diagram of the local $\mathbb{CP}^2$ free energy. This phase diagram encodes information about the transseries sectors that will be relevant for resummation. To have a proper expansion beyond perturbation theory in g_s, the exponential monomials must vanish in the $g_s \to 0$ limit. Otherwise the concept of perturbative sector would be inconsistent. Therefore, the physical transseries expansion must only include sectors for which $\mathrm{Re}(A/g_s) > 0$. For all the other sectors the transseries coefficients σ_β must vanish as we explained in section 1.3.2. Since the instanton actions are moduli dependent the previous condition will be true only in some region in moduli space. We show two possible phase diagrams, depending on the value of g_s, where the relevant conifold sectors are depicted, in figure 4.14. The boundaries, where $\mathrm{Re}(A/g_s) = 0$, are called anti-Stokes lines, and

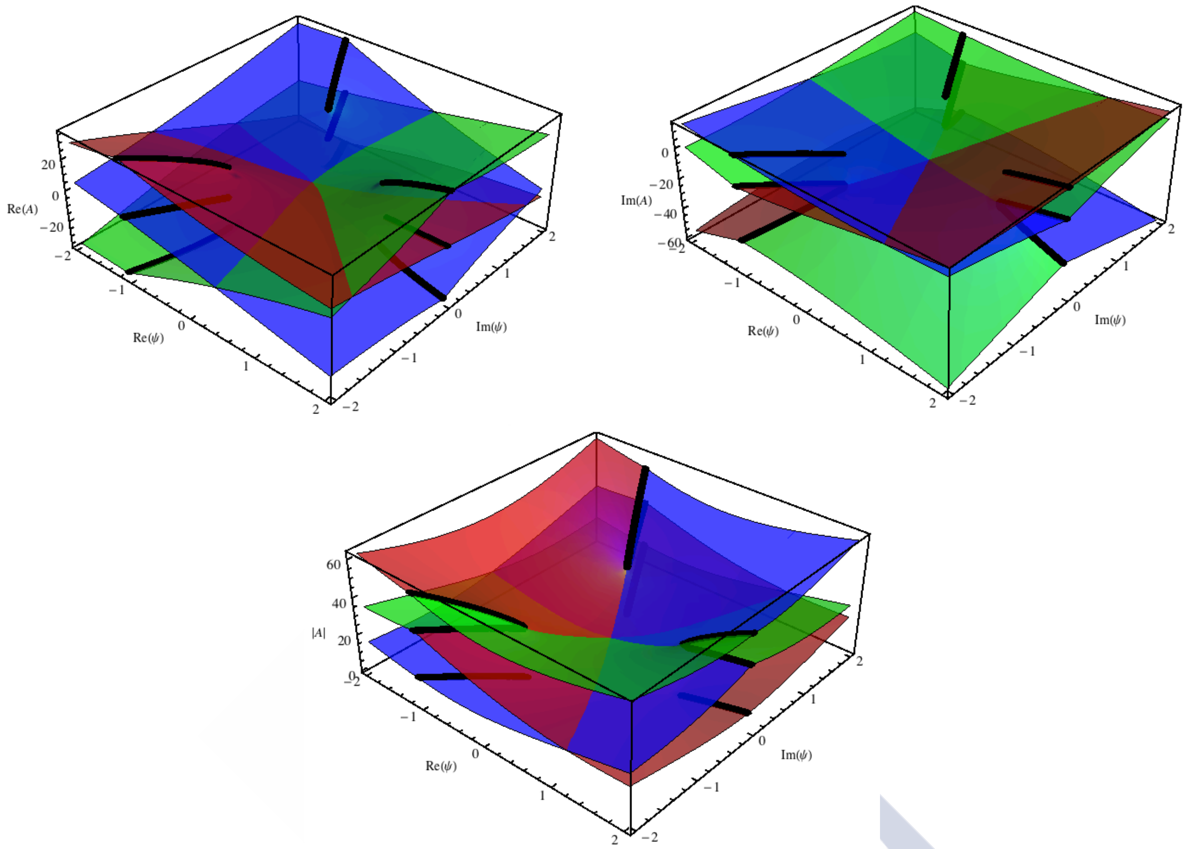


Figure 4.13: On top, real and imaginary parts of the conifold instanton actions in the ψ-plane. Below, the absolute values that show dominance and subdominance. Different instanton actions are joined along branched cuts marked with black lines.

they mark the region where some exponentially suppressed sectors are not suppressed anymore and a phase transition occurs. Further understanding of the phase diagrams, including other instanton actions and in other regions in moduli space, will be crucial to attempt a resummation of the transseries.

4.2 Large-order analysis of perturbation theory

The resurgent analysis of local $\mathbb{CP}^2$ presented in this chapter follows a kind of constructive approach. We start with data of which we are certain, the perturbative free energies, and extract all the information we can from it using numerical large-order analysis. This information regards higher instanton sectors of the transseries, starting with the one-instanton, two-instanton sectors, etc. We then proceed to understand the newly discovered free energies in terms of solutions of the holomorphic anomaly equations explained in chapter 3. Once all the numerical results are reproduced by analytic expressions we go on to analyze the large-order growth of the higher instanton sectors. This systematic approach would, in principle, allow us to to uncover the complete resurgent structure of the model and the

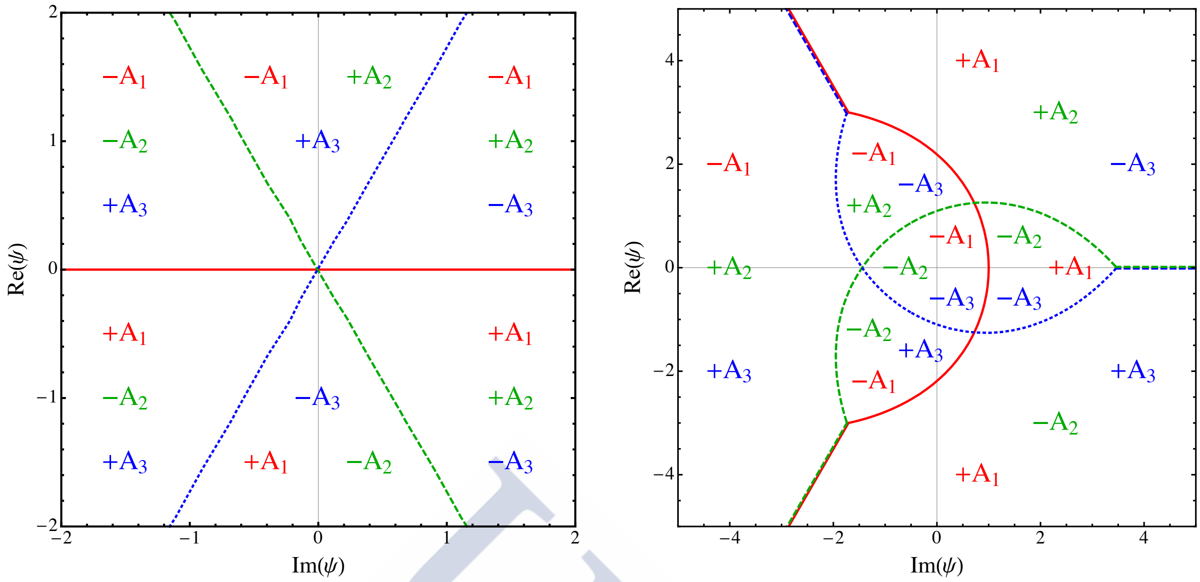


Figure 4.14: Phase diagrams for local $\mathbb{CP}^2$. On the left, $g_s \in \mathbb{R}^+$; on the right $g_s \in i\mathbb{R}^+$. The anti-Stokes phase boundaries satisfy $\mathrm{Re}\,(A(\psi)/g_s) = 0$ and in the plots we mark which instanton actions satisfy $\mathrm{Re}\,(A(\psi)/g_s) > 0$, in each region of the complex ψ plane. Straight double lines on the right plot indicate a branch-cut jump in $\mathrm{Re}\,(A(\psi)/g_s)$ from positive to negative value.

form of the transseries including all possible sectors. In practice, we restrict ourselves to the study of perturbative large-order growth to leading and subleading order ($\mathcal{O}(2^{-g})$) in this section, and the one-instanton large-order growth in the following section. We focus on conifold instanton sectors for which we have a strategy to fix the holomorphic ambiguities.

The notation used to refer to the different sectors of the transseries can become very long and distracting because we have many parameters or instanton actions. We order the instanton actions in the following way

$$A_1, \quad -A_1, \quad A_2, \quad -A_2, \quad A_3, \quad -A_3, \quad A_{\mathrm{K}}, \quad -A_{\mathrm{K}}, \quad \ldots \tag{4.24}$$

We allow for the possibility of further sectors associated to other instanton actions. Even though we do not find them in our analysis, we cannot discard their presence. In any case, their influence would remain deep in the large-order and it would be very subleading. In this section we only have to deal with pure sectors. These are of the form $(0|\cdots|0|n|0|\cdots|0)$. If the nonnegative integer n sits in the i-th position we use the notation

$$(n\boldsymbol{\varepsilon}_i) := (0|\cdots|0|n|0|\cdots|0). \tag{4.25}$$

Since we have a resonant pairing of $+A_i$ and $-A_i$ for all instanton actions and there is a symmetry between the corresponding free energies, as explained in section 1.5.3, we end up working with the free energies associated to $+A_i$. Therefore, we introduce the notation

$$(n\boldsymbol{e}_i) := (n\boldsymbol{\varepsilon}_{2i-1}). \tag{4.26}$$

For example, $(3\boldsymbol{\varepsilon}_3) = (3\boldsymbol{e}_2)$ corresponds to $(0|0\|3|0\|0|0\|0|0\|0\cdots)$, where we have used a double bar to show more clearly the separation between sectors of different instanton action up to a sign. In this section we will only encounter the following sectors,

$$(\boldsymbol{e}_1) = (1\boldsymbol{e}_1) = (1\boldsymbol{\varepsilon}_1) = (1|0\|0|0\|0|0\|0\cdots), \tag{4.27}$$

$$(\boldsymbol{e}_2) = (1\boldsymbol{e}_2) = (1\boldsymbol{\varepsilon}_3) = (0|0\|1|0\|0|0\|0\cdots), \tag{4.28}$$

$$(\boldsymbol{e}_3) = (1\boldsymbol{e}_3) = (1\boldsymbol{\varepsilon}_5) = (0|0\|0|0\|1|0\|0\cdots), \tag{4.29}$$

$$(2\boldsymbol{e}_1) = (2\boldsymbol{\varepsilon}_1) = (2|0\|0|0\|0|0\|0\cdots). \tag{4.30}$$

We learn in this section that the leading order growth of perturbation theory is controlled, in most of the moduli space near the conifold point $\psi = 1$, by the one-instanton sector $(\boldsymbol{e}_1)$ free energies. Subleading to this contribution there are several competing sectors: $(2\boldsymbol{e}_1)$, $(\boldsymbol{e}_2)$, and $(\boldsymbol{e}_3)$, depending on the particular value of ψ. Schematically,

$$\begin{aligned} F_g^{(0)} \sim{}& \frac{\Gamma(2g-1)}{A_1^{2g-1}}[(\boldsymbol{e}_1)] \\ &+ \max\left\{\frac{\Gamma(2g-1)}{(2A_1)^{2g-1}}[(2\boldsymbol{e}_1)], \frac{\Gamma(2g-1)}{A_2^{2g-1}}[(\boldsymbol{e}_2)], \frac{\Gamma(2g-1)}{A_3^{2g-1}}[(\boldsymbol{e}_3)]\right\}. \end{aligned} \tag{4.31}$$

However, on the boundaries of wedge 1, at $\arg(\psi) = \pm\pi/3$, either $(\boldsymbol{e}_2)$ or $(\boldsymbol{e}_3)$ also become dominant along with $(\boldsymbol{e}_1)$. For example, at $\arg(\psi) = +\pi/3$,

$$F_g^{(0)} \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}}[(\boldsymbol{e}_1)] + \frac{\Gamma(2g-1)}{A_2^{2g-1}}[(\boldsymbol{e}_2)], \tag{4.32}$$

and $|A_1| = |A_2|$. The goal of this section is to make this schematic formula precise, both at the level of numerical large-order computations—our principal guide in the study of resurgence—and at the level of analytical expressions from the extended holomorphic anomaly equations.

4.2.1 Leading contribution

We separate the discussion into the two qualitatively different large-order growths: the interior of wedge 1, $-\pi/3 < \arg(\psi) < +\pi/3$; and the boundaries, $\arg(\psi) \in \{+\pi/3, -\pi/3\}$.

Interior case around conifold point

In this regime $|A_1|$ is always smaller than $|A_2|$ and $|A_3|$, and all the other instanton actions of the transseries. We checked the dominance of A_1 in section 4.1.1. Here we go deeper in the large-order growth of the perturbative coefficients and find a structure very similar to what we described around (1.40). That equation was derived based only on the assumption of resurgence and a particular form for the bridge equation (1.31). The first condition of resurgence is still believed to hold, but we do not know what form of the bridge equation this model satisfies. Therefore we must take an exploratory approach. This means taking suitable numerical large g limits of particular combinations of the perturbative free energies, $F_g^{(\mathbf{0})}$, in order to uncover the g-expansion of these coefficients. We do not expect strong

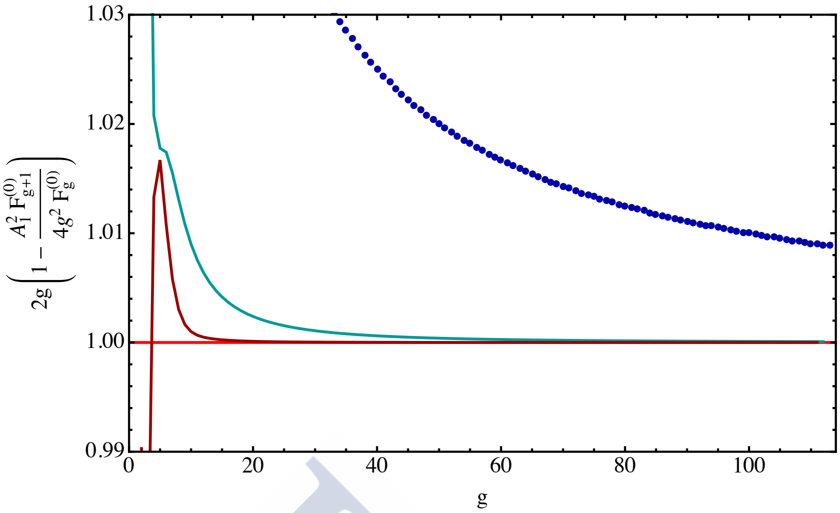


Figure 4.15: Numerical calculation of the limit $\lim_{g\to\infty} 2g\left(1-\frac{A_1^2}{4g^2}\frac{F_{g+1}^{(\mathbf{0})}}{F_g^{(\mathbf{0})}}\right)$ computed at $\psi = 2$ and $S^{zz} = S^{zz}_{[1],\text{hol}}$. The result equal to 1 implies a $\Gamma(2g-1)$ growth for the perturbative free energies.

deviations from (1.40); there will be Gamma functions of g_s, exponentials of g, and so on. What can differ is the precise role of the higher instanton coefficients, $F_h^{(\boldsymbol{n})}$, in this large-g expansion. We will explain the differences as they appear.

To first order we find the following asymptotics

$$F_g^{(\mathbf{0})} \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}}\frac{S_{1,1}}{\pi i}F_0^{(\boldsymbol{e}_1)}. \tag{4.33}$$

Before focusing on the one-instanton contribution, $F_0^{(\boldsymbol{e}_1)}$, let us note that the factorial dependence is $\Gamma(2g-1)$. In particular, the number -1 should be related to the starting powers $b^{(\mathbf{0})} = -2$ and $b^{(\boldsymbol{e}_1)}$. If the usual bridge equation (1.31) were to hold we would have a dependence like $\Gamma(2g + b^{(\mathbf{0})} - b^{(\boldsymbol{e}_1)})$, which would imply $b^{(\boldsymbol{e}_1)} = -1$. We will see later that this result is not consistent with other numerical calculations and with computations from the holomorphic anomaly equations. Notice that the dependence on $\Gamma(2g-1)$ is the one we found analytically in the previous chapter from equation (3.152). The large g limit there can be done numerically for the case of local $\mathbb{CP}^2$ (not necessarily in the holomorphic regime) and we obtain the same result. See figure 4.15.

The one-loop one-instanton contribution to the perturbative large-order growth is given by

$$\frac{S_{1,1}}{\pi i}F_0^{(\boldsymbol{e}_1)}. \tag{4.34}$$

Here $S_{1,1}$ is the Stokes constant coming from whatever bridge equation is at work. The denominator is one over πi instead of $2\pi i$ because (4.34) is actually the combination of symmetric contributions from the sectors $(\boldsymbol{\varepsilon}_1)$ and $(\boldsymbol{\varepsilon}_2)$, as in (1.62). $F_0^{(\boldsymbol{e}_1)}$ is found to be, after comparing to numerical results, the free energy one computes from the extended

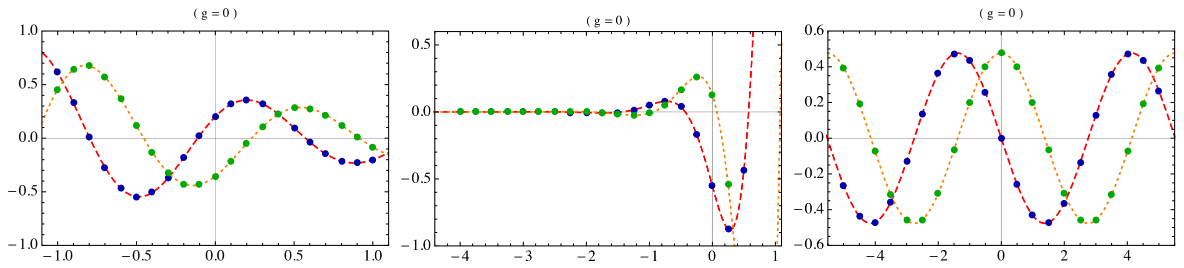
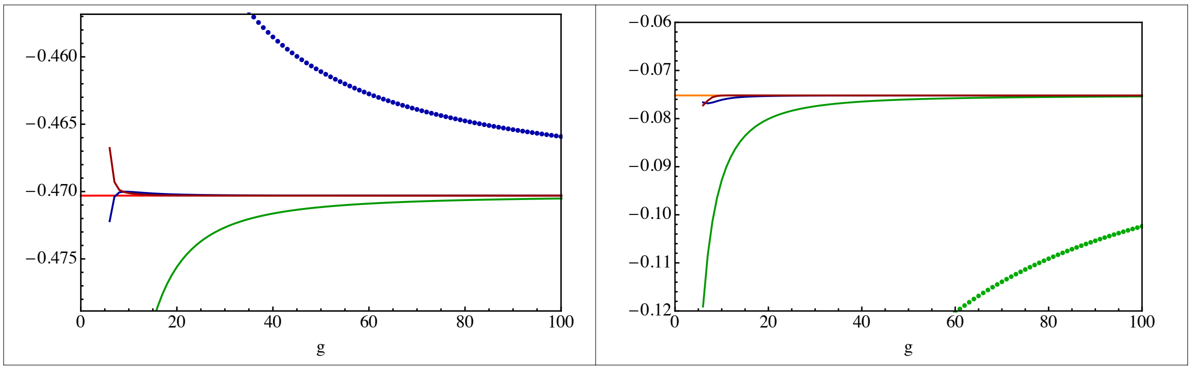


Figure 4.16: Large-order check of $\frac{S_{1,1}}{\pi i} F_0^{(\boldsymbol{e}_1)}$ at three points in moduli space: $\psi = \frac{1}{2}\mathrm{e}^{-i\pi/5}$ (left), $\psi = \frac{2}{2}\mathrm{e}^{+i\pi/4}$ (center), $\psi = 2$ (right). Variation on the horizontal x-axis is equivalent to changing the value of the propagator,S^{zz}, as $S^{zz} = S^{zz}_{[1],\mathrm{hol}} \cdot (1 + ix)$. Real and imaginary parts of the limit (4.35) are represented by blue and and green dots, respectively. The lines plot the analytic value of the one-instanton free energy.



$$2\,\frac{S_{1,1}}{2\pi i}\,F_0^{(e_1)} = -0.470\,302\,487 - 0.075\,185\,636\,i$$

$$3 \text{ Richardson Transforms} = -0.470\,302\,475 - 0.075\,185\,625\,i$$

Figure 4.17: Real (left) and imaginary (right) parts of the limit (4.35) along with three Richardson transforms and the analytic prediction. Here $\psi = 2$ and $S^{zz} = S^{zz}_{[1],\mathrm{hol}} \cdot (1 - 4i)$.

holomorphic anomaly equations. The numerical value of (4.34) is given by the limit

$$\frac{S_{1,1}}{\pi i}\,F_0^{(\boldsymbol{e}_1)} = \lim_{g\to\infty} \frac{A_1^{2g-1}}{\Gamma(2g-1)}\,F_g^{(\boldsymbol{0})}, \tag{4.35}$$

that can be evaluated for different values of ψ and S^{zz}. In figure 4.16 we show the propagator dependence of (4.35) for three different values of ψ. Each point in the plots is obtained by taking a few Richardson transforms on the g-sequence in order to reach enough significant figures. We show an example for a particular value of S^{zz} in figure 4.17 In both figures 4.16 and 4.17 we have included the exact result that the numerical limit tends to. It is computed form the holomorphic anomaly equations in the following way.

Since we are working with a pure one-instanton sector the distinction between one and multi-parameter transseries vanishes. Thus, we can borrow the description of the solution for $F_0^{(1)}$ in section 3.3.2 around (3.40). Using that the relevant instanton action is $A = A_1$

we can write

$$F_0^{(\boldsymbol{e}_1)} = f_0^{(\boldsymbol{e}_1)}(z)\mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 S^{zz}}, \tag{4.36}$$

where $f_0^{(\boldsymbol{e}_1)}(z)$ is the holomorphic ambiguity. To fix it we have to recall the general analysis of the perturbative free energies performed in the conifold holomorphic limit in section 3.7. There we found that due to the divergent nature of the gap condition (2.82) at the conifold point—in this case $\psi = 1$—the large-order growth can be spelled out in great detail to many orders. In particular we found

$$\mathcal{F}_g^{(\boldsymbol{e}_1)} \sim \frac{\Gamma(2g-1)}{A_{\mathrm{c}}^{2g-1}}\frac{A_{\mathrm{c}}}{2\pi^2}. \tag{4.37}$$

This is the piece of information we need to fix the ambiguity, up to the Stokes constant. We conclude

$$\frac{S_{1,1}}{\pi i}\mathcal{F}_0^{(\boldsymbol{e}_1)} = \frac{A_1}{2\pi^2}, \tag{4.38}$$

where we have specified that the particular conifold instanton action is A_1. For the one-instanton sector we find that we can drop the tilde in (3.151). The curly $\mathcal{F}$ on the left-hand-side indicates in this case the holomorphic regime with respect to the first conifold point, for which

$$S^{zz} \to S^{zz}_{[1],\mathrm{hol}}. \tag{4.39}$$

Putting all the ingredients together determines the full form of the one-loop one-instanton free energy

$$\frac{S_{1,1}}{\pi i}F_0^{(\boldsymbol{e}_1)} = \frac{A_1}{2\pi^2}\mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}. \tag{4.40}$$

Two comments on this result. First, the holomorphic limit (4.38), which is universal, can be checked for the example of local $\mathbb{CP}^2$ in figure 4.16 by looking at the value of the graphs crossing the vertical axis. Second, the exponential dependence on the propagator matches the large-order numerical calculations as shown in figure 4.16. If we did not have an extension of the holomorphic anomaly equations from which to compute (4.40) we could still have been able to guess the exponential dependence, and from that gain some information about the equation that $F_0^{(\boldsymbol{e}_1)}$ satisfies. This guessing game is not the most effective road to nonperturbative exploration but in some cases it could give a clue to continue.

The next correction to (4.33) is

$$F_g^{(\boldsymbol{0})} \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}}\frac{S_{1,1}}{\pi i}F_0^{(\boldsymbol{e}_1)} + \frac{\Gamma(2g-2)}{A_1^{2g-2}}\frac{S_{1,1}}{\pi i}F_1^{(\boldsymbol{e}_1)}. \tag{4.41}$$

The second term is subleading in $\frac{1}{g}$ with respect to the first one. Using the analytical expression for $\frac{S_{1,1}}{\pi i}F_0^{(\boldsymbol{e}_1)}$ we can perform the limit

$$\frac{S_{1,1}}{\pi i}F_1^{(\boldsymbol{e}_1)} = \lim_{g\to\infty}\frac{A_1^{2g-2}}{\Gamma(2g-2)}\left(F_g^{(\boldsymbol{0})} - \frac{\Gamma(2g-1)}{A_1^{2g-1}}\frac{S_{1,1}}{\pi i}F_0^{(\boldsymbol{e}_1)}\right). \tag{4.42}$$

We can explore this limit for various values of ψ and S^{zz}. All the numbers we find from Richardson transform on this sequence can be reproduced to great precision if $F_1^{(\boldsymbol{e}_1)}$ is computed from the holomorphic anomaly equations in the same way as before. In chapter 3 we learned that this free energy is the product of an exponential times a polynomial of degree 3,

$$F_1^{(\boldsymbol{e}_1)}(z, S^{zz}) = \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 \left(S^{zz} - S^{zz}_{[1],\mathrm{hol}}\right)} \left(f_1^{(\boldsymbol{e}_1)}(z) + R_1\, S^{zz} + R_2\, (S^{zz})^2 + R_3\, (S^{zz})^3 \right). \tag{4.43}$$

Here $R_i = R_i(z, A_1, \partial_z A_1, \partial_z^2 A_1)$ is rational as a function of z and polynomial in A_1, $\partial_z A_1$ and $\partial_z^2 A_1$. This is because in the equation for $F_1^{(\boldsymbol{e}_1)}$ there are several rational functions of z coming from the Yukawa coupling C_{zzz} in (2.62), $\tilde{f}^z_{zz}$ in (2.78) and f^{zz}_z in (2.79). Also, the dependence of the equation on $F_0^{(\boldsymbol{e}_1)}$ brings down powers of $S^{zz}_{[1],\mathrm{hol}}$ from the exponential. Due to the relation to $\partial_z \mathcal{F}_1^{[\mathrm{c}](\mathbf{0})}$ in (2.73) we can write

$$S^{zz}_{[1],\mathrm{hol}} = -\frac{1}{C_{zzz}} \left(\frac{\partial_z^2 A_1}{\partial_z A_1} - \tilde{f}^z_{zz} \right). \tag{4.44}$$

The derivative $\partial_z A_1$ in the denominator is always cancelled by a similar positive power coming from the quadratic term $(\partial_z A_1)^2$ in the exponential, leading to the final polynomial dependence.

The holomorphic ambiguity is fixed against (3.154), that is,

$$\frac{S_{1,1}}{\pi i} \mathcal{F}_1^{(\boldsymbol{e}_1)} = \frac{1}{2\pi^2}. \tag{4.45}$$

The higher order corrections to the factorial growth of $F_g^{(\mathbf{0})}$ can be written as

$$F_g^{(\mathbf{0})} \sim \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{A_1^{2g-1-h}} \frac{S_{1,1}}{\pi i} F_h^{(\boldsymbol{e}_1)}. \tag{4.46}$$

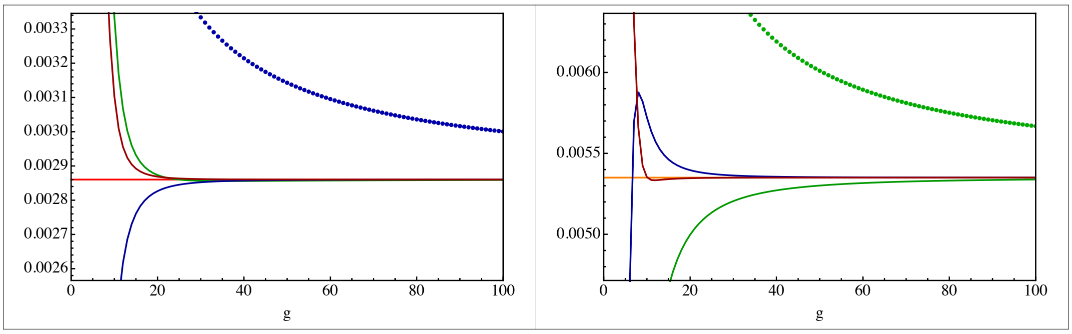
Each term $\frac{S_{1,1}}{\pi i} F_h^{(\boldsymbol{e}_1)}$ can be extracted by a large-g limit as explained in chapter 1,

$$\frac{S_{1,1}}{\pi i}\, F_h^{(\boldsymbol{e}_1)} = \lim_{g\to\infty} \frac{A_1^{2g-1-h}}{\Gamma(2g-1-h)} \left(F_g^{(\mathbf{0})} - \sum_{h'=0}^{h-1} \frac{\Gamma\left(2g-1-h'\right)}{A_1^{2g-1-h'}} \frac{S_{1,1}}{\pi i}\, F_{h'}^{(\boldsymbol{e}_1)} \right). \tag{4.47}$$

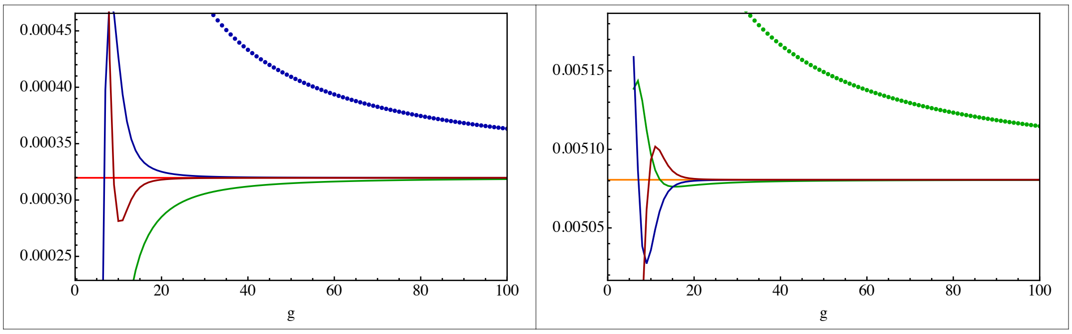
We show two examples of this limit, for $h = 2$ and $h = 4$ in figure 4.18. A visual check for the propagator dependence of the free energies is shown in figure 4.19 for $h = 1, 2, 3$. The difference between the numerical and analytical results can be made very small by increasing the number of Richardson transforms. The only constraint is the finite number of perturbative coefficients. Note that in order to keep sufficient numerical precision when going to high loop number h, we have to use the exact expressions for every function on the right-hand-side of (4.47). Otherwise errors propagate quickly and the numerical limit is impossible to take.

For $h \geq 2$ the holomorphic ambiguity $f_h^{(\boldsymbol{e}_1)}$ is fixed by requiring that the holomorphic limit of the free energies vanishes. This is in agreement with 4.19 and what we found in equation (3.161). For general h, the structure of the solution is

$$\frac{S_{1,1}}{\pi i} F_h^{(\boldsymbol{e}_1)} = \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 \left(S^{zz} - S^{zz}_{[1],\mathrm{hol}}\right)} \mathrm{Pol}(S^{zz}; 3g). \tag{4.48}$$



$$2\,\frac{S_{1,1}}{2\pi i}\,F_2^{(\boldsymbol{e}_1)} = 0.002\,860\,250\,8 + 0.005\,350\,260\,i$$
$$3\ \text{Richardson Transforms} = 0.002\,860\,257\,4 + 0.005\,350\,255\,i$$



$$2\,\frac{S_{1,1}}{2\pi i}\,F_4^{(\boldsymbol{e}_1)} = 0.000\,319\,688\,7 + 0.005\,080\,627\,80\,i$$
$$3\ \text{Richardson Transforms} = 0.000\,319\,687\,9 + 0.005\,080\,628\,13\,i$$

Figure 4.18: Real (left) and imaginary (right) parts of the limit (4.47) and the analytic prediction from the holomorphic anomaly equations. On top, $\psi = \frac{3}{2}\mathrm{e}^{i\pi/4}$, $S^{zz} = S^{zz}_{[1],\text{hol}} \cdot (1 - \frac{11}{4}i)$. In the bottom, $\psi = \frac{1}{2}\mathrm{e}^{-i\pi/5}$, $S^{zz} = S^{zz}_{[1],\text{hol}} \cdot (1 - \frac{7}{10}i)$.

The coefficients of the polynomial are rational functions of z and polynomials in A_1, $\partial_z A_1$ and $\partial_z^2 A_1$. No higher derivative terms appear because we can trade them for lower ones using the Picard–Fuchs equation. At the practical level this is a necessary step in order to have moderately compact formulae. For local $\mathbb{CP}^2$ we can push the computation to $h = 21$ but at a great cost in memory. See appendix C for more details on the structure of the free energies. We finish this part with several comments.

The large-order growth of the perturbative sector that we find has the same form as the one derived from a bridge equation like (1.31). The free energies appearing there have the interpretation of one-instanton coefficients of the transseries. They are computed from the extended holomorphic anomaly equations (3.85), for $(\boldsymbol{n}) = (\boldsymbol{e}_1)$. Notice that the quadratic term in the equations drops so there is no dependence of the solutions on the starting power $b^{(\boldsymbol{e}_1)}$. One has to go to higher sectors to see it. The complete computation of the free energies needs information about the resurgence relations in order to fix the ambiguity. This implies that only the combination $\frac{S_{1,1}}{\pi i} F_h^{(\boldsymbol{e}_1)}$ is known. The ambiguity on the value of the Stokes

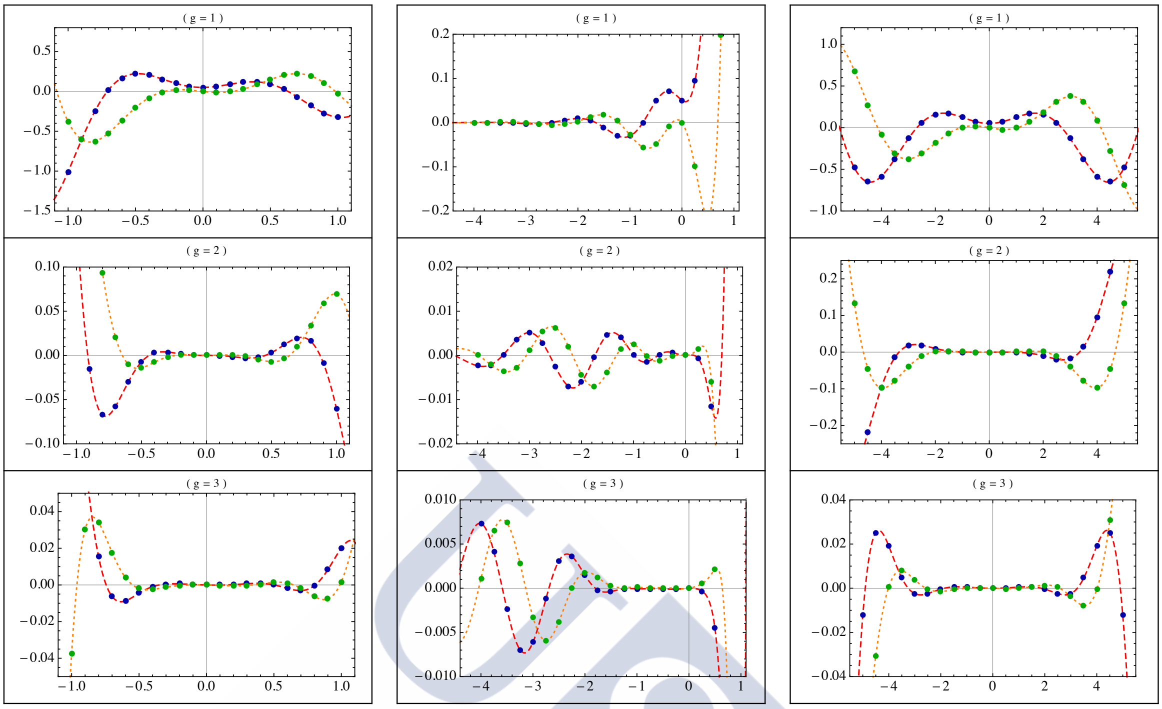


Figure 4.19: Numerical check of the propagator dependence of the one-instanton higher-loop free energies for three value of the modulus: $\psi = \frac{1}{2}\mathrm{e}^{-i\pi/5}$ (left), $\psi = \frac{2}{2}\mathrm{e}^{+i\pi/4}$ (center), $\psi = 2$ (right). As in figure 4.16, $S^{zz} = S^{zz}_{[1],\mathrm{hol}} \cdot (1 + ix)$. The lines plot the analytic value of the one-instanton free energies (4.48).

constants is not actually a problem by itself because the only situation where we need them is in the physical resummation of the transseries. But there we also need to worry about the value of the transseries parameter σ_1, and the dependence is such that only the product $\sigma_1/S_{1,1}$ has to be determined.

On the topic of the Stokes constants we have implicitly assumed that they are actually complex numbers. However, the general arguments that lead to a bridge equation allow for them to be dependent on any parameters in the problem, other than g_s. This means that, potentially, we could have

$$S_{1,1} = S_{1,1}(\psi, S^{zz}). \tag{4.49}$$

This turns out not to be the case and the evidence comes from the large-order numerics. First of all let us notice that the ambiguity fixing condition is valid in the holomorphic limit. So (4.38) is actually,

$$\frac{S_{1,1}\left(\psi, S^{zz}_{[1],\mathrm{hol}}\right)}{\pi i}\mathcal{F}_0^{(\boldsymbol{e}_1)} = \frac{A_1}{2\pi}. \tag{4.50}$$

Once the ambiguity is fixed, we can plug the result back in the asymptotic expression for

$F_g^{(\mathbf{0})}$ which involves the full Stokes "constant", $S_{1,1}(\psi, S^{zz})$. So (4.33) would read

$$F_g^{(\mathbf{0})} \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}} \left(\frac{S_{1,1}\left(\psi, S^{zz}\right)}{S_{1,1}\left(\psi, S^{zz}_{[1],\mathrm{hol}}\right)} \right) \frac{A_1}{2\pi^2} \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 \left(S^{zz} - S^{zz}_{[1],\mathrm{hol}}\right)}. \tag{4.51}$$

But the numerical analysis of the propagator dependence of this expression shows that

$$S_{1,1}(\psi, S^{zz}) = S_{1,1}\left(\psi, S^{zz}_{[1],\mathrm{hol}}\right), \tag{4.52}$$

so the Stokes "constant" is at least holomorphic. To show that it is a constant we have to notice that the equation for $F_1^{(\boldsymbol{e}_1)}$ involves z-derivatives $F_0^{(\boldsymbol{e}_1)}$ because of recursion. This would add an extra term for $F_1^{(\boldsymbol{e}_1)}$ involving $\partial_z S_{1,1}/S_{1,1}$ which we did not consider because the numerics is not compatible with it. Thus,

$$\partial_z S_{1,1} = 0. \tag{4.53}$$

A way to understand why the Stokes constants do not carry any parameter dependence, even when they are allowed to do it, is by looking at what happens in other examples. For differential equations there are usually no external parameters. However, in the computation of the transseries, the one-loop coefficient of the one-instanton sector is usually not determined by the recursion due to homogeneity, and it is set to unity by convention. The ambiguity is regarded as an integration constant associated to the transseries parameter σ. Analyzing the perturbative large-order would give in that case a numerical value for the Stokes constant since $F_0^{(1)}$ was set to one. So already here we see some ambiguity and freedom to set certain quantities to simple values and letting others carry important information. In our case the nature of the equations is different and $F_0^{(1)}$ is determined, up to the holomorphic ambiguity. It cannot be set to unity because it has modulus and propagator dependence. In this way it is not so surprising to see that $S_{1,1}$ is independent of ψ and S^{zz} because $F_0^{(\boldsymbol{e}_1)}$ is already carrying that information. In matrix models it is found that the Stokes constant can depend on the 't Hooft parameter.

Boundary case around conifold point

As we saw in the previous section, A_2 and A_3 are always subdominant to A_1 in wedge 1, except when

$$\arg(\psi) \in \{+\pi/3, -\pi/3\}. \tag{4.54}$$

In figure 4.10 one can see the takeover of dominance when leaving wedge 1 through the boundaries. For the sake of definiteness let us focus on $\arg(\psi) = +\pi/3$, where both A_1 and A_2 are dominant. More precisely,

$$A_1 = -A_2^*, \tag{4.55}$$

$$\partial_z A_1 = -\left(\partial_z A_2\right)^*, \tag{4.56}$$

$$\partial_z^2 A_1 = -\left(\partial_z^2 A_2\right)^*, \tag{4.57}$$

$$S^{zz}_{[1],\mathrm{hol}} = +\left(S^{zz}_{[2],\mathrm{hol}}\right)^*. \tag{4.58}$$

The last equation can be derived from the previous two by expressing the holomorphic limit of the propagator in terms of the instanton actions, as in (4.44).

In this region of moduli space, the perturbative free energies have the following asymptotics

$$F_g^{(\mathbf{0})} \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}} \frac{S_{1,1}}{\pi i} F_0^{(\boldsymbol{e}_1)} + \frac{\Gamma(2g-1)}{A_2^{2g-1}} \frac{S_{1,2}}{\pi i} F_0^{(\boldsymbol{e}_2)}, \tag{4.59}$$

which combines contributions from $(\boldsymbol{e}_1)$ and $(\boldsymbol{e}_2)$ on the same footing. $F_0^{(\boldsymbol{e}_2)}$ can be computed from the holomorphic anomaly equations equations in the same way that $F_0^{(\boldsymbol{e}_1)}$ was. The corresponding ambiguity is fixed up to the Stokes constant by the equation analogous to (4.38). The two solutions are formally the same. Using the explicit expressions for the free energies and (4.55)-(4.58) we find

$$F_g^{(\mathbf{0})} \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}} \frac{A_1}{2\pi^2} \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 \left(S^{zz} - S^{zz}_{[1],\mathrm{hol}}\right)} + \frac{\Gamma(2g-1)}{(A_1^*)^{2g-1}} \frac{A_1^*}{2\pi^2} \mathrm{e}^{\frac{1}{2}((\partial_z A_1)^*)^2 \left(S^{zz} - \left(S^{zz}_{[1],\mathrm{hol}}\right)^*\right)}. \tag{4.60}$$

The two contributions are almost complex conjugate to each other. This is the case if the propagator S^{zz} has a real value. The right-hand-side of (4.60) would then be real, agreeing with the left-hand-side—$\arg(\psi) = \pi/3$ implies $z \in \mathbb{R}^+$ so the perturbative free energies are real. Introducing some notation we can provide a compact expression for (4.60) that shows the dependence on g clearly. Let

$$A_1 = |A_1|\, \mathrm{e}^{i\theta_{A_1}}, \qquad \mu = |\mu|\, \mathrm{e}^{i\theta_\mu} := \frac{A_1}{2\pi^2} \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 \left(S^{zz} - S^{zz}_{[1],\mathrm{hol}}\right)}, \tag{4.61}$$

Then the perturbative free energies show an oscillatory behavior in g with frequency controlled by the argument of A_1, and amplitude and global phase controlled by $F_0^{(\boldsymbol{e}_1)}$,

$$F_g^{(\mathbf{0})} \sim \frac{\Gamma(2g-1)}{|A_1|^{2g-1}} 2|\mu| \cos\left(\theta_{A_1}(2g-1) - \theta_\mu\right). \tag{4.62}$$

In figure 4.20 we show an example of the oscillations and the prediction for large g. In this case there is no clean way to extract θ_{A_1}, θ_μ, etc, to check against the analytical result, so the comparison is qualitative. This oscillatory behavior, result of complex conjugate combination, is a consequence of having a complex instanton action but real perturbative coefficients. However the oscillations can be present in more general cases in which there is no reality condition. This is what happens when the propagator is chosen not to be real.

Let us define

$$\widetilde{\mu} = |\widetilde{\mu}|\, \mathrm{e}^{i\theta_{\widetilde{\mu}}} := \frac{A_1^*}{2\pi^2} \mathrm{e}^{\frac{1}{2}((\partial_z A_1)^*)^2 \left(S^{zz} - (S^{zz}_{[1],\mathrm{hol}})^*\right)}, \tag{4.63}$$

to find

$$\begin{aligned} F_g^{(\mathbf{0})} \sim& \frac{\Gamma(2g-1)}{|A_1|^{2g-1}} \Big\{ + |\mu| \cos\left(\theta_{A_1}(2g-1) - \theta_\mu\right) + |\widetilde{\mu}| \cos\left(\theta_{A_1}(2g-1) + \theta_{\widetilde{\mu}}\right) \Big\} + \\ &+ i\, \frac{\Gamma(2g-1)}{|A_1|^{2g-1}} \Big\{ - |\mu| \sin\left(\theta_{A_1}(2g-1) - \theta_\mu\right) + |\widetilde{\mu}| \sin\left(\theta_{A_1}(2g-1) + \theta_{\widetilde{\mu}}\right) \Big\}. \end{aligned} \tag{4.64}$$

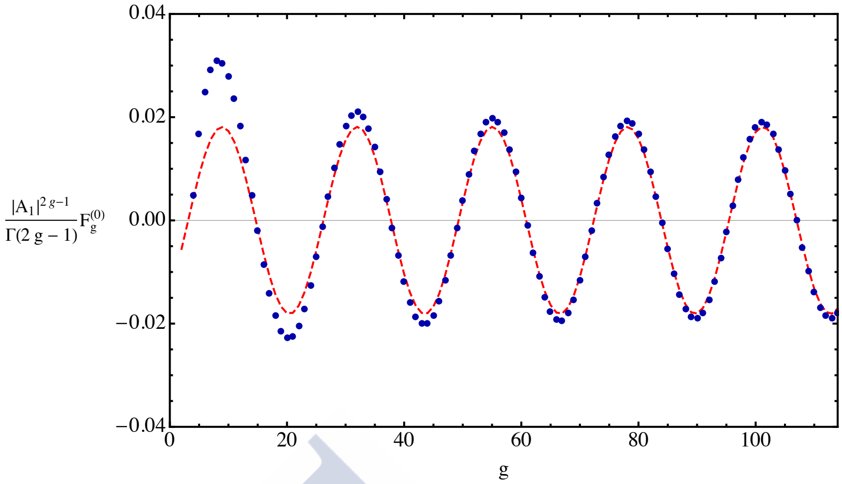


Figure 4.20: Same numerical data as in figure 4.9 for $\psi = 1.25e^{i\pi/3}$ and $S^{zz} = 10^{-5}$ along with the analytic prediction (4.62) represented by a dashed red line. This line is obtained by joining the values of (4.62) for integer g rather than letting g be real. The latter way of doing it would produce a wave of larger frequency.

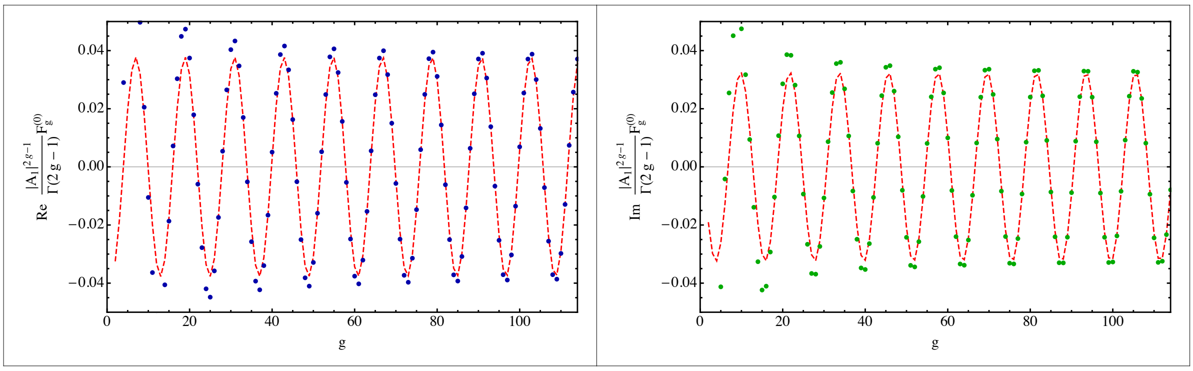


Figure 4.21: Large-order oscillatory behavior of the perturbative sector due to simultaneous contributions of A_1 and A_2, for a value of $\psi = 1.1\, e^{i\pi/3}$ and $S^{zz} = -(1+i)10^{-4}$. The free energies are complex because the propagator is. Dashed lines are given by (4.64)

Now there is a real and an imaginary part. We show an example in figure 4.21.

We have only considered for these tests the one-loop free energies, $F_0^{(\boldsymbol{e}_1)}$ and $F_0^{(\boldsymbol{e}_2)}$. Including higher terms would reproducing more and more accurately the oscillations shown in the figures, up until the optimal truncation point. This is because the $\frac{1}{g}$ series expansion of

$$\frac{|A_1|^{2g-1}}{\Gamma(2g-1)} F_g^{(\mathbf{0})} \tag{4.65}$$

is asymptotic. We will show that this is the case in section 4.2.2, on the subleading correction to the large-order growth.

4.2.2 Subleading contribution

Now that we have analyzed the leading contribution to the perturbative large-order growth, we can start looking at the subleading terms. This is a very rewarding task because beyond the sector $(\boldsymbol{e}_1)$ there are three other competing sectors: $(2\boldsymbol{e}_1)$, $(\boldsymbol{e}_2)$ and $(\boldsymbol{e}_3)$. We have represented this situation schematically in (4.31). In order to study the free energies associated to these sectors we need to do a resummation of the one-instanton contribution. This is an asymptotic series in $\frac{1}{g}$. In chapter 1 we have explained how to do this in various ways, depending on what the situation demands.

However, there is a special regime for a particular value of the propagator, for which resummation is trivial and coincides with the identity operator. In the holomorphic limit $S^{zz} \to S^{zz}_{[1],\text{hol}}$ the $(\boldsymbol{e}_1)$ free energies truncate after just two terms. The holomorphic limit is quite rich, though. Since the instanton actions are holomorphic they can be studied entirely in this regime. Also, we know from the analysis in section 3.7 what the holomorphic limit of the subleading contribution near $\psi = 1$ should be. A numerical check for this confirms the ambiguity fixing condition that will be used in the general nonholomorphic situation.

We separate this subsection in the natural holomorphic and nonholomorphic cases.

Holomorphic case

The holomorphic limit here refers to the one with respect to the first conifold point,

$$S^{zz} \to S^{zz}_{[1],\text{hol}}. \tag{4.66}$$

From here on, the curly $\mathcal{F}$ will always denote this limit, even if we do not include the corresponding frame label.

In this situation the perturbative free energies have the asymptotic behavior

$$\mathcal{F}^{(\mathbf{0})}_g \sim \frac{\Gamma(2g-1)}{A_1^{2g-1}} \frac{S_{1,1}}{\pi i} \mathcal{F}^{(\boldsymbol{e}_1)}_0 + \frac{\Gamma(2g-2)}{A_1^{2g-2}} \frac{S_{1,1}}{\pi i} \mathcal{F}^{(\boldsymbol{e}_1)}_1 \tag{4.67}$$

$$= \frac{\Gamma(2g-1)}{A_1^{2g-1}} \frac{A_1}{2\pi^2} + \frac{\Gamma(2g-2)}{A_1^{2g-2}} \frac{1}{2\pi^2} = \frac{\Gamma(2g-1)}{A_1^{2g-1}} \frac{A_1}{2\pi^2} \left(1 + \frac{1}{2g-2}\right). \tag{4.68}$$

Subleading contributions to the asymptotics are exponentially suppressed in g with respect to the leading one. To study them we define the following quantity,

$$\mathcal{X}^{(\boldsymbol{e}_1)}_g := \frac{A_1^{2g-1}}{\Gamma(2g-1)} \left(\mathcal{F}^{(\mathbf{0})}_g - \sum_{h=0}^{1} \frac{\Gamma(2g-1-h)}{A_1^{2g-1-h}} \frac{S_{1,1}}{\pi i} \mathcal{F}^{(\boldsymbol{e}_1)}_h \right). \tag{4.69}$$

The large-order behavior of this quantity has information about contributions from $(2\boldsymbol{e}_1)$, $(\boldsymbol{e}_2)$ or $(\boldsymbol{e}_3)$, depending on the value of the complex structure modulus ψ. Which sector it is depends on the ratio

$$\left(\frac{A_{\text{subl}}}{A_1}\right)^2 \quad \text{where} \quad A_{\text{subl}} \in \{2A_1, A_2, A_3\}. \tag{4.70}$$

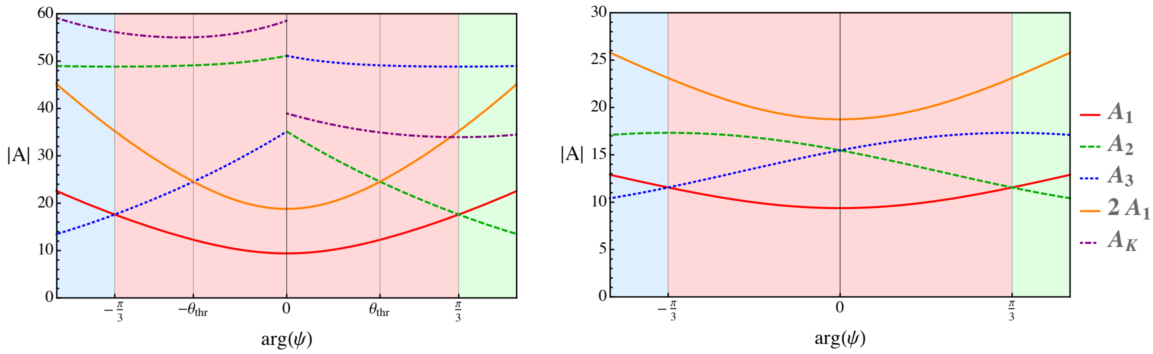


Figure 4.22: Absolute values of the three conifold points, A_i, $i = 1, 2, 3$, of $2A_1$ and the Kähler instanton action $A_{\rm K}$, for $|\psi| = 2$ (left) and $|\psi| = \frac{1}{4}$ (right). The subdominant instanton action, second smallest, depends on the point in moduli space. For $|\psi|$ small $2A_1$ is never subdominant, while for $|\psi|$ larger it can be around $\arg(\psi) = 0$. $A_{\rm K}$ does not appear in the right plot because it has moved to another Riemann sheet; see discussion around figure 4.7.

Using the analytic expression for the instanton actions we can immediately see which sector dominates in which region. We see in figure 4.22 that there are two different situations depending on the value of $|\psi|$. If this quantity is small enough then $2A_1$ is never the subleading instanton action, while it is the two-instanton sector dominates the central region around $\arg(\psi) = 0$. We can also see this separation of the ψ-plane is regions of subdominance in figure 4.23.

The general structure of $\mathcal{X}_g^{(\boldsymbol{e}_1)}$ is

$$\mathcal{X}_g^{(\boldsymbol{e}_1)} \sim \left(\frac{A_1}{A_{\rm subl}}\right)^{2g-1} \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{\Gamma(2g-1)} \frac{S_{1,{\rm subl}}}{\pi i} \mathcal{F}_h^{({\rm subl})}. \tag{4.71}$$

To extract $A_{\rm subl}$ we take the limit

$$\lim_{g\to\infty} \frac{\mathcal{X}_{g+1}^{(\boldsymbol{e}_1)}}{\mathcal{X}_g^{(\boldsymbol{e}_1)}} = \left(\frac{A_{\rm subl}}{A_1}\right)^2. \tag{4.72}$$

We select two values of ψ associated to two different subleading instanton actions, and show the perfect agreement with the analytic expressions in figure 4.24. The calculation of the value of A_2 outside the boundary ($\arg(\psi) = \pi/3$) of wedge 1 proves that the instanton sectors associated to the conifold points at $\mathrm{e}^{\pm i\pi/3}$ are as real as the one at $\psi = 1$, and that they all contribute to the large-order growth of perturbation theory.

We can be systematic in the computation of the limit (4.72) and the comparison with the right-hand-side, and display one on top of the other for different values of $|\psi|$ and $\arg(\psi)$ like in figure 4.22. See figure 4.25.

Let us explore now in more detail the different regions of this plot and uncover what is the form of the free energies involved in (4.71). The regions where A_2 and A_3 subdominate are essentially equivalent so we focus on A_2 and $\arg(\psi) > 0$.

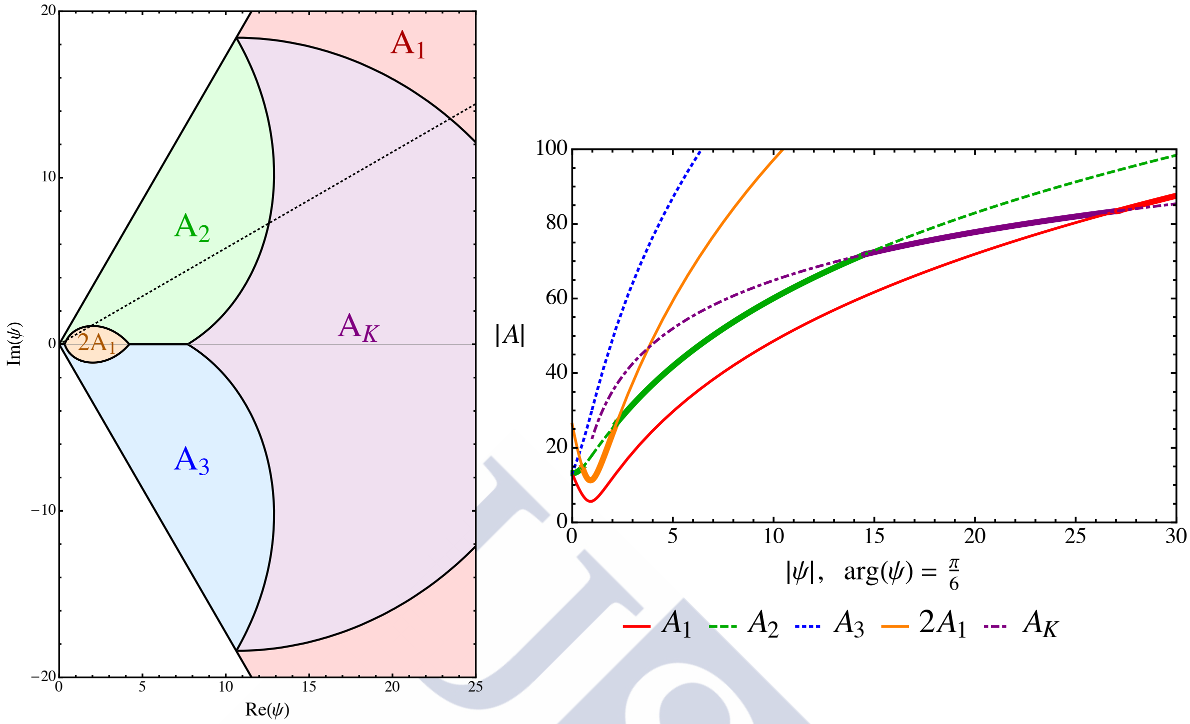


Figure 4.23: On the left plot we show the separation of the ψ-plane according to which instanton action controls the subleading growth of the perturbative sector. The conifold point $\psi = 1$ lies inside the orange region dominated by $2A_1$ because that instanton action vanishes there. Around that area A_2 and A_3 are subleading with respect to A_1, and further away it is $A_{\rm K}$. Beyond that A_K and A_1 interchange their subdominant and dominant roles. Plots in figure 4.22 show circular sections of this plot for $|\psi|$ constant and small. On the right plot we show the absolute value of the instanton actions along the dotted line and we stress the subleading contribution with thickened lines.

For (subl) $= (\boldsymbol{e}_2)$ the free energies, $\mathcal{F}_h^{(\rm subl)}$ are precisely those computed from the holomorphic anomaly equations and evaluated at $S^{zz} = S^{zz}_{[1],\rm hol}$. That is,

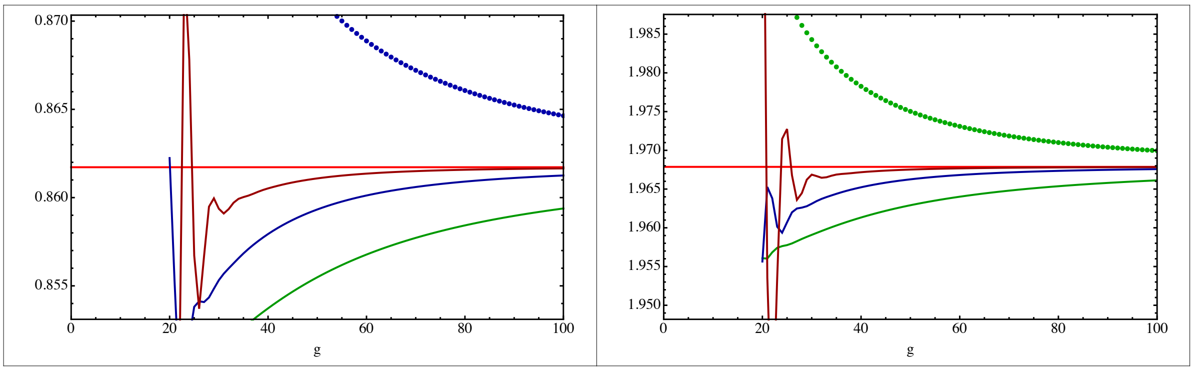
$$\mathcal{X}_g^{(\boldsymbol{e}_1)} \sim \left(\frac{A_1}{A_2}\right)^{2g-1} \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{\Gamma(2g-1)} A_2^h \frac{S_{1,2}}{\pi i} F_h^{(\boldsymbol{e}_2)}(z, S^{zz}_{[1],\rm hol}). \tag{4.73}$$

The form of $F_h^{(\boldsymbol{e}_2)}(z, S^{zz})$ is the same as that for $(\boldsymbol{e}_1)$ but with A_2 substituting A_1 in every instance. We have already seen before the case $h = 0$ when $\arg(\psi) = \pi/3$ and the same applies to higher loops. For general values of the propagator,

$$\frac{S_{1,2}}{\pi i} F_h^{(\boldsymbol{e}_2)} = \mathrm{e}^{\frac{1}{2}(\partial_z A_2)^2 \left(S^{zz} - S^{zz}_{[2],\rm hol}\right)} \mathrm{Pol}(S^{zz}; 3h), \tag{4.74}$$

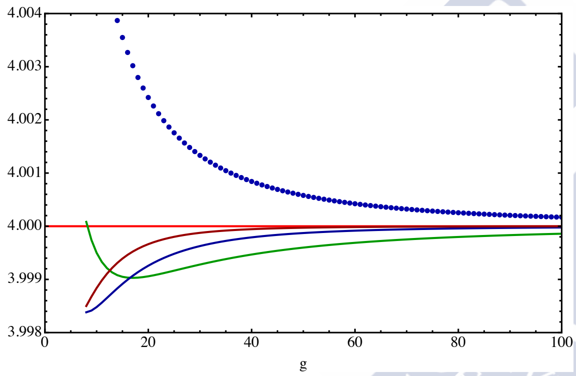
where the fixing of the ambiguity has been done by comparing against the natural holomorphic limit $S^{zz} \to S^{zz}_{[2],\rm hol}$. At that value we recover (3.159)-(3.161) for $n = 1$ and $A_{\rm c} = A_2$. These free energies have to be evaluated not at $S^{zz}_{[2],\rm hol}$, but at $S^{zz}_{[1],\rm hol}$,

$$\mathcal{F}_h^{(\boldsymbol{e}_2)} = F_h^{(\boldsymbol{e}_2)}(z, S^{zz}_{[1],\rm hol}). \tag{4.75}$$



$$\left(\frac{A_2}{A_1}\right)^2 = 0.861\,721\,3 + 1.967\,864\,55\,i$$

$$\text{5 Richardson Transforms} = 0.861\,720\,5 + 1.967\,864\,37\,i$$



$$\left(\frac{2A_1}{A_1}\right)^2 = 4.000\,000\,0$$

$$\text{3 Richardson Transforms} = 3.999\,997\,3$$

Figure 4.24: Numerical value for the limit (4.72) for $\psi = 2\,\mathrm{e}^{i\pi/4}$ and $\psi = 2$. In the first plot we plot both real and imaginary parts, along with the first Richardson transforms. The agreement is better than one part in 10^6. In the second the exact result must be exactly 4.

This means that there is no truncation of the large-order growth in this region of moduli space. We show the agreement between the numerical free energies and the ones computed from the holomorphic anomaly equations in figure 4.26 for $h = 0, 1, 2$.

We turn to the region in moduli space where $2A_1$ is the subdominant instanton action. In this case we need to reproduce the analytical results that we obtained in section 3.7.1,

$$\mathcal{X}_g^{(\boldsymbol{e}_1)} \sim \frac{1}{2^{2g-1}} \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{\Gamma(2g-1)} (2A_1)^h \frac{(S_{1,1})^2}{\pi i} \widetilde{\mathcal{F}}_h^{(2\boldsymbol{e}_1)}, \tag{4.76}$$

where

$$\frac{(S_{1,1})^2}{\pi i} \widetilde{\mathcal{F}}_0^{(2\boldsymbol{e}_1)} = \frac{1}{2} \frac{A_1}{2\pi^2}, \qquad \frac{(S_{1,1})^2}{\pi i} \widetilde{\mathcal{F}}_1^{(2\boldsymbol{e}_1)} = \frac{1}{2^2} \frac{1}{2\pi^2}, \qquad \widetilde{\mathcal{F}}_{h\geq 2}^{(2\boldsymbol{e}_1)} = 0. \tag{4.77}$$

the reason for having put a tilde on top of the free energies, $\widetilde{F}_h^{(2\boldsymbol{e}_1)}$ can only be seen outside the holomorphic regime. For now we just need to take the appropriate large g limits to extract the objects in (4.77) and compare against (4.76). We do this for a particular value of ψ and show the result in figure 4.27.

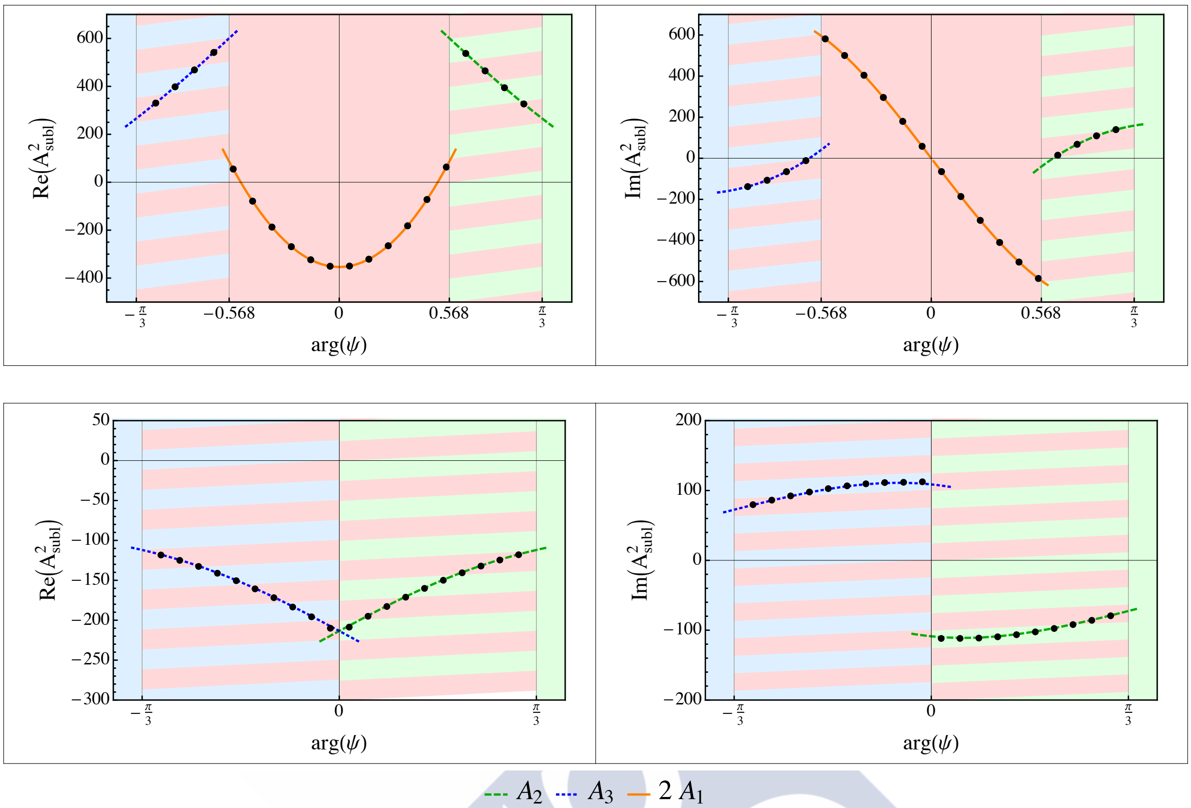


Figure 4.25: The real and imaginary parts of the subleading instanton action squared, A^2_{subl}, where A_{subl} can be $2A_1$ (red), A_2 (green) or A_3 (blue) depending on the point in moduli space. The numerical values are obtained from ratios of coefficients $\mathcal{X}_g^{(\boldsymbol{e}_1)}$, for $|\psi| = 2$ (top) and for $|\psi| = \frac{1}{4}$ (bottom). The results are in correspondence with figure 4.22. Jumps are not due to branch cuts—not even the bottom one since the cuts start at $|\psi| \geq 1$—but to a change in dominance of the subleading instanton action.

Nonholomorphic case

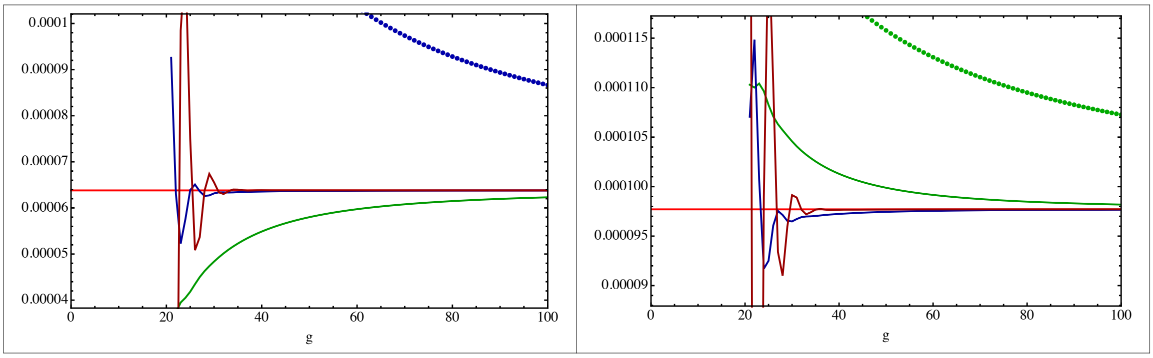
For any value of the propagator, other than $S^{zz}_{[1],\text{hol}}$, the one-instanton contribution to the perturbative growth does not truncate. We have to worry then about the problem of performing the resummation of the series

$$I(g) := \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{\Gamma(2g-1)} A_1^h \frac{S_{1,1}}{\pi i} F_h^{(\boldsymbol{e}_1)}, \tag{4.78}$$

which we can use to define the nonholomorphic extension of $\mathcal{X}_g^{(\boldsymbol{e}_1)}$,

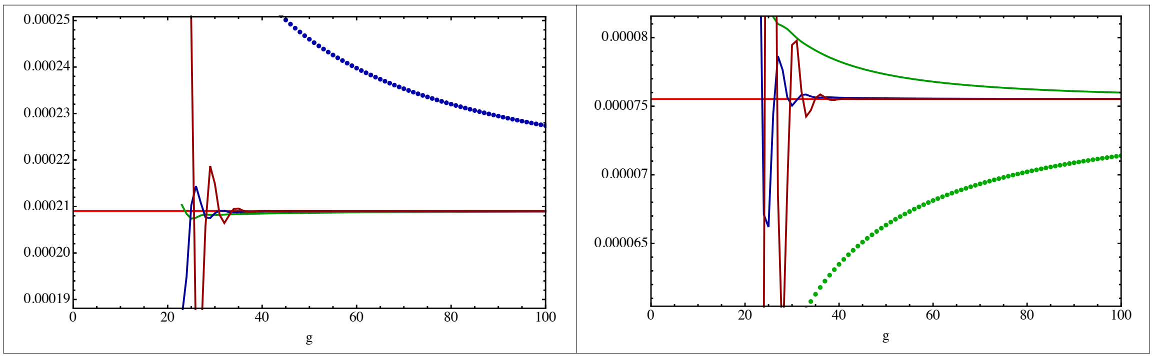
$$X_g^{(\boldsymbol{e}_1)} := \frac{A_1^{2g-1}}{\Gamma(2g-1)} F_g^{(\boldsymbol{0})} - \text{Resum}[I(g)]. \tag{4.79}$$

The choice of resummation technique is based on the comparison between the error of the resummation and the order of magnitude of the subleading contribution, which is the information we want to access. See section 1.5.2 for more details.



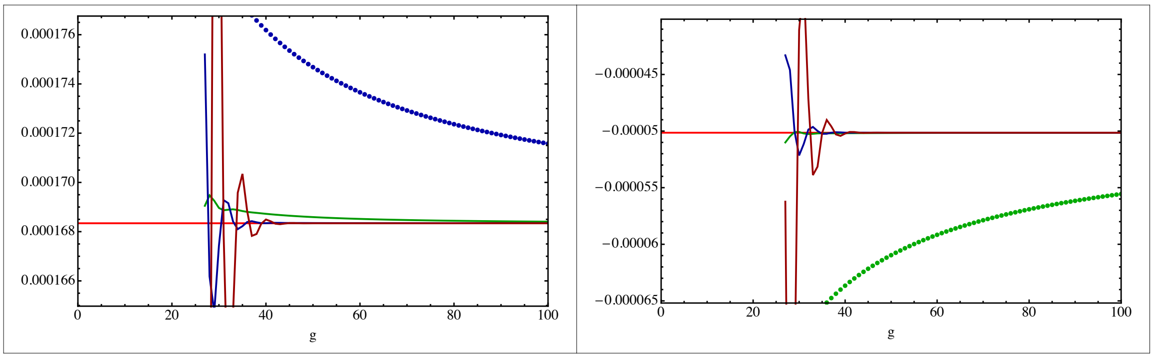
$$\frac{S_{1,2}}{\pi i}\,\mathcal{F}_0^{(e_2)} = 0.000\,063\,797\,45 + 0.000\,097\,703\,4\,i$$

$$\text{3 Richardson Transforms} = 0.000\,063\,797\,90 + 0.000\,097\,702\,4\,i$$



$$\frac{S_{1,2}}{\pi i}\,\mathcal{F}_1^{(e_2)} = 0.000\,209\,011\,81 + 0.000\,075\,524\,080\,i$$

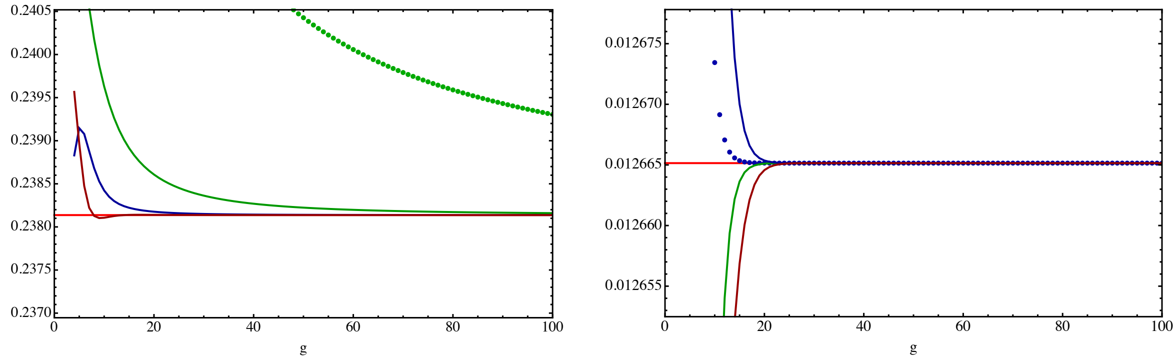
$$\text{3 Richardson Transforms} = 0.000\,209\,011\,64 + 0.000\,075\,524\,095\,i$$



$$\frac{S_{1,2}}{\pi i}\,\mathcal{F}_2^{(e_2)} = 0.000\,168\,340\,320 - 0.000\,050\,145\,41\,i$$

$$\text{3 Richardson Transforms} = 0.000\,168\,340\,333 - 0.000\,050\,145\,39\,i$$

Figure 4.26: (e_2) free energy coefficients, at $\psi = 2\,e^{i\pi/4}$, up to three loops. Numerical results after several Richardson transforms are compared to analytic expressions (4.75). The holomorphic limit is with respect to the *first* conifold point.



$\frac{(S_{1,1})^2}{\pi i}\,\widetilde{\mathcal{F}}_0^{(2\boldsymbol{e}_1)} = 0.238\,137\,202\,745\,i$

3 Richardson Transforms $= 0.238\,137\,208\,909\,i$

$\frac{(S_{1,1})^2}{\pi i}\,\widetilde{\mathcal{F}}_1^{(2\boldsymbol{e}_1)} = 0.012\,665\,147\,955$

No Richardson Transforms $= 0.012\,665\,147\,955$

Figure 4.27: Numerical tests of the holomorphic free-energy coefficients (4.77), for the $(2\boldsymbol{e}_1)$ sector, with $\psi = 2$. Numerical results after several or no Richardson transforms are compared against the analytic expressions. All higher-loop coefficients, $h \geq 2$, are zero.

In the particular case that we are considering we can distinguish between the two situations in which we separated the discussion in the holomorphic case. When the subleading contribution is $(\boldsymbol{e}_2)$ or $(\boldsymbol{e}_3)$, optimal truncation (OT) on $I(g)$ is enough to study the large-order growth of $X_g^{(\boldsymbol{e}_1)}$. Indeed, consider a particular choice of parameters, $\psi = 2\mathrm{e}^{2\pi i/9}$ and $S^{zz} = 10^{-8}$, along with $g = 50$. Let

$$\text{EXACT} = \frac{A_1^{2g-1}}{\Gamma(2g-1)} F_g^{(\boldsymbol{0})}, \tag{4.80}$$

$$\text{OT} = \sum_{h=0}^{h_{\text{OT}}} \frac{a_h}{g^h}, \tag{4.81}$$

where we have expanded $I(g)$ around $g = \infty$ as in (1.55) and truncated the sum to its optimal value. Then we can calculate

$$\text{EXACT} - \text{OT} \simeq 10^{-24}, \quad \text{OT ERROR} \simeq 10^{-33}, \tag{4.82}$$

where the optimal truncation error is estimated by the next term in the series as in (1.7). This error is much smaller than the order of magnitude of the subleading contribution, estimated by

$$\left(\frac{A_1}{A_2}\right)^{2g-1} \simeq 10^{-24}. \tag{4.83}$$

On the other hand, when the subleading term is controlled by the $(2\boldsymbol{e}_1)$ free energies and $2A_1$, the order of magnitude of this contribution is

$$\left(\frac{A_1}{2A_1}\right)^{2g-1} \simeq 10^{-30}, \tag{4.84}$$

that is large but still rivals the optimal truncation error. One has, then, to use a more powerful resummation method, like Borel–Padé, as explained in section 1.5.2.

When ψ is such that $|A_1| < |A_2| < \cdots$ the situation described around equation (4.73) generalizes very mildly to

$$X_g^{(\boldsymbol{e}_1)} \sim \left(\frac{A_1}{2A_1}\right)^{2g-1} \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{\Gamma(2g-1)} A_2^h \frac{S_{1,2}}{\pi i} F_h^{(\boldsymbol{e}_2)}. \tag{4.85}$$

Now we can evaluate the propagator to any value of choice. See figure 4.28 for some examples.

The situation in which the subleading contribution is given by $2A_1$ is more complicated and new features appear that had not been seen before. The large-order behavior of $X_g^{(\boldsymbol{e}_1)}$ is in this case,

$$X_g^{(\boldsymbol{e}_1)} \sim \frac{1}{2^{2g-1}} \sum_{h=0}^{\infty} \frac{\Gamma(2g-1-h)}{\Gamma(2g-1)} (2A_1)^h \frac{(S_{1,1})^2}{\pi i} \widetilde{F}_h^{(2\boldsymbol{e}_1)}. \tag{4.86}$$

Because we are not in the holomorphic limit anymore the sum over loops extends all the way to infinity instead of truncating at $h = 1$. The other, more important surprising, difference is that the free energies sitting in (4.86) do not directly have the interpretation of elements of the transseries that we can compute from the holomorphic anomaly equations. What we know about the objects $\widetilde{F}_h^{(2\boldsymbol{e}_1)}$ is that in the holomorphic limit they reduce to (4.77). However if we start computing two-instanton free energies out the holomorphic anomaly equations and fix the holomorphic ambiguities against this limit, we cannot reproduce the propagator dependence of $\widetilde{F}_h^{(2\boldsymbol{e}_1)}$ computed numerically out of (4.86). Note that the holomorphic anomaly equation (3.85) for $(\boldsymbol{n}) = (2\boldsymbol{e}_1)$ depends on the value

$$B(\boldsymbol{e}_1, 2\boldsymbol{e}_1) = 2b^{(\boldsymbol{e}_1)} - b^{(2\boldsymbol{e}_1)}. \tag{4.87}$$

We do not know what the value of this number is but we find disagreement for any choice. The only situation in which the free energy computed out the holomorphic anomaly equation and fixed with (4.77) agrees with the numerics is when B is such that the quadratic term in (3.85) drops for each g. But this would put a value of infinity of B, which cannot be.

Fortunately, even though the function $\widetilde{F}_h^{(2\boldsymbol{e}_1)}$ cannot be written as a single free energy, it can in fact be written as the combination of two,

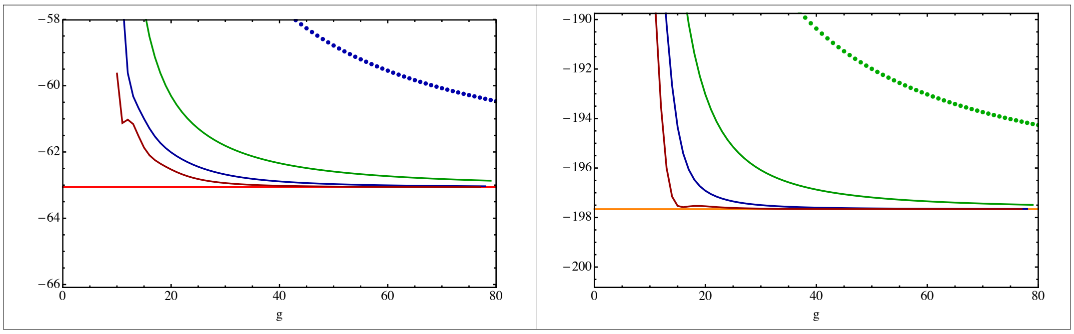
$$\widetilde{F}_h^{(2\boldsymbol{e}_1)} = F_h^{(2\boldsymbol{e}_1)} - \widehat{F}_h^{(2\boldsymbol{e}_1)}. \tag{4.88}$$

Both functions $F_h^{(2\boldsymbol{e}_1)}$ and $\widehat{F}_h^{(2\boldsymbol{e}_1)}$ are calculated from the holomorphic anomaly equations as two-instanton elements of the transseries, with

$$B(2\boldsymbol{e}_1, \boldsymbol{e}_1) = 0. \tag{4.89}$$

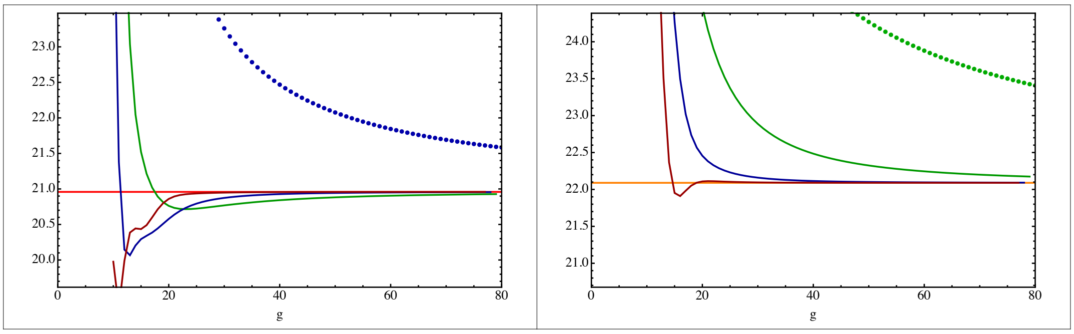
For example, for $g = 0$ the holomorphic anomaly equation reads

$$\left(\partial_{S^{zz}} - \frac{1}{2}4(\partial_z A_1)^2\right) F_0^{(2\boldsymbol{e}_1)} = \frac{1}{2}(\partial_z A_1)^2 \left(F_0^{(\boldsymbol{e}_1)}\right)^2, \tag{4.90}$$



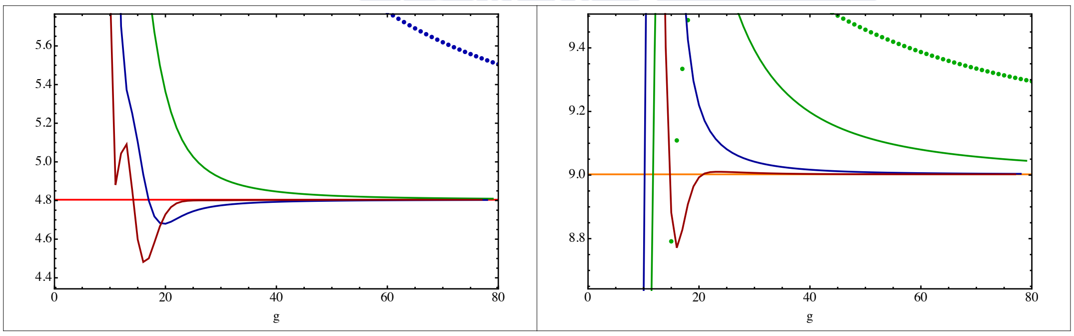
$$\frac{S_{1,2}}{\pi i}\, F_0^{(\boldsymbol{e}_2)} = -63.059\,684 - 197.660\,388\,9\, i$$

$$5 \text{ Richardson Transforms} = -63.059\,646 - 197.660\,384\,1\, i$$



$$\frac{S_{1,2}}{\pi i}\, F_1^{(\boldsymbol{e}_2)} = 20.958\,451\,4 + 22.089\,526\,167\, i$$

$$5 \text{ Richardson Transforms} = 20.958\,453\,5 + 22.089\,526\,107\, i$$



$$\frac{S_{1,2}}{\pi i}\, F_2^{(\boldsymbol{e}_2)} = 4.804\,186\,3 + 9.002\,264\,0\, i$$

$$5 \text{ Richardson Transforms} = 4.804\,185\,4 + 9.002\,263\,2\, i$$

Figure 4.28: Nonholomorphic counterpart of figure 4.26 for the sector $(\boldsymbol{e}_2)$. Here $\psi = 2\,\mathrm{e}^{i\pi/4}$ and $S^{zz} = 10^{-5}$.

whose solution is a combination of two exponentials. The holomorphic ambiguities, $f_0^{(2\boldsymbol{e}_1)}$ and $\widehat{f}_0^{(2\boldsymbol{e}_1)}$, are fix differently. $f_0^{(2\boldsymbol{e}_1)}$ is calculated like $f_0^{(\boldsymbol{e}_1)}$, by comparing the holomorphic limit of $F_0^{(2\boldsymbol{e}_1)}$ against $\frac{1}{2}\frac{A_1}{2\pi^2}$. On the other hand, $\widehat{f}_0^{(2\boldsymbol{e}_1)}$ is obtained from the condition that $\widehat{\mathcal{F}}_0^{(2\boldsymbol{e}_1)}$ vanishes. The end result is such that

$$\widetilde{F}_0^{(2\boldsymbol{e}_1)} = F_0^{(2\boldsymbol{e}_1)} - \widehat{F}_0^{(2\boldsymbol{e}_1)} = \frac{\pi i}{(S_{1,1})^2}\frac{1}{2}\frac{A_1}{2\pi^2}\mathrm{e}^{4\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}. \tag{4.91}$$

Higher loop free energies are computed and fixed in a similar way. See appendix C for a more detailed description on the structure of the solutions. We can compare some numerical values against these functions. See figure 4.29. The numerical convergence acceleration is not as effective for higher loops as it was in the previous cases due to loss of precision in the resummation process.

4.3 Large-order analysis of the one-instanton sector

In this section we explore the asymptotic behavior of the one-instanton sector ($\boldsymbol{e}_1$) around $\psi = 1$. We restrict to this sector and domain in moduli space because it is the most natural continuation of the analysis done so far, and it contains many interesting properties needed to understand resurgence in local $\mathbb{CP}^2$. These are resonance, triviality of the holomorphic limit, and the connection to the subleading contribution to $F_g^{(\boldsymbol{0})}$ incarnated in $\widetilde{F}_g^{(2\boldsymbol{e}_1)}$.

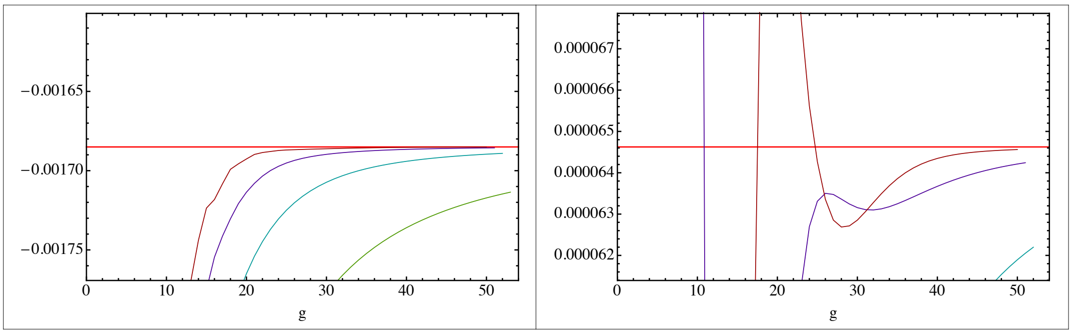
To study the large-order growth of the free energies $F_g^{(\boldsymbol{e}_1)}(z, S^{zz})$ we must have a sufficient number of them. That is, we must be able to compute recursively out of the holomorphic anomaly equations free energies up to large enough g. This turns out not to be an easy task in comparison with what is needed in the perturbative sector. In that case the size of the free energies $F_g^{(\boldsymbol{0})}$ did not grow too fast with g. The reason is that the free energies have the form

$$F_g^{(\boldsymbol{0})} = (C_{zzz})^{2g-2}\,\mathrm{Pol}(S^{zz}; 3g-3) \tag{4.92}$$

and the coefficients are polynomials in z of degree at most $8(g-1)$. This combination of linear growths can be maintained under control on the computer. For the one-instanton free energies, on the other hand, we have an extra element to work with which makes their size impossible to handle after a few orders, namely, the instanton action. The structure of the functions $F_g^{(\boldsymbol{e}_1)}$ is similar to that of the perturbative free energies,

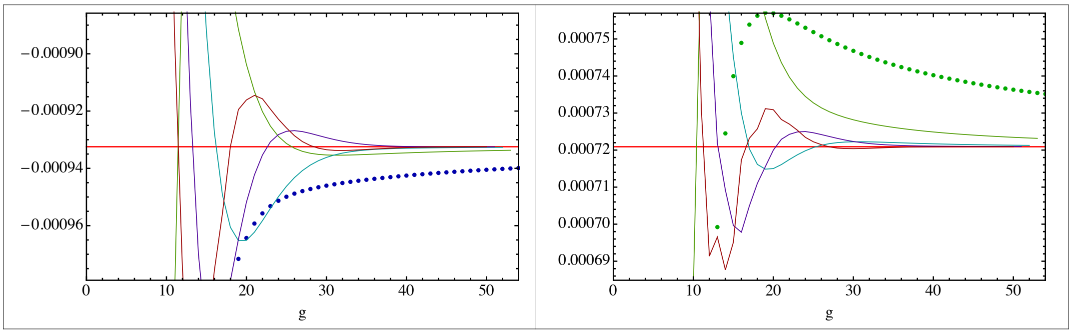
$$F_g^{(\boldsymbol{e}_1)} = \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}\mathrm{Pol}(S^{zz}; 3g), \tag{4.93}$$

but now the coefficients have more structure due to the presence of the instanton action A_1 and its derivatives. The size of the coefficients grows very fast with g and the free energies quickly become too big. The recursive nature of the computation makes this even worse, having to store in memory very large functions at a time. Some improvement comes after substituting higher derivatives of A_1 in terms of lower ones, as we mentioned once before, and seeing several cancellations happen. This allows us to compute up to around $g = 20$, but still it does not provide us with a sufficient number of free energies to perform a large-order analysis (some computations to leading order can be attempted but the results are poor). To solve this bottleneck we must turn to a numerical approach. To be precise we want to fix a point in moduli space, that is, a value of z or ψ, and evaluate as many free energies at that



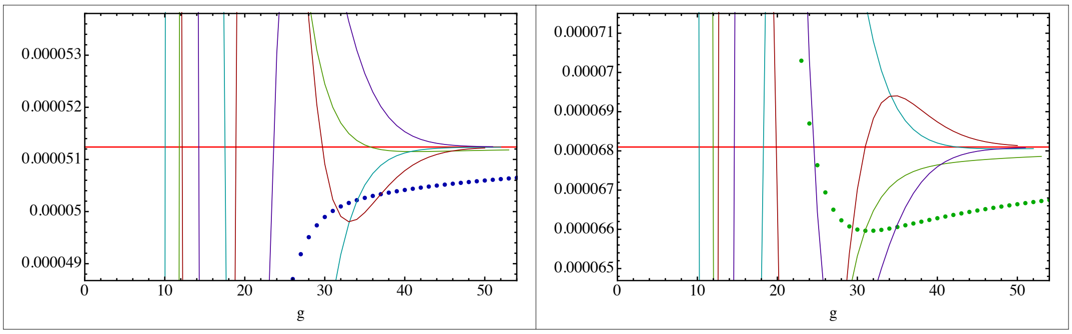
$$\frac{(S_{1,1})^2}{\pi i}\,\widetilde{F}_0^{(2e_1)} = -0.001\,685\,034 + 0.000\,064\,624\,8\,i$$

$$5 \text{ Richardson Transforms} = -0.001\,685\,043 + 0.000\,064\,622\,0\,i$$



$$\frac{(S_{1,1})^2}{\pi i}\,\widetilde{F}_1^{(2e_1)} = -0.000\,932\,452 + 0.000\,720\,907\,i$$

$$5 \text{ Richardson Transforms} = -0.000\,932\,426 + 0.000\,720\,911\,i$$



$$\frac{(S_{1,1})^2}{\pi i}\,\widetilde{F}_2^{(2e_1)} = 0.000\,051\,238 + 0.000\,068\,099\,i$$

$$5 \text{ Richardson Transforms} = 0.000\,051\,285 + 0.000\,068\,018\,i$$

Figure 4.29: Numerical checks of $\frac{(S_{1,1})^2}{\pi i}\,\widetilde{F}_h^{(2e_1)}$, for $h = 0, 1, 2$. Here $\psi = 2\,\mathrm{e}^{-i\pi/12}$, $S^{zz} = 10^{-8}$.

point as we can. The dependence on the propagator remains symbolic, hence exact, because the antiholomorphic structure stays simple and under control as g grows. Since we have to perform up to second order derivatives in z to solve the holomorphic anomaly equations we need to carry along not only the value of the free energies at z, but also at values close to it, so that we can take those derivatives numerically as finite differences. Precision drops as the order g grows and it does so linearly, so we must start with precise enough initial data, that is, values of z, A, $\partial_z A$, perturbative free energies, etc. Because we already know the propagator structure of the solution form general results, and we have checked it against analytic calculations and large-order analysis, we can use it in the design of the algorithm. The form of the free energies (4.93) can be spelled out as

$$F_g^{(1)} = c\,\mathrm{e}^{b\,S^{zz}}\left(a_0 + a_1\,S^{zz} + a_2(S^{zz})^2 + \cdots + a_{3g}(S^{zz})^{3g}\right), \tag{4.94}$$

where $b = \frac{1}{2}(\partial_z A)^2$, c is a number involving the Stokes constant S_1 and factors of 2, π and i, that we separate for convenience. The coefficients of the polynomial, a_k, are functions of z evaluated at the point of choice. See appendix C for the general form of these coefficients. We store this information in lists,

$$F_g^{(1)} \mapsto L := \{c, b, \{a_0, a_1, \ldots, a_{3g}, 0, \ldots, 0\}\} \tag{4.95}$$

that will be manipulated as the algorithm runs. Because we have to keep the values of the free energies for nearby values of z we need several, say N, of such L-lists. We collected them in another list,

$$\Lambda = \{L_1, L_2, \ldots, L_N\}. \tag{4.96}$$

Working with numerical lists has the advantage of being able to apply fast operations on them. These operations represent actions of usual functions such as taking a derivative with respect to z or S^{zz}, adding or multiplying two free energies, or any operation appearing in the holomorphic anomaly equations. The first part of the algorithm translates all these into actions on L and Λ lists. The second part implements the integration and fixing of the holomorphic ambiguity in this syntax. With respect to the latter, the fixing condition can be carried out in this way because it only implies the comparison of a holomorphic limit—easy to implement in this language—and a prescribed function at a single point in moduli space, namely (3.159)-(3.161). The whole procedure would break at this point if the fixing of the ambiguity involved a global condition as is the case in perturbation theory. To check that the numerical integration is correct we can compare the first loop free energies against the ones computed in the traditional way. The complexity of the algorithm seems to be quadratic in the order, based on experience, which allows us to reach up to $g = 80$ in about one day and with small memory impact.

Even though we have only implemented the seminumerical algorithm for the one-instanton sectors, it must be possible to extend it to higher sectors without too big a penalty on speed or memory consumption. The main disadvantage of the method is that it has to be carried out one point in moduli space at a time. But, on the other hand, we are able to retain full dependence on the propagator.

From the examples of Painlevé I and II and the quartic matrix model [38, 40, 41], we expect the following asymptotic behavior for the one-instanton sector, see (1.64),

$$F_g^{(\boldsymbol{e}_1)} \sim \frac{\Gamma(g+c)}{(+A_1)^{g+c}}\,\mu_0(2\boldsymbol{e}_1) + \frac{\Gamma(g+c)}{(-A_1)^{g+c}}\,\mu_0(\boldsymbol{e}_{1,1}). \tag{4.97}$$

The notation here is the following. c is a constant that has to be found with the appropriate large-order relation, see (4.100), and should be a function of the starting powers $b^{(\boldsymbol{e}_1)}$, $b^{(2\boldsymbol{e}_1)}$, and $b^{(\boldsymbol{e}_{1,1})}$. We will discuss this point in the next section. $(\boldsymbol{e}_{1,1})$ is shorthand for the mixed sector of $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$,

$$(\boldsymbol{e}_{1,1}) = (1|1\|0|0\|0|0\|0\cdots). \tag{4.98}$$

$\mu_0(2\boldsymbol{e}_1)$ and $\mu_0(\boldsymbol{e}_{1,1})$ are quantities that, in the light of the examples mentioned above, are expected to be proportional to the corresponding one-loop ($h=0$) free energies. We have not yet met the free energy $F_0^{(\boldsymbol{e}_{1,1})}$ but we have already encountered $\widetilde{F}_0^{(2\boldsymbol{e}_1)}$ in (4.91). However, this is not the free energy that corresponds to $\mu_0(2\boldsymbol{e}_1)$. The simplest reason is that their holomorphic limits are different. Indeed, $\widetilde{\mathcal{F}}_0^{(2\boldsymbol{e}_1)} \neq 0$, see equation (4.77), while $\mu_0(2\boldsymbol{e}_1) \to 0$ as $S^{zz} \to S^{zz}_{[1],\text{hol}}$. This has to be the case because the left-hand-side of (4.97) is zero in the holomorphic limit for $g \geq 2$. By consistency $\mu_0(2\boldsymbol{e}_1)$ and $\mu_0(\boldsymbol{e}_{1,1})$ must also be zero in that limit. So we see that the resurgence relations that hold for the Painlevé equations and the quartic matrix model must be modified for the case of local $\mathbb{CP}^2$. See next section for more details. Let us stress that at this point we are not certain $\mu_0(2\boldsymbol{e}_1)$ or $\mu_0(\boldsymbol{e}_{1,1})$ should exactly be. A study of the resurgence relation (4.97) should tell us.

About (4.97) we must also mention that the factorial growth is in g, not $2g$, as expect from a sector, $(\boldsymbol{e}_1)$, whose asymptotic series is in g_s, not g_s^2. Also the presence of both A_1 and $-A_1$ as dominant instanton actions shows explicitly the phenomenon of resonance, first found in [38]. Because both signs of A_1 appear the large-order is oscillatory in g. However, letting g be either even or odd eliminates any numerical obstacle associated with this issue.

Let us then go on and extract the value of the number c and also check that the dominant instanton action is A_1. In these calculations we have chosen $\psi = 2$ as the moduli space to work with. A_1 can be calculated by the usual ratio of free energies, adapted to even or odd orders,

$$A_1^2 = \lim_{g\to\infty} 4g^2\,\frac{F_{2g}^{(\boldsymbol{e}_1)}}{F_{2g+2}^{(\boldsymbol{e}_1)}} = \lim_{g\to\infty} 4g^2\,\frac{F_{2g+1}^{(\boldsymbol{e}_1)}}{F_{2g+3}^{(\boldsymbol{e}_1)}}. \tag{4.99}$$

We perform the limit for a nonholomorphic value of the propagator showing once again the holomorphicity of the instanton action. The value of c is computed from the expression,

$$-2c-1 = \lim_{g\to\infty} 2g\left(1 - \frac{A_1^2}{4g^2}\frac{F_{2g+2}^{(\boldsymbol{e}_1)}}{F_{2g}^{(\boldsymbol{e}_1)}}\right). \tag{4.100}$$

We show both limits in figure (4.30).

The numerical values of $\mu_0(2\boldsymbol{e}_1)$ and $\mu_0(\boldsymbol{e}_{1,1})$ are calculate from the following two limits,

$$\frac{S_{1,1}}{\pi i}\left(\mu_0(2\boldsymbol{e}_1) - \mu_0(\boldsymbol{e}_{1,1})\right) = \lim_{g\to\infty}\frac{A_1^{2g+1}}{\Gamma(2g+1)}\frac{S_{1,1}}{\pi i}F_{2g}^{(\boldsymbol{e}_1)}, \tag{4.101}$$

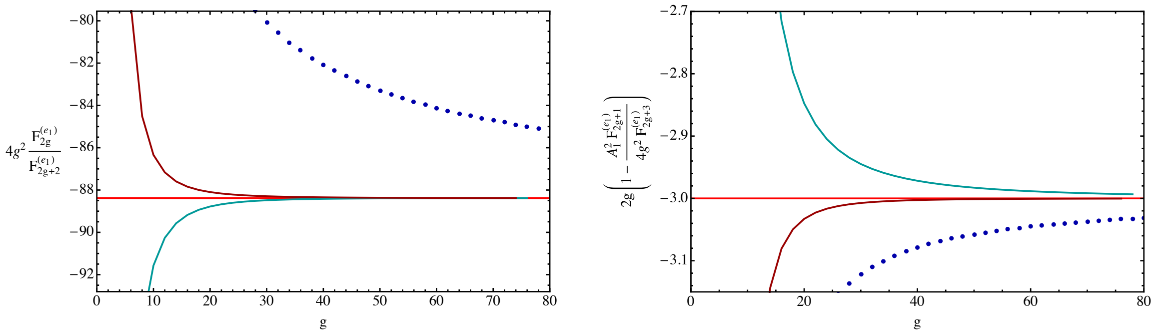


Figure 4.30: On the left, the limit (4.99) for $\psi = 2$ and $S^{zz} = -2\, S^{zz}_{[1],\mathrm{hol}}$. The value of the propagator can take any value except for the holomorphic one, since in that case the free energies are zero and the ratio under consideration would become indeterminate. On the right, the limit (4.100), with the same values of ψ and S^{zz}. It implies that $c = 1$ in (4.97). On both plots, original data and two Richardson transforms are shown.

$$\frac{S_{1,1}}{\pi i}\left(\mu_0(2\boldsymbol{e}_1) + \mu_0(\boldsymbol{e}_{1,1})\right) = \lim_{g\to\infty} \frac{A_1^{2g+2}}{\Gamma(2g+2)} \frac{S_{1,1}}{\pi i} F^{(\boldsymbol{e}_1)}_{2g+1}, \tag{4.102}$$

where we have multiplied $F^{(\boldsymbol{e}_1)}_g$ by $\frac{S_{1,1}}{\pi i}$ since that is the combination that we can determine. The results, for different values of the propagator are shown in figure 4.31, where $\psi = 2\mathrm{e}^{-i\pi/36}$. After looking at for a while at this plot and similar ones for other values of ψ, we can figure out what the antiholomorphic dependence is. With a little more effort we can guess the rest of the functional dependence. It is

$$\frac{S_{1,1}}{\pi i}\,\mu_0(2\boldsymbol{e}_1) = -\frac{1}{2}\left(\frac{A_1}{2\pi^2}\right)^2 \mathrm{e}^{2\cdot\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)} + \frac{1}{2}\left(\frac{A_1}{2\pi^2}\right)^2 \mathrm{e}^{4\cdot\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}, \tag{4.103}$$

$$\frac{S_{1,1}}{\pi i}\,\mu_0(\boldsymbol{e}_{1,1}) = +\frac{1}{2}\left(\frac{A_1}{2\pi^2}\right)^2 \mathrm{e}^{2\cdot\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)} - \frac{1}{2}\left(\frac{A_1}{2\pi^2}\right)^2. \tag{4.104}$$

Let us notice the quadratic dependence on A_1, the coefficients 2 and 4 in the exponentials of (4.103) and the constant term in (4.104). Also, as expected, the holomorphic limit vanishes for both functions.

The question now is, do these functions have an interpretation as elements of the transseries and can they be computed from the holomorphic anomaly equations? The answer to the second question is yes, they can, but their role in the transseries is more subtle. We will discuss it in section 4.4. With respect to $\mu_0(\boldsymbol{e}_{1,1})$ it can be calculated as an $(\boldsymbol{e}_{1,1})$, $h = 0$, free energy $\widehat{F}^{(\boldsymbol{e}_{1,1})}_0$ from the equations with $B(\boldsymbol{\varepsilon}_1, \boldsymbol{e}_{1,1}) = B(\boldsymbol{\varepsilon}_2, \boldsymbol{e}_{1,1}) = 0$, and the holomorphic ambiguity fixed by a vanishing limit condition. The hat in $\widehat{F}^{(\boldsymbol{e}_{1,1})}_0$ serves to indicate the way in which the ambiguity is fixed. The equation for $\widehat{F}^{(\boldsymbol{e}_{1,1})}_0$ is

$$(\partial_{S^{zz}} - 0)\widehat{F}^{(\boldsymbol{e}_{1,1})}_0 = -(\partial_z A_1)^2 F^{(\boldsymbol{\varepsilon}_1)}_0 F^{(\boldsymbol{\varepsilon}_2)}_0, \tag{4.105}$$

from which we calculate

$$\frac{1}{2}\frac{S_{1,1}}{\pi i}\frac{\widetilde{S}_{-1,1}}{\pi i}\widehat{F}^{(\boldsymbol{e}_{1,1})}_0 = \frac{1}{2}\left(\frac{A_1}{2\pi^2}\right)^2 \mathrm{e}^{2\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)} + \frac{1}{2}\frac{S_{1,1}}{\pi i}\frac{\widetilde{S}_{-1,1}}{\pi i} f^{(\boldsymbol{e}_{1,1})}_0. \tag{4.106}$$

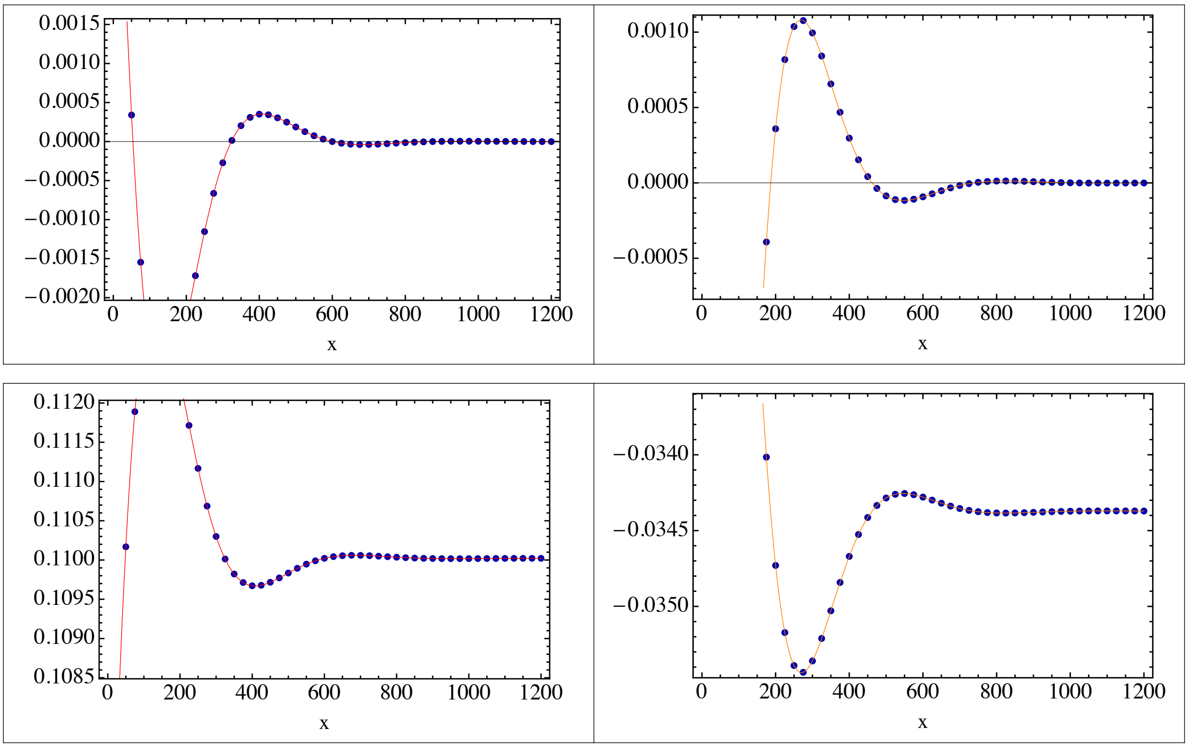


Figure 4.31: Real and imaginary parts of $\frac{(S_{1,1})^2}{(\pi i)^2}\,\widehat{F}_0^{(2\boldsymbol{e}_1)}$, following equation (4.103) (top), and $\frac{S_{1,1}}{\pi i}\,\frac{\widetilde{S}_{-1,1}}{2\pi i}\,\widehat{F}_0^{(\boldsymbol{e}_{1,1})}$, following equation (4.104) (bottom), for $\psi = 2\,\mathrm{e}^{-i\pi/36}$ and varying propagator $S^{zz} = 10^{-8}\,(1+ix)$, compared against numerical data from the large-order growth in (4.97).

The ambiguity $f_0^{(\boldsymbol{e}_{1,1})}$ is fixed by imposing that $\mathcal{F}_0^{(\boldsymbol{e}_{1,1})} = 0$. So

$$\mu_0(\boldsymbol{e}_{1,1}) = \frac{1}{2}\frac{\widetilde{S}_{-1,1}}{\pi i}\widehat{F}_0^{(\boldsymbol{e}_{1,1})}. \tag{4.107}$$

The extra factor $\frac{1}{2}$ is natural from the point of view of a resurgent relation derived from a standard bridge equation, see (1.64). $\widetilde{S}_{-1,1}$ is the Stokes constant associated to $F_0^{(\boldsymbol{\varepsilon}_2)}$. This free energy is the same as $F_0^{(\boldsymbol{\varepsilon}_1)}$ but with the sign of A_1 reversed. The zero in (4.105) corresponds to $A^{(\boldsymbol{e}_{1,1})} = A_1 + (-A_1) = 0$.

With respect to $\mu_0(2\boldsymbol{e}_1)$ we can calculate its functional form as a two-instanton sector $\widehat{F}_0^{(2\boldsymbol{e}_1)}$ satisfying a holomorphic anomaly equation with $B(\boldsymbol{e}_1, 2\boldsymbol{e}_1) = 0$, just like (4.90). The condition $\widehat{\mathcal{F}}_0^{(2\boldsymbol{e}_1)}$ fixes the holomorphic ambiguity. This identifies $\widehat{F}_0^{(2\boldsymbol{e}_1)}$ with the free energy of the same name in (4.91) in the context of the subleading contribution to perturbation theory. Thus,

$$\mu_0(2\boldsymbol{e}_1) = \frac{S_{1,1}}{\pi i}\widehat{F}_0^{(2\boldsymbol{e}_1)}. \tag{4.108}$$

Identifying $\mu_0(2\boldsymbol{e}_1)$ and $\mu_0(\boldsymbol{e}_{1,1})$ as free energies that can be computed from the holomorphic anomaly equations and that can have their ambiguities fixed efficiently, is the crucial step to then be able to check the more complicated higher order contributions to (4.97). In practice we find the following asymptotics for the one-instanton coefficients,

$$F_g^{(\boldsymbol{e}_1)} \sim \sum_{h=0}^{\infty} \left\{ \frac{\Gamma(g+1-h)}{(+A_1)^{g+1-h}} \frac{S_{1,1}}{\pi i} \widehat{F}_h^{(2\boldsymbol{e}_1)} + \frac{\Gamma(g+1-h)}{(-A_1)^{g+1-h}} \frac{1}{2} \frac{\widetilde{S}_{-1,1}}{\pi i} \widehat{F}_h^{(\boldsymbol{e}_{1,1})} \right\}. \tag{4.109}$$

The free energies $\widehat{F}_h^{(2\boldsymbol{e}_1)}$ have the general form

$$\widehat{F}_h^{(2\boldsymbol{e}_1)} = \mathrm{e}^{2\,\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}\,\mathrm{Pol}\left(S^{zz};3h\right) + \mathrm{e}^{4\,\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}\,\mathrm{Pol}\left(S^{zz};3h\right). \tag{4.110}$$

The exponents 2 and 4 are the ones dictated by the general combinatorial solution described in theorem 2.

The functions $\widehat{F}_h^{(\boldsymbol{e}_{1,1})}$ have a different structure due to resonance as we anticipated in section 3.5.2. Since it is a sector with total instanton action it has only even coefficients,

$$\widehat{F}_h^{(\boldsymbol{e}_{1,1})} = \mathrm{e}^{2\,\frac{1}{2}(\partial_z A_1)^2\left(S^{zz}-S^{zz}_{[1],\mathrm{hol}}\right)}\,\mathrm{Pol}\left(S^{zz};5\frac{h}{2}\right) + \mathrm{Pol}\left(S^{zz};3\frac{h}{2}-1\right), \quad h \text{ even}, \tag{4.111}$$

while $\widehat{F}_h^{(\boldsymbol{e}_{1,1})} = 0$ for h odd. See appendix C for more details. In order to check (4.109) we can do the exercise of summing the right-hand-side to low order and observe the agreement with the left-hand-side. More precisely, rewrite (4.109) as

$$\frac{A_1^{g+1}}{\Gamma(g+1)} \frac{S_{1,1}}{\pi i} F_g^{(\boldsymbol{e}_1)} \sim \sum_{h=0}^{h^*} \frac{\Gamma(g+1-h)}{\Gamma(g+1)} A_1^h \left\{ \frac{S_{1,1}}{\pi i} \frac{S_{1,1}}{\pi i} \widehat{F}_h^{(2\boldsymbol{e}_1)} + (-1)^{g+1-h} \frac{S_{1,1}}{\pi i} \frac{\widetilde{S}_{-1,1}}{2\pi i} \widehat{F}_h^{(\boldsymbol{e}_{1,1})} \right\}, \tag{4.112}$$

and compute the left-hand-side and right-hand-side for fixed $g = 75$, $\psi = 2$, and $S^{zz} = 10^{-8}$, and increasing values of h^*. We expect better and better agreement between the two, up until the optimal truncation point—the right-hand-side is an asymptotic series in $\frac{1}{g}$. We show the comparison in the following table,

h^*	RHS(h^*)	LHS − RHS(h^*)
0	0.112 257 517 800	$+8 \cdot 10^{-4}$
1	0.113 083 826 046	$-3 \cdot 10^{-5}$
2	0.113 054 511 927	$+3 \cdot 10^{-9}$
3	0.113 054 512 813	$+2 \cdot 10^{-9}$
4	0.113 054 514 589	$-7 \cdot 10^{-11}$
5	0.113 054 514 517	$+1 \cdot 10^{-12}$
LHS	0.113 054 514 518	

Note that the agreement is bettered the most for h^* even. That is because there is no contribution for $\widehat{F}_{\mathrm{odd}}^{(\boldsymbol{e}_{1,1})}$ due to resonance.

4.4 General discussion

In this chapter we have explored in detail some of the resurgence relations between free energies. Anticipating discrepancies with respect to the standard results out of a simple bridge equation, we decided to follow a route of discovery, based on a numerical analysis, rather than checking analytic predictions from chapter 3. The resurgent analysis of topological strings on local $\mathbb{CP}^2$ was approached with the example of the quartic matrix model in mind. There all the resurgence relations that were numerically checked could be derived from first principles out of a simple bridge equation like (1.31), or its generalization to a two-parameter resonant transseries. The validity of such a bridge equation was justified, although not rigorously proved, on the basis of the string equation for the free energy (technically for the coefficients r_n in the three-term recursion for orthogonal polynomials). Such a starting point is not available in local $\mathbb{CP}^2$ which should warn us about the possibility of deviations from the, let us call it, standard situation. Let us compare both cases to see where the differences are and what options we have to explain them.

We focus on the asymptotic growth of the perturbative and one-instanton sectors which are the ones we have studied in this thesis. In the standard situation, the bridge equation provides a way to compute all the alien derivatives for every sector in the transseries. Note that we must have a sufficiently general ansatz for the transseries if we want to accommodate for every type of large-order relation we find. In chapter 1 we reviewed the derivation of the large-order growth of the perturbative coefficients of a one-parameter transseries, ending with (1.62),

$$a_{2g}^{(0|0)} \sim \sum_{l=1} \frac{(S_1)^l}{\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(2g+b^{(0|0)}-b^{(l|0)}-h)}{(lA)^{2g+b^{(0|0)}-b^{(l|0)}-h}} a_h^{(l|0)}. \tag{4.113}$$

For multiparameter transseries with resonance in $\pm A_i$ the formula generalizes straightforwardly by summing over sectors. This formula provides a role for the higher instanton coefficients $a_h^{(\boldsymbol{e}_i)}$, because it says where to put each element of the transseries in the asymptotic formula. It even lays the starting powers $b^{(\boldsymbol{n})}$ in their appropriate place. Similarly, the large-order relation for the one-instanton sector has the form shown in (1.64) that we repeat here,

$$a_g^{(1|0)} \sim \frac{\Gamma(g+b^{(1|0)}-b^{(2|0)})}{A^{g+b^{(1|0)}-b^{(2|0)}}} \frac{S_1}{\pi i} a_0^{(2|0)} + \frac{\Gamma(g+b^{(1|0)}-b^{(1|1)})}{(-A)^{g+b^{(1|0)}-b^{(1|1)}}} \frac{1}{2} \frac{\widetilde{S}_{-1}}{\pi i} a_0^{(1|1)}. \tag{4.114}$$

Note again that the appearance of the starting powers. We have seen in this chapter that most of the structure of (4.113) and (4.114) is respected in the case of local $\mathbb{CP}^2$: the $2g$-factorial growth for $F_g^{(\boldsymbol{0})}$, the g-factorial growth for $F_g^{(\boldsymbol{e}_1)}$, the role of the instanton actions, and, in part, the higher instanton coefficients on the right-hand-sides of the relations. However, not everything matches. The most important, fundamental discrepancy is the role of the two-instanton sectors in both relations (4.113) and (4.114), or rather, in (4.86) and (4.109). In the standard case the two-instanton free enrgies in both asymptotic relations are the same, but in local $\mathbb{CP}^2$ they are different. More precisely, $\widetilde{F}_h^{(2\boldsymbol{e}_1)}$ in (4.86) is made out of two free energies, one of which is $\widehat{F}_h^{(2\boldsymbol{e}_1)}$ that appears in (4.109). Besides this discrepancy, the standard prediction for the starting powers $b^{(\boldsymbol{0})} = -2$, $b^{(\boldsymbol{e}_1)}$, $b^{(2\boldsymbol{e}_1)}$ does not agree with what is found. If the standard case were right we would need to match the following factorial growths,

$$\Gamma\left(2g + b^{(\mathbf{0})} - b^{(\boldsymbol{e}_1)}\right) = \Gamma(2g-1),, \tag{4.115}$$
$$\Gamma\left(2g + b^{(\mathbf{0})} - b^{(2\boldsymbol{e}_1)}\right) = \Gamma(2g-1), \tag{4.116}$$
$$\Gamma\left(g + b^{(\boldsymbol{e}_1)} - b^{(2\boldsymbol{e}_1)}\right) = \Gamma(g+1), \tag{4.117}$$

which does not have a consistent solution. The standard situation based on a simple bridge equation is not valid anymore. A new resurgent framework has to be found in which every detail discovered in the numerical analysis is naturally explained.

Let us go over the most natural and simple generalizations of the standard case. We are not certain whether any of them will be part of the final explanatory framework, but at least they can suggest new routes to explore.

First we devise a situation in which the compositeness of $\widetilde{F}_h^{(2\boldsymbol{e}_1)}$ can be accommodated, at least in part. For this we need to go back to the language of alien derivatives and the Stokes automorphism that we reviewed in chapter 1. In the standard case the following alien derivatives hold,

$$\Delta_{A_1} F^{(\mathbf{0})} = a\, F^{(\boldsymbol{e}_1)}, \quad \Delta_{2A_1} F^{(\mathbf{0})} = 0, \tag{4.118}$$
$$\Delta_{A_1} F^{(\boldsymbol{e}_1)} = c\, F^{(2\boldsymbol{e}_1)}, \tag{4.119}$$

where a and c are some constants. Here we have focused on the sector associated to A_1 only to simplify the discussion. To derive the large-order relations one has to use the Stokes automorphism (1.28) that in this schematic version reads

$$\underline{\mathfrak{S}} - \mathbf{1} = \mathrm{e}^{-A_1/g_s}\,\Delta_{A_1} + \mathrm{e}^{-2A_1/g_s}\left(\Delta_{2A_1} + \frac{1}{2}\Delta_{A_1}^2\right) + \cdots. \tag{4.120}$$

The first term in (4.120) generates the leading contribution to large-order involving the one-instanton sector $F^{(\boldsymbol{e}_1)}$. The second term generates subleading corrections in $(2A_1)^{2g}$, and contains two parts. The first is zero in the standard case and the second gives

$$\Delta_{A_1}^2 F^{(\mathbf{0})} = a\, c\, F^{(2\boldsymbol{e}_1)}. \tag{4.121}$$

Complementary to this, the asymptotics of $F^{(\boldsymbol{e}_1)}$ to leading order depends only on (4.119) (again, we are ignoring contributions from Δ_{-A_1}). To generalize the standard case we could allow $\Delta_{2A_1} F^{(\mathbf{0})}$ to be nonzero. And if we want to be as close to the local $\mathbb{CP}^2$ asymptotics we can postulate,

$$\Delta_{A_1} F^{(\mathbf{0})} = a\, F^{(\boldsymbol{e}_1)}, \quad \Delta_{2A_1} F^{(\mathbf{0})} = b\, F^{(2\boldsymbol{e}_1)}, \tag{4.122}$$
$$\Delta_{A_1} F^{(\boldsymbol{e}_1)} = c\, \widehat{F}^{(2\boldsymbol{e}_1)}, \tag{4.123}$$

where we have imposed $\widehat{F}^{(2\boldsymbol{e}_1)}$ on the last line so that now the second term in (1.28) gives

$$\mathrm{e}^{-2A_1/g_s}\left(b\, F^{(2\boldsymbol{e}_1)} + \frac{1}{2} a\, c\, \widehat{F}^{(2\boldsymbol{e}_1)}\right), \tag{4.124}$$

which may have a chance to produce $\widetilde{F}^{(2\boldsymbol{e}_1)}$ in (4.88). The constants a, b, c should be related to the Stokes constants as $a^2 = b = (S_{1,1})^2$ and $c = -2S_{1,1}$, but the minus sign propagates to the large-order of $F^{(\boldsymbol{e}_1)}$ and there should be none there, see (4.109). However, a bigger problem than this sign is the mismatch with the starting powers we have already mentioned. The inclusion of a right-hand-side to $\Delta_{2A_1} F^{(\boldsymbol{0})}$ does not cure this issue. A new ingredient must be included to make everything work.

A natural extension, but still unsuccessful, is to allow the asymptotic series to be resurgent functions but not simple ones. Simple resurgent functions satisfy (1.22) and (1.23). A mild extension that allows for higher order poles of the Borel transform suggests,

$$\Delta_{A_1} F^{(\boldsymbol{0})} = a\, g_s^{\alpha}\, F^{(\boldsymbol{e}_1)}, \quad \Delta_{2A_1} F^{(\boldsymbol{0})} = b\, g_s^{\beta}\, F^{(2\boldsymbol{e}_1)}, \tag{4.125}$$

$$\Delta_{A_1} F^{(\boldsymbol{e}_1)} = c\, g_s^{\gamma}\, \widehat{F}^{(2\boldsymbol{e}_1)}. \tag{4.126}$$

This would change the constraints (4.115)-(4.117) into

$$\Gamma\left(2g + b^{(\boldsymbol{0})} - b^{(\boldsymbol{e}_1)} - \alpha\right) = \Gamma(2g-1), \tag{4.127}$$

$$\Gamma\left(2g + b^{(\boldsymbol{0})} - b^{(2\boldsymbol{e}_1)} - \beta\right) = \Gamma(2g-1), \tag{4.128}$$

$$\Gamma\left(2g + b^{(\boldsymbol{0})} - \hat{b}^{(2\boldsymbol{e}_1)} - \alpha - \gamma\right) = \Gamma(2g-1), \tag{4.129}$$

$$\Gamma\left(g + b^{(\boldsymbol{1})} - \hat{b}^{(2\boldsymbol{e}_1)} - \gamma\right) = \Gamma(g+1), \tag{4.130}$$

but still this system has no solutions for the starting powers and α, β, γ. Even if this does not immediately solve the problem with the starting powers it should be kept in mind as a possible ingredient of the final framework that explains the resurgent structure.

Another possibility that should be considered is the presence of sectors which are more complicated than $(2\boldsymbol{e}_1)$ but have the same total instanton action. These sectors cannot be discarded from first principles because their contribution does not violate any resurgence principle. Also one can see that their propagator dependence is compatible with the results we have obtained from large-order. We can consider two examples. For the first,

$$\boldsymbol{n} = (1|0\|0|1\|0|1\|0\cdots\|0|1), \tag{4.131}$$

where the last two entries entries correspond to sectors associated to the instanton actions $\pm 4\pi^2 i$. This sector has total instanton action $2A_1$. The calculation of $F_0^{(\boldsymbol{n})}$ from the holomorphic anomaly equations is a little bit lengthy because the recursion goes through several intermediate sectors. A difficulty we encounter is that we do not know how to fix the holomorphic ambiguities of all those sectors, so they are carried along during the calculation. The final answer for the free energy consists of five exponentials with different exponents. Two of them are the same as those appearing in $F_0^{(2\boldsymbol{e}_1)}$ and $\widetilde{F}_0^{(2\boldsymbol{e}_1)}$, that is $\exp 4\omega$ and $\exp 2\omega$ with $\omega = \frac{1}{2}(\partial_z A_1)^2 S^{zz}$. The other three have coefficients that depend on the holomorphic ambiguities just mentioned. It is not difficult to imagine that the three coefficients are zero due to cancellations between ambiguities, and one ends with a valid candidate for an element of the perturbative asymptotics. Another example is $\boldsymbol{n} = (2|0\|0|1\|0|0\|0\cdots\|1|1)$ for which $F_0^{(\boldsymbol{n})}$ has only one extra exponential.

It is clear that the standard situation is not the correct framework to explain all the details of the resurgence relations the local $\mathbb{CP}^2$ free energies satisfy. Moreover, it is not clear what is the role in the transseries of some of the two-instanton sectors that we find in the large-order growth, or if those should actually have the interpretation of more higher instanton free energies like (4.131). At the end of the day there must be a consistent and comprehensive picture that explains both the transseries in all its resonant complexity, and the web of large-order relations. In the best case scenario there is a generalization of the simple bridge equation (1.31) from which the relations can be derived. Understanding the transseries and understanding the resurgent structure are two different, but of course related, problems. In the case of local $\mathbb{CP}^2$ both problems have to be considered at the same time so that we can use information gained from one in the other and back. This is the strategy that we followed in sections 4.2.2 and 4.3.

Besides the possible generalizations that we discussed above, there are two features of resurgence in local $\mathbb{CP}^2$ that will most likely play an important role in understanding the overall structure. These are resonance and the multisheeted structure of the Borel plane.

Having a look a the different instanton actions that we found in this chapter, A_1, A_2, A_3, $4\pi^2 i$, $A_{\rm K}$, along with their opposites, and the various relations they satisfy, we can expect a large amount of resonant behavior in the large-order relations. First of all one needs to devise a method to distinguish which instanton actions have transseries sectors associated to them from those that are simply combinations of more fundamental instanton actions. That is, we need to specify the number of the parameters of the transseries and the amount of resonance. At this point the most direct strategy requires a careful combination of numerical large-order analysis and solutions of the holomorphic anomaly equations, along with the constraint of self-consistency. One problem is that fixing the holomorphic ambiguity and decoding the large-order relations require sometimes some guessing that may be successful or not. In the sectors that we have analyzed in this chapter we were able to provide analytic expressions for every number out of the computer, so one may be able to go quite deep in the resurgent structure this way. From an overall perspective the requirement of self-consistency applied to as many resurgent relations as possible may provide new information to help put the pieces of the puzzle in place.

The multisheeted structure of the Borel plane and its relation to higher order Stokes phenomenon will need more study in the future. For now we can say that it will very likely play an important part in understanding the structure of the large-order relations, especially as we move around in moduli space. We have already seen how varying the propagator and tuning it its holomorphic values can have drastic effects in the resurgent structure of the theory. However, the dependence on ψ, or z, seems to be more subtle and interesting.

There is another source of information to understand the resurgence in local $\mathbb{CP}^2$ that we have not considered, and that is resummation. The physical resummation of the transseries must provide the full nonperturbative value of the topological string free energy. In the case of local $\mathbb{CP}^2$ we have nothing to compare this result to, expect for other nonperturbative proposals for the same model. However, for other geometries for which there is a dual nonperturbative definition, such comparison should be possible. The physical resummation considers sectors that are exponentially suppressed as g_s goes to zero, so this is a criterion for their presence in the transseries. At the practical level, however, it is difficult to think that we could go past the one or two-instanton sectors and still be sensitive to their contribution.

To end on a more optimistic note, there are still several avenues that have not been explored yet including contributions beyond subleading to the perturbative sector, subleading corrections to the one-instanton free energy large-order growth, exploration of moduli space further from the conifold points. Also, the content and results of this thesis need to be put on more rigorous grounds, both from the point of view of topological string theory and complex geometry, and of resurgence theory. An analysis from the perspective and language of modular forms should be pursued, along with the inclusion of all the propagators. The same techniques we developed here must be applied to fairly well-understood examples like polynomial matrix models or ABJM. Progress in any of these areas will provide useful insight for the other and for the general problem of defining topological string theory nonperturbatively.

Chapter 5

Summary and conclusions

The main theme of this thesis is the study of nonperturbative aspects of topological string theory. The tools we use are the theory of resurgence and the holomorphic anomaly equations that the topological string free energies satisfy.

The theory of resurgence, developed by Écalle, has become in recent years the natural framework in which to discuss and analyze nonperturbative computations. It is most powerful when applied to solutions of differential equations, but it has found room in problems in quantum field theory and string theory. The best example should be matrix models, a theory a random matrices that finds connections with all of the examples above and provides intuition to approach new problems. One of them is topological string theory due to its large N duality with matrix models. Topological string theory is defined perturbatively in the string coupling constant, but as it happens in numerous physical systems, this series expansion is asymptotic and does not converge. The origin of this divergence is the existence of other nonperturbative sectors. Resurgence theory provides a way to collect this nonperturbative results into a single object called the transseries which, after a process of resummation, will yield the full nonperturbative definition of the theory. This long term goal starts with the computation and understanding of the new nonperturbative sectors and their connection to perturbation theory. This connection has a quantitative description in resurgence theory. This thesis focuses on the computation of the transseries, exploiting the holomorphic anomaly of topological string theory and studies analytically, and numerically the resurgence relations linking its components.

In **chapter 1** we present a short introduction to the theory of resurgence, focusing on the main ingredients necessary for the derivation of the large-order growth of the coefficients in the transseries.

The language of resurgence and resurgent transseries is the natural one when describing the solutions to differential equations. It is often found here how a complete family of solutions involves not only powers series in the equation variable, x, but also nonanalytic functions like $\mathrm{e}^{-A/x}$, where A is a number called the instanton action. The existence of a singularity at the origin is represented in the asymptotic nature of the power series solution and the factorial growth of their coefficients. This behavior is actually ubiquitous not only in mathematics but also in physics. The formal solutions to a problem involving the monomials

x and $e^{-A/x}$, are called transseries. In order to move this solution from the space of formal series to the space of functions a resummation procedure is needed. Borel resummation is a natural option because it deals with the divergent series by first removing the factorial growth and working with the so-called Borel transform. Resummation may not always be well-defined due to the existence of singularities in the Borel plane where the Borel transform lives. This leads to the so-called nonperturbative ambiguity which is cured when all the sectors of the transseries are taken into account. Actually, these singularities know about the factorial growth because they carry information about other sectors. This *resurgence* of the function in different sectors is formalized with the notion of alien derivative. This operator captures precisely the singularities in the Borel plane and thus, the relation between different sectors of the transseries. Going one step further leads to the concept of Stokes automorphism, an operator that can be written in terms of the alien derivative and codifies the difference between resummations for different values of x. The Stokes automorphism has an important practical purpose. It is the key ingredient that allows to write the various formulae describing exactly how the coefficients of a given sector of the transseries grow for large order. The precise formula involves very explicitly many of all the other sectors of the transseries. The first example of such a large-order relation expresses the factorial growth of the perturbative coefficients in the solution of a given problem, in terms of nonperturbative coefficients. For the construction of the large-order relations it is necessary to know the action of the alien derivatives on the elements of the transseries. This difficult problem is sometimes overcome with a bridge equation that links alien and usual derivatives. However, the form of the bridge equation is not always known and the details of the resurgence relations must be discovered by studying the large-order of known sectors. This can be the case in topological string theory. Nevertheless, the existence of large-order relations coming from resurgence and linking all sectors of the transseries stands as a very useful tool to extract nonperturbative information when only perturbation theory is available.

In **chapter 2** we review the main aspects of topological string theory and present the example of local $\mathbb{CP}^2$. Topological string theory is defined as a conformal topological field theory coupled to two-dimensional gravity. Like physical string theory it is concerned with the dynamics of maps from a Riemann surface, the worldsheet the string describes as it moves, to a target space. In topological string theory this target space is a complex three-dimensional Calabi–Yau manifold. This manifold appears in the compactification of physical string theory and the topological string free energies define the couplings of the effective supergravity theory.

There are two types of topological string theory, A and B, depending on the type of underlying topological theory. Both types are related to each other by mirror symmetry in the sense that a type A topological string theory on a Calabi–Yau manifold is equivalent to a B-type theory on a mirror Calabi–Yau. Type A and B theories depend on the Kähler and complex structure of the Calabi–Yau manifold, respectively. This dependence is inherited from the underlying topological supersymmetric sigma model and, at that level, still holomorphic. However, the coupling to worldsheet gravity produces a holomorphic anomaly that makes the observables of the theory depend nonholomorphically on the appropriate moduli space. These observables are, among others, the free energies. Their definition depends on the genus of the Riemann surface and they can be put together in a generating function with parameter equal to the string coupling constant, g_s. This defines the perturbative topological string free energy. The free energies of mirror models are equal but the calculation is easier for the B-type working with complex structure dependence. The main and most efficient technique to compute B-type free energies on noncompact Calabi–Yau manifolds is the holomorphic anomaly equations. These equations quantitatively describe how a free energy of genus g fails to be holomorphic. A detailed computation shows that it depends on lower genus free energies. This allows for a recursive integration of the the free energies to high order once the holomorphic ambiguities, linked to the integration process, are conveniently fixed. The procedure is made easier using a carefully chosen antiholomorphic variable called the propagator. The dependence of the free energies on this variable is polynomial. The final part of this chapter is devoted to the computation of a particular example: the mirror of local $\mathbb{CP}^2$. This is a classical example, simple enough to allow for the computation of over a hundred perturbative free energies. Along with the free energies we review the computation of the periods. These are integrals on the Calabi–Yau geometry, they depend on the complex structure moduli, and they have a fundamental role as instanton actions controlling the large-order growth of the perturbative free energies. For this geometry the complex structure moduli space is one-dimensional and has points of special relevance: a large-radius point, and three copies of the conifold point, related by a $\mathbb{Z}_3$ orbifold symmetry. The conifold points are singular in the sense that perturbative free energies become infinite there. This behavior imposes constraints on the large-order of the perturbative free energies that are found useful in the nonperturbative context.

In **chapter 3** we develop the construction of transseries expressions for the nonperturbative topological string free energy based on a natural extension of the holomorphic anomaly equations that govern perturbation theory.

The tower of holomorphic anomaly equations can be packaged into a single differential equation with respect to the complex structure moduli and the propagators, in which the string coupling constant, g_s, is explicitly present. This equation is solved by the perturbative free energy asymptotic series. Such master equation was consider before in the literature and further rewriting prepares it to accept not only a perturbative series in the string coupling but a full transseries. The transseries ansatz solving this equation may have several parameters, that is, several instanton actions sitting in the nonanalytic exponential monomials, e^{-A/g_s}. The holomorphic anomaly equations are not differential equations on g_s but on the moduli and this reduces the computation power of the equations at the expense of being extremely general. This means that this lack of determination must be replaced by resurgent constraining in the form of large-order relations.

The simplest situation is the one involving a one-parameter transseries, although many of the properties of the solution generalize to multiparameter ones. The transseries is composed by a perturbative part and a series of multi-instanton nonperturbative contributions. The free energies in all these sectors are found to satisfy an extension of the holomorphic anomaly equations. The structure is still recursive: the antiholomorphic derivative of a nonperturbative free energy, at instanton level n and order g, depends on lower order free energies of the same and lower instanton levels. The presence of the instanton action is manifest and a subset of equations addresses it directly. These equations are all equivalent and imply that the instanton action is holomorphic. This conclusion is important because it retains the natural interpretation from matrix models of instanton actions as geometrical objects. In fact, instanton actions are found to be combinations of periods that can be computed from Picard–Fuchs equations.

The integration of the equations is analogous to the perturbative case. A careful analysis of the antiholomorphic (propagator) dependence shows that the polynomial structure associated to perturbative free energies is here generalized to linear combinations of products of exponentials and polynomials. The degree of these polynomials and the particular coefficients appearing in the exponentials can be characterized precisely in terms of a generating function. A detailed proof for this is provided based on induction on the instanton sector and the order. Generalizations of these results to multiparameter transseries, in which several instanton actions appear, is presented next. The latter are found to be holomorphic even in this generalized situation. The structure of the higher instanton free energies is analyzed in the same manner as before and a generalized generating function is provided to describe their antiholomorphic dependence.

The phenomenon of resonance, in which two sectors of the transseries have the same total instanton action, is considered by introducing logarithmic blocks in the transseries. It was found in matrix models and string related differential equations that the transseries must include an instanton action and its negative and also logarithms in the string coupling. The holomorphic anomaly equations admit such solutions, even if logarithms are replaced by other nonanalytic monomials. After describing the structure of the solutions, we focus on the diagonal sectors of the transseries, those with vanishing total instanton action. In the models mentioned above such sectors presented, due to resonance, a topological expansion in g_s, that is, they are power series in g_s^2 We show that this behavior is possible in the context of topological string theory. The proof relies on the fact that a change in sign on the instanton action can be balanced by another change in the coupling, g_s. This is what occurs at the perturbative level, where the topological expansion is recovered, at the level of large-order relations, by having both A and $-A$ sectors in the transseries.

The large-order growth of the perturbative sector can be studied at the analytical level and exploited to obtain nonperturbative results in closed form. This general approach to extract nonperturbative information out of large-order is difficult in general (without relying on numerical methods) but in some situations it can be done. An example is an independent proof of the holomorphicity of the instanton action based only on large-order growth of perturbation theory and the holomorphic anomaly equations these free energies satisfy.

The final aspect we cover in this chapter is the important problem of determining the holomorphic ambiguity associated to nonperturbative sectors. At the perturbative level, the ambiguities are fixed by looking at the behavior of the free energies at the conifold and large-radius points, and comparing against independently known behavior. This is not a possibility at the nonperturbative level, so we have to turn to the only tool we have, resurgent large-order analysis. By exploiting the singular behavior of the perturbative free energies at the conifold point we can determine, analytically, the precise large order growth of the free energies close to this point. Since this calculation is actually done in the holomorphic limit, and large-order relations involve other sectors of the transseries, this must provide information to fix the holomorphic ambiguities associated to conifold sectors of the transseries. This is checked for the case of local $\mathbb{CP}^2$ in the following chapter.

In **chapter 4** we work out the details of the resurgent and large-order properties of local $\mathbb{CP}^2$. The B-type topological string theory on the mirror of this geometry depends on a one-dimensional complex structure moduli space. The antiholomorphic dependence is captured by a single propagator. For a particular value of the propagator the holomorphic limit of the free energies is recovered. This holomorphic limit is not unique, but it is actually dependent on the frame, a label related to modular symmetry. Special points in moduli space have preferred frames associated to them.

We start the resurgent analysis by studying the large-order growth of the perturbative free energies. The first important element we focus on is the dominant instanton action. From the previous chapter we know this is a holomorphic, propagator independent, quantity. The label dominant refers to the smallest, in absolute value, of all the instanton actions in the transseries. As we explore moduli space the dominant instanton action changes. A numerical analysis shows strong evidence of holomorphicity, and we find two distinct dominant instanton actions. One is associated to a conifold point. The other is constant and coming from a universal constant contribution to the free energies. Removing it uncovers another instanton action dominating near the large-radius point. Both conifold and large-radius instanton actions are proportional to the corresponding flat coordinates around these points in moduli space. They are periods of the geometry. They are not the only instanton actions in the transseries. Two other instanton actions associated to the second and third conifold points exist. They can be first detected in a section of moduli space where their absolute value is equal to the first instanton action. The behavior of these free energies is oscillatory there due to the combination of two conjugate contributions coming from the first and second (or third) conifold instanton actions. All these four, three conifold and one large-radius, instanton actions and their negatives indicate a very complicated transseries with a large amount of potential resonance. From an analysis of the Borel plane of the perturbative sector we show how the pole associated to the large-radius instanton action disappears, changing to another Riemann sheet, as we move in moduli space towards the orbifold point where this instanton action should dominate. This is evidence that the Borel plane and the resurgence structure is more intricate than what was found in other examples in the past.

We turn next to the study of the higher instanton sectors appearing in the large-order relations. The first is the one-instanton sector associated to the first conifold point. It controls the leading factorial growth of the perturbative free energies. In the corresponding holomorphic limit the large-order series truncates, in agreement with the general analysis done in chapter 3. This allows for the fixing of the holomorphic ambiguity of the one-instanton free energies computed from the extended holomorphic anomaly equations as described in that chapter. We make numerical tests on both holomorphic and antiholomorphic dependence of this sector, using accelerating techniques for convergence like the Richardson transform, and find excellent agreement with the theoretical results. The same exercise gives evidence that the Stokes constants cannot depend on the moduli, so they really are complex numbers. Since the large-order relations are used to fix the holomorphic ambiguities, there is no remaining information to compute the Stokes constants. The oscillatory behavior of the free energies signaling the presence of other conifold instanton actions can be reproduced by a large-order formula involving both one-instanton sectors. Amplitude and frequency of the oscillations are matched by the theoretical predictions.

A study of the other one-instanton sectors associated to conifold points can be done by resumming the leading contribution to perturbation growth. The resummation technique we need to use has to be more or less powerful depending on the particular subleading sector. Depending on the point in moduli space we also find a subleading two-instanton contribution associated to the first conifold point. We perform numerical checks both on the holomorphic limit (where resummation is actually not needed due to truncation) and for general values of the propagator. All results are matched by theoretical predictions involving the appropriate free energies computed from the holomorphic anomaly equations.

The one-instanton free energies can be computed from the equations to high order with a seminumerical integration process. This procedure fixes a complex structure while leaving the propagator dependence free. An analytical computation in both variables becomes impractical after around twenty free energies, which makes subsequent studies impossible, or very deficient. The large-order growth of this sector reveals clearly the presence of resonance between sectors with instanton actions of opposite signs. A two-instanton and a mixed sector, involving A and $-A$, control the growth. The corresponding free energies can be computed from the holomorphic anomaly equations. Their ambiguities are fixed by noting that the one-instanton free energies are zero in the holomorphic limit (for high order), so the same must be the case with the former free energies. High precision numerical checks support these conclusions.

At this point it becomes apparent that the pair of two-instanton free energies found in the large-order growth of the one-instanton sector and of perturbation theory, respectively, are not the same. That they were would have been the expected result if the usual bridge equations were valid but the resurgent structure is slightly different here. We present different possibilities that could have a role here, from the consideration of non-simple resurgent functions to the existence of similar sectors with different conditions to fix the holomorphic ambiguity.

The conclusions of this thesis are the following:

- The holomorphic anomaly equations can be extended beyond perturbation theory and can be applied to a transseries ansatz.
- The generality of the holomorphic anomaly equations leaves undetermined a number of ambiguities: holomorphic ambiguities at all instanton orders, concrete structure of the transseries, number of parameters, nonstandard monomials in the coupling constant, instanton actions and starting powers. They all must be calculated from the constraints that resurgence imposes.
- The structure of the solutions generalizes that of the perturbative sector and can be described using combinatorial numbers out of a generating function. The detailed holomorphic dependence requires more effort and no general patterns can be found.

- The instanton actions are holomorphic. This is proved with the extended holomorphic anomaly equations and directly from a large-order argument involving only perturbative information.

- Resonant transseries include sectors with topological expansions.

- The singular and universal behavior of the perturbative free energies around the conifold point determines their large-order growth near that special point. This analysis becomes the substitute for the holomorphic ambiguity fixing condition in the nonperturbative sectors associated to the conifold points. This approach should be explored further on other special points in moduli space and other types of singularities.

- The transseries describing the topological string free energy of the mirror of local $\mathbb{CP}^2$ involves a number of instanton actions, coming in both signs to respect the topological expansion of the perturbative sector. There are three conifold instanton actions, related by $\mathbb{Z}_3$-symmetry, and another instanton action associated to the large-radius point. They are periods of the geometry, proportional to flat coordinates around the corresponding points.

- The large-order growth of the nonholomorphic perturbative free energies can be understood, around the conifold point, using higher-instanton free energies computed out of the extended holomorphic anomaly equations. The two-instanton sector is a combination of free energies whose holomorphic ambiguities are fixed differently. Different sectors of the transseries compete for the subleading contribution depending on the region in moduli space.

- One-instanton free energies must be computed seminumerically in the modulus and propagator to reach a sufficiently high number of them. The large-order growth exhibits explicit resonance and a mixed instanton sector with a topological expansion.

- Every numerical quantity obtained from large-order computations can be reproduced from an analytic function derived from the extended holomorphic anomaly equations. This is done for several resurgence relations on and off the holomorphic limit.

- The resurgent relations found do not completely agree with the naive predictions based on a simple bridge equation and inspired by related models. Different scenarios for a resurgent framework are suggested including dropping the requirement of simple resurgence, or taking into account the influence of resonance between conifold and large-radius sectors. No final coherent solution is obtained. Further study in local $\mathbb{CP}^2$ and other geometries is needed in order to understand not only the underlying resurgence structure but also the general problem of fixing the holomorphic ambiguity, and eventually the resummation of the transseries into a nonperturbative function.

Chapter 6

Resumo e conclusións

O tema principal desta tese é o estudo de aspectos non perturbativos da teoría de cordas topolóxica. As ferramentas que utilizamos son a teoría da resurxencia e as ecuacións de anomalía holomorfa que satisfán as enerxías libres de cordas topolóxicas.

A teoría da resurxencia, desenvolvida por Écalle, tense convertido nos últimos anos no marco natural onde discutir e analizar cálculos non perturbativos. Acada a súa maior efectividade cando se aplica á solucións de ecuacións diferenciais pero tamén atopa o seu espazo en problemas de teoría cuántica de campos de teoría de cordas. O mellor exemplo é o de modelos de matrices, unha teoría de matrices aleatorias que ten conexións con todos os examplos mencionados é proporciona intuición para atacar novos problemas. Un deles é a teoría de cordas topolóxica debido á súa dualidade de gran N con modelos de matrices. A teoría de cordas topolóxica está definida perturbativamente na constante de acoplamento de cordas, pero tal e como ocorre en numerosos sistemas físicos, esta expansión en serie é asintótica e non converxe. A orixe desta diverxencia atópase na existencia doutros sectores non perturbativos. A teoría da resurxencia ofrece unha forma de recoller estes resultados non perturbativos nun único obxecto chamado transerie que, despois dun proceso de resumación, fornecerá a definición completa e non perturbativa da teoría. Este obxectivo a longo prazo comeza co cálculo e comprensión dos novos sectores non perturbativos e a súa conexión coa teoría perturbativa. Esta conexión ten unha descrición cuantitativa na teoría da resurxencia. Esta tese céntrase no cálculo da transerie, tirando proveito da anomalía holomorfa da teoría de cordas topolóxica, e estuda analítica e numericamente as relacións de resurxencia que unen as súas compoñentes.

No **capítulo 1** preséntase unha pequena introdución á teoría da resurxencia, centrándose nos ingredientes necesarios para a derivación do crecemento de alta orde dos coeficientes da transerie.

A linguaxe da resurxencia e das transeries resurxentes é a máis natural cando se queren describir solucións de ecuacións diferenciais. Encóntrase a miúdo que unha familia completa de solucións involucra non só unha serie de potencias na variable da ecuación, x, senón tamén funcións non analíticas como $\mathrm{e}^{-A/x}$, onde A é un número nomeado como a acción de instantón. A existencia dunha singularidade na orixe ven representada pola natureza asintótica da solución en serie de potencias e no crecemento factorial dos seus coeficientes.

Este comportamento é de feito ubicuo non só en matemáticas senón tamén en física. As solucións formais a un problema que están compostas de monomios x e $\mathrm{e}^{-A/x}$ denomínanse transeries. Para transferir esta solución do espazo das series formais ao espazo de funcións necesítase un proceso de resumación. A resumación de Borel é unha opción natural porque no seu tratamento de series diverxentes elimina primeiro o crecemento factorial deixando o que se coñece como transformada de Borel. A resumación non ten sempre por que estar ben definida debido a existencia de singularidades no plano de Borel onde vive a transformada de Borel. Isto leva á chamada ambigüidade non perturbativa que se cura cando todos os sectores da transerie son tidos en conta. De feito estas singularidades coñecen o crecemento factorial porque levan información sobre outros sectores. Esta *resurxencia* da función nos distintos sectores ven formalizada pola noción de derivada estraña. Este operador captura con precisión as singularidades no plano de Borel e desta maneira a relación entre os distintos sectores da transerie. Indo un paso máis aló chégase ao concepto de automorfismo de Stokes, un operador que pode ser escrito en termos da derivada estraña e codifica a diferenza entre resumacións para diferentes valores de x. O automorfismo de Stokes ten unha aplicación práctica importante. É a peza fundamental que permite escribir as distintas fórmulas que describen con exactitude como crecen os coeficientes dun sector da transerie para alta orde. A fórmula precisa involucra explicitamente moitos dos outros sectores da transerie. O primeiro exemplo destas relacións de alta orde expresa o crecemento factorial dos coeficientes perturbativos na solución dun problema dado en termos de coeficientes non perturbativos. Para a construción de relacións de alta orde é necesario coñecer a acción das derivadas estrañas nos elementos da transerie. Este problema é difícil e ás veces pódese superar cunha ecuación ponte que relacione derivadas estrañas e usuais. Con todo, a forma da ecuación ponte non se coñece sempre e os detalles das relacións de resurxencia deben descubrirse estudando o comportamento a alta orde de sectores coñecidos. Este pode ser o caso na teoría de cordas topolóxica. De calquer modo, a existencia de relacións de alta orde provenientes da resurxencia e que ligan todos os sectores da transerie son unha ferramenta moi útil para a extracción de información non perturbativa cando soamente se está a disposición da parte perturbativa.

No **capítulo 2** repasamos os principais aspectos da teoría de cordas topolóxica e presentamos o exemplo de $\mathbb{CP}^2$ local. A teoría de cordas topolóxica defínese como unha teoría de campos conforme e topolóxica, acoplada á gravidade en dúas dimensións. Como a teoría de cordas física preocúpase da dinámica de funcións dunha superficie de Riemann, a superficie de universo que describe a corda ao moverse, a un espazo final. En teoría de cordas topolóxica este espazo final é unha variedade de Calabi–Yau de tres dimensións complexas. Esta variedade aparece na compactificación da teoría de cordas físicas e as enerxías libres de cordas topolóxicas definen os acoplamentos da teoría efectiva de supergravidade.

Existen dous tipos de teoría de cordas topolóxica, A e B, dependendo do tipo de teoría topolóxica subxacente. Os dous tipos están relacionados entre si pola simetría espello no senso de que unha teoría de cordas topolóxica de tipo A nunha variedade de Calabi–Yau é equivalente a unha teoría de tipo B nun Calabi–Yau especular. As teorías de tipo A e B dependen da estrutura de Kähler e da estrutura complexa da variedade de Calabi–Yau, respectivamente. Esta dependencia hérdase do modelo sigma supersimétrico e topolóxico subxacente e, a ese nivel, é aínda holomorfa. Non obstante, o acoplamento coa gravidade na superficie de universo produce unha anomalía holomorfa que fai que os observables da teoría dependan de forma non holomorfa do correspondente espazo de módulos. Estes observables son, entre outros, as enerxías libres. A súa definición depende do xénero da superficie de Riemann e pódense xuntar todas elas nunha función xeratriz con parámetro igual a constante de acoplamento de cordas, g_s. Isto define a enerxía libre perturbativa da teoría de cordas topolóxica. As enerxías libres de modelos especulares son iguais pero o cálculo e máis sinxelo para o tipo B onde se traballa coa dependencia da estrutura complexa. A técnica principal e máis eficiente para calcular enerxías libres de tipo B en variedades de Calabi–Yau non compactas son as ecuacións de anomalía holomorfa. Estas ecuacións describen cuantitativamente como unha enerxía libre de xénero g non consegue ser holomorfa. O cálculo detallado revela que depende das enerxías libres de menor xénero. Isto permite integrar recursivamente as enerxías libres até alta orde unha vez que as ambigüidades holomorfas ligadas ao proceso de integración sexan convenientemente fixadas. Pódese facilitar este procedemento usando unha variable antiholomorfa escollida axeitadamente e denominada propagador. A dependencia das enerxías libres nesta variable é polinómica. A parte final deste capítulo está dedicado ao cálculo dun exemplo particular: a xeometría especular de $\mathbb{CP}^2$ local. Este é un exemplo clásico, suficientemente simple como para poder calcular algo máis dun cento de enerxías libres perturbativas. Xunto coas enerxías libres tamén revisamos o cálculo dos períodos. Estes son integrais na xeometría de Calabi–Yau, dependen dos módulos de estrutura complexa, e teñen un papel fundamental como accións de instantón que controlan o crecemento de alta orde das enerxías libres perturbativas. Para esta xeometría o espazo de módulos de estrutura complexa é unidimensional e ten puntos de especial relevancia: un punto de gran radio, e tres copias do punto de *conifold*, relacionadas por unha simetría de *orbifold* $\mathbb{Z}_3$. Os puntos de conifold son singulares no sentido de que as enerxías libres perturbativas toman valor infinito neles. Este comportamento impón restricións no comportamento de alta orde das enerxías libres perturbativas que son de utilidade no contexto non perturbativo.

No **chapter 3** desenvolvemos a construción da transerie para a enerxía libre non perturbativa da teoría de cordas topolóxica baseada nunha extensión natural das ecuacións de anomalía holomorfa que gobernan a teoría de perturbacións.

A torre de ecuacións de anomalía holomorfa pódese compactar nunha soa ecuación diferencial con respecto aos módulos de estrutura complexa e aos propagadores, e na que a estrutura de acoplamento de cordas, g_s, está presente explicitamente. A serie asintótica que describe a enerxía libre perturbativa resolve esta ecuación. Tal ecuación mestra xa foi considerada na literatura e reescribíndoa un pouco máis déixaa preparada para recibir non só unha serie perturbativa senón tamén unha transerie completa. A proposta de transerie que resolve a ecuación pode ter varios parámetros, é dicir, varias accións de instantón que aparecen nos monomios non analíticos de tipo exponencial, e^{-A/g_s}. As ecuacións de anomalía holomorfa non son ecuacións diferenciais en g_s senón nos módulos e isto reduce o poder de cálculo das ecuacións a expensas de ser extremadamente xerais. Isto significa que a falta de determinación ten que ser substituída por restricións de tipo resurxente na forma de relacións de orde alta.

A situación máis sinxela é a que involucra unha transerie dun parámetro, aínda que moitas das propiedades desta solución xeneralízanse para o caso multiparamétrico. A transerie está composta por unha parte perturbativa e por unha serie de contribucións non perturbativas de multiinstantón. As enerxías libres de todos estes sectores cumpren unha extensión das ecuacións de anomalía holomorfa. A estrutura é aínda recursiva: a derivada antiholomorfa da enerxía libre non perturbativa, a nivel de instantón n e orde g, depende de enerxías libres de orde menor e de nivel de instantón igual ou menor. A presencia da acción de instantón é manifesta e un subconxunto de ecuacións alude a ela explicitamente. Todas estas ecuacións son equivalentes entre si e implican que a acción de instantón é holomorfa. Esta conclusión é importante porque permite reter a interpretación natural de modelos de matrices de accións de instantón como obxectos xeométricos. De feito, encóntrase que as accións de instantón son combinacións de períodos que se poden calcular como solucións das ecuacións de Picard–Fuchs.

A integración das ecuacións é análoga ao caso perturbativo. Unha análise coidadosa da dependencia antiholomorfa no propagador amosa que a estrutura polinómica propia das enerxías libres perturbativas ten aquí unha xeneralización a combinacións lineais de produtos de exponenciais e polinomios. O grao destes polinomios e os coeficientes particulares que aparecen nas exponenciais pódense caracterizar de forma precisa en termos dunha función xeratriz. Inclúese unha proba detallada baseada en indución sobre o sector de instantón e a orde. De seguido se presentan xeneralizacións destes resultados a transeries con múltiples parámetros onde aparecen varias accións de instantón. Tamén estas son holomorfas incluso neste caso máis xeral. Analízase a estrutura de enerxías libres de maior orde de instantón da mesma forma que antes e xeneralízase a función xeratriz que describe a dependencia antiholomorfa.

Considérase o fenómeno de resonancia, no que dous sectores da transerie teñen a mesma acción de instantón total, introducindo bloques logarítmicos na transerie. Atopouse en modelos de matrices e en ecuacións diferencias asociadas á teoría de cordas que as transeries deben incluír unha acción de instantón e a súa oposta ademais de logaritmos na constante de acoplamento de cordas. As ecuacións de anomalía holomorfa admiten estas solucións, incluso cando os logaritmos son substituídos por outros monomios non analíticos. Despois de describir a estrutura das solucións, centrámonos nos sectores diagonais da transerie, aqueles cunha acción de instantón total igual a cero. Nos modelos mencionados antes estes sectores presentan, debido á resonancia, unha expansión topolóxica en g_s, é dicir, teñen unha expansión en potencias de g_s^2. Amosamos que este comportamento é posible no contexto da teoría de cordas topolóxica. A proba baséase no feito de un cambio de signo na acción de instantón pódese compensar por outro cambio na constante de acoplamento, g_s. Isto é o que acontece a nivel perturbativo, onde a expansión topolóxica pódese recuperar, a nivel das relacións de alta orde, tendo tanto sectores con A e $-A$ na transerie.

O crecemento a orde alta do sector perturbativo pódese estudar a nivel analítico e tirar partido diso para obter resultados non perturbativos de forma exacta. Este enfoque xeral para extraer información non perturbativa das relacións de alta orde é complicado en xeral (sen basearse en métodos numéricos) mais en certas situacións pódese levar a cabo. Un exemplo é unha proba independente da holomorfía da acción de instantón baseada soamente no crecemento a orde alta da teoría perturbativa e nas ecuacións de anomalía holomorfa que estas enerxías libres satisfán.

O aspecto final que tratamos neste capítulo é o importante problema de determinar a ambigüidade holomorfa asociada cos sectores non perturbativos. A nivel perturbativo, as ambigüidades fíxanse mirando o comportamento das enerxías libres nos puntos de conifold e de gran radio, e comparando co comportamento destas obtido de maneira independente. Isto non é posible a nivel non perturbativo, así que temos que volvernos cara á única ferramenta dispoñible, a análise resurxente de alta orde. Aproveitando a dependencia singular das enerxías libres perturbativas no punto de conifold podemos determinar analiticamente o crecemento preciso a alta orde das enerxías libres preto deste punto. Xa que este cálculo realízase no límite holomorfo, e as relacións de alta orde involucran outros sectores da transerie, isto debe proporcionar información coa que fixar as ambigüidades holomorfas asociadas os sectores de conifold da transerie. Isto é o que se comproba para o caso de $\mathbb{CP}^2$ local no capítulo seguinte.

No **capítulo 4** explicamos con detalle as propiedades de resurxencia a alta orde de $\mathbb{CP}^2$ local. A teoría de cordas topolóxica de tipo B no espello desta xeometría depende dun espazo de módulos de estrutura complexa unidimensional. A dependencia antiholomorfa é capturada por un único propagador. Para un valor particular do propagador recupérase o límite holomorfo das enerxías libres. Este límite holomorfo non é único e é, de feito, dependente do marco, unha etiqueta asociada á simetría modular. Os puntos especiais no espazo de módulos teñen marcos preferentes asociados a eles.

Comezamos a análise resurxente estudando o comportamento a alta orde das enerxías libres perturbativas. O primeiro elemento importante no que nos centramos é a acción de instantón dominante. Polo capítulo anterior sabemos que é unha cantidade holomorfa, independente do propagador. A etiqueta de dominante refire ao menor, en valor absoluto, de todas as accións de instantón da transerie. Segundo exploramos o espazo de módulos a acción de instantón dominante cambia. Unha análise numérica amosa fortes evidencias de holomorfía e atopamos dúas accións dominantes distintas. Unha está asociada a un punto de conifold. A outra é constante e provén dunha contribución universal constante ás enerxías libres. Eliminándoa atópase outra acción de instantón que domina preto do punto de gran radio. Tanto a acción de instantón de conifold como do punto de gran radio son proporcionais ás correspondentes coordenadas planas sobre estes puntos no espazo de módulos. Son períodos da xeometría. Non son as únicas accións de instantón da transerie. Existen outras dúas accións asociadas a un segundo e terceiro punto de conifold. Pódense detectar primeiro nunha sección do espazo de módulos para a que as accións de instantón teñen o mesmo valor absoluto. O comportamento das enerxías libres é oscilatoria alí debido a combinación de dúas contribucións conxugadas das accións do primeiro e segundo (ou terceiro) puntos de conifold. Estas catro accións de instantón, tres de conifold e un de gran radio, e as mesmas co signo trocado indican unha transerie moi complicada con abundante cantidade de resonancia potencial. Dunha análise do plano de Borel do sector perturbativo amosamos como o polo asociado á acción de gran radio desaparece, mudando a outra folla de Riemann, segundo nos movemos no espazo de módulos cara ao punto de orbifold onde esta acción debería ser a dominante. Isto é evidencia de que o plano de Borel e a estrutura resurxente son máis intricados que o que se encontrou noutros exemplos no pasado.

Viramos a atención cara ao estudo de sectores de maior instantón que aparecen nas relacións de alta orde. O primeiro é o sector dun instantón asociado ao primeiro punto de conifold. Este controla o crecemento factorial principal das enerxías libres perturbativas. No límite holomorfo correspondente, a serie de alta orde trúncase, de acordo coa análise xeral realizada no capítulo 3. Isto permite fixar a ambigüidade holomorfa das enerxías libres dun instantón calculadas coas ecuacións de anomalía holomorfa estendidas como se describiu nese capítulo. Facemos probas numéricas da dependencia holomorfa e non holomorfa deste sector, utilizando técnicas de aceleración para a converxencia como a transformada de Richardson, e achamos excelentes concordancias cos resultados teóricos. O mesmo exercicio fornece evidencias de que as constantes de Stokes non poden depender dos módulos, así que son realmente números complexos. Dado que as relacións de alta orde xa se utilizaron para fixar as ambigüidades holomorfas non resta información para calcular as constantes de Stokes. O comportamento oscilatorio das enerxías libres, que sinalaban a presenza de accións de instantón doutros puntos de conifold, pódese reproducir mediante unha fórmula de alta orde que involucra dous sectores dun instantón. A amplitude e a frecuencia das oscilacións coinciden coas predicións teóricas.

Pódese facer un estudo dos outro sectores dun instantón asociados a puntos de conifold resumando primeiro a contribución principal ao crecemento perturbativo. A técnica de resumación necesaria ten que ser máis ou menos potente dependendo do sector que veña despois do principal. Dependendo do punto no que esteamos no espazo de módulos tamén podemos atopar contribucións secundarias de dous instantóns asociadas ao primeiro punto de conifold. Realizamos comprobacións numéricas tanto no límite holomorfo (onde a resumación xa non é necesaria debido ao truncamento da serie) e para valores xerais do propagador. Todos os resultados coinciden coas predicións teóricas das correspondentes enerxías libres calculadas coas ecuacións de anomalía holomorfa.

As enerxías libres dun instantón pódense calcular das ecuacións até alta orde cun proceso de integración seminumérico. Este procedemento fixa unha estrutura complexa e deixa a dependencia no propagador libre. Unha computación analítica nas dúas variables vólvese impracticable despois de vinte enerxías libres, aproximadamente, o que fai o subseguinte estudo imposible ou moi deficiente. O crecemento a alta orde deste sector revela claramente a presenza de resonancia entre sectores con accións de instantón de signo oposto. Un sector de dous instantóns e outro mixto, con A e $-A$, controlan este crecemento. As enerxías libres correspondentes pódense calcular coas ecuacións de anomalía holomorfa. As súas ambigüidades son fixadas reparando en que as enerxías libres dun instantón son cero no límite holomorfo (a orde alta), así que o mesmo ten que pasar coas enerxías libres anteriores. Comprobacións numéricas de alta precisión validan estas conclusións.

Neste punto resulta claro que a parella de enerxías libres de dous instantóns que se atoparon no crecemento a orde alta de sector dun instantón e do sector perturbativo, respectivamente, non son iguais. Que o foran sería o resultado agardado se as ecuacións ponte usuais fosen válidas mais a estrutura resurxente é lixeiramente diferente neste caso. Presentamos diferentes posibilidades que poderían ter algún papel, desde a consideración de funcións resurxentes non-simples á existencia de sectores similares con diferentes condicións para fixar a ambigüidade holomorfa.

As conclusións desta tese son as seguintes:

- As ecuacións de anomalía holomorfa pódense estender máis aló da teoría de perturbacións e pódense aplicar a unha solución en forma de transerie.
- A xeneralidade das ecuacións de anomalía holomorfa deixan sen determinar una serie de ambigüidades: as ambigüidades holomorfas a toda orde de instantón, a estrutura concreta da transerie, o número de parámetros, monomios non estándar na constante de acoplamento, accións de instantón e potencias iniciais. Todas elas deben ser calculadas a partir das restricións que impón a resurxencia.
- A estrutura das solucións xeneraliza a do sector perturbativo e pode describirse utilizando números combinatorios que saen dunha función xeratriz. A dependencia detallada da parte holomorfa require un esforzo maior e non se encontraron patróns xerais.
- As accións de instantón son holomorfas. Isto próbase coas ecuacións estendidas de anomalía holomorfa e directamente cun argumento de orde alta usando soamente información perturbativa.
- Transeries con resonancia inclúen sectores con expansións topolóxicas.
- O comportamento singular e universal das enerxías libres perturbativas arredor do punto de conifold determina o seu crecemento a orde alta preto dese punto especial. Esta análise convértese no substituto da condición para fixar a ambigüidade holomorfa nos sectores non perturbativos asociados a puntos de conifold. Este enfoque tería que ser explorado mellor noutros puntos especiais do espazo de módulos e para outros tipos de singularidade.
- A transerie que describe a enerxía libre de cordas topolóxicas da xeometría especular de $\mathbb{CP}^2$ local involucra varias accións de instantón que veñen a pares con signos opostos para respectar a expansión topolóxica do sector perturbativo. Hai tres accións de instantón de conifold, relacionadas por unha simetría $\mathbb{Z}_3$, e outra acción asociada ao punto de gran radio. Son períodos da xeometría e proporcionais ás coordenadas planas arredor dos puntos correspondentes.
- O crecemento a orde alta das enerxías libres perturbativas e non holomorfas pódese comprender, arredor do punto de conifold, utilizando enerxías libres de maior instantón calculadas dunha extensión das ecuacións de anomalía holomorfa. O sector de dous instantóns é unha combinación de enerxías libres cuxas ambigüidades holomorfas fíxanse de forma distinta. Diferentes sectores da transerie compiten pola contribución que segue á principal dependendo da rexión no espazo de módulos.
- As enerxías libres dun instantón teñen que ser calculadas seminumericamente no módulo e no propagador para obter un número suficientemente alto delas. O crecemento de orde alta amosa resonancia explicitamente e un sector de instantón mixto con expansión topolóxica.

- Toda cantidade numérica obtida de cálculos a alta orde pode ser reproducido por unha función analítica derivada das ecuacións estendidas de anomalía holomorfa. Isto faise para varias relacións de resurxencia tanto no límite holomorfo como fóra del.

- As relacións de resurxencia encontradas non se corresponden completamente coas predicións baseadas nunha ecuación ponte sinxela e inspiradas en modelos relacionados. Suxírense diversos escenarios para un marco resurxente que inclúen eliminar o requirimento de resurxencia simple, ou ter en conta a influencia de resonancia entre os sectores de conifold e de gran radio. Non se obtén unha solución final coherente. Precísase un estudo máis profundo de $\mathbb{CP}^2$ local e doutras xeometrías para entender non só a estrutura de resurxencia subxacente, senón tamén o problema xeral da fixación das ambigüidades holomorfas, e finalmente a resumación da transerie nunha función non perturbativa.

Appendix A

Holomorphic anomaly equations for multidimensional moduli space

All calculations and derivations done in chapter 3 have been oriented to the case of only one complex structure modulus because that is true for local $\mathbb{CP}^2$. However, the same exercise can be done for multidimensional moduli spaces. In this appendix we perform a detailed derivation of the holomorphic anomaly equations for a general multiparameter transseries with logarithmic sectors. Instanton actions are labelled by greek indices α, β, whereas complex structure moduli are labelled by latin indices, i, j, running from 1 to $h^{2,1}$, the dimension of moduli space. We take the opportunity to introduce new notation in terms of moduli derivatives that extend the usual covariant derivates by including an instanton action contribution. Their geometrical interpretation is an open problem.

The starting point is the holomorphic anomaly equation (3.14),

$$\frac{\partial F}{\partial S^{ij}} + \frac{1}{2}\left(U_i\, D_j F + U_j\, D_i F\right) - \frac{1}{2} g_s^2 \left(D_i D_j F + D_i F\, D_j F\right) = g_s^{-2} W_{ij} + V_{ij}. \tag{A.1}$$

The transseries ansatz is,

$$F = \sum_{\boldsymbol{n}\in\mathbb{N}^p} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-A^{(\boldsymbol{n})}/g_s} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k (g_s)\; F^{(\boldsymbol{n})[k]}(g_s), \tag{A.2}$$

where all ingredients were already explained in chapters 1 and 3. The covariant derivatives that appear in the master equation (A.1) are easy to compute,

$$D_j F = \sum_{\boldsymbol{n}} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-A^{(\boldsymbol{n})}/g_s} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k (g_s) \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})[k]}, \tag{A.3}$$

$$\begin{aligned} D_i D_j F = &\sum_{\boldsymbol{n}} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-A^{(\boldsymbol{n})}/g_s} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k (g_s) \times \\ &\times \left\{ \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{n})}\right)\left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n})}\right) - \gamma_{ij}^k \left(\partial_k - \frac{1}{g_s}\partial_k A^{(\boldsymbol{n})}\right)\right\} F^{(\boldsymbol{n})[k]}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} D_iF\,D_jF = \sum_{\boldsymbol{n}} \boldsymbol{\sigma}^{\boldsymbol{n}}\,\mathrm{e}^{-A^{(\boldsymbol{n})}/g_s} \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}} \sum_{k=0}^{k_{\max}^{(\boldsymbol{m})}+k_{\max}^{(\boldsymbol{n}-\boldsymbol{m})}} \log^k(g_s) \times \\ \times \sum_{\ell=\max\left(0,k-k_{\max}^{(\boldsymbol{n}-\boldsymbol{m})}\right)}^{\min\left(k,k_{\max}^{(\boldsymbol{m})}\right)} \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})[\ell]} \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n}-\boldsymbol{m})}\right) F^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}. \end{aligned} \tag{A.5}$$

In the last equation we have used $A^{(\boldsymbol{m})} + A^{(\boldsymbol{n}-\boldsymbol{m})} = A^{(\boldsymbol{n})}$ to group the exponential terms in one. From these expressions, the holomorphic anomaly equation (A.1) reads

$$\begin{aligned} \sum_{\boldsymbol{n}} \boldsymbol{\sigma}^{\boldsymbol{n}}\,\mathrm{e}^{-A^{(\boldsymbol{n})}/g_s} \Bigg\{ \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k(g_s) \Bigg[\left(\partial_{S^{ij}} - \frac{1}{g_s}\partial_{S^{ij}} A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})[k]} + \\ + \frac{1}{2} U_i \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})[k]} + \frac{1}{2} U_j \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})[k]} - \\ - \frac{1}{2} g_s^2 \left(\left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{n})}\right) \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n})}\right) - \gamma_{ij}^k \left(\partial_k - \frac{1}{g_s}\partial_k A^{(\boldsymbol{n})}\right) \right) F^{(\boldsymbol{n})[k]} \Bigg] - \\ - \frac{1}{2} g_s^2 \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}} \sum_{k=0}^{k_{\max}^{(\boldsymbol{m})}+k_{\max}^{(\boldsymbol{n}-\boldsymbol{m})}} \log^k(g_s) \sum_{\ell} \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})[\ell]} \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n}-\boldsymbol{m})}\right) F^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - \\ - \delta_{\boldsymbol{n},\boldsymbol{0}} \left(\frac{1}{g_s^2} W_{ij} + V_{ij} \right) \Bigg\} = 0. \end{aligned} \tag{A.6}$$

As explained in section 3.2, the perturbative sector $\boldsymbol{n} = \boldsymbol{0}$ gives back the ordinary holomorphic anomaly equations (2.39), as long as (3.11–3.13) are satisfied. We focus on sectors with $\boldsymbol{n} \neq \boldsymbol{0}$. For that we need to separate the terms corresponding to $\boldsymbol{m} = \boldsymbol{0}$ and $\boldsymbol{m} = \boldsymbol{n}$ from the sum in $\boldsymbol{m}$, and collect them together with the U_i. We find, for each $\boldsymbol{n} \neq \boldsymbol{0}$,

$$\begin{aligned} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \log^k(g_s) \Bigg[\left(\partial_{S^{ij}} - \frac{1}{g_s}\partial_{S^{ij}} A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})[k]} - \\ - \frac{1}{2} g_s^2 \Bigg\{ \partial_i \widehat{F}^{(\boldsymbol{0})} \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n})}\right) + \partial_j \widehat{F}^{(\boldsymbol{0})} \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{n})}\right) + \\ + \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{n})}\right) \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n})}\right) - \gamma_{ij}^k \left(\partial_k - \frac{1}{g_s}\partial_k A^{(\boldsymbol{n})}\right) \Bigg\} F^{(\boldsymbol{n})[k]} \Bigg] = \\ = \frac{1}{2} g_s^2 \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{k=0}^{k_{\max}^{(\boldsymbol{m})}+k_{\max}^{(\boldsymbol{n}-\boldsymbol{m})}} \log^k(g_s) \sum_{\ell} \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})[\ell]} \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n}-\boldsymbol{m})}\right) F^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}, \end{aligned} \tag{A.7}$$

where we have defined $\partial_i \widehat{F}^{(\boldsymbol{0})} := \partial_i F^{(\boldsymbol{0})} - \frac{1}{g_s^2} U_i$. We now introduce the symmetrization of two indices in the usual way $T_{(ij)} := \frac{1}{2}\left(T_{ij} + T_{ji}\right)$. If we also define $F^{(\boldsymbol{n})[k]} := 0$ if $k < 0$ or $k > k_{\max}^{(\boldsymbol{n})}$, we can collect similar powers of $\log g_s$ together,

$$\left(\partial_{S^{ij}} - \frac{1}{g_s}\partial_{S^{ij}} A^{(\boldsymbol{n})}\right) F^{(\boldsymbol{n})[k]} -$$

$$-\frac{1}{2}g_s^2\left(D_{(i|}-\frac{1}{g_s}\partial_{(i|}A^{(\boldsymbol{n})}+2\,\partial_{(i|}\widehat{F}^{(\boldsymbol{0})}\right)\left(\partial_{|j)}-\frac{1}{g_s}\partial_{|j)}A^{(\boldsymbol{n})}\right)F^{(\boldsymbol{n})[k]}=$$
$$=\frac{1}{2}g_s^2\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}'\sum_{\ell}\left(\partial_i-\frac{1}{g_s}\partial_iA^{(\boldsymbol{m})}\right)F^{(\boldsymbol{m})[\ell]}\left(\partial_j-\frac{1}{g_s}\partial_jA^{(\boldsymbol{n}-\boldsymbol{m})}\right)F^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}. \tag{A.8}$$

Here we have used that the Christoffel symbols, Γ_{ij}^k inside the covariant derivative D_i, are symmetric in the lower indices. As we announced at the beginning, the form of the differential operators in the equation above suggests the definition of a derivative extending D_i. It has a label on the instanton sector $(\boldsymbol{n})$ because it depends on the corresponding total instanton action.

$$\nabla_i^{(\boldsymbol{n})}F^{(\boldsymbol{n})[k]}:=\left(\partial_i-\frac{1}{g_s}\partial_iA^{(\boldsymbol{n})}\right)F^{(\boldsymbol{n})[k]}, \tag{A.9}$$

$$\nabla_i^{(\boldsymbol{n})}\nabla_j^{(\boldsymbol{n})}F^{(\boldsymbol{n})[k]}:=\left(D_{(i|}-\frac{1}{g_s}\partial_{(i|}A^{(\boldsymbol{n})}+2\,\partial_{(i|}\widehat{F}^{(\boldsymbol{0})}\right)\left(\partial_{|j)}-\frac{1}{g_s}\partial_{|j)}A^{(\boldsymbol{n})}\right)F^{(\boldsymbol{n})[k]}= \tag{A.10}$$
$$=\left(\delta_{(i|}^k\left(\partial_{|j)}-\frac{1}{g_s}\partial_{|j)}A^{(\boldsymbol{n})}+2\,\partial_{|j)}\widehat{F}^{(\boldsymbol{0})}\right)-\gamma_{ij}^k\right)\left(\partial_k-\frac{1}{g_s}\partial_kA^{(\boldsymbol{n})}\right)F^{(\boldsymbol{n})[k]},$$

$$\nabla_{S^{ij}}^{(\boldsymbol{n})}F^{(\boldsymbol{n})[k]}:=\left(\partial_{S^{ij}}-\frac{1}{g_s}\partial_{S^{ij}}A^{(\boldsymbol{n})}\right)F^{(\boldsymbol{n})[k]}. \tag{A.11}$$

With the new notation the holomorphic anomaly equations read,

$$\left(\nabla_{S^{ij}}^{(\boldsymbol{n})}-\frac{1}{2}g_s^2\,\nabla_i^{(\boldsymbol{n})}\nabla_j^{(\boldsymbol{n})}\right)F^{(\boldsymbol{n})[k]}=\frac{1}{2}g_s^2\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}'\sum_{\ell}\nabla_i^{(\boldsymbol{m})}F^{(\boldsymbol{m})[\ell]}\,\nabla_i^{(\boldsymbol{n}-\boldsymbol{m})}F^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}. \tag{A.12}$$

This is a recursive equation in the indices $\boldsymbol{n}$ and ℓ for the different asymptotic series that make up the transseries. The objects $F^{(\boldsymbol{m})[\ell]}$ depend on the propagators, the complex structure moduli, and also g_s.

Now we expand the equation (A.12) in g_s. As we did in chapter 3, we introduce a set of differential operators for the g_s-series expansion of the left-hand-side of (A.12),

$$\left(\nabla_{S^{ij}}^{(\boldsymbol{n})}-\frac{1}{2}g_s^2\,\nabla_i^{(\boldsymbol{n})}\nabla_j^{(\boldsymbol{n})}\right)=:\mathcal{D}_{ij}^{(\boldsymbol{n})}(g_s)=\sum_{g=-1}^{+\infty}g_s^g\,\mathcal{D}_{ij;g}^{(\boldsymbol{n})}. \tag{A.13}$$

For $g=-1$, $\mathcal{D}_{ij;-1}^{(\boldsymbol{n})}:=-\partial_{S^{ij}}A^{(\boldsymbol{n})}$, which is actually the zero operator because the instanton actions are holomorphic. The powers series expansion of the free energies is,

$$F^{(\boldsymbol{n})[k]}(g_s)=\sum_{g=0}^{+\infty}g_s^{g+b^{(\boldsymbol{n})[k]}}F_g^{(\boldsymbol{n})[k]}(z_i,S^{ij}). \tag{A.14}$$

The left-hand-side of (A.12) expands to

$$\mathcal{D}_{ij}^{(\boldsymbol{n})}F^{(\boldsymbol{n})[k]}=g^{b^{(\boldsymbol{n})[k]}}\sum_{g=-1}^{+\infty}g_s^g\left(-\partial_{S^{ij}}A^{(\boldsymbol{n})}\,F_{g+1}^{(\boldsymbol{n})[k]}+\sum_{h=0}^{g}\mathcal{D}_{ij;h}^{(\boldsymbol{n})}F_{g-h}^{(\boldsymbol{n})[k]}\right), \tag{A.15}$$

while the right-hand-side is

$$g_s^2 \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{\ell} \left(\partial_i - \frac{1}{g_s}\partial_i A^{(\boldsymbol{m})}\right) F^{(\boldsymbol{m})[\ell]} \left(\partial_j - \frac{1}{g_s}\partial_j A^{(\boldsymbol{n}-\boldsymbol{m})}\right) F^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} = \tag{A.16}$$

$$= \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{\ell} g_s^{b^{(\boldsymbol{m})[\ell]}+b^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}} \times$$

$$\times \sum_{g=0}^{+\infty} g_s^g \sum_{h=0}^{g} \left(\partial_i F_{h-1}^{(\boldsymbol{m})[\ell]} - \partial_i A^{(\boldsymbol{m})} F_h^{(\boldsymbol{m})[\ell]}\right) \left(\partial_j F_{g-1-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - \partial_j A^{(\boldsymbol{n}-\boldsymbol{m})} F_{g-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}\right).$$

We collect similar powers of g_s and end up with

$$-\partial_{S^{ij}} A^{(\boldsymbol{n})}\, F_{g+1}^{(\boldsymbol{n})[k]} + \sum_{h=0}^{g} \mathcal{D}_{ij;h}^{(\boldsymbol{n})} F_{g-h}^{(\boldsymbol{n})[k]} = \tag{A.17}$$

$$= \frac{1}{2} \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{\ell} \sum_{h=0}^{g-B} \left(\partial_i F_{h-1}^{(\boldsymbol{m})[\ell]} - \partial_i A^{(\boldsymbol{m})} F_h^{(\boldsymbol{m})[\ell]}\right) \left(\partial_j F_{g-1-B-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - \partial_j A^{(\boldsymbol{n}-\boldsymbol{m})} F_{g-B-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]}\right) = 0,$$

for $g = -1, 0, 1, 2, \ldots$, $k = 0, 1, 2, \ldots$, $1 \leq i \leq j \leq h^{2,1}$, and $\boldsymbol{n} \neq \boldsymbol{0}$. We have introduced the combination of starting powers,

$$B(\boldsymbol{n}, \boldsymbol{m})[k, \ell] := b^{(\boldsymbol{m})[\ell]} + b^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - b^{(\boldsymbol{n})[k]}. \tag{A.18}$$

The explicit form the derivatives on the left-hand-side of (A.17) is

$$\mathcal{D}_{ij;0}^{(\boldsymbol{n})} = \partial_{S^{ij}} - \frac{1}{2}\partial_i A^{(\boldsymbol{n})} \partial_j A^{(\boldsymbol{n})}, \tag{A.19}$$

$$\mathcal{D}_{ij;1}^{(\boldsymbol{n})} = \frac{1}{2} D_i \partial_j A^{(\boldsymbol{n})} + \partial_{(i|} A^{(\boldsymbol{n})} \left(\partial_{|j)} + \partial_{|j)} F_1^{(\boldsymbol{0})}\right), \tag{A.20}$$

$$\mathcal{D}_{ij;2}^{(\boldsymbol{n})} = -\frac{1}{2} D_i \partial_j - \partial_{(i|} F_1^{(\boldsymbol{0})} D_{|j)}, \tag{A.21}$$

$$\mathcal{D}_{ij;2h-1}^{(\boldsymbol{n})} = \partial_{(i|} F_h^{(\boldsymbol{0})} \partial_{|j)} A^{(\boldsymbol{n})}, \qquad h = 2, 3, \ldots, \tag{A.22}$$

$$\mathcal{D}_{ij;2h}^{(\boldsymbol{n})} = -\partial_{(i|} F_h^{(\boldsymbol{0})} \partial_{|j)}, \qquad h = 2, 3, \ldots. \tag{A.23}$$

The holomorphicity of the instanton action is derived from the first equation of the tower, for $\boldsymbol{n} = (0|\cdots|1|\cdots|0)$, $k = 0$, and $g = -1$,

$$\partial_{S^{ij}} A^{(0|\cdots|1|\cdots|0)}\, F_0^{(0|\cdots|1|\cdots|0)[0]} = 0. \tag{A.24}$$

By construction $F_0^{(0|\cdots|1|\cdots|0)[0]} \neq 0$, so $\partial_{S^{ij}} A_\alpha = 0$ for all i, j, and α. This simplifies the holomorphic anomaly equations to

$$\left(\partial_{S^{ij}} - \frac{1}{2}\partial_i A^{(\boldsymbol{n})} \partial_j A^{(\boldsymbol{n})}\right) F_g^{(\boldsymbol{n})[k]} = -\sum_{h=1}^{g} \mathcal{D}_{ij;h}^{(\boldsymbol{n})} F_{g-h}^{(\boldsymbol{n})[k]} + \tag{A.25}$$

$$+\frac{1}{2}\sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{\ell} \sum_{h=0}^{g-B} \left(\partial_i F_{h-1}^{(\boldsymbol{m})[\ell]} - \partial_i A^{(\boldsymbol{m})} F_h^{(\boldsymbol{m})[\ell]} \right) \left(\partial_j F_{g-1-B-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - \partial_j A^{(\boldsymbol{n}-\boldsymbol{m})} F_{g-B-h}^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} \right).$$

Besides a few more indices in the notation, the structure of the equations is the same that we discussed in chapter 3. They are recursive equations in the instanton sector $\boldsymbol{n}$, logarithmic sector k, and the order g. The range of ℓ is the same as in (A.5), and B is defined by (A.18). The prime in the sum over $\boldsymbol{m}$ means that $\boldsymbol{m} = \boldsymbol{0}$ and $\boldsymbol{m} = \boldsymbol{n}$ are excluded.

The structure of the solutions is completely analogous to the one described in theorem 2, and is summarized in the following

Theorem 4. *For any $\boldsymbol{n} \neq \boldsymbol{0}$, $k \in \{0, \dots, k_{\max}^{(\boldsymbol{n})}\}$, and $g \geq 0$, the structure of the nonperturbative free energies has the form*

$$F_g^{(\boldsymbol{n})[k]} = \sum_{\{\gamma_{\boldsymbol{n}}\}} \mathrm{e}^{\frac{1}{2}\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}}) \sum_{i,j=1}^{h^{2,1}} \partial_i A_\alpha\, \partial_j A_\beta\, S^{ij}}\; Pol\left(S^{ij}; 3\left(g + b^{(\boldsymbol{n})[k]} - \lambda_{b,k_{\max}}^{[k]}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\right)\right), \tag{A.26}$$

where the set of numbers $\{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\}$ and $\left\{\lambda_{b,k_{\max}}^{[k]}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\right\}$ are read from the generating function

$$\begin{aligned} \Phi_{b,k_{\max}} &= \prod_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty}{}' \prod_{\ell=0}^{k_{\max}^{(\boldsymbol{m})}} \frac{1}{1 - \varphi^{b^{(\boldsymbol{m})[\ell]}}\, \psi^\ell \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{m_\alpha m_\beta} \prod_{\alpha=1}^{p} \rho_\alpha^{m_\alpha}} = \\ &= \sum_{\boldsymbol{n}=\boldsymbol{0}}^{+\infty} \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{k=0}^{k_{\max}^{(\boldsymbol{n})}} \psi^k \sum_{\{\gamma_{\boldsymbol{n}}\}} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\, \varphi^{\lambda_{b,k_{\max}}^{[k]}(\boldsymbol{n};\gamma_{\boldsymbol{n}})} \mathcal{O}(\varphi^0). \end{aligned} \tag{A.27}$$

Here $Pol(S^{ij}; d)$ stands for a polynomial of total degree d in the variables $\{S^{ij}\}$ and whose coefficients have a dependence in $\{z_i\}$. Whenever $d < 0$, the polynomial is taken to be identically zero. We are assuming that $b^{(\boldsymbol{m})[\ell]} + b^{(\boldsymbol{n}-\boldsymbol{m})[k-\ell]} - b^{(\boldsymbol{n})[k]} \geq 0$, and $k_{\max}^{(\boldsymbol{n})} - k_{\max}^{(\boldsymbol{m})} - k_{\max}^{(\boldsymbol{n}-\boldsymbol{m})} \geq 0$.

The proof analogous to that of theorem 1, but one has to keep track of the dependence on the indices i, j and the logarithmic sectors. Let us notice that the multidimensionality of the moduli space does not change the structure of the solutions or the combinatorial coefficients. This is because all the propagators are on equal footing, which is not the case for the instanton sectors.

Appendix B

Proof of set recursion lemma

In this appendix we give a proof for the natural generalization of lemma 1 to multiparameter transseries. It provides the equivalence between two ways of describing the combinatorial numbers $a(\boldsymbol{n};\gamma_{\boldsymbol{n}})$ and $\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})$, in terms of a generating function or recursively. It is used to prove theorems 1 and 2 in chapter 3.

Lemma 4. *The set of numbers* $\{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\}$ *and* $\{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})\}$, *and the range of the labels* $\{\gamma_{\boldsymbol{n}}\}$, *which appear in*

$$\begin{aligned}\Phi_b &= \prod_{\boldsymbol{m}=\boldsymbol{0}}^{\infty}{}' \frac{1}{1-\varphi^{b(\boldsymbol{m})}\prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{m_\alpha m_\beta}\prod_{\alpha=1}^{p}\rho_\alpha^{m_\alpha}} = \\ &= \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty}\boldsymbol{\rho}^{\boldsymbol{n}}\sum_{\{\gamma_{\boldsymbol{n}}\}}\prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\,\varphi^{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\,\mathcal{O}(\varphi^0).\end{aligned} \tag{B.1}$$

are determined by the recursions

$$\{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})\}_{\gamma_n\neq\widehat{\gamma}_n} = \bigcup_{\boldsymbol{m},\gamma_{\boldsymbol{m}},\gamma_{\boldsymbol{n}-\boldsymbol{m}}}{}'\{a_{\alpha\beta}(\boldsymbol{m};\gamma_{\boldsymbol{m}})+a_{\alpha\beta}(\boldsymbol{n}-\boldsymbol{m};\gamma_{\boldsymbol{n}-\boldsymbol{m}})\} \tag{B.2}$$

and

$$\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}}) = \min\{\lambda_b(\boldsymbol{m};\gamma_{\boldsymbol{m}})+\lambda_b(\boldsymbol{n}-\boldsymbol{m};\gamma_{\boldsymbol{n}-\boldsymbol{m}})\}, \qquad \forall\gamma_{\boldsymbol{n}}\neq\widehat{\gamma}_{\boldsymbol{n}}, \tag{B.3}$$

where min *ranges over* $\boldsymbol{m}\in\{\boldsymbol{0},\dots,\boldsymbol{n}\}'$, *and* $\gamma_{\boldsymbol{m}}$, $\gamma_{\boldsymbol{n}-\boldsymbol{m}}$ *are such that*

$$a_{\alpha\beta}(\boldsymbol{m};\gamma_{\boldsymbol{m}})+a_{\alpha\beta}(\boldsymbol{n}-\boldsymbol{m};\gamma_{\boldsymbol{n}-\boldsymbol{m}}) = a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}}), \qquad \forall\alpha,\beta. \tag{B.4}$$

The prime means that $\boldsymbol{m}=\boldsymbol{0}$ *and* $\boldsymbol{m}=\boldsymbol{n}$ *are excluded. Further, we have to specify the initial data:*

$$a_{\alpha\beta}(\boldsymbol{n};\widehat{\gamma}_{\boldsymbol{n}}) = n_\alpha n_\beta, \qquad \forall\,\boldsymbol{n},\alpha,\beta, \tag{B.5}$$

$$\lambda_b(\boldsymbol{n};\widehat{\gamma}_{\boldsymbol{n}}) = b^{(\boldsymbol{n})}, \qquad \forall\,\boldsymbol{n}. \tag{B.6}$$

Proof. The proof involves calculating $(\Phi_b-1)^2$ in two different ways and comparing the results. In the first way we square the generating function as a formal power series,

$$(\Phi_b-1)^2 = \left(\sum_{\boldsymbol{n}=\boldsymbol{0}}^{+\infty}{}' \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\{\gamma_{\boldsymbol{n}}\}} E^{a(\boldsymbol{n};\gamma_{\boldsymbol{n}})}\, \varphi^{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})} \mathcal{O}(\varphi^0)\right)^2 = \tag{B.7}$$

$$= \sum_{\boldsymbol{n}=\boldsymbol{0}}^{+\infty}{}'' \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\boldsymbol{m}=\boldsymbol{0}}^{\boldsymbol{n}}{}' \sum_{\{\gamma_{\boldsymbol{m}},\gamma_{\boldsymbol{n}-\boldsymbol{m}}\}} E^{a(\boldsymbol{m};\gamma_{\boldsymbol{m}})+a(\boldsymbol{n}-\boldsymbol{m};\gamma_{\boldsymbol{n}-\boldsymbol{m}})}\, \varphi^{\min\{\lambda_b(\boldsymbol{m};\gamma_{\boldsymbol{m}})+\lambda_b(\boldsymbol{n}-\boldsymbol{m};\gamma_{\boldsymbol{n}-\boldsymbol{m}})\}}\, \mathcal{O}(\varphi^0).$$

Here we have used the shorthand notation

$$E^{a(\boldsymbol{m};\gamma_{\boldsymbol{m}})} := \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{a_{\alpha\beta}(\boldsymbol{n};\gamma_{\boldsymbol{n}})}. \tag{B.8}$$

The double–prime ($''$) means that perturbative and one-instanton sectors are excluded from the sum, that is $\|\boldsymbol{n}\| \neq 0,1$.

In the second way we first expand the square and then the generating function,

$$(\Phi_b-1)^2 = \Phi_b^2 - 2\Phi_b + 1, \tag{B.9}$$

with

$$\Phi_b^2 = \prod_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty}{}' \frac{1}{\left(1-\varphi^{b(\boldsymbol{m})} \prod_{\alpha,\beta=1}^{p} E_{\alpha\beta}^{m_\alpha m_\beta} \prod_{\alpha=1}^{p} \rho_\alpha^{m_\alpha}\right)^2}. \tag{B.10}$$

We use the formal expansion $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$ and the similar manipulations to the ones done below (3.48),

$$\Phi_b^2 = \sum_{\boldsymbol{n}=\boldsymbol{0}}^{+\infty} \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\{\gamma_{\boldsymbol{n}}\}} E^{a(\boldsymbol{n};\gamma_{\boldsymbol{n}})} \sum_{\{r_{\boldsymbol{m}}\}\in\gamma_{\boldsymbol{n}}} \varphi^{\sum_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty} r_{\boldsymbol{m}}\, b(\boldsymbol{m})} \prod_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty} (r_{\boldsymbol{m}}+1)\,. \tag{B.11}$$

Since (B.7) does not have terms with $\|\boldsymbol{n}\| = 0,1$ we must check that (B.9) does not have them either. For $\|\boldsymbol{n}\| = 0$,

$$\Phi_b^2\big|_{\|\boldsymbol{n}\|=0} = 1, \quad \Phi_b\big|_{\|\boldsymbol{n}\|=0} = 1 \qquad \Rightarrow \qquad (\Phi_b-1)^2\big|_{\|\boldsymbol{n}\|=0} = 1-2\cdot 1+1 = 0. \tag{B.12}$$

For $\|\boldsymbol{n}\| = 1$ there is only one class, $(\widehat{\gamma}_{\boldsymbol{n}})$, so

$$\Phi_b^2\big|_{\|\boldsymbol{n}\|=1} = \sum_{\alpha=1}^{p} \rho_\alpha\, E_{\alpha\alpha}\, \varphi^{b(\boldsymbol{n})}\,(1+1)\,,\ \ \Phi_b\big|_{\|\boldsymbol{n}\|=1} = \sum_{\alpha=1}^{p} \rho_\alpha\, E_{\alpha\alpha}\, \varphi^{b(\boldsymbol{n})} \quad \Rightarrow \quad (\Phi_b-1)^2\big|_{\|\boldsymbol{n}\|=1} = 0. \tag{B.13}$$

Altogether we can write $(\Phi_b-1)^2$ as

$$\sum_{\boldsymbol{n}=\boldsymbol{0}}^{+\infty}{}'' \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\{\gamma_{\boldsymbol{n}}\}} E^{a(\boldsymbol{n};\gamma_{\boldsymbol{n}})} \sum_{\{r_{\boldsymbol{m}}\}\in\gamma_{\boldsymbol{n}}} \varphi^{\sum_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty} r_{\boldsymbol{m}}\, b(\boldsymbol{m})} \left(\prod_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty} (r_{\boldsymbol{m}}+1) - 2\right). \tag{B.14}$$

Finally, the coefficient at the end of this expression satisfies

$$\prod_{\boldsymbol{m}=\boldsymbol{0}}^{+\infty} (r_{\boldsymbol{m}} + 1) - 2 \geq 0, \tag{B.15}$$

where the equality only holds for the special class $\widehat{\gamma}_{\boldsymbol{n}}$. Thus,

$$(\Phi_b - 1)^2 = \sum_{\boldsymbol{n}=\boldsymbol{0}}^{+\infty}{}'' \boldsymbol{\rho}^{\boldsymbol{n}} \sum_{\{\gamma_{\boldsymbol{n}}\}}{}' E^{a(\boldsymbol{n};\gamma_{\boldsymbol{n}})} \varphi^{\lambda_b(\boldsymbol{n};\gamma_{\boldsymbol{n}})} \mathcal{O}\left(\varphi^0\right). \tag{B.16}$$

Comparing (B.7) and (B.16) concludes the proof. □

A similar result can be proved when logarithmic sectors are included.

Appendix C

Structure of free energies in local $\mathbb{CP}^2$

In this appendix we provide a description of the holomorphic and antiholomorphic dependence of the free energies in local $\mathbb{CP}^2$ with low instanton number. These are the free energies that have appeared in the resurgent study of chapter 4. The propagator dependence was analyzed in detail in chapter 3 but the dependence of the complex modulus is more complicated and, even though there is some structure, it is not simple to describe. The skeleton of the free energies is a combination of exponentials and polynomials in the propagators. The exponents and the degrees of the polynomials are determined by theorem 2. The coefficients are rational functions of z and polynomials in the the instanton actions, their derivatives, and the holomorphic limit of the propagator. Because of the Picard–Fuchs equation we can make it so that only second order derivatives of A_i appear. Also, $S^{zz}_{i,\text{hol}}$ can be expressed in terms of the instanton actions and rational functions of z,

$$S^{zz}_{i,\text{hol}} = -\frac{1}{C_{zzz}(z)}\left(\frac{A_i''(z)}{A_i'(z)} - \tilde{f}^z_{zz}(z)\right), \tag{C.1}$$

with C_{zzz} and $\tilde{f}^z_{zz}$ given by (2.62) and (2.78), respectively. The inverse power of $A_i'(z)$ in (C.1) always gets cancelled in the final expressions for the nonperturbative free energies due to the term $(\partial_z A_i)^2$ in the exponentials.

One-instanton free energies

The one-instanton free energies associated to any of the instanton actions, $\pm A_i$, are the product of an exponential in the propagator and a polynomial. We use simplified notation where we drop the index i and we denote by A any of the instanton actions. We have,

$$F_g^{(1)}(z, S^{zz}) = \mathrm{e}^{\frac{1}{2}(\partial_z A)^2\left(S^{zz} - S^{zz}_{\text{hol}}\right)} \sum_{k=0}^{3g} p^{(1)}_{g,k}(z)[A, \partial_z A, \partial_z^2 A]\, (S^{zz})^k\,, \tag{C.2}$$

where the coefficients are

$$p^{(1)}_{g,k}(z)[A, \partial_z A, \partial_z^2 A] = \sum_{\eta\in\mathcal{X}^{(1)}_{g,k}} p^{(1)}_{g,k,\eta}(z)\, A^{\eta_0}\,(\partial_z A)^{\eta_1}\left(\partial_z^2 A\right)^{\eta_2}. \tag{C.3}$$

The $p^{(1)}_{g,k,\eta}(z)$ are rational functions of z, and each $\eta \in \mathcal{X}^{(1)}_{g,k}$ is written in the form $\eta = (k;\eta_0,\eta_1,\eta_2)$. $\mathcal{X}^{(1)}_{g,k}$ is a set of such vectors for which we can give a heuristic description. Start with $k=3g$ where there is only one $\eta=(3g;1,3g,0)\in\mathcal{X}^{(1)}_{g,k=3g}$, and apply the recursion

$$\left\{\eta \mid \eta\in\mathcal{X}^{(1)}_{g,k}\right\} = \Big\{\tilde\eta+\lambda \mid \tilde\eta\in\mathcal{X}^{(1)}_{g,k+1},\\ \lambda\in\{(-1;0,0,0),(-1;0,-2,0),(-1;0,-1,+1),(-1;-1,+1,0)\}\Big\}^{\star}. \tag{C.4}$$

The star (*) indicates that η's of the form

$$(k;1,0,0),\quad (k;0,1,0),\quad (k;0,0,1), \tag{C.5}$$

$$(k;0,\eta_1,\eta_2)\quad\text{if}\quad \eta_1+\eta_2=3g+1,\quad (k;0,0,0)\quad\text{if}\quad k\le 3\left[\frac{g}{2}\right], \tag{C.6}$$

have to be discarded, along with η's that have negative components. For $g=1$ we have to discard $(0;1,1,0)$, as well. Let us display the $g=1$ free energy is a schematic way, where we *omit* the rational functions $p^{(1)}_{1,k,\eta}(z)$ and show only the polynomials in A, A' and A''.

$$F^{(1)}_1(z,S^{zz}) \simeq \mathrm{e}^{\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{\text{hol}}\right)}\Big\{\left(S^{zz}\right)^3 AA'^3+\left(S^{zz}\right)^2\left(AA'+AA'^3+AA'^2A''\right)+\\ +\left(S^{zz}\right)\left(AA'+AA'^3+AA''+AA'^2A''+AA'A''^2\right)+\\ +1+A'^2+AA'^3+A'A''+AA'^2A''+AA'A''^2+AA''^3\Big\}. \tag{C.7}$$

From the definition of the set $\mathcal{X}^{(1)}_{g,k}$ we can derive how the free energies change as we flip the sign of the instanton action. If we do the transformation $A\to -A$ in C.3 then we can show by induction on k that $F^{(1)}_g$ acquires a factor $(-1)^{g+1}$. The base case is $k=3g$, for which $(-1)^{\sum_{i=0}^2\eta_i}=(-1)^{3g+1}=(-1)^{g+1}$. Because no addition of a vector λ in (C.4) changes the result holds for all k. Thus,

$$F^{(1)}_g\big|_{A\to -A} = (-1)^{g+1}\,F^{(1)}_g. \tag{C.8}$$

If we restore the notation in terms of $(\boldsymbol{\varepsilon}_{2i-1})$ and $(\boldsymbol{\varepsilon}_{2i})$ we conclude

$$\frac{S_{1,i}}{2\pi i}\,F^{(\boldsymbol{\varepsilon}_{2i-1})}_g = (-1)^{g+1}\,\frac{\widetilde{S}_{-1,i}}{2\pi i}\,F^{(\boldsymbol{\varepsilon}_{2i})}_g. \tag{C.9}$$

This is one of the necessary conditions to recover a genus expansion for the perturbative sector out of the large-order growth, (1.61).

Using a program in *Mathematica* we computed one-instanton free energies for all the conifold sectors up to $g=21$ in closed analytic form and with the holomorphic ambiguity fixed. The computation becomes impossible around this order due to the enormous requirement in memory—the size of $\mathcal{X}^{(1)}_{g,k}$ increases quickly with g. To overcome this one has to keep track of the holomorphic dependence numerically while leaving the propagator dependence analytic. This allows us to go as high as $g=80$. The caveat is that the computation has to be done for a particular point in moduli space at a time.

Two-instanton and mixed free energies

The large-order of the one-instanton free energies is controlled, to leading order, by the sectors $\widehat{F}_h^{(2\boldsymbol{e}_1)}$ and $\widehat{F}_h^{(\boldsymbol{e}_{1,1})}$, see (4.112). Both have a vanishing holomorphic limit, indicated by the hat. In what follows A and S^{zz}_{hol} refer to the first conifold point.

The two-instanton free energies $\widehat{F}_g^{(2\boldsymbol{e}_1)}$ have the structure,

$$\widehat{F}_g^{(2\boldsymbol{e}_1)} = \mathrm{e}^{2\cdot\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{\text{hol}}\right)} \sum_{k=0}^{3g} p^{(2\boldsymbol{e}_1)}_{2;g,k}(z)[A,\partial_z A,\partial_z^2 A]\,\left(S^{zz}\right)^k + \tag{C.10}$$

$$+\,\mathrm{e}^{4\cdot\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{\text{hol}}\right)} \sum_{k=0}^{3g} p^{(2\boldsymbol{e}_1)}_{4;g,k}(z)[A,\partial_z A,\partial_z^2 A]\,\left(S^{zz}\right)^k, \tag{C.11}$$

with

$$p^{(2\boldsymbol{e}_1)}_{r;g,k}(z)[A,\partial_z A,\partial_z^2 A] = \sum_{\eta\in\mathcal{X}^{(2\boldsymbol{e}_1)}_{r;g,k}} p^{(2\boldsymbol{e}_1)}_{r;g,k,\eta}(z)\,A^{\eta_0}\,(\partial_z A)^{\eta_1}\left(\partial_z^2 A\right)^{\eta_2}, \qquad r=2,4. \tag{C.12}$$

The sets $\mathcal{X}^{(2\boldsymbol{e}_1)}_{r;g,k}$ are similar to $\mathcal{X}^{(1)}_{g,k}$ but we have not found a recursive definition as before. However, it seems that $\mathcal{X}^{(2\boldsymbol{e}_1)}_{2;g,k} = \mathcal{X}^{(2\boldsymbol{e}_1)}_{4;g,k}$. In schematic notation the $g=1$ free energy is

$$\begin{aligned}\widehat{F}_1^{(2\boldsymbol{e}_1)} \simeq\ & \mathrm{e}^{2\,\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{[1],\text{hol}}\right)}\Big\{ \left(S^{zz}\right)^3 A^2 A'^3 + \left(S^{zz}\right)^2\left(A^2A' + A^2A'^3 + A^2A'^2A''\right) + \\ & + \left(S^{zz}\right)\left(A^2A' + AA'^2 + A^2A'^3 + A^2A'' + A^2A'^2A'' + A^2A'A''^2\right) + \\ & + A + AA'^2 + A^2A'^3 + AA'A'' + A^2A'^2A'' + A^2A'A''^2 + A^2A''^3\Big\} + \\ & + \mathrm{e}^{4\cdot\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{\text{hol}}\right)}\Big\{\text{ same as above }\Big\}.\end{aligned} \tag{C.13}$$

The mixed sector $\widehat{F}^{(\boldsymbol{e}_{1,1})}$ has the general structure

$$\widehat{F}_0^{(\boldsymbol{e}_{1,1})} = \frac{\pi i}{S_{1,1}}\,\frac{\pi i}{\widetilde{S}_{-1,1}}\left(\frac{A}{2\pi^2}\right)^2\left(\mathrm{e}^{2\,\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{[1],\text{hol}}\right)} - 1\right), \tag{C.14}$$

$$\widehat{F}_{2g}^{(\boldsymbol{e}_{1,1})} = \mathrm{e}^{2\cdot\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{[1],\text{hol}}\right)} \sum_{k=0}^{5g} p^{(\boldsymbol{e}_{1,1})}_{2;2g,k}(z)[A,\partial_z A,\partial_z^2 A]\,\left(S^{zz}\right)^k + \tag{C.15}$$

$$+\sum_{k=0}^{3g-1} p^{(\boldsymbol{e}_{1,1})}_{0;2g,k}(z)[A,\partial_z A,\partial_z^2 A]\,\left(S^{zz}\right)^k, \qquad g>0, \tag{C.16}$$

and $\widehat{F}^{(\boldsymbol{e}_{1,1})}_{\text{odd}} = 0$ due to resonance. We have not found a heuristic description for the coefficients $p^{(\boldsymbol{e}_{1,1})}_{2;2g,k}(z)$ or $p^{(2\boldsymbol{e}_1)}_{2;2g,k}(z)$. The first nontrivial example of free energy is for $g=2$, that has the rather long expression,

$$\widehat{F}_2^{(\boldsymbol{e}_{1,1})} \simeq \mathrm{e}^{2\cdot\frac{1}{2}(\partial_z A)^2\left(S^{zz}-S^{zz}_{[1]\text{hol}}\right)}\Big\{ \left(S^{zz}\right)^5 A^2A'^4 + \left(S^{zz}\right)^4\left(A^2A'^2 + A^2A'^4 + A^2A'^3A''\right) +$$

$$
\begin{aligned}
&+ (S^{zz})^3 \left(A^2A' + A^2A'^3 + A^2A'^4 + A^2A'A'' + A^2A'^3A'' + A^2A'^2A''^2\right) + \\
&+ (S^{zz})^2 \left(AA' + A^2A'^2 + AA'^3 + A'^4 + A^2A'^4 + A^2A'A'' + \right.\\
&\qquad \left. + AA'^2A'' + A^2A'^3A'' + A^2A''^2 + A^2A'^2A''^2 + A^2A'A''^3\right) + \\
&+ S^{zz} \left(AA' + A'^2 + A^2A'^2 + AA'^3 + A'^4 + A^2A'^4 + AA'' + A^2A'A'' + AA'^2A'' + \right.\\
&\qquad \left. + A'^3A'' + A^2A'^3A'' + A^2A''^2 + AA'A''^2 + A^2A'^2A''^2 + A^2A'A''^3 + A^2A''^4\right) + \\
&+ 1 + A'^2 + A^2A'^2 + AA'^3 + A'^4 + A^2A'^4 + A'A'' + A^2A'A'' + AA'^2A'' + A'^3A'' + \\
&+ A^2A'^3A'' + A^2A''^2 + AA'A''^2 + A'^2A''^2 + A^2A'^2A''^2 + AA''^3 + A^2A'A''^3 + A^2A''^4 \Big\} + \\
&+ \Big\{ (S^{zz})^2 AA' + S^{zz}\left(AA' + A'^2 + AA''\right) + 1 + A'^2 + A'A'' \Big\}. \qquad \text{(C.17)}
\end{aligned}
$$

In practice, we have computed the nonperturbative free energies, $\widehat{F}_g^{(2\boldsymbol{e}_1)}$ and $\widehat{F}_g^{(\boldsymbol{e}_1,1)}$, up to genus $g = 8$. Just like for the one-instanton sector it is necessary to introduce a seminumerical method to reach higher orders.

The two-instanton contribution that appears in the large-order growth of the perturbation theory at subleading order, $\widetilde{F}_g^{(2\boldsymbol{e}_1,1)}$, is a combination of two-instanton free energies computed out of the holomorphic anomaly equations. One is $\widehat{F}_g^{(2\boldsymbol{e}_1)}$ that has been described above. The other, $F_g^{(2\boldsymbol{e}_1)}$, is calculated in the same way but with a nonvanishing holomorphic limit—see (4.77). The antiholomorphic dependence of $F_g^{(2\boldsymbol{e}_1)}$ is the same as that of $\widehat{F}_g^{(2\boldsymbol{e}_1)}$. Their difference, $\widetilde{F}_g^{(2\boldsymbol{e}_1,1)} = F_g^{(2\boldsymbol{e}_1)} - \widehat{F}_g^{(2\boldsymbol{e}_1)}$, has only one exponential term,

$$
\widetilde{F}_g^{(2\boldsymbol{e}_1,1)} = \mathrm{e}^{4\cdot\frac{1}{2}(\partial_z A)^2\left(S^{zz} - S^{zz}_{[1],\mathrm{hol}}\right)} \sum_{k=0}^{3g} \widetilde{p}^{(2\boldsymbol{e}_1)}_{4;g,k}(z)[A, \partial_z A, \partial_z^2 A] \, (S^{zz})^k . \qquad \text{(C.18)}
$$

We have computed these free energies up to order $g = 8$. They satisfy a similar symmetric property as (C.8) when the sign of A is changed.

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