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GEOMETRIC CONSEQUENCES

OF

INTRINSIC AND EXTRINSIC

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Introduction

When studying the geometric properties of a semi–Riemannian manifold, the starting point usually comes from some invariants of the metric structure. Among those invariants, the curvature tensor is perhaps the simplest and most natural one. In the words of R. Osserman [109]

> The notion of curvature is one of the central concepts of differential geometry; one could argue that it is the central one, distinguishing the geometrical core of the subject from those aspects that are analytical, algebraic or topological. In the words of M. Berger, curvature is the "Number 1 Riemannian invariant and the most natural. Gauss and then Riemann saw it instantly".

Curvature, however, can be studied from several points of view. On the one hand, an essential problem in differential geometry is to relate properties of the curvature tensor to the underlying geometry of the manifold. Another point of view is to consider different kinds of objects naturally associated with the metric structure of the manifold and relate the curvature of the manifold to the properties of these natural constructions.

When dealing with a complicated object such as the curvature tensor, it is interesting to decompose it in more elementary constituents. Usually, these smaller parts give a simplified picture and a deeper insight into the whole problem. In Chapter 2 we show that the curvature tensor may be decomposed in terms of some simple algebraic curvature tensors. This is of special importance when considering Osserman–like problems.

Furthermore, the fact that the whole curvature tensor is very difficult to handle derived the investigation to the consideration of geometric objects naturally associated with the curvature. Typical examples are the sectional curvature, the Ricci tensor or the scalar curvature. Part I of this thesis fits into this philosophy. Among the different operators that can be defined from the curvature tensor, we are specially interested in the Jacobi operator, which encodes important geometric information and whose properties strongly influence the underlying geometry of the manifold. The Jacobi operator and Jacobi vector field theory are important tools in semi–Riemannian geometry. They provide a good description of curvature, behavior of geodesics and geometry of certain kinds of submanifolds. Thus, understanding the Jacobi operator of a semi–Riemannian manifold allows us to characterize the geometry of the manifold in several cases. Chapter 3 of this thesis is devoted to the investigation of the Jacobi operator in relation to the so–called Osserman problem. In Chapter 3 we focus on the Osserman problem in dimension four. Our main goal is to show the existence of Osserman metrics whose Jacobi operators are non–nilpotent and non–diagonalizable. This answers in the negative a conjecture on the non–existence of such manifolds. Moreover, a complete local description of any such metrics is given.

As it was stated above, another approach is to enlighten our understanding of a manifold by investigating the relation between curvatures of geometric objects naturally associated with the metric structure of the manifold and the curvature of the manifold itself. Examples of these structures are small geodesic spheres, geodesic disks and tubes around significant submanifolds.

Part II of this thesis is devoted to the study of some of the previously mentioned objects. In particular we study scalar curvature invariants of geodesic spheres in Chapter 4. Scalar curvature invariants are of main interest and many important geometries can be characterized in terms of these functions. We consider them in relation to geodesic spheres. In this chapter, we integrate the scalar curvature invariants of geodesic spheres and disks, obtaining the first terms in their power series expansions as a function of the radius. This leads to some characterizations of the two–point homogeneous spaces among Riemannian manifolds with adapted holonomy.

Inspired by the construction of geodesic disks in Riemannian geometry, we define geodesic celestial spheres in the Lorentzian setting. It turns out that this family of objects is adapted to the consideration of volume comparison results in the Lorentzian framework, which suffered from a lack of geometric constructions analogous to the Riemannian geodesic spheres and tubes. Chapter 5 is devoted to the investigation of volume properties of geodesic celestial spheres as well as their total scalar curvatures. This allows us to characterize isotropic Lorentzian manifolds.

Geodesic spheres and tubes are somehow level sets of the Riemannian distance function and hence they are closely related to the metric structure. Other objects in Riemannian manifolds which are related to the metric structure are those submanifolds invariant under the isometries of the ambient manifold. Orbits of cohomogeneity one actions are examples of this situation. Furthermore, a principal orbit of a cohomogeneity one action is geometrically a tube around a singular orbit of that action. This involves again the Riemannian distance function and the Jacobi operator, which is the main tool for calculating the geometry of geodesic spheres and tubes.

The geometry of orbits of cohomogeneity one actions is more interesting from the extrinsic point of view. Thus, it is the second fundamental form what we study in this case. Part III of this work is devoted to the investigation of real hypersurfaces with constant principal curvatures in the complex hyperbolic space. Orbits of cohomogeneity one actions are the main candidates for these hypersurfaces and are the only known examples so far. In Chapter 6 we study the shape operator of the orbits of cohomogeneity one actions on the complex hyperbolic space. We take advantage of this study and give in Chapter 7 a complete classification of real hypersurfaces with three distinct constant principal curvatures.

Chapter 1

Preliminaries and conventions

We introduce some of the basic notions in semi–Riemannian geometry. The notations and conventions described in this chapter are used throughout this monograph unless otherwise stated. This concepts can be found in most of the introductory books to Riemannian and semi–Riemannian geometry. Well–known references are for example [108], [114], [116].

In Section 1.1 we introduce the concept of semi–Riemannian manifold and provide our sign convention for the curvature tensor. In Section 1.2 we briefly state a few properties of geodesics and the semi–Riemannian exponential map. Section 1.3 is devoted to the description of some facts about submanifold geometry. We finish this chapter with a brief overview of some special kinds of manifolds which are used later in this work. This is accomplished in Section 1.4.

1.1 Semi–Riemannian manifolds

Let M be an n–dimensional differentiable manifold of class C^∞. Throughout this thesis all manifolds are assumed to satisfy the second countability axiom. Thus, all manifolds are para–compact. For each $m \in M$ we denote by $T_m M$ the tangent space of M at m. The tangent bundle is denoted by TM and we write $\Gamma(TM)$ for the module of sections of TM. As usual, an element in $\Gamma(TM)$ is called a smooth vector field.

A symmetric bilinear tensor ω in a vector space is said to be non–degenerate if $\omega(x, y) = 0$ for all y implies $x = 0$. Any non–degenerate symmetric bilinear tensor in a vector space is linearly congruent to a diagonal matrix $diag(1, \overset{r}{\cdots}, 1, -1, \overset{s}{\cdots}, -1)$. The pair of numbers (r, s) is called the *signature* of the tensor.

A *semi–Riemannian manifold* is a pair (M, g) where M is manifold and g is a non–degenerate symmetric covariant bilinear tensor field of type $(0, 2)$ and constant signature. Then, each tangent space is equipped with a non–degenerate symmetric bilinear tensor g_m. If the signature of g_m is (r, s), then (M, g) is said to have *signature* (r, s).

If a semi–Riemannian manifold M has signature $(n, 0)$, then M is called a *Riemannian manifold*. If the signature is $(n-1, 1)$, M is a *Lorentzian manifold*. Riemannian manifolds are the direct generalization of Gauss' theory of surfaces and Lorentzian manifolds appear in relation to the theory of general relativity.

While there is a natural way to differentiate smooth functions on a smooth manifold, there is no such natural way to differentiate smooth vector fields. The theory of connections studies the various possibilities for such a differentiation process. In a semi–Riemannian manifold (M, g) there is a unique torsion–free metric connection which is determined by the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X), \end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. This connection is known as the *Levi–Civita connection* or covariant derivative. The Levi–Civita connection acts as a tensor derivation on smooth vector fields in the usual way.

The most important concept of semi–Riemannian geometry is curvature. There are several kinds of curvature of great interest. All of them can be obtained from the *Riemann curvature tensor* which we define with the following sign convention:

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

We also define the $(0,4)$ Riemannian curvature tensor as $R_{XYVW} = g(R_{XY}V, W)$.

The Riemannian curvature tensor satisfies the following algebraic properties

$$R_{XYVW} = -R_{YXVW} = -R_{XYWV} = R_{VWXY},$$

$$R_{XYVW} + R_{YVXW} + R_{VXYW} = 0.$$

The last equality is known as the algebraic Bianchi identity. The curvature tensor also satisfies the differential Bianchi identity

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0.$$

Among all the possible curvatures that can be defined in a semi–Riemannian manifold we emphasize the Ricci tensor and the scalar curvature. The *Ricci tensor*, ρ_{XY}, is defined as the trace of the linear map $Z \mapsto R_{XZ}Y$. The algebraic identities of the curvature tensor show that the Ricci tensor is a self–adjoint bilinear map. The *scalar curvature* τ is the smooth function on M obtained by contracting the Ricci tensor.

1.2 Geodesics and the exponential map

Let $c : I \subset \mathbb{R} \to M$ be a smooth curve in a semi–Riemannian manifold. We denote by $c'(t)$ the tangent vector of c at t. The notion of covariant derivative can be defined along a curve. Such covariant derivative along curves maps smooth vector fields along c to smooth vector fields along c. Let X be a vector field along c. We denote by $X'(t)$ the covariant derivative of X with respect to $c'(t)$ at $c(t)$.

A smooth curve c is called a *geodesic* if it satisfies $c'' = \nabla_{c'}c' = 0$. Geodesics arise in Riemannian geometry as the curves which minimize distance between two given points. These curves do not exist in general but if they do (for example when the two points are sufficiently close), they are the solutions of the above variational problem. Geodesics can also be seen as curves with zero acceleration. This interpretation makes sense in the general semi–Riemannian setting.

The condition $c'' = 0$, when written in coordinates, translates into a system of second order differential equations. The basic theory of differential equations implies that, for each point $m \in M$ and each tangent vector $v \in T_mM$, there exists a unique maximal geodesic $c : I \subset \mathbb{R} \to M$ such that $c(0) = m$ and $c'(0) = v$. This maximal geodesic is often denoted by c_v.

A more general theorem of ordinary differential equations implies that geodesics vary in a differentiable way with respect to the initial conditions. Namely, there exists an open set $\mathcal{U}$ with $M \subset \mathcal{U} \subset TM$ such that the map

$$\begin{array}{rcl} \exp : \mathcal{U} & \longrightarrow & M \\ v & \mapsto & \exp(v) = c_v(1), \end{array}$$

is well–defined and differentiable. This map is called the *exponential map* of (M, g). Taking the fiber at a point we have the exponential map at a point.

Let $m \in M$. The *exponential map* at $m \in M$ is given by $\exp_m(v) = c_v(1)$ for any $v \in T_mM$. Such a map is defined in a star–shaped neighborhood of $0 \in T_mM$. The exponential map is a differentiable map and $\exp_{m*0}$ is the identity map of T_mM if we identify T_0T_mM with T_mM. Therefore, there exist an open neighborhood $\mathfrak{U}$ of $o \in T_mM$ and a neighborhood $\mathfrak{V}$ of $m \in M$ such that $\exp_m : \mathfrak{U} \to \mathfrak{V}$ is a diffeomorphism. A neighborhood $\mathfrak{V}$ as above is called a normal neighborhood of m.

Let $x \in T_mM$ be a tangent vector. We define $R_x : T_mM \to T_mM$ as $R_x(y) = R_{xy}x$. The algebraic identities of the Riemannian curvature tensor imply that R_x is a self–adjoint map and $R_x(x) = 0$. Hence, it can be restricted to $R_x : x^\perp \to x^\perp$, where $x^\perp$ denotes the orthogonal complement of the real span of x. This operator is known as the *Jacobi operator*.

The Jacobi operator turns out to be very important in semi–Riemannian geometry. We present now one of its applications. Let $c : I \subset \mathbb{R} \to M$ be a geodesic parametrized by arc length. A vector field X along c is called a *Jacobi vector field* if it satisfies the linear second order differential equation

$$X'' + R_{c'}(X) = 0,$$

which is known as the *Jacobi equation*. Basic theory of differential equations implies that Jacobi vector fields are defined in the whole interval I. Moreover, the Jacobi vector fields along a geodesic form a $2n$–dimensional vector space. Thus, any Jacobi vector field is determined by the initial values $X(0)$ and $X'(0)$.

There is an interesting interplay between Jacobi vector fields and geodesic variations. A *variation* of a curve $c : I \to M$ is a differentiable map $F : I \times (-\epsilon, \epsilon) \to M$ such that

$F(s,0) = c(s)$ for all s. For fixed s_0 and t_0, the curve $F(s_0, \cdot)$ is called transversal and the curve $F(\cdot, t_0)$ longitudinal. A variation is called a *geodesic variation* if every longitudinal curve is a geodesic. The variational vector field of F along c is the vector field X such that $X(s)$ is the velocity of the transversal curve trough $c(s)$.

Jacobi vector fields arise geometrically as the variational vector fields of a geodesic variation. Hence, a Jacobi vector field measures the infinitesimal behavior of nearby geodesics. Jacobi vector fields can be also used to describe the differential of the exponential map. If X is a Jacobi vector field along the geodesic c with $X(0) = 0$ and $X'(0) = v$ then

$$X(s) = \exp_{c(0)*sc'(0)}(sv),$$

for all the values of s along the geodesic c.

1.3 Submanifold geometry

Let $(\bar{M}, g)$ be a semi–Riemannian manifold and M an embedded submanifold. The restriction of g to M provides a symmetric bilinear tensor field on M. However, this tensor field can be degenerate. When it is not, that is, when M is itself a semi–Riemannian manifold, M is called a *semi–Riemannian submanifold* of $\bar{M}$. We follow [13] and [108].

The *normal bundle* of M, that is, the bundle of vectors orthogonal to the tangent space of M, is denoted by $T^\perp M$. By $\Gamma(T^\perp M)$ we denote the module of all normal vector fields to M. A canonical isomorphism holds at each point $m \in M$, namely, $T_m\bar{M} = T_m M \oplus T_m^\perp M$. Given a vector field X of $\bar{M}$ along M we denote by $X^\top$ the orthogonal projection of X onto TM and by $X^\perp$ the orthogonal projection onto $T^\perp M$.

If V is a vector space with inner product g and $W \subset V$ is a vector subspace, we denote by $V \ominus W$ the orthogonal complement of W in the inner product vector space V. For example, with the above notation we have $T_m^\perp M = T_m\bar{M} \ominus T_m M$. If E and F are two vector subbundles of the tangent bundle of $\bar{M}$ such that $F \subset E$, we denote by $E \ominus F$ the vector subbundle such that at each point m we have $(E \ominus F)_m = E_m \ominus F_m$. In particular, $T^\perp M = T\bar{M} \ominus TM$.

The definition of the Riemannian curvature tensor can be given for any Riemannian manifold. The curvature tensor is said to be an intrinsic geometric invariant. The intrinsic geometry of both $\bar{M}$ and M may be studied. Nonetheless, one can also study the geometry of M in relation to the geometry of $\bar{M}$. This is the extrinsic geometry of M. The extrinsic geometry of a submanifold is encoded in its second fundamental form.

Let us denote by $\bar{\nabla}$ and $\bar{R}$ the Levi–Civita connection and the Riemannian curvature tensor of $\bar{M}$, respectively, and by ∇ and R the corresponding objects in M. When studying submanifolds, this convention is assumed throughout this memory unless otherwise stated. The *second fundamental form* of M is defined by the Gauss formula

$$\nabla_X Y = \bar{\nabla}_X Y + II(X,Y)$$

for any $X, Y \in \Gamma(TM)$. Hence, $II(X,Y) = -(\bar{\nabla}_X Y)^\perp$. Let $\xi \in \Gamma(T^\perp M)$ be a unit normal vector field. The *shape operator* of M associated with ξ is the self–adjoint operator on M

defined by $g(S_\xi X, Y) = g(I\!I(X,Y),\xi)$, where $X, Y \in \Gamma(TM)$. Moreover, denote by $\nabla^\perp$ the normal connection of M, that is, $\nabla^\perp_X \xi = (\bar\nabla_X \xi)^\perp$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$. Then we have the Weingarten formula

$$\bar\nabla_X \xi = S_\xi X + \nabla^\perp_X \xi.$$

The relation between the Riemannian curvature tensors of $\bar M$ and M is given by means of the second fundamental form. This relation is known as the *Gauss equation*:

$$\bar R_{XYVW} = R_{XYVW} - g(I\!I(X,V), I\!I(Y,W)) + g(I\!I(X,W), I\!I(Y,V)).$$

The *Codazzi equation* is also of great important in our work

$$(\bar R_{XY} Z)^\perp = (\nabla^\perp_X I\!I)(Y,Z) - (\nabla^\perp_Y I\!I)(X,Z),$$

where the covariant derivative of the second fundamental form is given by

$$(\nabla^\perp_X I\!I)(Y,Z) = \nabla^\perp_X I\!I(Y,Z) - I\!I(\nabla_X Y, Z) - I\!I(Y, \nabla_X Z).$$

For the sake of completeness we also give the *Ricci equation*

$$\bar R_{XY\xi\eta} = g(R^\perp_{XY}\xi,\eta) + g([S_\xi, S_\eta]X, Y),$$

where $X, Y \in \Gamma(TM)$, $\xi, \eta \in \Gamma(T^\perp M)$ and $R^\perp$ is the curvature tensor of the normal vector bundle of M defined by $R^\perp_{XY}\xi = \nabla^\perp_{[X,Y]}\xi - [\nabla^\perp_X, \nabla^\perp_Y]\xi$.

We say that a submanifold is *totally geodesic* if its second fundamental form vanishes, $I\!I = 0$. This is equivalent to saying that every geodesic in M is also a geodesic in $\bar M$. If M is complete and totally geodesic we have that $M = \exp_m(T_m M)$ for any $m \in M$.

A submanifold is said to be *umbilical* if there exists a constant λ such that $I\!I = \lambda\, g$. Clearly, if $\lambda = 0$, then M is totally geodesic.

The *mean curvature vector* H of a semi–Riemannian submanifold is defined as the trace of the second fundamental form. Hence, with respect to a local orthonormal basis $\{E_i\}$ of TM we may write $H = \sum_i g(E_i, E_i) I\!I(E_i, E_i)$.

A submanifold is said to be *minimal* if and only if its mean curvature vector vanishes. Minimal submanifolds appear in a natural way as the critical points of the volume functional and they are a topic of current interest in differential geometry.

We say that M is a *spherical manifold* or an extrinsic sphere if M is umbilical and its mean curvature vector is parallel with respect to the normal connection of M, that is, $I\!I = \lambda\, g$ for some constant λ and $\bar\nabla^\perp H = 0$.

An umbilical submanifold of a space of constant curvature is also spherical. Umbilical submanifolds of spaces with constant curvature have been classified. See [13] for a more detailed discussion.

Assume now that M is a hypersurface of $\bar M$, that is, an embedded submanifold of codimension one. Then, up to sign, there exists a unique unit normal vector field $\xi \in T^\perp M$.

We write $\epsilon = g(\xi, \xi) \in \{-1, 1\}$. Hence the second fundamental form II is a multiple of ξ. We define the *scalar second fundamental form* σ of M by the equality $II(X,Y) = \epsilon\, \sigma(X,Y)\xi$ for $X, Y \in \Gamma(TM)$, that is, $\sigma(X,Y) = g(II(X,Y), \xi)$.

We denote by $S = S_\xi$ the shape operator with respect to ξ. With respect to the scalar second fundamental form we have $g(SX,Y) = \sigma(X,Y)$. The Gauss formula and the Weingarten equation can be written as

$$\nabla_X Y = \bar{\nabla}_X Y + \epsilon\, g(SX,Y)\xi \qquad \text{and} \qquad \bar{\nabla}_X \xi = SX.$$

Then, the Gauss and Codazzi equation reduce to

$$\bar{R}_{XYVW} = R_{XYVW} - \epsilon\, g(SX,V)g(SY,W) + \epsilon\, g(SX,W)g(SY,V),$$

$$(\nabla_X S)Y - (\nabla_Y S)X = -\bar{R}_{XY}\xi,$$

whereas the Ricci equation does not give further information for hypersurfaces.

The mean curvature vector H is proportional to the vector ξ. We define the *scalar mean curvature* h by the equation $H = h\,\xi$.

We say that $\lambda : M \to \mathbb{R}$ is a *principal curvature* of M (associated with ξ) if there exists a vector field $X \in \Gamma(TM)$ such that $SX = \lambda X$. If $\bar{M}$ is a Riemannian manifold, the shape operator S is diagonalizable at every point because it is a self–adjoint map and the metric is positive definite.

If λ is a principal curvature we denote by $T_\lambda(p)$ the eigenvector space of $\lambda(p)$ and call it the *principal curvature space* associated with $\lambda(p)$. If $X \in T_\lambda(p)$, $X \neq 0$ we say that X is a *principal curvature vector* of λ at p. We emphasize here that, in general, the principal curvature spaces associated with a principal curvature λ do not always have the same dimension.

A connected hypersurface is said to have *constant principal curvatures* if the shape operator is diagonalizable and its eigenvalues are the same at every point. In this case the principal curvature spaces associated with an eigenvalue λ have the same dimension at any point. We denote by m_λ the dimension of any of the vector spaces $T_\lambda(p)$ and call this number the *multiplicity* of λ. By T_λ we denote the distribution on M formed by the principal curvature spaces of λ and by $\Gamma(T_\lambda)$ we denote the set of all sections of T_λ, that is, the vector fields $X \in \Gamma(TM)$ such that $SX = \lambda X$.

1.4 Some special classes of semi–Riemannian manifolds

We introduce a few kinds of manifolds which will be of special relevance in this thesis. The description is not intended to be thorough and we restrict ourselves to those types which are going to be used later. A wider study of structures on manifolds can be found for example in [134].

1.4.1 Two–point homogeneous spaces

A connected Riemannian manifold M is called *two–point homogeneous* if the isometry group of M acts transitively on equidistant pairs of points. This means that for any $p_1, p_2, q_1, q_2 \in M$ with $d(p_1, q_1) = d(p_2, q_2)$, where d is the Riemannian distance function of M, there is an isometry Φ of M such that $\Phi(p_1) = p_2$ and $\Phi(q_1) = q_2$. This definition clearly implies that a two–point homogeneous space is homogeneous and complete.

Let M be a semi–Riemannian manifold and $p \in M$. The manifold M is said to be *isotropic at* p if the isotropy group of the isometry group of M at p acts transitively on the unit pseudo–sphere bundle. The manifold is called *isotropic* if it is isotropic at every point, or equivalently, if for each point $p \in M$ and any non–null vectors $x, y \in T_m M$ with $g(x, x) = g(y, y)$ there exists an isometry Φ of M such that $\Phi(p) = p$ and $\Phi_{*p}(x) = \Phi_{*p}(y)$. The notion of locally isotropic manifold can be defined in an analogous way.

If M is a Riemannian manifold, then M is two–point homogeneous if and only if it is isotropic. Any two–point homogeneous space is symmetric [122]. Indeed, a simply connected two–point homogeneous space is a flat space, an irreducible symmetric space of rank one or one of its non–compact duals. Hence, a simply connected two–point homogenous space is isometric to one of the following manifolds:

(i) The Euclidean space $\mathbb{R}^n$.

(ii) The sphere $\mathbb{S}^n = SO(n+1)/SO(n)$, the real projective space $\mathbb{R}P^n = SO(n+1)/O(n)$ or the real hyperbolic space $\mathbb{R}H^n = SO^0(1, n)/SO(n)$.

(iii) The complex projective space $\mathbb{C}P^n = SU(n + 1)/U(n)$ or the complex hyperbolic space $\mathbb{C}H^n = SU(1, n)/S(U(1)U(n))$.

(iv) The quaternionic projective space $\mathbb{H}P^n = Sp(n+1)/Sp(1)Sp(n)$ or the quaternionic hyperbolic space $\mathbb{H}H^n = Sp(1, n)/Sp(1)Sp(n)$.

(v) The Cayley projective plane $\mathbb{O}P^2 = F_4/Spin(9)$ or the Cayley hyperbolic plane $\mathbb{O}H^2 = F_4^{-20}/Spin(9)$.

The examples in (ii) are called real space forms, the examples in (iii) are called complex space forms and the examples in (iv) are called quaternionic space forms. These three constructions can be generalized to the general semi–Riemannian setting. We briefly describe them in what follows.

Indefinite real space forms

A semi–Riemannian manifold (M^n, g) of signature (r, s) is called a *real space form* if (M, g) has constant sectional curvature. If (M, g) is a real space form of constant curvature $\lambda \in \mathbb{R}$, the curvature tensor of (M, g) is given by

$$R_{xy}z = \lambda\big(g(x, z)y - g(y, z)x\big),$$

for all $x, y, z \in TM$.

A complete and simply connected real space form is isometric to

$$\mathbb{R}P_s^n = SO(s, r+1)/O(s, r), \quad \mathbb{R}H_s^n = SO^0(s+1, r)/SO(s, r) \quad \text{or} \quad \mathbb{R}_s^n$$

according to whether the sectional curvature is positive, negative or zero [133].

Indefinite complex space forms

Let (M, J) be an almost complex manifold with almost complex structure J, that is, J is a $(1,1)$–tensor field on M satisfying $J^2 = -\,\mathrm{Id}$. A semi–Riemannian metric tensor g of signature $(2r, 2s)$ is said to be Hermitian if $g(JX, Y) + g(X, JY) = 0$ for all $X, Y \in \Gamma(TM)$. If the metric tensor is integrable, that is, if $[J, J] = 0$ where $[J, J](X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$, then J is said to be a complex structure.

The triple (M^{2n}, g, J) is said to be a *Kähler manifold* if J is a complex structure and the 2–form $\Omega(X, Y) = g(X, JY)$ is closed. This couple of conditions can be equivalently described by $\nabla J = 0$, where ∇ is the Levi–Civita connection of g.

A plane π is called *holomorphic* if it remains invariant under the complex structure ($J\pi \subset \pi$), and the *holomorphic sectional curvature* is defined as the restriction of the sectional curvature to non–degenerate holomorphic planes. A Kähler manifold (M, g, J) is called a *complex space form* if (M, g, J) is of constant holomorphic sectional curvature. If this constant is μ, then the curvature tensor of (M, g, J) is given by,

$$R_{xy}z = \frac{\mu}{4}\Big(g(x,z)y - g(y,z)x + g(Jx,z)Jy - g(Jy,z)Jx + 2g(Jx,y)Jz\Big)$$

for all $x, y, z \in TM$. Let $z \in TM$ be a unit vector. The Jacobi operator of z is given by

$$R_z = \begin{cases} \mu\, g(z,z)\ \mathrm{Id}, & \text{if} \quad z \in \mathbb{R}Jz, \\ \frac{\mu}{4} g(z,z)\ \mathrm{Id}\,, & \text{if} \quad z \in \mathbb{C}z^{\perp}. \end{cases}$$

The model spaces of non–zero constant holomorphic sectional curvature are given by the symmetric spaces

$$\mathbb{C}P_s^n = SU(s, r+1)/U(s, r) \qquad \text{and} \qquad \mathbb{C}H_s^n = SU(s+1, r)/S(U(s+1)U(r)).$$

Indefinite quaternionic space forms

An *almost quaternionic manifold* is a manifold M equipped with a 3–dimensional vector bundle $\mathbb{Q}$ of $(1,1)$–tensor fields on M such that there exists a local basis $\{J_1, J_2, J_3\}$ of $\mathbb{Q}$ satisfying $J_i^2 = -\,\mathrm{Id}$, $i = 1, 2, 3$, and $J_i J_j = J_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Such a local basis $\{J_1, J_2, J_3\}$ is called a *canonical local basis* of $\mathbb{Q}$ and $\mathbb{Q}$ is referred to as an *almost quaternionic structure* on M. A semi–Riemannian metric tensor g of signature $(4r, 4s)$ is said to be adapted to the almost quaternionic structure $\mathbb{Q}$ if $g(\phi X, Y) + g(X, \phi Y) = 0$ for all $\phi \in \mathbb{Q}$ and $X, Y \in \Gamma(TM)$.

Let $(M, g, \mathbb{Q})$ be an almost quaternionic manifold and $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathbb{Q}$. For each $i \in \{1, 2, 3\}$, we put $\Phi_i(X, Y) = g(X, J_iY)$, where $X, Y \in \Gamma(TM)$. Then, Φ_i is a locally defined 2–form such that $\Omega = \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3$ gives rise to a globally defined 4–form on M. A quaternionic metric structure $(g, \mathbb{Q})$ is said to be *Kähler* if Ω is parallel (or equivalently, if $\mathbb{Q}$ is parallel) with respect to the Levi–Civita connection ∇ of g.

Let $(M, g, \mathbb{Q})$ be a quaternionic Kähler manifold. Then M has signature $(4r, 4s)$. Any vector $x \in T_pM$ determines a 4–dimensional subspace $\mathbb{Q}(x) = \mathbb{R}x \oplus \mathbb{R}J_1x \oplus \mathbb{R}J_2x \oplus \mathbb{R}J_3x$ which remains invariant under the action of the quaternionic structure. We call it the $\mathbb{Q}$*–section* determined by x. If the sectional curvature of planes in $\mathbb{Q}(x)$ is a constant $\nu(x)$, where $x \in TM$ is non–null, we call this constant $\nu(x)$ the *quaternionic sectional curvature* of (M, g) with respect to x.

A quaternionic Kähler manifold $(M, g, \mathbb{Q})$ is called a *quaternionic space form* if $(M, g, \mathbb{Q})$ is of constant quaternionic sectional curvature. Then its curvature tensor is given by

$$R_{xy}z = \frac{\nu}{4}\Big\{g(x,z)y - g(y,z)x + \sum_{i=1}^{3}\Big(g(J_ix,z)J_iy - g(J_iy,z)J_ix + 2g(J_ix,y)J_iz\Big)\Big\},$$

for all $x, y, z \in TM$, and where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathbb{Q}$.

A non–flat quaternionic space form is isometric to one of the following symmetric spaces

$$\mathbb{H}P^n_s = Sp(s, r+1)/Sp(1)Sp(s, r) \qquad \text{or} \qquad \mathbb{H}H^n_s = Sp(s+1, r)/Sp(1)Sp(s, r).$$

1.4.2 Para–complex space forms

In addition to the well–known examples of semi–Riemannian manifolds described above, there are some other examples which have no Riemannian analog. However, they may be considered as a kind of real version of complex manifolds.

A *para–Kähler manifold* is a symplectic manifold locally diffeomorphic to a product of Lagrangian submanifolds. Such a product induces a decomposition of the tangent bundle TM into a Whitney sum of Lagrangian subbundles L and L', that is, $TM = L \oplus L'$. By generalizing this definition, an *almost para–Hermitian manifold* is defined to be an almost symplectic manifold (M, Ω) whose tangent bundle splits into a Whitney sum of Lagrangian subbundles. This implies that the $(1, 1)$–tensor field J defined by $J = \sigma_L - \sigma_{L'}$ is an almost para–complex structure ($J^2 = \mathrm{Id}$) on M such that $\Omega(JX, JY) = -\Omega(X, Y)$ for all $X, Y \in \Gamma(TM)$, where σ_L and σ'_L are the projections of TM onto L and L', respectively. The 2–form Ω induces a non–degenerate $(0, 2)$–tensor field g on M defined by $g(X, Y) = \Omega(X, JY)$, where $X, Y \in \Gamma(TM)$. Now, by using the relation between the almost para–complex and the almost symplectic structures on M, it follows that g defines a semi–Riemannian metric tensor of signature (n, n) on M and $g(JX, Y) + g(X, JY) = 0$, where $X, Y \in \Gamma(TM)$. The special significance of the para–Kähler condition is equivalently stated in terms of the parallelizability of the para–complex structure with respect to the Levi–Civita connection of g, that is, $\nabla J = 0$ [35].

A plane π is called *para–holomorphic* if it is left–invariant by the action of the para–complex structure J, that is, $J\pi \subset \pi$. The *para–holomorphic sectional curvature* is defined by the restriction of the sectional curvature to para–holomorphic non–degenerate planes. A para–Kähler manifold (M, g, J) is called a *para–complex space form* if (M, g, J) is of constant para–holomorphic sectional curvature. Hence, the curvature tensor of (M, g, J) is determined by

$$R_{xy}z = \frac{\mu}{4}\Big(g(x,z)y - g(y,z)x - g(Jx,z)Jy + g(Jy,z)Jx - 2g(Jx,y)Jz\Big),$$

for all $x, y, z \in T_pM$ and some constant $\mu \in \mathbb{R}$. Then, the Jacobi operator with respect to a unit vector $z \in TM$ is given by

$$R_z = \begin{cases} \mu\, g(z,z)\ \mathrm{Id}, & \text{if} \quad z \in \mathbb{R}Jz, \\ \frac{\mu}{4} g(z,z)\ \mathrm{Id}\,, & \text{if} \quad z \in (\mathbb{R}z \oplus \mathbb{R}Jz)^{\perp}. \end{cases}$$

Non–flat complete and simply connected para–complex space forms are isometric to the symmetric spaces $SL(n,\mathbb{R})/SL(n-1,\mathbb{R}) \times \mathbb{R}$.

1.4.3 Einstein manifolds and k–stein manifolds

A semi–Riemannian manifold (M^n, g) is called an *Einstein manifold* if the Ricci tensor is proportional to the metric, that is, if there exists a constant $\lambda \in \mathbb{R}$ such that $\rho = \lambda\, g$. Taking traces we easily see that $\lambda = \tau/n$ and hence the scalar curvature is constant. If $n > 2$ and there exists a function $f : M \to \mathbb{R}$ such that $\rho = f\, g$ then, the Schur lemma implies that f is constant and thus the manifold is Einstein. If a semi–Riemannian manifold M has dimension 2 or 3 then, M is Einstein if and only if M has constant sectional curvature.

A semi–Riemannian manifold is said to be k–stein, for $k \geq 1$, if there exists a constant λ such that $\operatorname{tr} R_x^k = \lambda g(x,x)^k$ for all $x \in TM$, where R_x^k is the k–power of the Jacobi operator. Note that a manifold is 1–stein if and only if it is Einstein.

We are specially interested in the 2–stein condition, which plays an important role in Part II. With respect to an orthonormal basis $\{e_i\}$ the 2–stein condition may be written as

$$\sum_{i,j=1}^{n} g(e_i, e_i) g(e_j, e_j) R^2_{xe_ixe_j} = \lambda\, g(x,x)^2.$$

A manifold M is said to be *super–Einstein* if

$$\sum_{i,j,k=1}^{n} g(e_i, e_i) g(e_j, e_j) g(e_k, e_k) R^2_{xe_ie_je_k} = \mu\, g(x,x),$$

for some constant μ. It was shown in [33] that 2–stein manifolds are super–Einstein although the converse is not true. For instance, irreducible symmetric spaces are super–Einstein but not necessarily 2–stein.

Part I

Geometric consequences of algebraic properties of the curvature tensor

A central problem in differential geometry is to relate algebraic properties of the curvature tensor to the underlying geometry of the manifold. From an algebraic point of view the space of algebraic curvature tensors on an n–dimensional vector space V is a vector space $\mathcal{R}(V)$ of dimension $n^2(n^2-1)/12$, which makes it very difficult to manipulate. Hence, the investigation focused many times on trying to find suitable bases or sets of generators allowing some simplifications. A typical example is the Singer–Thorpe basis in dimension four (see also [92] for higher dimensions).

Recently, the work of B. Fiedler [59] and P. Gilkey [68] showed the existence of nice sets of generators of $\mathcal{R}(V)$ constructed from symmetric and skew–symmetric bilinear forms, which seems to be useful in understanding some curvature conditions. Our approach to this problem, based on the use of the Nash embedding theorem and the possibility of realizing geometrically any algebraic curvature tensor, has two main advantages. The first one is that it allows us to obtain some sharper (although not optimal) estimates for the number of generators of $\mathcal{R}(V)$. Secondly, it shows that each algebraic curvature tensor can also be seen from an extrinsic point of view as the second fundamental form of a suitable embedding. All these discussions are carried out in Chapter 2.

Another purpose of this part is to study the influence of algebraic properties of natural operators associated with the curvature tensor on the manifold geometry. More precisely, our attention is mainly devoted to the investigation of the Jacobi operator by focusing on the structure of four–dimensional Osserman metrics.

A semi–Riemannian manifold is said to be Osserman if the eigenvalues of the Jacobi operators are independent of the direction and the base point. Since the group of local isometries of an isotropic space acts transitively on the unit pseudo–sphere bundles, it is clear that any isotropic space is Osserman. No other examples may exist in the Riemannian ($\dim \neq 16$) and Lorentzian settings but there exist non–symmetric and even non–locally homogeneous Osserman metrics in any signature (p, q) with $p, q \geq 2$.

Four–dimensional Osserman metrics are of particular interest. First of all, four is the first non–trivial dimension to be considered in the investigation of the Osserman problem (note that any Osserman metric is Einstein, and thus of constant sectional curvature in dimensions 2 and 3), and moreover, four is the lowest possible dimension which supports metrics of neutral non–Lorentzian signature, where the first non–symmetric Osserman metrics were discovered.

Due to curvature identities, for any non–null vector $x \in TM$, the Jacobi operator acts as a self–adjoint operator in $x^{\perp}$, which has induced metric of Lorentzian signature in the $(2,2)$ setting. Osserman metrics with diagonalizable Jacobi operators have been characterized by N. Blažić, N. Bokan and Z. Rakić [21], who also showed the non–existence of Osserman metrics in dimension four whose Jacobi operators have complex eigenvalues. However, the Lorentzian signature of $x^{\perp}$ supports two other possibilities corresponding to a double or triple root of the minimal polynomial of the Jacobi operators. The fact that all known examples in those situations have nilpotent Jacobi operators and that four–dimensional symmetric Osserman spaces have diagonalizable or two–step nilpotent Jacobi operators motivated a conjecture that Osserman metrics whose Jacobi operators are not diagonalizable must have nilpotent Jacobi operators.

Our purpose in Chapter 3 is to answer the above conjecture in the negative by showing explicit examples of Osserman metrics whose Jacobi operators are neither diagonalizable nor nilpotent. Finally, a complete description of such metrics is given in Section 3.3.

Chapter 2

Algebraic curvature tensors and natural operators

In this chapter we discuss some algebraic properties of the curvature tensor and its covariant derivatives. When studying curvature it is sometimes convenient to work in the algebraic setting. This often simplifies calculations and allows one to distinguish between purely geometric or topological properties and those properties which are imposed by the linear nature of most of the objects that can be defined in a manifold.

Section 2.1 is devoted to the study of algebraic curvature tensors. Geometric realizability turns this concept into a very powerful notion when studying manifolds where the curvature tensor verifies some algebraic property. Hence, it is interesting to be capable of decomposing the curvature tensor into more elementary parts which can be studied in an easier way. Theorems 2.3 and 2.4 contribute to this philosophy giving somehow an upper bound of the complexity of the curvature tensor. Some other results are given in relation to this decomposition of the curvature tensor.

Section 2.2 deals with certain natural operators that can be defined from the curvature tensor of a semi–Riemannian manifold. We give the basic definitions and results that will be used in the following chapter.

2.1 Algebraic curvature tensors

Let V be an n-dimensional vector space with an inner product g. An *algebraic curvature tensor* is a tensor $F \in \otimes^4(V^*)$ satisfying the algebraic identities of the Riemannian curvature tensor, that is,

$$F(x,y,v,w) = -F(y,x,v,w) = -F(x,y,w,v) = F(v,w,x,y),$$

$$F(x,y,v,w) + F(y,v,x,w) + F(v,x,y,w) = 0.$$

Let us denote by $\mathcal{R}(V)$ the vector space of algebraic curvature tensors of V. This vector space has dimension $n^2(n^2-1)/12$.

Given a symmetric bilinear form ϕ in V we define the algebraic curvature tensor F^ϕ by

$$F^\phi(x,y,v,w) = \phi(x,v)\phi(y,w) - \phi(y,v)\phi(x,w).$$

Now let ψ be a skew–symmetric bilinear form on V. Then, F^ψ defined by

$$F^\psi(x,y,v,w) = \psi(x,v)\psi(y,w) - \psi(y,v)\psi(x,w) - 2\psi(x,y)\psi(v,w)$$

is an algebraic curvature tensor.

We define $\mathcal{S}(V)$ as the span of all F^ϕ where ϕ is a symmetric bilinear tensor. Analogously we define $\mathcal{A}(V)$ as the span of all F^ψ where ψ is a skew–symmetric bilinear tensor. The following theorem was proved by B. Fiedler [59] using group representation theory and by P. Gilkey and R. Ivanova [68], [72] using linear algebra.

Theorem 2.1. *Let (V^n, g) be an n–dimensional vector space with an inner product g. Then, $\mathcal{R}(V) = \mathcal{S}(V) = \mathcal{A}(V)$.*

Now we turn our attention to the covariant derivative of the Riemannian curvature tensor. As before, we work in the algebraic setting. Let (V^n, g) be an inner product vector space. An *algebraic covariant derivative curvature tensor* F_1 is a tensor $F_1 \in \otimes^5(V^*)$ verifying both the algebraic identities of a Riemannian curvature tensor and the differential Bianchi identity, namely,

$$\begin{gathered} F_1(z,x,y,v,w) = -F_1(z,y,x,v,w) = -F_1(z,x,y,w,v) = F_1(z,v,w,x,y), \\ F_1(z,x,y,v,w) + F_1(z,y,v,x,w) + F_1(z,v,x,y,w) = 0, \\ F_1(z,x,y,v,w) + F_1(x,y,z,v,w) + F_1(y,z,x,v,w) = 0. \end{gathered}$$

We point out that the first entry of the tensor stands for derivation when considering the covariant derivative of the Riemannian tensor of a semi–Riemannian manifold. Let $\mathcal{R}_1(V)$ be the vector space of algebraic covariant derivative curvature tensors.

Let ϕ be a symmetric bilinear tensor and ϕ_1 a symmetric 3–linear tensor in V. Then, the tensor $F_1^{\phi,\phi_1} \in \otimes^5(V^*)$ defined by

$$\begin{aligned} F_1^{\phi,\phi_1}(z,x,y,v,w) &= \phi_1(z,x,v)\phi(y,w) + \phi(x,v)\phi_1(z,y,w) \\ &\quad -\phi_1(z,x,w)\phi(y,v) - \phi(x,w)\phi_1(z,y,w) \end{aligned}$$

is an algebraic covariant derivative curvature tensor. If one thinks of ϕ_1 as the symmetrized covariant derivative of ϕ, then F_1^{ϕ,ϕ_1} can be regarded, at least formally speaking, as the covariant derivative of F^ϕ.

Again, B. Fiedler used group representation theory to prove the analog of Theorem 2.1 for algebraic covariant derivative curvature tensors [59], [60].

Theorem 2.2. *Let (V^n, g) be an inner product vector space. Then the linear span of the tensors F_1^{ϕ,ϕ_1} coincides with the vector space of algebraic covariant derivative curvature tensors $\mathcal{R}_1(V)$.*

Theorem 2.1 (resp. Theorem 2.2) shows that algebraic curvature tensors (resp. algebraic covariant derivative curvature tensors) can be written as a linear combination of simple algebraic curvature tensors (resp. algebraic covariant derivative curvature tensors). In order to simplify the latter linear combinations further, it is interesting to find the minimum number of addends. We partially respond to the question.

Let us take $F \in \mathcal{R}(V)$ and $F_1 \in \mathcal{R}_1(V)$. We denote by $\mu(F)$ and $\mu_1(F_1)$ the minimum integer number so that there exist symmetric bilinear tensors ϕ_i and ψ_j, symmetric 3–linear tensors $\psi_{1,j}$ and constants $\lambda_i, \lambda_{1,j}$ such that

$$F = \sum_{i=1}^{\mu(F)} \lambda_i F^{\phi_i} \qquad \text{and} \qquad F_1 = \sum_{j=1}^{\mu_1(F_1)} \lambda_{1,j} F_1^{\psi_j, \psi_{1,j}}.$$

We define the following constants depending on the dimension

$$\mu(n) = \sup_{F \in \mathcal{R}(V)} \mu(F) \qquad \text{and} \qquad \mu_1(n) = \sup_{F_1 \in \mathcal{R}_1(V)} \mu(F_1),$$

where V is any inner product vector space of dimension n.

We give upper and lower bounds for these quantities in the following section.

2.1.1 Decomposition of algebraic curvature tensors

The proof of Theorem 2.1 as given in [68] or [72] is constructive and relies on basic linear algebra. By following that proof one may estimate the number of the distinct symmetric tensors needed to express a given algebraic curvature tensor. Let F be an algebraic curvature tensor and decompose it as $F = \sum_{r=1}^{\mu} \lambda_r F^{\phi_r}$. Choose an orthonormal basis $\{e_i\}$. Then ϕ_r belongs to one of the following:

(i) For $i < j$ we define $\phi(e_i, e_j) = \phi(e_j, e_i) = 1$, $\phi(e_a, e_b) = 0$ otherwise.

(ii) For $j \neq i \neq k, j < k$ one defines $\phi(e_i, e_j) = \phi(e_j, e_i) = \phi(e_i, e_k) = \phi(e_k, e_i) = 1$, $\phi(e_a, e_b) = 0$ otherwise.

(iii) For distinct i, j, k, l, one considers $\phi(e_i, e_k) = \phi(e_k, e_i) = \phi(e_j, e_l) = \phi(e_l, e_j) = 1$, $\phi(e_a, e_b) = 0$ otherwise.

A simple calculation shows that the number of different symmetric tensors ϕ needed to express any given algebraic curvature tensor is at most $n(n-1)(n^2-n+2)/8$. Our purpose is to provide an alternative proof of $\mathcal{R}(V) = \mathcal{S}(V)$ that gives a better (although not optimal) estimate [40].

Theorem 2.3. *Let (V^n, g) be an n–dimensional vector space with an inner product g. Then, for each algebraic curvature tensor $F \in \otimes^4(V^*)$ there exist at most $n(n+1)/2$ symmetric tensors ϕ on V such that F is a linear combination of the associated algebraic curvature tensors F^{ϕ}.*

Proof. Any algebraic curvature tensor F is geometrically realizable, that is, there exists a smooth manifold M and a metric g on M such that the curvature tensor of (M, g) at some point $m \in M$ is exactly F. More explicitly, there exists a linear isometry of inner product vector spaces $\Phi : (V, g) \to (T_m M, g_m)$ such that $F = \Phi^* R_m$, where R is the curvature tensor of (M, g). This can be achieved, for example, by defining the following metric in a neighborhood of the origin of $\mathbb{R}^n$, $g_{ij}(x^1, \dots, x^n) = \delta_{ij} - (1/3)\sum_{\alpha,\beta=1}^n F_{i\alpha j\beta} x^\alpha x^\beta$, where $F_{ijkl} = F(e_i, e_j, e_k, e_l)$, $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^n$ and δ denotes the Kronecker delta. Then, using the previous basis to identify $V = \mathbb{R}^n$, we get $R_m = F$ at the origin.

It follows from the Nash embedding theorem [102] that M can be isometrically embedded in $\mathbb{R}^{n+\kappa}$ for sufficiently large κ. As usual, let us denote by II the second fundamental form of the embedding. Let $\{e_1, \dots, e_\kappa\}$ be an orthonormal basis of the normal space $T_m^\perp M$. We define the symmetric bilinear tensors ϕ_i by $\phi_i(x, y) = g(II(x, y), e_i)$ for all $i \in \{1, \dots, \kappa\}$. Then, for any $x, y \in T_m M$ we have $II(x, y) = \sum_{i=1}^\kappa \phi_i(x, y) e_i$. Using the Gauss equation (note that $\bar{R} = 0$) and the above expression for the second fundamental form II we get

$$\begin{aligned} F(x,y,v,w) &= R_m(x,y,v,w) = g\left(II(x,v), II(y,w)\right) - g\left(II(x,w), II(y,v)\right) \\ &= \sum_{i=1}^{\kappa} \{\phi_i(x,v)\phi_i(y,w) - \phi_i(x,w)\phi_i(y,v)\} = \sum_{i=1}^{\kappa} F^{\phi_i}(x,y,v,w). \end{aligned}$$

In order to obtain the bound $\kappa = n(n+1)/2$, we note that the dimension in the Nash embedding theorem can be reduced provided that the manifold is analytic and the embedding is local [84]. □

We have a similar result for covariant derivative curvature tensors. See also [41].

Theorem 2.4. *Let (V^n, g) be an n–dimensional vector space with an inner product g. For any covariant derivative algebraic curvature tensor $F_1 \in \mathcal{R}_1(V)$ there exist at most $n(n+1)/2$ symmetric tensors $\phi_i \in \otimes^2(V^*)$ and $\phi_{1,i} \in \otimes^3(V^*)$ such that F_1 is a linear combination of the associated algebraic curvature tensors $F^{\phi_i, \phi_{1,i}}$.*

Proof. Again, we assume that F_1 is the covariant derivative of the Riemannian curvature tensor of certain Riemannian manifold M at some point m. For example this can be achieved by defining the metric in $\mathbb{R}^n$

$$g_{ij}(x^1, \dots, x^n) = \delta_{ij} - \frac{1}{6} \sum_{\alpha,\beta,\gamma=1}^{n} F_1(e_\alpha, e_\beta, e_i, e_\gamma, e_j) x^\alpha x^\beta x^\gamma,$$

with respect to some basis $\{e_i\}$ at the origin. By virtue of the Nash embedding theorem [102] we may assume that M is isometrically embedded in $\mathbb{R}^{n+\kappa}$ for some κ. Taking covariant derivatives in the Gauss equation, using the definition of the covariant derivative of the second fundamental form and the fact that $\bar{R} = 0$ we get

$$\begin{aligned} F_1(z,x,y,v,w) &= (\nabla_z R)_m(x,y,v,w) \\ &= g\left((\nabla_z^\perp II)(x,v), II(y,w)\right) + g\left(II(x,v), (\nabla_z^\perp II)(y,w)\right) \\ &\quad - g\left((\nabla_z^\perp II)(x,w), II(y,v)\right) - g\left(II(x,w), (\nabla_z^\perp II)(y,v)\right), \end{aligned}$$

for any tangent vectors $z, x, y, v, w \in V = T_mM$. Let $\{e_1, \dots, e_\kappa\}$ be an orthonormal basis of $T_m^\perp M$. We define the tensors $\phi_i \in \otimes^2(T_m^*M)$ and $\phi_{1,i} \in \otimes^3(T_m^*M)$ by $\phi_i(x,y) = g(I\!I(x,y), e_i)$ and $\phi_{1,i}(z,x,y) = g((\nabla_z^\perp I\!I)(x,y), e_i)$ for all $i \in \{1, \dots, \kappa\}$. As in the previous theorem, ϕ_i is symmetric and $I\!I(x,y) = \sum_i \phi_i(x,y)e_i$. For all $x, y, z \in T_mM$, $\phi_{1,i}(z,x,y) = \phi_{1,i}(z,y,x)$ by the symmetry of $I\!I$. Moreover, the Codazzi equation in $\mathbb{R}^n$ reads $(\nabla_z^\perp I\!I)(x,y) - (\nabla_x^\perp I\!I)(z,y) = 0$. Hence $\phi_{1,i}(z,x,y) = \phi_{1,i}(x,z,y)$ for all $x, y, z \in T_mM$. Altogether, this means that $\phi_{1,i}$ is a symmetric tensor for all $i \in \{1, \dots, \kappa\}$ and $(\nabla_z^\perp I\!I)(x,y) = \sum_i \phi_{1,i}(z,x,y)e_i$. Therefore, the above expression becomes

$$F_1(z,x,y,v,w) = \sum_{i=1}^{\kappa} F_1^{\phi_i,\phi_{1,i}}(z,x,y,v,w).$$

The bound $\kappa = n(n+1)/2$ can be obtained by taking a local embedding of M in a neighborhood of m as M may be supposed analytic [84]. □

Now, we turn our attention to the lower bounds of $\mu(n)$ and $\mu_1(n)$.

Let (V^n, g) be an n–dimensional inner product vector space, $F \in \mathcal{R}(V)$ and $F_1 \in \mathcal{R}_1(V)$. We define the curvature operators $\mathcal{K}_F$ and $\mathcal{K}_{F_1}$ associated with F and F_1 by the identities

$$\begin{aligned} g(\mathcal{K}_F(x,y)v, w) &= F(x,y,v,w), \\ g(\mathcal{K}_{F_1}(z,x,y)v, w) &= F_1(z,x,y,v,w), \end{aligned}$$

for arbitrary $z, x, y, v, w \in V$. Thus, once we fix $z, x, y \in V$, both $\mathcal{K}_F(x,y)$ and $\mathcal{K}_{F_1}(z,x,y)$ are endomorphisms of V.

Lemma 2.5. *Let (V^n, g) be an inner product vector space. Let $\phi \in \otimes^2(V^*)$ and $\phi_1 \in \otimes^3(V^*)$ be symmetric tensors and $z, x, y \in V$ arbitrary vectors. Then* $\operatorname{rank}\{\mathcal{K}_{F^\phi}(x,y)\} \leq 2$ *and* $\operatorname{rank}\{\mathcal{K}_{F_1^{\phi,\phi_1}}(z,x,y)\} \leq 2$.

Proof. Let Φ and Φ_1 be the associated self–adjoint endomorphism characterized by the identities $g(\Phi x, y) = \phi(x,y)$ and $g(\Phi_1(z)x, y) = \phi_1(z,x,y)$. Then

$$\begin{aligned} \mathcal{K}_{F^\phi}(x,y)v &= \big\{\phi(x,v)\,\Phi\big\}y - \big\{\phi(y,v)\,\Phi\big\}x, \\ \mathcal{K}_{F_1}^{\phi,\phi_1}(z,x,y)v &= \big\{\phi_1(z,x,v)\Phi + \phi(x,v)\Phi_1(z)\big\}y - \big\{\phi(y,v)\Phi_1(z) + \phi_1(z,y,v)\phi\big\}x, \end{aligned}$$

for any $z, x, y, v \in V$ and the result follows. □

Corollary 2.6. *Let (V^n, g) be an inner product vector space. Let $F \in \mathcal{R}(V)$ and $F_1 \in \mathcal{R}_1(V)$. Then, for any $z, x, y \in V$ we have the relation* $\operatorname{rank}\{\mathcal{K}_F(x,y)\} \leq 2\mu(F)$ *and* $\operatorname{rank}\{\mathcal{K}_{F_1}(z,x,y)\} \leq 2\mu_1(F_1)$.

Proof. By definition of $\mu(F)$ and $\mu_1(F_1)$, we may write

$$F = \sum_{i=1}^{\mu(F)} \alpha_i F^{\phi_i} \qquad \text{and} \qquad F_1 = \sum_{j=1}^{\mu_1(F_1)} \beta_j F_1^{\psi_j,\psi_{1,j}}$$

for certain symmetric tensors $\phi_i, \psi_j \in \otimes^2(V^*)$, $\psi_{1,j} \in \otimes^3(V^*)$ and constants $\alpha_i, \beta_j \in \mathbb{R}$. Clearly, we have $\mathcal{K}_F = \sum_i \alpha_i \mathcal{K}_{F^{\phi_i}}$ and $\mathcal{K}_{F_1} = \sum_j \beta_j \mathcal{K}_{F_1^{\psi_j,\psi_{1,j}}}$. Then Lemma 2.5 implies

$$\begin{aligned} \text{rank } \mathcal{K}_F &= \text{rank}\left\{\sum_{i=1}^{\mu(F)} \alpha_i \mathcal{K}_{F^{\phi_i}}\right\} \leq \sum_{i=1}^{\mu(F)} \text{rank } \mathcal{K}_{F^{\phi_i}} \leq 2\mu(F), \\ \text{rank } \mathcal{K}_{F_1} &= \text{rank}\left\{\sum_{j=1}^{\mu_1(F_1)} \beta_j \mathcal{K}_{F_1^{\psi_j,\psi_{1,j}}}\right\} \leq \sum_{j=1}^{\mu_1(F_1)} \text{rank } \mathcal{K}_{F_1^{\psi_j,\psi_{1,j}}} \leq 2\mu_1(F_1), \end{aligned}$$

which proves the result. □

Lemma 2.7. *Let V be a vector space of dimension $n = 2\tilde{n}$ or $n = 2\tilde{n}+1$. There exist $F \in \mathcal{R}(V)$, $F_1 \in \mathcal{R}_1(V)$ and vectors $z, x, y \in V$ such that*

$$\text{rank}\{\mathcal{K}_F(x,y)\} = 2\tilde{n} \quad \textit{and} \quad \text{rank}\{\mathcal{K}_{F_1}(z,x,y)\} = 2\tilde{n}.$$

Proof. If $n = 2\tilde{n}$, let $\{e_1, \dots, e_{\tilde{n}}, f_1, \dots, f_{\tilde{n}}\}$ be an orthonormal basis of V; if n is odd, the argument is similar and we simply extend F and F_1 to be trivial on the additional basis vector. Define $\phi_i \in \otimes^2(V^*)$ and $\phi_{1,i} \in \otimes^3(V^*)$ by

$$\phi_i(e_j, e_k) = \phi_i(f_j, f_k) = \delta_{ij}\,\delta_{ik}, \qquad \phi_i(e_j, f_k) = 0,$$

$$\phi_{1,i}(e_j, e_k, e_l) = \phi_{1,i}(f_j, f_k, f_l) = \delta_{ij}\,\delta_{ik}\,\delta_{il}, \qquad \phi_{1,i}(e_j, e_k, f_k) = \phi_{1,i}(e_j, f_j, f_k) = 0,$$

for $i \in \{1, \dots, \tilde{n}\}$. We consider the following algebraic curvature tensors

$$F = \sum_{i=1}^{\tilde{n}} F^{\phi_i} \qquad \text{and} \qquad F_1 = \sum_{i=1}^{\tilde{n}} F_1^{\phi_i,\phi_{1,i}}.$$

We also define the vectors $x = e_1 + \cdots + e_{\tilde{n}}$, $y = f_1 + \cdots + f_{\tilde{n}}$ and $z = x + y$. We have

$$\begin{aligned} &\mathcal{K}_F(x,y)e_i = \mathcal{K}_{F^{\phi_i}}(e_i, f_i)e_i = -f_i, \\ &\mathcal{K}_F(x,y)f_i = \mathcal{K}_i(e_i, f_i)f_i = e_i, \\ &\mathcal{K}_{F_1}(z,x,y)e_i = \mathcal{R}_{F_1^{\phi_i,\phi_{1,i}}}(e_i, f_i, e_i + f_i)e_i = -2f_i \\ &\mathcal{K}_{F_1}(z,x,y)f_i = \mathcal{R}_{F_1^{\phi_i,\phi_{1,i}}}(e_i, f_i, e_i + f_i)f_i = 2e_i. \end{aligned}$$

The statement now follows. □

We can now prove the main theorem of this section [41].

Theorem 2.8. *Let $n \geq 2$. Then*

$$\frac{n}{2} \leq \mu(n) \leq \frac{n(n+1)}{2} \qquad \textit{and} \qquad \frac{n}{2} \leq \mu_1(n) \leq \frac{n(n+1)}{2}.$$

Proof. The upper bounds for $\mu(n)$ and $\mu_1(n)$ follow immediately from Theorems 2.3 and 2.4. On the other hand, Corollary 2.6 shows that for any $F \in \mathcal{R}(V)$ (resp. $F_1 \in \mathcal{R}_1(V)$), $\mu(F) \geq \frac{1}{2}\operatorname{rank}\{\mathcal{K}_F(x,y)\}$ (resp. $\mu_1(F_1) \geq \frac{1}{2}\operatorname{rank}\{\mathcal{K}_{F_1}(z,x,y)\}$). But Lemma 2.7 shows that the value $n/2$ is attained for certain F (resp. F_1). Thus $\mu(n) \geq n/2$ and $\mu_1(n) \geq n/2$. □

For low dimension we provide the exact value of μ [40].

Proposition 2.9. *Let F be an algebraic curvature tensor in a 3–dimensional vector space. One of the following two possibilities holds:*

(a) There exists exactly one symmetric tensor $\phi \in \otimes^3(V^*)$ such that $F = F^\phi$.

(b) There exist exactly two distinct symmetric tensors $\phi_1, \phi_2 \in \otimes^2(V^*)$ and constants κ_1 and κ_2 such that $F = \kappa_1 F^{\phi_1} + \kappa_2 F^{\phi_2}$.

The second case occurs if and only if the Ricci tensor has eigenvalues $\lambda_1 \neq 0 \neq \lambda_2$ and $\lambda_3 = \lambda_1 + \lambda_2$.

Proof. Let F be an algebraic curvature tensor in a 3–dimensional vector space V with inner product g. Let ρ^F denote the Ricci tensor and τ^F the scalar curvature of F. Then F can be written as

$$\begin{aligned} F(x,y,v,w) &= \frac{\tau^F}{2}\Big(g(x,v)g(y,w) - g(x,w)g(y,v)\Big) \\ &\quad -\Big(\rho^F(x,v)g(y,w) + \rho^F(y,w)g(x,v) - \rho^F(x,w)g(y,v) - \rho^F(y,v)g(x,w)\Big). \end{aligned}$$

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis diagonalizing ρ^F and put $\rho^F(e_i, e_i) = \lambda_i$. Then we have

$$F(e_i, e_j, e_k, e_l) = \left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{2} - \lambda_i - \lambda_j\right) F^\delta(e_i, e_j, e_k, e_l),$$

where $F^\delta(e_i, e_j, e_k, e_l) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ and δ is the Kronecker delta. Define

$$\begin{aligned} \alpha_1 &= \lambda_1 - \lambda_2 - \lambda_3 &&= 2F(e_2, e_3, e_2, e_3), \\ \alpha_2 &= -\lambda_1 + \lambda_2 - \lambda_3 &&= 2F(e_1, e_3, e_1, e_3), \\ \alpha_3 &= -\lambda_1 - \lambda_2 + \lambda_3 &&= 2F(e_1, e_2, e_1, e_2). \end{aligned}$$

We consider several possibilities.

Assume α_1, α_2 and α_3 are different from zero. Let $\epsilon_i = \pm 1$ denote the sign of α_i, put $\epsilon = \epsilon_1\epsilon_2\epsilon_3$ and $\beta = \sqrt{\epsilon\,\alpha_1\alpha_2\alpha_3/2}$. We define the symmetric tensor ϕ with respect to the above basis by $\phi_{ij} = (\beta/\alpha_i)\delta_{ij}$. Then $F = \epsilon\, F^\phi$.

Assume $\alpha_1 \neq 0$ and $\alpha_2 = \alpha_3 = 0$. We define the symmetric tensor ϕ by the trivial bilinear extension of $\phi(e_2, e_2) = 1$ and $\phi(e_3, e_3) = \alpha_1/2$. It is straightforward to check that $F = F^\phi$.

Assume $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Then $F = 0$.

Finally *assume* $\alpha_1, \alpha_2 \neq 0$ *and* $\alpha_3 = 0$. We show that it is not possible to express the given algebraic curvature tensor as $F = \gamma F^\phi$. On the contrary, assume this can be achieved for certain γ and ϕ. Since $\alpha_1, \alpha_2 \neq 0$ we have $F \neq 0$ and hence $\gamma \neq 0$. Then $F = \gamma F^\phi$ implies

$$\begin{array}{rclrclrcl} \phi_{11}\phi_{22} - \phi_{12}^2 & = & 0, & \phi_{11}\phi_{33} - \phi_{13}^2 & = & \dfrac{\alpha_2}{2\gamma}, & \phi_{22}\phi_{33} & = & \dfrac{\alpha_1}{2\gamma}, \\ \phi_{11}\phi_{23} & = & \phi_{13}\phi_{12}, & \phi_{12}\phi_{33} & = & \phi_{13}\phi_{23}, & \phi_{12}\phi_{23} & = & \phi_{13}\phi_{22}. \end{array}$$

Straightforward calculations show that the above system of equations has no solution.

Nevertheless, it is possible to write $F = F^{\phi_1} + F^{\phi_2}$. For example take

$$\phi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\alpha_1}{2} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2}{2} \end{pmatrix}$$

and the equality follows after a simple calculation. □

Corollary 2.10. *We have* $\mu(2) = 1$ *and* $\mu(3) = 2$.

Proof. First, we observe that for any two–dimensional manifold, the curvature tensor is expressed in terms of the Ricci tensor and thus, any algebraic curvature tensor on a two–dimensional vector space is completely determined by exactly one F^ϕ. The second assertion is an immediate consequence of Proposition 2.9. □

Remark 2.11. Theorem 2.3 provides a criteria for the non–existence of embeddings of a given manifold into a Euclidean space. For instance, no Riemannian 3–dimensional manifold whose curvature tensor is as in Proposition 2.9 (b) at some point can be isometrically embedded as a hypersurface in a flat space.

2.2 Natural curvature operators

When investigating algebraic properties of the curvature tensor, one usually focus on different kinds of natural operators defined from the curvature, with special attention to their spectrum. Among those operators, the Jacobi operator is probably the most natural and widely investigated. Nevertheless, many interesting information is encoded by other operators such as the Szabó operator or the skew–symmetric curvature operator. We recall the definitions and some relevant results related to the associated Osserman–like problems.

2.2.1 The Jacobi operator

Let M be a semi–Riemannian manifold of signature (p, q) and dimension $n = p + q$. Let $\mathbb{S}^+(M)$ be the bundle of unit spacelike tangent vectors and $\mathbb{S}^-(M)$ the bundle of unit timelike tangent vectors. Also, $\mathbb{S}(M)$ is defined by $\mathbb{S}_p(M) = \mathbb{S}_p^+(M) \cup \mathbb{S}_p^-(M)$ for all $p \in M$.

We recall that the *Jacobi operator* R_x for $x \in TM$ is the self-adjoint endomorphism of $x^\perp$ characterized by the identity

$$g(R_x(y), z) = R(x, y, x, z).$$

One says that M is *spacelike Osserman* (resp. *timelike Osserman*) if the eigenvalues of the Jacobi operator are constant on $\mathbb{S}^+(M)$ (resp. $\mathbb{S}^-(M)$). It turns out that these two notions are equivalent and such a manifold is simply said to be *Osserman.* A manifold M is said to be *pointwise Osserman* if the eigenvalues of the Jacobi operator are independent of the direction, although they may change from point to point.

A manifold is pointwise Osserman if and only if it is k–stein for all $k \geq 1$. In particular, every Osserman manifold is Einstein.

The local isometries of any isotropic space act transitively on the unit pseudo–sphere bundles and thus the eigenvalues of the Jacobi operator are constant on $\mathbb{S}(M)$, which shows that M is Osserman. R. Osserman [109] wondered whether the converse holds. This question has been called the Osserman conjecture by subsequent authors. This conjecture has been answered in the affirmative in the Riemannian setting if $n \neq 16$ by the work of Q. S. Chi [34] and Y. Nikolayevsky [104], [105], [106].

In the Lorentzian setting ($p = 1$), an Osserman manifold has constant sectional curvature [19], [61]. In the higher signature setting ($p > 1$, $q > 1$) the situation is much more complicated since many non–symmetric examples exist [64]. See for example [62], [68] and the references therein for more information. Moreover, the fact that the spectrum does not completely determine a self–adjoint operator in the indefinite setting suggested the consideration of the Jordan normal form rather than just the eigenvalue structure. Then, one says that (M, g) is *spacelike Jordan–Osserman* (resp. *timelike Jordan–Osserman*) if the Jordan normal form of the Jacobi operator is constant on $\mathbb{S}^+(M)$ (resp. $\mathbb{S}^-(M)$). These two notions are not equivalent if $n \geq 5$. The structure of a Jordan–Osserman algebraic curvature tensor strongly depends on the signature (p, q) of the metric tensor. Indeed, it has been shown in [71] that the spacelike Jacobi operators of a spacelike Jordan–Osserman algebraic curvature tensor are necessarily diagonalizable whenever $p < q$, but they can be arbitrarily complicated in the neutral case ($p = q$) [70].

Example 2.12. [41] Let $(\vec{x}, \vec{y})$ for $\vec{x} = (x_1, ..., x_p)$ and $\vec{y} = (y_1, ..., y_p)$ be coordinates on $\mathbb{R}^{2p}$ where $p \geq 3$. Let $f : \mathbb{R}^p \to \mathbb{R}$ be a differentiable function. We define a semi–Riemannian metric g_f of signature (p, p) on $\mathbb{R}^{2p}$ by

$$g_f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^i}, \quad g_f\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0 \quad \text{and} \quad g_f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \delta_{ij}\,.$$

Let ϕ be the Euclidean Hessian

$$\phi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 f}{\partial x^i \partial x^j}, \quad \phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0 \quad \text{and} \quad \phi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \phi\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i}\right) = 0.$$

Then, the curvature tensor of g_f is $R = F^\phi$. We assume that the restriction of ϕ to $\operatorname{span}\{\partial/\partial x^i\}$ is positive definite henceforth. Then M is a complete semi–Riemannian manifold that is spacelike and timelike Jordan–Osserman. Similarly define ϕ_1 by the trivial bilinear extension of

$$\phi_1\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\right)=\frac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}.$$

One has $\nabla R = F^{\phi,\phi_1}$. Thus if f is not quadratic, M is not a locally symmetric space. With a bit more work one can show that for such a generic f, M is curvature homogeneous but not locally affine homogeneous. We refer to [52], [75] for further details.

2.2.2 The higher order Jacobi operator

Let (M,g) be a semi–Riemannian manifold and let $Gr_{r,s}(T_pM)$ be the Grassmannian of all subspaces $E\subset T_pM$ such that the restriction of g to E is a non–degenerate inner product of signature (r,s). Let $\{e_i\}$ be an orthonormal basis of $E\in Gr_{r,s}(T_pM)$. We define the *higher order Jacobi operator* by

$$\mathcal{J}(E)=\sum_{i,j=1}^{r+s}\epsilon_{e_i}R_{e_i}.$$

where $\epsilon_i = g(e_i,e_i)$. A semi–Riemannian manifold (M,g) is said to be (r,s)–*Osserman* at $p\in M$ if the coefficients of the characteristic polynomial of $\mathcal{J}(E)$ are independent of $E\in Gr_{r,s}(T_pM)$ (see [68], [78], [120] and the references therein). An interesting observation is that only the value $r+s$ is important in the previous definition. Moreover, any k–Osserman manifold is of constant sectional curvature in the Riemannian (for $k>1$) and Lorentzian settings (see [69], [79]). Again, the situation is more complex in the higher signature case, where many non–symmetric k–Osserman metrics exist (see for example [22], [68]).

2.2.3 The Szabó operator

There is an analogous operator to the Jacobi operator which is defined for ∇R. The *Szabó operator* $\mathcal{J}_1(x)$ is the self–adjoint endomorphism of TM characterized by

$$g(\mathcal{J}_1(x)y,z)=(\nabla R)(x,x,y,x,z)=(\nabla_x R)(x,y,x,z).$$

One says that M is *spacelike Szabó* (resp. *timelike Szabó*) if the eigenvalues of $\mathcal{J}_1(\cdot)$ are constant on $\mathbb{S}^+(M)$ (resp. $\mathbb{S}^-(M)$). These notions are equivalent and such a manifold is simply said to be *Szabó*. The notion *spacelike Jordan–Szabó* (resp. *timelike Jordan–Szabó*) is defined similarly.

In his study of 2–point homogeneous spaces, Z. I. Szabó [122] gave a topological argument showing that any Riemannian Szabó manifold is necessarily a locally symmetric space, that is, $\nabla R=0$. This result was subsequently extended to the Lorentzian case [79]. In the higher signature setting, the situation is unclear again. The metric g_f described in Example 2.12 defines a Szabó semi–Riemannian manifold of signature (p,p).

Even in the algebraic setting, there are no known non–zero elements $F_1 \in \mathcal{R}_1(V)$ which are spacelike Jordan–Szabó. It has been shown in [73] that if F_1 is a spacelike Jordan–Szabó algebraic covariant derivative curvature tensor on a vector space of signature (p,q), where $q \equiv 1 \pmod 2$ and $p < q$ or where $q \equiv 2 \pmod 4$ and $p < q-1$, then $F_1 = 0$. This algebraic result yields an elementary proof of the geometrical fact that any pointwise totally isotropic semi–Riemannian manifold with such a signature is locally symmetric. The general question of finding non–trivial spacelike Jordan Szabó covariant algebraic curvature tensors or showing non–existence remains open.

2.2.4 The skew-symmetric curvature operator

Let $\{e_1, e_2\}$ be an orthonormal basis for an oriented spacelike (resp. timelike) 2–plane π. The *skew–symmetric curvature operator* $\tilde{\mathcal{R}}(\pi)$ is characterized by the identity

$$g(\tilde{\mathcal{R}}(\pi)y, z) = R(e_1, e_2, y, z).$$

This definition is independent of the particular choice of orthonormal basis. One says that M is *spacelike Ivanov–Petrova* (resp. *timelike Ivanov–Petrova*) if the eigenvalues of $\tilde{\mathcal{R}}(\cdot)$ are constant on the Grassmannian of oriented spacelike (resp. timelike) 2–planes. These two notions are equivalent and such a manifold is simply said to be *Ivanov–Petrova*. The notions *spacelike Jordan–Ivanov–Petrova* and *timelike Jordan Ivanov–Petrova* are defined similarly and are not equivalent.

The Riemannian Ivanov–Petrova manifolds have been classified in [74], [107]. They have also been classified in the Lorentzian setting [135] if $n \geq 10$. For all these manifolds, the curvature tensors have the form $R = F_\phi$ where ϕ is an idempotent isometry and $\tilde{\mathcal{R}}(\pi)$ has rank 2. Conversely, in the algebraic setting, if R is a spacelike Jordan–Ivanov–Petrova algebraic curvature tensor on a vector space of signature (p,q) where $q \geq 5$ and where $\operatorname{rank} \tilde{\mathcal{R}}(\cdot) = 2$, then there exist λ and ϕ such that $R = \lambda F^\phi$. The situation in the indefinite setting is again quite different. There exist spacelike Ivanov–Petrova manifolds of signature $(p, 2p)$ where $\tilde{\mathcal{R}}(\pi)$ has rank 4 and where the curvature tensor does not have the form $R = F^\phi$. We refer to [76] for further details.

Chapter 3

Four–dimensional Osserman metrics

Any Osserman metric is Einstein and thus of constant sectional curvature in dimension two and three. Therefore, dimension four is the lowest dimensional non–trivial case to study Osserman metrics. Moreover, it supports metrics of neutral signature $(2,2)$ where the Jacobi operator exhibits a completely different behavior with respect to both the Riemannian and Lorentzian settings and enjoys some special features of four–dimensional geometry.

Considering the curvature tensor R as an endomorphism of $\Lambda^2(M)$, we have the following $O(2,2)$–decomposition for $(2,2)$–metrics

$$R = \frac{\tau}{12}\,\mathrm{Id}_{\Lambda^2} + \rho^0 + W : \ \Lambda^2 \to \Lambda^2$$

where ρ^0 denotes the traceless Ricci tensor, $\rho^0(X,Y) = \rho(X,Y) - (\tau/4)\,g(X,Y)$ and W denotes the Weyl conformal curvature tensor given by

$$\begin{aligned} W(X,Y,V,W) \;=\;& R(X,Y,V,W) + \frac{\tau}{(n-1)(n-2)}\Big\{g(X,V)g(Y,W) - g(Y,V)g(X,W)\Big\} \\ & -\frac{1}{n-2}\Big\{\rho(X,V)g(Y,W) - \rho(Y,V)g(X,W) \\ & \qquad + \rho(Y,W)g(X,V) - \rho(X,W)g(Y,V)\Big\}. \end{aligned}$$

The Hodge star operator $*: \Lambda^2 \to \Lambda^2$ associated with any $(2,2)$ metric induces a further splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, where $\Lambda^2_\pm$ denotes the ± 1–eigenspaces of the Hodge star operator, that is, $\Lambda^2_\pm = \{\alpha \in \Lambda^2(M) : *\alpha = \pm\alpha\}$. Then, the curvature tensor decomposes as

$$R = \frac{\tau}{12}\,\mathrm{Id}_{\Lambda^2} + \rho^0 + W^+ + W^-,$$

where $W^\pm = (W \pm *W)/2$. Recall that a semi–Riemannian four–dimensional manifold is called *self–dual* (resp. *anti–self–dual*) if $W^- = 0$ (resp. $W^+ = 0$).

An interesting feature of four–dimensional Osserman metrics comes from the fact that an algebraic curvature tensor in a four–dimensional vector space is Osserman if and only if it is Einstein and self-dual for an appropriate orientation of the underlying vector space.

Therefore, *pointwise Osserman metrics* in dimension four are those which are Einstein and self–dual or anti–self–dual.

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis with e_1 and e_2 spacelike vectors and e_3 and e_4 timelike vectors. Local bases of the spaces of self–dual and anti–self–dual two–forms may be constructed as

$$\Lambda^2_\pm = \operatorname{span}\left\{E_1^\pm, E_2^\pm, E_3^\pm\right\},$$

where

$$E_1^\pm = \frac{e^1 \wedge e^2 \pm e^3 \wedge e^4}{\sqrt{2}}, \quad E_2^\pm = \frac{e^1 \wedge e^3 \pm e^2 \wedge e^4}{\sqrt{2}}, \quad E_3^\pm = \frac{e^1 \wedge e^4 \mp e^2 \wedge e^3}{\sqrt{2}}.$$

We observe that the Hodge star operator satisfies

$$e^i \wedge e^j \wedge \star(e^k \wedge e^l) = \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right) \varepsilon_i \varepsilon_j \, e^1 \wedge e^2 \wedge e^3 \wedge e^4,$$

where $\varepsilon_i = g(e_i, e_i)$. Note that $\langle E_1^\pm, E_1^\pm \rangle = 1$, $\langle E_2^\pm, E_2^\pm \rangle = \langle E_3^\pm, E_3^\pm \rangle = -1$. Then, with respect to the above bases the self–dual and the anti–self–dual Weyl curvature operators $W^\pm : \Lambda^2_\pm \to \Lambda^2_\pm$ have the matrix representation

$$W^\pm = \begin{pmatrix} W^\pm_{11} & W^\pm_{12} & W^\pm_{13} \\ -W^\pm_{12} & -W^\pm_{22} & -W^\pm_{23} \\ -W^\pm_{13} & -W^\pm_{23} & -W^\pm_{33} \end{pmatrix},$$

where $W^\pm_{ij} = W(E_i^\pm, E_j^\pm)$ and $W(e^i \wedge e^j, e^k \wedge e^l) = W(e_i, e_j, e_k, e_l)$.

For any non–null vector x in the $(2,2)$ setting, the induced metric on $\mathbb{R}x^\perp$ is of Lorentzian signature, and hence, the eigenvalue structure does not completely characterize the Jacobi operator R_x. The consideration of the Jordan normal form led to introduce the so–called Jordan–Osserman metrics (see Subsection 2.2.1). Four–dimensional Jordan–Osserman metrics were initially investigated by N. Blažić, N. Bokan and Z. Rakić [21] who considered four different possibilities according to the behavior of the Jordan normal form of the Jacobi operators. These four types are:

(Ia) The Jacobi operator is diagonalizable, $R_x = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}$.

(Ib) The Jacobi operator has a complex eigenvalue, $R_x = \begin{pmatrix} \alpha & -\beta & \\ \beta & \alpha & \\ & & \gamma \end{pmatrix}$.

(II) The minimal polynomial of the Jacobi operator has a double root, $R_x = \begin{pmatrix} \alpha & & \\ & \beta & \\ & 1 & \beta \end{pmatrix}$.

(III) The minimal polynomial of the Jacobi operator has a triple root, $R_x = \begin{pmatrix} \alpha & & \\ 1 & \alpha & \\ & 1 & \alpha \end{pmatrix}$.

Moreover, there is a one to one correspondence between the different possibilities of the Jacobi operators of Types Ia, Ib, II, III and the Jordan normal form of the (anti–)self–dual part of the Weyl conformal curvature tensor [62].

It has been shown in [21] that four–dimensional Osserman metrics with diagonalizable Jacobi operators are locally isometric to a real, complex or para–complex space form and that Type Ib metrics cannot occur. Moreover, a locally symmetric Osserman $(2,2)$ metric has diagonalizable Jacobi operators or it is isometric with some Type II metric to nilpotent Jacobi operators [65].

The fact that all known examples of non–symmetric Osserman metrics had 2–step or 3–step nilpotent Jacobi operators suggested that no other examples exist [62], [68]. This was conjectured by several authors. Our purpose is to show the existence of such metrics (Section 3.1) and to give a complete description of them (Sections 3.2 and 3.3).

3.1 New examples of Osserman metrics with non–diagonalizable Jacobi operators

Let us take the usual coordinates (x_1, x_2, x_3, x_4) in $M = \mathbb{R}^4$. For any arbitrary real–valued function f and any non–zero constant k we define the metric [48]

$$\begin{aligned} g \;=\; & dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2 + \left(4kx_1^2 - \frac{1}{4k} f(x_4)^2\right) dx^3 \otimes dx^3 \\ & + 4kx_2^2 dx^4 \otimes dx^4 + \left(4kx_1x_2 + x_2 f(x_4) - \frac{1}{4k} f'(x_4)\right)(dx^3 \otimes dx^4 + dx^4 \otimes dx^3). \end{aligned}$$

The following two lemmas can be obtained after some tedious but straightforward calculations from the definition of the metric g.

Lemma 3.1. *The Christoffel symbols associated with g are*

$$\Gamma_{13}^1 = -\Gamma_{33}^3 = 4kx_1, \qquad \Gamma_{13}^2 = \Gamma_{14}^1 = -\Gamma_{34}^3 = \frac{1}{2}\Gamma_{24}^2 = -\frac{1}{2}\Gamma_{44}^4 = 2kx_2,$$

$$\Gamma_{23}^2 = \Gamma_{24}^1 = -\Gamma_{34}^4 = \frac{1}{2}\big(4kx_1 + f(x_4)\big), \qquad \Gamma_{33}^1 = 16k^2x_1^3 - x_1 f(x_4)^2,$$

$$\Gamma_{33}^2 = x_1\big(16k^2x_1x_2 - f'(x_4)\big) + f(x_4)\left(4kx_1x_2 + \frac{f'(x_4)}{4k}\right),$$

$$\Gamma_{34}^1 = 16k^2x_1^2x_2 + 4kx_1x_2 f(x_4) - \frac{1}{2}x_1 f'(x_4) - \frac{3f(x_4)f'(x_4)}{8k},$$

$$\Gamma_{34}^2 = \frac{1}{2}\,x_2\big(32k^2x_1x_2 + 8kx_2 f(x_4) - f'(x_4)\big),$$

$$\Gamma_{44}^1 = 16k^2x_1x_2^2 + 4kx_2^2 f(x_4) - \frac{f''(x_4)}{4k}, \qquad \Gamma_{44}^2 = 16k^2x_2^3.$$

From the previous lemma we get

Lemma 3.2. *The curvature tensor of g is determined by*

$$
\begin{aligned}
R_{1313} &= R_{2424} = -4k, & R_{1324} &= R_{1423} = -2k,\\
R_{1334} &= kx_2\big(4kx_1 + f(x_4)\big), & R_{1434} &= 4k^2x_2^2,\\
R_{2334} &= \frac{f(x_4)^2}{4} - 4k^2x_1^2, & R_{2434} &= \frac{f'(x_4)}{2} - kx_2\big(4kx_1 + f(x_4)\big),\\
R_{3434} &= \frac{f'(x_4)^2}{4k} + 2kx_1x_2f'(x_4) - 2kx_2^2f(x_4)^2 - x_1f''(x_4)\\
&\quad -f(x_4)\left(8k^2x_1x_2^2 - \frac{5}{2}x_2f'(x_4) - \frac{f''(x_4)}{4k}\right).
\end{aligned}
$$

Now, we calculate the Jacobi operator associated with g. We have the following theorem [48].

Theorem 3.3. *For any function f, the metric g is Osserman of signature $(2,2)$ with eigenvalues $\{0, 4k, k, k\}$. Moreover, the Jacobi operators are diagonalizable if and only if*

$$24kf(x_4)f'(x_4)x_2 - 12kf''(x_4)x_1 + 3f(x_4)f''(x_4) + 4f'(x_4)^2 = 0.$$

Otherwise, k is a double root of the minimal polynomial of the Jacobi operators and (M, g) is Jordan–Osserman on the open set where the above equation does not hold.

Proof. The eigenvalues of the Jacobi operator of an Osserman metric change sign when passing from timelike to spacelike directions. Thus, for the purpose of studying the Osserman property, it is convenient to consider the normalized Jacobi operator $J_R(X) = g(X,X)^{-1}R_X$ associated with each non–null vector X, whose eigenvalues are constant if and only if (M, g) is Osserman. Let $X = \sum_{i=1}^{4}\alpha_i\partial_i$ be a non–null vector, where $\{\partial_i = \partial/\partial x^i\}$ denotes the coordinate basis. The associated Jacobi operator $R_X = R(X, \cdot)X$ can be expressed with respect to the coordinate basis $\{\partial_i\}$ as

$$
R_X = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14}\\
a_{21} & a_{22} & a_{23} & a_{24}\\
-4k\alpha_3^2 & -4k\alpha_3\alpha_4 & a_{33} & a_{34}\\
-4k\alpha_3\alpha_4 & -4k\alpha_4^2 & a_{43} & a_{44}
\end{pmatrix}, \tag{3.1}
$$

with

$$
\begin{aligned}
a_{11} &= 5kx_2f(x_4)\alpha_3\alpha_4 - f(x_4)^2\alpha_3^2 + 2k\big(2\alpha_1\alpha_3 + \alpha_2\alpha_4\\
&\quad +2k(4x_1^2\alpha_3^2 + 5x_1x_2\alpha_3\alpha_4 + x_2^2\alpha_4^2)\big) - \alpha_3\alpha_4f'(x_4),\\
a_{12} &= \frac{1}{4}\alpha_4\big(12kx_2f(x_4)\alpha_4 - 3f(x_4)^2\alpha_3 + 8k(\alpha_1 + 6kx_1(x_1\alpha_3 + x_2\alpha_4)) - 2\alpha_4f'(x_4)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_{13} &= \frac{1}{16k}\big(f(x_4)^2\alpha_3(16k\alpha_1+\alpha_4 f'(x_4))+4f(x_4)\alpha_4(7kx_2\alpha_4 f'(x_4)-16k^2x_2\alpha_1\\
&\quad +\alpha_4 f''(x_4))-2(4k\alpha_4(2kx_1(x_1\alpha_3+x_2\alpha_4)-\alpha_1)f'(x_4)\\
&\quad -3\alpha_4^2 f'(x_4)^2+8k(4k\alpha_1(\alpha_1+4kx_1(x_1\alpha_3+x_2\alpha_4))+x_1\alpha_4^2 f''(x_4)))\big),\\
a_{14} &= -\frac{1}{16k}\big(f(x_4)^2\alpha_3(\alpha_3 f'(x_4)-12k\alpha_2)+4f(x_4)(4k^2x_2(\alpha_1\alpha_3+3\alpha_2\alpha_4)\\
&\quad +7kx_2\alpha_3\alpha_4 f'(x_4)+\alpha_3\alpha_4 f''(x_4))+2(-4k(\alpha_1\alpha_3+\alpha_2\alpha_4\\
&\quad +2kx_1\alpha_3(x_1\alpha_3+x_2\alpha_4))f'(x_4)+3\alpha_3\alpha_4 f'(x_4)^2+8k(4k(3kx_1\alpha_2(x_1\alpha_3+x_2\alpha_4)\\
&\quad +\alpha_1(\alpha_2+kx_2(x_1\alpha_3+x_2\alpha_4)))-x_1\alpha_3\alpha_4 f''(x_4)))\big),\\
a_{21} &= \alpha_3\big(3kx_2 f(x_4)\alpha_3+2k(\alpha_2+6kx_2(x_1\alpha_3+x_2\alpha_4))-\alpha_3 f'(x_4)\big),\\
a_{22} &= 2k\alpha_1\alpha_3-\frac{1}{4}f(x_4)^2\alpha_3^2+4k^2x_1^2\alpha_3^2+4k\alpha_2\alpha_4+5kf(x_4)x_2\alpha_3\alpha_4\\
&\quad +20k^2x_1x_2\alpha_3\alpha_4+16k^2x_2^2\alpha_4^2-\frac{3}{2}\alpha_3\alpha_4 f'(x_4),\\
a_{23} &= -\frac{1}{4k}\big(4k(kx_2\alpha_4(x_1\alpha_3+x_2\alpha_4)-\alpha_1\alpha_3)f'(x_4)-kf(x_4)^2\alpha_2\alpha_3+\alpha_3\alpha_4 f'(x_4)^2\\
&\quad +f(x_4)(4k^2x_2(3\alpha_1\alpha_3+\alpha_2\alpha_4)+9kx_2\alpha_3\alpha_4 f'(x_4)+\alpha_3\alpha_4 f''(x_4))\\
&\quad +4k(4k(kx_1\alpha_2(x_1\alpha_3+x_2\alpha_4)+\alpha_1(\alpha_2+3kx_2(x_1\alpha_3+x_2\alpha_4)))-x_1\alpha_3\alpha_4 f''(x_4)\big),\\
a_{24} &= \frac{1}{4k}\big(2k\alpha_3(3\alpha_2+2kx_2(x_1\alpha_3+x_2\alpha_4))f'(x_4)+\alpha_3^2 f'(x_4)^2+f(x_4)\alpha_3(-16k^2x_2\alpha_2\\
&\quad +9kx_2\alpha_3 f'(x_4)+\alpha_3 f''(x_4))-4k(4k\alpha_2(\alpha_2+4kx_2(x_1\alpha_3+x_2\alpha_4))+x_1\alpha_3^2 f''(x_4))\big),\\
a_{33} &= k\big(4\alpha_1\alpha_3+x_2 f(x_4)\alpha_3\alpha_4+2\alpha_4(\alpha_2+2kx_2(x_1\alpha_3+x_2\alpha_4))\big),\\
a_{34} &= -k\alpha_3\big(-2\alpha_2+x_2 f(x_4)\alpha_3+4kx_2(x_1\alpha_3+x_2\alpha_4)\big),\\
a_{43} &= \frac{1}{4}\alpha_4\big(f(x_4)^2\alpha_3-4kx_2 f(x_4)\alpha_4+2(4k(\alpha_1-2kx_1(x_1\alpha_3+x_2\alpha_4))+\alpha_4 f'(x_4))\big),\\
a_{44} &= kx_2 f(x_4)\alpha_3\alpha_4-\frac{1}{4}f(x_4)^2\alpha_3^2+2k(\alpha_1\alpha_3+2(\alpha_2\alpha_4+kx_1\alpha_3(x_1\alpha_3+x_2\alpha_4)))\\
&\quad -\frac{1}{2}\alpha_3\alpha_4 f'(x_4).
\end{aligned}
$$

Using the above expressions we get that the characteristic polynomial of $J_R(X)$ is given by $p_{J_R(X)}(\lambda)=\lambda(\lambda-4k)(\lambda-k)^2$, and thus the metric g is Osserman with eigenvalues $\{0,4k,k,k\}$. In order to analyze the diagonalizability of the Jacobi operators, we consider the minimal polynomials $m_{J_R(X)}(\lambda)$. It follows after some calculations that

$$
J_R(X)\cdot(J_R(X)-4kId)\cdot(J_R(X)-kId)=\frac{k}{4}g(X,X)^{-1}\,\Xi\begin{pmatrix}0&0&-\alpha_4^2&\alpha_3\alpha_4\\0&0&\alpha_3\alpha_4&-\alpha_3^2\\0&0&0&0\\0&0&0&0\end{pmatrix},
$$

where $\Xi = 3f(x_4)(8kx_2f'(x_4)+f''(x_4))+4(f'(x_4)^2-3kx_1f''(x_4))$. This shows that $\Xi = 0$ is the necessary and sufficient condition for diagonalizability of the Jacobi operators. Finally, in the open set where Ξ does not vanish (M, g) is Jordan–Osserman and k is a double root of the minimal polynomials $m_{J_R(X)}(\lambda)$. □

Remark 3.4. It was shown in [65] that the Jacobi operators of a locally symmetric four–dimensional Osserman metric are either diagonalizable or two–step nilpotent. Therefore, the metric g cannot be locally symmetric unless their Jacobi operators diagonalize. It follows after some calculations that the covariant derivative of the curvature tensor vanishes at a point $(x_1, \ldots, x_4)$ if and only if

$$\begin{aligned} f''(x_4) &= 0, & f(x_4)f'(x_4) &= 0, \\ f'(x_4)^2 x_1 &= 0, & 24kx_2f'(x_4)^2 + f'''(x_4)(f(x_4) - 4kx_1) &= 0. \end{aligned}$$

Hence $(\mathbb{R}^4, g)$ is locally symmetric if and only if the function f is constant, and thus the Jacobi operators are diagonalizable by Theorem 3.3. Furthermore, it follows from the work in [20] that any four–dimensional Jordan–Osserman manifold has isotropic covariant derivative of the curvature tensor, that is, $\|\nabla R\| = 0$, although ∇R may be non–zero.

The existence of timelike, spacelike and null vectors on any indefinite inner product vector space suggested the consideration of the Osserman problem separately. However, it was shown in [62] that the spacelike and timelike Osserman conditions are equivalent and moreover, any of them implies the null Osserman condition (since eigenvalues of the Jacobi operators change sign from spacelike to timelike directions). On the other hand it is known that the spacelike and timelike Jordan–Osserman conditions are not equivalent [68] and even both of them do not imply the null Jordan–Osserman condition as shown in the following

Theorem 3.5. *For any function f, the metric g is null Osserman with two–step nilpotent null Jacobi operators.*

Proof. First of all, observe that a vector $U = \sum_{i=1}^4 \alpha_i \partial_i$ is null if and only if

$$2\alpha_1\alpha_3 + 2\alpha_2\alpha_4 + \alpha_3^2\left(4kx_1^2 - \frac{f(x_4)^2}{4k}\right) + \alpha_3\alpha_4\left(2f(x_4)x_2 + 8kx_1x_2 - \frac{f'(x_4)}{2k}\right) + 4kx_2^2\alpha_4^2 = 0.$$

A tedious but straightforward calculation from (3.1) shows that

$$R_U^2 = g(U,U)\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ -16k^2\alpha_3^2 & -16k^2\alpha_3\alpha_4 & b_{33} & b_{34} \\ -16k^2\alpha_3\alpha_4 & -16k^2\alpha_4^2 & b_{43} & b_{44} \end{pmatrix},$$

where

$$
\begin{aligned}
b_{11} &= k\big(16k\alpha_1\alpha_3 - 4f(x_4)^2\alpha_3^2 + 17kf(x_4)x_2\alpha_3\alpha_4 \\
&\quad +2k(\alpha_2\alpha_4 + 2k(x_1\alpha_3 + x_2\alpha_4)(16x_1\alpha_3 + x_2\alpha_4)) - 3\alpha_3\alpha_4 f'(x_4)\big), \\
b_{12} &= \frac{k}{4}\alpha_4(56k\alpha_1 - 15(f(x_4) + 4kx_1)((f(x_4) - 4kx_1)\alpha_3 - 4kx_2\alpha_4) - 10\alpha_4 f'(x_4)), \\
b_{13} &= -16k^2\alpha_1^2 + \frac{k}{2}\alpha_1(8(f(x_4) + 4kx_1)((f(x_4) - 4kx_1)\alpha_3 - 4kx_2\alpha_4) + 3\alpha_4 f'(x_4)) \\
&\quad +\frac{1}{16}\alpha_4\big(f'(x_4)(5f(x_4)^2\alpha_3 + 44kf(x_4)x_2\alpha_4 - 80k^2x_1(x_1\alpha_3 + x_2\alpha_4) \\
&\quad +14\alpha_4 f'(x_4)) + 8(f(x_4) - 4kx_1)\alpha_4 f''(x_4)\big), \\
b_{14} &= \frac{1}{16}\big(-960k^3x_1^2\alpha_2\alpha_3 - 960k^3x_1x_2\alpha_2\alpha_4 + 80k^2x_1^2\alpha_3^2 f'(x_4) + 40k\alpha_2\alpha_4 f'(x_4) \\
&\quad +80k^2x_1x_2\alpha_3\alpha_4 f'(x_4) - 14\alpha_3\alpha_4 f'(x_4)^2 - 5f(x_4)^2\alpha_3(-12k\alpha_2 + \alpha_3 f'(x_4)) \\
&\quad +8k\alpha_1(-32k\alpha_2 - 2kx_2((f(x_4) + 4kx_1)\alpha_3 + 4kx_2\alpha_4) + 3\alpha_3 f'(x_4)) \\
&\quad +32kx_1\alpha_3\alpha_4 f''(x_4) + 4f(x_4)\alpha_4(-kx_2(60k\alpha_2 + 11\alpha_3 f'(x_4)) - 2\alpha_3 f''(x_4))\big), \\
b_{21} &= k\alpha_3\big(14k\alpha_2 + 15kx_2((f(x_4) + 4kx_1)\alpha_3 + 4kx_2\alpha_4) - 5\alpha_3 f'(x_4)\big), \\
b_{22} &= \frac{k}{4}(8k\alpha_1\alpha_3 - f(x_4)^2\alpha_3^2 + 68kf(x_4)x_2\alpha_3\alpha_4 \\
&\quad +16k(4\alpha_2\alpha_4 + k(x_1\alpha_3 + x_2\alpha_4)(x_1\alpha_3 + 16x_2\alpha_4)) - 22\alpha_3\alpha_4 f'(x_4)), \\
b_{23} &= \frac{1}{4}\big(kf(x_4)^2\alpha_2\alpha_3 - 16k^3x_1^2\alpha_2\alpha_3 - 16k^3x_1x_2\alpha_2\alpha_4 - 4k\alpha_2\alpha_4 f'(x_4) \\
&\quad -20k^2x_1x_2\alpha_3\alpha_4 f'(x_4) - 20k^2x_2^2\alpha_4^2 f'(x_4) - \alpha_3\alpha_4 f'(x_4)^2 + 4k\alpha_1(-16k\alpha_2 \\
&\quad -15kx_2((f(x_4) + 4kx_1)\alpha_3 + 4kx_2\alpha_4) + 5\alpha_3 f'(x_4)) + 8kx_1\alpha_3\alpha_4 f''(x_4) \\
&\quad +f(x_4)\alpha_4(-kx_2(4k\alpha_2 + 21\alpha_3 f'(x_4)) - 2\alpha_3 f''(x_4))\big), \\
b_{24} &= \frac{1}{4}\big(-64k^2\alpha_2^2 + 2k\alpha_2(-32kx_2((f(x_4) + 4kx_1)\alpha_3 + 4kx_2\alpha_4) + 13\alpha_3 f'(x_4)) \\
&\quad +\alpha_3(f'(x_4)(kx_2(21f(x_4)\alpha_3 + 20k(x_1\alpha_3 + x_2\alpha_4)) + \alpha_3 f'(x_4)) \\
&\quad +2(f(x_4) - 4kx_1)\alpha_3 f''(x_4))\big), \\
b_{33} &= k\big(16k\alpha_1\alpha_3 + \alpha_4(2k\alpha_2 + kx_2((f(x_4) + 4kx_1)\alpha_3 + 4kx_2\alpha_4) + \alpha_3 f'(x_4))\big), \\
b_{34} &= -k\alpha_3\big(-14k\alpha_2 + kx_2((f(x_4) + 4kx_1)\alpha_3 + 4kx_2\alpha_4) + \alpha_3 f'(x_4)\big), \\
b_{43} &= -\frac{k}{4}\alpha_4\big(-56k\alpha_1 - (f(x_4) + 4kx_1)((f(x_4) - 4kx_1)\alpha_3 - 4kx_2\alpha_4) - 6\alpha_4 f'(x_4)\big), \\
b_{44} &= \frac{k}{4}\big(8k\alpha_1\alpha_3 - f(x_4)^2\alpha_3^2 + 4kf(x_4)x_2\alpha_3\alpha_4 \\
&\quad +16k(4\alpha_2\alpha_4 + kx_1\alpha_3(x_1\alpha_3 + x_2\alpha_4)) - 6\alpha_3\alpha_4 f'(x_4)\big).
\end{aligned}
$$

Since U is a null vector we clearly have $R_U^2 = 0$. Moreover, it follows from (3.1) that if $R_U = 0$ then $\alpha_3 = \alpha_4 = 0$ and the Jacobi operator reduces to

$$R_U = -4k \begin{pmatrix} 0 & 0 & \alpha_1^2 & \alpha_1\alpha_2 \\ 0 & 0 & \alpha_1\alpha_2 & \alpha_2^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which shows that R_U vanishes if and only if $U = 0$. This proves that (M, g) is null Osserman with two–step nilpotent null Jacobi operators. □

Remark 3.6. Although the null Jacobi operators are two–step nilpotent, their Jordan normal form is not necessarily constant on the null cone since the corresponding minimal polynomials may admit one or two double roots. For instance, $U = \alpha_1\partial_1 + \alpha_2\partial_2$ is a null vector whose associated Jacobi operator is

$$R_U = -4k \begin{pmatrix} 0 & 0 & \alpha_1^2 & \alpha_1\alpha_2 \\ 0 & 0 & \alpha_1\alpha_2 & \alpha_2^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with Jordan normal form} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

On the other hand, for any function f with $f(0) = 0$, $V = \partial_3$ is a null vector at $(0, x_2, x_3, 0)$. Moreover, in such a case the associated Jacobi operator satisfies

$$(R_V)_{(0,x_2,x_3,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -f'(0) & 0 & 0 & \frac{f'(0)^2}{4k} \\ -4k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with Jordan normal form} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

whenever $f'(0) \neq 0$. Hence the null Osserman and the null Jordan–Osserman conditions are not equivalent for $(2, 2)$ metrics at the algebraic level, in contrast to the non–null Osserman conditions. The above example shows that, although the algebraic Osserman condition implies the null Osserman condition, there exist Jordan–Osserman algebraic curvature tensors which are not null Jordan–Osserman.

3.1.1 Some geometrical properties

Next, we show that there exist four–dimensional Szabó metrics such that the degree of nilpotency of the associated Szabó operators changes depending on the direction. In contrast to what happens with the Jacobi operator, the Szabó and the Jordan–Szabó algebraic conditions are not equivalent in dimension four.

Theorem 3.7. *For any function f, the metric g is Szabó of signature $(2, 2)$ with zero eigenvalues. The minimal polynomial of the Szabó operators $\mathcal{J}_1(X)$ depends on the direction X at each point and hence the metric g is not pointwise Jordan–Szabó.*

Proof. Let $X = \sum_{i=1}^4 \alpha_i \partial_i$ be a non–null vector. The associated Szabó operator, when expressed in the coordinate basis has the form

$$\mathcal{J}_1(X) = \begin{pmatrix} A & B \\ 0 & {}^tA \end{pmatrix}, \qquad A = \Psi \begin{pmatrix} \alpha_3\alpha_4 & \alpha_4^2 \\ -\alpha_3^2 & -\alpha_3\alpha_4 \end{pmatrix},$$

where $\Psi = 2\alpha_3 f(x_4) f'(x_4) + \alpha_4 f''(x_4)$. Hence, the characteristic polynomial of the Szabó operators is $p_{\mathcal{J}_1(X)}(\lambda) = \lambda^4$, independently of the 2×2–matrix B.

Since the degree of nilpotency depends on B, in order to show that the Szabó and Jordan Szabó algebraic conditions are not equivalent, we make the special choice $f(x_4) = x_4$. If X and Y are the unit vectors in the direction of $\partial_1 + \partial_3$ and $\partial_2 + \partial_4$, respectively, one has

$$\mathcal{J}_1(X) = \begin{pmatrix} 0 & 0 & 0 & 2(\varepsilon_X - 1)x_4 \\ -2x_4 & 0 & 2x_4 & 4(x_1 - \frac{x_4}{8k} + x_2x_4(8kx_1 + x_4)) \\ 0 & 0 & 0 & -2x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{J}_1(Y) = \begin{pmatrix} 0 & 0 & 6x_2 + 2(3\varepsilon_Y - 5)x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\varepsilon_Z = g(Z, Z) = \pm 1$. This shows that $\mathcal{J}_1(X)$ is three–step nilpotent at most points while $\mathcal{J}_1(Y)$ is two–step nilpotent. $\square$

Remark 3.8. It follows from the work in [22] that a four–dimensional metric is 1-Osserman and 2-Osserman if and only if it is either of constant curvature or the Jacobi operators are two–step nilpotent. Therefore, the metric g is not Osserman of higher order.

Jordan–Osserman metrics which are also Ivanov–Petrova but not of constant sectional curvature have been constructed by P. Gilkey and S. Nikčević by using the so–called generalized wave metrics in neutral signature $(2, 2)$ [77]. All such examples have nilpotent Jacobi operators which seems to be a specific feature of the intersection between Ivanov–Petrova and Jordan–Osserman metrics.

Theorem 3.9. *For any function f, the metric g is Osserman but not Ivanov–Petrova.*

Proof. Note that $\pi = \operatorname{span}\{\partial_1, \partial_3\}$ is a non–degenerate plane whose skew-symmetric operator satisfies

$$\tilde{\mathcal{R}}(\pi) = \begin{pmatrix} 4k & 0 & 16k^2x_1^2 - f(x_4)^2 & 3kx_2(4kx_1 + f(x_4)) - \frac{1}{2}f'(x_4) \\ 0 & 2k & 3kx_2(4kx_1 + f(x_4)) - f'(x_4) & 8k^2x_2^2 \\ 0 & 0 & -4k & 0 \\ 0 & 0 & 0 & -2k \end{pmatrix}$$

and hence, it has constant eigenvalues $\{2k, 4k, -2k, -4k\}$ independently of the function f. On the other hand, for any function f with $f'(0) \neq 0$, it follows that $\pi = \mathrm{span}\{\partial_3, \partial_4\}$ is a non–degenerate plane at the origin whose skew–symmetric operator satisfies

$$\tilde{\mathcal{R}}(\pi) = \left|\frac{k}{f'(0)}\right| \begin{pmatrix} 0 & f(0)^2 & -\frac{f(0)^2 f'(0)}{4k} & -\frac{3f'(0)^2+2f(0)f''(0)}{2k} \\ 0 & 2f'(0) & \frac{f'(0)^2+f(0)f''(0)}{k} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f(0)^2 & -2f'(0) \end{pmatrix}$$

and has eigenvalues $\{0, 0, 2k, -2k\}$. This shows that for any function f with $f'(0) \neq 0$, the metric g is not Ivanov–Petrova on planes of signature $(-+)$ at the origin.

The eigenspace corresponding to the double eigenvalue k of the Jacobi operator is of Lorentzian signature (see Remark 3.11), and thus the curvature tensor at each point is completely determined by the diagonalizability of the Jacobi operator, independently of the function f. In fact, at any point where the Jacobi operators diagonalize (resp. are not diagonalizable) there exist orthonormal bases where the (algebraic) curvature tensor is expressed in terms of the eigenvalues of the Jacobi operators, independently of the function f (see [21], [62, Thm. 4.2.2]).

Next, observe that it is possible to give functions f satisfying $f'(0) \neq 0$ and $3f(0)f''(0) + 4f'(0)^2 = 0$ (see Theorem 3.3) and therefore the Jacobi operators are diagonalizable at the origin. Also, there exist functions with $f'(0) \neq 0$ and $3f(0)f''(0) + 4f'(0)^2 \neq 0$ and hence the corresponding metric g has non–diagonalizable Jacobi operators at the origin. From the eigenvalue structure of the skew–symmetric curvature operators corresponding to the planes discussed above, it follows that none of the corresponding (algebraic) curvature tensors can be Ivanov–Petrova, which shows that the metric g is not Ivanov–Petrova at any point. □

Remark 3.10. It was proved in [21] that any four–dimensional Osserman algebraic curvature tensor is Jordan–Osserman. The existence of Osserman metrics that are not Jordan–Osserman was already pointed out in [64]. Indeed, the Jordan normal form of the Jacobi operators (3.1) corresponding to the metric g changes from diagonalizable to non–diagonalizable according to the statement of Theorem 3.3. Since $24kf(x_4)f'(x_4)x_2 - 12kf''(x_4)x_1 + 3f(x_4)f''(x_4) + 4f'(x_4)^2$ defines a polynomial on x_1, x_2, the metric g, when considered as globally defined in $\mathbb{R}^4$, changes its Jordan normal form, and hence, it is Osserman but not Jordan–Osserman. However, it restricts to Jordan–Osserman metrics on suitable open sets.

Since the metric g is not Jordan–Osserman in general, it is not curvature homogeneous, and thus it cannot be locally homogeneous. Even when we restrict to open sets where g defines a Jordan–Osserman metric (and hence 0-curvature homogeneous), the metric is not necessarily locally homogeneous. Indeed, for the special choice of $f(x_4) = x_4$, $(\mathbb{R}^4, g)$ is Jordan–Osserman in the open set defined by $6kx_2x_4 \neq -1$. However, it is not locally homogeneous, since ∇R vanishes at any point $(0, 0, x_3, 0)$ and it is different from zero at those points $(0, 0, x_3, x_4)$ with $x_4 \neq 0$. This shows that it is not 1-curvature homogeneous.

Remark 3.11. Different kinds of Osserman manifolds may share the same eigenvalue structure. Indeed, the Jacobi operators of indefinite complex and para–complex space forms have the same spectrum as those of the metric g. A straightforward calculation shows that g has exactly the same second, fourth and sixth degree scalar curvature invariants as the symmetric models. The main difference between complex and para–complex space forms from the point of view of their Jacobi operators, is that the restriction of the metric to the subspace $E_{4k}(X) = \mathbb{R}X \oplus \ker(R_X - 4k\,\mathrm{Id})$ is definite in the complex case and indefinite in the para–complex setting [23]. The metric g induces a Lorentzian inner product on E_{4k}, because the Jacobi operators are non–diagonalizable. It follows from the expression of the Jacobi operator associated with any non–null vector $X = \sum \alpha_i \partial_i$ that $-\alpha_4\partial_1 + \alpha_3\partial_2$ is a null eigenvector of R_X corresponding to the double eigenvalue k.

3.2 Osserman para-Hermitian metrics

In order to give some motivation for the metrics g discussed in the previous section we show that they appear naturally in the study of Walker para–Hermitian structures [49], [50].

A starting point in the search of Osserman spaces with non–diagonalizable Jacobi operators is the known fact that in the case of two different eigenvalues α and β (α with multiplicity two) we have $\beta = 4\alpha$ (see [21]). As it was discussed in Remark 3.11, an important difference between complex and para–complex space forms from the point of view of their Jacobi operators is that the restriction of the metric to the subspace $E_\beta = \mathbb{R}X \oplus \ker(R_X - \beta\,\mathrm{Id})$ is definite in the complex case and indefinite in the para–complex setting [23]. In the case of two distinct eigenvalues, the non–diagonalizability of the Jacobi operators implies that the metric induces a Lorentzian inner product on E_β. This fact turns our attention to para–Kähler structures and, by extension, to Walker manifolds.

A *Walker manifold* is a triple $(M, g, \mathcal{D})$ where M is an n–dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r–dimensional parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximum dimension ($r = n/2$). Since the dimension of a null plane is $r \leq n/2$, the lowest dimensional case of a Walker metric is that of $(2, 2)$–manifolds admitting a field of parallel null two–planes. For such metrics a canonical form was obtained by A. G. Walker [129]. He showed the existence of suitable coordinates $(x_1, \ldots, x_4)$ where the metric is expressed as

$$g_{(x_1,x_2,x_3,x_4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}$$

for some functions a, b and c depending on the variables $(x_1, \ldots, x_4)$.

If a four–dimensional Walker manifold is assumed to be Osserman para–Kähler, then it is a Ricci flat manifold or a para–complex space form, and hence this kind of manifolds does not provide the new desired examples of Osserman manifolds whose Jacobi operators are neither diagonalizable nor nilpotent. This motivates the study of a more general situation: Walker four–dimensional manifolds equipped with a para–Hermitian structure, which we tackle in the following section.

3.2.1 Einstein para-Hermitian structures on Walker manifolds

Let g be a Walker metric expressed in the above coordinates $(x_1, \ldots, x_4)$. There is a natural almost para–Hermitian structure J defined by

$$\begin{aligned} J\partial_1 &= -\partial_1, & J\partial_2 &= \partial_2, \\ J\partial_3 &= -a\partial_1 + \partial_3, & J\partial_4 &= b\partial_2 - \partial_4, \end{aligned}$$

where as usual, $\partial_i = \partial/\partial x^i$. Throughout this section we use subscripts to denote partial derivatives of functions, that is, for each function h depending on $(x_1, \ldots, x_4)$ we write $h_i = \partial h/\partial x_i$.

After doing some straightforward calculations we determine the Levi–Civita connection of a Walker metric.

Lemma 3.12. *The non–vanishing components of the Levi–Civita connection are*

$$\begin{aligned} \nabla_{\partial_1}\partial_3 &= \frac{1}{2}a_1\partial_1 + \frac{1}{2}c_1\partial_2, \qquad \nabla_{\partial_1}\partial_4 = \frac{1}{2}c_1\partial_1 + \frac{1}{2}b_1\partial_2, \\ \nabla_{\partial_2}\partial_3 &= \frac{1}{2}a_2\partial_1 + \frac{1}{2}c_2\partial_2, \qquad \nabla_{\partial_2}\partial_4 = \frac{1}{2}c_2\partial_1 + \frac{1}{2}b_2\partial_2, \\ \nabla_{\partial_3}\partial_3 &= \frac{1}{2}(aa_1 + ca_2 + a_3)\partial_1 + \frac{1}{2}(ca_1 + ba_2 - a_4 + 2c_3)\partial_2 - \frac{a_1}{2}\partial_3 - \frac{a_2}{2}\partial_4, \\ \nabla_{\partial_3}\partial_4 &= \frac{1}{2}(a_4 + ac_1 + cc_2)\partial_1 + \frac{1}{2}(b_3 + cc_1 + bc_2)\partial_2 - \frac{c_1}{2}\partial_3 - \frac{c_2}{2}\partial_4, \\ \nabla_{\partial_4}\partial_4 &= \frac{1}{2}(ab_1 + cb_2 - b_3 + 2c_4)\partial_1 + \frac{1}{2}(cb_1 + bb_2 + b_4)\partial_2 - \frac{b_1}{2}\partial_3 - \frac{b_2}{2}\partial_4. \end{aligned}$$

By analyzing the almost para–Hermitian structure J we obtain the following

Theorem 3.13. *The Walker metric g equipped with the almost para–Hermitian structure J is para–Hermitian if and only if $a_2 = b_1 = 0$. Moreover, the almost para–Kähler condition holds if and only if $c_1 = c_2 = 0$ and hence the para–Kähler condition is equivalent to $a_2 = b_1 = c_1 = c_2 = 0$.*

Proof. We write $J\partial_i = \sum_j J_i^j \partial_j$. The components of the Nijenhuis tensor are determined by

$$N_{jk}^i = 2\sum_{l=1}^{4}\left(J_j^l \frac{\partial J_k^i}{\partial x_l} - J_k^l \frac{\partial J_j^i}{\partial x_l} - J_l^i \frac{\partial J_k^l}{\partial x_j} + J_j^l \frac{\partial J_j^l}{\partial x_k} \right).$$

The non–zero components are

$$N_{14}^2 = 4b_1, \qquad N_{23}^1 = 4a_2, \qquad N_{43}^1 = -2ba_2, \qquad N_{43}^2 = 2ab_1.$$

Hence, the integrability of J is characterized by $a_2 = b_1 = 0$. On the other hand, the second part of the result is obtained after a direct and straightforward calculation from Lemma 3.12. □

In the rest of this section we study four–dimensional Walker metrics equipped with the para–Hermitian structure J. We obtain a classification of Einstein para–Hermitian Walker metrics as a first step to analyze the Osserman condition for Walker manifolds.

Using Lemma 3.12, we calculate the Riemannian curvature tensor after some tedious calculations:

Lemma 3.14. *The curvature tensor of the Walker metric g is given by*

$$\begin{aligned}
&R_{1313} = -\frac{1}{2}a_{11}, \qquad R_{1314} = -\frac{1}{2}c_{11}, \qquad R_{1323} = -\frac{1}{2}a_{12}, \qquad R_{1324} = -\frac{1}{2}c_{12},\\
&R_{1414} = -\frac{1}{2}b_{11}, \qquad R_{1423} = -\frac{1}{2}c_{12}, \qquad R_{1424} = -\frac{1}{2}b_{12}, \qquad R_{2424} = -\frac{1}{2}b_{22},\\
&R_{2323} = -\frac{1}{2}a_{22}, \qquad R_{2324} = -\frac{1}{2}c_{22}, \qquad R_{2434} = \frac{1}{4}\left(a_2b_1 - c_1c_2 - 2b_{23} + 2c_{24}\right),\\
&R_{1334} = \frac{1}{4}\left(-a_2b_1 + c_1c_2 + 2a_{14} - 2c_{13}\right),\\
&R_{1434} = \frac{1}{4}\left(-c_1^2 + a_1b_1 - b_1c_2 + b_2c_1 - 2b_{13} + 2c_{14}\right),\\
&R_{2334} = \frac{1}{4}\left(c_2^2 - a_2b_2 - a_1c_2 + a_2c_1 + 2a_{24} - 2c_{23}\right),\\
&R_{3434} = \frac{1}{4}\big(-ac_1^2 - bc_2^2 + aa_1b_1 + ca_1b_2 - a_1b_3 + 2a_1c_4 + ca_2b_1 + ba_2b_2 + a_2b_4\\
&\qquad\qquad + a_3b_1 - a_4b_2 - 2a_4c_1 + 2b_2c_3 - 2b_3c_2 - 2cc_1c_2 - 2a_{44} - 2b_{33} + 4c_{34}\big).
\end{aligned}$$

Using the previous result we calculate the Ricci tensor and the scalar curvature.

Lemma 3.15. *The Ricci tensor of the four–dimensional Walker metric g is given by*

$$\begin{aligned}
&\rho_{13} = \frac{1}{2}\left(a_{11} + c_{12}\right), \qquad \rho_{14} = \frac{1}{2}\left(b_{12} + c_{11}\right),\\
&\rho_{23} = \frac{1}{2}\left(a_{12} + c_{22}\right), \qquad \rho_{24} = \frac{1}{2}\left(b_{22} + c_{12}\right),\\
&\rho_{33} = \frac{1}{2}\left(-c_2^2 + a_1c_2 + a_2b_2 - a_2c_1 + aa_{11} + 2ca_{12} + ba_{22} + 2c_{23} - 2a_{24}\right),\\
&\rho_{34} = \frac{1}{2}\left(-a_2b_1 + c_1c_2 + a_{14} + b_{23} + ac_{11} + 2cc_{12} - c_{13} + bc_{22} - c_{24}\right),\\
&\rho_{44} = \frac{1}{2}\left(-c_1^2 + a_1b_1 - b_1c_2 + b_2c_1 + ab_{11} + 2cb_{12} - 2b_{13} + bb_{22} + 2c_{14}\right).
\end{aligned}$$

As a consequence, the scalar curvature is $\tau = a_{11} + b_{22} + 2c_{12}$.

Corollary 3.16. *The traceless Ricci tensor $\rho^0 = \rho - (\tau/4)g$ of the Walker metric g is determined by*

$$\rho^0_{13} = -\rho^0_{24} = \frac{1}{4}(a_{11} - b_{22}), \qquad \rho^0_{14} = \frac{1}{2}(b_{12} + c_{11}), \qquad \rho^0_{23} = \frac{1}{2}(a_{12} + c_{22}),$$

$$\rho^0_{33} = \frac{1}{4}\left(2a_1c_2 + 2a_2b_2 - 2a_2c_1 - 2c_2^2 + a(a_{11} - b_{22}) + 4ca_{12} + 2ba_{22} - 4a_{24} - 2ac_{12} + 4c_{23}\right),$$

$$\rho^0_{44} = \frac{1}{4}\left(2a_1b_1 - 2b_1c_2 + 2b_2c_1 - 2c_1^2 - b(a_{11} - b_{22}) + 2ab_{11} + 4cb_{12} - 4b_{13} - 2bc_{12} + 4c_{14}\right),$$

$$\rho^0_{34} = \frac{1}{4}(-2a_2b_1 + 2c_1c_2 - c(a_{11} - 2c_{12} + b_{22}) + 2a_{14} + 2b_{23} + 2ac_{11} - 2c_{13} + 2bc_{22} - 2c_{24}).$$

We are now ready to characterize Einstein para–Hermitian Walker metrics.

Theorem 3.17. *The four dimensional Walker metric g equipped with the almost para–Hermitian structure J is Einstein para-Hermitian if and only if the defining functions a, b and c are any of the following types:*

(A) The scalar curvature τ vanishes and a, b and c can be written as

$$\begin{aligned} a &= a(x_1, x_3, x_4) &&= x_1P(x_3, x_4) + \xi(x_3, x_4), \\ b &= b(x_2, x_3, x_4) &&= x_2Q(x_3, x_4) + \eta(x_3, x_4), \\ c &= c(x_1, x_2, x_3, x_4) &&= x_1S(x_3, x_4) + x_2T(x_3, x_4) + \gamma(x_3, x_4), \end{aligned}$$

where ξ, η and γ are arbitrary smooth functions, and P, Q, S, T are smooth functions satisfying

$$PT - T^2 + 2T_3 = 0, \qquad QS - S^2 + 2S_4 = 0, \qquad ST + Q_3 - S_3 + P_4 - T_4 = 0.$$

(B) The scalar curvature τ is non–zero and a, b and c satisfy

$$\begin{aligned} a &= a(x_1, x_3, x_4) &&= \frac{\tau}{4}x_1^2 + x_1P(x_3, x_4) + \xi(x_3, x_4), \\ b &= b(x_2, x_3, x_4) &&= \frac{\tau}{4}x_2^2 + x_2Q(x_3, x_4) + \eta(x_3, x_4), \\ c &= c(x_3, x_4) &&= \frac{2}{\tau}(P_4(x_3, x_4) + Q_3(x_3, x_4)), \end{aligned}$$

where P, Q, ξ and η are arbitrary smooth functions.

(C) The scalar curvature τ is non–zero and a, b and c can be written as

$$\begin{aligned} a &= a(x_1, x_3, x_4) &&= \frac{\tau}{6}x_1^2 + x_1P + \frac{6}{\tau}\left(PT - T^2 + 2T_3\right), \\ b &= b(x_2, x_3, x_4) &&= \frac{\tau}{6}x_2^2 + x_2Q + \frac{6}{\tau}\left(QS - S^2 + 2S_4\right), \\ c &= c(x_1, x_2, x_3, x_4) &&= \frac{\tau}{6}x_1x_2 + x_1S + x_2T + \frac{6}{\tau}(ST + Q_3 - S_3 + P_4 - T_4), \end{aligned}$$

for any smooth functions P, Q, S and T depending on (x_3, x_4).

Proof. Since J is para–Hermitian, we have $a_2 = b_1 = 0$ by Theorem 3.13. Hence, $a = a(x_1, x_3, x_4)$ and $b = b(x_2, x_3, x_4)$. Since g is Einstein, we have $\rho^0 = 0$. Using the previous fact for the functions a and b and Corollary 3.16 we get

$$
\begin{aligned}
& a_{11} - b_{22} \;=\; c_{11} \;=\; c_{22} \;=\; 0, \\
& 2a_1c_2 - 2c_2^2 - 2ac_{12} + 4c_{23} \;=\; 2b_2c_1 - 2c_1^2 - 2bc_{12} + 4c_{14} \;=\; 0, \\
& 2c_1c_2 - ca_{11} + 2a_{14} - cb_{22} + 2b_{23} + 2cc_{12} - 2c_{13} - 2c_{24} \;=\; 0.
\end{aligned}
\tag{3.2}
$$

We separate the proof of this theorem in three steps.

Claim 3.18. *The functions a, b and c defining the metric g satisfy*

$$
\begin{aligned}
a &= a(x_1, x_3, x_4) &&= x_1^2\kappa(x_3, x_4) + x_1P(x_3, x_4) + \xi(x_3, x_4), \\
b &= b(x_2, x_3, x_4) &&= x_2^2\kappa(x_3, x_4) + x_2Q(x_3, x_4) + \eta(x_3, x_4), \\
c &= c(x_1, x_2, x_3, x_4) &&= x_1x_2\alpha(x_3, x_4) + x_1S(x_3, x_4) + x_2T(x_3, x_4) + \gamma(x_3, x_4)
\end{aligned}
$$

where $\kappa(x_3, x_4)$, $P(x_3, x_4)$, $Q(x_3, x_4)$, $\xi(x_3, x_4)$, $\eta(x_3, x_4)$, $\alpha(x_3, x_4)$, $S(x_3, x_4)$, $T(x_3, x_4)$ and $\gamma(x_3, x_4)$ are arbitrary functions.

The first equation in (3.2) and $a_2 = b_1 = 0$ implies $a_{111} = b_{222} = 0$ and hence a (resp. b) is a quadratic function of x_1 (resp. x_2) with parameters x_3 and x_4. Then, we can express a and b as stated in the first two equations of Claim 3.18. On the other hand, the last two equalities of the first equation in (3.2) imply that c is a linear function with respect to x_1 and x_2, taking the form of the third equation of Claim 3.18.

Claim 3.19. *The functions a, b and c can be written as*

$$
\begin{aligned}
a &= a(x_1, x_3, x_4) &&= \kappa\, x_1^2 + x_1P(x_3, x_4) + \xi(x_3, x_4), \\
b &= b(x_2, x_3, x_4) &&= \kappa\, x_2^2 + x_2Q(x_3, x_4) + \eta(x_3, x_4), \\
c &= c(x_1, x_2, x_3, x_4) &&= \left(\frac{\tau}{2} - 2\kappa\right) x_1x_2 + x_1S(x_3, x_4) + x_2T(x_3, x_4) + \gamma(x_3, x_4),
\end{aligned}
$$

where κ is a constant and $P(x_3, x_4)$, $Q(x_3, x_4)$, $\xi(x_3, x_4)$, $\eta(x_3, x_4)$, $S(x_3, x_4)$, $T(x_3, x_4)$ and $\gamma(x_3, x_4)$ are arbitrary functions. Moreover, one of the following three possibilities can occur: $\kappa = \tau = 0$ or, in case $\tau \neq 0$, either $\kappa = \frac{\tau}{4}$ or $\kappa = \frac{\tau}{6}$.

Lemma 3.15 combined with Claim 3.18 implies $\tau = 4\kappa(x_3, x_4) + 2\alpha(x_3, x_4)$. Hence, $\alpha(x_3, x_4) = \tau/2 - 2\kappa(x_3, x_4)$. We recall that, since g is Einstein, the scalar curvature τ is constant. Differentiating the second equation in (3.2) twice with respect to x_1, we get

$$\tau^2 - 10\tau\kappa(x_3, x_4) + 24\kappa(x_3, x_4)^2 = 0.$$

Thus, $\kappa(x_3, x_4)$ must be constant and Claim 3.19 follows.

We are now ready to finish the proof of Theorem 3.17. We analyze the three different possibilities which arise in Claim 3.19 separately.

Assume $\kappa = \tau = 0$. This is the simplest case, because the expression in Claim 3.19 reduces to

$$\begin{array}{rclcl} a & = & a(x_1, x_3, x_4) & = & x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b & = & b(x_2, x_3, x_4) & = & x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c & = & c(x_1, x_2, x_3, x_4) & = & x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4). \end{array}$$

Furthermore, for such functions the last two equations in (3.2) transform into

$$PT - T^2 + 2T_3 = 0, \quad QS - S^2 + 2S_4 = 0, \quad ST + Q_3 - S_3 + P_4 - T_4 = 0,$$

which is exactly case (A) of Theorem 3.17.

Assume $\kappa = \tau/4 \neq 0$. In this case, the expression of Claim 3.19 transforms into

$$\begin{array}{rclcl} a & = & a(x_1, x_3, x_4) & = & \frac{\tau}{4} x_1^2 + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b & = & b(x_2, x_3, x_4) & = & \frac{\tau}{4} x_2^2 + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c & = & c(x_1, x_2, x_3, x_4) & = & x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4). \end{array}$$

The second equation in (3.2) reduces to

$$\begin{array}{l} \big(\tau x_1 + 2P(x_3, x_4)\big) T(x_3, x_4) - 2T(x_3, x_4)^2 + 4T_3(x_3, x_4) = 0, \\ \big(\tau x_2 + 2Q(x_3, x_4)\big) S(x_3, x_4) - 2S(x_3, x_4)^2 + 4S_4(x_3, x_4) = 0, \end{array}$$

which hold if and only if $T(x_3, x_4) = S(x_3, x_4) = 0$. Using this condition, the last equation in (3.2) leads to $\tau\gamma(x_3, x_4) - 2(P_4(x_3, x_4) + Q_3(x_3, x_4)) = 0$ and therefore we can determine γ by

$$\gamma(x_3, x_4) = \frac{2}{\tau}(P_4(x_3, x_4) + Q_3(x_3, x_4)).$$

Altogether this implies case (B) of Theorem 3.17.

Assume $\kappa = \tau/6 \neq 0$. In this case, the expression in Claim 3.19 yields

$$\begin{array}{rclcl} a & = & a(x_1, x_3, x_4) & = & \frac{\tau}{6} x_1^2 + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b & = & b(x_2, x_3, x_4) & = & \frac{\tau}{6} x_2^2 + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c & = & c(x_1, x_2, x_3, x_4) & = & \frac{\tau}{6} x_1 x_2 + x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4), \end{array}$$

and a straightforward calculation shows that the last two equations in (3.2) transform into

$$\begin{array}{l} \frac{\tau}{6}\, \xi(x_3, x_4) - (P(x_3, x_4) T(x_3, x_4) - T(x_3, x_4)^2 + 2T_3(x_3, x_4)) = 0, \\ \frac{\tau}{6}\, \eta(x_3, x_4) - (Q(x_3, x_4) S(x_3, x_4) - S(x_3, x_4)^2 + 2S_4(x_3, x_4)) = 0, \\ \frac{\tau}{6}\, \gamma(x_3, x_4) - (S(x_3, x_4) T(x_3, x_4) + Q_3(x_3, x_4) - S_3(x_3, x_4) + P_4(x_3, x_4) - T_4(x_3, x_4)) = 0, \end{array}$$

from where we can determine $\xi(x_3,x_4)$, $\eta(x_3,x_4)$ and $\gamma(x_3,x_4)$ as follows

$$\begin{aligned}
\xi(x_3,x_4) &= \frac{6}{\tau}(P(x_3,x_4)T(x_3,x_4) - T(x_3,x_4)^2 + 2T_3(x_3,x_4)),\\
\eta(x_3,x_4) &= \frac{6}{\tau}(Q(x_3,x_4)S(x_3,x_4) - S(x_3,x_4)^2 + 2S_4(x_3,x_4)),\\
\gamma(x_3,x_4) &= \frac{6}{\tau}(S(x_3,x_4)T(x_3,x_4) + Q_3(x_3,x_4) - S_3(x_3,x_4) + P_4(x_3,x_4) - T_4(x_3,x_4)).
\end{aligned}$$

Altogether this implies case (C) in Theorem 3.17, which finishes the proof. □

3.2.2 Osserman para–Hermitian structures on Walker manifolds

In this section we analyze the Osserman condition for the three families of Einstein para–Hermitian Walker metrics determined in Theorem 3.17. We study each case separately.

Einstein para–Hermitian metrics of type (A)

Einstein para–Hermitian Walker metrics of type (A) defined in Theorem 3.17 are Osserman, but they do not provide the new desired examples. Indeed, if $X = \sum_{i=1}^4 \alpha_i\partial_i$ is an arbitrary vector then the associated Jacobi operator, when expressed in the coordinate basis, has the form

$$R_X = \begin{pmatrix} A & B \\ 0 & {}^tA \end{pmatrix}, \qquad \text{where} \qquad A = \frac{\Psi}{4}\begin{pmatrix} -\alpha_3\alpha_4 & -\alpha_4^2 \\ \alpha_3^2 & \alpha_3\alpha_4 \end{pmatrix}$$

and $\Psi = Q_3 + S_3 - P_4 - T_4$. Hence the characteristic polynomial of the Jacobi operators is $p_{R_X}(\lambda) = \lambda^4$ (independently of the 2×2-matrix B). Therefore the Jacobi operators are either vanishing or nilpotent.

Einstein para–Hermitian metrics of type (B)

Metrics in the family (B) of Theorem 3.17 are not Osserman. To see this, recall that a four–dimensional semi–Riemannian manifold is pointwise Osserman if and only if there is a choice of orientation such that the manifold is Einstein self–dual (or anti–self–dual). See [3], [62].

Given the Walker metric g, we have that

$$\begin{aligned}
e_1 &= \frac{1}{2}(1-a)\partial_1 + \partial_3, & e_2 &= -c\partial_1 + \frac{1}{2}(1-b)\partial_2 + \partial_4,\\
e_3 &= -\frac{1}{2}(1+a)\partial_1 + \partial_3, & e_4 &= -c\partial_1 - \frac{1}{2}(1+b)\partial_2 + \partial_4
\end{aligned}$$

defines an orthonormal basis of the tangent space. Local bases of the spaces of self–dual and anti–self–dual two–forms can be constructed as $\Lambda^2_\pm = \operatorname{span}\left\{E_1^\pm, E_2^\pm, E_3^\pm\right\}$, where

$$E_1^\pm = \frac{e^1\wedge e^2 \pm e^3\wedge e^4}{\sqrt{2}}, \qquad E_2^\pm = \frac{e^1\wedge e^3 \pm e^2\wedge e^4}{\sqrt{2}}, \qquad E_3^\pm = \frac{e^1\wedge e^4 \mp e^2\wedge e^3}{\sqrt{2}}.$$

A long but direct and straightforward calculation using Lemmas 3.14 and 3.15 and the definition of the Weyl tensor shows that $W^+_{22} = W^-_{22} = -\tau/6$, and hence para–Hermitian Walker metrics of type (B) cannot be Osserman. See Lemma 3.23 for details.

Einstein para–Hermitian metrics of type (C)

This last family of Einstein para–Hermitian Walker metrics will provide the desired examples of Osserman spaces. In particular, we have the following

Theorem 3.20. *An Einstein para–Hermitian Walker metric of type (C) is Osserman of signature* $(2,2)$ *with eigenvalues* $\{0, \tau/6, \tau/24, \tau/24\}$.

Proof. After a long but straightforward calculation one gets that (see Lemma 3.23)

$$W^- = 0, \qquad W^+ = \begin{pmatrix} W^+_{11} & W^+_{12} & W^+_{11} + \frac{\tau}{12} \\ -W^+_{12} & \frac{\tau}{6} & -W^+_{12} \\ -(W^+_{11} + \frac{\tau}{12}) & -W^+_{12} & -(W^+_{11} + \frac{\tau}{6}) \end{pmatrix},$$

and hence it follows that W^+ has eigenvalues $\{\tau/6, -\tau/12, -\tau/12\}$. As a consequence, any Einstein para–Hermitian Walker metric defined by Theorem 3.17 (C) is Osserman (Einstein self–dual) and thus the eigenvalues of the self–dual operator W^+ determine the eigenvalues of the Jacobi operators, which turn out to be $\{0, \tau/6, \tau/24, \tau/24\}$. □

Remark 3.21. Note that the metric studied in Section 3.1 is a particular case of the general family of Einstein para–Hermitian Walker metrics of Theorem 3.17 (C).

3.3 General description of Osserman metrics whose Jacobi operators have two distinct non-zero eigenvalues

The purpose of this section is to clarify the situation of Type II Jordan–Osserman metrics by proving the following [49]

Theorem 3.22. *Let* (M, g) *be a four–dimensional Type II Jordan–Osserman manifold. Then the Jacobi operators are either two–step nilpotent or there exist local coordinates* $(x_1, \dots, x_4)$ *such that the metric is given by*

$$dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2 + \sum_{i,j=3}^{4} s_{ij}\, dx^i \otimes dx^j$$

for some functions $s_{ij}(x_1, \ldots, x_4)$ *which can be written as*

$$\begin{aligned}
s_{33} &= x_1^2 \frac{\tau}{6} + x_1 P + x_2 Q + \frac{6}{\tau}\big\{Q(T-U) + V(P-V) - 2(Q_4 - V_3)\big\}, \\
s_{44} &= x_2^2 \frac{\tau}{6} + x_1 S + x_2 T + \frac{6}{\tau}\big\{S(P-V) + U(T-U) - 2(S_3 - U_4)\big\}, \\
s_{34} &= s_{43} = x_1 x_2 \frac{\tau}{6} + x_1 U + x_2 V + \frac{6}{\tau}\big\{-QS + UV + T_3 - U_3 + P_4 - V_4\big\},
\end{aligned}$$

where P*,* Q*,* S*,* T*,* U *and* V *are arbitrary functions depending on the coordinates* (x_3, x_4)*.*

The proof of Theorem 3.22 is based on the following facts:

1. A four–dimensional semi–Riemannian manifold is pointwise Osserman if and only if it is Einstein self–dual (or anti–self–dual) [3], [80].

2. A Type II Jordan–Osserman metric is either Ricci flat (that is, $\alpha = \beta = 0$) or $\beta = 4\alpha \neq 0$ [21, Corollary 8.3].

3. A Type II Jordan–Osserman metric whose Jacobi operators are not nilpotent (that is, $\alpha = 4\beta \neq 0$) admits a local parallel field of two–dimensional planes [21, Proposition 8.4].

Therefore, we investigate Walker metrics (which are those admitting a locally defined two–dimensional degenerate parallel distribution) in detail in Subsection 3.3.1, with special attention to the (anti–)self–dual Weyl curvature tensors. A complete description of self–dual Walker metrics is given in Subsection 3.3.2. The integration of the Einstein equation for a self–dual Walker metric, which lets us determine all pointwise Osserman self–dual Walker metrics, is carried out in Subsection 3.3.3. This leads to the proof of Theorem 3.22.

3.3.1 Self-duality and anti-self-duality conditions

In this section we obtain the expression of the self–dual and the anti–self–dual Weyl conformal curvature tensors for the Walker metric g given at the beginning of Section 3.2 with respect to an orthonormal basis $\{e_1, \ldots, e_4\}$ where

$$\begin{aligned}
e_1 &= \frac{1}{2}(1-a)\partial_1 + \partial_3, \qquad & e_2 &= -c\partial_1 + \frac{1}{2}(1-b)\partial_2 + \partial_4, \\
e_3 &= -\frac{1}{2}(1+a)\partial_1 + \partial_3, \qquad & e_4 &= -c\partial_1 - \frac{1}{2}(1+b)\partial_2 + \partial_4.
\end{aligned}$$

A long but straightforward calculation using Lemma 3.14 and the expressions for the Ricci tensor and the scalar curvature in Lemma 3.15 implies the following lemma.

Lemma 3.23. *With respect to the above basis, the components of W^- are given by*

$$
\begin{aligned}
W^-_{11} &= -\frac{1}{12}(a_{11}+3a_{22}+3b_{11}+b_{22}-4c_{12}), & W^-_{22} &= -\frac{1}{6}(a_{11}+b_{22}-4c_{12}),\\
W^-_{33} &= \frac{1}{12}(a_{11}-3a_{22}-3b_{11}+b_{22}-4c_{12}), & W^-_{12} &= \frac{1}{4}(a_{12}+b_{12}-c_{11}-c_{22}),\\
W^-_{13} &= \frac{1}{4}(a_{22}-b_{11}), & W^-_{23} &= -\frac{1}{4}(a_{12}-b_{12}+c_{11}-c_{22}).
\end{aligned}
$$

The components of W^+ are determined by W^+_{11}, W^+_{12} and the scalar curvature as follows

$$
W^+_{22} = -\frac{\tau}{6}, \qquad W^+_{33} = W^+_{11} + \frac{\tau}{6}, \qquad W^+_{13} = W^+_{11} + \frac{\tau}{12}, \qquad W^+_{23} = W^+_{12}.
$$

Finally, we have the expressions for W^+_{11} and W^+_{12}:

$$
\begin{aligned}
W^+_{11} = \frac{1}{12}\Big(& 6ca_1b_2 - 6a_1b_3 - 6ba_1c_2 + 12a_1c_4 - 6ca_2b_1 + 6a_2b_4 + 6ba_2c_1 + 6a_3b_1 - 6a_4b_2\\
& - 12a_4c_1 + 6ab_1c_2 - 6ab_2c_1 + 12b_2c_3 - 12b_3c_2 - a_{11} - 12c^2a_{11} - 12bca_{12}\\
& + 24ca_{14} - 3b^2a_{22} + 12ba_{24} - 12a_{44} - 3a^2b_{11} + 12ab_{13} - b_{22} - 12b_{33}\\
& + 12acc_{11} - 2c_{12} + 6abc_{12} - 24cc_{13} - 12ac_{14} - 12bc_{23} + 24c_{34}\Big),\\
W^+_{12} = \frac{1}{4}(& -2ca_{11} - ba_{12} + 2a_{14} + ab_{12} - 2b_{23} + ac_{11} - 2cc_{12} - 2c_{13} - bc_{22} + 2c_{24}).
\end{aligned}
$$

Remark 3.24. The connection between Einstein (anti–)self–dual and pointwise Osserman manifolds goes further to the Jordan normal forms of the non–zero part of the Weyl curvature tensor $W^\pm$ and the Jacobi operators (see [62]). Pointwise Osserman manifolds whose Jacobi operators are of Type Ia, Ib, II or III correspond to self–dual (or anti–self–dual) Einstein manifolds whose self–dual (or anti–self–dual) Weyl curvature tensor is of Type Ia, Ib, II or III, respectively.

Lemma 3.23 shows that

$$
W^+ = \begin{pmatrix} W^+_{11} & W^+_{12} & W^+_{11} + \frac{\tau}{12} \\ -W^+_{12} & \frac{\tau}{6} & -W^+_{12} \\ -(W^+_{11} + \frac{\tau}{12}) & -W^+_{12} & -(W^+_{11} + \frac{\tau}{6}) \end{pmatrix},
$$

and, as a consequence, the eigenvalues of W^+ are $\{\tau/6, -\tau/12, -\tau/12\}$. Since the induced metric on Λ^2_+ has Lorentzian signature, the structure of W^+ is determined by its Jordan normal form, which may correspond to Type Ia or Type II/III, depending on whether W^+ is diagonalizable or not. A straightforward calculation shows that

$$
\left(W^+ - \frac{\tau}{6}Id\right)\cdot\left(W^+ + \frac{\tau}{12}Id\right) = \frac{\tau^2 + 12\tau W^+_{11} + 48\left(W^+_{12}\right)^2}{48}\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},
$$

from where we have the following:

(i) If $\tau \neq 0$, we have that W^+ has non–zero eigenvalues $\{\tau/6, -\tau/12, -\tau/12\}$ and the equality $\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0$ is the necessary and sufficient condition for the diagonalizability of W^+. If the above equation does not hold, then $-\tau/12$ is a double root of the minimal polynomial of W^+.

(ii) If $\tau = 0$, then W^+ vanishes if and only if $W_{11}^+ = W_{12}^+ = 0$ and moreover

1. W^+ is two–step nilpotent if and only if $W_{11}^+ \neq 0$ and $W_{12}^+ = 0$,
2. W^+ is three–step nilpotent if and only if $W_{12}^+ \neq 0$.

On the other hand, taking into account the eigenvalues of W^+, any anti–self–dual Walker metric has vanishing scalar curvature and hence Einstein anti–self–dual Walker metrics are Ricci flat.

3.3.2 Explicit form of self-dual Walker metrics

Our main purpose is to obtain a description of non–Ricci flat Type II Jordan–Osserman four–dimensional manifolds. As a consequence of Remark 3.24 we may restrict our analysis to self–dual Walker metrics. In this section we give a complete description of self–dual Walker metrics by integrating the partial differential equations obtained from Lemma 3.23.

Theorem 3.25. *A Walker metric g is self–dual if and only if the defining functions a, b and c are given by*

$$a = x_1^3\mathcal{A} + x_1^2\mathcal{B} + x_1^2x_2\mathcal{C} + x_1x_2\mathcal{D} + x_1P + x_2Q + \xi,$$

$$b = x_2^3\mathcal{C} + x_2^2\mathcal{E} + x_1x_2^2\mathcal{A} + x_1x_2\mathcal{F} + x_1S + x_2T + \eta,$$

$$c = \frac{1}{2}x_1^2\mathcal{F} + \frac{1}{2}x_2^2\mathcal{D} + x_1^2x_2\mathcal{A} + x_1x_2^2\mathcal{C} + \frac{1}{2}x_1x_2(\mathcal{B}+\mathcal{E}) + x_1U + x_2V + \gamma,$$

where P, Q, S, T, U, V, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, ξ, η and γ are functions depending on the coordinates (x_3, x_4).

Proof. Using Lemma 3.23 the self–duality can be initially characterized by means of the following five equations

$$a_{22} = b_{11} = a_{12} - c_{22} = b_{12} - c_{11} = a_{11} + b_{22} - 4c_{12} = 0. \tag{3.3}$$

Claim 3.26. *We have*

$$\begin{aligned} a(x_1,x_2,x_3,x_4) &= x_2A(x_1,x_3,x_4) + B(x_1,x_3,x_4), \\ b(x_1,x_2,x_3,x_4) &= x_1C(x_2,x_3,x_4) + D(x_2,x_3,x_4), \\ c(x_1,x_2,x_3,x_4) &= \frac{1}{2}x_2^2A_1(x_1,x_3,x_4) + x_2E(x_1,x_3,x_4) + F(x_1,x_3,x_4). \end{aligned}$$

for differentiable functions A, B, C, D, E and F.

The first and second equalities in (3.3) imply that a and b can be written as

$$\begin{aligned} a(x_1,x_2,x_3,x_4) &= x_2 A(x_1,x_3,x_4) + B(x_1,x_3,x_4), \\ b(x_1,x_2,x_3,x_4) &= x_1 C(x_2,x_3,x_4) + D(x_2,x_3,x_4) \end{aligned}$$

Hence the third equation in (3.3) reads $c_{22}(x_1,x_2,x_3,x_4) = A_1(x_1,x_3,x_4)$ which implies that

$$c(x_1,x_2,x_3,x_4) = \frac{1}{2}x_2^2 A_1(x_1,x_3,x_4) + x_2 E(x_1,x_3,x_4) + F(x_1,x_3,x_4).$$

This finishes the proof of Claim 3.26.

Claim 3.27. *We have*

$$\begin{aligned} a(x_1,x_2,x_3,x_4) &= x_1^3 x_2 G + x_1^2 x_2 \mathcal{C} + x_1 x_2 \mathcal{D} + x_2 Q + B, \\ b(x_1,x_2,x_3,x_4) &= x_1 x_2^3 G + x_1 x_2^2 \mathcal{A} + x_1 x_2 \mathcal{F} + x_1 \mathcal{S} + D, \\ c(x_1,x_2,x_3,x_4) &= \frac{3}{2}x_1^2 x_2^2 G + \frac{1}{2}x_1^2 \mathcal{F} + \frac{1}{2}x_2^2 \mathcal{D} + x_1^2 x_2 \mathcal{A} + x_1 x_2^2 \mathcal{C} + x_1 x_2 I + x_1 U + x_2 V + \gamma. \end{aligned}$$

where G, I, Q, U, V, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{F}$, $\mathcal{S}$ and γ are functions depending on (x_3,x_4) and B and D are functions depending on (x_2,x_3,x_4).

The fourth equality in (3.3) ($c_{11} - b_{12} = 0$) and Claim 3.26 imply

$$\frac{1}{2}x_2^2 A_{111}(x_1,x_3,x_4) + x_2 E_{11}(x_1,x_3,x_4) + F_{11}(x_1,x_3,x_4) - C_2(x_2,x_3,x_4) = 0.$$

Taking derivatives with respect to x_1 yields $A_{1111}(x_1,x_3,x_4) = 0$, $E_{111}(x_1,x_3,x_4) = 0$ and $F_{111}(x_1,x_3,x_4) = 0$. Hence

$$\begin{aligned} A(x_1,x_3,x_4) &= x_1^3 G(x_3,x_4) + x_1^2 \mathcal{C}(x_3,x_4) + x_1 \mathcal{D}(x_3,x_4) + Q(x_3,x_4), \\ E(x_1,x_3,x_4) &= x_1^2 H(x_3,x_4) + x_1 I(x_3,x_4) + V(x_3,x_4), \\ F(x_1,x_3,x_4) &= x_1^2 J(x_3,x_4) + x_1 U(x_3,x_4) + \gamma(x_3,x_4). \end{aligned}$$

Using the last two equations we get

$$3x_2^2 G(x_3,x_4) + 2x_2 \mathcal{A}(x_3,x_4) + 2J(x_3,x_4) - C_2(x_2,x_3,x_4) = 0,$$

from where, taking derivatives with respect to x_2, we get $C_{22}(x_2,x_3,x_4) = 6x_2 G(x_3,x_4) + 2\mathcal{A}(x_3,x_4)$ and hence

$$C(x_2,x_3,x_4) = x_2^3 G(x_3,x_4) + x_2^2 \mathcal{A}(x_3,x_4) + x_2 \mathcal{F}(x_3,x_4) + \mathcal{S}(x_3,x_4).$$

Finally the above equation reads $-2J(x_3,x_4) + \mathcal{F}(x_3,x_4) = 0$, that is, $J = \mathcal{F}/2$. Altogether this implies Claim 3.27.

We now finish the proof of Theorem 3.25. Plugging the expression of Claim 3.27 in the last equality of (3.3) and differentiating with respect to x_1 and x_2 leads to $G(x_3, x_4) = 0$ and hence that equality reduces to

$$6x_1\mathcal{A}(x_3, x_4) + 6x_2\mathcal{C}(x_3, x_4) + 4I(x_3, x_4) - B_{11}(x_1, x_3, x_4) - D_{22}(x_2, x_3, x_4) = 0.$$

Differentiation with respect to x_1 gives $B_{111}(x_1, x_2, x_3) = 6\mathcal{A}(x_3, x_4)$ and hence

$$B(x_1, x_3, x_4) = x_1^3\mathcal{A}(x_3, x_4) + x_1^2\mathcal{B}(x_3, x_4) + x_1P(x_3, x_4) + \xi(x_3, x_4).$$

As a consequence, the last equality becomes

$$6x_2\mathcal{C}(x_3, x_4) + 4I(x_3, x_4) - 2\mathcal{B}(x_3, x_4) - D_{22}(x_2, x_3, x_4) = 0,$$

from where $D_{222}(x_2, x_3, x_4) = 6\mathcal{C}(x_3, x_4)$ and hence

$$D(x_2, x_3, x_4) = x_2^3\mathcal{C}(x_3, x_4) + x_2^2\mathcal{E}(x_3, x_4) + x_2T(x_3, x_4) + \eta(x_3, x_4).$$

Thus, the last equation turns into $2I(x_3, x_4) - \mathcal{B}(x_3, x_4) - \mathcal{E}(x_3, x_4) = 0$ and we can write $I = (\mathcal{B} + \mathcal{E})/2$. Altogether this finishes the proof of Theorem 3.25. □

Remark 3.28. A four–dimensional Walker metric is said to be *strict* if it admits two orthogonal parallel null vector fields rather than a parallel two–dimensional null distribution. It follows from the work by Walker [129] that any strict Walker metric is given by

$$\begin{aligned} g_{(x_1,x_2,x_3,x_4)} &= dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2 + a(x_3, x_4)dx^3 \otimes dx^3 \\ &\quad + b(x_3, x_4)dx^4 \otimes dx^4 + c(x_3, x_4)\big(dx^3 \otimes dx^4 + dx^4 \otimes dx^3\big). \end{aligned}$$

Thus, Lemma 3.15 and Theorem 3.25 imply that any strict Walker metric is Ricci flat and self–dual, and hence Osserman. Moreover, Remark 3.24 shows that the Jacobi operators are either identically zero or two–step nilpotent (depending on whether $W_{11}^+ = 2c_{34} - a_{44} - b_{33}$ vanishes or not).

3.3.3 Proof of Theorem 3.22

Our purpose is to obtain a local description of Type II Jordan–Osserman metrics whose Jacobi operators have non–zero eigenvalues. In such a case, the eigenvalues must be in a ratio 1 : 4 and the underlying metric is a Walker metric. Therefore, in order to achieve the desired result, only Osserman metrics on Walker manifolds deserve further consideration. It immediately follows from Remark 3.24 that we may restrict to self–dual Walker metrics. In Theorem 3.29 we obtain a complete description of self–dual Einstein Walker metrics from where Theorem 3.22 is derived.

Theorem 3.29. *A Walker metric is pointwise Osserman self–dual if and only if one of the following holds:*

(i) The scalar curvature τ *is non–zero and the metric tensor is completely determined by the functions* a, b *and* c *as follows*

$$\begin{aligned} a &= x_1^2\frac{\tau}{6} + x_1P + x_2Q + \frac{6}{\tau}\Big\{Q(T-U) + V(P-V) - 2(Q_4 - V_3)\Big\},\\ b &= x_2^2\frac{\tau}{6} + x_1S + x_2T + \frac{6}{\tau}\Big\{S(P-V) + U(T-U) - 2(S_3 - U_4)\Big\},\\ c &= x_1x_2\frac{\tau}{6} + x_1U + x_2V + \frac{6}{\tau}\Big\{-QS + UV + T_3 - U_3 + P_4 - V_4\Big\}, \end{aligned}$$

where P, Q, S, T, U *and* V *are arbitrary functions depending on* (x_3, x_4). *In this case, the Jacobi operators have eigenvalues* $\{0, \tau/6, \tau/24, \tau/24\}$ *and they are diagonalizable if and only if* $\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0$. *Otherwise,* $\tau/24$ *is a double root of the minimal polynomial of the Jacobi operators and the Walker manifold is Jordan–Osserman on the open set where* $\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0$ *does not hold.*

(ii) The scalar curvature vanishes and the metric tensor is given by

$$\begin{aligned} a &= x_1P + x_2Q + \xi,\\ b &= x_1S + x_2T + \eta,\\ c &= x_1U + x_2V + \gamma, \end{aligned}$$

where P, Q, S, T, U, V, ξ, η *and* γ *are smooth functions depending only on* (x_3, x_4) *and satisfying*

$$\begin{aligned} &2(Q_4 - V_3) = Q(T-U) + V(P-V),\\ &2(S_3 - U_4) = S(P-V) + U(T-U),\\ &T_3 - U_3 + P_4 - V_4 = QS - UV. \end{aligned}$$

In this case, the Jacobi operators have zero eigenvalues and one of the following possibilities holds:

(a) The Jacobi operators vanish (which corresponds to the diagonalizable case Type Ia) if and only if $T_3 + U_3 - P_4 - V_4 = 0$ *and* $W_{11}^+ = 0$ *(see Lemma 3.23).*

(b) The Jacobi operators are two–step nilpotent (which corresponds to Type II) if and only if $T_3 + U_3 - P_4 - V_4 = 0$ *and* $W_{11}^+ \neq 0$.

(c) The Jacobi operators are three–step nilpotent (that is, Type III) if and only if $T_3 + U_3 - P_4 - V_4 \neq 0$.

Proof. Since the manifold is self–dual, the functions defining the Walker metric are completely determined by Theorem 3.25. Since the manifold is Einstein, the traceless Ricci tensor vanishes. Then, using Lemma 3.16, $\rho^0_{13} = \rho^0_{14} = \rho^0_{23} = 0$ becomes

$$2x_1\mathcal{A} - 2x_2\mathcal{C} + \mathcal{B} - \mathcal{E} = 0, \qquad 2x_2\mathcal{A} + \mathcal{F} = 0, \qquad 2x_1\mathcal{C} + \mathcal{D} = 0,$$

from where $\mathcal{A} = \mathcal{C} = \mathcal{D} = \mathcal{F} = 0$ and $\mathcal{E} = \mathcal{B}$. Hence, Lemma 3.15 implies that the (constant) scalar curvature is $\tau = 6\mathcal{B}$. As a consequence, the expression in Theorem 3.25 reduces to

$$\begin{aligned} a &= x_1^2\frac{\tau}{6} + x_1 P + x_2 Q + \xi, \\ b &= x_2^2\frac{\tau}{6} + x_1 S + x_2 T + \eta, \\ c &= x_1 x_2\frac{\tau}{6} + x_1 U + x_2 V + \gamma. \end{aligned}$$

Using Lemma 3.16 we have that $\rho^0_{33} = \rho^0_{34} = \rho^0_{44} = 0$ transforms into

$$\begin{aligned} &\frac{\tau}{6}\xi - \{Q(T-U) + V(P-V) - 2(Q_4 - V_3)\} = 0, \\ &\frac{\tau}{6}\gamma - \{-QS + UV + T_3 - U_3 + P_4 - V_4\} = 0, \\ &\frac{\tau}{6}\eta - \{S(P-V) + U(T-U) - 2(S_3 - U_4)\} = 0. \end{aligned}$$

If the scalar curvature τ does not vanish, we can determine ξ, η and γ from above. If $\tau = 0$ we get exactly the system of equations in Theorem 3.29 (ii).

Finally, the eigenvalues and the minimal polynomial of the Jacobi operators for the two cases are obtained as a direct application of Remark 3.24 since the eigenvalues and the minimal polynomial of the self–dual Weyl tensor W^+ determine the behavior of the Jacobi operators of a pointwise Osserman self–dual manifold. □

As a consequence of Theorem 3.29 and Remark 3.24 we have the following characterization of Jordan–Osserman Walker metrics.

Theorem 3.30. *Let (M, g) be a Jordan–Osserman Walker 4–dimensional manifold. Then, one of the following holds:*

(i) If the Jacobi operators are diagonalizable, then (M, g) is either flat or locally isometric to a para–complex space form.

(ii) If the Jacobi operators are non–diagonalizable, then one of the following two possibilities holds:

  (a) The Jacobi operators are two–step or three–step nilpotent.

  (b) The metric is given by Theorem 3.22.

Proof. Four–dimensional Jordan–Osserman manifolds with diagonalizable Jacobi operators have been classified in [21], where it is shown that they correspond to real, complex or para–complex space forms. Real and complex space forms do not support a Walker metric unless they are flat. Indeed, any space of constant curvature is locally conformally flat and hence $W^+ = 0$ implies that any such Walker metric is flat. Analogously, Kähler metrics of constant holomorphic sectional curvature have zero Bochner tensor, which shows that

$W^+ = 0$ [25], [63]. Hence, no Kähler metric of constant holomorphic sectional curvature is a Walker metric unless it is flat.

On the other hand, as Type Ib cannot occur [21], if the Jacobi operators are non–diagonalizable, they are of Type II or III. Since anti–self–dual Jordan–Osserman Walker metrics have vanishing scalar curvature, the corresponding Jacobi operators are either two–step or three–step nilpotent. The only remaining case is that of self–dual Jordan–Osserman Walker metrics, which corresponds to Theorem 3.22. This finishes the proof. □

Remark 3.31. Para–Kähler manifolds of constant para–holomorphic sectional curvature α may be easily described as Walker manifolds. For instance, let a, b and c be the coordinate functions given by

$$a(x_1, x_2, x_3, x_4) = \alpha x_1^2, \qquad b(x_1, x_2, x_3, x_4) = \alpha x_2^2, \qquad c(x_1, x_2, x_3, x_4) = \alpha x_1 x_2,$$

and J the para–complex structure determined by

$$J\partial_1 = -\partial_1, \qquad J\partial_2 = -\partial_2, \qquad J\partial_3 = -a\partial_1 - c\partial_2 + \partial_3, \qquad J\partial_4 = -c\partial_1 - b\partial_2 + \partial_4.$$

Then, $(\mathbb{R}^4, g, J)$ is a para–Kähler manifold of constant para–holomorphic sectional curvature α.

Remark 3.32. From Remark 3.24 we see that any anti–self–dual Jordan–Osserman Walker metric has nilpotent Jacobi operators. Although many nilpotent Jordan–Osserman metrics are known, none of the previous examples were anti–self–dual and all of them corresponded to special cases of Theorem 3.29. The general expression of W_{11}^+ given in Lemma 3.23 is untractable and hence it is very difficult to obtain the general form of anti–self–dual Walker metrics. However, for the special choice of $a(x_1, x_2, x_3, x_4) = b(x_1, x_2, x_3, x_4) = c(x_1, x_2, x_3, x_4)$, anti–self–dual Einstein metrics are characterized by

$$a_{11} = a_{22} = -a_{12}, \qquad a_{13} = a_{14}, \qquad a_{23} = a_{24}, \qquad a_{33} + a_{44} = 2a_{34}.$$

After some calculations it follows that

$$\begin{aligned} a(x_1, x_2, x_3, x_4) &= (x_2 - x_1)P(x_2 - x_1, x_3 + x_4) \\ &\quad + (x_1 + x_2)\alpha(x_3 + x_4) + x_3\beta(x_3 + x_4) + x_4\gamma(x_3 + x_4) + \delta(x_+ x_4), \end{aligned}$$

for some function P depending on two variables and some real–valued functions α, β, γ and δ. The corresponding Walker metric is an Osserman anti–self–dual Walker metric whose Jacobi operators are vanishing or two–step nilpotent, depending on whether the expression $2P_1 + (x_2 - x_1)P_{11}$ is zero or not,

Remark 3.33. Any Type III Jordan–Osserman Walker metric is Ricci flat, and thus the Jacobi operators are three–step nilpotent. The existence of non–Ricci flat Type III metrics is still an open problem.

Open problems

The following questions remain open.

- Obtain an optimal bound for $\mu(n)$ and $\mu_1(n)$. Our upper bound is obtained from variations of the Nash embedding theorem and we might expect that, under weaker conditions, the codimension of the submanifold could be lowered.

- Existence of non–nilpotent Osserman metrics with non–diagonalizable Jacobi operators in dimension higher than four. It is interesting to know whether our construction can be generalized for higher dimensions. We might expect this to be possible in other neutral signature settings.

- Obtain a description of Type III four–dimensional Osserman metrics. The complete solution of the Osserman problem is still an open question in dimension four, where the existence of non–nilpotent Type III Osserman metrics is unclear. It is not even known under which conditions these examples may exist.

Part II

Curvature invariants of geodesic spheres and geodesic celestial spheres

In order to study the geometry of a Riemannian manifold (M, g) it is often useful to consider objects naturally associated with the metric structure of (M, g). These can be special hypersurfaces such as small geodesic spheres and tubes, bundles with (M, g) as base manifold or families of transformations reflecting symmetry properties of (M, g) [128]. In this part of the thesis we focus on the study of geodesic spheres and their curvature in relation to the curvature of the ambient manifold. Indeed, the existence of a relationship between the curvature of a Riemannian manifold and the volume of its geodesic spheres and tubes led some authors to state the following question: "To what extent is the curvature or the geometry of a given Riemannian manifold influenced, or even determined, by the properties of certain naturally defined families of geometric objects in M?". This problem seems very difficult to handle in such a generality. However, when one looks at manifolds with a high degree of symmetry (for example two–point homogeneous spaces), these geometric objects have nice properties and one may expect to obtain characterizations of those spaces by means of such properties. By comparing a Riemannian manifold with a model space such as a two–point homogenous space we get an idea of its geometry. Thus, by understanding the geometry of spaces with a high degree of symmetry and why their properties are characteristic of them, we get a better insight into the geometry of a Riemannian manifold.

Since geodesic spheres are compact submanifolds, it makes sense to calculate their volume. A. Gray and L. Vanhecke calculated the first terms in the power series expansion of the volume of geodesic spheres [83]. They conjectured that the volume of geodesic spheres can be used to characterize Euclidean geometry. More specifically, if each geodesic sphere of a Riemannian manifold has the same volume as a Euclidean sphere of the same radius, then the manifold is flat. Although the answer is known to be affirmative in several special cases, the problem remains open in full generality.

Further work on geodesic spheres involved the investigation of their geometric properties and how they influence the geometry of the ambient manifold. Certain types of manifolds can be characterized by properties of geodesic spheres [33]. In this work, B.–Y. Chen and L. Vanhecke study intrinsic and extrinsic curvatures of geodesic spheres. It turns out that in many cases curvature properties provide a better understanding of geometry than volume properties.

The fact that the curvature tensor of a manifold is very difficult to handle motivated the study of several kinds of simplifications of this object. We are specially interested in the so–called scalar curvature invariants. Apart from their ubiquity in Riemannian geometry, specially when studying geodesic spheres and related objects, they are of interest by themselves. See for example [113] where a nice characterization of homogeneous spaces using scalar curvature invariants is given.

Our aim in Chapter 4 is to investigate curvature invariants of geodesic spheres. By integrating a scalar curvature invariant along every geodesic sphere of a manifold we get a good interplay between curvature and volume–like properties. The volume conjecture of A. Gray and L. Vanhecke may be generalized to these new objects. We see in Subsection 4.2.3 that in certain cases two–point homogeneous spaces can be characterized by the integrals of scalar curvature invariants of geodesic spheres. We emphasize that it suffices one single curvature invariant to characterize these model spaces. See Subsection 4.3.1 for examples of such curvature invariants.

In addition to geodesic spheres, other objects may be considered in Riemannian geometry which are also related to the Riemannian distance function: tubes around submanifolds and disks. The former are studied in Chapter 4 and they are of interest in the last part of this thesis. Geodesic disks are the main concern of Subsection 4.3.2. They were previously investigated by O. Kowalski and L. Vanhecke with special attention to their volume properties [93], [94], [95]. In this subsection we are interested in the intrinsic geometry of the boundaries of these disks and we devote our attention to the study of their total scalar curvatures obtained by integrating the scalar curvature and the quadratic scalar curvature invariants along these boundaries. Our main result is that two–point homogeneous spaces are characterized by some of the total curvatures of the boundaries of geodesic disks among Riemannian manifolds with adapted holonomy.

When the attention is turned from Riemannian manifolds to space–times, various difficulties emerge. An important characteristic of Riemannian manifolds is that they have a Riemannian distance function which is continuous and whose induced topology is the same as the topology of the manifold itself. Thus, several geometric objects such as geodesic spheres may be defined, at least locally, by means of this function. These objects are also Riemannian manifolds. They have nice properties, such as compactness and an acceptable behavior with respect to other constructions. When dealing with general semi–Riemannian manifolds there is no “semi–Riemannian distance” function. In fact, a distance–like function is only defined for space–times, but even in this case its properties are completely different from those in the Riemannian setting [7]. For example, the “Lorentzian distance” may not be continuous or bounded and geometric objects defined from it usually have awkward properties. Moreover, level sets of the Lorentzian distance function with respect to a given point are not compact and although some properties of those sets have been previously investigated, they do not seem to be adequate for the investigation of volume properties.

In Chapter 5 we consider a new family of geometric objects in Lorentzian geometry, namely geodesic celestial spheres. Roughly speaking, they are the set of points reached after a fixed distance travelling along radial geodesics emanating from a point which are orthogonal to a given timelike direction. In Relativity, a unit timelike vector represents an instantaneous observer and the vector subspace which is orthogonal to it is called the infinitesimal rest–space, that is, the infinitesimal Newtonian universe where the observer perceives particles as Newtonian particles relative to his rest position. Then, a geodesic celestial sphere is nothing but the image by the exponential map of a celestial sphere in the infinitesimal rest–space.

Following the idea of characterizing spaces with high degree of symmetry by means of volume properties of geometric objects, we carry out in Section 5.2 the calculation of the volume of geodesic celestial spheres. This depends on the radius, the base point and the instantaneous observer employed to define it. Nonetheless, in an isotropic Lorentzian manifold, this measure depends only on the radius. We see that this property is characteristic of locally isotropic Lorentzian manifolds. We discuss volume comparison results and give Bishop–Günther and Gromov type theorems for these objects in Section 5.2. Finally, in Section 5.3 we accomplish the characterization of locally isotropic Lorentzian manifolds using integrals of scalar curvature invariants of geodesic celestial spheres in the spirit of Chapter 4. We take advantage of the results of Subsection 4.2.3 to get this characterization.

In this part we tried to keep calculations to a minimum in order to make the work more readable. A package implementing the basic identities of curvature tensors has been developed by the author [39]. This package allows us to perform calculations involving scalar curvature invariants and integration along geodesic spheres. We can obtain both explicit expressions in two–point homogeneous spaces and power series expansions in general Riemannian manifolds.

Chapter 4

The Riemannian setting

Let $p, q \in M$ be two points of a Riemannian manifold M and $c : [a, b] \to M$ a curve joining p and q, that is, $c(a) = p$ and $c(b) = q$. The length of c is given by $L(c) = \int_a^b \|c'(t)\| dt$. The *Riemannian distance* between the points p and q is defined as

$$d(p, q) = \inf\{L(c) : c \text{ joins } p \text{ and } q\}.$$

This function is indeed a distance in M and the induced topology of d coincides with the topology of M as a topological manifold. We emphasize at this point that the above definition is characteristic of Riemannian geometry and cannot be directly generalized to the indefinite signature setting.

Given a point $m \in M$, the geodesic spheres of M centered at m are the level sets of the Riemannian distance function with respect to m, that is, $\{p \in M : d(p, m) = r\}$ for each radius $r > 0$. For sufficiently small radius r, these level sets are Riemannian hypersurfaces of M. Nevertheless, for some radii it might happen that these level sets have codimension greater that one or they fail to be submanifolds of M. The latter case is not of interest to us in this chapter and we restrict our definition of geodesic spheres to sufficiently small radii so that they are compact Riemannian submanifolds.

A geodesic sphere can also be defined as the image by the exponential map of a Euclidean sphere of the tangent space at a point. The fact that the Riemannian metric is positive definite ensures a good interplay between the exponential map and the Riemannian distance function. This definition is more operative for our purposes and provides us a good setting to perform the calculations needed in this chapter. The following chapter takes advantage of these ideas and proposes a new family of objects in Lorentzian manifolds whose properties may be used to characterize isotropy.

As it was stated before, we focus on the study of scalar curvature invariants of geodesic spheres, thus contributing to the investigation of how the curvature of geodesic spheres is related to the curvature of the ambient manifold.

This chapter is organized as follows. In Section 4.1 we introduce the main concepts of Jacobi vector field theory which are used both in this part and Part III. Then, we particularize this study to geodesic spheres in Section 4.2. We also introduce in this section the concept of simple Weyl invariant. By integrating simple Weyl invariants along geodesic spheres we get the so–called total curvatures of geodesic spheres. After giving some properties of these objects, we focus on the characterization of two–point homogeneous spaces in Subsection 4.2.3. Finally, Section 4.3 takes advantage of our previous work to provide examples of total curvatures of geodesic spheres and disks which may be used to characterize real, complex and quaternionic space forms.

4.1 Tubes and Jacobi vector field theory

Let $\bar{M}$ be a Riemannian manifold of dimension n and $M \subset \bar{M}$ a Riemannian submanifold of $\bar{M}$. For fixed $r > 0$, we define the set

$$G_M(r) = \{\exp(r\xi) : \xi \in T^{\perp}M,\ g(\xi,\xi) = 1\}.$$

In general $G_M(r)$ is not a Riemannian submanifold of $\bar{M}$. If M is a compact embedded submanifold of $\bar{M}$ it turns out that $G_M(r)$ is a compact hypersurface of $\bar{M}$ for sufficiently small radius. Thus, for a sufficiently small neighborhood of any point $p \in M$, a tube of sufficiently small radius around that neighborhood is a hypersurface. If $G_M(r)$ is a hypersurface then we say that $G_M(r)$ is the *tube* of radius r around M. We follow [13].

If $G_M(r)$ is a Riemannian submanifold of $\bar{M}$ of codimension greater than one, then $G_M(r)$ is called a *focal manifold* of M at distance r.

Let $p \in M$ and $c : I \to \bar{M}$ a geodesic parametrized by arc length with $c(0) = p$ and $c'(0) \in T_p^{\perp}M$. Let $F(s,t) = c_s(t)$ be a geodesic variation of $c = c_0$ such that $\gamma(s) = F(s,0) = c_s(0) \in M$ for all s and let us define $\xi(s) = c_s'(0) \in T^{\perp}M$. Let ζ be the variational vector field of F. Then ζ is a solution of the initial value problem

$$\zeta'' + \bar{R}_{c'}(\zeta) = 0, \qquad \zeta(0) = \gamma'(0) \in T_pM, \qquad \zeta'(0) = S_{\xi(0)}\zeta(0) + \nabla^{\perp}_{\zeta(0)}\xi.$$

A Jacobi vector field ζ along c verifying $\zeta(0) \in T_{c(0)}M$ and $\zeta'(0) - S_{c'(0)}\zeta(0) \in T^{\perp}_{c(0)}M$ is called an *M–Jacobi vector field*.

We say that $c(r)$ is a *focal point* of M along c if there exists an M–Jacobi vector field ζ along c such that $\zeta(r) = 0$. A focal point arising from a Jacobi vector field ζ such that $\zeta(0) = 0$, $\zeta'(0) \in T_p^{\perp}M$ and $\zeta(r) = 0$ is a conjugate point of p in $\bar{M}$ along c.

Assume now that $G_M(r)$ is a submanifold of $\bar{M}$. Let ξ be a smooth curve in $T^{\perp}M$ with $\xi(0) = c'(0)$ such that $g(\xi(t),\xi(t)) = 1$ for all t. Then $F(s,t) = \exp\big(t\,\xi(s)\big)$ is a smooth geodesic variation of c consisting of geodesics intersecting M perpendicularly. Let ζ be the corresponding M–Jacobi vector field which is the variational vector field of F. Then ζ is determined by the initial values $\zeta(0) = \gamma'(0)$ and $\zeta'(0) = \xi'(0)$, where $\gamma(s) = F(s,0)$. For any r, the curve $\gamma_r(s) = F(r,s) = \exp(r\,\xi(s))$ is a smooth curve in $G_M(r)$. Then,

$$T_{c(r)}G_M(r) = \{\zeta(r) : \zeta \text{ is an } M\text{–Jacobi vector field along } c\}.$$

Let us denote by $S(r)$ the shape operator of $G_M(r)$. Then it follows that

$$S(r)_{c'(r)}\zeta(r) = \zeta'(r)^{\top}.$$

If $G_M(r)$ is a tube, that is, if $G_M(r)$ is a hypersurface, its shape operator can be described in an efficient way.

Let $X \in T_{c(0)}\bar{M} \ominus \mathbb{R}\, c'(0)$, where $\ominus$ denotes the orthogonal complement. We introduce the following notation. By B_X we denote the parallel translation of X along the geodesic c. We define ζ_X as the M–Jacobi vector field along c given by the following initial conditions

$$\zeta_X(0) = X, \quad \zeta_X'(0) = S_{c'(0)}X, \qquad \text{if } X \in T_{c(0)}M,$$
$$\zeta_X(0) = 0, \quad \zeta_X'(0) = X, \qquad \text{if } X \in T^\perp_{c(0)}M \ominus \mathbb{R}\, c'(0).$$

We define $D(r)$ by $D(r)B_X(r) = \zeta_X(r)$ for all $X \in T_{c(0)}\bar{M} \ominus \mathbb{R}\, c'(0)$. Then D is a $T_{c(r)}\bar{M} \ominus \mathbb{R}c'(r)$ endomorphism–valued tensor field along c determined by the following initial value problem

$$D'' + \bar{R}_{c'} \circ D = 0, \qquad D(0) = \begin{pmatrix} \mathrm{Id}_{T_pM} & 0 \\ 0 & 0 \end{pmatrix}, \qquad D'(0) = \begin{pmatrix} S_{c'(0)} & 0 \\ 0 & \mathrm{Id}_{T_p^\perp M \ominus \mathbb{R}\, c'(0)} \end{pmatrix}.$$

The endomorphism $D(r)$ is singular if and only if $c(r)$ is a focal point of M along c. If this is not the case, $G_M(r)$ is a tube and its shape operator in the direction of $c'(r)$ is given by

$$S(r)_{c'(r)} = D'(r)\, D(r)^{-1}.$$

Of special interest is the case when M is just a single point. This is the main concern of the rest of this chapter. Another interesting situation occurs when M is hypersurface. We deal with this in what follows.

Let $M \subset \bar{M}$ be a hypersurface. The next calculations are local, so we may assume that M is an oriented submanifold and its orientation is given by a global unit normal vector field ξ. Let $r > 0$ and define the map

$$\begin{array}{rcl} \Phi_r : M & \longrightarrow & \bar{M} \\ p & \mapsto & \Phi_r(p) = \exp(r\, \xi_p). \end{array}$$

We denote by η the vector field along Φ_r such that $\eta_r(p) = c_p'(r)$, where c_p is the geodesic of $\bar{M}$ determined by the initial conditions $c_p(0) = p$ and $c_p'(0) = \xi_p$. The map Φ_r is smooth and parametrizes the tube of radius r around M, $G_M(r)$. Obviously, $G_M(r)$ is an immersed submanifold of $\bar{M}$ if and only if Φ_r is an immersion. It may happen, nonetheless, that $G_M(r)$ is a focal manifold. The fact that $G_M(r)$ has higher codimension depends on the rank of Φ_r.

Let ζ_X be an M–Jacobi vector field. We have $X = \zeta_X(0) \in TM$ and $\zeta_X'(0) = SX$ because ξ has unit length and the normal bundle of M has rank one. Then it follows that

$$\Phi_{r*}X = \zeta_X(r), \qquad \bar{\nabla}_v \eta_r = \zeta_X'(r).$$

Thus, Φ_r is not an immersion at $p \in M$ if and only if $\Phi_r(p)$ is a focal point of M along the geodesic c_p. The dimension of the kernel of Φ_{r*p} is called the *multiplicity* of the focal point. If there exists a positive integer k such that $\Phi_r(q)$ is a focal point of M along c_q with multiplicity k for all q in some open neighborhood $\mathcal{U}$ of p, then, if $\mathcal{U}$ is sufficiently small, $\Phi_{r|\mathcal{U}}$ parametrizes and embedded $(n-1-k)$–dimensional submanifold of $\bar{M}$, the focal manifold of M in $\bar{M}$. If $\Phi_r(q)$ is not a focal point of M along c_q, for a sufficiently small neighborhood $\mathcal{U}$ of p, $\Phi_{r|\mathcal{U}}$ parametrizes an embedded hypersurface of $\bar{M}$, which is called an *equidistant hypersurface* to M in $\bar{M}$.

If $G_M(r)$ is a hypersurface, its shape operator can be calculated by using the endomorphism D defined above. In this case the initial conditions simplify slightly and D is determined by the initial value problem

$$D'' + \bar{R}_{c'} \circ D = 0, \qquad D(0) = \mathrm{Id}_{T_pM}, \qquad D'(0) = S_{\xi_p}.$$

4.2 Geodesic spheres in Riemannian manifolds

Of particular interest is the case of a tube around a single point $M = \{m\}$. In this situation $G_m(r)$ is called the *geodesic sphere* centered at m with radius r. For sufficiently small radius we have $G_m(r) = \exp_m(\mathbb{S}^{n-1}(r))$, where $\mathbb{S}^{n-1}(r)$ is the Euclidean sphere of radius r centered at the origin of T_mM and of dimension $n-1$. We always assume that $r < i(m)$, where $i(m)$ is the injectivity radius at the point m. Hence, geodesic spheres are the level sets of the Riemannian distance function, that is, $G_m(r) = \{p \in M : d(m,p) = r\}$.

Throughout this chapter, it is convenient to introduce the following notation. We use the symbol $\tilde{}$ for the geometric objects of $G_m(r)$. For the geometric objects of M we just use the usual symbols (without bars). We perform most of the calculations with respect to an orthonormal basis $\{e_i\}$. We define $\epsilon_i = g(e_i, e_i) \in \{-1, 1\}$. We also set $\epsilon_{i_1 \cdots i_k} = \epsilon_{i_1} \cdots \epsilon_{i_k}$ for the sake of simplicity in our notation. The notation $\omega_{i_1 \cdots i_k}$ means $\omega_{e_{i_1} \cdots e_{i_k}}$ for any tensor field ω and $\nabla_{ij \ldots}$ means $\nabla_{e_i e_j \ldots}$. Finally, we define $\nabla^0 \omega = \omega$ and we write $\nabla^k_{i_1 \cdots i_k} \omega_{j_1 \cdots j_l}$ for $(\nabla^k_{e_{i_1} \cdots e_{i_k}} \omega)_{e_{j_1} \cdots e_{j_l}}$. This will simplify considerably our writing in the long formulas appearing in this chapter.

Let $m \in M$ and $u \in T_mM$. The shape operator $S(r)$ of a geodesic sphere is given by

$$S(r)(\exp_m(ru)) = D'(r)D(r)^{-1},$$

where D is the above endomorphism which in this particular case is determined by the initial value problem

$$D'' + R_{c'} \circ D = 0, \qquad D(0) = 0, \qquad D'(0) = \mathrm{Id}\,.$$

along the geodesic $c(t) = \exp_m(tu)$.

We define the *volume density function* θ_m as

$$\theta_m(p) = \sqrt{\det(g_p)}.$$

It can be proved (see [33], for example) that $\theta_m(\exp_m(ru)) = (\det D(r))/r^{n-1}$. Hence, the mean curvature h_m of a geodesic sphere centered at m is [33], [128]

$$h_m(\exp_m(ru)) = \operatorname{tr} S(r)(\exp_m(ru)) = \frac{(\det D)'(r)}{(\det D)(r)} = \frac{n-1}{r} + \frac{\theta'_m(\exp_m(ru))}{\theta_m(\exp_m(ru))}.$$

The map $r \mapsto S(r)$ is an endomorphism–valued tensor field along the geodesic c. Taking derivatives in the equality $S(r)D(r) = D'(r)$ we get $S'(r)D(r) + S(r)D'(r) = D''(r)$ and using again $S(r)D(r) = D'(r)$ we obtain the *Riccati equation*

$$S' + S^2 + R_{c'} = 0.$$

We define $C(r) = rS(r)$. It can be proved that C is a differentiable endomorphism–valued tensor field in a neighborhood of $0 \in \mathbb{R}$. Then the Riccati equation is equivalent to $rC' + C^2 - C + r^2 R_{c'} = 0$. Taking the k–th derivative of the latter equation and evaluating at $r = 0$, we obtain the *Ledger recursion formula* [33], [128]

$$(k-1)C^{(k)}(0) = -k(k-1)R^{(k-2)}(0) - \sum_{i=0}^{k} \binom{n}{k} C^{(i)}(0)C^{(k-i)}(0), \quad k \in \mathbb{N},$$

where $g(R^{(i)}(0)x, y) = \nabla^{i-2}_{u\cdots u} R_{uxuy}(m)$ for all $x, y \in T_mM \ominus \mathbb{R}u$ and $i \in \mathbb{N}$.

The Ledger recursion formula allows us to calculate power series expansions of geometric objects defined in geodesic spheres. This has already been done by several authors. See for example [33], [82], [83], [128] where the first terms of power series expansions of several intrinsic and extrinsic curvature tensors of geodesic spheres are given. However, we are interested in a more detailed description. In the following section we introduce the concepts and notation needed to achieve this.

4.2.1 Curvature and Weyl invariants

A *scalar curvature invariant* is a polynomial in the components of the curvature tensor and its covariant derivatives which does not depend on the choice of orthonormal basis used in its construction [83], [113]. The *degree* of a scalar curvature invariant is the number of derivatives of the metric tensor involved in its construction. Since the curvature tensor has two derivatives of the metric tensor, a scalar curvature invariant has always even degree.

Let us denote by $I(\nu, n)$ the vector space of scalar curvature invariants of degree 2ν in a manifold of dimension n.

It is well known that for $n \geq 2$, $I(1, n)$ is a vector space of dimension 1 generated by the scalar curvature τ.

If $n \geq 4$, $I(2, n)$ is a vector space of dimension 4 spanned by

$$(4.1) \qquad \tau^2, \qquad \|R\|^2 = \sum \varepsilon_{ijkl}\, R^2_{ijkl}, \qquad \|\rho\|^2 = \sum \varepsilon_{ij}\, \rho^2_{ij}, \qquad \Delta\tau = \sum \epsilon_i \nabla^2_{ii}\tau.$$

For $n \geq 6$, the vector space $I(3,n)$ has dimension 17 and is spanned by the following basis:

$$
\begin{array}{ll}
\tau^3, & \\
\tau\|\rho\|^2, & \\
\tau\|R\|^2, & \\
\check{\rho} & = \sum \varepsilon_{ijkl}\, \rho_{ij}\rho_{ik}\rho_{jk}, \\
\langle \rho\otimes\rho, \bar{R}\rangle & = \sum \varepsilon_{ijkl}\, \rho_{ij}\rho_{kl}R_{ikjl}, \\
\langle \rho, \dot{R}\rangle & = \sum \varepsilon_{ijklp}\, \rho_{ij}R_{iklp}R_{jklp}, \\
\check{R} & = \sum \varepsilon_{ijklpq}\, R_{ijkl}R_{ijpq}R_{klpq}, \\
\check{\check{R}} & = \sum \varepsilon_{ijklpq}\, R_{ijkl}R_{ipkq}R_{jplq},
\end{array}
\qquad
\begin{array}{ll}
\|\nabla\tau\|^2 & = \sum \epsilon_i \, (\nabla_i\tau)^2, \\
\|\nabla\rho\|^2 & = \sum \epsilon_{ijk} \, (\nabla_i\rho_{jk})^2, \\
\alpha(\rho) & = \sum \epsilon_{ijk}\, \nabla_i\rho_{jk}\nabla_j\rho_{ik}, \\
\|\nabla R\|^2 & = \sum \epsilon_{ijklp} \, (\nabla_i R_{jklp})^2, \\
\tau\Delta\tau, & \\
\langle \rho, \Delta\rho\rangle & = \sum \epsilon_{ijk}\, \rho_{ij}\nabla^2_{kk}\rho_{ij}, \\
\langle \nabla^2\tau, \rho\rangle & = \sum \epsilon_{ij}\, \rho_{ij}\nabla^2_{ij}\tau, \\
\langle R, \Delta R\rangle & = \sum \epsilon_{ijklp}\, R_{ijkl}\nabla^2_{pp}R_{ijkl}, \\
\Delta^2\tau. &
\end{array}
\tag{4.2}
$$

Scalar curvature invariants are a powerful tool in Riemannian geometry, but they may become useless when the metric is allowed to have indefinite signature [22], [27].

We explain some notions from the theory of invariants in a more general context mainly following [47], [113].

Let $\mathcal{F}M = (\mathcal{F}M, \pi, M, Gl(n,\mathbb{R}))$ be the bundle of linear frames over (M,g). For $k \geq 1$ we shall denote by $T^kM = \cup_{m\in M}(T_mM \times \cdots \times T_mM)$ the bundle over M with standard fibre $\mathbb{R}^n \times \overset{k}{\cdots} \times \mathbb{R}^n$ and structure group $Gl(n,\mathbb{R})$ which is associated with the principal bundle $\mathcal{F}M$. If $k=0$ we set $T^0M := M$.

A *partial Weyl invariant*, W, with k degrees of freedom is a map

$$
\begin{array}{llll}
W: & T^kM & \longrightarrow & \mathbb{R} \\
 & (v_1,\dots,v_k) & \mapsto & \operatorname{tr}(g\otimes\cdots\otimes g\otimes \nabla^{l_1}R\otimes\cdots\otimes\nabla^{l_\nu}R)(v_1,\dots,v_k)
\end{array}
\tag{4.3}
$$

where $l_j \in \mathbb{N}\cup\{0\}$, $j\in\{1,\dots,\nu\}$, $\nu\in\mathbb{N}$, and tr is a product of traces [9] with respect to some permutation of the indices. Two partial Weyl invariants W_1 and W_2 are equal if and only if $W_1(v_1,\dots,v_k) = W_2(v_1,\dots,v_k)$ for any $(v_1,\dots,v_k)\in T^kM$ and every semi–Riemannian manifold (M,g).

We say that a partial Weyl invariant W is *simple* if its construction does not involve covariant derivatives of the curvature tensor, that is, $W = \operatorname{tr}(g\otimes\cdots\otimes g\otimes R\otimes\cdots\otimes R)$.

In particular, a *Weyl invariant*, as defined in [113], is a partial Weyl invariant with zero degrees of freedom, that is, $k=0$.

We define the *degree* of a partial Weyl invariant given by (4.3) as

$$\deg W = l_1 + \cdots + l_\nu + 2\nu.$$

We point out that other authors define the degree (or order) of a curvature invariant as half this number. Equivalently, the degree of a partial Weyl invariant is the number of derivatives of the metric tensor involved in its construction. Clearly, if W_1 and W_2 are two partial Weyl invariants, then W_1W_2 can be considered as another partial Weyl invariant in the obvious way and

$$\deg W_1W_2 = \deg W_1 + \deg W_2.$$

For instance, the curvature tensor R and the Ricci tensor ρ are simple partial Weyl invariants of degree 2, the former with 4 degrees of freedom and the latter with 2.

By definition, a *partial scalar curvature invariant* is a linear combination of partial Weyl invariants. If all partial Weyl invariants involved in the construction of a partial scalar curvature invariant have the same degree d then this partial scalar curvature invariant is said to have degree d.

Given a tangent vector $u \in TM$ and a partial scalar curvature invariant W with k degrees of freedom, we say that $W(u, \overset{k}{\ldots}, u)$ is a *partial directional curvature invariant* or to be more specific, a *partial curvature invariant in the direction of* u.

It follows from Weyl's theory of invariants [9], [132], that the scalar curvature invariants are precisely the traces of the curvature tensor and its covariant derivatives. As a consequence, a scalar curvature invariant is a linear combination of Weyl invariants. Alternatively, a scalar curvature invariant is a partial scalar curvature invariant with zero degrees of freedom.

Simple curvature invariants and simple Weyl invariants are of special interest and they constitute one of our main concerns in this chapter. Up to multiplication by a constant, there exists a unique simple curvature invariant of degree 2, which is the scalar curvature τ. The space of simple curvature invariants of degree 4 has dimension 3 and is spanned by τ^2, $\|\rho\|^2$ and $\|R\|^2$. A basis for the vector space of simple scalar curvature invariants of degree 6 is given by the left–hand side column of (4.2).

Curvature invariants of degree 4 are important from a geometric point of view because they may be used to characterize certain types of manifolds. The following result can be found, for example, in [9], [33].

Lemma 4.1. *We have:*

(a) For any n–dimensional Riemannian manifold, $\|\rho\|^2 \geq \dfrac{1}{n}\tau^2$, with equality if and only if the manifold is an Einstein space.

(b) For any n–dimensional Riemannian manifold, $\|R\|^2 \geq \dfrac{2}{n-1}\|\rho\|^2$, with equality if and only if the manifold has constant sectional curvature.

(c) For a $2n$–dimensional Kähler manifold, $\|R\|^2 \geq \dfrac{4}{n+1}\|\rho\|^2$, with equality if and only if M has constant holomorphic sectional curvature.

(d) For a $4n$–dimensional quaternionic Kähler manifold, $\|R\|^2 \geq \dfrac{5\,n+1}{(n+2)^2}\|\rho\|^2$, with equality precisely for quaternionic space forms.

Remark 4.2. Let $W = \operatorname{tr}(R \otimes \overset{\nu}{\cdots} \otimes R)$ be a Weyl invariant of degree 2ν. In a $(n-1)$–dimensional Riemannian manifold of constant sectional curvature λ the curvature tensor may be written as $R = \lambda R^0$, where R^0 can be expressed with respect to an orthonormal basis as

$$R^0_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Then, $W = \operatorname{tr}(R \otimes \overset{\nu}{\cdots} \otimes R) = \lambda^\nu \operatorname{tr}(R^0 \otimes \overset{\nu}{\cdots} \otimes R^0) = \bar{A}_W(n-1)\lambda^\nu$ where $\bar{A}_W$ is a polynomial that depends only on W. Moreover, if $n \in \{1, 2\}$, then $R = 0$ and hence we have $\bar{A}_W(0) = \bar{A}_W(1) = 0$. Thus, $\bar{A}_W$ may be written as $\bar{A}_W(n-1) = (n-1)(n-2)A_W(n-1)$, where A_W is another polynomial. Therefore, for the constant curvature case we have

$$W = (n-1)(n-2)A_W(n-1)\lambda^\nu.$$

The polynomial A_W will be used latter in this work and plays an important role to determine several curvatures of geodesic spheres.

Example 4.3. The polynomials A_W corresponding to the simple Weyl invariants appearing in (4.1) and (4.2) can be explicitly given as follows. Suppose (M^{n-1}, g) has constant sectional curvature λ. First, we have $\tau = (n-1)(n-2)\lambda$, and thus,

$$A_\tau(n-1) = 1.$$

Also, for the simple Weyl invariants of degree 4,

$$A_{\|R\|^2}(n-1) = 2, \qquad A_{\|\rho\|^2}(n-1) = n-2, \qquad A_{\tau^2}(n-1) = (n-1)(n-2).$$

The expressions corresponding to the simple invariants of degree 6 are summarized in the following table:

W	$A_W(n-1)$	W	$A_W(n-1)$
τ^3	$(n-1)^2(n-2)^2$	$\langle \rho \otimes \rho, \bar{R} \rangle$	$(n-2)^2$
$\tau\|\rho\|^2$	$(n-1)(n-2)^2$	$\langle \rho, \dot{R} \rangle$	$2(n-2)$
$\tau\|R\|^2$	$2(n-1)(n-2)$	$\check{R}$	4
$\check{\rho}$	$(n-2)^2$	$\check{\check{R}}$	$n-3$

We consider a $(n-1)$–dimensional manifold because this is what we will need afterwards.

The concept of partial Weyl invariant allows us to describe in a convenient way several geometric objects in a geodesic sphere. First, we calculate the scalar second fundamental form. Then, the Gauss formula allows us to calculate its curvature tensor. See [33] for explicit calculations of the first terms in its power series expansion. We follow [47].

Lemma 4.4. *Let σ denote the second fundamental form of the geodesic sphere $G_m(r)$. We have the power series expansion*

$$\sigma_{ij}\left(\exp_m(ru)\right) = \sum_{\alpha=-1}^{s-1} \frac{r^\alpha}{(\alpha+1)!}\, \sigma_{ij}^{\alpha+1}(u) + O\left(r^s\right),$$

where $\sigma_{ij}^{\alpha}(u)$, $\alpha \geq 2$, is a partial scalar curvature invariant of M at m with $\alpha+2$ degrees of freedom and degree α. The first terms of this expansion are

$$\begin{aligned}\sigma_{ij}\left(\exp_m(ru)\right) &= \frac{1}{r}\delta_{ij} - \frac{r}{3}R_{uiuj}(m) - \frac{r^2}{4}\nabla_u R_{uiuj}(m) \\ &\quad - r^3\left(\frac{1}{45}\sum_{a=1}^{n} R_{uiua}R_{ujua} + \frac{1}{10}\nabla_{uu}^2 R_{uiuj}\right)(m) + O\left(r^4\right).\end{aligned}$$

Proof. Using the Ledger recursion formula and the fact $\sigma(\exp_m(ru))(x,y) = g(C(r)x,y)$, with the notation of that formula we get

$$\sigma_{ij}^0(u) = \delta_{ij}, \qquad \sigma_{ij}^1(u) = 0,$$
$$\sigma_{ij}^{\alpha}(u) = -\frac{\alpha(\alpha-1)}{\alpha+1}\nabla_{u\ldots u}^{\alpha-2}R_{uiuj}(m) - \frac{1}{\alpha+1}\sum_{\beta=2}^{\alpha-2}\binom{\alpha}{\beta}\sum_{\gamma=2}^{n}\sigma_{i\gamma}^{\beta}(u)\sigma_{\gamma j}^{\alpha-\beta}(u),$$

for all $\alpha \geq 2$. The result now follows by induction. □

The first terms in the power series expansion of the curvature tensor of a geodesic sphere where obtained in [33], [44].

Lemma 4.5. *Let $\widetilde{R}$ denote the curvature tensor of a geodesic sphere $G_m(r)$. Then*

$$\widetilde{R}_{ijkl}\left(\exp_m(ru)\right) = \sum_{\alpha=-2}^{s-2} r^{\alpha}\widetilde{R}_{ijkl}^{\alpha+2}(u) + O\left(r^{s-1}\right),$$

where $\widetilde{R}_{ijkl}^{\alpha}(u)$, $\alpha \geq 2$, is a partial curvature invariant at m of degree α such that for all the Weyl invariants used in its construction the number of degrees of freedom has the same parity as α. More specifically

$$\begin{aligned}\widetilde{R}_{ijkl}\left(\exp_m(ru)\right) &= \frac{1}{r^2}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big) \\ &+\Big(R_{ijkl} - \frac{1}{3}\delta_{ik}R_{ujul} + \frac{1}{3}\delta_{il}R_{ujuk} + \frac{1}{3}\delta_{jk}R_{uiul} - \frac{1}{3}\delta_{jl}R_{uiuk}\Big)(m) \\ &+r\left(\nabla_u R_{ijkl} - \frac{1}{4}\delta_{jl}\nabla_u R_{uiuk} + \frac{1}{4}\delta_{jk}\nabla_u R_{uiul} + \frac{1}{4}\delta_{il}\nabla_u R_{ujuk} - \frac{1}{4}\delta_{ik}\nabla_u R_{ujul}\right)(m) \\ &+r^2\Big(-\frac{1}{9}R_{uiul}R_{ujuk} + \frac{1}{9}R_{uiuk}R_{ujul} - \frac{1}{45}\delta_{ik}\sum_{a=1}^{n}R_{ujua}R_{ulua} + \frac{1}{45}\delta_{il}\sum_{a=1}^{n}R_{ujua}R_{ukua} \\ &\quad + \frac{1}{45}\delta_{jk}\sum_{a=1}^{n}R_{uiua}R_{ulua} - \frac{1}{45}\delta_{jl}\sum_{a=1}^{n}R_{uiua}R_{ukua} + \frac{1}{2}\nabla_{uu}^2 R_{ijkl} - \frac{1}{10}\delta_{jl}\nabla_{uu}^2 R_{uiuk} \\ &\quad + \frac{1}{10}\delta_{jk}\nabla_{uu}^2 R_{uiul} + \frac{1}{10}\delta_{il}\nabla_{uu}^2 R_{ujuk} - \frac{1}{10}\delta_{ik}\nabla_{uu}^2 R_{ujul}\Big)(m) + O\left(r^3\right).\end{aligned}$$

Proof. The Gauss equation may be written as $\widetilde{R}_{xyvw} = R_{xyvw} + \sigma_{xv}\sigma_{yw} - \sigma_{xw}\sigma_{yv}$ in this case. Using the power series expansion along a geodesic with respect to a parallel basis, $R_{ijkl}(\exp_m(ru)) = R_{ijkl}(m) + r\nabla_u R_{ijkl}(m) + (r^2/2!)\nabla^2_{uu}R_{ijkl}(m) + \cdots$ and plugging the above expression and the formula of Lemma 4.4 into the Gauss equation we have

$$\begin{aligned}
\widetilde{R}^0_{ijkl}(u) &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} = R^0_{ijkl}, \qquad \widetilde{R}^1_{ijkl}(u) = 0,\\
\widetilde{R}^\alpha_{ijkl}(u) &= \frac{1}{(\alpha-2)!}\nabla^{\alpha-2}_{u\ldots u}R_{ijkl}(m)\\
&\quad + \frac{1}{\alpha!}\sum_{\beta=1}^{\alpha-3}\binom{\alpha}{\beta+1}\left(\sigma_{ik}^{\beta+1}(u)\sigma_{jl}^{\alpha-\beta-1}(u) - \sigma_{il}^{\beta+1}(u)\sigma_{jk}^{\alpha-\beta-1}(u)\right),
\end{aligned}$$

for all $\alpha \geq 2$. Hence, the first part of the result follows by induction. Finally, in the last equality of the above formula there are two clearly different terms. The first one $\nabla^{\alpha-2}_{u\cdots u}R_{ijkl}(m)/(\alpha-2)!$ is a partial scalar curvature invariant with $\alpha+2$ degrees of freedom. The second addend is another partial curvature invariant with $\alpha+4$ degrees of freedom. The last statement then follows from Lemma 4.4. $\square$

The following lemma is a technical result that will be needed in Theorem 4.7.

Lemma 4.6. *Let $(V, \langle\,,\,\rangle)$ be an inner product vector space of dimension $n > 2$ and* tr *a total trace in the space of covariant tensors of order 4ν over V. If R is an algebraic curvature tensor on V, $Sc(R)$ its scalar curvature and W the algebraic invariant defined by $W = \mathrm{tr}(R \otimes \overset{\nu}{\cdots} \otimes R)$, then*

$$\mathrm{tr}\left(\sum_{\alpha=1}^{\nu} R^0 \otimes \cdots \otimes \overset{\overset{\alpha}{\downarrow}}{R} \otimes \cdots \otimes R^0\right) = \nu A_W(n) Sc(R).$$

Proof. Clearly, $\mathrm{tr}(\sum_{\alpha=1}^{\nu} R^0 \otimes \cdots \otimes R \otimes \cdots \otimes R^0)$ is a scalar curvature invariant of degree one, and hence it is a multiple of the scalar curvature. Write

$$a\, Sc(R) = \mathrm{tr}\left(\sum_{\alpha=1}^{\nu} R^0 \otimes \cdots \otimes \overset{\overset{\alpha}{\downarrow}}{R} \otimes \cdots \otimes R^0\right).$$

The above formula is true for each algebraic curvature tensor R in V. If we take $R = R^0$ we have

$$\begin{aligned}
a\, n(n-1) &= \sum_{\alpha=1}^{\nu} \mathrm{tr}\left(R^0 \otimes \cdots \otimes \overset{\overset{\alpha}{\downarrow}}{R^0} \otimes \cdots \otimes R^0\right)\\
&= \nu\ \mathrm{tr}\,(R^0 \otimes \cdots \otimes R^0) = \nu\, n(n-1)A_W(n).
\end{aligned}$$

Thus $a = \nu\, A_W(n)$. $\square$

We are now ready to give a power series expansion of a simple Weyl invariant in a geodesic sphere.

Theorem 4.7. *Let $\widetilde{W}$ be a simple intrinsic Weyl invariant of degree 2ν, $\nu > 1$, in a geodesic sphere $G_m(r)$. We have*

$$\widetilde{W}\left(\exp_m(ru)\right) = \sum_{\alpha=-2\nu}^{s-2\nu} r^{\alpha} \widetilde{W}_{\alpha+2\nu}(u) + O\left(r^{s-2\nu+1}\right),$$

where $\widetilde{W}_{\alpha}(u)$ is a partial curvature invariant of degree α in the direction of u such that the degree of freedom of each Weyl invariant involved in its construction has the same parity as α. More specifically, we have

$$\begin{aligned}
\widetilde{W}_0(u) &= (n-1)(n-2)A_W(n-1),\\
\widetilde{W}_1(u) &= 0,\\
\widetilde{W}_2(u) &= \nu\, A_W(n-1)\left(\tau - \frac{2(n+1)}{3}\rho_{uu}\right)(m),\\
\widetilde{W}_3(u) &= \nu\, A_W(n-1)\left(\nabla_u\tau - \frac{n+2}{2}\nabla_u\rho_{uu}\right)(m),\\
\widetilde{W}_4(u) &= \omega_4(u) + \frac{\nu}{2}A_W(n-1)\left(\nabla^2_{uu}\tau - \frac{2(n+3)}{5}\nabla^2_{uu}\rho_{uu}\right)(m),
\end{aligned}$$

where $\omega_4(u)$ is a simple directional curvature invariant of degree four given by

$$\begin{aligned}
\omega_4(u) = \Bigg\{ & \nu\, A_W(n-1)\left(-\frac{2n+1}{45}\sum_{a,b=1}^{n} R^2_{uaub} + \frac{1}{9}\rho^2_{uu}\right)\\
& + B^1_W(n-1)\left(\|R\|^2 - 4\sum_{a,b,c=1}^{n} R^2_{uabc} + \frac{4(n+12)}{9}\sum_{a,b=1}^{n} R^2_{uaub} - \frac{8}{3}\sum_{a,b=1}^{n}\rho_{ab}R_{uaub} + \frac{4}{9}\rho^2_{uu}\right)\\
& + B^2_W(n-1)\left(\|\rho\|^2 + \frac{n^2}{9}\sum_{a,b=1}^{n} R^2_{uaub} - \frac{2n}{3}\sum_{a,b=1}^{n}\rho_{ab}R_{uaub} - 2\sum_{a=1}^{n}\rho^2_{ua} + \frac{3n+14}{9}\rho^2_{uu} - \frac{2}{3}\tau\rho_{uu}\right)\\
& + B^3_W(n-1)\left(\tau - \frac{2(n+1)}{3}\rho_{uu}\right)^2\Bigg\}(m),
\end{aligned}$$

and B^1_W, B^2_W and B^3_W are polynomials satisfying

$$2\,B^1_W(n-1) + (n-2)B^2_W(n-1) + (n-1)(n-2)B^3_W(n-1) = \binom{\nu}{2} A_W(n-1).$$

Proof. Using the notation of Lemma 4.5, we have

$$(\widetilde{R}\otimes\cdots\otimes\widetilde{R})_{i_1j_1k_1l_1\cdots i_\nu j_\nu k_\nu l_\nu} = \sum_{\alpha=-2\nu}^{s-2\nu} r^{\alpha}\left(\sum_{\beta_1+\cdots+\beta_\nu=\alpha}\widetilde{R}^{\beta_1+2}_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^{\beta_\nu+2}_{i_\nu j_\nu k_\nu l_\nu}(u)\right) + O\left(r^{s-2\nu+1}\right).$$

By taking traces in the above expression, the result of Lemma 4.5 and the rule to compute the degrees, imply the first statement of Theorem 4.7.

Now we turn our attention to the explicit expressions of Theorem 4.7. As $\widetilde{R}$ is the curvature tensor of the geodesic sphere $G_m(r)$, the coefficients of its power series expansion given by Lemma 4.5 are algebraic curvature tensors in $u^\perp = T_mM \ominus \mathbb{R}u$. Using Lemma 4.5 and Remark 4.2 we get

$$\begin{aligned}\widetilde{W}_0(u) &= \operatorname{tr}\left(\widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right) = \operatorname{tr}\left(R^0\otimes\overset{\nu}{\cdots}\otimes R^0\right)\\ &= (n-1)(n-2)A_W(n-1).\end{aligned}$$

Using the notation of Lemma 4.5, since $\widetilde{R}^1=0$, we have

$$\widetilde{W}_1(u) = \operatorname{tr}\left(\sum_{\alpha=1}^{\nu}\widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^1_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right) = 0.$$

Lemmas 4.5 and 4.6 yield

$$\begin{aligned}\widetilde{W}_2(u) &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu}\widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^2_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right)\\ &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu}R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^2_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right) = \nu A_W(n-1)Sc(\widetilde{R}^2),\end{aligned}$$

and from the definition of $\widetilde{R}^2$ in Lemma 4.5 we get $Sc(\widetilde{R}^2) = \tau - \frac{2(n+1)}{3}\rho_{uu}$. Hence

$$\widetilde{W}_2(u) = \nu A_W(n-1)\left(\tau - \frac{2(n+1)}{3}\rho_{uu}\right)(m).$$

Similarly, using Lemmas 4.5 and 4.6 and the fact that $Sc(\widetilde{R}^3) = \nabla_u\tau - \frac{n+2}{2}\nabla_u\rho_{uu}$, we obtain

$$\begin{aligned}\widetilde{W}_3(u) &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu}\widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^3_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right)\\ &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu}R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^3_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right)\\ &= \nu\, A_W(n-1)\left(\nabla_u\tau - \frac{n+2}{2}\nabla_u\rho_{uu}\right)(m).\end{aligned}$$

Finally, using the expression of $\widetilde{R}\otimes\cdots\otimes\widetilde{R}$ at the beginning of this proof, we get for $\nu>1$,

$$\begin{aligned}\widetilde{W}_4(u) &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu}R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^4_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right)\\ &\quad+\operatorname{tr}\left(\sum_{\alpha<\beta}R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^2_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\gamma j_\gamma k_\gamma l_\gamma}(u)\cdots\widetilde{R}^2_{i_\beta j_\beta k_\beta l_\beta}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right).\end{aligned}$$

For the first term of the above equality we again use Lemmas 4.5 and 4.6 to get

$$\nu\, A_W(n-1)\Big(-\frac{2n+1}{45}\sum_{a,b=1}^{n} R^2_{uaub}+\frac{1}{9}\rho^2_{uu}+\frac{1}{2}\nabla^2_{uu}\tau-\frac{n+3}{5}\nabla^2_{uu}\rho_{uu}\Big)(m).$$

Now, we briefly discuss the second term of $\widetilde{W_4}(u)$, which is a simple directional curvature invariant of degree 4. Using the method of Lemma 4.6 to write the second addend of $\widetilde{W_4}(u)$ as a linear combination of curvature invariants of degree 4 associated with $\widetilde{R}^2$ (see the basis of curvature invariants of degree 4 (4.1)) we get the expression for $\omega_4(u)$ and the relation among the polynomials B^1_W, B^2_W and B^3_W. We delete the details. □

Remark 4.8. If $\nu=1$ in the previous theorem, we essentially have to deal with the scalar curvature $\widetilde{\tau}$. In this case, the second addend of $\widetilde{W_4}(u)$ does not appear and $\omega_4(u)=0$. Then, $\widetilde{\tau}(\exp_m(ru))=\sum_{\alpha=-2}^{s-2} r^\alpha Sc(\widetilde{R}^{\alpha+2})+O\left(r^{s-1}\right)$. See [33] and [44] for an explicit power series expansion.

Example 4.9. The coefficients B^1_W and B^2_W in the expression of $\omega_4(u)$ for Weyl invariants of degree 4 and 6 can be given as follows [39]

W	$\|R\|^2$	$\|\rho\|^2$	τ^2
$B^1_W(n-1)$	1	0	0
$B^2_W(n-1)$	0	1	0

W	τ^3	$\tau\|\rho\|^2$	$\tau\|R\|^2$	$\check{\rho}$	$\langle\rho\otimes\rho,\bar{R}\rangle$	$\langle\rho,\dot{R}\rangle$	$\check{R}$	$\check{\check{R}}$
$B^1_W(n-1)$	0	0	$(n-1)(n-2)$	0	0	$n-2$	6	$-3/2$
$B^2_W(n-1)$	0	$(n-2)(n-1)$	0	$3(n-2)$	$2n-5$	4	0	3

These coefficients can also we found in [37], [38] and [44].

Now, we derive some geometrical consequences of the expansions in Theorem 4.7. We first need the following technical result. See for example [45].

Lemma 4.10. *Let (M,g) be an n–dimensional Einstein manifold. If*

$$a\,\|R\|^2+b\sum_{i,j,k=1}^{n} R^2_{uijk}+c\sum_{i,j=1}^{n} R^2_{uiuj}=k$$

for some real constants a, b, c, k with $(n+4)b+3c\neq 0$, $c\neq 0$ and for all unit vectors $u\in TM$, then (M,g) is 2–stein.

Proof. We define the tensors

$$\omega_{xyvw}=\sum_{i,j=1}^{n} R_{xiyj}R_{viwj} \qquad \text{and} \qquad \eta_{xy}=\sum_{i,j,k=1}^{n} R_{xijk}R_{yijk}.$$

For all vectors $x, y \in T_m M$ and all constants $\alpha, \beta \in \mathbb{R}$, it follows from the assumption that $a\,\|R\|^2 g^2_{\alpha x+\beta y,\alpha x+\beta y} + b\,\eta_{\alpha x+\beta y,\alpha x+\beta y} g_{\alpha x+\beta y,\alpha x+\beta y} + c\,\omega_{\alpha x+\beta y,\ldots,\alpha x+\beta y} \;=\; k\,g^2_{\alpha x+\beta y,\alpha x+\beta y}.$ Now, we expand the previous expression and take the coefficient of $\alpha^2\beta^2$. Putting $y = e_i$ and taking the trace we obtain

$$\begin{aligned}2a\,\|R\|^2(n+2)g(x,x) + b\Big(\,\|R\|^2 g(x,x) + (n+4)\eta_{xx}\Big)&\\ +2c\left(\sum_{i,j=1}^n \rho_{ij}R_{xixj} + \frac{3}{2}\eta_{xx}\right) &= 2(n+2)kg(x,x).\end{aligned}$$

Since (M,g) is assumed to be an Einstein manifold the previous equation becomes

$$\big\{b(n+4)+3c\big\}\eta_{xx} = -\Big\{2(n+2)a\,\|R\|^2 + b\,\|R\|^2 + \frac{2c\tau^2}{n^2} - 2(n+2)k\Big\}g_{xx}$$

and taking traces this gives

$$\big\{b(n+4)+3c\big\}\,\|R\|^2 = -n\Big\{2(n+2)a\,\|R\|^2 + b\,\|R\|^2 + \frac{2c\tau^2}{n^2} - 2(n+2)k\Big\}.$$

The last two equations and the fact that $b(n+4)+3c \neq 0$ imply $\eta_{xx} = (\,\|R\|^2/n)g_{xx}$ and thus polarization gives $\eta = (\,\|R\|^2/n)g$. Hence, it follows from the assumption that

$$\omega_{xxxx} = -\frac{1}{c}\left(\frac{na+b}{n}\,\|R\|^2 - k\right)g^2_{xx}$$

which shows that (M,g) is 2–stein. □

Proposition 4.11. *Let (M^n, g) a Riemannian manifold and W a simple Weyl invariant of degree 2ν, $\nu > 1$, such that*

$$\begin{aligned}&A_W(n-1) \neq 0,\\ &(2n+1)\nu A_W(n-1) - 20(n+12)B^1_W(n-1) + 5n^2 B^2_W(n-1) \neq 0,\\ &(2n+1)\nu A_W(n-1) + 40nB^1_W(n-1) + 5n^2 B^2_W(n-1) \neq 0,\end{aligned}$$

If the corresponding Weyl invariants of geodesic spheres $\widetilde{W}(\exp_m(ru))$ depend neither on the center m nor on the direction u, then M is 2–stein.

Proof. Since $A_W(n-1) \neq 0$, using the coefficient $\widetilde{W}_2(u)$ given in Theorem 4.7, we get that $\tau - 2\rho_{uu}(n+1)/3$ is independent of m and u. This implies that the manifold M is Einstein. Now, the coefficient $\widetilde{W}_4$ is also constant by hypothesis. Using the fact that M is Einstein, we obtain

$$\begin{aligned}&B^1_W(n-1)\,\|R\|^2 - 4B^1_W(n-1)\sum_{a,b,c=1}^n R^2_{uabc}\\ &-\left(\frac{2n+1}{45}\nu A_W(n-1) - \frac{4(n+12)}{9}B^1_W(n-1) - \frac{n^2}{9}B^2_W(n-1)\right)\sum_{a,b=1}^n R^2_{uaub} \;=\; \textit{constant}.\end{aligned}$$

The above equation has the form of that of Lemma 4.10. The last two conditions of Proposition 4.11 ensure that the latter lemma can be applied and the result follows. □

Remark 4.12. Let W be a simple Weyl invariant such that $A_W(n-1) \neq 0$. If M is a Riemannian manifold such that the corresponding Weyl invariants of geodesic spheres $\widetilde{W}(\exp_m(ru))$ depend only on the radius, then the above proof shows that M is an Einstein manifold.

Remark 4.13. It was proved in [33] that if $\widetilde{\tau}(\exp_m(ru))$ depends neither on the center m nor on the direction u, then the manifold is 2–stein. Moreover, if the manifold is assumed to be analytic, then it is harmonic.

Example 4.14. It can easily be shown that the conditions in Proposition 4.11 hold for all the curvature invariants of Example 4.9. Hence, those may be used to characterize 2–stein manifolds.

Proposition 4.15. *Let (M, g) be a complete analytic Riemannian manifold with constant Weyl invariants and such that all its small geodesic spheres have constant scalar curvature. Then, M is locally isometric to a two–point homogeneous manifold or a Damek–Ricci space.*

Proof. As all the Weyl invariants of M are constant, M is locally homogeneous [113]. Since all the small geodesic spheres have constant scalar curvature and the manifold is analytic, M is harmonic [33]. Complete homogenous harmonic manifolds have been classified in [87]. According to this paper, M is locally isometric to a two–point homogeneous manifold or a Damek–Ricci space. □

Corollary 4.16. *Let (M, g) be a complete analytic Riemannian manifold with constant Weyl invariants such that all its small geodesic spheres have also constant Weyl invariants. Then, M is locally isometric to a two–point homogeneous space.*

Proof. Using the previous proposition, M is locally isometric to a two–point homogeneous space or a Damek–Ricci space. On the other hand, all the small geodesic spheres of M are homogeneous, as they also have constant Weyl invariants. Hence, M is Osserman [80]. But a Damek–Ricci space cannot be Osserman unless it is symmetric [17]. The result follows because locally symmetric Damek–Ricci spaces are locally isometric to a two–point homogeneous space. □

4.2.2 Total scalar curvatures of geodesic spheres

Since a geodesic sphere is a compact Riemannian submanifold, one may consider the integral of a curvature invariant W for geodesic spheres. Following, for example, [33], we define the *total scalar curvature $\mathcal{W}$ associated with the scalar curvature invariant W* by

$$\mathcal{W}(m,r) = \int_{G_m(r)} \widetilde{W} = r^{n-1} \int_{\mathbb{S}^{n-1}} (\widetilde{W}\theta_m)(\exp_m(ru))du,$$

where $\widetilde{W}$ is the corresponding curvature invariant of $G_m(r)$, θ_m is the volume density function at m and du is the volume element of $\mathbb{S}^{n-1}$. Then, $\mathcal{W}$ is a function depending on the base point and the radius of the geodesic sphere.

Example 4.17. When (M,g) is a Riemannian manifold of constant sectional curvature $\lambda > 0$, each geodesic sphere $G_m(r)$ has constant sectional curvature $\widetilde{\lambda} = \lambda/\sin^2 r\sqrt{\lambda}$ [83] (here, we only consider the positive curvature case; similar expressions can be obtained for negative and zero curvature). We now compute the total scalar curvature associated with a Weyl invariant W of degree 2ν. From Remark 4.2 we get

$$\widetilde{W} = (n-1)(n-2)A_W(n-1)\left(\frac{\lambda}{\sin^2 r\sqrt{\lambda}}\right)^{\nu}.$$

In a space of constant sectional curvature $\lambda > 0$ the volume density function is

$$\theta_m\left(\exp_m(ru)\right) = \left(\frac{\sin r\sqrt{\lambda}}{r\sqrt{\lambda}}\right)^{n-1}$$

(see for example [82], [128]) and we have the exact expression for the total scalar curvature associated with W

$$\int_{G_m(r)} \widetilde{W} = c_{n-1}(n-1)(n-2)A_W(n-1)\left(\frac{\sin r\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1-2\nu}.$$

We emphasize that the above total scalar curvature does not depend on the base point m.

In order to obtain a power series expansion of a total scalar curvature we need the volume density function of a Riemannian manifold [33].

Lemma 4.18. *Let θ_m be the volume density function at a point m. Then we have*

$$\theta_m\left(\exp_m(ru)\right) = \sum_{\alpha=0}^{s} r^\alpha \theta_\alpha(u) + O\left(r^{s+1}\right),$$

where $\theta_\alpha(u)$, $\alpha \geq 2$, is a partial curvature invariant of degree α in the direction of u with α degrees of freedom. The first terms of this power series expansion are

$$\begin{aligned}\theta_m\left(\exp_m(ru)\right) &= 1 - \frac{1}{6}\rho_{uu}(m)r^2 - \frac{1}{12}\nabla_u\rho_{uu}(m)r^3 \\ &\quad + \Big(-\frac{1}{180}\sum_{a,b=1}^{n} R^2_{uaub} + \frac{\rho^2_{uu}}{72} - \frac{1}{40}\nabla^2_{uu}\rho_{uu}\Big)(m)r^4 + O\left(r^5\right).\end{aligned}$$

Proof. First, we consider the mean curvature function of a geodesic sphere which is easily obtained from Lemma 4.4 taking traces

$$h_m(\exp_m(ru)) = \sum_{\alpha=-1}^{s-1} r^\alpha h_{\alpha+1}(u) + O\left(r^s\right),$$

where $h_\alpha(u)$, $\alpha \geq 0$ are partial scalar curvature invariants of degree α in the direction of u and with α degrees of freedom. The first terms in the power series expansion of h_m are well–known [33], [128]

$$h_m(\exp_m(ru)) = \left\{ \frac{n-1}{r} - \frac{r}{3}\rho_{uu} - \frac{r^2}{4}\nabla_u\rho_{uu} - r^3\left(\frac{1}{45}\sum_{a,b=1}^{n} R^2_{uaub} + \frac{1}{10}\nabla^2_{uu}\rho_{uu}\right)\right\}(m) + O\left(r^4\right).$$

Now, using the relation

$$h_m\left(\exp_m(ru)\right) = \frac{n-1}{r} + \frac{\partial}{\partial r}\log\theta_m\left(\exp_m(ru)\right),$$

we obtain

$$\theta_\alpha(u) = \sum_{\beta=1}^{[\alpha/2]} \frac{1}{\beta!}\left(\sum_{\gamma_1+\cdots+\gamma_\beta=\alpha} \frac{h_{\gamma_1}(u)\cdots h_{\gamma_\beta}(u)}{\gamma_1\cdots\gamma_\beta}\right), \quad \alpha \geq 2,$$

where [ ] denotes the integer part of a real number. The result follows from the above formula after considering the properties and explicit expressions of the terms h_α. □

Using standard arguments of calculus one can show that the volume of a Euclidean sphere of radius 1 in $\mathbb{R}^n$ is given by

$$c_{n-1} = \frac{n\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)},$$

where Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt = \int_{-\infty}^{\infty} e^{-t^2}|t|^{2\alpha-1}dt$.

The following result is a technical lemma which will be used in the proof of the Theorem 4.20. We only point out the main steps of the proof.

Lemma 4.19. *Let ω be a covariant tensor of order 2ν. Then,*

$$\int_{\mathbb{S}^{n-1}} \omega_{u\cdots u}du = \frac{c_{n-1}}{2^\nu \nu! \prod_{\alpha=0}^{\nu-1}(n+2\alpha)} \sum_{\alpha_1\cdots\alpha_{2\nu}=1}^{n} \delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2\nu-1}\alpha_{2\nu}} \sum_{\sigma\in\mathfrak{S}_{2\nu}} \omega_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(2\nu)}}.$$

where $\mathfrak{S}_{2\nu}$ is the group of permutations of 2ν elements.

Proof. We proceed by induction. If $\nu = 1$, the expression of Lemma 4.19 is a well–known fact. See, for example, [83]. Next, let ω be a covariant tensor of order $2(\nu+1)$. Choose an orthonormal basis $\{e_i\}$ at the origin of $\mathbb{R}^n$ and write the unit vector u with respect to that basis as $u = \sum_i x_i e_i$. Then

$$\int_{\mathbb{S}^{n-1}} \omega_{u\cdots u}du = \sum_{\alpha_1\cdots\alpha_{2\nu+2}=1}^{n} \omega_{\alpha_1\cdots\alpha_{2\nu+2}} \int_{\mathbb{S}^{n-1}} x_{\alpha_1}\cdots x_{\alpha_{2\nu+2}}du.$$

We recall the formula for integrating polynomials along Euclidean spheres [82]:

$$\int_{\mathbb{S}^{n-1}} x_1^{\beta_1} \cdots x_n^{\beta_n} du = c_{n-1} \frac{\beta_1) \cdots \beta_n)}{n(n+2)\cdots(n+\beta_1+\cdots+\beta_n+2)},$$

where

$$2\beta) = (2\beta-1)(2\beta-3)\cdots 3\cdot 1, \quad 2\beta-1) = 0, \quad \text{for all } \beta \in \mathbb{N} \qquad \text{and} \qquad 0) = 1.$$

Concentrating on the last index $\alpha_{2\nu+2}$ and using the previous formula, we get

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \omega_{u\cdots u} du = & \sum_{\alpha_1\ldots\alpha_{2\nu+1}=1}^{n} \sum_{\beta=1}^{2\nu+1} \left(\frac{1}{\sum_{\gamma=1}^{2\nu+1} \delta_{\alpha_\beta \alpha_\gamma}} \, \omega_{\alpha_1\ldots \underset{\beta}{\underset{\downarrow}{\alpha_{2\nu+1}}} \ldots\alpha_{2\nu+1}} \int_{\mathbb{S}^{n-1}} x_{\alpha_1}\ldots x_{\alpha_{2\nu}} x_{\alpha_{2\nu+1}}^2 du \right) \\
= & \frac{1}{n+2\nu} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{2\nu+1} \left(\int_{\mathbb{S}^{n-1}} \omega(u,\ldots,u,\underset{\beta}{\underset{\downarrow}{e_\alpha}},u,\ldots,u,e_\alpha) du \right).
\end{aligned}$$

Now the inner integrand is a tensor of order 2ν and we can apply our induction hypothesis to get

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \omega_{u\cdots u} du = & \frac{c_{n-1}}{2^\nu \nu! \prod_{\alpha=0}^{\nu}(n+2\alpha)} \sum_{\alpha_1\ldots\alpha_{2\nu+1}=1}^{n} \sum_{\beta=1}^{2\nu+1} \left(\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2\nu-1}\alpha_{2\nu}} \sum_{\sigma\in\mathfrak{S}_{2\nu}} \omega_{\alpha_{\sigma(1)}\ldots \underset{\beta}{\underset{\downarrow}{\alpha_{2\nu+1}}} \ldots\alpha_{\sigma(2\nu)}\alpha_{2\nu+1}} \right) \\
= & \frac{c_{n-1}}{2^\nu \nu! \prod_{\alpha=0}^{\nu}(n+2\alpha)} \sum_{\alpha_1\ldots\alpha_{2\nu+2}=1}^{n} \left(\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2\nu+1}\alpha_{2\nu+2}} \sum_{\substack{\sigma\in\mathfrak{S}_{2\nu+2} \\ \sigma(2\nu+2)=2\nu+2}} \omega_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(2\nu+2)}} \right),
\end{aligned}$$

from where the result follows. □

Theorem 4.20. *Let W be a simple Weyl invariant of degree 2ν. The total scalar curvature associated with W has a power series expansion*

$$\mathcal{W}(m,r) = \int_{G_m(r)} \widetilde{W} = c_{n-1} r^{n-1-2\nu} \left(\sum_{\alpha=0}^{[s/2]} r^{2\alpha} \frac{\mathcal{W}_{2\alpha}(m)}{\prod_{\beta=0}^{\alpha-1}(n+2\beta)} + O\left(r^{s+1}\right) \right),$$

where $\mathcal{W}_{2\alpha}(m)$, $\alpha \geq 1$, is a scalar curvature invariant of M at m of degree α, [ ] denotes the integer part of a real number and

$$\begin{aligned}
\mathcal{W}_0(m) &= (n-1)(n-2)A_W(n-1), \\
\mathcal{W}_2(m) &= -\frac{(n-2)(n-2\nu-1)A_W(n-1)}{6}\tau(m), \\
\mathcal{W}_4(m) &= \Big(C_W^1(n-1)\,\|R\|^2 + C_W^2(n-1)\,\|\rho\|^2 + C_W^3(n-1)\,\tau^2 \\
&\qquad - \frac{(n-2)(n-2\nu-1)A_W(n-1)}{20}\,\Delta\tau \Big)(m).
\end{aligned}$$

Moreover, we have the relation

$$2C_W^1(n-1)+(n-1)C_W^2(n-1)+n(n-1)C_W^3(n-1)$$
$$=\frac{(n-2)(n+2)(n-1-2\nu)(5n-10\nu-7)}{360}A_W(n-1).$$

Proof. By definition we have

$$\mathcal{W}(m,r)=\int_{G_m(r)}\widetilde{W}=r^{n-1}\int_{\mathbb{S}^{n-1}}\left(\widetilde{W}\theta_m\right)(\exp_m(ru))du.$$

Using Theorem 4.7 and Lemma 4.18, we get

$$\left(\widetilde{W}\theta_m\right)(\exp_m(ru))=\sum_{\alpha=-2\nu}^{s-2\nu}r^{\alpha}\bar{W}_{\alpha+2\nu}(u)+O\left(r^{s-2\nu+1}\right),$$

where $\bar{W}_\alpha(u)$, $\alpha\geq 2$, is a partial curvature invariant in the direction of u with degree α such that for all the Weyl invariants involved in its construction their number of degrees of freedom has the same parity as α. In fact, we have $\bar{W}_\alpha(u)=\sum_{\beta=0}^{\alpha}\widetilde{W}_\beta(u)\theta_{\alpha-\beta}(u)$, $\alpha\geq 2$, and in particular, using Theorem 4.7 and Lemma 4.18

$$\begin{aligned}
\bar{W}_0(u) &= (n-1)(n-2)A_W(n-1),\\
\bar{W}_1(u) &= 0,\\
\bar{W}_2(u) &= A_W(n-1)\left(\nu\tau-\frac{4\nu(n+1)+(n-1)(n-2)}{6}\rho_{uu}\right)(m),\\
\bar{W}_4(u) &= \bar{\omega}_4(u)+\frac{A_W(n-1)}{2}\left(\nu\nabla^2_{uu}\tau-\frac{40\nu(n+3)+(n-1)(n-2)}{20}\nabla^2_{uu}\rho_{uu}\right)(m),
\end{aligned}$$

where $\bar{\omega}_4(u)$ is a simple directional curvature invariant of degree 4. Integration gives

$$\mathcal{W}(m,r)=r^{n-1}\left(\sum_{\alpha=-2\nu}^{s-2\nu}r^{\alpha}\int_{\mathbb{S}^{n-1}}\bar{W}_{\alpha+2\nu}(u)du+O\left(r^{s-2\nu+1}\right)\right).$$

If α is odd, $\bar{W}_{\alpha+2\nu}(u)$ is a linear combination of Weyl invariants in the direction of u with an odd number of degrees of freedom. Each one of them is an odd function on a sphere, and thus its integral vanishes. Hence, we have

$$\mathcal{W}(m,r)=r^{n-1}\left(\sum_{\alpha=-\nu}^{\left[\frac{s-2\nu}{2}\right]}r^{2\alpha}\int_{\mathbb{S}^{n-1}}\bar{W}_{2\alpha+2\nu}(u)du+O\left(r^{s-2\nu+1}\right)\right).$$

The problem of integrating $\bar{W}_{2\alpha}(u)$, with $\alpha\geq 1$, reduces to the integration of directional Weyl invariants of degree 2α in the direction of u with an even number of degrees of freedom.

This number is at most 2α. For such an invariant, Lemma 4.19 asserts that its integral is a linear combination of products of total traces, divided by certain polynomial. This immediately implies that $\int_{\mathbb{S}^{n-1}} \bar{W}_{2\alpha+2\nu}(u)du$ is a curvature invariant at m with degree 2α, and the first statement of Theorem 4.20 follows.

For the explicit expressions of $\bar{W}_0$, $\bar{W}_2$ and $\bar{W}_4$, we may use the general method described in this proof and just do the calculations taking into account the formulas for the known $\bar{W}_{2\alpha}(u)$. Examples of those may be found in [33], [82] and [83]. Also, the calculations can be obtained automatically using the program [39]. Finally, we integrate $\bar{\omega}_4(u)$ which is a simple curvature invariant at m of degree 4. Lemma 4.19 implies that $\int_{\mathbb{S}^{n-1}} \bar{\omega}_4(u)du$ is a simple curvature invariant of degree 4. Hence, we may write

$$\int_{\mathbb{S}^{n-1}} \bar{\omega}_4(u)du = \frac{1}{n(n+2)}\Big(C_W^1(n-1)\,\|R\|^2 + C_W^2(n-1)\,\|\rho\|^2 + C_W^3(n-1)\,\tau^2\Big)(m),$$

and from here we get the expression for $\mathcal{W}_4(m)$. Doing the Taylor power series expansion of the function in Example 4.17 we get in a space of constant sectional curvature

$$\begin{aligned}\int_{G_m(r)} \widetilde{W} \;&=\; c_{n-1}(n-1)(n-2)A_W(n-1)r^{n-1-2\nu}\Bigg(1 - \frac{n-1-2\nu}{6}\lambda r^2\\ &\qquad + \frac{(n-1-2\nu)(5n-10\nu-7)}{360}\lambda^2 r^4 + O\left(r^6\right)\Bigg).\end{aligned}$$

Since for an n–dimensional space of constant sectional curvature λ we have $\tau = n(n-1)\lambda$, $\|R\|^2 = 2n(n-1)\lambda^2$ and $\|\rho\|^2 = n(n-1)^2\lambda^2$ the integral $\int_{\mathbb{S}^{n-1}} \bar{\omega}_4(u)du$ becomes

$$\int_{\mathbb{S}^{n-1}} \bar{\omega}_4(u)du = \frac{n-1}{n+2}\Big(2C_W^1(n-1) + (n-1)C_W^2(n-1) + n(n-1)C_W^3(n-1)\Big)\lambda^2.$$

But $\nabla^2_{uu}\tau = \nabla^2_{uu}\rho_{uu} = 0$ in a space of constant curvature and hence the last two equations imply the desired result. □

4.2.3 Homogeneity and two–point homogeneous spaces

If M is a locally homogeneous Riemannian manifold, its total scalar curvatures $\mathcal{W}(m,r) = \int_{G_m(r)} \widetilde{W}$ do not depend on the point m and thus one may wonder whether the converse is also true. The answer is known to be positive for several special cases, but the problem remains open in its full generality. In our general context, positive answers can be given in a similar way as a consequence of Theorem 4.20 and the following result (we omit the details, which are similar to those in [26]).

Proposition 4.21. *Let W be a Weyl invariant such that $A_W(n-1) \neq 0$. If a Riemannian manifold (M^n, g) of dimension $n > 2$, $n \neq 2\nu + 1$ satisfies that $\mathcal{W}(m,r)$ is independent of the point m, then the scalar curvature and the quadratic invariant $C_W^1(n-1)\|R\|^2 + C_W^2(n-1)\|\rho\|^2$ are constant.*

In particular, if $W = 1$, $\mathcal{S}(m,r) = \int_{G_m(r)} 1$ is the volume of the geodesic sphere $G_m(r)$. A manifold having the property that the volume of geodesic spheres is independent of the center is called *ball-homogeneous* [26], [57]. Also, a Riemannian manifold is said to be *scalar curvature homogeneous* if $\mathcal{T}(m,r) = \int_{G_m(r)} \widetilde{\tau}$ is independent of m [26], [81] (this is also a particular case of our context for $W = \tau$). Next, we show that both notions above are equivalent for Einstein manifolds, thus answering a question stated in [26].

Theorem 4.22. *Ball–homogeneity and scalar curvature homogeneity are equivalent in the class of Einstein manifolds.*

Proof. As usual, we denote by $'$ the derivative with respect to the radius r. We use the following relation all over this proof

$$h_m(\exp_m(ru)) = \frac{n-1}{r} + \frac{\theta'_m(\exp_m(ru))}{\theta_m(\exp_m(ru))}.$$

Taking derivatives and using the above relation we get

$$\mathcal{S}'(m,r) = \frac{d}{dr}\left[r^{n-1}\int_{\mathbb{S}^{n-1}} \theta_m du\right] = r^{n-1}\int_{\mathbb{S}^{n-1}} (h_m\,\theta_m) du.$$

Again, taking derivatives with respect to the radius and using the relation between h_m and θ_m we obtain

$$\begin{aligned}
\mathcal{S}''(m,r) &= \frac{d}{dr}\left[r^{n-1}\int_{\mathbb{S}^{n-1}} (h_m\,\theta_m) du\right] = r^{n-1}\int_{\mathbb{S}^{n-1}} \left(\frac{n-1}{r}\,h_m\,\theta_m + h'_m\,\theta_m + h_m\,\theta'_m\right) du \\
&= r^{n-1}\int_{\mathbb{S}^{n-1}} \left(\left(h_m^2 + h'_m\right)\theta_m\right) du.
\end{aligned}$$

Taking traces in the Gauss equation we get the scalar curvature of a geodesic sphere $G_m(r)$. This is $\widetilde{\tau} = \tau - 2\,\rho_{uu} + h_m^2 - \|\sigma_m\|^2$. Next, we consider the Riccati equation, $\sigma' + \sigma^2 + R_u = 0$, and we take traces to get $h'_m + \|\sigma_m\|^2 + \rho_{uu} = 0$. Therefore, the scalar curvature becomes $\widetilde{\tau} = \tau - \rho_{uu} + h_m^2 + h'_m$. Hence, $\mathcal{S}''(m,r)$ turns into

$$\mathcal{S}''(m,r) = \mathcal{T}(m,r) - r^{n-1}\int_{\mathbb{S}^{n-1}} ((\tau - \rho_{uu})\,\theta_m).$$

In the class of Einstein manifolds $\tau - \rho_{uu} = \frac{n-1}{n}\,\tau$ is constant. Thus, we get

$$\mathcal{S}''(m,r) = \mathcal{T}(m,r) - \frac{n-1}{n}\,\tau\,\mathcal{S}(m,r).$$

Since τ is constant, $\mathcal{T}$ depends on m if and only if $\mathcal{S}$ depends on m. □

Now, we turn our attention to the characterization of two–point homogeneous spaces using the total curvatures of geodesic spheres associated with simple Weyl invariants.

Theorem 4.23. *Let (M^n, g) be a Riemannian manifold of dimension $n > 2$. Assume that the total scalar curvature associated with a simple Weyl invariant of degree 2ν is the same as in a Riemannian manifold of constant sectional curvature λ. If*

$$n \neq 2\nu + 1, \qquad A_W(n-1) \neq 0, \qquad C_W^1(n-1) \neq 0,$$
$$C_W^1(n-1)\left(C_W^2(n-1) + \frac{2}{n-1} C_W^1(n-1)\right) \geq 0,$$

then M is a Riemannian manifold of constant sectional curvature λ.

Proof. As we have already seen, in a manifold of constant sectional curvature λ, we have

$$\begin{aligned} \int_{G_m(r)} \widetilde{W} &= c_{n-1}(n-1)(n-2)A_W(n-1)r^{n-1-2\nu}\left(1 - \frac{n-1-2\nu}{6}\lambda\, r^2\right. \\ &\qquad \left. + \frac{(n-1-2\nu)(5n-10\nu-7)}{360}\lambda^2 r^4 + O\left(r^6\right)\right). \end{aligned} \tag{4.4}$$

Since $A_W(n-1) \neq 0$ and $n \neq 2\nu+1$, comparing (4.4) with the expression in Theorem 4.20 we immediately get $\tau = n(n-1)\lambda$. Then, the formula of Theorem 4.20 becomes

$$\begin{aligned} \mathcal{W}(m,r) &= c_{n-1}r^{n-1-2\nu}\Big\{(n-1)(n-2)A_W(n-1) - \frac{r^2}{6}(n-2)(n-1-2\nu)A_W(n-1)\lambda \\ &+ \frac{r^4}{n(n+2)}\left(C_W^1(n-1)\left\|R\right\|^2 + C_W^2(n-1)\left\|\rho\right\|^2 + n^2(n-1)^2 C_W^3(n-1)\lambda^2\right) + O\left(r^6\right)\Big\}(m). \end{aligned}$$

Comparing again with (4.4) we easily get

$$C_W^1(n-1)\left(\left\|R\right\|^2 - \frac{2}{n-1}\left\|\rho\right\|^2\right) + \left(C_W^2(n-1) + \frac{2}{n-1}C_W^1(n-1)\right)\left(\left\|\rho\right\|^2 - \frac{1}{n}\tau^2\right) = 0.$$

The hypotheses of this theorem and Lemma 4.1 (a)–(b) imply that both terms of the left–hand side of the above equation are simultaneously non–negative or non–positive (depending on the sign of $C_W^1(n)$). Then, both addends must be zero and hence $\|R\|^2 = \frac{2}{n-1}\|\rho\|^2$. Thus, M has constant sectional curvature by Lemma 4.1 (a)–(b). Since $\tau = n(n-1)\lambda$, the sectional curvature is precisely λ. □

We state similar theorems for the other two–point homogeneous spaces. See [33] or [83] for more information. The proof is similar to the above theorem using Lemma 4.1 (c)–(d) instead of Lemma 4.1 (b). We delete the details.

Theorem 4.24. *Let (M^{2n}, g, J) be a Kähler manifold of complex dimension $n > 1$. Assume that the total scalar curvature associated with a simple Weyl invariant of degree 2ν is the same as in a Kähler manifold of constant holomorphic sectional curvature λ. If*

$$A_W(2n-1) \neq 0, \qquad C_W^1(2n-1) \neq 0,$$
$$C_W^1(2n-1)\left(C_W^2(2n-1) + \frac{4}{n+1}C_W^1(2n-1)\right) \geq 0,$$

then M is a Kähler manifold of constant holomorphic sectional curvature λ.

Theorem 4.25. *Let M^{4n} be a quaternionic Kähler manifold of real dimension $4n$. Assume that the total scalar curvature associated with a simple Weyl invariant of degree 2ν is the same as in a quaternionic Kähler manifold of constant Q–sectional curvature λ. If*

$$A_W(4n-1) \neq 0, \qquad C^1_W(4n-1) \neq 0,$$

then M is a quaternionic Kähler manifold of constant Q–sectional curvature λ.

If the holonomy group of a Riemannian manifold is contained in $Spin(9)$, then the manifold is locally isometric to the Cayley projective or hyperbolic plane [2]. Combining the previous results and this fact we get

Theorem 4.26. *Let M be an n–dimensional Riemannian manifold, $n \geq 2$, whose holonomy group is contained in the holonomy group of one of the two–point homogeneous spaces. Assume that the total scalar curvature associated with a simple Weyl invariant of degree 2ν is the same as in the corresponding two–point homogeneous space. If*

$$n \neq 2\nu + 1, \qquad A_W(n-1) \neq 0, \qquad C^1_W(n-1) \neq 0,$$
$$C^1_W(n-1)\left(C^2_W(n-1) + \frac{2}{n-1}C^1_W(n-1)\right) \geq 0,$$

then M is locally isometric to that space.

Remark 4.27. If $n = 3$, the Gauss–Bonnet theorem gives $\mathcal{T}(m,r) = 8\pi$. Hence, $\mathcal{T}$ is a topological invariant. Generalizations of the Gauss–Bonnet theorem show that some total scalar curvatures have no geometrical meaning in certain dimensions [44]. Now, let W be a simple Weyl invariant of order 2ν. Consider a Riemannian manifold of constant sectional curvature and dimension $2\nu+1$. Then, Example 4.17 shows that $\mathcal{W}(m,r) = 2\nu(2\nu-1)c_{2\nu}$. Thus, $\mathcal{W}(m,r)$ is a topological invariant for $(2\nu+1)$–dimensional manifolds of constant sectional curvature and therefore it cannot be used to determine the curvature.

Remark 4.28. The third condition in Theorem 4.26 can be dropped if the manifold is assumed to be Einstein or locally conformally flat (see [44] or [83] for similar situations).

4.3 Applications

The purpose of this section is to employ Theorem 4.26 and its variations to characterize two–point homogeneous spaces. It turns out that for certain important scalar curvature invariants the hypotheses of Theorem 4.26 are satisfied and thus these invariants provide essential information of two–point homogeneous spaces.

In what follows, we provide an immediate application of Theorem 4.26 for the simple Weyl invariants of degree 4 and 6 given by (4.1) and (4.2). Then, we discuss further results that can be obtained under assumptions which are not satisfied in Theorem 4.26.

Subsection 4.3.2 deals with total scalar curvatures of boundaries of geodesic disks in a Riemannian manifold. The results of this section are consequence of those obtained in Subsection 4.3.1. One may obtain more general results in the spirit of Theorem 4.26 but we content ourselves with just a few examples.

4.3.1 Total scalar curvatures of geodesic spheres

It is possible to give explicit expressions of C_W^1 and C_W^2 and C_W^3 for simple Weyl invariants of low degree. This was achieved for example in [37], [38], [39] and [44]. Since C_W^3 can be obtained from C_W^1 and C_W^2 from the relation of Theorem 4.20, we do not include it in the following tables.

First, for the scalar curvature we have

W	$C_W^1(n-1)$	$C_W^2(n-1)$
τ	$-\dfrac{(n+2)(n+3)}{120}$	$\dfrac{n^2+5n+21}{45}$

For the simple Weyl invariant of degree 4 in (4.1) we have

W	$C_W^1(n-1)$	$C_W^2(n-1)$
$\Vert R\Vert^2$	$\dfrac{59n^2-93n-10}{60}$	$\dfrac{2(n^2-37n+60)}{45}$
$\Vert\rho\Vert^2$	$-\dfrac{n^3-9n^2-16n-20}{120}$	$\dfrac{n^3+31n^2-16n-120}{45}$
τ^2	$-\dfrac{(n-2)(n-1)(n^2+13n+10)}{120}$	$\dfrac{n^4+10n^3+43n^2-14n+120}{45}$

Finally, for simple Weyl invariants of degree 6 in (4.2),

W	$C_W^1(n-1)$	$C_W^2(n-1)$
τ^3	$-\dfrac{(n-1)^2(n-2)^2(n^2+21n+14)}{120}$	$\dfrac{(n-1)(n-2)(n^4+18n^3+118n^2+105n+238)}{45}$
$\tau\Vert\rho\Vert^2$	$-\dfrac{(n-1)(n-2)(n^3-n^2-28n-28)}{120}$	$\dfrac{(n-2)(n^4+38n^3+28n^2+15n+238)}{45}$
$\tau\Vert R\Vert^2$	$\dfrac{(n-1)(n-2)(59n^2-101n-14)}{60}$	$\dfrac{2(n^4-32n^3+248n^2-135n+238)}{45}$
$\check{\rho}$	$-\dfrac{(n-2)(n^3-41n^2-28n-28)}{120}$	$\dfrac{(n-2)(n^3+79n^2-73n-238)}{45}$
$\langle\rho\otimes\rho,\bar{R}\rangle$	$-\dfrac{n^4-23n^3+34n^2+28n+56}{120}$	$\dfrac{n^4+57n^3-141n^2-2n+476}{45}$
$\langle\rho,\dot{R}\rangle$	$\dfrac{59n^3-179n^2+188n+28}{60}$	$\dfrac{2(n^3+9n^2+77n-238)}{45}$
$\check{R}$	$\dfrac{179n^2-261n-14}{30}$	$\dfrac{4(n^2-129n+119)}{45}$
$\check{\check{R}}$	$-\dfrac{n^3+138n^2-289n-42}{120}$	$\dfrac{n^3+78n^2+56n-357}{45}$

Example 4.29. Using the expressions of Example 4.3 and the coefficients of C_W^1 and C_W^2 just calculated, we may check the conditions of Theorem 4.26. We give a table with those simple Weyl invariants which can be used for characterizing the two–point homogeneous spaces and the dimension n for which the conditions of Theorem 4.26 hold.

W	$C_W^2(n-1) + \dfrac{2}{n-1} C_W^1(n-1)$	n
$\|R\|^2$	$\dfrac{4n^3+25n^2+109n-270}{90(n-1)}$	$n>2$, $n\neq 5$
$\|\rho\|^2$	$\dfrac{4n^4+117n^3-161n^2-368n+540}{180(n-1)}$	$3\leq n\leq 10$, $n\neq 5$
$\tau\|\rho\|^2$	$\dfrac{(n-2)(4n^4+149n^3+115n^2+144n+1036)}{180}$	$3\leq n\leq 6$
$\tau\|R\|^2$	$\dfrac{4n^4+49n^3+335n^2+24n+1036}{90}$	$n>2$, $n\neq 7$
$\check{\rho}$	$\dfrac{(n-2)(4n^4+309n^3-485n^2-576n+1036}{180(n-1)}$	$3\leq n\leq 41$, $n\neq 7$
$\langle\rho\otimes\rho,\bar{R}\rangle$	$\dfrac{4n^5+221n^4-723n^3+454n^2+1828n-2072}{180(n-1)}$	$3\leq n\leq 21$, $n\neq 7$
$\langle\rho,\dot{R}\rangle$	$\dfrac{4n^4+209n^3-265n^2-696n+1036}{90(n-1)}$	$n>2$, $n\neq 7$
$\check{R}$	$\dfrac{(n-2)(4n^2+25n+259)}{45(n-1)}$	$n>2$, $n\neq 7$

In particular, we emphasize the following result which gives an answer to a question posed in [33] of whether a single invariant might be used for the characterization of the two–point homogeneous spaces. Characterizations with two invariants have already been obtained in [33].

Theorem 4.30. *Let M a Riemannian manifold of dimension $n\geq 2$ whose holonomy group is contained in the holonomy of one of the two–point homogenous spaces. The following statements are equivalent:*

(i) For each sufficiently small geodesic sphere, $\int_{G_m(r)}\|\widetilde{R}\|^2$ is the same as in the two–point homogeneous space ($n\neq 5$).

(ii) For each sufficiently small geodesic sphere, $\int_{G_m(r)}\widetilde{\tau}\|\widetilde{R}\|^2$ is the same as in the two–point homogeneous space ($n\neq 7$).

(iii) For each sufficiently small geodesic sphere, $\int_{G_m(r)}\langle\widetilde{\rho},\dot{\widetilde{R}}\rangle$ is the same as in the two–point homogeneous space ($n\neq 7$).

(iv) *For each sufficiently small geodesic sphere,* $\int_{G_m(r)} \check{\widetilde{R}}$ *is the same as in the two–point homogeneous space* ($n \neq 7$).

(v) M *is locally isometric to that two–point homogeneous space.*

Remark 4.31. If the manifold is assumed to be Einstein or locally conformally flat then all the simple Weyl invariants of degree 4 and 6 appearing in (4.1) and (4.2) can be used to characterize the two–point homogeneous spaces.

Dimension 5 (resp. 7) is a singular case for simple scalar curvature invariants of degree 4 (resp. 6) since the corresponding total curvatures of geodesic spheres are topological invariants in manifolds of constant sectional curvature (see Remark 4.27). Nonetheless, we can still detect constant curvature manifolds [37], [44] although we cannot determine the exact value of their sectional curvature.

Theorem 4.32. *Let* M *a* 5*–dimensional Riemannian manifold. The following statements are equivalent:*

(i) *For each sufficiently small geodesic sphere,* $\int_{G_m(r)} \|\widetilde{R}\|^2$ *is the same as in a manifold of constant sectional curvature.*

(ii) *For each sufficiently small geodesic sphere,* $\int_{G_m(r)} \|\widetilde{\rho}\|^2$ *is the same as in a manifold of constant sectional curvature.*

(iii) M *has constant sectional curvature.*

Proof. Using Theorem 4.20 and the formulas for $C^1_{\|R\|^2}$ and $C^2_{\|R\|^2}$ at the beginning of this section we get

$$\int_{G_m(r)} \|\widetilde{R}\|^2 = 24c_4 + \frac{r^4}{35}c_4 \left\{ \frac{50}{3}\left(\|R\|^2 - \frac{1}{2}\|\rho\|^2 \right) + \frac{35}{9}\left(\|\rho\|^2 - \frac{1}{5}\tau^2 \right) \right\}(m) + O\left(r^6\right).$$

By Remark 4.27 we obtain that in a manifold of constant sectional curvature $\int_{G_m(r)} \|\widetilde{R}\|^2 = 24c_4$. Hence, comparing this two expressions we get

$$\left(\|R\|^2 - \frac{1}{2}\|\rho\|^2 \right) + \frac{35}{9}\left(\|\rho\|^2 - \frac{1}{5}\tau^2 \right) = 0,$$

and it follows from Lemma 4.1 that M has constant sectional curvature. For $\|\rho\|^2$ we proceed in an analogous way taking into account that

$$\int_{G_m(r)} \|\widetilde{\rho}\|^2 = 36c_4 + \frac{r^4}{35}c_4 \left\{ \frac{5}{3}\left(\|R\|^2 - \frac{1}{2}\|\rho\|^2 \right) + \frac{298}{18}\left(\|\rho\|^2 - \frac{1}{5}\tau^2 \right) \right\}(m) + O\left(r^6\right).$$

and using Lemma 4.1 once again. □

In a similar way we obtain the following result. We delete the details [38].

Theorem 4.33. *Let M a 7–dimensional Riemannian manifold. The following statements are equivalent:*

(i) For each sufficiently small geodesic sphere, $\int_{G_m(r)} \widetilde{\tau}\|\widetilde{R}\|^2$ is the same as in a manifold of constant sectional curvature.

(ii) For each sufficiently small geodesic sphere, $\int_{G_m(r)} \check{\widetilde{\rho}}$ is the same as in a manifold of constant sectional curvature.

(iii) For each sufficiently small geodesic sphere, $\int_{G_m(r)} \langle\widetilde{\rho}\otimes\widetilde{\rho}, \bar{\bar{\widetilde{R}}}\rangle$ is the same as in a manifold of constant sectional curvature.

(iv) For each sufficiently small geodesic sphere, $\int_{G_m(r)} \langle\widetilde{\rho}, \dot{\widetilde{R}}\rangle$ is the same as in a manifold of constant sectional curvature.

(v) For each sufficiently small geodesic sphere, $\int_{G_m(r)} \check{\widetilde{R}}$ is the same as in a manifold of constant sectional curvature.

(vi) M has constant sectional curvature.

4.3.2 Total scalar curvatures of boundaries of geodesic disks

Geodesic disks were introduced by O. Kowalski and L. Vanhecke as a generalization of two–dimensional disks in the Euclidean space $\mathbb{R}^3$. In a series of papers [93], [94], [95], they investigated their volume properties in relation to local homogeneity and gave a characterization of two–point homogeneous spaces by means of the volumes of their small geodesic disks. Since the boundaries of small geodesic disks are compact submanifolds, we are interested in their total scalar curvatures obtained by integrating the corresponding scalar curvature invariants.

The *geodesic disk*, $\bar{D}^\xi_m(r)$, of sufficiently small radius r, centered at $m \in M$ and orthogonal to $\xi \in T_mM$, is defined by

$$\begin{aligned}\bar{D}^\xi_m(r) &= \{\exp_m(su) : u \in T_mM,\ \|u\| = 1,\ g(u,\xi) = 0,\ 0 \le s \le r\}\\ &= \{p \in M : d(m,p) \le r\} \cap \exp_m(\mathbb{R}\xi^\perp).\end{aligned}$$

For the purpose of this section and the investigation of total scalar curvatures, we consider the *boundaries of geodesic disks*

$$D^\xi_m(r) = \{p \in M : d(m,p) = r\} \cap \exp_m(\mathbb{R}\xi^\perp).$$

The boundary of a geodesic disk is nothing but a geodesic sphere of the (local) manifold $\exp_m(\mathbb{R}\xi^\perp)$ centered at m for sufficiently small radius. Throughout this section we use the following notation. The objects of M are denoted by R, ρ and so on, the objects of $\exp_m(\mathbb{R}\xi^\perp)$ are denoted by $\widetilde{R}$, $\widetilde{\rho}$, ... and the objects of boundaries of geodesic disks are denoted by $\hat{R}$, $\hat{\rho}$ and so on.

In what follows we consider the total scalar curvatures of boundaries of geodesic disks associated with scalar curvature invariants of degree 2 and 4. Hence, it suffices to study the scalar curvature τ and the scalar curvature invariants in (4.1). We do not consider the Laplacian of the scalar curvature since $\int_{G_m(r)} \widetilde{\Delta}\widetilde{\tau} du = 0$ by the divergence theorem.

In order to obtain the first terms in the power series expansions of total curvatures of boundaries of geodesic disks, we need the following result relating scalar curvature invariants of degree 2 and 4 of $\exp_m(\mathbb{R}\xi^{\perp})$ with the corresponding objects in M.

Lemma 4.34. *Let (M, g) be an n–dimensional Riemannian manifold and $\xi \in T_mM$ a unit vector. Then, the following relation holds at m:*

$$\begin{aligned}
\|\widetilde{R}\|^2 &= \|R\|^2 + 4\sum_{i,j=1}^{n} R^2_{\xi i \xi j} - 4\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\,,\\
\|\widetilde{\rho}\|^2 &= \|\rho\|^2 + \rho^2_{\xi\xi} - 2\sum_{i=1}^{n}\rho^2_{\xi i} + \sum_{i,j=1}^{n} R^2_{\xi i\xi j} - 2\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i \xi j}\,,\\
\widetilde{\tau} &= \tau - 2\rho_{\xi\xi}\,,\\
\widetilde{\Delta}\widetilde{\tau} &= \Delta\tau - 2\Delta\rho_{\xi\xi} + 2\nabla^2_{\xi\xi}\rho_{\xi\xi} - \nabla^2_{\xi\xi}\tau + \frac{4}{9}\rho^2_{\xi\xi} - \frac{4}{9}\sum_{i=1}^{n}\rho^2_{\xi i} + \frac{4}{3}\sum_{i,j=1}^{n}R^2_{\xi i\xi j} - \frac{2}{3}\sum_{i,j,k=1}^{n}R^2_{\xi ijk}.
\end{aligned}$$

Proof. It follows from the work in [93], after some calculations. □

The first terms in the power series expansions of the total curvatures of the boundaries of geodesic disks are obtained from the corresponding ones of geodesic spheres. We use the results of Section 4.3.1 in conjunction to Lemma 4.34. We omit the calculations which are straightforward and immediately state the different expansions separately in the following

Proposition 4.35. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_mM$ a unit vector. Then, for sufficiently small radius r, one has the following power series expansions*

$$\begin{aligned}
\int_{D^{\xi}_m(r)} \hat{\tau} = c_{n-2}r^{n-2}\Bigg\{ &\frac{(n-2)(n-3)}{r^2} - \frac{(n-3)(n-4)}{6(n-1)}\left(\tau - 2\rho_{\xi\xi}\right)\\
&+\frac{r^2}{(n-1)(n+1)}\Bigg(\frac{n^2-9n+2}{72}\tau^2 - \frac{(n+2)(n+1)}{120}\|R\|^2 + \frac{n^2+3n+17}{45}\|\rho\|^2\\
&-\frac{(n-3)(n-4)}{20}\Delta\tau + \frac{(n-3)(n-4)}{20}\left(\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}\right) - \frac{n^2-9n+2}{18}\tau\rho_{\xi\xi}\\
&-\frac{(n+2)(n+11)}{45}\sum_{i=1}^{n}\rho^2_{\xi i} - \frac{(n-4)(7n-11)}{90}\sum_{i,j=1}^{n}R^2_{\xi i\xi j} + \frac{n^2-2n+7}{15}\sum_{i,j,k=1}^{n}R^2_{\xi ijk}\\
&-\frac{2(n^2+3n+17)}{45}\sum_{i,j=1}^{n}R_{\xi i\xi j}\rho_{ij} + \frac{(n-1)(n-4)}{18}\rho^2_{\xi\xi}\Bigg) + O\left(r^3\right)\Bigg\}(m),
\end{aligned}$$

$$
\begin{aligned}
\int_{D_m^\xi(r)} \hat{\tau}^2 &= c_{n-2} r^{n-2} \Bigg\{ \frac{(n-2)^2(n-3)^2}{r^4} - \frac{(n-3)^2(n-6)(n-2)}{6(n-1)r^2} (\tau - 2\rho_{\xi\xi}) \\
&+ \frac{1}{(n-1)(n+1)} \Bigg(\frac{n^4 - 18n^3 + 77n^2 - 164n - 84}{72} \tau^2 - \frac{(n-3)^2(n-6)(n-2)}{20} \Delta\tau \\
&+ \frac{n^4 + 6n^3 + 19n^2 - 74n + 168}{45} \|\rho\|^2 - \frac{(n-2)(n-3)(n^2+11n-2)}{120} \|R\|^2 \\
&- \frac{n^4 + 26n^3 - 31n^2 - 4n + 228}{45} \sum_{i=1}^n \rho_{\xi i}^2 + \frac{(n-2)(n-3)(n^2+n+8)}{15} \sum_{i,j,k=1}^n R_{\xi ijk}^2 \\
&- \frac{n^4 - 18n^3 + 77n^2 - 164n - 84}{18} \tau\rho_{\xi\xi} - \frac{7n^4 - 78n^3 + 223n^2 - 488n + 276}{90} \sum_{i,j=1}^n R_{\xi i\xi j}^2 \\
&- \frac{2(n^4 + 6n^3 + 19n^2 - 74n + 168)}{45} \sum_{i,j=1}^n R_{\xi i\xi j}\rho_{ij} + \frac{n^4 - 10n^3 + 57n^2 - 136n - 60}{18} \rho_{\xi\xi}^2 \\
&+ \frac{(n-3)^2(n-6)(n-2)}{20} \left(\nabla_{\xi\xi}^2\tau - 2\nabla_{\xi\xi}^2\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}\right) \Bigg) + O(r) \Bigg\}(m),
\end{aligned}
$$

$$
\begin{aligned}
\int_{D_m^\xi(r)} \|\hat{\rho}\|^2 &= c_{n-2} r^{n-2} \Bigg\{ \frac{(n-2)(n-3)^2}{r^4} - \frac{(n-3)^2(n-6)}{6(n-1)r^2} (\tau - 2\rho_{\xi\xi}) \\
&+ \frac{1}{(n-1)(n+1)} \Bigg(\frac{n^3 - 16n^2 + 13n + 46}{72} \tau^2 - \frac{n^3 - 12n^2 + 5n - 14}{120} \|R\|^2 \\
&+ \frac{n^3 + 28n^2 - 75n - 74}{45} \|\rho\|^2 - \frac{(n-3)^2(n-6)}{20} \Delta\tau + \frac{n^3 - 35n + 38}{18} \rho_{\xi\xi}^2 \\
&+ \frac{(n-3)^2(n-6)}{20} \left(\nabla_{\xi\xi}^2\tau - 2\nabla_{\xi\xi}^2\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}\right) - \frac{n^3 - 16n^2 + 13n + 46}{18} \tau\rho_{\xi\xi} \\
&- \frac{2(n^3 + 28n^2 - 75n - 74)}{45} \sum_{i,j=1}^n R_{\xi i\xi j}\rho_{ij} - \frac{7n^3 - 164n^2 + 435n - 218}{90} \sum_{i,j=1}^n R_{\xi i\xi j}^2 \\
&- \frac{n^3 + 68n^2 - 195n - 94}{45} \sum_{i=1}^n \rho_{\xi i}^2 + \frac{n^3 - 12n^2 + 25n - 34}{15} \sum_{i,j,k=1}^n R_{\xi ijk}^2 \Bigg) + O(r) \Bigg\}(m),
\end{aligned}
$$

$$
\begin{aligned}
\int_{D_m^\xi(r)} \|\hat{R}\|^2 &= c_{n-2} r^{n-2} \Bigg\{ \frac{2(n-2)(n-3)}{r^4} - \frac{(n-3)(n-6)}{3(n-1)r^2} (\tau - 2\rho_{\xi\xi}) \\
&+ \frac{1}{(n-1)(n+1)} \Bigg(\frac{n^2 - 13n + 14}{36} \tau^2 + \frac{59n^2 - 211n + 142}{60} \|R\|^2 - \frac{(n-3)(n-6)}{10} \Delta\tau \\
&+ \frac{2(n^2 - 39n + 98)}{45} \|\rho\|^2 + \frac{173n^2 - 657n + 514}{45} \sum_{i,j=1}^n R_{\xi i\xi j}^2 - \frac{4(n^2 - 39n + 98)}{45} \sum_{i,j=1}^n R_{\xi i\xi j}\rho_{ij} \\
&- \frac{2(29n^2 - 101n + 62)}{15} \sum_{i,j,k=1}^n R_{\xi ijk}^2 + \frac{(n-3)(n-6)}{10} \left(\nabla_{\xi\xi}^2\tau - 2\nabla_{\xi\xi}^2\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}\right) \\
&- \frac{2(n^2 - 69n + 178)}{45} \sum_{i=1}^n \rho_{\xi i}^2 - \frac{n^2 - 13n + 14}{9} \tau\rho_{\xi\xi} + \frac{(n-2)(n-23)}{9} \rho_{\xi\xi}^2 \Bigg) + O(r) \Bigg\}(m).
\end{aligned}
$$

As an application of the expansions in Theorem 4.35 we are now ready to obtain characterizations of the two–point homogeneous spaces by means of the total curvatures of the boundaries of geodesic disks.

Lemma 4.36. *Let M be an n–dimensional Riemannian manifold. Assume that one of the following holds:*

(i) We have $n > 4$ and $\int_{D^{\xi}_m(r)} \hat{\tau}$ coincides with the corresponding one in an Einstein manifold.

(ii) We have $3 < n \neq 6$ and any of $\int_{D^{\xi}_m(r)} \hat{\tau}^2$, $\int_{D^{\xi}_m(r)} \|\hat{\rho}\|^2$ or $\int_{D^{\xi}_m(r)} \|\hat{R}\|^2$ coincides with the corresponding one in an Einstein manifold.

Then M is an Einstein manifold with the same scalar curvature as the model space.

Proof. In case (i) it follows from the constant term of the power series expansion of $\int_{D^{\xi}_m(r)} \hat{\tau}$ in Proposition 4.35. In case (ii) it follows from the coefficient of r^{-2} in the power series expansion of $\int_{D^{\xi}_m(r)} \hat{\tau}^2$, $\int_{D^{\xi}_m(r)} \|\hat{\rho}\|^2$ or $\int_{D^{\xi}_m(r)} \|\hat{R}\|^2$ in Proposition 4.35. □

Lemma 4.37. *Let M be an n–dimensional Riemannian manifold. Assume that one of the following holds:*

(i) We have $n > 4$ and $\int_{D^{\xi}_m(r)} \hat{\tau}$ does not depend on the normal direction ξ.

(ii) We have $3 < n \neq 6$ and one of $\int_{D^{\xi}_m(r)} \hat{\tau}^2$, $\int_{D^{\xi}_m(r)} \|\hat{\rho}\|^2$ or $\int_{D^{\xi}_m(r)} \|\hat{R}\|^2$ does not depend on the normal direction ξ.

Then, M is 2–stein.

Proof. Assume that (i) holds. Since the total scalar curvatures of the boundaries of geodesic disks of the manifold M do not depend on the normal direction, the constant term and the coefficient of r^2 in its power series expansion given by Proposition 4.35 are independent of the unit $\xi \in TM$. From the independent term it follows that $\tau - 2\rho_{\xi\xi}$ is constant and hence M is an Einstein space. Moreover, for an Einstein manifold, the coefficient of r^2 in the power series expansion of $\int_{D^{\xi}_m(r)} \hat{\tau}$ becomes

$$\frac{1}{(n-1)(n+1)} \left\{ \frac{(n-4)(5n^3-37n^2+62n+92)}{360n^2}\,\tau^2 - \frac{(n+2)(n+1)}{120}\,\|R\|^2 \right. \\ \left. - \frac{(n-4)(7n-11)}{90} \sum_{i,j=1}^{n} R^2_{\xi i \xi j} + \frac{n^2-2n+7}{15} \sum_{i,j,k=1}^{n} R^2_{\xi ijk} \right\}.$$

Therefore, using Lemma 4.10 we get that M is a 2–stein space.

If (ii) holds, we get the result in a similar way. From the coefficient of r^{-2} in the power series expansions of $\int_{D^\xi_m(r)} \hat{\tau}^2$, $\int_{D^\xi_m(r)} \|\hat{\rho}\|^2$ and $\int_{D^\xi_m(r)} \|\hat{R}\|^2$ we deduce that M is Einstein, and then, rewriting the independent terms of those power series expansions we get

$$\frac{1}{(n-1)(n+1)}\left\{\frac{5n^6-102n^5+789n^4-2712n^3+3352n^2+1520n-5712}{360n^2}\tau^2\right.$$
$$-\frac{(n-2)(n-3)(n^2+11n-2)}{120}\|R\|^2-\frac{7n^4-78n^3+223n^2-488n+276}{90}\sum_{i,j=1}^n R^2_{\xi i\xi j}$$
$$\left.+\frac{(n-2)(n-3)(n^2+n+8)}{15}\sum_{i,j,k=1}^n R^2_{\xi ijk}\right\},$$

$$\frac{1}{(n-1)(n+1)}\left\{-\frac{7n^3-164n^2+435n-218}{90}\sum_{i,j=1}^n R^2_{\xi i\xi j}-\frac{n^3-12n^2+5n-14}{120}\|R\|^2\right.$$
$$\left.+\frac{5n^5-92n^4+605n^3-1622n^2+548n+2696}{360n^2}\tau^2+\frac{n^3-12n^2+25n-34}{15}\sum_{i,j,k=1}^n R^2_{\xi ijk}\right\},$$

$$\frac{1}{(n-1)(n+1)}\left\{\frac{5n^4-77n^3+14n^2+1180n-2072}{180n^2}\tau^2+\frac{59n^2-211n+142}{60}\|R\|^2\right.$$
$$\left.+\frac{173n^2-657n+514}{45}\sum_{i,j=1}^n R^2_{\xi i\xi j}-\frac{2(29n^2-101n+62)}{15}\sum_{i,j,k=1}^n R^2_{\xi ijk}\right\},$$

for $\int_{D^\xi_m(r)} \hat{\tau}^2$, $\int_{D^\xi_m(r)} \|\hat{\rho}\|^2$ and $\int_{D^\xi_m(r)} \|\hat{R}\|^2$, respectively. Applying Lemma 4.10 one eventually gets the result. □

Now we are ready to derive the desired characterizations of the two–point homogeneous spaces for $n > 4$.

Theorem 4.38. *Let M be an n–dimensional Riemannian manifold whose holonomy group is contained in the holonomy group of a two–point homogeneous space. If $n > 4$ and $\int_{D^\xi_m(r)} \hat{\tau}$ coincides with that of the two–point homogeneous space for sufficiently small radius, then M is locally isometric to that two–point homogeneous space.*

Proof. It follows from Lemma 4.37 (i) that M is 2–stein and thus super–Einstein [33], from where we get that

$$\sum_{i,j=1}^n R^2_{\xi i\xi j}=\frac{1}{n(n+2)}\left(\frac{3}{2}\|R\|^2+\frac{1}{n}\tau^2\right)\qquad\text{and}\qquad\sum_{i,j,k=1}^n R^2_{\xi ijk}=\frac{1}{n}\|R\|^2.$$

Then, the coefficient of r^2 in the power series expansion of $\int_{D^\xi_m(r)} \hat{\tau}$ given by Proposition 4.35 becomes

$$\frac{n-4}{n(n^2-1)(n+2)}\left\{\frac{5n^4-27n^3-12n^2+188n+228}{360n}\tau^2-\frac{n^3+n^2+26n+6}{120}\|R\|^2\right\}.$$

Now the result is obtained just by comparing this with the corresponding coefficient in the model space and using Lemma 4.1. □

It is worthwhile to emphasize that dimension four is excluded in the previous theorem. Since the boundaries of the geodesic disks in a 4–dimensional manifold are compact surfaces, the total curvature $\int_{D^\xi_m(r)} \hat{\tau}$ is the Gauss Bonnet integral, and thus a topological invariant.

Theorem 4.39. *Let M be an n–dimensional Riemannian manifold whose holonomy group is contained in the holonomy group of a two–point homogeneous space. If $3 < n \neq 6$ and one of $\int_{D^\xi_m(r)} \hat{\tau}^2$, $\int_{D^\xi_m(r)} \|\hat{\rho}\|^2$ or $\int_{D^\xi_m(r)} \|\hat{R}\|^2$ coincides with that of the two–point homogeneous space for sufficiently small radius, then M is locally isometric to that two–point homogeneous space.*

Proof. We proceed as in the previous theorem. Using Lemma 4.37 (ii) we get that M is 2–stein. Then, the independent terms of the power series expansions of $\int_{D^\xi_m(r)} \hat{\tau}^2$, $\int_{D^\xi_m(r)} \|\hat{\rho}\|^2$ or $\int_{D^\xi_m(r)} \|\hat{R}\|^2$ given by Proposition 4.35 become

$$\frac{1}{n(n^2-1)(n+2)}\left\{\frac{5n^7-92n^6+585n^5-1162n^4-1760n^3+7332n^2-720n-12528}{360n}\tau^2 \right.$$
$$\left. -\frac{n^6-9n^4-190n^3+714n^2-840n-216}{120}\|R\|^2\right\},$$

$$\frac{n-3}{n(n^2-1)(n+2)}\left\{\frac{5n^5-67n^4+220n^3+220n^2-1380n-2088}{360n}\tau^2 \right.$$
$$\left. -\frac{(n^2-14n-2)(n^2-n+18)}{120}\|R\|^2\right\},$$

$$\frac{1}{n(n^2-1)(n+2)}\left\{\frac{(n-3)(5n^4-52n^3-296n^2+1012n+696}{180n}\tau^2 \right.$$
$$\left. +\frac{(n-3)(59n^3-148n^2-34n-12)}{60}\right\},$$

respectively. Now the result follows by comparing these coefficients with the corresponding ones in the model spaces and using Lemma 4.1. □

Chapter 5

Geodesic celestial spheres in Lorentzian manifolds

In the previous chapter we have seen that every Riemannian manifold carries a so–called Riemannian distance function whose level sets with respect to a point are exactly the geodesic spheres of the manifold. Moreover, geodesic spheres can also be seen as the image by the exponential map of Euclidean spheres in the tangent space at a point. In the general semi–Riemannian setting, such a distance is not defined and the fact that the pseudo–spheres of the tangent space are not compact leads us to believe that their image by the exponential map is not suitable for study.

A distance–like function $d : M \times M \to [0, \infty]$ may be defined for space–times. For any $p, q \in M$ we have $d(p, q) = 0$ if and only if q is not in the causal future of p and $d(p, q) = \sup\{L(c) : c \text{ is a future directed non–spacelike curve from } p \text{ to } q\}$ if q is in the causal future of p. However, the properties of this distance function are completely different from those in the Riemannian setting [7]. For example, the "Lorentzian distance" may fail to be continuous or finite–valued and its level sets with respect to a given point are not compact. Some properties of those sets have been previously investigated [4], [56], but they do not seem to be adequate for the investigation of volume properties. Therefore, different families of objects have been considered for this purpose in Lorentzian geometry. We give an overview of these constructions in Section 5.1.

The concept of geodesic celestial sphere which we introduce in this chapter is somehow an extension of the concept of geodesic disk to Lorentzian geometry. Given a unit timelike vector, its orthogonal complement in the tangent space has definite signature. In Relativity, a unit timelike vector ξ is called an instantaneous observer and its orthogonal complement in the tangent space is called the infinitesimal rest–space of ξ. In Special Relativity the rest space of an instantaneous observer corresponds to the Newtonian universe perceived by this observer. A geodesic celestial sphere is the image by the exponential map of a sphere centered at the origin of the rest–space associated with certain instantaneous observer.

It is clear that every geodesic celestial sphere (for sufficiently small radius) is a compact Riemannian submanifold. We see in this chapter that geodesic celestial spheres are closely related to geodesic spheres and they inherit most of their properties. Thus, volume properties can be discussed and in many cases we get analogous results to those of the Riemannian setting.

This chapter is organized as follows. Section 5.1 reviews some constructions in Riemannian and Lorentzian geometry which we attempt to generalize. In Section 5.2 we study the volume of geodesic celestial spheres and give the necessary background to prove the main theorems of this section, namely, Theorems 5.16 and 5.17 (which compare the volumes of sufficiently small geodesic celestial spheres in a Lorentzian manifold with the corresponding ones in a Lorentzian space form) and Theorem 5.22 (which characterizes locally isotropic space–times). Finally, in Section 5.3 we state some results analogous to Theorem 5.22, showing that local isotropy can be detected by considering the total curvatures of geodesic celestial spheres. We also give examples of scalar curvature invariants which may be used for this characterization.

5.1 Volume comparison results

Any Riemannian manifold (M^{n+1}, g) carries a Riemannian distance function which has a very nice behavior with respect to the underlying structure of the manifold. Therefore, a natural family of subregions of a Riemannian manifold to be considered is that defined by the level sets of the Riemannian distance function with respect to a base point (that is, geodesic spheres) or with respect to some topologically embedded submanifolds (that is, tubes around a submanifold).

For sufficiently small radii $r > 0$, geodesic spheres $G_m(r)$ are obtained by projecting the Euclidean spheres $\mathbb{S}^n(r)$ centered at $0 \in T_mM$ via the exponential map. Therefore, they are a nice family of hypersurfaces and their volume can be calculated as

$$\mathcal{S}(m,r) = \mathrm{vol}\,(G_m(r)) = r^n \int_{\mathbb{S}^n} \theta_m(exp_m(ru))du.$$

Comparison theorems for the volumes of subregions of Riemannian manifolds under some curvature hypotheses have played an important role in Riemannian geometry. For instance, the Bishop–Günther inequalities show lower (resp. upper) bounds for volumes of geodesic balls and tubes by imposing upper (resp. lower) bounds on the sectional curvature. These inequalities have been improved by assuming weaker conditions on the Ricci tensor or by considering the ratio between the volumes of geodesic balls in the manifold and the model spaces (see for example [82] and the references therein).

The basic idea behind the Bishop–Günter and Gromov comparison theorems [82], is that under suitable curvature conditions the Riccati differential equation

$$S' + S^2 + R_u = 0.$$

becomes an inequality and its solutions give upper or lower bounds for the volume density function θ_m in terms of the corresponding function in the model space via

$$h_m(\exp_m(ru)) = \frac{n}{r} + \frac{\partial}{\partial r} \log \theta_m(\exp_m(ru)).$$

Finally, an integration process from the Riccati equations leads to [18], [86]

Theorem 5.1. *Let (M^{n+1}, g) be a complete Riemannian manifold and assume that r is not greater than the distance between m and its cut locus. Let K^M denote the sectional curvature of (M, g).*

(i) If $K^M \geq \lambda$, then $\mathrm{vol}^M(G_m(r)) \leq \mathrm{vol}^{M(\lambda)}(G_{\tilde{m}}(r))$.

(ii) If $K^M \leq \lambda$, then $\mathrm{vol}^M(G_m(r)) \geq \mathrm{vol}^{M(\lambda)}(G_{\tilde{m}}(r))$.

Here, $M(\lambda)$ is a model space of constant sectional curvature λ and $\tilde{m} \in M(\lambda)$. Moreover, equalities hold for (i) or (ii) and some radii if and only if $G_m(r)$ is isometric to the corresponding geodesic sphere in the model space.

A sharper result involving the Ricci curvature instead of the sectional curvature was proved by R. L. Bishop [18].

Theorem 5.2. *Let (M^{n+1}, g) be a complete Riemannian manifold. Assume that r is not greater that the distance between m and its cut locus and the Ricci curvature ρ^M of (M, g) satisfies $\rho^M(v, v) \geq n\,\lambda$ for all vectors $v \in TM$.*

Then $\mathrm{vol}^M(G_m(r)) \leq \mathrm{vol}^{M(\lambda)}(G_{\tilde{m}}(r))$, where $M(\lambda)$ is a space of constant sectional curvature λ. The equality holds if and only if $G_m(r)$ is isometric to the corresponding geodesic sphere in the model space.

A further generalization of Theorem 5.2 was obtained by M. Gromov as follows [85].

Theorem 5.3. *Let (M^{n+1}, g) be a complete Riemannian manifold. Assume that r is not greater that the distance between m and its cut locus and that the Ricci curvature ρ^M of (M, g) satisfies $\rho^M(v, v) \geq n\,\lambda$ for all vectors v. Then the function*

$$r \mapsto \frac{\mathrm{vol}^M(G_m(r))}{\mathrm{vol}^{M(\lambda)}(G_{\tilde{m}}(r))},$$

where $M(\lambda)$ is a space of constant sectional curvature λ, is non–increasing.

When the attention is turned from Riemannian manifolds to space–times, various difficulties emerge. For example, conditions on bounds for the sectional curvature (resp. the Ricci tensor) easily produce manifolds of constant sectional curvature (resp. Einstein) [7], [108]. This demands a revision of such conditions [5] (see Section 5.2.2). However, a more difficult task is related to the consideration of the regions under investigation. This is mainly due to the fact that when dealing with general semi–Riemannian manifolds there is no "semi–Riemannian distance function". In fact, a distance–like function is only defined for space–times, but even in this case its properties are completely different from those in the Riemannian setting (see [7]). For instance, level sets of the Lorentzian distance function with respect to a given point are not compact and they do not seem to be adequate for the investigation of volume properties. Therefore, different families of objects have been considered in Lorentzian geometry for the purpose of investigating their volume properties. Among those, truncated light cones, compact distance wedges in the chronological future of some point, and more generally some neighborhoods covered by timelike geodesics emanating from a given point have been investigated. In what follows we review some known results of the geometry of those families. Then, we introduce geodesic celestial spheres, which is the main object of study throughout this chapter.

5.1.1 Truncated light cones

Truncated light cones were defined in [66], [67] where the authors studied the link between the volume of the light cones and the curvature of a Lorentzian manifold.

Let ξ be an instantaneous observer. The *truncated light cone of* (sufficiently small) *height* T *and axis* ξ is the set

$$L_\xi(T) = \big\{ \exp_m(u) : \langle u, u\rangle < 0,\ \ 0 \le -\langle u, \xi\rangle \le T \big\}.$$

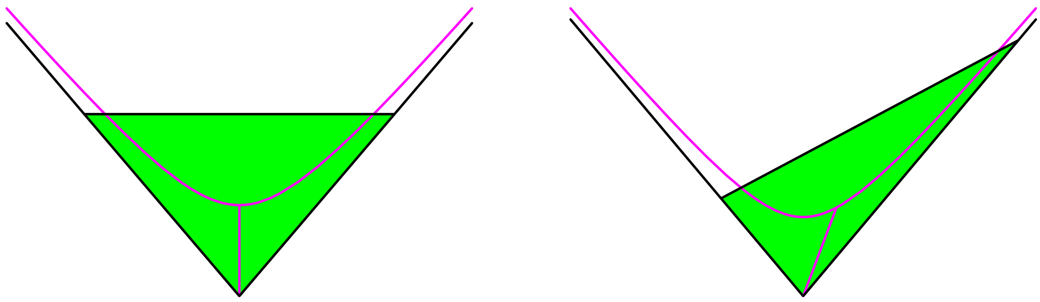

Figure 5.1: Truncated light cones in $\mathbb{R}^2_1$ with height $T = 3$ and axes $\xi_1 = (0, 1)$ and $\xi_2 = (1, \sqrt{2})$.

It is easy to see that the volume of a truncated light cone in the four–dimensional Minkowski space–time is given by $\mathrm{vol}(L_\xi(T)) = \pi T^4/3$. The investigation of whether this volume property is characteristic of the Minkowski space motivated further work by R. Schimming [117], [118], who proved the following result. See also [66].

Theorem 5.4. *Let* (M, g) *be a Lorentzian manifold such that every truncated light cone has the same volume as in the Minkowski space–time. Then* (M, g) *is locally flat.*

5.1.2 Compact distance wedges

Let E denote the set of future pointing unit timelike vectors in T_mM such that the exponential map is well defined. Let K be a compact subset of E and put $\mathfrak{K} = \exp_m(t_0K)$, which is a compact subset of the level set $d_m^{-1}(t_0)$ of the Lorentzian distance function with respect to $m \in M$ and which is well defined for sufficiently small t_0.

The $\mathfrak{K}$–*distance wedge* $B_m^{\mathfrak{K}}(t_0)$ is defined by [54]

$$B_m^{\mathfrak{K}}(t_0) = \{\exp_m(tv) : v \in K,\, 0 \le t \le t_0\}.$$

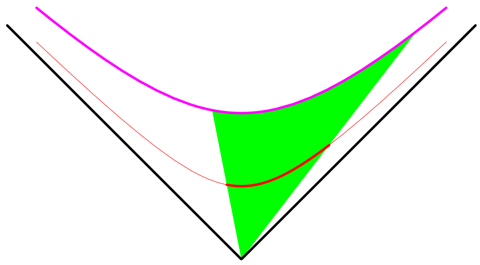

Figure 5.2: $\mathfrak{K}$–distance wedge centered at m up to a distance t_0.

In order to study volume comparison results with model spaces, one needs a method to relate distance wedges on M and the model space. One proceeds as follows. Choose a point $\tilde{m}$ in the model space of constant sectional curvature $M(-\lambda)$ and define a differentiable map Ψ by $\Psi = \exp_{\tilde{m}}^{M(-\lambda)} \circ \psi \circ (\exp_m^M)^{-1}$, where $\psi : T_mM \to T_{\tilde{m}}M(-\lambda)$ is a linear isometry. Then, given a distance wedge $B_m^{\mathfrak{K}}(t_0)$ and using the timelike vectors $\psi(K)$ in $T_{\tilde{m}}M(-\lambda)$ to construct the corresponding wedge $B_{\tilde{m}}^{\Psi(\mathfrak{K})}(t_0)$ in $M(-\lambda)$, we have $B_{\tilde{m}}^{\Psi(\mathfrak{K})}(t_0) = \Psi(B_m^{\mathfrak{K}}(t_0))$ for sufficiently small t_0.

By making use of the Riccati equation and comparison of the Jacobi equations, the following volume comparison results for compact distance wedges have been obtained by P. Ehrlich, Y.–T. Jung and S.–B. Kim [54] as an analogous of the Günter–Bishop and Gromov theorems.

Theorem 5.5. *Let (M^{n+1}, g) be a globally hyperbolic space–time satisfying $\rho^M(v,v) \ge n\lambda > 0$ for all timelike unit vectors v. Then for all $0 \le r_0 \le inj_{K(m)}$,*

$$\mathrm{vol}^M(B_m^{\mathfrak{K}}(r_0)) \le \mathrm{vol}^{M(-\lambda)}(B_{\psi(m)}^{\Psi(\mathfrak{K})}(r_0))$$

and equality holds at some $r_0 > 0$ if and only if $B_m^{\mathfrak{K}}(r)$ and $B_{\psi(m)}^{\Psi(\mathfrak{K})}(r)$ are isometric for all $0 < r \le r_0$.

Theorem 5.6. *Let (M^{n+1}, g) be a globally hyperbolic space–time satisfying $\rho^M(v,v) \ge n\lambda > 0$ for all timelike unit vectors v. Then for all $0 \le r_0 < r_1 \le inj_{\overline{K}(m)}$,*

$$\frac{vol^M(B_m^{\mathfrak{K}}(r_0))}{vol^{M(-\lambda)}(B_{\psi(m)}^{\Psi(\mathfrak{K})}(r_0))} \ge \frac{vol^M(B_m^{\mathfrak{K}}(r_1))}{vol^{M(-\lambda)}(B_{\psi(m)}^{\Psi(\mathfrak{K})}(r_1))}.$$

Moreover, equality holds for some $0 \le r_0 < r_1 \le inj_{\overline{K}(m)}$ if and only if $B_m^{\mathfrak{K}}(r)$ and $B_{\psi(m)}^{\Psi(\mathfrak{K})}(r)$ are isometric for all $0 < r \le r_1$.

Note that, when comparing with the corresponding results in the Riemannian setting, inequalities in the previous theorems are with respect to a space of constant sectional curvature $-\lambda$. This is due to the fact that the assumption on the Ricci tensor in Theorems 5.2 and 5.3 gives reversed inequalities when considering the Ricatti equation.

5.1.3 SCLV sets

A further generalization of the distance wedges was obtained in [55], where the authors considered a more general family of subsets of a Lorentzian manifold. Let $m \in M$ and take $U \subset T_m M$ an open subset in the causal future of the origin, $U \subset J^+(0)$ such that U is star–shaped from the origin and the exponential map $\exp_{m|U}$ is a diffeomorphism onto its image $\mathfrak{U} = \exp_m U$. We also assume that the closure of U is compact.

A subset $\mathfrak{U}$ as above is called *standard for comparison of Lorentzian volumes* (SCLV set) at the base point $m \in M$ [55].

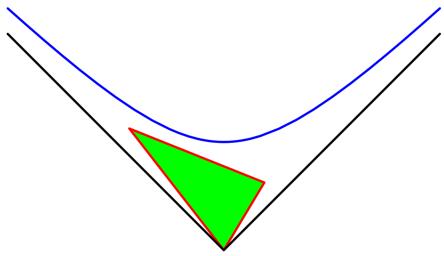

Figure 5.3: A SCLV set.

In order to state some comparison results with spaces of constant sectional curvature $M(\lambda)$, a transplantation process is also needed as before. Let $\psi : T_m M \to T_{\tilde{m}} M(\lambda)$ be a linear isometry, and define the transplantation map Ψ on a sufficiently small open set as $\Psi = \exp_{\tilde{m}}^{M(\lambda)} \circ \psi \circ (\exp_m^M)^{-1}$. For any $U \subset T_m M$ put $U_\lambda = \psi(U)$ and $\mathfrak{U}_\lambda = \exp_{\tilde{m}}^{M(\lambda)}(U_\lambda) = \Psi(\mathfrak{U})$ which makes possible a volume comparison between SCLV sets in M and $M(\lambda)$. Then we have [55]

Theorem 5.7. *Let (M, g) be a $(n+1)$–dimensional Lorentzian manifold and assume that $\rho^M(v, v) \geq n\,\lambda\, g(v, v)$ for all timelike vector fields $v = \frac{d}{dt} exp_m(tv_m)|_{t=t_0}$ tangent to $\mathfrak{U}$ at $m \in M$. If $\mathfrak{U}$ is a SCLV set at m, then*

$$\mathrm{vol}^M(\mathfrak{U}) \leq \mathrm{vol}^{M(\lambda)}(\mathfrak{U}_\lambda)$$

and the equality holds if and only if $\Psi : \mathfrak{U} \to \mathfrak{U}_\lambda$ is an isometry.

A comparison result in the spirit of Bishop–Gromov Theorem can also be stated for SCLV sets, but it requires some previous conventions. For each $r > 0$ put $U(r) = r \cdot U = \{ru : u \in U\}$, $U_\lambda(r) = r \cdot U_\lambda$, $\mathfrak{U}(r) = \exp_m^M(U(r))$, $\mathfrak{U}_\lambda(r) = \exp_{\tilde{m}}^{M(\lambda)}(U_\lambda(r))$. Note that the star–shaped form of SCLV sets ensures the possibility of constructing the above sets for $r > 0$ sufficiently small. Then, we have the following result [55].

Theorem 5.8. *Let (M^{n+1}, g) be a Lorentz manifold such that $\rho(v,v) \geq n\,\lambda\, g(v,v)$ for all timelike vector fields $v = \frac{d}{dt}\exp_m(tv_m)|_{t=t_0}$ tangent to a SCLV set $\mathfrak{U}$ based at $m \in M$. Assume that $c = 0$ or the cut function $c_{\mathfrak{U}}$ of $\mathfrak{U}$ is constant. Then, the function $r \mapsto \mathrm{vol}^M(\mathfrak{U}(r))/\mathrm{vol}^{M(\lambda)}(\mathfrak{U}_\lambda(r))$ is non–increasing. Moreover if there exists $r_1 < r_2$ such that $vol^M(\mathfrak{U}(r_1))/vol^{M(\lambda)}(\mathfrak{U}_\lambda(r_1)) = vol^M(\mathfrak{U}(r_2))/vol^{M(\lambda)}(\mathfrak{U}_\lambda(r_2))$ then $\mathfrak{U}(r)$ and $\mathfrak{U}_\lambda(r)$ are isometric.*

5.1.4 Geodesic celestial spheres

In this subsection we consider a different family of geometric objects from those presented so far, the so–called geodesic celestial spheres.

In Relativity, a unit timelike vector $\xi \in T_mM$ is called an *instantaneous observer*, and $\mathbb{R}\xi^\perp$ is called the *infinitesimal rest–space* of ξ, that is, the infinitesimal Newtonian universe where the observer perceives particles as Newtonian particles relative to his rest position. The *celestial sphere of radius* r perpendicular to a unit timelike vector ξ is defined as the set $\mathbb{S}^\xi(r) = \{x \in \xi^\perp : \|x\| = r\}$ (see [114]). If $\mathfrak{U}$ is a sufficiently small neighborhood of the origin in T_mM, $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$ is an embedded Riemannian submanifold of M. We denote by $\widetilde{\nabla}$ the Levi–Civita connection of $\widetilde{M}$, by $\widetilde{R}$ its curvature tensor and, in general, we use the symbol $\widetilde{\ }$ to denote the geometric objects of $\widetilde{M}$.

We define the *geodesic celestial sphere of radius r associated with ξ* as [46]

$$G^\xi_m(r) = \exp_m\left(\{x \in \xi^\perp : \|x\| = r\}\right) = \exp_m(\mathbb{S}^\xi(r)).$$

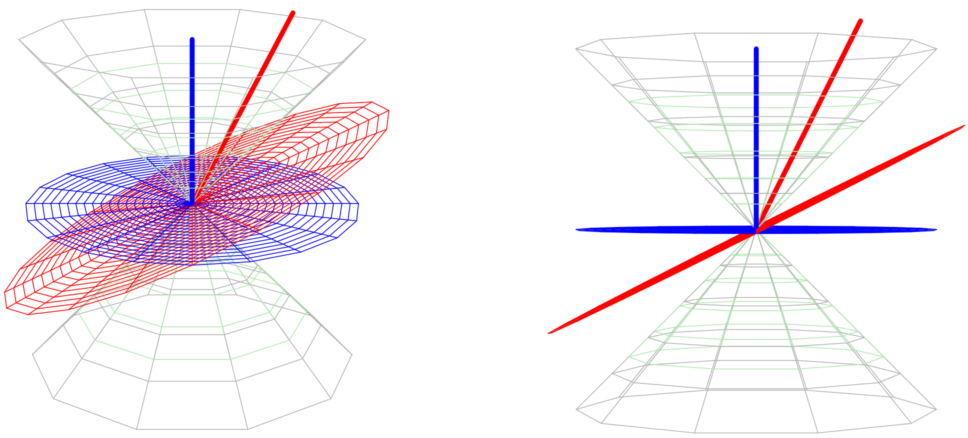

Figure 5.4: A plot of geodesic celestial spheres in the Minkowski space–time $(\mathbb{R}^3, -dt^2 + dx^2 + dy^2)$ with center $(1,1,1)$ associated with the instantaneous observers $(1,0,0)$ and $(2/\sqrt{3}, 1/\sqrt{3}, 0)$.

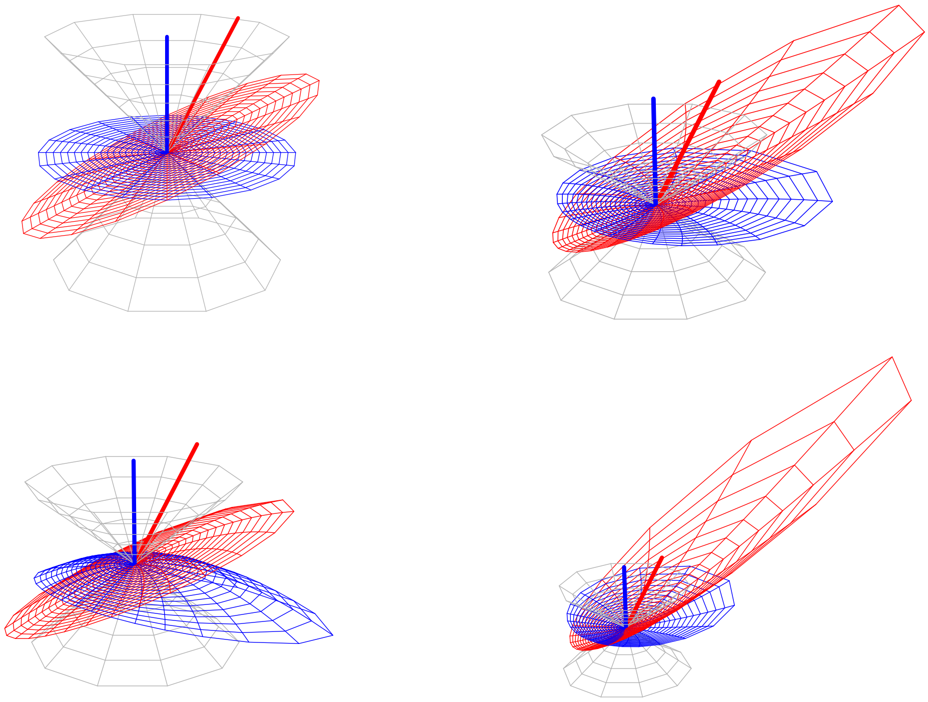

Figure 5.5: Geodesic celestial spheres in a space–time of constant sectional curvature $\left(\mathbb{R}^3, \frac{-dt^2+dx^2+dy^2}{(1+\frac{1}{4}(-t^2+x^2+y^2))^2}\right)$ with centers at $(0,0,0)$, $(0,1,1)$, $(1,1,1)$ and $(-1/2,1,1)$ associated with the instantaneous observers $(1,0,0)$ and $(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$.

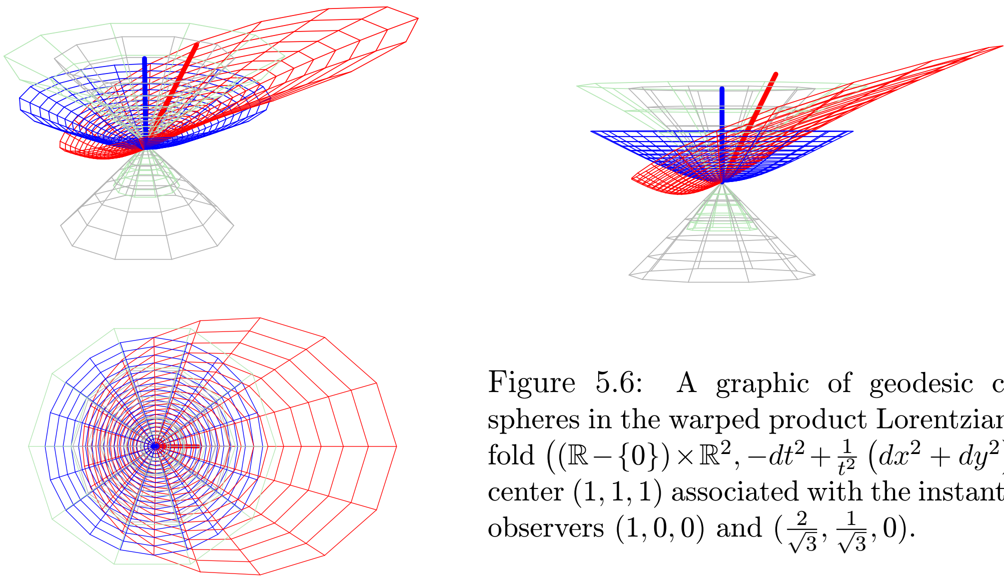

Figure 5.6: A graphic of geodesic celestial spheres in the warped product Lorentzian manifold $\left((\mathbb{R}-\{0\})\times\mathbb{R}^2, -dt^2+\frac{1}{t^2}\left(dx^2+dy^2\right)\right)$ with center $(1,1,1)$ associated with the instantaneous observers $(1,0,0)$ and $(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$.

For sufficiently small radius r, $G_m^\xi(r)$ is a compact submanifold of $\widetilde{M}$. Furthermore, by definition, a geodesic celestial sphere $G_m^\xi(r)$ is nothing but a geodesic sphere $G_m(r)$ of radius r centered at m in the submanifold $\widetilde{M}$. Therefore, by studying the volumes of geodesic celestial spheres in comparison with the volumes of the corresponding celestial spheres one obtains a measure of how the exponential map distorts volumes on spacelike directions.

5.2 Volume of geodesic celestial spheres

In this section we study the volume of geodesic celestial spheres. Our objectives are two–fold. On the one hand, we are interested in volume comparison results in the spirit of the Bishop–Günther and Gromov theorems previously discussed. Then we characterize locally isotropic Lorentzian manifolds by means of the volume of geodesic celestial spheres.

From the very definition of geodesic celestial sphere it is clear that for a given radius, the volume of geodesic celestial spheres depends both on the observer field $\xi \in T_mM$ and the center point $m \in M$. However, if (M, g) is assumed to be of constant sectional curvature, then the volumes depend only on the radii, as Lorentzian space forms are locally isotropic [133]. The converse result is also true. Indeed, one may compute the volume of geodesic celestial spheres in a Lorentzian space form [46].

Theorem 5.9. *Let $M^{n+1}(\lambda)$ be a Lorentzian manifold of constant sectional curvature λ. Then, for each point $m \in M$ and each instantaneous observer $\xi \in T_mM$, the volume of the geodesic celestial sphere $G_m^\xi(r)$ satisfies*

$$\mathrm{vol}_{n-1}\left(G_m^\xi(r)\right) = \begin{cases} c_{n-1}\left(\dfrac{\sin r\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1}, & \lambda > 0, \\ c_{n-1}r^{n-1}, & \lambda = 0, \\ c_{n-1}\left(\dfrac{\sinh r\sqrt{-\lambda}}{\sqrt{-\lambda}}\right)^{n-1}, & \lambda < 0. \end{cases}$$

Proof. Consider the manifold $\widetilde{M} = \exp_m\left(\mathfrak{U} \cap \mathbb{R}\xi^\perp\right)$ defined above, which is an embedded Riemannian submanifold of M. Since M has constant sectional curvature λ, $\widetilde{M}$ has also sectional curvature λ (see for example [116]). As it was noticed before, the geodesic celestial sphere $G_m^\xi(r)$ is the geodesic sphere of radius r centered at m of the n–dimensional Riemannian submanifold $\widetilde{M}$. The volume of geodesic spheres in constant curvature Riemannian manifolds is well known [33], [83], [128], which gives the formula of the statement of this theorem. □

If $N(\lambda)$ is a Lorentzian manifold of constant sectional curvature λ, by Theorem 5.9, the volume of a geodesic celestial sphere is independent of the base point $m \in N(\lambda)$ and the instantaneous observer $\xi \in T_m N(\lambda)$. Thus, in this case we can use the unambiguous notation

$$\mathrm{vol}_{n-1}\left(G(r)\right) = \mathrm{vol}_{n-1}\left(G_m^\xi(r)\right).$$

For the purpose of the comparison results below, we will also denote by $\mathrm{vol}_{n-1}^M\left(G_m^\xi(r)\right)$ the $(n-1)$–dimensional volume of the geodesic celestial sphere $G_m^\xi(r)$ of radius r and center m associated with the instantaneous observer ξ in the manifold M.

In the following subsection we give the technical background to prove the main theorems of this chapter, which are in Subsections 5.2.2 and 5.2.3.

5.2.1 Power series expansions

The technique we use to prove the main results of this chapter relies on the possibility of writing down the first terms in the power series expansion of the function $r \mapsto \mathrm{vol}_{n-1}^M\left(G_m^\xi(r)\right)$, for sufficiently small r.

From now on we assume the following notation. For fixed $m \in M$ and $\xi \in T_m M$ we consider the Riemannian submanifold $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$, where $\mathfrak{U}$ is a sufficiently small neighborhood of m. Objects of $\widetilde{M}$ are denoted by $\widetilde{\ }$. We choose an orthonormal basis $\{e_0 = \xi, e_1, \ldots, e_n\}$ at m.

Lemma 5.10. *With the above notation, the first and second order curvature invariants of M and $\widetilde{M}$ at the base point m satisfy*

$$\begin{aligned}
\|\widetilde{R}\|^2 &= \|R\|^2 + 4\sum_{i,j,k=1}^n R_{\xi ijk}^2 - 4\sum_{i,j=1}^n R_{\xi i\xi j}^2,\\
\|\widetilde{\rho}\|^2 &= \|\rho\|^2 + 2\sum_{i=1}^n \rho_{\xi i}^2 - \rho_{\xi\xi}^2 + \sum_{i,j=1}^n R_{\xi i\xi j}^2 + 2\sum_{i,j=1}^n \rho_{ij}R_{\xi i\xi j},\\
\widetilde{\tau} &= \tau + 2\rho_{\xi\xi},\\
\widetilde{\Delta}\widetilde{\tau} &= \Delta\tau + 2\Delta\rho_{\xi\xi} + \nabla_{\xi\xi}^2\tau + 2\nabla_{\xi\xi}^2\rho_{\xi\xi} + \frac{4}{9}\sum_{i=1}^n \rho_{\xi i}^2 + \frac{2}{3}\sum_{i,j,k=1}^n R_{\xi ijk}^2.
\end{aligned}$$

Proof. Denote by ξ a local extension of $\xi \in T_m M$ to the normal bundle of $\widetilde{M}$. If c is a radial geodesic in $\widetilde{M}$ starting at m, then $\widetilde{\nabla}_{c'}c' = \nabla_{c'}c' = II(c', c') = 0$ since $\widetilde{M} = \exp_m(\mathfrak{U} \cap \mathbb{R}\xi^\perp)$. Thus, taking covariant derivatives and evaluating at m, we get $\widetilde{\nabla}_{u\cdots u}^k \sigma_{uu} = 0$, $k \geq 0$, for all $u \in T_m\widetilde{M}$. For $k = 0$ we immediately get by polarization that

$$\sigma_{uv} = 0, \quad \text{for all } u, v \in T_m\widetilde{M}.$$

Now put $k = 1$ and take arbitrary $a, b, c \in \mathbb{R}$ and $u, v, w \in T_m\widetilde{M}$. We have

$$0 = \widetilde{\nabla}_{au+bv+cw}\sigma_{au+bv+cw,au+bv+cw} = \cdots + 2abc\left(\widetilde{\nabla}_u\sigma_{vw} + \widetilde{\nabla}_v\sigma_{uw} + \widetilde{\nabla}_w\sigma_{uv}\right) + \cdots$$

and hence $\widetilde{\nabla}_u\sigma_{vw} + \widetilde{\nabla}_v\sigma_{uw} + \widetilde{\nabla}_w\sigma_{uv} = 0$. Then it follows from the Riccati equation that

$$R_{uvw\xi} = \widetilde{\nabla}_u\sigma_{vw} - \widetilde{\nabla}_v\sigma_{uw} \qquad \text{and} \qquad R_{uwv\xi} = \widetilde{\nabla}_u\sigma_{vw} - \widetilde{\nabla}_w\sigma_{uv}.$$

Therefore, we can express $\widetilde{\nabla}\sigma$ in terms of the curvature tensor of the ambient manifold M as follows

$$\widetilde{\nabla}_u\sigma_{vw} = \frac{1}{3}\left(R_{uvw\xi} + R_{uwv\xi}\right), \quad \text{for all } u, v, w \in T_m\widetilde{M}.$$

We now determine the curvature tensor of $\widetilde{M}$ at m. An immediate application of the Gauss equation and $\sigma_{uv} = 0$ shows that

$$\widetilde{R}_{xyvw} = R_{xyvw}, \tag{5.1}$$

for all $x, y, v, w \in T_m\widetilde{M}$.

Taking covariant derivatives in the Gauss equation we get

$$\begin{aligned}\widetilde{\nabla}_Z\widetilde{R}_{XYVW} = {} & \nabla_Z R_{XYVW} + \sigma_{ZX}R_{\xi YVW} + \sigma_{ZY}R_{X\xi VW} + \sigma_{ZV}R_{XY\xi W} + \sigma_{ZW}R_{XYV\xi} \\ & -\sigma_{YW}\widetilde{\nabla}_Z\sigma_{XV} - \sigma_{XV}\widetilde{\nabla}_Z\sigma_{YW} + \sigma_{YV}\widetilde{\nabla}_Z\sigma_{XW} + \sigma_{XW}\widetilde{\nabla}_Z\sigma_{YV}\end{aligned} \tag{5.2}$$

for all $X, Y, Z, V, W \in \Gamma(T\widetilde{M})$. Using $\sigma_{uv} = 0$ we get

$$\widetilde{\nabla}_z\widetilde{R}_{xyvw} = \nabla_z R_{xyvw} \tag{5.3}$$

for all $z, x, y, v, w \in T_m\widetilde{M}$.

Finally, taking covariant derivatives in (5.2) we obtain

$$\begin{aligned}\widetilde{\nabla}^2_{XX}\widetilde{R}_{YZYZ} = {} & \nabla^2_{XX}R_{YZYZ} + \sigma_{XX}\nabla_\xi R_{YZYZ} + 2\,\sigma^2_{XY}R_{\xi X\xi Z} + 2\,\sigma^2_{XZ}R_{\xi Y\xi Y} \\ & -4\,\sigma_{XY}\sigma_{XZ}R_{\xi Y\xi Z} + 2\,\sigma_{XY}R_{TXZYZ} + 2\,\sigma_{XZ}R_{YTXYZ} + 2\,\widetilde{\nabla}_X\sigma_{XY}R_{\xi ZYZ} \\ & +2\,\widetilde{\nabla}_X\sigma_{XZ}R_{Y\xi YZ} - \sigma_{YY}\widetilde{\nabla}_{XX}\sigma_{ZZ} - \sigma_{ZZ}\widetilde{\nabla}_{XX}\sigma_{YY} + \sigma_{YZ}\widetilde{\nabla}_{XX}\sigma_{YZ} \\ & -2\widetilde{\nabla}_X\sigma_{YY}\widetilde{\nabla}_X\sigma_{ZZ} + 2\left(\widetilde{\nabla}_X\sigma_{YZ}\right)^2 + 4\sigma_{XY}\nabla_X R_{\xi ZYZ} + 4\sigma_{XZ}\nabla_X R_{Y\xi YZ},\end{aligned}$$

and using the expression for σ and $\widetilde{\nabla}\sigma$ at m we get

$$\begin{aligned}\widetilde{\nabla}^2_{xx}\widetilde{R}_{yzyz} = {} & \nabla^2_{xx}R_{yzyz} + \tfrac{2}{3}R_{xyx\xi}R_{yz\xi z} + \tfrac{2}{3}R_{xzx\xi}R_{yzy\xi} \\ & -\tfrac{8}{9}R_{xy\xi y}R_{xz\xi z} + \tfrac{2}{9}R^2_{xyz\xi} + \tfrac{2}{9}R^2_{xzy\xi} + \tfrac{4}{9}R_{xyz\xi}R_{xzy\xi},\end{aligned} \tag{5.4}$$

for all $x, y, z \in T_m\widetilde{M}$.

Lemma 5.10 follows from (5.1), (5.3), (5.4) and the definitions of τ, $\|R\|^2$, $\|\rho\|^2$ and $\Delta\tau$ after doing some straightforward calculations. $\square$

Theorem 5.11. *Let (M^{n+1}, g) be a Lorentzian manifold and $\xi \in T_mM$ an instantaneous observer. The $(n-1)$–dimensional volume of the geodesic celestial spheres associated with $\xi \in T_mM$ satisfies*

$$\mathrm{vol}_{n-1}\left(G_m^{\xi}(r)\right) = c_{n-1}r^{n-1}\left(1 + \frac{A(\xi)}{n}r^2 + \frac{B(\xi)}{n(n+2)}r^4 + O(r^6)\right)(m),$$

where

$$\begin{aligned} A(\xi) &= -\frac{1}{6}(\tau + 2\rho_{\xi\xi}), \\ B(\xi) &= -\frac{1}{120}\,\|R\|^2 + \frac{1}{45}\,\|\rho\|^2 + \frac{1}{72}\,\tau^2 - \frac{1}{20}\,\Delta\tau - \frac{1}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk} + \frac{2}{45}\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i\xi j} \\ &\quad + \frac{1}{18}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{1}{45}\sum_{i=1}^{n}\rho^2_{\xi i} + \frac{1}{30}\rho^2_{\xi\xi} + \frac{1}{18}\tau\rho_{\xi\xi} - \frac{1}{10}\Delta\rho_{\xi\xi} - \frac{1}{20}\nabla^2_{\xi\xi}\tau - \frac{1}{10}\nabla^2_{\xi\xi}\rho_{\xi\xi}. \end{aligned}$$

Proof. Since radial geodesics starting from m orthogonally to ξ are the same for M and $\widetilde{M}$, it is clear that the geodesic celestial sphere $G_m^{\xi}(r)$ of M associated with the instantaneous observer $\xi \in T_mM$ coincides with the geodesic sphere $G_m^{\widetilde{M}}(r)$ of radius r centered at m in the Riemannian manifold $\widetilde{M}$ for sufficiently small radius. Now, the first terms in the power series expansion of the volume of sufficiently small geodesic spheres are well known [83]. This is also a special case of Theorem 4.20 for the Weyl invariant of degree 0, $W = 1$:

$$\mathrm{vol}\left(G_m^{\widetilde{M}}(r)\right) = c_{n-1}r^{n-1}\left\{1 - \frac{\widetilde{\tau}}{6n}r^2 - \frac{r^4}{n(n+2)}\left(\frac{\|\widetilde{R}\|^2}{120} - \frac{\|\widetilde{\rho}\|^2}{45} - \frac{\widetilde{\tau}^2}{72} + \frac{\widetilde{\Delta}\widetilde{\tau}}{20}\right) + O(r^6)\right\}(m).$$

Using the relations in Lemma 5.10 the result follows. □

We also state here some algebraic preliminaries.

Lemma 5.12. *Let $(V, \langle\, ,\, \rangle)$ a Lorentzian vector space and let W denote a covariant tensor of type $(0, 2k)$. If $W_{\zeta\cdots\zeta} = 0$ for all ζ with $\langle\zeta, \zeta\rangle = -1$, then $W_{x\cdots x} = 0$ for all $x \in V$.*

Proof. If ζ is a timelike vector, we have, by hypothesis

$$0 = W\left(\frac{\zeta}{\sqrt{-\langle\zeta,\zeta\rangle}}, \dots, \frac{\zeta}{\sqrt{-\langle\zeta,\zeta\rangle}}\right) = \left(-\langle\zeta,\zeta\rangle\right)^{-k} W_{\zeta\cdots\zeta},$$

and thus $W_{\zeta\cdots\zeta} = 0$. Now, if x is an arbitrary vector, for sufficiently small ϵ, $\zeta + \epsilon x$ is timelike if ζ is timelike. Then, $0 = W(\zeta + \epsilon x, \dots, \zeta + \epsilon x) = W_{\zeta\cdots\zeta} + \cdots + \epsilon^{2k}W_{x\cdots x}$. Taking into account that ϵ is arbitrary, this immediately implies that $W_{x\cdots x} = 0$ which proves the result. □

Lemma 5.13. *Let (M^{n+1}, g) be a Lorentzian manifold and let a, b, c be real numbers with $b \neq 0$. If $a\,\tau + b\,\rho_{\zeta\zeta} = c$ at some point $m \in M$ for all vector $\zeta \in T_mM$ with $g(\zeta, \zeta) = -1$, then the manifold is Einstein at m.*

Proof. The hypothesis can equivalently be written as $-ag(\zeta,\zeta)\tau + b\rho_{\zeta\zeta} + cg(\zeta,\zeta) = 0$, which using Lemma 5.12 implies $-ag(x,x)\tau + b\rho_{xx} + cg(x,x) = 0$ for all $x \in T_mM$. The result follows from linearity and symmetry of the Ricci tensor. □

For the purpose of analyzing the coefficient $B(\xi)$ in Theorem 5.11, we define the following two tensors

$$\begin{aligned} \eta(x,y) &= \sum_{i,j,k=0}^{n} \epsilon_i\epsilon_j\epsilon_k R(x,e_i,e_j,e_k)R(y,e_i,e_j,e_k), \\ \omega(x,y,v,w) &= \sum_{i,j=0}^{n} \epsilon_i\epsilon_j R(x,e_i,y,e_j)R(v,e_i,w,e_j), \end{aligned}$$

where, as usual $\epsilon_i = g(e_i,e_i)$ and $x,y,v,w \in T_mM$. Note that the definitions above are independent of the orthonormal basis chosen, and thus ω and η are well defined tensors at a given point $m \in M$. We have the following result.

Lemma 5.14. *Let (M^{n+1},g) be an Einstein Lorentzian manifold. If there exist constants $a,b,c,k \in \mathbb{R}$ with $c \neq 0$ and $3c \neq (n+5)b$ such that*

$$a\,\|R\|^2 + b\,\eta_{\zeta\zeta} + c\,\omega_{\zeta\zeta\zeta\zeta} = k$$

for all vectors $\zeta \in T_mM$ with $g(\zeta,\zeta) = -1$, then M has constant sectional curvature at m.

Proof. Using Lemma 5.12, the hypothesis can be rewritten as

$$a\,\|R\|^2 g(x,x)^2 - bg(x,x)\eta(x,x) + c\,\omega(x,x,x,x) = kg(x,x)^2 \tag{5.5}$$

for all $x \in T_mM$. For arbitrary $\alpha,\beta \in \mathbb{R}$ and tangent vectors x and y we get

$$a\,\|R\|^2 g^2_{\alpha x+\beta y,\alpha x+\beta y} - b\,g_{\alpha x+\beta y,\alpha x+\beta y}\eta_{\alpha x+\beta y,\alpha x+\beta y} + c\,\omega_{\alpha x+\beta y,\dots,\alpha x+\beta y} \;=\; k\,g^2_{\alpha x+\beta y,\alpha x+\beta y}.$$

Since α and β are arbitrary, expanding the above equality and comparing the coefficients of $\alpha^2\beta^2$ we get,

$$\begin{aligned} 2a\,\|R\|^2(g_{xx}g_{yy} + 2g_{xy}^2) - b(g_{xx}\eta_{yy} + 4g_{xy}\eta_{xy} + g_{yy}\eta_{xx}) & \\ +2c(\omega_{xxyy} + \omega_{xyxy} + \omega_{xyyx}) &= 2k(g_{xx}g_{yy} + 2g_{xy}^2). \end{aligned}$$

Setting $y = e_i$ in the above equality and contracting we have

$$2a(n+3)\,\|R\|^2 g_{xx} - b\,(\|R\|^2 g_{xx} + (n+5)\eta_{xx}) + c\left(3\eta_{xx} + 2\sum_{i,j=0}^{n} \epsilon_i\epsilon_j\rho_{ij}R_{xixj}\right) = 2(n+3)k\,g_{xx}.$$

Since M is Einstein,

$$\sum_{i,j=0}^{n} \epsilon_i\epsilon_j\rho_{ij}R_{xixj} = \frac{\tau^2}{(n+1)^2}\,g_{xx},$$

and the above equation becomes

$$(-b(n+5)+3c)\eta_{xx} = \left(-2a(n+3)\,\|R\|^2 + b\,\|R\|^2 - 2c\frac{\tau^2}{(n+1)^2} + 2k\,(n+3)\right) g_{xx}.$$

Contracting again,

$$(-b(n+5)+3c)\,\|R\|^2 = \left(-2a(n+3)\,\|R\|^2 + b\,\|R\|^2 - 2c\frac{\tau^2}{(n+1)^2} + 2k\,(n+3)\right)(n+1).$$

Hence, $\eta_{xx} = \|R\|^2 g_{xx}/(n+1)$, which by the symmetry of η and the metric tensor, it is equivalent to $\eta = \|R\|^2 g/(n+1)$. As a consequence, using (5.5) we have

$$\omega_{xxxx} = \frac{1}{c}\left(-a\,\|R\|^2 + b\,\frac{\|R\|^2}{n+1} + k\right) g(x,x)^2. \tag{5.6}$$

Next, we show that the above equation is an equivalent condition to constant sectional curvature for Lorentzian manifolds. Let $\pi \subset T_mM$ be a plane of signature $(-+)$ and let $\{\zeta, \vartheta\}$ be an orthonormal basis of π with $g(\zeta,\zeta) = -g(\vartheta,\vartheta) = -1$. The Jacobi operator $R_\zeta(x) = R(\zeta, x)\zeta$ is self–adjoint when restricted to $\zeta^\perp$, and thus it is diagonalizable with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\zeta^\perp$ with eigenvalues $\lambda_1(\zeta), \ldots, \lambda_n(\zeta)$. Now, with respect to the orthonormal basis of T_mM, $\{e_0 = \zeta, e_1, \ldots, e_n\}$, Equation (5.6) gives

$$\sum_{i,j=1}^{n} R^2_{\zeta e_i \zeta e_j} = \frac{1}{c}\left(\frac{b-(n+1)a}{n+1}\,\|R\|^2 + k\right).$$

Hence, the the eigenvalues $\lambda_\alpha(\zeta)$ are bounded independently of the timelike unit ζ because

$$\lambda_\alpha(\zeta)^2 = R^2_{\zeta e_\alpha \zeta e_\alpha} \leq \sum_{i,j=1}^{n} R^2_{\zeta e_i \zeta e_j} = \frac{1}{c}\left(\frac{b-(n+1)a}{n+1}\,\|R\|^2 + k\right),$$

for all $\alpha \in \{1, \ldots, n\}$. Writing $\vartheta = \sum_{i=1}^n \vartheta^i e_i$ with respect to the basis above, one has that the sectional curvature of π satisfies

$$K(\pi) = -R_{\zeta\vartheta\zeta\vartheta} = -\sum_{i,j=1}^{n} \vartheta^i\vartheta^j R_{\zeta e_i \zeta e_j} = -\sum_{i=1}^{n} (\vartheta^i)^2 \lambda_i(\zeta).$$

Since $\langle \vartheta, \vartheta\rangle = 1 = \sum_{i=1}^n (\vartheta^i)^2$, one has $|K(\pi)| \leq \sum_{i=1}^n (\vartheta^i)^2 |\lambda_i(\zeta)| \leq C$ for some constant C. This shows that the sectional curvature is bounded on planes of signature $(+-)$ and therefore, M has constant curvature at m (see [7], [96], [108]). □

Remark 5.15. Equation (5.6) is equivalent to the 2–stein condition. See [79] for a different proof that 2–stein Lorentzian manifolds have constant curvature.

5.2.2 Volume comparison theorems

It is well known that the sectional curvature of a semi–Riemannian manifold is bounded from above or from below if and only if it is constant [7], [108]. In order to derive volume comparison theorems we need a revision of the boundedness conditions on the sectional curvature. It seems natural to impose such curvature bounds on the curvature tensor itself rather than on the sectional curvature. Following [5], we write $R \geq \lambda$ or $R \leq \lambda$ if and only if for all $x, y \in TM$

$$
\begin{aligned}
R(x,y,x,y) &\geq \lambda\left(g(x,x)g(y,y) - g(x,y)^2\right) \quad \text{or} \\
R(x,y,x,y) &\leq \lambda\left(g(x,x)g(y,y) - g(x,y)^2\right),
\end{aligned}
$$

respectively. Note that the first condition above (resp. the second condition) is equivalent to requiring the sectional curvature to be bounded from below (resp. from above) on planes of signature $(++)$ and from above (resp. from below) on planes of signature $(+-)$.

Examples of semi–Riemannian manifolds whose curvature tensor is bounded as above can easily be produced as follows:

- Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds with non–negative $K^{M_1} \geq 0$ and non–positive $K^{M_2} \leq 0$ sectional curvature, respectively. Then the product manifold $(M_1 \times M_2, g_1 - g_2)$ is a semi–Riemannian manifold whose curvature tensor satisfies $R \geq 0$. See [5] for related examples.

- A more general construction of Lorentzian manifolds with bounded curvature is as follows. Let (M, g) be a conformally flat Lorentz manifold whose Ricci tensor is diagonalizable, $\rho = diag(\mu_0, \mu_1, \ldots, \mu_n)$, where the distinguished eigenvalue μ_0 corresponds to a timelike eigenspace. If $\mu_0 \geq max\{\mu_1, \ldots, \mu_n\}$ (resp. $\mu_0 \leq min\{\mu_1, \ldots, \mu_n\}$) then $R \leq \lambda$ (resp. $R \geq \lambda$) for some constant λ. Note that the previous construction applies to Robertson–Walker space–times as well as to locally conformally flat static space–times whose rest–spaces are of constant sectional curvature [24].

Although it is not possible to obtain direct information of the Ricci tensor from the boundedness conditions above, an important observation for the purpose of studying volume properties of geodesic celestial spheres is the following. Let ξ be an instantaneous observer at $m \in M$ and complete it to an orthonormal basis $\{e_0 = \xi, e_1, \ldots, e_n\}$ of T_mM. Then $\tau + 2\rho_{\xi\xi} = \sum_{i,j=1}^{n} R_{ijij}$. Hence by assuming $R \geq \lambda$ (resp. $R \leq \lambda$), we have $\tau + 2\rho_{\xi\xi} \geq n(n-1)\lambda$ (resp. $\tau + 2\rho_{\xi\xi} \leq n(n-1)\lambda$).

We are now ready to prove a Bishop–Günther type theorem [43], [46].

Theorem 5.16. *Let (M^{n+1}, g) be a $(n+1)$–dimensional Lorentzian manifold and $N^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ. The following statements hold:*

(i) If $R \geq \lambda$, then

$$\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right) \leq \mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r)\right),$$

for all sufficiently small r and all instantaneous observers $\xi \in T_mM$.

(ii) If $R \leq \lambda$, then

$$\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right) \geq \mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r)\right),$$

for all sufficiently small r and all instantaneous observers $\xi \in T_mM$.

Moreover, the equality holds in (i) or (ii) for all $\xi \in T_mM$ if and only if M has constant sectional curvature λ at m.

Proof. Assume that $R \geq \lambda$. If $R \leq \lambda$ the result is obtained in a similar way. As usual, let $\{e_0 = \xi, e_1, \ldots, e_n\}$ be an orthonormal basis of T_mM. As we have already seen, $\tau + 2\rho_{\xi\xi} = \sum_{i,j=1}^{n} R_{ijij}$. Hence, $\tau + 2\rho_{\xi\xi} \geq n(n-1)\lambda$. Thus, by Theorems 5.9 and 5.11, we have for sufficiently small r

$$\begin{aligned}\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right) &= c_{n-1}r^{n-1}\left(1 - \frac{\tau + 2\rho_{\xi\xi}}{6n}r^2 + O(r^4)\right)\\ &\leq c_{n-1}r^{n-1}\left(1 - \frac{n-1}{6}\lambda\, r^2 + O(r^4)\right) = \mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r)\right),\end{aligned}$$

which proves the first part of the assertion.

Now, assume that the equality holds for sufficiently small r and all $\xi \in T_mM$. Then, $\tau + 2\rho_{\xi\xi} = n(n-1)\lambda$ for all $\xi \in T_mM$. We prove that this implies that the sectional curvature K is constant $K = \lambda$ on planes of signature $(++)$. Given π a plane of signature $(++)$, we take an orthonormal basis $\{x, y\}$ of π and we complete it to an orthonormal basis $\{e_0, e_1 = x, e_2 = y, \ldots, e_n\}$ of T_mM with e_0 timelike. Then

$$\sum_{i,j=1}^{n} R_{ijij} = \tau + 2\rho_{e_0e_0} = n(n-1)\lambda$$

and since $R_{ijij} \geq \lambda$ by assumption, it follows that $K(\pi) = \lambda$. Now the constancy of the sectional curvature at m follows from [108]. □

The previous theorem shows that $\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right) / \mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r)\right) \leq 1$ (resp. ≥ 1) if $R \geq \lambda$ (resp. $R \leq \lambda$). A more precise result in the spirit of Gromov's theorem can be stated as follows [43], [46]

Theorem 5.17. *Let (M^{n+1}, g) be a $(n+1)$–dimensional Lorentzian manifold and $N^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ.*

(i) If $R \geq \lambda$, then

$$r \mapsto \frac{\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r)\right)}$$

is non–increasing for sufficiently small r and all instantaneous observers $\xi \in T_mM$.

(ii) If $R \leq \lambda$, then

$$r \mapsto \frac{\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r)\right)}$$

is non–decreasing for sufficiently small r and all instantaneous observers $\xi \in T_m M$.

Proof. By using the results in Theorem 5.9 and Theorem 5.11 one gets the first terms in the power series expansion of the quotient

$$\frac{\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(S(r)\right)} = 1 + \left(\frac{n(n-1)\lambda - (\tau + 2\rho_{\xi\xi})}{6n}\right) r^2 + O(r^4).$$

Therefore, if $R > \lambda$, we have $\tau + 2\rho_{\xi\xi} > n(n-1)\lambda$. Hence the derivative of the quotient is negative for small r, and thus the quotient is decreasing, which shows (i), since in case $R = \lambda$ the quotient above is constant for sufficiently small r. The proof of (ii) is completely analogous. □

Remark 5.18. Under the hypothesis of Theorem 5.17, if there exists $0 < r_0 < r_1$ such that

$$\frac{\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r_0)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r_0)\right)} = \frac{\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r_1)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(G(r_1)\right)},$$

then the sectional curvature is constant. Indeed, since the quotient above is monotone, then it must be constant and thus $R = \lambda$ (see proof of Theorem 5.17).

Remark 5.19. We point out here that the proofs of Theorem 5.16 and 5.17 only require the boundedness conditions to hold for spacelike planes.

5.2.3 Characterization of locally isotropic Lorentzian manifolds

We recall that a Lorentzian manifold is said to be *locally isotropic* if for each point $m \in M$ and all pair of non–null vectors $x, y \in T_m M$ with $g(x,x) = g(y,y)$ there exists a local isometry of (M,g) fixing m and transforming x into y.

If M is locally isotropic, $\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right)$ does not depend on the instantaneous observer $\xi \in T_m M$. Moreover, since locally isotropic Lorentzian manifolds are locally homogeneous, it follows that $\mathrm{vol}_{n-1}^{M}\left(G_m^{\xi}(r)\right)$ does not depend on the center m. The following theorem shows that local isotropy can be recovered from the properties of the volume of geodesic celestial spheres [46].

Theorem 5.20. *Let $\left(M^{n+1}, g\right)$ be a Lorentzian manifold. If the volume of the geodesic celestial spheres $G_m^{\xi}(r)$ is independent of the observer field $\xi \in TM$, then M has constant sectional curvature.*

Proof. If the volume of each geodesic celestial sphere $G_m^{\xi}(r)$ is independent of the instantaneous observer $\xi \in T_m M$, then the coefficients $A(\xi)$ and $B(\xi)$ in the power series expansion of $\mathrm{vol}_{n-1}(G_m^{\xi}(r))$ in Theorem 5.11 are independent of ξ. Now, as $-(\tau + 2\rho_{\xi\xi})/6 = A(\xi)$ is constant, using Lemma 5.13, one has that that M is Einstein, and thus $\rho = \frac{\tau}{n+1} g$. Hence, it follows from the second coefficient $B(\xi)$ in Theorem 5.11 that

$$constant \;=\; B(\xi) \;=\; -\frac{1}{120}\,\|R\|^2 + \frac{5n^2+38n+61}{360(n+1)^2}\,\tau^2 + \frac{1}{18}\sum_{i,j=1}^{n} R^2_{\xi i \xi j} - \frac{1}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}.$$

With the notation of Lemma 5.14 we have

$$\omega_{\xi\xi\xi\xi} = \sum_{i,j=0}^{n} \epsilon_i\epsilon_j R^2_{\xi i\xi j} = \sum_{i,j=1}^{n} R^2_{\xi i\xi j} \quad \text{and} \quad \eta_{\xi\xi} = \sum_{i,j,k=0}^{n} \epsilon_i\epsilon_j\epsilon_k R^2_{\xi ijk} = \sum_{i,j,k=1}^{n} R^2_{\xi ijk} - 2\sum_{i,j=1}^{n} R^2_{\xi i\xi j},$$

and it follows from that lemma that the sectional curvature of M is constant. □

Corollary 5.21. *Let (M^{n+1}, g) be a Lorentzian manifold. If the volume of each geodesic celestial sphere of sufficiently small radius coincides with the corresponding one of a geodesic celestial sphere of the same radius in a space of constant sectional curvature λ, then M has constant sectional curvature λ.*

Proof. Theorem 5.9 implies that in a Lorentzian manifold of constant sectional curvature the volume of geodesic celestial spheres does not depend on the instantaneous observer. Using Theorem 5.20 we deduce that M has constant sectional curvature λ. Doing the power series expansion of the formula in Theorem 5.9 we get

$$\mathrm{vol}_{n-1}\left(G^{N(\lambda)}(r)\right) = c_{n-1} r^{n-1} \left\{ 1 - \frac{n-1}{6}\lambda\, r^2 + O\left(r^4\right) \right\}.$$

Comparing the coefficient of r^2 in the above power series expansion with the corresponding one in the formula of Theorem 5.11 we get that the sectional curvature is exactly λ. □

Since the concept of local isotropy is equivalent to constant sectional curvature for Lorentzian manifolds, the results of this section can be condensed for $n+1 \geq 3$ in the following

Theorem 5.22. *A Lorentzian manifold is locally isotropic if and only if the volume of its geodesic celestial spheres is independent of the instantaneous observer.*

5.3 Total curvatures of geodesic celestial spheres

The main purpose of this section is to investigate the curvature of geodesic celestial spheres by focusing on the properties of their total curvatures associated with simple Weyl invariants. We show that Lorentzian manifolds of constant sectional curvature can be characterized by means of total curvatures of geodesic celestial spheres (see Theorem 5.26).

From now on, all geometric objects defined on a geodesic celestial sphere will be denoted using the symbol $\hat{}$. Let W be a simple Weyl invariant. We define the *total scalar curvature of the geodesic celestial sphere $G_m^\xi(r)$ associated with the simple Weyl invariant W* as [42]

$$\mathcal{W}_m(\xi, r) = \int_{G_m^\xi(r)} \hat{W}.$$

As stated before, $\hat{W}$ denotes the corresponding simple Weyl invariant in the geodesic celestial sphere $G_m^\xi(r)$.

As it was the case for the volume of geodesic celestial spheres, it is clear from the definition that the total scalar curvature associated with a simple Weyl invariant depends on the radius r, the base point m, the instantaneous observer $\xi \in T_m M$ and the Weyl invariant W involved in its construction. If the manifold has constant sectional curvature there is no dependence on the point or the instantaneous observer. In fact, an exact formula may be obtained.

Theorem 5.23. *Let $\left(M^{n+1}, g\right)$ a Lorentzian manifold of constant sectional curvature λ. For each point $m \in M$ and each instantaneous observer $\xi \in T_m M$ the total scalar curvature $\mathcal{W}_m(\xi, r)$ associated with the simple Weyl invariant W of degree 2ν is*

$$\operatorname{vol}_{n-1}\left(G_m^\xi(r)\right) = \begin{cases} c_{n-1}(n-1)(n-2)A_W(n-1)\left(\dfrac{\sin r\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1-2\nu}, & \lambda > 0, \\ c_{n-1}(n-1)(n-2)A_W(n-1)r^{n-1-2\nu}, & \lambda = 0, \\ c_{n-1}(n-1)(n-2)A_W(n-1)\left(\dfrac{\sinh r\sqrt{-\lambda}}{\sqrt{-\lambda}}\right)^{n-1-2\nu}, & \lambda < 0, \end{cases}$$

where A_W is the polynomial given in Remark 4.2.

Proof. Since the manifold M has constant curvature, the submanifold $\widetilde{M} = \exp_m\left(\mathfrak{U} \cap \xi^\perp\right)$ defined in the previous section has also sectional curvature λ. The geodesic celestial sphere $G_m^\xi(r)$ is the geodesic sphere of radius r centered at m of the n–dimensional Riemannian submanifold $\widetilde{M}$. Then, the total scalar curvature of the geodesic celestial sphere $G_m^\xi(r)$ associated with W is the total scalar curvature of the geodesic sphere $G_m(r)$ of $\widetilde{M}$ associated with W. The latter was given in Example 4.17 for positive curvature. For zero and negative curvature we get analogous expressions, the ones appearing in the statement of this theorem. $\square$

In order to derive a characterization of locally isotropic Lorentzian manifolds we need the first terms of the power series expansion of $r \mapsto \mathcal{W}_m(\xi, r)$. These are calculated in what follows. We assume that $\{e_0 = \xi, e_1, \dots, e_n\}$ is an orthonormal basis of $T_m M$.

Theorem 5.24. *Let (M^{n+1}, g) be a Lorentzian manifold and let W be a simple Weyl invariant of order 2ν. Then, we have :*

$$
\begin{aligned}
\mathcal{W}_m(\xi, r) \;=\; & c_{n-1} r^{n-1-2\nu} \Big\{ (n-1)(n-2) A_W(n-1) \\
& -\frac{r^2}{6n}(n-2)(n-2\nu-1) A_W(n-1)\, \{\tau + 2\rho_{\xi\xi}\} \\
& +\frac{r^4}{n(n+2)} \Big(C_W^1(n-1)\, \|R\|^2 + C_W^2(n-1)\, \|\rho\|^2 + C_W^3(n-1)\, \tau^2 \\
& \quad -\frac{(n-2)(n-2\nu-1)}{20} A_W(n-1)\, \Delta\tau + 2C_W^2(n-1) \sum_{i,j=1}^{n} \rho_{ij} R_{\xi i \xi j} \\
& \quad +\Big(4C_W^1(n-1) - \frac{(n-2)(n-2\nu-1)}{30} A_W(n-1)\Big) \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 \\
& \quad +\Big(2C_W^2(n-1) - \frac{(n-2)(n-2\nu-1)}{45} A_W(n-1)\Big) \sum_{i=1}^{n} \rho_{\xi i}^2 + 4C_W^3(n-1)\tau\rho_{\xi\xi} \\
& \quad +\Big(-C_W^2(n-1) + 4C_W^3(n-1)\Big)\rho_{\xi\xi}^2 + \Big(-4C_W^1(n-1) + C_W^2(n-1)\Big) \sum_{i,j=1}^{n} R_{\xi i \xi j}^2 \\
& \quad -\frac{(n-2)(n-2\nu-1)}{20} A_W(n-1)\Big(2\Delta\rho_{\xi\xi} + \nabla_{\xi\xi}^2\tau + 2\nabla_{\xi\xi}^2\rho_{\xi\xi}\Big)\Big) + O\left(r^6\right)\Big\}(m),
\end{aligned}
$$

where A_W, C_W^1, C_W^2 and C_W^3 are the same polynomials as in Theorem 4.20.

Proof. It follows from Theorem 4.20 and the fact that the geodesic celestial sphere $G_m^\xi(r)$ is the geodesic sphere centered at m in the Riemannian submanifold $\exp_m(\mathfrak{U}\cap\xi^\perp)$ [42]. $\square$

Remark 5.25. Of particular interest is the case $W = 1$, where we consider W as a simple Weyl invariant of order 0. Then $\mathcal{W}_m(\xi, r)$ is nothing but the volume of a geodesic celestial sphere, and one has Theorem 5.11 as a particular case.

In what follows we generalize Theorem 5.22 to total scalar curvatures of geodesic celestial spheres associated with simple Weyl invariants. Afterwards, we deal with the usual lower degree simple Weyl invariants and we see that the abstract conditions in the following theorem can be dropped.

Theorem 5.26. *Let (M^{n+1}, g) be a Lorentzian manifold with $n+1 > 3$ and W a simple Weyl invariant of order 2ν. Assume $\mathcal{W}_m(\xi, r)$ is independent of the infinitesimal observer $\xi \in TM$ and that the following relations hold*

$$
\begin{aligned}
& 2\nu + 1 \neq n, \qquad\qquad A_W(n-1) \neq 0, \\
& 4C_W^1(n-1) + C_W^2(n-1) - \frac{(n-2)(n-1-2\nu)}{15} A_W(n-1) \neq 0, \\
& 4(n+2)C_W^1(n-1) - 3C_W^2(n-1) - \frac{(n-2)(n-1)(n-1-2\nu)}{30} A_W(n-1) \neq 0.
\end{aligned}
$$

Then, M has constant sectional curvature.

Proof. Since $\mathcal{W}_m(\xi, r)$ is independent of $\xi \in TM$, it is clear that the coefficient of r^2 in the power series expansion of Theorem 5.24 must be constant. Since $n+1 > 3$ using the first two conditions we have $0 \neq (n-2)(n-2\nu-1)A_W(n-1)$ and thus $\tau + 2\rho_{\xi\xi}$ is constant. Then Lemma 5.13 implies that M is Einstein.

Since M is Einstein, the power series expansion of Theorem 5.24 reduces to

$$\begin{aligned}
\mathcal{W}(\xi, r) &= c_{n-1} r^{n-1-2\nu} \Big\{ (n-1)(n-2)A_W(n-1) \\
&\quad - \frac{r^2}{6\,n(n+1)}(n-2)(n-2\nu-1)(n-1)A_W(n-1)\tau \\
&\quad + \frac{r^4}{n(n+2)} \Big\{ C^1_W(n-1)\,\|R\|^2 + \frac{1}{(n+1)^2}\Big((n-2)C^2_W(n-1) + (n-1)^2 C^3_W(n-1)\Big)\tau^2 \\
&\quad + \Big(4C^1_W(n-1) - \frac{(n-2)(n-2\nu-1)}{30}A_W(n-1)\Big)\sum_{i,j,k=1}^{n} R^2_{\xi ijk} \\
&\quad + \Big(-4C^1_W(n-1) + C^2_W(n-1)\Big)\sum_{i,j=1}^{n} R^2_{\xi i\xi j} \Big\} + O\left(r^6\right) \Big\}(m).
\end{aligned}$$

Proceeding as before, the coefficient of r^4 must also be constant and thus

$$\begin{aligned}
\mathit{constant} &= C^1_W(n-1)\,\|R\|^2 + \Big(4C^1_W(n-1) - \tfrac{(n-2)(n-2\nu-1)}{30}A_W(n-1)\Big)\sum_{i,j,k=1}^{n} R^2_{\xi ijk} \\
&\quad + \Big(-4C^1_W(n-1) + C^2_W(n-1)\Big)\sum_{i,j=1}^{n} R^2_{\xi i\xi j}.
\end{aligned}$$

The last two conditions of the statement of the theorem ensure that Lemma 5.14 can be applied and thus, M has constant sectional curvature. □

Corollary 5.27. *Let (M^{n+1}, g) be a Lorentzian manifold and W a simple Weyl invariant. If for each small radius r and each $\xi \in TM$, $\mathcal{W}_m(\xi, r)$ is the same as the corresponding one in an $(n+1)$–Lorentzian manifold of constant sectional curvature λ and the conditions in Theorem 5.26 hold, then M is a Lorentzian manifold of constant sectional curvature λ.*

Proof. A Lorentzian manifold of constant sectional curvature is locally isotropic, so the total scalar curvatures of geodesic celestial spheres do not depend on the infinitesimal observer ξ. Thus, from Theorem 5.26 it follows that M has also constant sectional curvature. From Theorem 5.23 we get that the power series expansion of $\mathcal{W}_m(\xi, r)$ for a Lorentzian manifold of constant sectional curvature becomes

$$\begin{aligned}
\mathcal{W}_m(\xi, r) &= c_{n-1}\,(n-1)(n-2)A_W(n-1)\,r^{n-1-2\nu}\Big\{1 - \frac{(n-2\nu-1)}{6}\,\lambda\,r^2 \\
&\quad + \frac{(n-1-2\nu)(5n-10\nu-7)}{360}\,\lambda^2\,r^4 + O\left(r^6\right)\Big\}.
\end{aligned}$$

Comparing the coefficient of r^2 in the power series expansion of Theorem 5.24 with the corresponding one in the equation above gives $\tau = n(n+1)\lambda$, and hence the curvature is exactly λ. □

As an application of Theorem 5.26 we show how the simple Weyl invariants of lower degree can be used for characterizing the Lorentzian manifolds of constant sectional curvature. First of all, when $n = 2$ geodesic celestial spheres are flat, and hence all scalar curvature invariants vanish. When $n = 3$ geodesic celestial spheres are 2–dimensional Riemannian manifolds. Therefore, by Gauss–Bonnet Theorem $\int_{G^\xi_m(r)} \hat{\tau} = 8\pi$, which makes τ useless for the purpose of characterizing Lorentzian manifolds by means of total scalar curvatures. However, for higher dimension we have

Corollary 5.28. *Let (M^{n+1}, g) be a Lorentzian manifold with $n \geq 4$ such that $\int_{G^\xi_m(r)} \hat{\tau}$ depends only on the radius. Then, M has constant sectional curvature.*

Proof. It follows from Theorem 5.26 taking into account that $A_\tau(n-1) = 1$, $C^1_\tau(n-1) = -(n+2)(n+3)/120$ and $C^2_\tau(n-1) = (n^2+5n+21)/45$, [42], [44]. See Example 4.3 and Section 4.3.1 for details. □

The space of simple Weyl invariants of degree four is generated by τ^2, $\|R\|^2$ and $\|\rho\|^2$. Using Example 4.3 and Section 4.3.1 we get the following result. We delete the details.

Corollary 5.29. *Let (M^{n+1}, g) be a Lorentzian manifold with $n \neq 5$. The following statements are equivalent:*

(i) $\int_{G^\xi_m(r)} \|\hat{R}\|^2$ *depends only on the radius.*

(ii) $\int_{G^\xi_m(r)} \|\hat{\rho}\|^2$ *depends only on the radius.*

(iii) $\int_{G^\xi_m(r)} \hat{\tau}^2$ *depends only on the radius.*

(iv) *M has constant sectional curvature.*

A similar characterization is obtained for simple Weyl invariants of degree six. A basis of this vector space is the first column of (4.2).

Corollary 5.30. *Let (M^{n+1}, g) be a Lorentzian manifold with $n \neq 7$. The following statements are equivalent:*

(i) $\int_{G^\xi_m(r)} \hat{\tau}^2$ *depends only on the radius.*

(ii) $\int_{G^\xi_m(r)} \hat{\tau}\|\hat{\rho}\|^2$ *depends only on the radius.*

(iii) $\int_{G^\xi_m(r)} \hat{\tau}\|\hat{R}\|^2$ *depends only on the radius.*

(iv) $\int_{G^\xi_m(r)} \check{\hat{\rho}}$ *depends only on the radius.*

(v) $\int_{G_m^\xi(r)} \langle \hat{\rho} \otimes \hat{\rho}, \bar{\hat{R}} \rangle$ *depends only on the radius.*

(vi) $\int_{G_m^\xi(r)} \langle \hat{\rho}, \dot{\hat{R}} \rangle$ *depends only on the radius.*

(vii) $\int_{G_m^\xi(r)} \check{\hat{R}}$ *depends only on the radius.*

(viii) $\int_{G_m^\xi(r)} \check{\bar{\hat{R}}}$ *depends only on the radius.*

(ix) M *has constant sectional curvature.*

Open problems

We are interested in the following problems:

- The volume conjecture of A. Gray and L. Vanhecke remains open in its full generality. In view of the examples given in [83] where the authors build manifolds such that the first terms of the power series expansion of the volume of their geodesic spheres vanish, it seems that this power series expansion approach is not sufficient to attack the problem at this stage. Hence, a new method or a more powerful description of these power series expansions is needed. On the other hand, similar questions can be stated for total scalar curvatures and the same comments apply. Nonetheless, the fact that certain curvature invariants can be used to characterize two–point homogenous spaces whereas others with the same degree cannot, poses the following question: what is the significance of those total curvatures which can be used to detect two–point homogeneous spaces and why do they provide such a characterization?

- A similar problem can be stated for ball–homogeneity. It is not known whether ball–homogeneity implies homogeneity or whether the notions of W–homogeneity (that is, the fact that the total curvatures associated with W do not depend on the base point) are equivalent for different scalar curvature invariants W. Similar questions can be stated for disk–homogeneity.

- In Chapter 5 we carried out the characterization of Lorentzian manifolds of constant sectional curvature. It is an interesting question to determine whether other manifolds such as Robertson–Walker or Schwarzschild space–times can be detected using geometric features associated with geodesic celestial spheres.

Part III

Real hypersurfaces in the complex hyperbolic space

The aim of submanifold geometry is to understand geometric invariants of submanifolds and to classify submanifolds according to given geometric data. In Riemannian geometry, the structure of a submanifold is encoded in the second fundamental form and its geometry is controlled by the equations of Gauss, Codazzi and Ricci. The situation simplifies for hypersurfaces, as the Ricci equation is trivial and the second fundamental form can be written in terms of a self–adjoint tensor field, the shape operator. The eigenvalues of the shape operator, the so–called principal curvatures, are the simplest geometric invariants of a hypersurface. Two basic problems in submanifold geometry are to understand the geometry of hypersurfaces for which the principal curvatures are constant, and to classify them. This problem has a long history and over the years many surprising features have been discovered.

É. Cartan [28] showed that in spaces of constant curvature a hypersurface has constant principal curvatures if and only if it is isoparametric. There is a remarkable interplay between the geometry and topology of isoparametric hypersurfaces in spheres S^n. Using methods from algebraic topology, H. F. Münzner [101] proved that the number g of distinct principal curvatures of an isoparametric hypersurface in S^n is 1, 2, 3, 4 or 6. In a series of papers, [28], [29], [30], [31], É. Cartan investigated isoparametric hypersurfaces in spheres and classified those for which g is at most three. It follows from his work that all isoparametric hypersurfaces in spheres with $g \leq 3$ are open parts of homogeneous hypersurfaces. It is obvious that homogeneous hypersurfaces have constant principal curvatures. The homogeneous hypersurfaces in spheres have been classified by W.–Y. Hsiang and H. B. Lawson [89]. It follows from this classification that homogeneous hypersurfaces with $g = 6$ exist only in spheres of dimension 7 and 13. U. Abresch [1] then proved that isoparametric hypersurfaces with $g = 6$ exist only in S^7 and S^{13}. This naturally leads to the conjecture that any isoparametric hypersurface in a sphere with $g = 6$ is an open part of a homogeneous hypersurface. This was answered affirmatively by J. Dorfmeister and E. Neher [51] for $n = 7$, but for $n = 13$ the problem is still open. Surprisingly, for $g = 4$ there are inhomogeneous isoparametric hypersurfaces. The first such examples were constructed by H. Ozeki and M. Takeuchi [110]. D. Ferus, H. Karcher and H.–F. Münzner

[58] then constructed series of inhomogeneous isoparametric hypersurfaces in spheres using representations of real Clifford algebras. A remarkable result by S. Stolz [121] says that the principal curvatures and their multiplicities of any isoparametric hypersurface with $g = 4$ in a sphere coincide with the ones of either the homogeneous hypersurfaces or the hypersurfaces constructed by D. Ferus, H. Karcher and H.–F. Münzner. T. Cecil, Q.–S. Chi and G. Jensen [32] recently proved that with 10 possible exceptions all isoparametric hypersurfaces in spheres with $g = 4$ are among the known homogeneous or inhomogeneous examples.

Whereas the classification problem of isoparametric hypersurfaces in spheres is rather involved, it is much simpler in its non–compact dual, the real hyperbolic space $\mathbb{R}H^n$. In fact, using the Gauss and Codazzi equations, É Cartan [28] showed that the number g of distinct principal curvatures of an isoparametric hypersurface in $\mathbb{R}H^n$ is either 1 or 2. This easily leads to a complete classification: geodesic hyperspheres, horospheres, totally geodesic hyperplanes and its equidistant hypersurfaces, tubes around totally geodesic subspaces of dimension greater or equal than one. As a consequence, all isoparametric hypersurfaces in real hyperbolic spaces are open parts of homogeneous hypersurfaces.

The isoparametric hypersurfaces in Euclidean spaces were classified by T. Levi–Civita [97] for dimension 3 and by B. Segre [119] for arbitrary dimensions. Also here all isoparametric hypersurfaces are open parts of homogeneous hypersurfaces.

In complex space forms the notions of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures are not the same [127]. In fact, Q.–M. Wang [130] gave an example of an isoparametric hypersurface in complex projective space $\mathbb{C}P^n$ with non–constant principal curvatures. The current state of the classification problem of hypersurfaces with constant principal curvatures in complex space forms is as follows. We continue to denote by g the number of distinct principal curvatures. To emphasize that the real codimension of the hypersurface is one (and not two as it is for a complex hypersurface) we use the notion of a real hypersurface. Y. Tashiro and S. I. Tachibana [126] proved that there are no totally umbilical real hypersurfaces in non–flat complex space forms. Thus the case $g = 1$ cannot occur. If ξ is a (local) unit normal field of a real hypersurface M in a complex space form $\bar{M}$, and J denotes the complex structure of $\bar{M}$, then $J\xi$ is tangent to M everywhere. The vector field $J\xi$ is called the Hopf vector field on M. The hypersurface M is said to be a Hopf hypersurface if $J\xi$ is a principal curvature vector of M everywhere. We assume $n \geq 2$.

Using the classification of homogeneous hypersurfaces in spheres and the Hopf map $S^{2n+1} \to \mathbb{C}P^n$, R. Takagi [123] derived the classification of homogeneous real hypersurfaces in complex projective spaces. All of them are Hopf hypersurfaces, and the number g of distinct principal curvatures is either 2, 3 or 5. R. Takagi then proved in [124] and [125] that every real hypersurface with two or three distinct constant principal curvatures in $\mathbb{C}P^n$ is an open part of a homogeneous hypersurface. The case $g = 3$ and $n = 2$ was omitted by R. Takagi and settled later by Q.-M. Wang [131]. M. Kimura [91] showed that every Hopf hypersurface in $\mathbb{C}P^n$ with constant principal curvatures is an open part of a homogeneous hypersurface in $\mathbb{C}P^n$. It is still unknown whether for any real hypersurface with constant principal curvatures in $\mathbb{C}P^n$ the number g is necessarily 2, 3 or 5. Also, there is no known example of a real hypersurface with constant principal curvatures in $\mathbb{C}P^n$ which is not an open part of a homogeneous real hypersurface.

S. Montiel [99] proved that every real hypersurface with two distinct constant principal curvatures in complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$, is an open part of a geodesic sphere, of a horosphere, of a tube around a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$, or of a tube with radius $\log(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. All these real hypersurfaces are homogeneous Hopf hypersurfaces. For $n = 2$ this problem is still open. J. Berndt derived in [10] the classification of all Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$. Any such hypersurface is an open part of a horosphere, of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some $k \in \{0, \ldots, n-1\}$, or of a tube around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. All these tubes and horospheres are homogeneous hypersurfaces. This naturally leads to the question whether all homogeneous real hypersurfaces in $\mathbb{C}H^n$ are necessarily Hopf hypersurfaces. The answer to this question is negative. In [11] J. Berndt constructed homogeneous hypersurfaces in $\mathbb{C}H^n$ which are not Hopf hypersurfaces.

J. Berndt and M. Brück constructed in [12] new examples of homogeneous real hypersurfaces in $\mathbb{C}H^n$. J. Berndt and Tamaru [16] showed recently that these new examples, together with the above mentioned homogeneous real hypersurfaces, provide the complete classification of homogeneous real hypersurfaces in $\mathbb{C}H^n$. The number of distinct principal curvatures of all these homogeneous real hypersurfaces is either 2, 3, 4 or 5. No examples are known of real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ which are not an open part of a homogeneous real hypersurface. It is also not known whether for any real hypersurface with constant principal curvatures in $\mathbb{C}H^n$ the number g of distinct principal curvatures must necessarily be 2, 3, 4 or 5.

In Chapter 6 we study cohomogeneity one actions on the complex hyperbolic space. Based on the classification given by J. Berndt and H. Tamaru, we focus on the geometry of the orbits of the cohomogeneity one actions described in [16]. This is accomplished in Section 6.3. In Chapter 7 we carry out the classification of real hypersurfaces in $\mathbb{C}H^n$ with three distinct constant principal curvatures. In particular, our result implies that real hypersurfaces in complex hyperbolic spaces with at most three constant principal curvatures are homogeneous submanifolds.

Chapter 6

Cohomogeneity one actions on the complex hyperbolic space

In this chapter we study the geometry of the orbits of a cohomogeneity one action on $\mathbb{C}H^n$ [15]. In Section 6.1 we give the basic definitions and concepts needed to describe cohomogeneity one actions. We explain some facts of cohomogeneity one actions on Hadamard manifolds and give an overview of the situation in $\mathbb{R}^n$ and $\mathbb{R}H^n$. Then, Section 6.2 is devoted to presenting a suitable description of $\mathbb{C}H^n$. The conventions and results explained throughout this section are used in the rest of the chapter, sometimes without explicit mention to them. Finally, Section 6.3 carries out the study of cohomogeneity one actions on the complex hyperbolic space with special attention to the description of the singular orbits of cohomogeneity one actions with one non–totally geodesic singular orbit. In particular we emphasize Theorems 6.8 and 6.16 as they are used in the following chapter.

6.1 Preliminaries

Let M be a Riemannian manifold and G a Lie group. A *G–action* on M or an action of G on M is a map

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (g,p) & \mapsto & gp \end{array}$$

such that $ep = p$ for all $p \in M$, where e is the identity of G, and $g(hp) = (gh)p$ for all $g, h \in G$ and $p \in M$. If $p \in M$, then $G \cdot p = \{gp : g \in G\}$ is the *orbit* of G through p and $G_p = \{g \in G : gp = p\}$ is the *isotropy group* of G at p. If $M = G \cdot p$ for some $p \in M$, then the action of G is said to be *transitive* and M is called a *homogeneous G–space*. Homogeneous spaces are of great interest in differential geometry. See [88] for a comprehensive introduction to the subject.

An *isometric action* of G on M is a G–action such that for any fixed $g \in G$, the map $p \mapsto gp$ is an isometry of M. From now on, we assume that G is a connected closed subgroup of the isometry group of M acting on M in the usual way.

We denote by M/G the set of orbits of the action of G on M and equip M/G with the quotient topology relative to the canonical projection $p \in M \to G \cdot p \in M/G$. Since G is a closed subgroup of the isometry group of M, the quotient space M/G is a Hausdorff space and each orbit $G \cdot p$ is a closed embedded submanifold [90]. Moreover, G_p is compact, $G \cdot p$ is a Riemannian homogeneous space $G \cdot p = G/G_p$ and G acts transitively on $G \cdot p$ by isometries.

D. Montgomery and C. T. Yang introduced in [98] the concept of a slice. This notion provides the technical machinery which allows us to define a partial ordering on the set of orbit types. We say that two orbits $G \cdot p$ and $G \cdot q$ have the same orbit type if G_p and G_q are conjugate in G. This defines an equivalence relation among the orbits of G. We denote by $[G \cdot p]$ the corresponding equivalence class of $G \cdot p$ and we call $[G \cdot p]$ the *orbit type* of $G \cdot p$. We introduce a partial ordering on the moduli space of orbit types. We put $[G \cdot p] \leq [G \cdot q]$ if and only if G_q is conjugate in G to some subgroup of G_p. There exists a largest orbit type in the moduli space of orbit types. Each representative of this largest orbit type is called a *principal orbit*. The union of all principal orbits forms a dense and open subset of M. Each principal orbit is an orbit of maximal dimension. A non–principal orbit with the same dimension as a principal orbit is called an *exceptional orbit*. An orbit whose dimension is less that the dimension of a principal orbit is called a *singular orbit*.

A *cohomogeneity one action* of G on a manifold M is an isometric action of G on M such that the codimension of each principal orbit is one. We say that two cohomogeneity one actions are *orbit equivalent* if there is an isometry of M that maps the orbits of one action onto the orbits of the other action.

An embedded submanifold of a Riemannian manifold M is said to be *extrinsically homogeneous*, if there exists an isometry of M that acts transitively on the submanifold and leaves it invariant. Cohomogeneity one actions are intimately related to extrinsically homogeneous hypersurfaces. Indeed the classification problem of cohomogeneity one actions up to orbit equivalence is equivalent to the classification of extrinsically homogeneous hypersurfaces up to isometry congruence.

P. S. Mostert [100] and L. Bérard Bergery [8] proved that the orbit space M/G of a cohomogeneity one action is homeomorphic to $\mathbb{R}$, $\mathbb{S}^1$, $[0,1]$ or $[0,\infty)$. This result implies that a cohomogeneity one action has at most two singular or exceptional orbits corresponding to the boundary points of M/G. If there exists one singular orbit, each principal orbit is geometrically a tube around the singular orbit. If there are no singular or exceptional orbits, in which case M/G is homeomorphic either to $\mathbb{R}$ or $\mathbb{S}^1$, the orbits of the action of G on M form a Riemannian foliation on M. Moreover, since principal orbits are always homeomorphic to each other, the projection $M \to M/G$ is a fiber bundle.

Assume G is a connected closed subgroup of the isometry group of M acting on M with cohomogeneity one. Let F be a singular or exceptional orbit of the action. Then, the isotropy group G_p at $p \in F$ acts transitively on the unit sphere of the normal space of F at p. This implies that any singular or exceptional orbit of a cohomogeneity one action is minimal [12]. Moreover, if $\dim(G \cdot p) < (\dim M - 1)/2$ then $G \cdot p$ is totally geodesic in M [112].

From now on we assume that M is a Hadamard manifold, that is, a connected, simply connected, complete Riemannian manifold of non–positive curvature. As M is simply connected, M/G cannot be homeomorphic to $\mathbb{S}^1$. This follows from the exact homotopy sequence of a fiber bundle with connected fibers and base space $\mathbb{S}^1$

$$\cdots \to \pi_1(M) \to \pi_1(M/G) \to \pi_0(F) \to \cdots .$$

where F is the fiber. A cohomogeneity one action on a Hadamard manifold cannot have exceptional orbits and it can have at most one singular orbit. Therefore, M/G is homeomorphic to $\mathbb{R}$ or to $[0, \infty)$.

The above assertions can be improved in the following way (see [12] and [111]).

Theorem 6.1. *Let G be a connected closed subgroup of the isometry group of an n–dimensional Hadamard manifold M acting on M with cohomogeneity one. Then one of the following two possibilities holds:*

(a) All orbits are principal and the isotropy group at any point is a maximal compact subgroup of G. Any orbit is diffeomorphic to $\mathbb{R}^{n-1}$ and there exists a solvable connected closed subgroup of G acting simply transitively on each orbit.

(b) There exists exactly one singular orbit F and the isotropy group at any point of F is a maximal compact subgroup of G. The singular orbit is diffeomorphic to $\mathbb{R}^k$ for some $k \in \{0, \ldots, n-2\}$ and there exists a solvable connected closed subgroup of G acting simply transitively on F. Any principal orbit is a tube around F and thus diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$.

Among all Hadamard manifolds, of special interest are the Euclidean space and all rank one symmetric spaces of non–compact type. The cohomogeneity one actions on the Euclidean space were classified by T. Levi–Civita [97] and B. Segre [119].

Theorem 6.2. *Let G be a Lie subgroup of the isometry group of $\mathbb{R}^n$, $\mathbb{R}^n \times_\tau O(n)$, acting on $\mathbb{R}^n$ with cohomogeneity one. Then the action of G is orbit equivalent to one of the following actions:*

(i) The action of $SO(n) \subset \mathbb{R}^n \times_\tau O(n)$. The singular orbit is a point and the principal orbits are spheres.

(ii) The action of $\mathbb{R}^k \times_\tau SO(n-k) \subset \mathbb{R}^n \times_\tau O(n)$ for some $k \in \{1, \ldots, n-2\}$. There is one singular orbit which is a totally geodesic $\mathbb{R}^k \subset \mathbb{R}^n$ and the principal orbits are tubes around it.

(iii) The action of $\mathbb{R}^{n-1} \subset \mathbb{R}^n \times_\tau O(n)$. All orbits are principal and totally geodesic hyperplanes.

The classification of cohomogeneity one actions on the real hyperbolic space follows from the work by É. Cartan [28], where he classified all the hypersurfaces with constant principal curvatures in the real hyperbolic space $\mathbb{R}H^n$. Every principal orbit of a cohomogeneity one action has constant principal curvatures. Hence, Cartan's result applies and we get

Theorem 6.3. *Every cohomogeneity one action on the the real hyperbolic space $\mathbb{R}H^n = SO^0(1,n)/SO(n)$ is orbit equivalent to one of the following cohomogeneity one actions:*

(i) The action of $SO(n) \subset SO^0(1,n)$. The singular orbit is a point and the principal orbits are geodesic spheres.

(ii) The action of $SO^0(1,k) \times SO(n-k) \subset SO^0(1,n)$ for some $k \in \{1,\dots,n-2\}$. The singular orbit is a totally geodesic $\mathbb{R}H^k \subset \mathbb{R}H^n$ and the principal orbits are tubes around it.

(iii) The action of $SO^0(1,n-1) \subset SO^0(1,n)$. All the orbits are principal, one orbit is a totally geodesic $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$ and the others are equidistant hypersurfaces to it.

(iv) The action of the nilpotent subgroup in an Iwasawa decomposition of $SO^0(1,n)$. All the orbits are principal and the resulting foliation is the horosphere foliation on $\mathbb{R}H^n$.

As we have just seen, every singular orbit of a cohomogeneity one action on $\mathbb{R}^n$ or $\mathbb{R}H^n$ is totally geodesic. This is no longer true in the other rank one symmetric spaces of non–compact type. Examples of cohomogeneity one actions on $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ with one non–totally geodesic singular orbit were given in [12]. Moreover, the moduli space of orbit equivalent cohomogeneity one actions on $\mathbb{R}^n$ and $\mathbb{R}H^n$ is finite. This does not hold for the other hyperbolic spaces either.

J. Berndt and H. Tamaru derived in [16] the classification of cohomogeneity one actions on the complex hyperbolic space. We devote the rest of this chapter to the study of the geometry of the orbits of that list.

6.2 The complex hyperbolic space as a solvable Lie group

Let $\mathbb{C}H^n$, $n \geq 2$, denote the n–dimensional complex hyperbolic space equipped with the Fubini–Study metric of constant holomorphic sectional curvature -1 which we denote by $g = \langle\cdot,\cdot\rangle$. Let J be its complex structure. Thus the curvature tensor of the complex hyperbolic space can be written as

$$\bar{R}_{XY}Z = -\frac{1}{4}\Big(\langle X,Z\rangle Y - \langle Y,Z\rangle X + \langle JX,Z\rangle JY - \langle JY,Z\rangle JX + 2\langle JX,Y\rangle JZ\Big)$$

for any $X,Y,Z \in \Gamma(T\mathbb{C}H^n)$. Hence the Jacobi equation is written as

$$\zeta''(t) - \frac{1}{4}\Big(\zeta(t) + 3\langle\zeta(t), Jc'_\xi(t)\rangle Jc'_\xi(t)\Big) = 0,$$

along a unit speed geodesic c_ξ determined by the initial condition $c'_\xi(0) = \xi \in T\mathbb{C}H^n$.

We denote by $\mathbb{C}H^n(\infty)$ the ideal boundary of $\mathbb{C}H^n$. Each element x of $\mathbb{C}H^n(\infty)$ is an equivalence class of asymptotic geodesics in $\mathbb{C}H^n$. Two geodesic c_1 and c_2 are asymptotic if $\lim_{t\to\infty} d(c_1(t), c_2(t)) \leq C$ for some constant $C > 0$, where d is the Riemannian distance function of $\mathbb{C}H^n$. We equip $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ with the cone topology. Then, $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ becomes homeomorphic to a closed ball in the Euclidean space $\mathbb{R}^{2n}$. For any $p \in \mathbb{C}H^n$ and any $x \in \mathbb{C}H^n(\infty)$ there exists a unique unit speed geodesic c through p such that $\lim_{t\to\infty} c(t) = x$. Thus, the choice of a point at infinity $x \in \mathbb{C}H^n(\infty)$ is equivalent to the choice of a unit geodesic vector field on $\mathbb{C}H^n$: the one whose integral curves are geodesics that converge to x. See [53] for more details.

Let KAN be the Iwasawa decomposition of the identity component of the isometry group of $\mathbb{C}H^n$, which we denote by $I^0(\mathbb{C}H^n)$, with respect to some point $o \in \mathbb{C}H^n$ and some point x in the ideal boundary $\mathbb{C}H^n(\infty)$ of $\mathbb{C}H^n$. The Lie group K coincides with the isotropy group of $I^0(\mathbb{C}H^n)$ at o and the orbit through o of the one–dimensional Lie group A is a geodesic in $\mathbb{C}H^n$ belonging to equivalence class determined by the point at infinity x. It is known that AN is a connected, simply connected, solvable Lie group that acts simply transitively on $\mathbb{C}H^n$. Thus, we may identify $\mathbb{C}H^n$ with the Lie group AN equipped with the left–invariant Riemannian metric $\langle\cdot,\cdot\rangle$. We now describe in more detail the Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ of AN. We follow [17].

Let us consider $\mathfrak{z} = \mathbb{R}$ endowed with the quadratic form $q(x) = -x^2$, and denote by J the standard representation of the Clifford algebra $Cl(\mathfrak{z}, q) \cong \mathbb{C}$ on the vector space $\mathfrak{v} = \mathbb{C}^{n-1}$, $J : Z \in Cl(\mathfrak{z}, q) \to J_Z \in End(\mathfrak{v})$. We define an inner product $\langle\cdot,\cdot\rangle$ on the vector space direct sum $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ by requiring that the induced quadratic form on $\mathfrak{z}$ is just $-q$, the vector spaces $\mathfrak{z}$ and $\mathfrak{v}$ are orthogonal and J_1 is an orthogonal transformation with respect to the induced inner product on $\mathfrak{v}$. Such inner product exists and is unique. We define a skew–symmetric bilinear map $[\cdot,\cdot] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ by the equation

$$\langle [X + U, Y + V], Z + W\rangle = \langle J_Z U, V\rangle,$$

where $X, Y, Z \in \mathfrak{z}$ and $U, V, W \in \mathfrak{v}$. Then, $(\mathfrak{n}, [\,,\,])$ becomes a two–step nilpotent Lie algebra with center $\mathfrak{z}$, called the *Heisenberg algebra* (of dimension $2n - 1$). The connected, simply connected, nilpotent Lie group N, with Lie algebra $\mathfrak{n}$ is called the *Heisenberg group* (of dimension $2n - 1$). It is isomorphic to the nilpotent Lie group in the above Iwasawa decomposition of $I^0(\mathbb{C}H^n)$. We equip N with the left–invariant Riemannian metric determined by $\langle\cdot,\cdot\rangle$.

Let us denote by $\mathrm{Exp}_{\mathfrak{n}}$ the Lie exponential map of N. Since N is connected, simply connected and nilpotent, the Lie exponential map $\mathrm{Exp}_{\mathfrak{n}}$ is a diffeomorphism [88]. This implies that N is diffeomorphic to $\mathbb{R}^{2n-1}$. The Campbell–Hausdorff formula simplifies in the case of a two–step nilpotent Lie group and gives in our case

$$\mathrm{Exp}_{\mathfrak{n}}(X + U) \cdot \mathrm{Exp}_{\mathfrak{n}}(Y + V) = \mathrm{Exp}_{\mathfrak{n}}\Big(X + Y + U + V + \frac{1}{2}[U, V]\Big),$$

for any $X + U$, $Y + V \in \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$.

The Lie algebra $\mathfrak{a}$ is one–dimensional. Choose $A \in \mathfrak{a}$. We extend the previous inner product $\langle\cdot,\cdot\rangle$ to the vector space direct sum $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ by requiring the following three conditions. The vector A is a unit vector of $\mathfrak{a}$. The Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ are orthogonal.

The restriction of this inner product to $\mathfrak{n}$ is just the previously defined one. Again, such an inner product exists and is unique. We extend the Lie algebra structure of $\mathfrak{n}$ to the vector space direct sum $\mathfrak{a} \oplus \mathfrak{n}$ by defining

$$[A, X] = X, \qquad [A, U] = \frac{1}{2}U,$$

where $X \in \mathfrak{z}$ and $U \in \mathfrak{v}$. Thus $\mathfrak{a} \oplus \mathfrak{n}$ becomes a solvable Lie algebra. The connected, simply connected solvable Lie group AN equipped with the left–invariant Riemannian metric determined by $\langle \cdot, \cdot \rangle$ is isometric to $\mathbb{C}H^n$ and is isomorphic to the solvable Lie group in the above Iwasawa decomposition of $I^0(\mathbb{C}H^n)$.

We use the following notation in what follows. Since, $\mathfrak{z}$ is one dimensional, we may choose a unit vector $Z \in \mathfrak{z}$ such that J_Z is the complex structure J of $\mathbb{C}H^n$ acting on $\mathfrak{v}$. We may assume as well $Z = JA$. Then, the Lie algebra structure of $\mathfrak{n}$ is determined by

$$[U, V] = \langle JU, V \rangle Z \qquad \text{and} \qquad [Z, U] = 0,$$

for any $U, V \in \mathfrak{v}$.

The definition of the Lie algebra structure on $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ implies that $\mathfrak{s}$ is a semi–direct sum of $\mathfrak{a}$ and $\mathfrak{n}$ with respect to the algebra homomorphism $f : \mathfrak{a} \to \mathrm{der}(\mathfrak{n})$ given by $f(A)(X+U) = X + \frac{1}{2}U$, where $\mathrm{der}(\mathfrak{n})$ is the Lie algebra of the derivations on $\mathfrak{n}$ and $X + U \in \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Since the Lie group $A = \mathrm{Exp}_{\mathfrak{a}}(\mathbb{R}A)$ is one–dimensional, we may identify $\mathbb{R} = A$ using the isomorphism $t \mapsto \mathrm{Exp}_{\mathfrak{a}}(tA)$. Thus AN is the semi–direct product of $A = \mathbb{R}$ and the Heisenberg group N, $\mathbb{R} \times_F N$, where $F(t)(\mathrm{Exp}_{\mathfrak{n}}(X+U)) = \mathrm{Exp}_{\mathfrak{n}}(e^t X + e^{t/2}U)$. Then the group structure of $AN = \mathbb{R} \times_F N$ is determined by

$$\Big(a, \mathrm{Exp}_{\mathfrak{n}}(xZ + U)\Big) \cdot \Big(b, \mathrm{Exp}_{\mathfrak{n}}(yZ + V)\Big) = \Big(a + b, \mathrm{Exp}_{\mathfrak{n}}\Big(xZ + e^a Y + U + e^{a/2}V + \frac{1}{2}[U, V]\Big)\Big).$$

In particular, this implies that AN is diffeomorphic to $\mathbb{R}^{2n}$ as we already knew. To describe the Lie exponential map $\mathrm{Exp}_{\mathfrak{s}}$ of $\mathfrak{s}$ we first define the function $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(s) = \begin{cases} \dfrac{e^s - 1}{s} & , \text{ if } s \neq 0, \\ 1 & , \text{ if } s = 0. \end{cases}$$

The function ρ is analytic in $\mathbb{R}$. Then we have

$$\mathrm{Exp}_{\mathfrak{s}}\big(aA + xZ + U\big) = \Big(a, \mathrm{Exp}_{\mathfrak{n}}\Big(\rho(a)x\, Z + \rho\Big(\frac{a}{2}\Big)\, U\Big)\Big)$$

and the Lie exponential map $\mathrm{Exp}_{\mathfrak{s}}$ is a diffeomeorphism.

The standard method for calculating the Levi–Civita connection $\bar{\nabla}$ of a Lie group equipped with a left–invariant metric yields the following expression in our particular case:

$$\bar{\nabla}_{aA + xZ + U}\big(bA + yZ + V\big) = \Big(\frac{1}{2}\langle U, V \rangle + x\,y\Big)A + \Big(\frac{1}{2}\,\langle JU, V \rangle - b\,x\Big)Z - \frac{b}{2}\,U - \frac{y}{2}\,JU - \frac{x}{2}\,JV,$$

where $a, b, x, y \in \mathbb{R}$ are real numbers and $U, V \in \mathfrak{v}$. All vector fields are considered to be left–invariant.

Let $aA + xZ + U$ be a unit vector in $\mathfrak{s}$ and $c : \mathbb{R} \to \mathbb{C}H^n = AN$ the geodesic in $\mathbb{C}H^n$ such that $c(0) = o$ and $c'(0) = aA + xZ + U$. Then, c lies in a suitable totally geodesic $\mathbb{C}H^2 \subset \mathbb{C}H^n$ and we have the explicit expression [17]

$$c = \left(\log\left(\frac{1-\theta^2}{\chi}\right), \mathrm{Exp}_{\mathfrak{n}}\left(\frac{2\theta(1-a\,\theta)}{\chi}U + \frac{2x\,\theta^2}{\chi}JU + \frac{2x\,\theta}{\chi}Z\right)\right),$$

where

$$\theta(t) = \tanh\frac{t}{2} \qquad \text{and} \qquad \chi(t) = (1 - a\,\theta(t))^2 + x^2\theta(t)^2.$$

Moreover, the tangent vector of the geodesic is given by

$$c' = \frac{\sqrt{h}}{\chi}\left\{(1-a\,\theta)^2 - x^2\theta^2\right\}U + \frac{2x\sqrt{h}}{\chi}\theta(1-a\,\theta)JU + x\,h\,Z + (\log h)'A,$$

with $h(t) = (1 - \theta(t)^2)/\chi(t)$.

We refer to [17], where a comprehensive study of the geometry of Damek–Ricci spaces is presented. Non–symmetric Damek–Ricci spaces are counterexamples to the Lichnerowicz conjecture on harmonic spaces [36]. The symmetric Damek–Ricci spaces are the rank one symmetric spaces of non–compact type. Thus, Damek–Ricci spaces provide a unified description of all hyperbolic spaces over the real division algebras.

6.3 Cohomogeneity one actions on $\mathbb{C}H^n$

The study of cohomogeneity one actions on the complex hyperbolic space relies on the following classification result given by J. Berndt and H. Tamaru [16].

Theorem 6.4. *Let G be a connected closed subgroup of the isometry group of the complex hyperbolic space acting on $\mathbb{C}H^n$, $n \geq 2$, with cohomogeneity one. Then the action of G is orbit equivalent to one of the following cohomogeneity one actions:*

(i) The action of $S(U(1,k) \times U(n-k)) \subset SU(1,n)$ for some $k \in \{0, \ldots, n-1\}$. The singular orbit is a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ and the principal orbits are tubes around it .

(ii) The action of $SO^0(1,n) \subset SU(1,n)$. The singular orbit is a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$ and the principal orbits are tubes around it.

(iii) The action of N. Each orbit is a horosphere in $\mathbb{C}H^n$. The orbits of N form a Riemannian foliation on $\mathbb{C}H^n$ called the horosphere foliation of $\mathbb{C}H^n$.

(iv) *The action of the connected, simply connected Lie subgroup H of AN whose Lie algebra is $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$, where $\mathfrak{w}$ is a linear hyperplane of $\mathfrak{v}$. All the orbits of this action are principal, the orbit $H \cdot o$ is minimal and any other orbit is an equidistant hypersurface to this minimal one.*

(v) *The action of the group $N_K^0(H)H$ where H is the connected, simply connected Lie subgroup H of AN whose Lie algebra is $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$, where $\mathfrak{w}^\perp$ is a real linear subspace of $\mathfrak{v}$ of codimension $k \in \{2, \ldots, n-1\}$. The singular orbit is minimal and non–totally geodesic. The principal orbits are tubes around the singular one.*

(vi) *The action of the group $N_K^0(H)H$, where H is the connected, simply connected Lie subgroup H of AN whose Lie algebra is $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$, where $\mathfrak{w}^\perp$ is a linear subspace of $\mathfrak{v}$ of constant Kähler angle $\varphi \in (0, \pi/2)$. The singular orbit is minimal and non–totally geodesic. The principal orbits are tubes around the singular one.*

Examples (i) and (ii) correspond to cohomogeneity one actions with one totally geodesic singular orbit. The families (iii) and (iv) are foliations in $\mathbb{C}H^n$ and as a consequence the corresponding cohomogeneity one actions do not have singular orbits. The families (v) and (vi) are cohomogeneity one actions with one non–totally geodesic singular orbit.

Let $(\bar{M}, J)$ be a Hermitian manifold and M a real hypersurface with (local) unit vector field ξ. Obviously, $J\xi$ is everywhere tangent to M. The vector field $J\xi$ is called the *Hopf vector field* of M. We say that M is a *Hopf hypersurface* if the integral curves of $J\xi$ are geodesics in M. If $\bar{M}$ is a Kähler manifold this is equivalent to the condition that $J\xi$ is a principal curvature vector of M at every point. Principal orbits of examples (i), (ii) and (iii) in the above theorem are Hopf hypersurfaces while principal orbits of (iv), (v) and (vi) are not.

The classification of Hopf hypersurfaces with constant principal curvatures is due to J. Berndt [10]. We use this result later so we state it here.

Theorem 6.5. *Let M be a connected Hopf real hypersurface of $\mathbb{C}H^n$, $n \geq 2$, with constant principal curvatures. Then M is holomorphically congruent to an open part of one of the following hypersurfaces:*

(i) *A tube around a totally geodesic $\mathbb{C}H^k$ for some $k \in \{0, \ldots, n-1\}$.*

(ii) *A tube around a totally geodesic $\mathbb{R}H^n$.*

(iii) *A horosphere in $\mathbb{C}H^n$.*

We emphasize that each hypersurface in Theorem 6.5 coincides with one of the principal orbits in cases (i), (ii) or (iii) of Theorem 6.4. Therefore, a connected complete Hopf real hypersurface is extrinsically homogeneous if and only if it has constant principal curvatures.

In what follows we discuss in detail the above cohomogeneity one actions on the complex hyperbolic space [15].

6.3.1 Cohomogeneity one actions with one totally geodesic singular orbit

Theorem 6.4 states the existence of two families of cohomogeneity one actions on $\mathbb{C}H^n$ with one totally geodesic singular orbit. They correspond to cases (i) and (ii) in that theorem.

The following result completely explains both the intrinsic and extrinsic geometry of totally geodesic submanifolds of $\mathbb{C}H^n$.

Theorem 6.6 (Rigidity of totally geodesic submanifolds in $\mathbb{C}H^n$). *Let M be a totally geodesic submanifold of $\mathbb{C}H^n$. Then M is holomorphically congruent to an open part of a real hyperbolic space $\mathbb{R}H^k$ for some $k \in \{1, \ldots, n\}$ or to a complex hyperbolic space $\mathbb{C}H^k$ for some $k \in \{1, \ldots, n-1\}$. Any two totally geodesic submanifolds of $\mathbb{C}H^n$ are locally holomorphically congruent to each other if and only if they are locally isometric.*

Theorem 6.4 implies that a totally geodesic $\mathbb{R}H^k$ with $k \in \{1, \ldots, n-1\}$ cannot be a singular orbit of a cohomogeneity one action. We briefly explain the reason [12]. For any totally geodesic $\mathbb{R}H^k \subset \mathbb{C}H^n$ there exists a totally geodesic $\mathbb{C}H^k$ such that $\mathbb{R}H^k \subset \mathbb{C}H^k$. Any isometry of $\mathbb{C}H^n$ leaving $\mathbb{R}H^k$ invariant leaves $\mathbb{C}H^k$ also invariant. The isotropy group of a cohomogeneity one action on $\mathbb{C}H^n$ acts transitively on the normal space of $\mathbb{R}H^k$ at any point. But normal vectors of $\mathbb{R}H^k$ which are tangent to $\mathbb{C}H^k$ remain tangent to $\mathbb{C}H^k$; this is only possible if $k = n$.

The other totally geodesic hyperbolic spaces of $\mathbb{C}H^n$ can be singular orbits of cohomogeneity one action as Theorem 6.4 shows. In what follows we study the geometry of the orbits of those actions. We briefly study the two families (i) and (ii) separately.

The action of $S(U(1,k) \times U(n-k))$

The group $G = S(U(1,k) \times U(n-k)) \subset SU(1,n)$ for some $k \in \{0, \ldots, n-1\}$ acts on $\mathbb{C}H^n$ with cohomogeneity one. This action has exactly one singular orbit which is a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$. Therefore, the second fundamental form is completely determined by $I\!I = 0$.

The procedure to construct such a totally geodesic $\mathbb{C}H^k$ is as follows. Let $o \in \mathbb{C}H^n$ and choose $V \subset T_o\mathbb{C}H^n$ a complex linear subspace of complex dimension k. Then, $\exp_o(V)$ is a totally geodesic $\mathbb{C}H^k$.

We turn our attention to the principal orbits of this action. If M is one principal orbit of the action of G then M is a tube of certain radius $r > 0$ around the singular orbit. Standard Jacobi vector field theory shows that M has three principal curvatures (we choose the outward unit normal vector field ξ so that the principal curvatures with respect to it are positive)

$$\alpha = \frac{1}{2}\tanh\frac{r}{2}, \qquad \beta = \frac{1}{2}\coth\frac{r}{2}, \qquad \gamma = \coth r,$$

with corresponding multiplicities

$$m_\alpha = 2(n-k-1), \qquad m_\beta = 2k, \qquad m_\gamma = 1.$$

Furthermore, T_α and T_β are complex distributions on M and the Hopf vector field is a principal curvature vector of γ at any point of M. Thus, M is a Hopf hypersurface. More specifically, let $p \in M$ and let c be the geodesic of $\mathbb{C}H^n$ defined by the initial condition $c'(0) = \xi_p$. Then $c(r)$ is a point in the totally geodesic $\mathbb{C}H^k$. The principal curvature vector subspace $T_\alpha(p)$ is the parallel translate of $T_{c(r)}\mathbb{C}H^k$ along the geodesic c and $T_\beta(p)$ is the parallel translate of $T^\perp_{c(r)}\mathbb{C}H^k \ominus \mathbb{R}Jc'(r)$ along the geodesic c.

In the above discussion if $k = 0$, that is, the singular orbit is a point, then the principal orbits are geodesic spheres and there are just two eigenvalues $\alpha = \frac{1}{2}\tanh\frac{r}{2}$ and $\gamma = \coth r$ with multiplicities $2(n-1)$ and 1 respectively.

Similarly, if $k = n-1$ then the singular orbit is a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$ and the tubes around it have only two constant principal curvatures $\beta = \frac{1}{2}\coth\frac{r}{2}$ and $\gamma = \coth r$ with corresponding multiplicities $2(n-1)$ and 1.

The action of $SO^0(1,n)$

The group $G = SO^0(1,n) \subset SU(1,n)$ acts on $\mathbb{C}H^n$ with cohomogeneity one. This action has one singular orbit which is a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. The second fundamental form is completely determined by $I\!I = 0$. Such a totally geodesic $\mathbb{R}H^n$ can be constructed as follows. Let $o \in \mathbb{C}H^n$ and choose $V \subset T_o\mathbb{C}H^n$ a real linear subspace of the tangent space of real dimension n. Then, $\exp_o(V)$ is a totally geodesic $\mathbb{R}H^n$.

We briefly discuss the geometry of the principal orbits of the action of G. Let M be one of these orbits. Then, M is a tube of certain radius $r > 0$ around the singular orbit. Using standard Jacobi vector field theory we get that M has three principal curvatures. We choose the outward unit normal vector field ξ so that the principal curvatures with respect to it are positive. The three principal curvatures are

$$\alpha = \frac{1}{2}\tanh\frac{r}{2}, \qquad \beta = \frac{1}{2}\coth\frac{r}{2}, \qquad \gamma = \tanh r,$$

with corresponding multiplicities

$$m_\alpha = n-1, \qquad m_\beta = n-1, \qquad m_\gamma = 1.$$

The distributions T_α and T_β are real and the Hopf vector field is a principal curvature vector of γ. Thus, M is a Hopf hypersurface. Let $p \in M$ and let c be the geodesic of $\mathbb{C}H^n$ defined by the initial condition $c'(0) = \xi_p$. Then $c(r)$ is a point in the totally geodesic $\mathbb{R}H^n$. The principal curvature vector subspace $T_\alpha(p)$ is the parallel translate of $T_{c(r)}\mathbb{R}H^n \ominus \mathbb{R}Jc'(r)$ along the geodesic c and the principal vector subspace $T_\beta(p)$ is the parallel translate of $T^\perp_{c(r)}\mathbb{R}H^n \ominus \mathbb{R}c'(r)$ along the geodesic c.

A special situation occurs when $r = \log(2+\sqrt{3})$. In this case $\beta = \gamma$ and there are just two principal curvatures α and γ with multiplicities $n-1$ and n. Both $T_\alpha(p)$ and $T_\gamma(p)$ keep being real and the Hopf vector field is a principal vector field.

6.3.2 Cohomogeneity one actions with no singular orbits

Cases (iii) and (iv) in Theorem 6.4 correspond to cohomogeneity one actions on the complex hyperbolic space with no singular orbits. These cohomogeneity one actions arise naturally from the Iwasawa decomposition of $SU(1,n)$. Different choices for the Iwasawa decomposition lead to congruent actions.

The horosphere foliation

Let KAN be an Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$ with respect to some point $o \in \mathbb{C}H^n$ and some point at infinity $x \in \mathbb{C}H^n(\infty)$.

The group N acts on $\mathbb{C}H^n$ with cohomogeneity one and all the orbits are principal. The resulting foliation is the well–known horosphere foliation. It contains the Heisenberg group N as a horosphere and any other orbit is a suitable left translate of it. This foliation is constructed in the following way. Let c be a unit speed geodesic with $c(0) = o$. We define the *Busemann function*, $\mathcal{B}_c : \mathbb{C}H^n \to \mathbb{R}$, with respect to c as

$$\mathcal{B}_c(p) = \lim_{t\to\infty}(d(p, c(t)) - t),$$

where d stands for the Riemannian distance function. The level sets of this function are called *horospheres*.

A horosphere has the following geometrical interpretation. Consider the geodesic sphere centered at $c(r)$ of radius r. This geodesic sphere contains o. In the complex hyperbolic space such geodesic spheres are defined for any $r > 0$. The limit set of these geodesic spheres when r tends to infinity is a horosphere. Different choices of o along c give all the different horospheres of the horosphere foliation determined by the point at infinity $x = \lim_{t\to\infty} c(t)$.

A horosphere has exactly two distinct principal curvatures

$$\alpha = \frac{1}{2} \qquad \text{and} \qquad \beta = 1,$$

with corresponding multiplicities

$$m_\alpha = 2(n-1) \qquad \text{and} \qquad m_\beta = 1.$$

As usual, we choose the unit normal vector ξ so that the principal curvatures are positive. The principal vector space of α is the orthogonal complement of the complex span of the unit normal vector ξ. Hence, T_α is a complex distribution. The Hopf vector field $J\xi$ is a principal curvature vector of the principal curvature β. Hence, every horosphere is a Hopf hypersurface with constant principal curvatures.

We have the following rigidity result. The proof is an easy consequence of Theorem 6.5. However, we sketch the proof as it was given in [10] because of its geometric interest.

Theorem 6.7 (Rigidity of horospheres in $\mathbb{C}H^n$). *Let M be a real hypersurface of the complex hyperbolic space with principal curvatures $1/2$ and 1. Then M is an open part of a horosphere.*

Sketch of the proof. Let ξ denote a local unit vector of M. Using the Gauss and Codazzi equations one can show that $J\xi$ is a principal vector associated with the eigenvalue 1. For any p, let c_p be the geodesic $c_p(t) = \exp_p(t\xi_p)$. For $r \geq 0$ let Φ_r be the map defined by $\Phi_r(p) = \exp_p(r\xi_p)$. Let $v \in T_pM$. Using Jacobi vector field theory one gets $\Phi_{r*}(v) = e^{-r/2}B_v(r)$ if $v \in T_{1/2}(p)$ and $\Phi_{r*}(v) = e^{-r}B_v(r)$ if $v \in \mathbb{R}J\xi_p$. This implies $\|\Phi_{r*}(v)\| \leq e^{-r/2}\|v\|$ for all $v \in TM$. Let d denote the Riemannian distance function of $\mathbb{C}H^n$ and d_M the Riemannian distance function of M. The above inequality shows that $d(c_p(r), c_q(r)) \leq e^{-r/2}d_M(p,q)$ for any $p, q \in M$ and $r \geq 0$. Using the triangle inequality we get

$$d(p, c_o(t)) - t = d(p, c_o(t)) - d(p, c_p(t)) \leq d(c_p(t), c_o(t)) \leq e^{-t/2}d_M(p, o).$$

Then the Busemann function verifies $\mathcal{B}_c(p) = 0$ for all $p \in M$, which proves that M is an open part of a horosphere. □

The solvable foliation

As usual, let $\mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{v}$ be the Lie algebra of the solvable part of the Iwasawa decomposition KAN with respect to some $o \in \mathbb{C}H^n$ and $x \in \mathbb{C}H^n(\infty)$. Let us take $\mathfrak{w}$ a linear hyperplane in $\mathfrak{v}$. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$ is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{v}$ of codimension one. If H is the connected, simply connected Lie subgroup whose Lie algebra is $\mathfrak{h}$, then H acts on $\mathbb{C}H^n$ with cohomogeneity one. The resulting cohomogeneity one action has no singular orbits and therefore induces a foliation on $\mathbb{C}H^n$. We call it the *solvable foliation* of $\mathbb{C}H^n$. Different choices of $\mathfrak{w}$ lead to congruent actions. We mainly follow [11].

The orbit $H \cdot o$ through o is the unique minimal orbit of this action. We study its geometry in more detail. The maximal complex subspace of $\mathfrak{c} \subset \mathfrak{h}$ is a Lie subalgebra and the connected, simply connected Lie subgroup whose Lie algebra is this maximal complex subspace $\mathfrak{c}$ acts on $H \cdot o$ by left translation. The resulting orbit through o is a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$. Indeed, $H \cdot o$ is ruled by totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$. The orthogonal complement $\mathfrak{h} \ominus \mathfrak{c}$ is one–dimensional and induces in $H \cdot o$ an integrable distribution $\mathfrak{D}$ by left translation of $\mathfrak{h} \ominus \mathfrak{c}$. Each integral curve of $\mathfrak{D}$ through $p \in H \cdot o$ is a horocycle in the totally geodesic $\mathbb{R}H^2$ determined by $\mathfrak{D}_p$ and $x \in \mathbb{C}H^n(\infty)$. By definition we denote by W^{2n-1} a manifold constructed in this way. As we stated before all W^{2n-1} are holomorphically congruent to each other. In Subsection 6.3.3 we generalize this construction and give more details about it. For the moment we content ourselves with the present description and study the geometry of the other orbits.

Any other orbit of the action of H is an equidistant hypersurface to this minimal one. Any two such orbits are congruent to each other if and only if their distance to $H \cdot o$ is the same. None of them is ruled by a totally geodesic $\mathbb{C}H^{n-1}$ in the above sense.

Let M denote an orbit of H at a distance $r \geq 0$ from $H \cdot o$. If $r = 0$ we consider the orbit $H \cdot o$ itself. The shape operator S of M has exactly three eigenvalues

$$\alpha = \frac{1}{2}\tanh\frac{r}{2}\,, \quad \beta = \frac{3}{4}\tanh\frac{r}{2} - \frac{1}{2}\sqrt{1 - \frac{3}{4}\tanh^2\frac{r}{2}}\,, \quad \gamma = \frac{3}{4}\tanh\frac{r}{2} + \frac{1}{2}\sqrt{1 - \frac{3}{4}\tanh^2\frac{r}{2}}\,.$$

with corresponding multiplicities

$$m_\alpha = 2n-3, \qquad m_\beta = 1, \qquad m_\gamma = 1.$$

The principal vector space of α is neither real nor complex. If ξ denotes the unit normal of M, the Hopf vector field $J\xi$ is not a principal vector field and hence M is not a Hopf hypersurface. Indeed, $J\xi$ has non–trivial orthogonal projections onto T_β and T_γ.

For the orbit $H \cdot o$ we have $r = 0$ and the principal curvatures become 0, $-1/2$ and $1/2$ with multiplicities $2n-3$, 1 and 1. We clearly see then that $H \cdot o$ is minimal. The following theorem shows that this eigenvalue structure is characteristic of this orbit [14]. It is a consequence of Theorem 6.16 which we prove later in Section 6.3.3. We also use some elementary results of the following chapter which we avoid repeating here to focus our attention on the main argument.

Theorem 6.8 (Rigidity of the submanifold W^{2n-1}). *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct principal curvatures* 0, $-1/2$ *and* $1/2$ *and multiplicities* $2n-3$, 1 *and* 1, *respectively. Then M is holomorphically congruent to an open part of the ruled real hypersurface W^{2n-1}.*

Proof. Let ξ be the corresponding unit normal vector of M. Let $p \in M$ and suppose that the orthogonal projection of $J\xi_p$ onto $T_0(p)$ is non–zero. Then $T_0(p)$ is a real subspace of $T_p\mathbb{C}H^n$ by Corollary 7.5. Since $\dim T_0(p) = 2n-3$, this is impossible for $n > 3$ and we must have $n = 3$. As $\xi_p \in T_0(p)^\perp$ it follows that $J\xi_p \in T_0(p)$. Since orthogonal projection onto subbundles is a continuous map, this must hold on an open neighborhood U of p in M. Therefore, U is a Hopf hypersurface in $\mathbb{C}H^3$ with three distinct constant principal curvatures 0, $-1/2$ and $1/2$. According to Theorem 6.5 such a hypersurface does not exist. We conclude that the orthogonal projection of the Hopf vector field $J\xi$ onto T_0 is zero everywhere.

Now define M^+ as the set of all points $p \in M$ at which the orthogonal projections of $J\xi_p$ onto $T_{-1/2}(p)$ and $T_{1/2}(p)$ are both non–zero. Clearly, M^+ is an open subset of M. Using again the classification of Theorem 6.5 we see that M^+ is non–empty.

Let X and Y be local unit vector fields on M with $X \in \Gamma(T_{-1/2})$ and $Y \in \Gamma(T_{1/2})$. Then we can write $J\xi = aX + bY$ with $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. We may assume that X and Y are chosen such that $a, b \geq 0$. As we have seen above, $T_0(p)$ cannot be a real subspace at any point $p \in M$. Thus there exists a non–zero vector field $U \in \Gamma(T_0)$ such that $JU \in \Gamma(T_0)$. Since $\bar{\nabla} J = 0$ we have $\bar{\nabla}_U J\xi = J\bar{\nabla}_U \xi = JSU = 0$, and thus Lemma 7.3 implies

$$0 = U\langle JU, J\xi\rangle = \langle \nabla_U JU, J\xi\rangle = a\langle \nabla_U JU, X\rangle + b\langle \nabla_U JU, Y\rangle = \frac{1}{2}(a^2 - b^2)\langle U, U\rangle\,.$$

This gives $a^2 = b^2$ and hence $a = b = 1/\sqrt{2}$. This shows that M^+ is a closed subset of M. As M^+ is open and non–empty, we see that $M^+ = M$. In particular, the length of the orthogonal projections of the Hopf vector field $J\xi$ onto $T_{-1/2}$ and $T_{1/2}$ is constant and equal to $1/\sqrt{2}$. We now define $Z = a(X - Y)$. Then the second fundamental form of M has the form of that in Theorem 6.16. Indeed,

$$\begin{aligned}
I\!I(Z, J\xi) &= -\langle \bar{\nabla}_Z J\xi, \xi\rangle\, \xi &&= \langle SZ, J\xi\rangle\, \xi &&= -\frac{a^2}{2}\langle X+Y, X+Y\rangle\, \xi &&= -\frac{1}{2}\,\xi\,,\\
I\!I(J\xi, J\xi) &= -\langle \bar{\nabla}_{J\xi} J\xi, \xi\rangle\, \xi &&= \langle SJ\xi, J\xi\rangle\, \xi &&= \frac{a^2}{2}\langle -X+Y, X+Y\rangle\, \xi &&= 0,\\
I\!I(Z, Z) &= -\langle \bar{\nabla}_Z Z, \xi\rangle\, \xi &&= \langle SZ, Z\rangle\, \xi &&= \frac{a^2}{2}\langle X+Y, X-Y\rangle\, \xi &&= 0,
\end{aligned}$$

and $I\!I(U, X) = -\langle \bar{\nabla}_U X, \xi\rangle = \langle X, SU\rangle = 0$ for any $U \in \Gamma(T_0)$ and $X \in \Gamma(TM)$. The result now follows from that theorem. □

6.3.3 Cohomogeneity one actions with one non–totally geodesic singular orbit

Let H be a closed subgroup of AN and consider the closed subgroup $N_K^0(H)H \subset KAN$, where $N_K^0(H)$ is the identity component of the normalizer $N_K(H) = \{k \in K : kHk^{-1} \subset H\}$ of H in K. Let $F = H \cdot o$ be the orbit of H through o. Then, $F = (N_K^0(H)H) \cdot o$ and the following result holds [12].

Theorem 6.9. *Let $\mathfrak{h}$ be the Lie algebra of H. Assume $\mathfrak{h}$ can be written in the form $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$, where $\mathfrak{w}^\perp$ is a linear vector subspace of $\mathfrak{v}$ of dimension ≥ 2 and constant Kähler angle φ. Then $N_K^0(H)H$ acts on $\mathbb{C}H^n$ with cohomogeneity one and F is a singular orbit of that action. Furthermore, if $\varphi \in (0, \pi/2]$ then F is not totally geodesic in $\mathbb{C}H^n$.*

Let $V \subset \mathbb{C}^n$ be a linear subspace. Let $v \in V$ be a non–zero vector. The Kähler angle of V with respect to v is the angle $\varphi(v) \in [0, \pi/2]$ between V and the real span of iv. Thus, $\varphi(v) \in [0, \pi/2]$ is determined by requiring that $(\cos\varphi(v))\|v\|$ is the length of the orthogonal projection of iv onto V. We say that V has *constant Kähler angle* φ if $\varphi(v) = \varphi$ for all non–zero vectors $v \in V$.

Linear subspaces with constant Kähler angle are of interest in what follows so we first derive some results that will be used later.

Let $V \subset \mathbb{C}^n$ a linear subspace with constant Kähler angle $\varphi \in [0, \pi/2]$. We denote by J the endomorphism of $\mathbb{C}^n$ consisting of multiplying by the imaginary unit, that is, $Jv = iv$ for all $v \in \mathbb{C}^n$. If $\varphi = 0$ then V is said to be a *complex subspace* of $\mathbb{C}^n$. This is equivalent to $JV \subset V$. If $\varphi = \pi/2$ then V is a *real subspace* of $\mathbb{C}^n$. In this case $JV \subset \mathbb{C}^n \ominus V$.

Let $\mathbb{C}V$ be the minimal complex vector subspace of $\mathbb{C}^n$ containing V and let $V^\perp = \mathbb{C}V \ominus V$. We denote by $\pi : \mathbb{C}V \to V$ and $\sigma : \mathbb{C}V \to V^\perp$ the orthogonal projections onto V and $V^\perp$, respectively. We define $P = \pi J$ and $F = \sigma J$. If $\varphi = 0$ we have $\mathbb{C}V = V$, $\pi = \mathrm{Id}_{\mathbb{C}V}$, $\sigma = 0$, $P = J$ and $F = 0$. If $\varphi = \pi/2$ we have the orthogonal direct sum decomposition $\mathbb{C}V = V \oplus JV$ and $P = J\sigma$, $F = J\pi$. In what follows we study the non–trivial case $\varphi \in (0, \pi/2)$.

Lemma 6.10. *Let $V \subset \mathbb{C}^n$ be a linear subspace with constant Kähler angle $\varphi \in (0, \pi/2)$. Then*

$$\begin{aligned} P^2 &= -(\cos^2\varphi)\pi - PF\sigma, & F^2 &= -FP\pi - (\cos^2\varphi)\sigma, \\ PF &= -(\sin^2\varphi)\pi + PF\sigma, & FP &= FP\pi - (\sin^2\varphi)\sigma. \end{aligned}$$

Proof. Let $x \in V$. Since V has constant Kähler angle φ, by definition, $\langle Px, Px\rangle = (\cos^2\varphi)\langle x, x\rangle$. Iterating the equality $\langle P^2x, P^2x\rangle = (\cos^2\varphi)\langle Px, Px\rangle = (\cos^4\varphi)\langle x, x\rangle$. On the other hand, $\langle P^2x, x\rangle = \langle JPx, x\rangle = -\langle Px, Jx\rangle = -\langle Px, Px\rangle = -(\cos^2\varphi)\langle x, x\rangle$. Since P^2x has the same length as the component of P^2x in the direction of x we obtain $P^2x = -(\cos^2\varphi)x$. Then, $P^2\pi = -(\cos^2\varphi)\pi$.

Now we have $-x = J^2x = P^2x + FPx + PFx + F^2x$. Taking the component in $V^\perp$ we get $F^2x = -FPx$, that is, $F^2\pi = -FP\pi$. Taking the component in V we get $-x = P^2x + PFx = -(\cos^2\varphi)x + PFx$. Thus, $PFx = -(\sin^2\varphi)x$, which implies $PF\pi = -(\sin^2\varphi)\pi$.

Using the above relations we have $JFPx = PFPx + F^2Px = -(\sin^2\varphi)Px + (\cos^2\varphi)Fx$. Also, $\langle Fx, FPx\rangle = -\langle x, JFPx\rangle = -\langle x, PFPx\rangle = (\sin^2\varphi)\langle x, Px\rangle = 0$. Altogether this means that for any non–zero vector $x \in V$ the vectors x, Px, Fx and FPx are orthogonal and span a complex vector subspace of $\mathbb{C}V$. Moreover, $x, Px \in V$ and $Fx, FPx \in V^\perp$.

A similar argument in $\mathbb{C}V \ominus (\mathbb{R}x \oplus \mathbb{R}Px \oplus \mathbb{R}Fx \oplus \mathbb{R}FPx)$ shows that there exist non–zero vectors $x_1, \ldots, x_k \in V$ such that $\{x_1, Px_1, \ldots, x_k, Px_k\}$ is an orthogonal basis of V and $\{Fx_1, FPx_1, \ldots, Fx_k, FPx_k\}$ is an orthogonal basis of $V^\perp$. In particular this implies that the dimension of V is even.

Now let $y \in V^\perp$. We observe that Fy is the projection of Jy onto $V^\perp$. The existence of the previous basis of $\mathbb{C}V$ shows that we can write $y = aFx + bFPx$ for some $x \in V$ and $a, b \in \mathbb{R}$. For any $z \in V$ we have $\langle Fz, Fz\rangle = \langle Jz, Jz\rangle - \langle Pz, Pz\rangle = (1 - \cos^2\varphi)\langle z, z\rangle = (\sin^2\varphi)\langle z, z\rangle$. This, the above results and the fact that $\langle FPx, Fx\rangle = 0$ implies

$$\begin{aligned} \langle Fy, Fy\rangle &= a^2\langle F^2x, F^2x\rangle + 2ab\langle F^2x, F^2Px\rangle + b^2\langle F^2Px, F^2Px\rangle \\ &= a^2(\sin^2\varphi)(\cos^2\varphi)\langle x, x\rangle + 2ab(\cos^2\varphi)\langle FPx, Fx\rangle + b^2(\cos^4\varphi)(\sin^2\varphi)\langle x, x\rangle \\ &= (\cos^2\varphi)\Big(a^2\langle Fx, Fx\rangle + b^2\langle FPx, FPx\rangle\Big) \;=\; (\cos^2\varphi)\langle y, y\rangle, \end{aligned}$$

which shows that $V^\perp$ has constant Kähler angle. Reversing the roles of P and F we get $F^2\sigma = -(\cos^2\varphi)\sigma$, $FP\sigma = -(\sin^2\varphi)\sigma$, $P^2\sigma = -FP\sigma$. Altogether this gives the result. □

The proof of the previous lemma implies

Corollary 6.11. *Let $V \subset \mathbb{C}^n$ be a vector subspace with constant Kähler angle $\varphi \in (0, \pi/2)$. Then V has even dimension, let us say $2k$ and there exist non–zero vectors $x_1, \ldots, x_k \in V$ such that $\{x_1, Px_1, \ldots, x_k, Px_k\}$ is an orthogonal basis of V. Moreover, $\mathbb{C}V \ominus V$ has also constant Kähler angle φ.*

An easy consequence of the definition allows us to calculate the inner product of the orthogonal projections of a vector onto V and $V^\perp$.

Corollary 6.12. *Let $V \subset \mathbb{C}^n$ a linear subspace with constant Kähler angle $\varphi \in [0, \pi/2]$. We have*

(i) If $x, y \in V$ then $\langle Px, Py\rangle = (\cos^2\varphi)\langle x, y\rangle$ and $\langle Fx, Fy\rangle = (\sin^2\varphi)\langle x, y\rangle$.

(ii) If $x, y \in V^\perp$ then $\langle Px, Py\rangle = (\sin^2\varphi)\langle x, y\rangle$ and $\langle Fx, Fy\rangle = (\cos^2\varphi)\langle x, y\rangle$.

Proof. For $\varphi = 0$ and $\varphi = \pi/2$ the result follows immediately. Let $\varphi \in (0, \pi/2)$. Since V has constant Kähler angle φ, for any $x \in V$ we have $\langle Px, Px\rangle = (\cos^2\varphi)\langle x, x\rangle$. Polarization of this equality implies $\langle Px, Py\rangle = (\cos^2\varphi)\langle x, y\rangle$ for all $x, y \in V$. Hence $\langle Fx, Fy\rangle = \langle Jx, Jy\rangle - \langle Px, Py\rangle = (1 - \cos^2\varphi)\langle x, y\rangle = (\sin^2\varphi)\langle x, y\rangle$. This proves (i). Statement (ii) follows easily after taking into account that $V^\perp$ has also constant Kähler angle φ and the roles of P and F are reversed. $\square$

We give a geometric construction of the singular orbits of cohomogeneity one actions in $\mathbb{C}H^n$ with one non–totally geodesic singular orbit [15].

Let KAN be the Iwasawa decomposition with respect to $o \in \mathbb{C}H^n$ and $x \in \mathbb{C}H^n(\infty)$, and let $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding decomposition on Lie algebra level. The nilpotent algebra $\mathfrak{n}$ is decomposed into $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ as described in Section 6.2.

Let $\mathfrak{w}$ be a linear subspace of $\mathfrak{v}$ such that $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has constant Kähler angle $\varphi \in [0, \pi/2]$. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{v}$ of codimension k. Denote by H the closed subgroup of AN with Lie algebra $\mathfrak{h}$ and by $N_K^0(H)$ the identity component of the normalizer of H in K. Then $G = N_K^0(H)H \subset KAN$ acts on $\mathbb{C}H^n$ with cohomogeneity one. We denote by W_φ^{2n-k} the orbit of G through o. For all $g \in N_K^0(H)$ we have $g(H \cdot o) = g(H \cdot g^{-1}o) = (gHg^{-1}) \cdot o \subset H \cdot o$, and hence $W_\varphi^{2n-k} = H \cdot o$.

If $k = 1$, then obviously $\varphi = \pi/2$, and the orbits of this action form a Riemannian foliation on $\mathbb{C}H^n$ which is the solvable foliation described in the previous subsection. In this case the ruled minimal orbit of this foliation W^{2n-1} is exactly $W_{\pi/2}^{2n-1}$. In general, if $\varphi = \pi/2$ we denote $W^{2n-k} = W_{\pi/2}^{2n-k}$. If $k > 1$, then W_φ^{2n-k} has codimension k, and all other orbits are the tubes around it. If $\varphi = 0$, then k is even, say $k = 2j$, and W_0^{2n-k} is a totally geodesic $\mathbb{C}H^{n-j} \subset \mathbb{C}H^n$. In this case the action of G is orbit equivalent to the action in Theorem 6.4 (i). For this reason we assume $\varphi > 0$ from now on.

Any two Iwasawa decompositions of KAN are conjugate, and any two linear subspaces of $\mathfrak{v}$ with the same dimension and the same Kähler angle are conjugate by $g_* = \mathrm{Ad}(g)$ for some g in the normalizer of A in K [6]. As a consequence any two submanifolds W_φ^{2n-k} and W_ϕ^{2n-j} with $k = j$ and $\varphi = \phi$ are holomorphically congruent.

We now study the geometry of W_φ^{2n-k} in more detail. The maximal complex subspace $\mathfrak{c}$ of $\mathfrak{h}$ is a subalgebra and the closed subgroup $H_\mathfrak{c}$ of H with Lie algebra $\mathfrak{c}$ acts on W_φ^{2n-k} isometrically by left translations. The orbit $H_\mathfrak{c} \cdot o$ is a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. Note that the complex dimension of $\mathfrak{c}$ is $n - k$. By identifying W_φ^{2n-k} with H equipped with the induced left–invariant Riemannian metric, it follows now that W_φ^{2n-k} is ruled by totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$.

The Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ can be decomposed orthogonally into

$$\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{c} \oplus \mathfrak{d} \oplus \mathfrak{w}^\perp.$$

We denote by $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{W}^\perp$ the corresponding left–invariant distributions on $\mathbb{C}H^n$ along W^{2n-k}_φ. As we have seen above, $\mathfrak{C}$ is autoparallel and the integral submanifolds are totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. The distribution $\mathfrak{W}^\perp$ is just the normal bundle $T^\perp W^{2n-k}_\varphi$.

We give a geometric description of the submanifold W^{2n-k}_φ.

Proposition 6.13. *The submanifold W^{2n-k}_φ has the following properties:*

(i) The maximal holomorphic subbundle $\mathfrak{C}$ of TW^{2n-k}_φ is integrable and the leaves of the induced foliation on W^{2n-k}_φ are totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$.

(ii) The following statements are equivalent:

(a) The distribution $\mathfrak{D}$ on W^{2n-k}_φ is integrable.

(b) The distribution $\mathbb{R}A \oplus \mathfrak{D}$ on W^{2n-k}_φ is integrable.

(c) $\varphi = \pi/2$.

In this case the leaves of the foliation on W^{2n-k} induced by $\mathbb{R}A \oplus \mathfrak{D}$ are totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$ and the leaves of the foliation on W^{2n-k} induced by $\mathfrak{D}$ are horospheres with center x in these totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$.

(iii) The left-invariant subbundle $\mathfrak{W}^\perp$ of $T\mathbb{C}H^n$ along W^{2n-k}_φ is the normal bundle of W^{2n-k}_φ.

(iv) For each non–zero $\xi \in \mathfrak{w}^\perp$ the left–invariant distribution $\mathbb{R}A \oplus \mathbb{R}P\xi$ along W^{2n-k}_φ is integrable and the leaves of the induced foliation on W^{2n-k}_φ are totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.

(v) For each non–zero $\xi \in \mathfrak{w}^\perp$ the left-invariant distribution $\mathbb{R}P\xi$ on W^{2n-k}_φ is integrable and the leaves of the induced foliation on W^{2n-k}_φ are horocycles with center x in the totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$ given by the distribution $\mathbb{R}A \oplus \mathbb{R}P\xi$.

Proof. We use the formulas and notations as described in Section 6.2.

Statement (i) follows immediately from the expression of the Levi–Civita connection of $\mathbb{C}H^n$ for left–invariant vector fields and the fact that the only complex totally geodesic submanifolds of $\mathbb{C}H^n$ are complex hyperbolic spaces.

Using the formulas for the Lie bracket of left–invariant vector fields in $\mathbb{C}H^n$ we get

$$[aA + U, bA + V] = \frac{a}{2}V - \frac{b}{2}U + [U, V] \qquad \text{and} \qquad [U, V] = \langle JU, V\rangle Z.$$

for all $aA+U$, $bA+V \in \mathbb{R}A\oplus\mathfrak{D}$. This shows that $\mathbb{R}A\oplus\mathfrak{D}$ is integrable if and only if $\mathfrak{D}$ is integrable if and only if $\mathfrak{D}$ is real, that is, $\varphi = \pi/2$. In this case the Levi–Civita connection yields

$$\bar{\nabla}_{aA+U}(bA + V) = \frac{1}{2}\langle U, V\rangle A - \frac{b}{2}U \in \mathbb{R}A \oplus \mathfrak{D}$$

for all $aA+U$, $bA+V \in \mathbb{R}A \oplus \mathfrak{D}$. This shows that $\mathbb{R}A \oplus \mathfrak{D}$ is autoparallel and its leaves are totally geodesic real submanifolds of $\mathbb{C}H^n$. The only real totally geodesic submanifolds of $\mathbb{C}H^n$ are the real hyperbolic spaces. For all $U, V \in \mathfrak{D}$ we have

$$\bar{\nabla}_U V = \frac{1}{2}\langle U, V\rangle A \qquad \text{and} \qquad \bar{\nabla}_U A = -\frac{1}{2}\,U,$$

which implies that the leaves of $\mathfrak{D}$ are spherical hypersurfaces of the corresponding real hyperbolic spaces (Theorem 6.6). Since the sectional curvature of a totally geodesic real hyperbolic subspace is $-1/4$, and the mean curvature vector of any leaf of $\mathfrak{D}$ is $(1/2)A$, it follows that the leaves of $\mathfrak{D}$ are horospheres centered at x in the real hyperbolic subspaces. This finishes the proof of (ii).

Statement (iii) holds by construction.

For any $aA + xP\xi$, $bA + yP\xi \in \mathbb{R}A \oplus \mathbb{R}P\xi$ we have

$$\bar{\nabla}_{aA+xP\xi}(bA + yP\xi) = \frac{xy}{2}\,(\sin^2\varphi)A - \frac{bx}{2}\,P\xi \in \mathbb{R}A \oplus \mathbb{R}P\xi.$$

From this, we easily get the assertion (iv) using Theorem 6.6.

Finally, define $U_\xi = P\xi/\sin(\varphi)$. Then the expression of the Levi–Civita connection for left–invariant metrics implies

$$\bar{\nabla}_{U_\xi} U_\xi = \frac{1}{2}\,A \qquad \text{and} \qquad \bar{\nabla}_{U_\xi}\bar{\nabla}_{U_\xi} U_\xi = -\frac{1}{4}\,U_\xi.$$

Since the real hyperbolic planes in (iv) have constant sectional curvature $-1/4$, this shows that the integral curves of U_ξ are horocycles with center x in the corresponding real hyperbolic planes. This proves (v). $\square$

The above properties are characteristic of W^{2n-k}_φ. Any other submanifold of $\mathbb{C}H^n$ with these properties is holomorphically congruent to some W^{2n-k}_φ as the following result shows. Afterwards we will see that, in fact, all the information of W^{2n-k}_φ is encoded in its second fundamental form.

Corollary 6.14. *Let $k \in \{1, \ldots, n-1\}$, and fix a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$ and points $o \in \mathbb{C}H^{n-k}$ and $x \in \mathbb{C}H^{n-k}(\infty)$. Let KAN be the Iwasawa decomposition of $SU(1,n)$ with respect to o and x, and let H' be the subgroup of AN which acts simply transitively on $\mathbb{C}H^{n-k}$. Next, let $\mathcal{V}$ be a subspace of $T_o^\perp\mathbb{C}H^{n-k}$ with constant Kähler angle $\varphi \in (0, \pi/2]$ such that $\mathbb{C}\mathcal{V} = T_o^\perp\mathbb{C}H^{n-k}$. Left translation of $\mathcal{V}$ by H' to all points in $\mathbb{C}H^{n-k}$ determines a subbundle $\mathfrak{V}$ of the normal bundle $T^\perp\mathbb{C}H^{n-k}$. At each point $p \in \mathbb{C}H^{n-k}$ attach the horocycles determined by x and the linear lines in $\mathfrak{V}_p$. The resulting subset M of $\mathbb{C}H^n$ is holomorphically congruent to the ruled submanifold W^{2n-k}_φ.*

Proof. Let W^{2n-k}_φ be the ruled minimal submanifold of $\mathbb{C}H^n$ constructed from the Iwasawa decomposition KAN associated with x and o and the choice of $\mathfrak{w}^\perp = T_o^\perp\mathbb{C}H^{n-k} \ominus \mathcal{V}$. We use the above notations. From Proposition 6.13 we already know that $M \subset W^{2n-k}_\varphi$. It suffices to prove that $W^{2n-k}_\varphi \subset M$.

Let $p \in W^{2n-k}_\varphi$. There exists an isometry $s \in H$ with $p = s(o)$. Then there is a unique vector X in the Lie algebra $\mathfrak{h}$ of H such that $s = \operatorname{Exp}_{\mathfrak{s}}(X)$. We can write $X = aA + zZ + U + V$ with some $U \in \mathfrak{c}$, $V \in \mathfrak{d}$ and $a, z \in \mathbb{R}$. Note that $[V, U] = 0$ because they are complex orthogonal. We now define

$$g = \operatorname{Exp}_{\mathfrak{s}}\left(\rho\left(\frac{a}{2}\right)V\right) \qquad \text{and} \qquad h = \operatorname{Exp}_{\mathfrak{s}}(aA + zZ + U).$$

Note that $h \in H'$. Using the description of $\mathbb{C}H^n$ given in Section 6.2 we get

$$\begin{aligned} gh &= \operatorname{Exp}_{\mathfrak{s}}\left(\rho\left(\frac{a}{2}\right)V\right)\operatorname{Exp}_{\mathfrak{s}}(aA + zZ + U) \\ &= \left(0, \operatorname{Exp}_{\mathfrak{n}}\left(\rho\left(\frac{a}{2}\right)V\right)\right) \cdot \left(a, \operatorname{Exp}_{\mathfrak{n}}\left(\rho(a)z\,Z + \rho\left(\frac{a}{2}\right)U\right)\right) \\ &= \left(a, \operatorname{Exp}_{\mathfrak{n}}\left(\rho(a)z\,Z + \rho\left(\frac{a}{2}\right)U + \rho\left(\frac{a}{2}\right)V + \frac{1}{2}[V,U]\right)\right) \\ &= \operatorname{Exp}_{\mathfrak{s}}(aA + zZ + U + V) \;=\; s. \end{aligned}$$

By construction, $h(o) \in \mathbb{C}H^{n-k}$ and $s(o) = g(h(o))$ is on the horocyle with center x through $h(o)$ tangent to $\mathbb{R}\mathcal{V}$. From this we conclude that $W^{2n-k}_\varphi \subset M$. Altogether this implies $M = W^{2n-k}_\varphi$ and the result follows. □

Next, we calculate the second fundamental form of W^{2n-k}_φ.

Proposition 6.15. *The second fundamental form of W^{2n-k}_φ is given by the formula*

$$I\!I\big(aA + xZ + U + P\xi, bA + yZ + V + P\eta\big) = -\frac{\sin^2\varphi}{2}\big(y\xi + x\eta\big)$$

for any $U, V \in TW^{2n-k}_\varphi \ominus (\mathbb{R}A \oplus \mathbb{R}Z)$, $\xi, \eta \in T^\perp W^{2n-k}_\varphi$ and $a, b, x, y \in \mathbb{R}$. Thus $I\!I$ is given by the trivial bilinear extension of $2I\!I(Z, P\xi) = -(\sin^2\varphi)\xi$ for any $\xi \in T^\perp W^{2n-k}_\varphi$.

Proof. Since A, Z, U, V, $P\xi$ and $P\eta$ are tangent to W^{2n-k}_φ, the normal component of the Levi–Civita connection reduces to

$$\begin{aligned} I\!I\big(aA + xZ + U + P\xi, bA + yZ + V + P\eta\big) &= -\left(\bar\nabla_{aA+xZ+U+P\xi}(bA + yZ + V + P\eta)\right)^\perp \\ &= \left(\frac{y}{2}JP\xi + \frac{x}{2}JP\eta\right)^\perp \;=\; \frac{y}{2}FP\xi + \frac{x}{2}FP\eta. \end{aligned}$$

Since $FP_{|T^\perp W^{2n-k}_\varphi} = -(\sin^2\varphi)\operatorname{Id}_{T^\perp W^{2n-k}_\varphi}$ by Lemma 6.10, the result follows. □

The second fundamental form of W^{2n-k}_φ and the fact that its normal bundle has constant Kähler angle φ are enough to characterize W^{2n-k}_φ among all the submanifolds of $\mathbb{C}H^n$.

Theorem 6.16 (Rigidity of the submanifold W^{2n-k}_φ). *Let M be a $(2n-k)$–dimensional connected submanifold in $\mathbb{C}H^n$ with normal bundle $T^\perp M \subset T\mathbb{C}H^n$ of constant Kähler angle $\varphi \in (0,\pi/2]$. Assume that there exists a unit vector field Z tangent to the maximal holomorphic distribution on M such that the second fundamental form $I\!I$ of M is given by the trivial bilinear extension of*

$$2I\!I(Z,P\xi) = -(\sin^2\varphi)\,\xi,$$

for all $\xi \in T^\perp M$. Then M is holomorphically congruent to an open part of the ruled minimal submanifold W^{2n-k}_φ.

Proof. We will first show the following:

(i) The maximal holomorphic subbundle $\mathfrak{C}$ of TM is integrable and each integral manifold is an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$.

(ii) For each unit normal vector field ξ of M the totally real subbundle $\mathbb{R}JZ \oplus \mathbb{R}P\xi$ of TM is integrable and each integral manifold is an open part of a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.

(iii) For each unit normal vector field ξ of M the image of any integral curve of $P\xi$ is an open part of the horocycle centered at the point at infinity determined by $-JZ$ in the corresponding totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$ as described in (ii).

To prove (i) we first note that $TM = \mathfrak{C} \oplus PT^\perp M$. For $U,V \in \Gamma(\mathfrak{C})$ and $\xi \in \Gamma(T^\perp M)$ the condition on $I\!I$ implies $\langle I\!I(U,V), F\xi\rangle = 0$ and thus

$$\langle \nabla_U V, P\xi\rangle = \langle \bar\nabla_U V, P\xi\rangle = \langle \bar\nabla_U V, J\xi\rangle - \langle \bar\nabla_U V, F\xi\rangle = -\langle J\bar\nabla_U V, \xi\rangle = \langle I\!I(U,JV),\xi\rangle = 0$$

and

$$\langle \bar\nabla_U V, \xi\rangle = -\langle I\!I(U,V),\xi\rangle = 0$$

by the condition on $I\!I$ once again. This shows that $\mathfrak{C}$ is an autoparallel subbundle of TM and each integral manifold is a totally geodesic submanifold of $\mathbb{C}H^n$. As $\mathfrak{C}$ is a complex subbundle of complex rank $n-k$, each of these integral manifolds must be an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$.

We turn our attention to (ii). Let $A = -JZ$, $X \in \Gamma(\mathfrak{D} \ominus \mathbb{R}JZ)$ and $\eta \in \Gamma(T^\perp M)$ be a local unit normal vector field on M. Using the explicit expression of $\bar R$, the Codazzi equation, the second fundamental form $I\!I$ and $\bar\nabla J = 0$ we get

$$\begin{aligned} 0 &= \bar R_{AP\eta JX\eta} = \langle (\nabla^\perp_A I\!I)(P\eta, JX) - (\nabla^\perp_{P\eta} I\!I)(A, JX), \eta\rangle \\ &= -\langle I\!I(P\eta, \nabla_A JX),\eta\rangle = -\langle \nabla_A JX, Z\rangle \langle I\!I(P\eta, Z),\eta\rangle \\ &= \frac{\sin^2\varphi}{2}\langle \bar\nabla_A JX, Z\rangle = -\frac{\sin^2\varphi}{2}\langle \nabla_A A, X\rangle. \end{aligned}$$

This implies $\langle \bar\nabla_A A, X\rangle = \langle \nabla_A A, X\rangle = 0$. Next, our assumption and $\bar\nabla J = 0$ yield

$$\langle \nabla_A A, P\eta\rangle = \langle \bar\nabla_A A, P\eta\rangle = \langle \bar\nabla_A A, J\eta - F\eta\rangle = \langle II(A,Z),\eta\rangle + \langle II(A,A), F\eta\rangle = 0.$$

Also, we easily get $\langle \bar\nabla_A A, \eta\rangle = -\langle II(A,A),\eta\rangle = 0$. Therefore we have $\bar\nabla_A A = 0$.

Now we study $\bar\nabla_A P\xi$. First,

$$\langle \nabla_A P\xi, X\rangle = \langle \bar\nabla_A P\xi, X\rangle = -\langle \bar\nabla_A X, J\xi - F\xi\rangle = -\langle II(A,JX),\xi\rangle - \langle II(A,X), F\xi\rangle = 0.$$

Our assumption of II implies $\langle S_\eta A, Y\rangle = \langle II(A,Y),\eta\rangle = 0$ for any $Y \in \Gamma(TM)$ and thus $S_\eta A = 0$. This gives

$$\langle \nabla_A P\xi, P\eta\rangle = \langle \bar\nabla_A P\xi, P\eta\rangle = \langle \bar\nabla_A P\xi, J\eta - F\eta\rangle = -\langle JS_\eta A, P\xi\rangle + \langle II(A,P\xi), F\eta\rangle = 0.$$

Moreover, $\langle \bar\nabla_A P\xi, \eta\rangle = -\langle II(A,P\xi),\eta\rangle = 0$. Thus $\bar\nabla_A P\xi = 0$.

We have $\langle \nabla_{JX} P\xi, Z\rangle = -\langle \nabla_{JX} Z, J\xi - F\xi\rangle = \langle II(JX,A),\xi\rangle - \langle II(JX,Z), F\xi\rangle = 0$. This, together with the explicit expression of $\bar R$, the Codazzi equation, the equation of II and $\bar\nabla J = 0$ implies

$$\begin{aligned} 0 &= \bar R_{P\xi JX P\xi\xi} = \langle (\nabla^\perp_{P\xi} II)(JX,P\xi) - (\nabla^\perp_{JX} II)(P\xi,P\xi),\xi\rangle \\ &= -\langle II(\nabla_{P\xi} JX, P\xi),\xi\rangle - 2\langle II(\nabla_{JX} P\xi, P\xi),\xi\rangle \\ &= -\langle \nabla_{P\xi} JX, Z\rangle\langle II(Z,P\xi),\xi\rangle - 2\langle \nabla_{JX} P\xi, Z\rangle\langle II(Z,P\xi),\xi\rangle \\ &= \frac{\sin^2\varphi}{2}\langle \bar\nabla_{P\xi} JX, Z\rangle + (\sin^2\varphi)\langle \nabla_{JX} P\xi, Z\rangle = -\frac{\sin^2\varphi}{2}\langle \nabla_{P\xi} A, X\rangle. \end{aligned}$$

Thus we get $\langle \bar\nabla_{P\xi} A, X\rangle = \langle \nabla_{P\xi} A, X\rangle = 0$. Next, our assumption implies

$$\langle \nabla_{P\xi} A, P\eta\rangle = -\langle \bar\nabla_{P\xi} JZ, J\eta - F\eta\rangle = \langle II(P\xi,Z),\eta\rangle + \langle II(P\xi,A), F\eta\rangle = -\frac{\sin^2\varphi}{2}\langle \xi,\eta\rangle.$$

If $\langle P\xi, P\eta\rangle = 0$, then $\langle \xi,\eta\rangle = 0$ because of Corollary 6.12, and hence $\langle \nabla_{P\xi} A, P\eta\rangle = 0$. We also have $\langle \bar\nabla_{P\xi} A, \eta\rangle = -\langle II(P\xi,A),\eta\rangle = 0$. Then $\bar\nabla_{P\xi} A = -\frac{\sin^2\varphi}{2} P\xi \in \Gamma(\mathbb{R}A \oplus \mathbb{R}P\xi)$.

We now consider the covariant derivative $\bar\nabla_{P\xi} P\xi$. We have

$$\begin{aligned} \langle \nabla_{P\xi} P\xi, X\rangle &= \langle \bar\nabla_{P\xi} P\xi, X\rangle = -\langle \bar\nabla_{P\xi} X, J\xi - F\xi\rangle \\ &= -\langle II(P\xi, JX),\xi\rangle - \langle X,Z\rangle\langle II(P\xi,Z), F\xi\rangle \\ &= -\frac{\sin^2\varphi}{2}\langle X,Z\rangle\langle \xi, F\xi\rangle = 0. \end{aligned}$$

For any $Y \in \Gamma(TM)$ we have $2\langle S_\eta P\xi, Y\rangle = 2\langle II(P\xi,Y),\eta\rangle = -(\sin^2\varphi)\langle \xi,\eta\rangle\langle Z,Y\rangle$ and hence $2S_\eta P\xi = -(\sin^2\varphi)\langle \xi,\eta\rangle Z$. This implies

$$\langle \nabla_{P\xi} P\xi, P\eta\rangle = \langle \bar\nabla_{P\xi} P\xi, J\eta - F\eta\rangle = -\langle JS_\eta P\xi, P\xi\rangle + \langle II(P\xi,P\xi), F\eta\rangle = 0.$$

Finally, $\langle \bar\nabla_{P\xi} P\xi, \eta\rangle = \langle II(P\xi,P\xi),\eta\rangle = 0$. Therefore $\bar\nabla_{P\xi} P\xi = \frac{\sin^2\varphi}{2} A \in \Gamma(\mathbb{R}A \oplus \mathbb{R}P\xi)$.

Altogether this shows that $\mathbb{R}JZ \oplus \mathbb{R}P\xi$ is integrable and each integral manifold is a totally geodesic submanifold of $\mathbb{C}H^n$. As $\mathbb{R}JZ \oplus \mathbb{R}P\xi$ is a totally real subbundle of rank 2, each of these totally geodesic submanifolds must be an open part of a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.

We now proceed with (iii). We define $U_\xi = P\xi/\|P\xi\| = P\xi/\sin\varphi$. From the above result for $\bar\nabla_{P\xi}P\xi$ we obtain $2\bar\nabla_{U_\xi}U_\xi = A$. Using this and $\bar\nabla_{P\xi}A$ we get

$$\bar\nabla_{U_\xi}\bar\nabla_{U_\xi}U_\xi + \langle\bar\nabla_{U_\xi}U_\xi, \bar\nabla_{U_\xi}U_\xi\rangle U_\xi = \frac{1}{2}\bar\nabla_{U_\xi}A + \frac{1}{4}\langle A, A\rangle U_\xi = 0.$$

From this we see that the integral curves of U_ξ are horocycles as described in (iii).

To finish our argument let $o \in M$ and F_o be the leaf of $\mathfrak{C}$ through o, which is an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. Let $c : I \to F_o$ be a curve with $c(0) = o$. The normal spaces of M along c are uniquely determined by the differential equation

$$\bar\nabla_{c'}X + \frac{\sin^2\varphi}{2}\langle c', Z\rangle JX = 0$$

along $c^*T^\perp\mathbb{C}H^{n-k}$. Indeed, if X is a vector field normal to F_o along c with $X_o \in T_o^\perp M$ and satisfying the above differential equation, then $X_{c(t)} \in T_{c(t)}^\perp M$ for any t. To prove this assertion we write $X = \xi + J\eta$ with $\xi, \eta \in \Gamma(\gamma^* T^\perp M)$ and $\eta_o = 0$. Using the assumption on the second fundamental form we get $\langle S_\xi c', X\rangle = \langle I\!I(c', X), \xi\rangle = \langle c', Z\rangle\langle X, P\xi\rangle\langle I\!I(Z, P\xi), \xi\rangle = -(\sin^2\varphi)\langle c', Z\rangle\langle P\xi, X\rangle/2$, which implies

$$\bar\nabla_{c'}\xi = -\frac{\sin^2\varphi}{2}\langle c', Z\rangle P\xi + \nabla^\perp_{c'}\xi.$$

Then using $\bar\nabla J = 0$ we get

$$\begin{aligned} 0 &= \bar\nabla_{c'}X + \frac{\sin^2\varphi}{2}\langle c', Z\rangle JX \;=\; \bar\nabla_{c'}\xi + J\bar\nabla_{c'}\eta + \frac{\sin^2\varphi}{2}\langle c', Z\rangle J\xi + \frac{\sin^2\varphi}{2}\langle c', Z\rangle J^2\eta \\ &= P\left(\nabla^\perp_{c'}\eta + \frac{\sin^2\varphi}{2}\langle c', Z\rangle F\eta\right) \\ &\quad +\nabla^\perp_{c'}\xi + \frac{\sin^2\varphi}{2}\langle c', Z\rangle F\xi + F\left(\nabla^\perp_{c'}\eta + \frac{\sin^2\varphi}{2}\langle c', Z\rangle F\eta\right). \end{aligned}$$

Taking the component tangent to M yields $2\nabla^\perp_{c'}\eta + (\sin^2\varphi)\langle c', Z\rangle F\eta = 0$. Since $\eta_{c(0)} = 0$, the uniqueness of solutions of ordinary differential equations implies $\eta_{c(t)} = 0$ for all t as desired. Conditions (i)–(iii), the rigidity of totally geodesic submanifolds of Riemannian manifolds (see for example [13, page 230]), and of horocycles in real hyperbolic planes (see for example [13, pages 24-26]), then imply the assertion. □

Remark 6.17. The proof of the above rigidity result shows that the differential equation

$$\bar\nabla_{c'}X + \frac{\sin^2\varphi}{2}\langle c', Z\rangle JX = 0$$

characterizes left translation by $S_\mathfrak{c}$ of the normal spaces of W^{2n-k}_φ.

The above study provides a fairly good description of the non–totally geodesic singular orbits of the cohomogeneity one actions on $\mathbb{C}H^n$ given by cases (v) and (vi) of Theorem 6.4. It also includes the ruled minimal orbit of case (iv) as a particular case. We now conclude this chapter by studying the geometry of the principal orbits of the cohomogeneity one actions given by Theorem 6.4 (v) and (vi). We recall here that these orbits are tubes around the singular ones so one actually has to study the geometry of tubes around W_φ^{2n-k}. We do this in two steps depending on the the Kähler angle of $T^\perp W_\varphi^{2n-k}$.

Constant Kähler angle $\varphi = \pi/2$

Using the notation above for the singular orbit W^{2n-k}, with $k \geq 2$, we have that $\mathfrak{w}^\perp$ has constant Kähler angle $\varphi = \pi/2$, that is, $\mathfrak{w}^\perp$ is real. This means that the normal bundle $T^\perp W^{2n-k}$ of W^{2n-k} is real.

We recall that the second fundamental form of W^{2n-k} is given by the trivial bilinear extension of

$$II(Z, J\xi) = -\frac{1}{2}\,\xi,$$

for all $\xi \in T^\perp W^{2n-k}$. Thus, with respect to a unit vector $\xi \in T^\perp W^{2n-k}$ the shape operator is determined by

$$S_\xi(Z) = -\frac{1}{2}\,J\xi, \qquad S_\xi(J\xi) = -\frac{1}{2}\,Z, \qquad S_\xi(X) = 0,$$

for all $X \in TW^{2n-k} \ominus (\mathbb{R}Z \oplus \mathbb{R}J\xi)$. The eigenvalues of the shape operator with respect to ξ are 0, $-1/2$ and $1/2$, with corresponding multiplicities $2n-2-k$, 1 and 1. The corresponding eigenspaces are $TW^{2n-k} \ominus (\mathbb{R}Z \oplus \mathbb{R}J\xi)$, $\mathbb{R}(Z + J\xi)$ and $\mathbb{R}(-Z + J\xi)$, respectively.

The above information allows us to calculate the shape operator of the principal orbits. Every principal orbit of this action is a tube around W^{2n-k}. We use Jacobi vector field theory. Let us fix a unit vector $\xi \in T^\perp W^{2n-k}$. We define the geodesic c_ξ by the initial conditions $c_\xi(0) = o$ and $c_\xi'(0) = \xi$. We follow the notation of Section 4.1. For any $X \in T_{c_\xi(0)}\mathbb{C}H^n$ we denote by B_X the parallel translation of the vector X along c_ξ. If $X \in TW^{2n-k}$ we denote by ζ_X the Jacobi vector field defined by the initial conditions $\zeta_X(0) = X$ and $\zeta_X'(0) = S_\xi(X)$. If $\eta \in T^\perp W^{2n-k}$ we define the Jacobi vector field ζ_η by the initial conditions $\zeta_\eta(0) = 0$ and $\zeta_\eta'(0) = \eta$.

The solution of the Jacobi equation yields

$$\zeta_X(t) = \begin{cases} \cosh(t/2)B_X(t) & , \quad \text{if } X \in TW^{2n-k} \ominus (\mathbb{R}Z \oplus \mathbb{R}J\xi), \\ \cosh(t/2)B_Z(t) - \dfrac{1}{2}\sinh(t)B_{J\xi}(t) & , \quad \text{if } X = Z, \\ -\sinh(t/2)B_Z(t) + \cosh(t)B_{J\xi}(t) & , \quad \text{if } X = J\xi, \\ 2\sinh(t/2)B_X(t) & , \quad \text{if } X \in T^\perp W^{2n-k} \ominus \mathbb{R}\xi. \end{cases}$$

Using the theory of Section 4.1 we get the shape operator $S(r)$ of the tube of radius r around W^{2n-k}, $G_{W^{2n-k}}(r)$. This is given by

$$\begin{aligned}
S(r)B_X(r) &= \frac{1}{2}\left(\tanh\frac{r}{2}\right)B_X(r), \qquad \text{if } X \in TW^{2n-k} \ominus (\mathbb{R}Z \oplus \mathbb{R}J\xi),\\
S(r)B_Z(r) &= \frac{1}{2}\left(\tanh^3\frac{r}{2}\right)B_Z(r) - \frac{1}{2}\left(\operatorname{sech}^3\frac{r}{2}\right)B_{J\xi}(r),\\
S(r)B_{J\xi}(r) &= -\frac{1}{2}\left(\operatorname{sech}^3\frac{r}{2}\right)B_Z(r) + \left(1+\frac{1}{2}\operatorname{sech}^2\frac{r}{2}\right)\left(\tanh\frac{r}{2}\right)B_{J\xi}(r),\\
S(r)B_X(r) &= \frac{1}{2}\left(\coth\frac{r}{2}\right)B_X(r), \qquad \text{if } X \in T^\perp W^{2n-k} \ominus \mathbb{R}\xi.
\end{aligned}$$

Hence, we have the matrix representation

$$S(r) = \frac{1}{2}\left(\begin{array}{c|c|c|c|c}
\tanh\frac{r}{2} & & & & \\ \hline
 & \tanh^3\frac{r}{2} & & -\operatorname{sech}^3\frac{r}{2} & \\ \hline
 & & \tanh\frac{r}{2}\ \mathrm{Id}_{2n-3-k} & & \\ \hline
 & -\operatorname{sech}^3\frac{r}{2} & & 2\left(1+\frac{1}{2}\operatorname{sech}^2\frac{r}{2}\right)\tanh\frac{r}{2} & \\ \hline
 & & & & \coth\frac{r}{2}\ \mathrm{Id}_{k-1}
\end{array}\right).$$

with respect to the following direct sum decomposition

$$T_{c_\xi(r)}G_{W^{2n-k}}(r) = B_{\mathbb{R}A}(r) \oplus B_{\mathbb{R}Z}(r) \oplus B_{T_oW^{2n-k}\ominus(\mathbb{R}A\oplus\mathbb{R}Z\oplus J\xi)}(r) \oplus B_{\mathbb{R}J\xi}(r) \oplus B_{T_o^\perp W^{2n-k}\ominus\mathbb{R}\xi}(r).$$

where B_V denotes the parallel translation of any vector subspace $V \subset T_o\mathbb{C}H^n$ along c_ξ.

A straightforward calculation shows that $G_{W^{2n-k}}(r)$ has four principal curvatures

$$\begin{aligned}
\alpha &= \frac{1}{2}\tanh\frac{r}{2}, & \beta &= \frac{1}{2}\coth\frac{r}{2},\\
\gamma &= \frac{3}{4}\tanh\frac{r}{2} - \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\frac{r}{2}}\,, & \delta &= \frac{3}{4}\tanh\frac{r}{2} + \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\frac{r}{2}}\,.
\end{aligned}$$

with corresponding multiplicities

$$m_\alpha = 2n-2-k, \qquad m_\beta = k-1, \qquad m_\gamma = 1, \qquad m_\delta = 1.$$

The Hopf vector field of $G_{W^{2n-k}}(r)$ is not a principal vector. In fact, it has non–trivial orthogonal projection onto T_γ and T_δ.

A special case occurs when $r = \log(2+\sqrt{3})$. In this case, $\beta = \delta$ and the principal curvatures are $\alpha = \sqrt{3}/6$, $\beta = \delta = \sqrt{3}/2$ and $\gamma = 0$ with multiplicities $2n-k-2$, k and 1. The Hopf vector field has non–trivial projection onto T_β and T_γ. We emphasize this fact here as it becomes important in Section 7.2.1.

The previous calculations show that the interesting part of the shape operator of both the singular and the principal orbits is the one concerning the vectors Z and $J\xi$. We make this statement more precise.

Let $\xi \in T_o^\perp W^{2n-k}$ be a unit vector. Consider $\tilde{\mathfrak{g}} = \mathbb{C}(A \oplus \xi) = \mathbb{R}A \oplus \mathbb{R}Z \oplus \mathbb{R}J\xi \oplus \mathbb{R}\xi$, which is a Lie subalgebra of $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{v}$, and let $\tilde{G} = \mathrm{Exp}(\tilde{\mathfrak{g}})$ be the connected, simply connected Lie subgroup of AN whose Lie algebra is $\tilde{\mathfrak{g}}$. Then, $\tilde{G} \cdot o$ is a totally geodesic $\mathbb{C}H^2$ in $\mathbb{C}H^n$. This $\mathbb{C}H^2$ defines a "slice" of $\mathbb{C}H^n$ through o.

Let $\tilde{\mathfrak{h}} = \mathfrak{s} \cap \tilde{\mathfrak{g}} = \mathbb{R}A \oplus \mathbb{R}Z \oplus \mathbb{R}J\xi$. Then $\tilde{\mathfrak{h}}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$ of codimension one. Let $\tilde{H} = \mathrm{Exp}(\tilde{\mathfrak{h}})$ be the connected, simply connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. Then, $\tilde{H}$ acts on $\mathbb{C}H^2 = \tilde{G} \cdot o$ with cohomogeneity one and all the orbits of this action are principal. This cohomogeneity one action on $\mathbb{C}H^2$ gives exactly the solvable foliation of $\mathbb{C}H^2$ described in the previous section.

We know that the orbits of the action of $\tilde{H}$ on $\mathbb{C}H^2$ are the equidistant hypersurfaces to the orbit $\tilde{H} \cdot o$. On the other hand, the intersection of the orbits of the cohomogeneity one action of G on $\mathbb{C}H^n$ with the slice $\mathbb{C}H^2$ also gives tubes around $\tilde{H} \cdot o$ because $\mathbb{C}H^2 = \tilde{G} \cdot o$ is totally geodesic in $\mathbb{C}H^n$. So, in order to study the geometry of the orbits of the action of G on $\mathbb{C}H^n$ in the slice $\mathbb{C}H^2$ it suffices to study the orbits of the action of $\tilde{H}$ on $\mathbb{C}H^2$. As it was said before, this has been accomplished in the previous section. See also [11].

Let c be the geodesic determined by the initial condition $c'(0) = \xi$. The shape operator of the orbit through $c(r)$ of the action of $\tilde{H}$ on $\mathbb{C}H^2$ with respect to the parallel basis along the geodesic c, $\{B_A(r), B_Z(r), B_{J\xi}(r)\}$, has the matrix representation

$$S(r) = \frac{1}{2} \begin{pmatrix} \tanh \frac{r}{2} & 0 & 0 \\ 0 & \tanh^3 \frac{r}{2} & -\operatorname{sech}^3 \frac{r}{2} \\ 0 & -\operatorname{sech}^3 \frac{r}{2} & 2\left(1 + \frac{1}{2}\operatorname{sech} \frac{r}{2}\right) \tanh \frac{r}{2} \end{pmatrix}$$

as previous calculations show. However, in this context it is more convenient to use left–invariant vector fields.

The tangent vector of the geodesic c can be written with respect to left–invariant vector fields as (see Section 6.2)

$$c'(t) = -\left(\tanh \frac{t}{2}\right) A + \left(\operatorname{sech} \frac{t}{2}\right) \xi.$$

Then, $\{\operatorname{sech}(r/2)A + \tanh(r/2)\xi, Z, J\xi\}$ is an orthonormal basis of the tangent space of $\tilde{H} \cdot c(r)$ at $c(r)$. With respect to this basis the shape operator has the matrix representation

$$S(r) = \frac{1}{2} \begin{pmatrix} \tanh \frac{r}{2} & 0 & 0 \\ 0 & 2\tanh \frac{r}{2} & -\operatorname{sech} \frac{r}{2} \\ 0 & -\operatorname{sech} \frac{r}{2} & \tanh \frac{r}{2} \end{pmatrix}.$$

The following result gives the relation between the above two bases. The second one is more suitable for calculations, so we use it in what follows.

Lemma 6.18. *The parallel basis along c, $\{B_A(r), B_Z(r), B_{J\xi}(r)\}$, and the left–invariant basis $\{\operatorname{sech}(r/2)A + \tanh(r/2)\xi, Z, J\xi\}$ are related by the linear transformation*

$$\Big(B_A(r), B_Z(r), B_{J\xi}(r)\Big) = \Big(\big(\operatorname{sech}\frac{r}{2}\big)A + \big(\tanh\frac{r}{2}\big)\xi,\ Z,\ J\xi\Big)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sech}\frac{r}{2} & \tanh\frac{r}{2} \\ 0 & -\tanh\frac{r}{2} & \operatorname{sech}\frac{r}{2} \end{pmatrix}.$$

Proof. First, we find a relation between the parallel basis $\{B_A(r), B_Z(r), B_{J\xi}(r), B_\xi(r)\}$ and the left–invariant basis $\{A, Z, J\xi, \xi\}$ along c.

Let $U \in T_o\mathbb{C}H^2$ and denote by $B_U(t)$ its parallel translation along c. We may write $B_U(t) = a_1(t)A + a_2(t)Z + a_3(t)J\xi + a_4(t)\xi$. Since $c'(t) = B_\xi(r) = -\tanh(t/2)A + \operatorname{sech}(t/2)\xi$, the formula for the Levi–Civita connection of $\mathbb{C}H^2$ yields

$$\begin{aligned} 0 = B_U'(t) &= \left(a_1'(t) + \frac{a_4(t)}{2}\operatorname{sech}\frac{t}{2}\right)A + \left(a_2'(t) + \frac{a_3(t)}{2}\operatorname{sech}\frac{t}{2}\right)Z \\ &\quad + \left(a_3'(t) - \frac{a_2(t)}{2}\operatorname{sech}\frac{t}{2}\right)J\xi + \left(a_4'(t) - \frac{a_1(t)}{2}\operatorname{sech}\frac{t}{2}\right)\xi. \end{aligned}$$

As a consequence, in order to express the parallel basis $\{B_A(t), B_Z(t), B_{J\xi}(t), B_\xi(t)\}$ in terms of the left–invariant basis $\{A, Z, J\xi, \xi\}$ one needs to solve the matrix differential equation

$$D'(t) + \frac{1}{2}\Big(\operatorname{sech}\frac{t}{2}\Big)C\,D(t) = 0, \quad D(0) = \mathrm{Id}, \qquad \text{where} \quad C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The solution is the change of basis matrix

$$D(t) = \begin{pmatrix} \operatorname{sech}\frac{t}{2} & 0 & 0 & -\tanh\frac{t}{2} \\ 0 & \operatorname{sech}\frac{t}{2} & -\tanh\frac{t}{2} & 0 \\ 0 & \tanh\frac{t}{2} & \operatorname{sech}\frac{t}{2} & 0 \\ \tanh\frac{t}{2} & 0 & 0 & \operatorname{sech}\frac{t}{2} \end{pmatrix}.$$

The relation between $\{B_A(r), B_Z(r), B_{J\xi}(r)\}$ and $\{\operatorname{sech}(r/2)A + \tanh(r/2)\xi, Z, J\xi\}$ follows easily from the previous transition matrix. □

We now focus our attention on some distributions of the orbits of the action of $\tilde{H}$ on $\mathbb{C}H^2$. Let $G_{W^3}(r)$ the orbit of the action of $\tilde{H}$ through the point $c(r)$. If $r = 0$, we just have $W^3 \subset \mathbb{C}H^2$, whose geometry has been studied in this section. Assume $r \neq 0$.

The non–trivial part of the shape operator of the orbits of $\tilde{H}$ concerns the vectors $B_Z(r)$ and $B_{J\xi}(r) = Jc'(r)$. Lemma 6.18 shows that the real span of $\{B_Z(r), B_{J\xi}(r)\}$ coincides with the real span of the left–invariant vector fields Z and $J\xi$ at $c(r)$. Then, the subbundle generated by $\{B_Z(r), B_{J\xi}(r)\}$ is an integrable distribution and using the expression of the Levi–Civita connection of $\mathbb{C}H^n$ we get

$$\bar{\nabla}_Z Z = A, \qquad \bar{\nabla}_{J\xi} J\xi = \frac{1}{2}A, \qquad \bar{\nabla}_Z J\xi = \frac{1}{2}\xi, \qquad \bar{\nabla}_{J\xi} Z = \frac{1}{2}\xi.$$

Thus, the shape operator with respect to the normal vector $B_A(r)$ of each integral manifold as a submanifold of $G_{W^3}(r) = \tilde{H}\cdot c(r)$, has the matrix representation

$$\frac{1}{2}\begin{pmatrix} 2\operatorname{sech}\frac{r}{2} & \tanh\frac{r}{2} \\ \tanh\frac{r}{2} & \operatorname{sech}\frac{r}{2} \end{pmatrix}.$$

Obviously, this implies that its mean curvature is $(3/2)\operatorname{sech}(r/2)$.

The distribution $\mathbb{R}Z \oplus \mathbb{R}J\xi$, when considered as a distribution on $\mathbb{C}H^2$, is integrable. The above formulas for the Levi–Civita connection show that

$$I\!I(Z,Z) = 2I\!I(J\xi, J\xi) = A, \qquad I\!I(Z, J\xi) = \frac{1}{2}\xi.$$

Hence, the mean curvature vector of each integral manifold is $(3/2)A$. With regard to the intrinsic geometry of the integral manifolds we have that the Gaussian curvature is $K = -(1/4) + \langle A, (1/2)A\rangle - \langle (1/2)\xi, (1/2)\xi\rangle = 0$.

On the other hand, using Lemma 6.18 we get

$$\bar{\nabla}_{B_A(r)} B_A(r) = \bar{\nabla}_{\left(\operatorname{sech}\frac{r}{2}\right)A + \left(\tanh\frac{r}{2}\right)\xi}\left(\left(\operatorname{sech}\frac{r}{2}\right)A + \left(\tanh\frac{r}{2}\right)\xi\right) = -\frac{1}{2}\left(\tanh\frac{r}{2}\right)B_\xi(r).$$

Hence, every integral curve of $B_A(r)$ is a geodesic in $G_{W^3}(r) = \tilde{H}\cdot c(r)$.

All in all, this means that $G_{W^3}(r) = \tilde{H}\cdot c(r)$ is diffeomorphic to $\mathbb{R}^3$ and it is foliated by one autoparallel one–dimensional distribution whose leaves are geodesics and one orthogonal integrable distribution whose leaves are isometric to $\mathbb{R}^2$.

Constant Kähler angle $\varphi \in (0, \pi/2)$

Let us consider the singular orbit $W_\varphi^{2n-\tilde{k}}$, with $\tilde{k} \geq 2$ and $\varphi \in (0, \pi/2)$. We have that $\mathfrak{w}^\perp$ (and hence $T^\perp W_\varphi^{2n-\tilde{k}}$) has constant Kähler angle φ. Corollary 6.11 implies that $\tilde{k}$ is an even number, so we write $\tilde{k} = 2k$. We use the notation and results above. For any point $o \in W_\varphi^{2(n-k)}$, we denote by $\pi : \mathbb{C}T_o^\perp W_\varphi^{2(n-k)} \to \mathbb{C}T_o^\perp W_\varphi^{2(n-k)} \ominus T_o^\perp W_\varphi^{2(n-k)}$ and $\sigma : \mathbb{C}T_o^\perp W_\varphi^{2(n-k)} \to T_o^\perp W_\varphi^{2(n-k)}$ the corresponding orthogonal projections and $P = \pi J$ and $F = \sigma J$ are the operators defined at the beginning of this section. Note that $\mathbb{C}T_o^\perp W_\varphi^{2(n-k)} \ominus T_o^\perp W_\varphi^{2(n-k)}$ is tangent to $W_\varphi^{2(n-k)}$. It is convenient to introduce the following notation

$$\bar{P}X = \frac{PX}{\|PX\|} \qquad \text{and} \qquad \bar{F}X = \frac{FX}{\|FX\|}.$$

for all unit vector $X \in \mathbb{C}T_o^\perp W_\varphi^{2(n-k)}$. Then, the second fundamental form of $W_\varphi^{2(n-k)}$ is given by the trivial bilinear extension of

$$II(Z, \bar{P}\xi) = -\frac{\sin\varphi}{2}\,\xi,$$

for all unit $\xi \in T^\perp W_\varphi^{2(n-k)}$. Thus, with respect to a unit vector $\xi \in T^\perp W_\varphi^{2(n-k)}$ the shape operator is described by

$$S_\xi(Z) = -\frac{\sin\varphi}{2}\,\bar{P}\xi, \qquad S_\xi(\bar{P}\xi) = -\frac{\sin\varphi}{2}\,Z, \qquad S_\xi(X) = 0,$$

for all $X \in TW_\varphi^{2(n-k)} \ominus (\mathbb{R}Z \oplus \mathbb{R}P\xi)$. The eigenvalues of the shape operator are 0, $-(\sin\varphi)/2$ and $(\sin\varphi)/2$, with corresponding multiplicities $2(n-k-1)$, 1 and 1. The corresponding eigenspaces are $TW_\varphi^{2(n-k)} \ominus (\mathbb{R}Z \oplus \mathbb{R}\bar{P}\xi)$, $\mathbb{R}(Z + \bar{P}\xi)$ and $\mathbb{R}(-Z + \bar{P}\xi)$, respectively.

Since the principal orbits of the cohomogeneity one action corresponding to Theorem 6.4 (vi) are tubes around the singular orbit $W_\varphi^{2(n-k)}$, we may calculate their shape operator using Jacobi vector field theory. We follow the usual notation for parallel translation and Jacobi vector field theory explained in Section 4.1. Let us fix a unit vector $\xi \in T^\perp W_\varphi^{2(n-k)}$ and let c_ξ be the geodesic determined by the initial condition $c_\xi'(0) = \xi$. Let $X \in T_{c_\xi(0)}\mathbb{C}H^n$. The solution of the Jacobi equation of the manifold $\mathbb{C}H^n$ gives

$$\begin{aligned}
\zeta_X(t) &= \cosh\frac{t}{2}\,B_X(t), \qquad \text{if } X \in TW^{2n-k} \ominus (\mathbb{R}Z \oplus \mathbb{R}\bar{P}\xi),\\
\zeta_Z(t) &= \cosh\frac{t}{2}\,B_Z(t) - \sin\varphi\left(\cos^2\varphi + \sin^2\varphi\cosh\frac{t}{2}\right)\sinh\frac{t}{2}\,B_{\bar{P}\xi}(t)\\
&\quad - \cos\varphi\,\sin^2\varphi\left(\cosh\frac{t}{2} - 1\right)\sinh\frac{t}{2}\,B_{\bar{F}\xi}(t),\\
\zeta_{\bar{P}\xi}(t) &= -\sin\varphi\,\sinh\frac{t}{2}\,B_Z(t) + \left(\cos^2\varphi\cosh\frac{t}{2} + \sin^2\varphi\cosh t\right)B_{\bar{P}\xi}(t)\\
&\quad - \sin\varphi\cos\varphi\left(\cosh\frac{t}{2} - \cosh t\right)B_{\bar{F}\xi}(t),\\
\zeta_{\bar{F}\xi}(t) &= 2\sin\varphi\,\cos\varphi\left(\cosh\frac{t}{2} - 1\right)\sinh\frac{t}{2}\,B_{\bar{P}\xi}(t)\\
&\quad +2\left(1 + \cos^2\varphi\left(\cosh\frac{t}{2} - 1\right)\right)\sinh\frac{t}{2}\,B_{\bar{F}\xi}(t),\\
\zeta_X(t) &= 2\sinh\frac{t}{2}\,B_X(t), \qquad \text{if } X \in T^\perp W^{2n-k} \ominus (\mathbb{R}\xi \oplus \mathbb{R}\bar{F}\xi).
\end{aligned}$$

Therefore, the shape operator $S(r)$ of the tube of radius r around $W_\varphi^{2(n-k)}$ can be retrieved from the above expressions using Jacobi vector field theory (see again Section 4.1). The explicit expressions are

$$\begin{aligned}
S(r)_{c'_\xi(r)}B_X(t) &= \frac{1}{2}\tanh\frac{r}{2}\,B_X(t), \qquad \text{if } X\in TW^{2n-k}\ominus(\mathbb{R}Z\oplus\mathbb{R}\bar{P}\xi),\\
S(r)_{c'_\xi(r)}B_Z(t) &= s_{11}(r)\,B_Z(r)+s_{12}(r)\,B_{\bar{P}\xi}(r)+s_{13}(r)\,B_{\bar{F}\xi}(r),\\
S(r)_{c'_\xi(r)}B_{\bar{P}\xi}(t) &= s_{21}(r)\,B_Z(r)+s_{22}(r)\,B_{\bar{P}\xi}(r)+s_{23}(r)\,B_{\bar{F}\xi}(r),\\
S(r)_{c'_\xi(r)}B_{\bar{F}\xi}(t) &= s_{13}(r)\,B_Z(r)+s_{23}(r)\,B_{\bar{P}\xi}(r)+s_{33}(r)\,B_{\bar{F}\xi}(r),\\
S(r)_{c'_\xi(r)}B_X(t) &= \frac{1}{2}\coth\frac{r}{2}B_X(r), \qquad \text{if } X\in T^\perp W^{2n-k}\ominus\mathbb{R}\xi.
\end{aligned}$$

where

$$\begin{aligned}
s_{11}(r) &= \frac{1}{2}\tanh\frac{r}{2}\,\operatorname{sech}^2\frac{r}{2}\left(\cos^2\varphi\,\cosh^2\frac{r}{2}+\sin^2\varphi\,\sinh^2\frac{r}{2}\right),\\
s_{12}(r) &= -\frac{1}{2}\operatorname{sech}^2\frac{r}{2}\,\sin\varphi\left(\cos^2\varphi+\sin^2\varphi\,\operatorname{sech}\frac{r}{2}\right),\\
s_{13}(r) &= \frac{1}{2}\operatorname{sech}^3\frac{r}{2}\,\sin^2\varphi\,\cos\varphi\left(\cosh\frac{r}{2}-1\right),\\
s_{22}(r) &= \frac{1}{2}\operatorname{sech}^2\frac{r}{2}\Big\{\cosh\frac{r}{2}\,\sinh\frac{r}{2}\,\cos^2\varphi+\tanh\frac{r}{2}\,\sin^4\varphi\left(1+2\cosh^2\frac{r}{2}\right)\\
&\quad -2\operatorname{csch}\frac{r}{2}\,\sin^2\varphi\,\cos^2\varphi\left(1-\cosh^3\frac{r}{2}\right)\Big\},\\
s_{23}(r) &= -\frac{1}{2}\operatorname{sech}^2\frac{r}{2}\,\operatorname{csch}\frac{r}{2}\,\sin\varphi\,\cos\varphi\Big\{\cos^2\varphi\left(1-\cosh^3\frac{r}{2}\right)\\
&\quad +\operatorname{sech}\frac{r}{2}\,\sin^2\varphi\left(1-\cosh\frac{r}{2}-\cosh^2\frac{r}{2}\,\sinh^2\frac{r}{2}\right)\Big\},\\
s_{33}(r) &= \frac{1}{2}\operatorname{sech}^2\frac{r}{2}\Big\{\coth\frac{r}{2}\,\cos^4\varphi\left(\cosh^2\frac{r}{2}+\sinh^2\frac{r}{2}\right)+\cosh^2\frac{r}{2}\,\coth\frac{r}{2}\,\sin^4\varphi\\
&\quad +\sin^2\varphi\,\cos^2\varphi\left(2\operatorname{csch}\frac{r}{2}+3\cosh\frac{r}{2}\,\sinh\frac{r}{2}+\tanh\frac{r}{2}\right)\Big\}.
\end{aligned}$$

As a consequence, we have the matrix representation

$$S(r)_{c'_\xi(r)}=\frac{1}{2}\left(\begin{array}{c|c|c|c|c|c}
\tanh\frac{r}{2} & & & & & \\ \hline
 & s_{11}(r) & & s_{12}(r) & & s_{13}(r)\\ \hline
 & & \tanh\frac{r}{2}\ \mathrm{Id}_{2n-3-2k} & & & \\ \hline
 & s_{12}(r) & & s_{22}(r) & & s_{23}(r)\\ \hline
 & & & & \coth\frac{r}{2}\ \mathrm{Id}_{2k-2} & \\ \hline
 & s_{13}(r) & & s_{23}(r) & & s_{33}(r)
\end{array}\right)$$

with respect to the direct sum decomposition

$$\begin{aligned} T_{c_\xi(r)}G_{W_\varphi^{2(n-k)}}(r) &= B_{\mathbb{R}A}(r)\oplus B_{\mathbb{R}Z}(r)\oplus B_{T_oW_\varphi^{2(n-k)}\ominus(\mathbb{R}A\oplus\mathbb{R}Z\oplus\mathbb{R}\bar{P}\xi)}(r)\\ &\quad\oplus B_{\mathbb{R}\bar{P}\xi}(r)\oplus B_{T_o^\perp W^{2(n-k)}\ominus(\mathbb{R}\xi\oplus\mathbb{R}\bar{F}\xi)}(r)\oplus B_{\mathbb{R}\bar{F}\xi}(r). \end{aligned}$$

We will see that $G_{W_\varphi^{2(n-k)}}(r)$ has five distinct constant principal curvatures when $k>1$ and four distinct principal curvatures when $k=1$. At this stage, however, we are not ready to study in detail the principal curvatures of the real hypersurface $G_{W_\varphi^{2(n-k)}}(r)$ due to the computational difficulty its shape operator presents. As in the previous subsection we focus on the non–trivial part of the shape operator of $G_{W_\varphi^{2(n-k)}}(r)$ and we introduce a new basis to simplify calculations.

Let $\mathfrak{v}_0\subset\mathfrak{v}$ be a two–dimensional vector subspace with constant Kähler angle φ. Then, $\tilde{\mathfrak{g}}=\mathfrak{a}\oplus\mathfrak{z}\oplus\mathbb{C}\,\mathfrak{v}_0$ is a Lie subalgebra of $\mathfrak{s}=\mathfrak{a}\oplus\mathfrak{z}\oplus\mathfrak{v}$. Let $\tilde{G}=\operatorname{Exp}(\tilde{\mathfrak{g}})$ be the connected, simply connected Lie subgroup of AN whose Lie algebra is $\tilde{\mathfrak{g}}$. Then, $\tilde{G}\cdot o$ is a totally geodesic $\mathbb{C}H^3$ in $\mathbb{C}H^n$ through o. The vector subspace $\tilde{\mathfrak{h}}=\mathfrak{s}\cap\tilde{\mathfrak{g}}=\mathfrak{a}\oplus\mathfrak{z}\oplus\mathfrak{v}_0$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$ of codimension two. Denote by $\tilde{H}=\operatorname{Exp}(\tilde{\mathfrak{h}})$ the connected, simply connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. We know that the Lie group $N_K^0(\tilde{H})\tilde{H}$ acts on $\tilde{G}\cdot o$ with cohomogeneity one and its orbit through o is exactly $\tilde{H}\cdot o$. This cohomogeneity one action is the one we have been describing throughout this subsection. We are interested in this particular case because, being it fully representative, it is the simplest of all.

We investigate the geometry of the orbits of the action of G on $\mathbb{C}H^n$ in the slice $\mathbb{C}H^3=\tilde{G}\cdot o$. Since the orbits of a cohomogeneity one action are tubes around the singular orbit and the slice $\mathbb{C}H^3=\tilde{G}\cdot o$ is totally geodesic, it suffices to study the action of $N_K^0(\tilde{H})\tilde{H}$ on $\mathbb{C}H^3$.

Let us denote $\mathfrak{v}_0^\perp=\mathbb{C}\,\mathfrak{v}_0\ominus\mathfrak{v}_0$. For any unit vector $\xi\in\mathfrak{v}_0$ the set $\{A,Z,\bar{P}\bar{F}\xi,\bar{P}\xi,\bar{F}\xi,\xi\}$ is a left–invariant orthonormal basis and $\{A,Z,\bar{P}\bar{F}\xi,\bar{P}\xi\}$ spans the tangent space of $\tilde{H}\cdot o$. Our previous study shows that, with respect to the above tangent basis, the shape operator S_ξ of $\tilde{H}\cdot o$ has the matrix representation

$$S_\xi=-\frac{\sin\varphi}{2}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},$$

and the shape operator of a principal orbit with respect to $c_\xi'(r)$, where c_ξ is the geodesic determined by $c_\xi(0)=\xi$, is given by the matrix

$$S(r)_{c_\xi'(r)}=\frac{1}{2}\begin{pmatrix}\tanh\frac{r}{2}&0&0&0&0\\0&s_{11}(r)&0&s_{12}(r)&s_{13}(r)\\0&0&\tanh\frac{r}{2}&0&0\\0&s_{12}(r)&0&s_{22}(r)&s_{23}(r)\\0&s_{13}(r)&0&s_{23}(r)&s_{33}(r)\end{pmatrix},$$

with respect to the basis $\{B_A(r), B_Z(r), B_{\bar{P}\bar{F}\xi}(r), B_{\bar{P}\xi}(r)\}$. In what follows we introduce a convenient basis to perform calculations.

For any unit vector $\eta \in T_p^{\perp} W_{\varphi}^4$ we denote by c_η the geodesic determined by the initial conditions $c_\eta(0) = p$, $c_\eta'(0) = \eta$ and by B^η the parallel translation along c_η. Using the fact that the normal exponential map of W_{φ}^4 is a diffeomorphism, we may define the vector fields $\widetilde{PF\xi}$, $\widetilde{P\xi}$, $\widetilde{F\xi}$ and $\widetilde{\xi}$ on $\mathbb{C}H^3 - \tilde{H}\cdot o$ by the formulas

$$\begin{aligned} \widetilde{PF\xi}_{c_\eta(t)} &= L_{c_\eta(t)*}\bar{P}\bar{F}\eta, & \widetilde{P\xi}_{c_\eta(t)} &= L_{c_\eta(t)*}\bar{P}\eta, \\ \widetilde{F\xi}_{c_\eta(t)} &= L_{c_\eta(t)*}\bar{F}\eta, & \widetilde{\xi}_{c_\eta(t)} &= L_{c_\eta(t)*}\eta. \end{aligned}$$

The following lemma gives the relation between the previously defined vector fields and the parallel vector fields.

Lemma 6.19. *The parallel basis along c_η, $\{B_A^\eta(r), B_Z^\eta(r), B_{\bar{P}\bar{F}\eta}^\eta(r), B_{\bar{P}\eta}^\eta(r), B_{\bar{F}\eta}^\eta(r), B_\eta^\eta(r)\}$ and the basis $\{A_{c_\eta(r)}, Z_{c_\eta(r)}, \widetilde{PF\xi}_{c_\eta(r)}, \widetilde{P\xi}_{c_\eta(r)}, \widetilde{F\xi}_{c_\eta(r)}, \widetilde{\xi}_{c_\eta(r)}\}$ are related by the linear transformation*

$$\begin{pmatrix} \operatorname{sech}\frac{r}{2} & 0 & 0 & 0 & 0 & -\tanh\frac{r}{2} \\ 0 & \operatorname{sech}\frac{r}{2} & 0 & -\sin\varphi\,\tanh\frac{r}{2} & \cos\varphi\,\tanh\frac{r}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sin\varphi\,\tanh\frac{r}{2} & 0 & \left(\cos^2\varphi\cosh\frac{r}{2} + \sin^2\varphi\right)\operatorname{sech}\frac{r}{2} & \sin\varphi\cos\varphi\left(\cosh\frac{r}{2} - 1\right)\operatorname{sech}\frac{r}{2} & 0 \\ 0 & -\cos\varphi\tanh\frac{r}{2} & 0 & \sin\varphi\cos\varphi\left(\cosh\frac{r}{2} - 1\right)\operatorname{sech}\frac{r}{2} & \left(\cos^2\varphi + \sin^2\varphi\cosh\frac{r}{2}\right)\operatorname{sech}\frac{r}{2} & 0 \\ \tanh\frac{r}{2} & 0 & 0 & 0 & 0 & \operatorname{sech}\frac{r}{2} \end{pmatrix}.$$

Proof. For simplicity let us denote $c = c_\eta$. Using the expression of the tangent vector of a geodesic in terms of left–invariant vector fields we get

$$c'(t) = -\tanh\frac{t}{2}\,A_{c(t)} + \operatorname{sech}\frac{t}{2}\,\eta_{c(t)} = -\tanh\frac{t}{2}\,A_{c(t)} + \operatorname{sech}\frac{t}{2}\,\widetilde{\xi}_{c(t)},$$

Let $U \in T_o\mathbb{C}H^2$. We may write the parallel translation of U along c as $B_U(t) = a_1(t)A_{c(t)} + a_2(t)Z_{c(t)} + a_3(t)\widetilde{PF\xi}_{c(t)} + a_4(t)\widetilde{P\xi}_{c(t)} + a_5(t)\widetilde{F\xi}_{c(t)} + a_6(t)\widetilde{\xi}_{c(t)}$. We recall that, by definition, along the geodesic c the vector fields $\{A_{c(r)}, Z_{c(r)}, \widetilde{PF\xi}_{c(r)}, \widetilde{P\xi}_{c(r)}, \widetilde{F\xi}_{c(r)}, \widetilde{\xi}_{c(r)}\}$ are left–invariant. The expression of the Levi–Civita connection on $\mathbb{C}H^n$ yields

$$\begin{aligned} 0 = B_U'(t) = {} & \left(a_1'(t) + \frac{a_6(t)}{2}\operatorname{sech}\frac{t}{2}\right)A_{c(t)} \\ & + \left(a_2'(t) + \frac{a_4(t)}{2}\sin\varphi\operatorname{sech}\frac{t}{2} - \frac{a_5(t)}{2}\cos\varphi\operatorname{sech}\frac{t}{2}\right)Z_{c(t)} \\ & + a_3'(t)\widetilde{PF\xi}_{c(t)} + \left(a_4'(t) - \frac{a_2(t)}{2}\sin\varphi\operatorname{sech}\frac{t}{2}\right)\widetilde{P\xi}_{c(t)} \\ & + \left(a_5'(t) + \frac{a_2(t)}{2}\cos\varphi\operatorname{sech}\frac{t}{2}\right)\widetilde{F\xi}_{c(t)} + \left(a_6'(t) - \frac{a_1'(t)}{2}\operatorname{sech}\frac{t}{2}\right)\widetilde{\xi}_{c(t)}. \end{aligned}$$

Hence, the proof reduces to solving the matrix differential equation

$$D'(t) + \frac{1}{2}\Big(\operatorname{sech}\frac{t}{2}\Big) C D(t) = 0,\ D(0) = \mathrm{Id}, \quad \text{where} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sin\varphi & -\cos\varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin\varphi & 0 & 0 & 0 & 0 \\ 0 & \cos\varphi & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The solution of this differential equation is exactly the matrix given in the statement of Lemma 6.19. □

Our aim in what follows consists of calculating the covariant derivatives of the above defined vector fields.

Lemma 6.20. *Let $c = c_\eta$ be the geodesic determined by the initial condition $c'(0) = \eta$, where $\eta \in \mathfrak{v}_0^\perp$ is a unit vector. Then we have*

$$\begin{aligned}
\bar\nabla_{A_{c(t)}} \widetilde{PF\xi} &= 0, & \bar\nabla_{A_{c(t)}} \widetilde{P\xi} &= 0,\\
\bar\nabla_{A_{c(t)}} \widetilde{F\xi} &= 0, & \bar\nabla_{A_{c(t)}} \widetilde{\xi} &= 0,\\
\bar\nabla_{Z_{c(t)}} \widetilde{PF\xi} &= -\frac{\cos\varphi}{2}\,\bar P\eta_{c(t)} + \frac{\sin\varphi}{2}\,\bar F\eta_{c(t)}, & \bar\nabla_{Z_{c(t)}} \widetilde{P\xi} &= \frac{\cos\varphi}{2}\,\bar P\bar F\eta_{c(t)} + \frac{\sin\varphi}{2}\,\eta_{c(t)},\\
\bar\nabla_{Z_{c(t)}} \widetilde{F\xi} &= -\frac{\sin\varphi}{2}\,\bar P\bar F\eta_{c(t)} + \frac{\cos\varphi}{2}\,\eta_{c(t)}, & \bar\nabla_{Z_{c(t)}} \widetilde{\xi} &= -\frac{\sin\varphi}{2}\,\bar P\eta_{c(t)} - \frac{\cos\varphi}{2}\,\bar F\eta_{c(t)},\\
\bar\nabla_{\bar P\bar F\eta_{c(t)}} \widetilde{PF\xi} &= \frac{1}{2}\,A_{c(t)}, & \bar\nabla_{\bar P\bar F\eta_{c(t)}} \widetilde{P\xi} &= \frac{\cos\varphi}{2}\,Z_{c(t)},\\
\bar\nabla_{\bar P\bar F\eta_{c(t)}} \widetilde{F\xi} &= -\frac{\sin\varphi}{2}\,Z_{c(t)}, & \bar\nabla_{\bar P\bar F\eta_{c(t)}} \widetilde{\xi} &= 0,\\
\bar\nabla_{\bar P\eta_{c(t)}} \widetilde{PF\xi} &= -\frac{\cos\varphi}{2}\,Z_{c(t)}, & \bar\nabla_{\bar P\eta_{c(t)}} \widetilde{P\xi} &= \frac{1}{2}\,A_{c(t)},\\
\bar\nabla_{\bar P\eta_{c(t)}} \widetilde{F\xi} &= 0, & \bar\nabla_{\bar P\eta_{c(t)}} \widetilde{\xi} &= -\frac{\sin\varphi}{2}\,Z_{c(t)},\\
\bar\nabla_{\bar F\eta_{c(t)}} \widetilde{PF\xi} &= -\frac{\sin\varphi}{2}\,Z_{c(t)} + \frac{1}{2}\operatorname{csch}\frac{t}{2}\,\bar P\eta_{c(t)}, & \bar\nabla_{\bar F\eta_{c(t)}} \widetilde{P\xi} &= -\frac{1}{2}\operatorname{csch}\frac{t}{2}\,\bar P\bar F_{c(t)},\\
\bar\nabla_{\bar F\eta_{c(t)}} \widetilde{F\xi} &= -\frac{1}{2}\,A_{c(t)} + \frac{1}{2}\operatorname{csch}\frac{t}{2}\,\eta_{c(t)}, & \bar\nabla_{\bar F\eta_{c(t)}} \widetilde{\xi} &= -\frac{\cos\varphi}{2}\,Z_{c(t)} + \frac{1}{2}\operatorname{csch}\frac{t}{2}\,\bar F\eta_{c(t)},\\
\bar\nabla_{\eta_{c(t)}} \widetilde{PF\xi} &= 0, & \bar\nabla_{\eta_{c(t)}} \widetilde{P\xi} &= \frac{\sin\varphi}{2}\,Z_{c(t)},\\
\bar\nabla_{\eta_{c(t)}} \widetilde{F\xi} &= \frac{\cos\varphi}{2}\,Z_{c(t)}, & \bar\nabla_{\eta_{c(t)}} \widetilde{\xi} &= \frac{1}{2}\,A_{c(t)},
\end{aligned}$$

Proof. Using the notation and results of Section 6.2, it is not difficult to calculate explicitly the normal exponential map of W_φ^4 as the following result shows.

Claim 6.21. *The normal exponential map of W^4_φ satisfies*

$$\begin{aligned}&\exp_{\left(a,\mathrm{Exp}_{\mathfrak{n}}(zZ+u\bar{P}\bar{F}\eta+v\bar{P}\eta)\right)}\left(h\,\bar{F}\eta+j\,\eta\right)\\ &\quad=\Big(a+\log\operatorname{sech}^2\frac{l}{2},\ \mathrm{Exp}_{\mathfrak{n}}\Big(\Big\{z-\frac{uh+vj}{l}\sin\varphi\, e^{a/2}\tanh\frac{l}{2}\Big\}Z+u\,\bar{P}\bar{F}\eta+v\,\bar{P}\eta\\ &\qquad+\frac{2h}{l}\,e^{a/2}\tanh\frac{l}{2}\bar{F}\eta+\frac{2j}{l}\,e^{a/2}\tanh\frac{l}{2}\eta\Big)\Big)\end{aligned}$$

where $l=\sqrt{h^2+j^2}$ and $\eta\in\mathfrak{v}_0^\perp$ is a unit normal vector.

The left–invariant vector field $N=(h\,\bar{F}\eta+j\,\eta)/\sqrt{h^2+j^2}$ is always a unit vector. Then the geodesic c_N whose initial conditions are $c_N(0)=o$ and $c_N'(0)=N$ is written as

$$c_N(t)=\left(\log\operatorname{sech}^2\frac{t}{2},\ \mathrm{Exp}_{\mathfrak{n}}\left(2\tanh\frac{t}{2}\,N\right)\right).$$

The point $g=\left(a,\mathrm{Exp}_{\mathfrak{n}}(zZ+u\bar{P}\bar{F}\eta+v\bar{P}\eta)\right)\in W^4_\varphi$ can be regarded as an isometry of $\mathbb{C}H^3$. Hence

$$\begin{aligned}\exp_{g\cdot o}(tN)&=g\cdot\exp_o(tN)=g\cdot c_N(t)\\ &=\Big(a+\log\operatorname{sech}^2\frac{t}{2},\ \mathrm{Exp}_{\mathfrak{n}}\Big(zZ+u\,\bar{P}\bar{F}\eta+v\,\bar{P}\eta+2e^{a/2}\tanh\frac{t}{2}N\\ &\qquad+\frac{1}{2}e^{a/2}\Big\{u\,\bar{P}\bar{F}\eta+v\,\bar{P}\eta,\ 2\tanh\frac{t}{2}N\Big\}\Big)\Big).\end{aligned}$$

Putting $t=\sqrt{h^2+j^2}$, plugging the value of N and calculating the expression for the above bracket using the usual formula for left–invariant vector fields, we get Claim 6.21.

In particular, Claim 6.21 shows that the principal orbit at distance r from W^4_φ of the cohomogeneity one action of $\tilde{H}$ on $\mathbb{C}H^3$ is the set

$$G_{W^4_\varphi}(r)=\left\{\exp_{\left(a,\mathrm{Exp}_{\mathfrak{n}}(zZ+u\bar{P}\bar{F}\xi+v\bar{P}\xi)\right)}\left(h\,\bar{F}\xi+j\,\xi\right):a,z,u,v,h,j\in\mathbb{R},\ h^2+j^2=r^2\right\}.$$

Now, Let X be a left–invariant vector field on $\mathbb{C}H^3$ (considered as a solvable Lie group) and denote by χ_X the integral curve of X trough $c(r)$. Then we have $\chi_X(s)=c(r)\,\mathrm{Exp}_{\mathfrak{s}}(sX)$. We know that

$$c(r)=\left(\log\operatorname{sech}^2\frac{r}{2},\ \mathrm{Exp}_{\mathfrak{n}}\left(2\tanh\frac{r}{2}\,\eta\right)\right).$$

Using this and the explicit expression of the Lie exponential map of $\mathbb{C}H^n$, $\mathrm{Exp}_{\mathfrak{s}}$, we easily obtain

Claim 6.22. *Let $X \in \{A, Z, \bar{P}\bar{F}\eta, \bar{P}\eta, \bar{F}\eta, \eta\}$ be a left–invariant vector field. Then the integral curve of X through $c(r)$ is given by*

$$\begin{aligned}
\chi_A(s) &= \left(s + \log \operatorname{sech}^2 \frac{t}{2}, \operatorname{Exp}_{\mathfrak{n}}\left(2 \tanh \frac{t}{2}\,\eta\right)\right),\\
\chi_Z(s) &= \left(\log \operatorname{sech}^2 \frac{t}{2}, \operatorname{Exp}_{\mathfrak{n}}\left(s \operatorname{sech}^2 \frac{t}{2}\, Z + 2 \tanh \frac{t}{2}\,\eta\right)\right),\\
\chi_{\bar{P}\bar{F}\eta}(s) &= \left(\log \operatorname{sech}^2 \frac{t}{2}, \operatorname{Exp}_{\mathfrak{n}}\left(s \operatorname{sech} \frac{t}{2}\, \bar{P}\bar{F}\eta + 2 \tanh \frac{t}{2}\,\eta\right)\right),\\
\chi_{\bar{P}\eta}(s) &= \left(\log \operatorname{sech}^2 \frac{t}{2}, \operatorname{Exp}_{\mathfrak{n}}\left(s \sin\varphi \tanh \frac{t}{2} \operatorname{sech} \frac{t}{2}\, Z + s \operatorname{sech} \frac{t}{2}\, \bar{P}\eta + 2 \tanh \frac{t}{2}\,\eta\right)\right),\\
\chi_{\bar{F}\eta}(s) &= \left(\log \operatorname{sech}^2 \frac{t}{2}, \operatorname{Exp}_{\mathfrak{n}}\left(-s \cos\varphi \tanh \frac{t}{2} \operatorname{sech} \frac{t}{2}\, Z + s \operatorname{sech} \frac{t}{2}\, \bar{F}\eta + 2 \tanh \frac{t}{2}\,\eta\right)\right),\\
\chi_\eta(s) &= \left(\log \operatorname{sech}^2 \frac{t}{2}, \operatorname{Exp}_{\mathfrak{n}}\left(\left(2 \tanh \frac{t}{2} + s \operatorname{sech} \frac{t}{2}\right)\eta\right)\right).
\end{aligned}$$

Since $\mathbb{C}H^3$ is a Hadamard manifold, the normal exponential map of W^4_φ is a diffeomorphism. Using the notation above, if χ_X denotes the integral curve of a left–invariant vector field X through $c(r)$, then we may write $\chi_X(s) = \exp_{g_X(s)\cdot o}(h_X(s)\bar{F}\eta + j_X(s)\eta)$ for some $g_X(s) \in \tilde{H}$ and $h_X(s), j_X(s) \in \mathbb{R}$. By construction the curves $g_X : \mathbb{R} \to \tilde{H}$ and $h_X, j_X : \mathbb{R} \to \mathbb{R}$ are differentiable. Comparing the formulas in Claims 6.21 and 6.22 one can obtain the explicit expression of those curves. We content ourselves with the derivative $h_X'(0)$ which is given in the following

Claim 6.23. *Assuming the notation above we have*

$$h_X'(0) = 0, \quad \text{if } X \in \mathfrak{s} \ominus \mathbb{R}\bar{F}\xi \qquad \text{and} \qquad h_X'(0) = \frac{t}{2} \operatorname{csch} \frac{t}{2}, \quad \text{if } X \in \mathbb{R}\bar{F}\xi.$$

We are now ready to conclude the proof of this lemma.

Assume that χ is a curve in $\mathbb{C}H^3$ such that $\chi(0) = c(r)$ and write χ as $\chi(s) = \exp_{g(s)\cdot o}\left(h(s)\bar{F}\eta + j(s)\eta\right)$. Then, by definition,

$$\begin{aligned}
\widetilde{PF\xi}_{\chi(s)} &= \frac{j(s)\bar{P}\bar{F}\eta_{\chi(s)} - h(s)\bar{P}\eta_{\chi(s)}}{\sqrt{h(s)^2 + j(s)^2}}, &\qquad \widetilde{P\xi}_{\chi(s)} &= \frac{h(s)\bar{P}\bar{F}\eta_{\chi(s)} + j(s)\bar{P}\eta_{\chi(s)}}{\sqrt{h(s)^2 + j(s)^2}},\\
\widetilde{F\xi}_{\chi(s)} &= \frac{j(s)\bar{F}\eta_{\chi(s)} - h(s)\eta_{\chi(s)}}{\sqrt{h(s)^2 + j(s)^2}}, &\qquad \tilde{\xi}_{\chi(s)} &= \frac{h(s)\bar{F}\eta_{\chi(s)} + j(s)\eta_{\chi(s)}}{\sqrt{h(s)^2 + j(s)^2}}.
\end{aligned}$$

Since $\chi(0) = c(r)$ we have $h(0) = 0$ and $j(0) = r$. This fact and the above expressions yield

$$\begin{aligned}
\bar{\nabla}_{\chi'(0)}\widetilde{PF\xi} &= \bar{\nabla}_{\chi'(0)}\bar{P}\bar{F}\eta + \frac{h'(0)}{r}\bar{P}\eta_{c(r)}, &\qquad \bar{\nabla}_{\chi'(0)}\widetilde{P\xi} &= \bar{\nabla}_{\chi'(0)}\bar{P}\eta - \frac{h'(0)}{r}\bar{P}\bar{F}\eta_{c(r)},\\
\bar{\nabla}_{\chi'(0)}\widetilde{F\xi} &= \bar{\nabla}_{\chi'(0)}\bar{F}\eta + \frac{h'(0)}{r}\eta_{c(r)}, &\qquad \bar{\nabla}_{\chi'(0)}\widetilde{F\xi} &= \bar{\nabla}_{\chi'(0)}\eta - \frac{h'(0)}{r}\bar{F}\eta_{c(r)}.
\end{aligned}$$

Finally, taking χ as the integral curve of one of the left–invariant vector fields A, Z, $\bar{P}\bar{F}\eta$, $\bar{P}\eta$, $\bar{F}\eta$ or η and using Claim 6.23 we get the result. $\square$

We now give a simpler expression of the shape operator of $G_{W^4_\varphi}(r)$. Let η be a unit normal vector of W^4_φ and c the geodesic determined by $c'(0)=\eta$. Lemma 6.19 implies that the set $\{B^\eta_A(r), Z_{c(r)}, \widetilde{PF\xi}_{c(r)}, \widetilde{P\xi}_{c(r)}, \widetilde{F\xi}_{c(r)}\}$ is a basis of the tangent space $T_{c(r)}G_{W^4_\varphi}(r)$. Note that $c'(r) = -\tanh(r/2)A_{c(r)} + \operatorname{sech}(r/2)\widetilde{\xi}_{c(r)}$. Then, Lemma 6.20 allows us to calculate the shape operator of $G_{W^4_\varphi}(r)$ with respect to that basis. This is given by the matrix representation

$$S(r)_{c'(r)} = \frac{1}{2}\begin{pmatrix} \tanh\frac{r}{2} & 0 & 0 & 0 & 0 \\ 0 & 2\tanh\frac{r}{2} & 0 & -\sin\varphi\operatorname{sech}\frac{r}{2} & \cos\varphi\operatorname{sech}\frac{r}{2} \\ 0 & 0 & \tanh\frac{r}{2} & 0 & 0 \\ 0 & -\sin\varphi\operatorname{sech}\frac{r}{2} & 0 & \tanh\frac{r}{2} & 0 \\ 0 & \cos\varphi\operatorname{sech}\frac{r}{2} & 0 & 0 & \operatorname{sech}\frac{r}{2}\left(\operatorname{csch}\frac{r}{2}+\sinh\frac{r}{2}\right) \end{pmatrix}.$$

The characteristic polynomial of the non–diagonal part of the above matrix may be written, after doing some calculations, as

$$\begin{aligned} p_{r,\varphi}(x) &= -x^3 + \frac{1}{2}\left(\operatorname{csch}\frac{r}{2}\operatorname{sech}\frac{r}{2} + 4\tanh\frac{r}{2}\right)x^2 - \frac{1}{4}\left(2\operatorname{sech}^2\frac{r}{2} + 5\tanh^2\frac{r}{2}\right)x \\ &\quad -\frac{1}{8}\left(\operatorname{csch}\frac{r}{2}\operatorname{sech}^3\frac{r}{2}\sin^2\varphi - \operatorname{sech}^2\frac{r}{2}\tanh\frac{r}{2} - 2\tanh^3\frac{r}{2}\right). \end{aligned}$$

We have

$$p_{r,\varphi}\left(\frac{1}{2}\tanh\frac{r}{2}\right) = -\frac{1}{8}\operatorname{csch}\frac{r}{2}\operatorname{sech}^3\frac{r}{2}\sin^2\varphi \quad\text{and}\quad p\left(\frac{1}{2}\coth\frac{r}{2}\right) = \frac{1}{8}\operatorname{csch}\frac{r}{2}\operatorname{sech}^3\frac{r}{2}\cos^2\varphi.$$

Hence, neither $(1/2)\tanh(r/2)$ nor $(1/2)\coth(r/2)$ are solutions of the above polynomial for any value of $r \neq 0$ and $\varphi \in (0,\pi/2)$. Since the matrix $(s_{ij}(r))$ has the same eigenvalues as the non–diagonal part of the above matrix we conclude that neither $(1/2)\tanh(r/2)$ nor $(1/2)\coth(r/2)$ are eigenvalues of $(s_{ij}(r))$. A study of the polynomial $p_{r,\varphi}$ using elementary calculus shows that $p_{r,\varphi}$ has three different roots when $r \neq 0$ and $\varphi \in (0,\pi/2)$. Hence, in the general context of $\mathbb{C}H^n$ we get that $G_{W^{2(n-k)}_\varphi}(r)$ has five distinct constant principal curvatures when $k>1$ and four distinct constant principal curvatures when $k=1$.

The following result is an easy consequence of Lemmas 6.19 and 6.20.

Proposition 6.24. *Let $G_{W^4_\varphi}(r)$ be the principal orbit of the action of $N^0_K(\tilde{H})\tilde{H}$ at a distance r from $\tilde{H}\cdot o = W^4_\varphi$. Then we have* *(i) The distribution* $c_\eta(r) \mapsto B_A^\eta(r)$ *is integrable and each of its integral curves is a geodesic in* $G_{W_\varphi^4}(r)$.

(ii) The distribution $c_\eta(r) \mapsto \mathbb{R}B_A^\eta(r) \oplus \mathbb{R}B_{\bar{P}\bar{F}\eta}(r)$ *is autoparallel and each integral manifold has constant sectional curvature* $-(1/4)\operatorname{sech}(r/2)$.

(iii) The distribution $c_\eta(r) \mapsto \mathbb{R}B_Z^\eta(r) \oplus \mathbb{R}\widetilde{PF}\xi_{c_\eta(r)} \oplus \mathbb{R}\widetilde{P}\xi_{c_\eta(r)} \oplus \mathbb{R}\widetilde{F}\xi_{c_\eta(r)}$ *is integrable but not autoparallel.*

Chapter 7

Real hypersurfaces with constant principal curvatures in the complex hyperbolic space

The purpose of this chapter is to obtain the following classification of real hypersurfaces with three distinct constant principal curvatures [14].

Theorem 7.1. *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures. Then M is holomorphically congruent to an open part of one of the following real hypersurfaces:*

(a) A tube of radius $r > 0$ around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some integer $k \in \{1, \ldots, n-2\}$.

(b) A tube of radius $r > 0$, $r \neq \log(2 + \sqrt{3})$, around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$.

(c) A ruled minimal real hypersurface $W^{2n-1} \subset \mathbb{C}H^n$ or one of the equidistant hypersurfaces to W^{2n-1}.

(d) A tube of radius $r = \log(2 + \sqrt{3})$ around a ruled minimal real submanifold $W^{2n-k} \subset \mathbb{C}H^n$ for some integer $k \in \{2, \ldots, n-1\}$.

The previous chapter was devoted, among other things, to the study of the geometry of the examples in the statement of the theorem above. We emphasize that cases (a) and (b) are Hopf hypersurfaces whereas cases (c) and (d) are not. From Theorem 6.5 it follows that in order to derive the above classification result we just need to focus on cases (c) and (d).

In [115] J. Saito gave a classification of connected real hypersurfaces of $\mathbb{C}H^n$ with three distinct constant principal curvatures using the classification of Hopf hypersurfaces of Theorem 6.5. In this paper J. Saito proves that when three different principal curvatures are assumed, the Hopf vector field is principal. Most unfortunately J. Saito's proof is incorrect. Cases (c) and (d), together with our study in the previous chapter, show that this is not true. J. Saito's mistake consists of getting wrong formulas ((4.42) and (4.59) in [115]) to get a contradiction when the Hopf vector field is not principal.

This chapter is organized as follows. In Section 7.1 we use the Gauss and Codazzi equations to get some general results for real hypersurfaces with constant principal curvatures in the complex hyperbolic space. An easy consequence of this, is Corollary 7.6 where we classify real hypersurfaces with two distinct constant principal curvatures. Section 7.2 is devoted to the proof of Theorem 7.1. To achieve this, we first obtain some information of the eigenvalue structure of the shape operator of the submanifold in Subsection 7.2.1. Then, in Subsection 7.2.2, using Jacobi vector field theory, we derive the classification result as a consequence of Theorems 6.8 and 6.16.

7.1 Formulas for constant principal curvatures

From now on, M denotes a real hypersurface with constant principal curvatures. This simplifies Gauss and Codazzi equations considerably and allows us to derive crucial formulas for the rest of our work.

Let α, β and γ constant real numbers and let $X \in \Gamma(T_\alpha)$, $Y \in \Gamma(T_\beta)$ and $Z \in \Gamma(T_\gamma)$. Using the Codazzi equation we get

$$\langle (\nabla_X S)\, Y - (\nabla_Y S)\, X, Z\rangle = -\langle \bar{R}_{XY}\xi, Z\rangle.$$

On the other hand,

$$\begin{aligned}\langle (\nabla_X S)\, Y - (\nabla_Y S)\, X, Z\rangle &= \langle \nabla_X SY - S\nabla_X Y - \nabla_Y SX + S\nabla_Y X, Z\rangle \\ &= (\beta-\gamma)\langle \nabla_X Y, Z\rangle - (\alpha-\gamma)\langle \nabla_Y X, Z\rangle.\end{aligned}$$

This proves the following result.

Lemma 7.2. *For all $X \in \Gamma(T_\alpha)$, $Y \in \Gamma(T_\beta)$ and $Z \in \Gamma(T_\gamma)$ we have*

$$(\beta-\gamma)\langle \nabla_X Y, Z\rangle + (\gamma-\alpha)\langle \nabla_Y X, Z\rangle = -\bar{R}_{XY\xi Z}.$$

Putting $\alpha = \gamma$ in the above lemma we get $(\beta-\alpha)\langle \nabla_X Y, Z\rangle = -\bar{R}_{XY\xi Z}$. Changing the role of Y and Z yields the following result whose importance in what follows cannot be understated.

Lemma 7.3. *For all $X, Y \in \Gamma(T_\alpha)$ and $Z \in \Gamma(T_\beta)$ with $\alpha \neq \beta$ we have*

$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= -\langle \nabla_X Z, Y\rangle = \frac{1}{\beta-\alpha}\,\bar{R}_{XZ\xi Y} \\ &= \frac{1}{4(\alpha-\beta)}\Big\{\langle JX, Y\rangle\langle Z, J\xi\rangle + \langle JY, Z\rangle\langle X, J\xi\rangle + 2\langle JX, Z\rangle\langle Y, J\xi\rangle\Big\}.\end{aligned}$$

Corollary 7.4. *For all $X \in \Gamma(T_\alpha)$ with $\langle X, J\xi\rangle = 0$, we have $\nabla_X X \in \Gamma(T_\alpha)$.*

We follow [125]. Assume $\alpha = \beta = \gamma$ in Lemma 7.2. Then $\bar{R}_{XY\xi Z} = 0$ for any $X, Y, Z \in T_\alpha(p)$. In particular, $0 = \bar{R}_{XY\xi X} = -\frac{3}{4}\langle JX, Y\rangle\langle X, J\xi\rangle$ for any $X, Y \in T_\alpha(p)$. Then, $0 = \langle X, J\xi\rangle \, \bar{R}_{XY\xi Z} = \frac{1}{4}\langle JY, Z\rangle \langle X, J\xi\rangle^2$. As a consequence, if $X, Y, Z \in T_\alpha(p)$, we have $\langle JX, Y\rangle \langle Z, J\xi\rangle = 0$. Hence, we have proved

Corollary 7.5. *If the projection of $J\xi_p$ onto $T_\alpha(p)$ is non–zero, then $T_\alpha(p)$ is a real subspace of $T_p\mathbb{C}H^n$, that is, $J\,T_\alpha(p) \subset T_p\mathbb{C}H^n \ominus T_\alpha(p)$.*

This immediately implies that the number of constant principal curvatures of M must be at least two. If this minimum is attained, then the classification of real hypersurfaces with two constant principal curvatures follows from the work by S. Montiel [99] for $n \geq 3$. We give a simple proof for $n \geq 2$ based on the classification of Hopf hypersurfaces with constant principal curvatures of Theorem 6.5.

Corollary 7.6. *Let M be a connected real hypersurface of $\mathbb{C}H^n$, $n \geq 2$, with two distinct constant principal curvatures. Then M is holomorphically congruent to an open part of one of the following real hypersurfaces:*

(i) A geodesic sphere of radius $r > 0$ in $\mathbb{C}H^n$.

(ii) A tube of radius $r > 0$ around a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$.

(iii) A tube of radius $r = \log(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$.

(iv) A horosphere in $\mathbb{C}H^n$.

Proof. It suffices to prove that M is Hopf hypersurface. The result then follows from Theorem 6.5 and the study of Section 6.3.

Assume M is not a Hopf hypersurface and call α and β the principal curvatures of M. Then, there exists a point $p \in M$ such that we can write $J\xi_p = X + Y$ for some non–zero vectors $X \in T_\alpha(p)$ and $Y \in T_\beta(p)$. According to Lemma 7.5, both $T_\alpha(p)$ and $T_\beta(p)$ are real, so $J\,T_\alpha(p) \subset T_p\mathbb{C}H^n \ominus T_\alpha(p) = T_\beta(p) \oplus \mathbb{R}\xi_p$ and $J\,T_\beta(p) \subset T_p\mathbb{C}H^n \ominus T_\beta(p) = T_\alpha(p) \oplus \mathbb{R}\xi_p$. Since $n \geq 2$, we can assume $\dim T_\alpha(p) \geq 2$.

We have $J(T_\alpha(p) \ominus \mathbb{R}X) \subset T_\beta(p)$, which implies $\dim T_\beta(p) \geq \dim J(T_\alpha(p) \ominus \mathbb{R}X) = \dim T_\alpha(p) - 1$. But Y is not an element of $J(T_\alpha(p) \ominus \mathbb{R}X)$ because $\langle Y, J\xi\rangle \neq 0$. Thus we have $\dim T_\beta(p) \geq \dim T_\alpha(p)$.

The previous equality shows that $\dim T_\beta(p) \geq 2$, so we can proceed with $T_\beta(p) \ominus \mathbb{R}Y$ in the same way to prove that $\dim T_\beta(p) \geq \dim T_\alpha(p)$. Therefore, $\dim T_\beta(p) = \dim T_\alpha(p)$. This implies that $\dim M = \dim T_\beta(p) + \dim T_\alpha(p)$ is even, which contradicts $\dim M = 2n-1$. $\square$

In the following lemma we rewrite the Gauss equation for principal curvature vectors associated with different principal curvatures. The resulting formula will be used in many cases throughout this chapter.

Lemma 7.7. *Let α and β be two distinct principal curvatures, $X \in \Gamma(T_\alpha)$ and $Y \in \Gamma(T_\beta)$. The following relation holds:*

$$\begin{aligned}
&\Big(\frac{1}{4} - \alpha\,\beta\Big)\langle X, X\rangle\langle Y, Y\rangle + \frac{1}{2}\langle JX, Y\rangle^2 + 2\langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_X X, \nabla_Y Y\rangle \\
&\quad + \frac{1}{4(\alpha-\beta)}\Big\{4\,\langle JX, Y\rangle\Big(X\langle Y, J\xi\rangle + Y\langle X, J\xi\rangle\Big) \\
&\qquad\qquad + \langle X, J\xi\rangle\Big(3\,Y\langle JX, Y\rangle + \langle \nabla_Y X, JY\rangle - 2\langle \nabla_X Y, JY\rangle\Big) \\
&\qquad\qquad + \langle Y, J\xi\rangle\Big(3\,X\langle JX, Y\rangle - \langle \nabla_X Y, JX\rangle + 2\langle \nabla_Y X, JX\rangle\Big)\Big\} = 0.
\end{aligned}$$

Proof. Using the Gauss equation we get

$$R_{XYXY} = \Big(\alpha\,\beta - \frac{1}{4}\Big)\langle X, X\rangle\langle Y, Y\rangle - \frac{3}{4}\langle JX, Y\rangle^2.$$

On the other hand, the definition of the intrinsic curvature tensor R yields

$$\begin{aligned}
R_{XYXY} &= \langle \nabla_{[X,Y]}X - \nabla_X\nabla_Y X + \nabla_Y\nabla_X X, Y\rangle \\
&= \langle \nabla_{[X,Y]}X, Y\rangle - X\langle \nabla_Y X, Y\rangle + \langle \nabla_Y X, \nabla_X Y\rangle + Y\langle \nabla_X X, Y\rangle - \langle \nabla_X X, \nabla_Y Y\rangle.
\end{aligned}$$

Using Lemma 7.3 we get

$$\begin{aligned}
X\langle \nabla_Y X, Y\rangle &= -\frac{3}{4(\alpha-\beta)}\Big(\langle Y, J\xi\rangle X\langle JX, Y\rangle + \langle JX, Y\rangle X\langle Y, J\xi\rangle\Big), \\
Y\langle \nabla_X X, Y\rangle &= \frac{3}{4(\alpha-\beta)}\Big(\langle X, J\xi\rangle Y\langle JX, Y\rangle + \langle JX, Y\rangle Y\langle X, J\xi\rangle\Big).
\end{aligned}$$

Using the Codazzi equation and the first Bianchi identity, we obtain

$$\begin{aligned}
(\alpha-\beta)\langle \nabla_{[X,Y]}X, Y\rangle &= \langle\big(\nabla_{[X,Y]}S\big)\,X, Y\rangle \;=\; \langle(\nabla_X S)\,[X,Y], Y\rangle - \langle \bar{R}_{[X,Y]X}\xi, Y\rangle \\
&= \langle(\nabla_X S)\,Y, \nabla_X Y\rangle - \langle(\nabla_X S)\,Y, \nabla_Y X\rangle - \bar{R}_{[X,Y]X\xi Y} \\
&= (\alpha-\beta)\langle \nabla_X Y, \nabla_Y X\rangle - \bar{R}_{XY\xi\nabla_X Y} - \bar{R}_{[X,Y]X\xi Y} \\
&= (\alpha-\beta)\langle \nabla_X Y, \nabla_Y X\rangle + \bar{R}_{X\nabla_Y XY\xi} - \bar{R}_{Y\nabla_X YX\xi}\,.
\end{aligned}$$

Now, the definition of the curvature tensor of $\mathbb{C}H^n$ and the Weingarten equation yield

$$\begin{aligned}
4(\bar{R}_{X\nabla_Y XY\xi} - \bar{R}_{Y\nabla_X YX\xi}) &= -(\alpha-\beta)\langle JX, Y\rangle^2 + \langle JX, Y\rangle\Big(X\langle Y, J\xi\rangle + Y\langle X, J\xi\rangle\Big) \\
&\quad + \langle X, J\xi\rangle\Big(\langle \nabla_Y X, JY\rangle - 2\langle \nabla_X Y, JY\rangle\Big) \\
&\quad + \langle Y, J\xi\rangle\Big(-\langle \nabla_X Y, JX\rangle + 2\langle \nabla_Y X, JX\rangle\Big).
\end{aligned}$$

Altogether this proves the result. $\square$

7.2 Proof of Theorem 7.1

Let M be a connected real hypersurface of the complex hyperbolic space with three constant principal curvatures. We denote the principal curvatures by α, β and γ. By ξ we denote a local unit normal vector field on M.

The Hopf vector field $J\xi$ is always tangent to the submanifold M. If $J\xi$ is also a principal curvature vector, then M is a Hopf hypersurface. As a consequence of Theorem 6.5, and taking into account the study of the number of principal curvatures carried out in the previous chapter, we have that M is congruent to a local part of a tube of radius $r > 0$ around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some integer $k \in \{1, \ldots, n-2\}$ or to a tube of radius $r > 0$, $r \neq \log(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. This takes care of cases (a) and (b) of Theorem 7.1.

Therefore, from now on we assume that M is not a Hopf hypersurface. We follow the next scheme:

1. Using the Gauss and Codazzi equations we get general relations for the eigenvalue structure of the shape operator of M. This is accomplished by using some facts from Section 7.1. More specifically, the Hopf vector field $J\xi$ cannot have non–trivial projection onto the three principal curvature spaces. In case it has two non–trivial projections, two different possibilities arise and we totally describe the eigenvalue structure of the shape operator in each case.

2. We study the two cases obtained in the previous point. For the first of them, the corresponding hypersurface has focal points at distance $r = \log(2+\sqrt{3})$. We determine the second fundamental form of the focal manifold. For the second possibility there are no focal points and hence equidistant hypersurfaces form locally a foliation. Exactly one of the leaves of that foliation is minimal. We study the shape operator of that minimal leave.

3. Both the focal set in the first possibility and the minimal equidistant hypersurface in the second have a second fundamental form as described in Theorems 6.8 and 6.16. Then, those rigidity results apply and the assertion follows.

7.2.1 Principal curvatures

This section is devoted to proving the following result.

Proposition 7.8. *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures α, β and γ. If the Hopf vector field $J\xi$ of M is not a principal vector everywhere, then $J\xi$ has non–trivial projection onto two principal curvature spaces. Assume these are T_β and T_γ. Then, we have $J\xi = a\,X + b\,Y$, where $X \in \Gamma(T_\beta)$ and $Y \in \Gamma(T_\gamma)$ are unit vectors, $a, b > 0$ and $a^2 + b^2 = 1$. There exists a unit vector field $A \in \Gamma(T_\alpha)$ such that $JA = b\,X - a\,Y$. The subbundle $\mathbb{R}X \oplus \mathbb{R}Y$ is real and the subbundle $\mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}\xi$ is complex. Moreover we may put $m_\gamma = 1$ and then one of the following two cases holds:*

(i) $m_\alpha = 2n - m_\beta - 2$ *and* $m_\beta > 1$. *The eigenvalues are* $\alpha = \sqrt{3}/6$, $\beta = \sqrt{3}/2$ *and* $\gamma = 0$. *Moreover* $a = 2\sqrt{2}/3$, $b = 1/3$, *the subbundle* $T_\beta \ominus \mathbb{R}X$ *is real and* $J(T_\beta \ominus \mathbb{R}X) \subset T_\alpha$.

(ii) $m_\alpha = 2n - 3$ *and* $m_\beta = m_\gamma = 1$. *The eigenvalues satisfy*

$$|\alpha| < 1/2, \qquad \beta = \frac{1}{2}\left(3\alpha - \sqrt{1-3\alpha^2}\right), \qquad \gamma = \frac{1}{2}\left(3\alpha + \sqrt{1-3\alpha^2}\right),$$

and a *and* b *are constant. In particular,*

$$a^2 = -\frac{(\alpha-\beta)(1+4\alpha^2-4\alpha\gamma)}{\beta-\gamma}, \qquad b^2 = \frac{(\alpha-\gamma)(1+4\alpha^2-4\alpha\beta)}{\beta-\gamma}.$$

We divide the proof in two major steps.

$J\xi$ has non–trivial projection onto three principal curvature spaces

Let ξ be the (local) unit vector field of the hypersurface M. We assume that there exists a point p such that $J\xi_p$ has non–trivial projection onto $T_\alpha(p)$, $T_\beta(p)$ and $T_\gamma(p)$. By continuity there exists an open neighborhood $\mathcal{U}$ of p such that these three projections must be non–trivial. Hence we may write

$$J\xi = a\,X + b\,Y + c\,Z,$$

for some local unit vector fields $X \in \Gamma(T_\alpha)$, $Y \in \Gamma(T_\beta)$, $Z \in \Gamma(T_\gamma)$ and functions $a, b, c : \mathcal{U} \to \mathbb{R}$ such that $a, b, c \neq 0$ in $\mathcal{U}$.

Lemma 7.9. *By taking a suitable orientation, in the neighborhood* $\mathcal{U}$ *we have*

$$JX = c\,Y - b\,Z - a\,\xi, \qquad JY = -c\,X + a\,Z - b\,\xi, \qquad JZ = b\,X - a\,Y - c\,\xi.$$

In particular, $\mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathbb{R}\xi$ *is a complex subbundle of* TM.

Proof. Since T_α, T_β and T_γ are real subbundles by Corollary 7.5, we may write

$$\begin{aligned}
JX &= \langle JX, Y\rangle Y + \langle JX, Z\rangle Z + V_1 + W_1 - a\,\xi,\\
JY &= -\langle JX, Y\rangle X + \langle JY, Z\rangle Z + U_2 + W_2 - b\,\xi,\\
JZ &= -\langle JX, Z\rangle X - \langle JY, Z\rangle Y + U_3 + V_3 - c\,\xi,
\end{aligned}$$

with $U_2, U_3 \in \Gamma(T_\alpha \ominus \mathbb{R}X)$, $V_1, V_3 \in \Gamma(T_\beta \ominus \mathbb{R}Y)$ and $W_1, W_2 \in \Gamma(T_\gamma \ominus \mathbb{R}Z)$. Taking into account that $a^2 + b^2 + c^2 = 1$ and using the above expressions we have

$$\begin{aligned}
-\xi = J^2\xi &= J(a\,X + b\,Y + c\,Z)\\
&= a\big(\langle JX, Y\rangle Y + \langle JX, Z\rangle Z + V_1 + W_1 - a\,\xi\big)\\
&\quad + b\big(-\langle JX, Y\rangle X + \langle JY, Z\rangle Z + U_2 + W_2 - b\,\xi\big)\\
&\quad + c\big(-\langle JX, Z\rangle X - \langle JY, Z\rangle Y + U_3 + V_3 - c\,\xi\big)\\
&= \big(-b\langle JX, Y\rangle - c\langle JX, Z\rangle\big)X + \big(a\langle JX, Y\rangle - c\langle JY, Z\rangle\big)Y + \big(a\langle JX, Z\rangle + b\langle JY, Z\rangle\big)Z\\
&\quad + \big(b\,U_2 + c\,U_3\big) + \big(a\,V_1 + c\,V_3\big) + \big(a\,W_1 + b\,W_2\big) - \xi.
\end{aligned}$$

Hence,

$$b\,U_2 + c\,U_3 = a\,V_1 + c\,V_3 = a\,W_1 + b\,W_2 = 0.$$

and a, b and c satisfy the following system of linear equations

$$\begin{pmatrix} 0 & \langle JX,Y\rangle & \langle JX,Z\rangle \\ -\langle JX,Y\rangle & 0 & \langle JY,Z\rangle \\ -\langle JX,Z\rangle & -\langle JY,Z\rangle & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7.1}$$

Let us call $\mathcal{A}$ the matrix of the above system. Since $\mathcal{A}$ is skew–symmetric, $\operatorname{rank}\mathcal{A} \in \{0,2\}$.

Assume that $U_2, U_3, V_1, V_3, W_1, W_2$ are all non–zero. Then U_2 and U_3 are both non–zero and collinear and the same happens with the pairs V_1, V_3 and W_1, W_2. For any $\tilde{V} \in T_\beta \ominus (\mathbb{R}Y \oplus \mathbb{R}V_1)$ we have

$$\begin{aligned} \langle JU_3, \tilde{V}\rangle &= \langle -Z + \langle JX,Z\rangle JX + \langle JY,Z\rangle JY - JV_3 + c\,J\xi, \tilde{V}\rangle \\ &= \langle JX,Z\rangle\langle\langle JX,Y\rangle Y + \langle JX,Z\rangle Z + V_1 + W_1 - a\,\xi, \tilde{V}\rangle \\ &\quad + \langle JY,Z\rangle\langle -\langle JX,Y\rangle X + \langle JY,Z\rangle Z + U_2 + W_2 - b\,\xi, \tilde{V}\rangle = 0, \end{aligned}$$

and hence $JU_3 \in \mathbb{R}Y \oplus \mathbb{R}V_1 \oplus \mathbb{R}Z \oplus \mathbb{R}W_1$. Proceeding in a similar way we also get $JV_1 \in \mathbb{R}X \oplus \mathbb{R}U_2 \oplus \mathbb{R}Z \oplus \mathbb{R}W_1$ and $JW_1 \in \mathbb{R}X \oplus \mathbb{R}U_3 \oplus \mathbb{R}Y \oplus \mathbb{R}V_1$. This means that the vector subbundle spanned by $\{X, U_2, Y, V_1, Z, W_1, \xi\}$ is complex. As a complex vector space cannot have odd dimension, one of the three vectors U_3, V_1 or W_1 must be zero. We may assume $U_2 = U_3 = 0$.

Now, if $\operatorname{rank}\mathcal{A} = 0$, then $\langle JX,Y\rangle = \langle JX,Z\rangle = \langle JY,Z\rangle = 0$, or equivalently, the subbundle spanned by $\{X,Y,Z\}$ is real. In this case, $JY = W_2 - b\,\xi$ which implies $J\xi = (JW_2 + Y)/b \in \Gamma(T_\alpha \oplus T_\beta)$, contradiction. Therefore, $\operatorname{rank}\mathcal{A} = 2$. Since a, b and c are are solutions of the system (7.1) we have

$$\langle JX,Y\rangle\langle JX,Z\rangle\langle JY,Z\rangle \neq 0.$$

because a, b and c are all non–zero.

Taking the expressions of JX, JY and JZ into account we obtain

$$\begin{aligned} 0 = \langle Z, V_3\rangle = \langle JZ, JV_3\rangle &= \langle -\langle JX,Z\rangle X - \langle JY,Z\rangle Y + V_3 - c\,\xi, JV_3\rangle \\ &= \langle JX,Z\rangle\langle\langle JX,Y\rangle Y + \langle JX,Z\rangle Z + V_1 + W_1 - a\,\xi, V_3\rangle \\ &\quad + \langle JY,Z\rangle\langle -\langle JX,Y\rangle X + \langle JY,Z\rangle Z + W_2 - b\,\xi, V_3\rangle \\ &= \langle JX,Z\rangle\langle V_1, V_3\rangle. \end{aligned}$$

Since V_1 and V_3 are collinear and $\langle JX,Z\rangle \neq 0$ we get $V_1 = V_3 = 0$. In a similar way we also get $W_1 = W_2 = 0$. Therefore $\mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathbb{R}\xi$ is a complex subbundle of TM.

Solving the system of equations (7.1) we see that the vector $(a,b,c) \in \mathbb{R}^3$ is in the real span of $(\langle JY,Z\rangle, -\langle JX,Z\rangle, \langle JX,Y\rangle) \in \mathbb{R}^3$. From $a^2+b^2+c^2 = 1$ and the expressions for JX, JY and JZ we get

$$3 = \langle JX,JX\rangle + \langle JY,JY\rangle + \langle JZ,JZ\rangle = 2\Big(\langle JY,Z\rangle^2 + \langle JX,Z\rangle^2 + \langle JX,Y\rangle^2\Big) + 1.$$

Thus $(\langle JY,Z\rangle, -\langle JX,Z\rangle, \langle JX,Y\rangle)$ is a unit vector in $\mathbb{R}^3$. By a suitable choice of orientation we may put $\langle JX,Y\rangle = c$, $\langle JX,Z\rangle = -b$ and $\langle JY,Z\rangle = a$ and Lemma 7.9 follows. $\square$

Throughout this subsection we assume the notation of Lemma 7.9. The following result describes the eigenvalue structure of the shape operator of M. This will allow us to prove that $J\xi$ cannot have non–trivial projections onto the three eigenspaces.

Lemma 7.10. *For any $U \in \Gamma(T_\alpha \ominus \mathbb{R}X)$, $V \in \Gamma(T_\beta \ominus \mathbb{R}Y)$ and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Z)$ we have*

$$\begin{aligned}
(2\alpha(\alpha-\beta)-1)\langle JU,V\rangle &= (2\beta(\alpha-\beta)+1)\langle JU,V\rangle &= 0,\\
(2\alpha(\alpha-\gamma)-1)\langle JU,W\rangle &= (2\gamma(\alpha-\gamma)+1)\langle JU,W\rangle &= 0,\\
(2\beta(\beta-\gamma)-1)\langle JV,W\rangle &= (2\gamma(\beta-\gamma)+1)\langle JV,W\rangle &= 0.
\end{aligned}$$

Proof. We recall that by Corollary 7.5, T_α, T_β and T_γ are real subbundles of TM. We divide the proof in several steps.

Claim 7.11. *We have*

$$\begin{aligned}
\langle\nabla_U V,Y\rangle &= \frac{b(a^2-2\alpha(\alpha-\beta))}{2(b^2+c^2)(\alpha-\beta)}\langle JU,V\rangle, & \langle\nabla_U V,Z\rangle &= \frac{c(a^2-2\alpha(\alpha-\beta))}{2(b^2+c^2)(\alpha-\beta)}\langle JU,V\rangle,\\
\langle\nabla_V U,X\rangle &= \frac{a(b^2+2\beta(\alpha-\beta))}{2(a^2+c^2)(\alpha-\beta)}\langle JU,V\rangle, & \langle\nabla_V U,Z\rangle &= \frac{c(b^2+2\beta(\alpha-\beta))}{2(a^2+c^2)(\alpha-\beta)}\langle JU,V\rangle,\\
\langle\nabla_U W,Z\rangle &= \frac{c(a^2-2\alpha(\alpha-\gamma))}{2(b^2+c^2)(\alpha-\gamma)}\langle JU,W\rangle, & \langle\nabla_U W,Y\rangle &= \frac{b(a^2-2\alpha(\alpha-\gamma))}{2(b^2+c^2)(\alpha-\gamma)}\langle JU,W\rangle,\\
\langle\nabla_W U,X\rangle &= \frac{a(c^2+2\gamma(\alpha-\gamma))}{2(a^2+b^2)(\alpha-\gamma)}\langle JU,W\rangle, & \langle\nabla_W U,Y\rangle &= \frac{b(c^2+2\gamma(\alpha-\gamma))}{2(a^2+b^2)(\alpha-\gamma)}\langle JU,W\rangle,\\
\langle\nabla_V W,Z\rangle &= \frac{c(b^2-2\beta(\beta-\gamma))}{2(a^2+c^2)(\beta-\gamma)}\langle JV,W\rangle, & \langle\nabla_V W,X\rangle &= \frac{a(b^2-2\beta(\beta-\gamma))}{2(a^2+c^2)(\beta-\gamma)}\langle JV,W\rangle,\\
\langle\nabla_W V,Y\rangle &= \frac{b(c^2+2\gamma(\beta-\gamma))}{2(a^2+b^2)(\beta-\gamma)}\langle JV,W\rangle, & \langle\nabla_W V,X\rangle &= \frac{a(c^2+2\gamma(\beta-\gamma))}{2(a^2+b^2)(\beta-\gamma)}\langle JV,W\rangle.
\end{aligned}$$

Since $V \in \Gamma(T_\beta \ominus \mathbb{R}Y)$ and T_β is a real subbundle of TM, we may write $JV = \tilde{U} + \tilde{W}$ with $\tilde{U} \in \Gamma(T_\alpha \ominus \mathbb{R}X)$ and $\tilde{W} \in \Gamma(T_\gamma \ominus \mathbb{R}Z)$. Taking into account that T_α is a real subbundle of TM and the fact that the complex structure is parallel, $\bar{\nabla}J = 0$, we get $\langle\tilde{U}, \bar{\nabla}_U J\xi\rangle = \alpha\langle\tilde{U}, JU\rangle = 0$. Using this and Lemma 7.3 we get

$$0 = U\langle\tilde{U}, J\xi\rangle = \langle\bar{\nabla}_U\tilde{U}, J\xi\rangle + \langle\tilde{U}, \bar{\nabla}_U J\xi\rangle = \langle\bar{\nabla}_U\tilde{U}, a\,X + b\,Y + c\,Z\rangle = a\langle\nabla_U\tilde{U}, X\rangle,$$

which implies $\langle\nabla_U X, \tilde{U}\rangle = 0$. Then, the previous equation and Lemma 7.3 give

$$\langle\nabla_U X, JV\rangle = \langle\nabla_U X, \tilde{U}\rangle + \langle\nabla_U X, \tilde{W}\rangle = \frac{a}{2(\alpha-\gamma)}\langle JU, \tilde{W}\rangle.$$

Since T_α is a real subbundle of TM we have $\langle JU, \tilde{W}\rangle = \langle JU, JV - \tilde{U}\rangle = 0$ and hence

$$\langle\nabla_U X, JV\rangle = 0.$$

Lemma 7.3 implies

$$\begin{aligned} 0 &= U\langle V, J\xi\rangle = \langle \bar\nabla_U V, a\,X + b\,Y + c\,Z\rangle + \langle V, \bar\nabla_U J\xi\rangle \\ &= \frac{a^2}{2(\alpha-\beta)}\langle JU, V\rangle + b\langle \nabla_U V, Y\rangle + c\langle \nabla_U V, Z\rangle - \alpha\langle JU, V\rangle. \end{aligned}$$

Using the expression for JX in Lemma 7.9, $\langle \nabla_U X, JV\rangle = 0$ and Lemma 7.3 we get

$$\begin{aligned} 0 &= U\langle V, JX\rangle = \langle \bar\nabla_U V, c\,Y - b\,Z - a\,\xi\rangle + \langle V, \bar\nabla_U JX\rangle \\ &= c\langle \nabla_U V, Y\rangle - b\langle \nabla_U V, Z\rangle - \langle \nabla_U X, JV\rangle = c\langle \nabla_U V, Y\rangle - b\langle \nabla_U V, Z\rangle. \end{aligned}$$

Thus, the last two equations provide a linear system whose unknowns are $\langle \nabla_U V, Y\rangle$ and $\langle \nabla_U V, Z\rangle$. This system has clearly one unique solution

$$\langle \nabla_U V, Y\rangle = \frac{b(a^2 - 2\alpha(\alpha-\beta))}{2(b^2+c^2)(\alpha-\beta)}\langle JU, V\rangle, \quad \langle \nabla_U V, Z\rangle = \frac{c(a^2 - 2\alpha(\alpha-\beta))}{2(b^2+c^2)(\alpha-\beta)}\langle JU, V\rangle.$$

which is the first pair of formulas of Claim 7.11. The other equations are obtained in a similar way.

Claim 7.12. *We have*

$$\begin{aligned} \langle \nabla_U V, Y\rangle &= -\frac{b}{2(\alpha-\beta)}\langle JU, V\rangle, & \langle \nabla_V U, X\rangle &= -\frac{a}{2(\alpha-\beta)}\langle JU, V\rangle, \\ \langle \nabla_U W, Z\rangle &= -\frac{c}{2(\alpha-\gamma)}\langle JU, W\rangle, & \langle \nabla_W U, X\rangle &= -\frac{a}{2(\alpha-\gamma)}\langle JU, W\rangle, \\ \langle \nabla_V W, Z\rangle &= -\frac{c}{2(\beta-\gamma)}\langle JV, W\rangle, & \langle \nabla_W V, Y\rangle &= -\frac{b}{2(\beta-\gamma)}\langle JV, W\rangle. \end{aligned}$$

Again, we write $JV = \tilde U + \tilde W$ with $\tilde U \in \Gamma(T_\alpha \ominus \mathbb{R}X)$ and $\tilde W \in \Gamma(T_\gamma \ominus \mathbb{R}Z)$. Using Lemma 7.3, Claim 7.11 and $\langle JU, \tilde W\rangle = 0$ we get

$$\langle V, \bar\nabla_U JZ\rangle = -\langle \nabla_U Z, JV\rangle = -\langle \nabla_U Z, \tilde U\rangle - \langle \nabla_U Z, \tilde W\rangle = -\frac{c(a^2-2\alpha(\alpha-\gamma))}{2(b^2+c^2)(\alpha-\gamma)}\langle JU, \tilde W\rangle = 0.$$

The last equation and Lemmas 7.3 and 7.9 give

$$\begin{aligned} 0 &= U\langle V, JZ\rangle = \langle \bar\nabla_U V, b\,X - a\,Y - c\,\xi\rangle + \langle V, \bar\nabla_U JZ\rangle \\ &= -\frac{a\,b}{2(\alpha-\beta)}\langle JU, V\rangle - a\langle \nabla_U V, Y\rangle - \langle \nabla_U Z, JV\rangle \\ &= -\frac{a\,b}{2(\alpha-\beta)}\langle JU, V\rangle - a\langle \nabla_U V, Y\rangle, \end{aligned}$$

which gives the first equality of Claim 7.12. The others are obtained in a similar way.

Comparing the first two equations of Claims 7.11 and 7.12 we get

$$\langle \nabla_U V, Y\rangle = \frac{b(a^2 - 2\alpha(\alpha-\beta))}{2(b^2+c^2)(\alpha-\beta)}\langle JU, V\rangle = -\frac{b}{2(\alpha-\beta)}\langle JU, V\rangle,$$

and hence $(2\alpha(\alpha-\beta)-1)\langle JU, V\rangle = 0$, which is the first equation of the assertion of Lemma 7.10. The other equalities are obtained in the same way comparing the first column of Claim 7.11 with the formulas of Claim 7.12. □

The previous lemma together with Lemma 7.7 implies that $J\xi$ cannot have non–trivial projection onto T_α, T_β and T_γ as the following result shows.

Proposition 7.13. *Let M be a real hypersurface of $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures α, β and γ. There is no point $p \in M$ such that $J\xi_p$ has non–trivial projection onto $T_\alpha(p)$, $T_\beta(p)$ and $T_\gamma(p)$.*

Proof. As $n \geq 3$, the vector subspace $(T_\alpha(p) \ominus \mathbb{R}X_p) \oplus (T_\beta(p) \ominus \mathbb{R}Y_p) \oplus (T_\gamma(p) \ominus \mathbb{R}Z_p)$ is non–zero and all the direct addends are real by Corollary 7.5. Possibly changing the roles of α, β and γ, we can choose $U_p \in T_\alpha(p)$, $V_p \in T_\beta(p)$ and $W_p \in T_\gamma(p)$ such that $\langle JU_p, V_p\rangle, \langle JU_p, W_p\rangle \neq 0$. Then, Lemma 7.10 implies

$$2\alpha(\alpha-\beta)-1 = 2\alpha(\alpha-\gamma)-1 = 2\beta(\alpha-\beta)+1 = 2\gamma(\beta-\gamma)+1 = 0$$

and a simple calculation shows that $\alpha^2 = \beta^2 = \gamma^2$. This is a contradiction because the principal curvatures are different.

Therefore, we may assume $m_\gamma = 1$, and as a consequence, $T_\alpha \ominus \mathbb{R}X = J(T_\beta \ominus \mathbb{R}Y)$. Using this fact, the non–trivial equations given by Lemma 7.10 are $2\alpha(\alpha-\beta)-1 = 2\beta(\alpha-\beta)+1 = 0$. This implies $\alpha^2 = \beta^2 = 1/4$. Changing the orientation of ξ, if necessary, we can put $\alpha = -1/2$ and $\beta = 1/2$.

Claim 7.14. *Let $U \in \Gamma(T_\alpha \ominus \mathbb{R}X)$ be a unit vector. Then, $\langle \nabla_U JU, \nabla_{JU} U\rangle = 1/4$.*

Since $T_\alpha \ominus \mathbb{R}X = J(T_\beta \ominus \mathbb{R}Y)$, we have that $JU \in \Gamma(T_\beta \ominus \mathbb{R}Y)$. Let $\tilde{U} \in \Gamma(T_\alpha \ominus \mathbb{R}X)$ and $\tilde{V} \in \Gamma(T_\beta \ominus \mathbb{R}Y)$. Using Lemma 7.3 and the fact that $\alpha = -\beta = -1/2$ we have

$$\begin{aligned} \langle \nabla_U JU, X\rangle &= -\frac{a}{2(\alpha-\beta)}\langle JU, JU\rangle = \frac{a}{2}, & \langle \nabla_U JU, \tilde{U}\rangle &= 0,\\ \langle \nabla_{JU} U, Y\rangle &= -\frac{b}{2(\alpha-\beta)}\langle JU, JU\rangle = \frac{b}{2}, & \langle \nabla_{JU} U, \tilde{V}\rangle &= 0. \end{aligned}$$

Taking into account that $a^2+b^2+c^2 = 1$ and putting $\alpha = -\beta = -1/2$ in the formulas of Claims 7.11 and 7.12 yields

$$\begin{aligned} \langle \nabla_U JU, Y\rangle &= -\frac{b}{2(\alpha-\beta)} = \frac{b}{2}, & \langle \nabla_U JU, Z\rangle &= \frac{c(a^2-2\alpha(\alpha-\beta))}{2(b^2+c^2)(\alpha-\beta)} = \frac{c}{2},\\ \langle \nabla_{JU} U, X\rangle &= -\frac{a}{2(\alpha-\beta)} = \frac{a}{2}, & \langle \nabla_{JU} U, Z\rangle &= \frac{c(b^2+2\beta(\alpha-\beta))}{2(a^2+c^2)(\alpha-\beta)} = \frac{c}{2}. \end{aligned}$$

Altogether this means $\langle \nabla_U JU, \nabla_{JU} U\rangle = \langle \nabla_U JU, X\rangle\langle \nabla_{JU} U, X\rangle + \langle \nabla_U JU, Y\rangle\langle \nabla_{JU} U, Y\rangle + \langle \nabla_U JU, Z\rangle\langle \nabla_{JU} U, Z\rangle = 1/4$ as Claim 7.14 states.

We now finish the proof of this proposition. Let $U \in \Gamma(T_\alpha \ominus \mathbb{R}X)$ be a unit vector. From Corollary 7.4 we get $\langle \nabla_U U, \nabla_{JU} JU \rangle = 0$. Claim 7.14 asserts $\langle \nabla_U JU, \nabla_{JU} U \rangle = 1/4$. Applying Lemma 7.7 for $U \in \Gamma(T_\alpha \ominus \mathbb{R}X)$ and $JU \in \Gamma(T_\beta \ominus \mathbb{R}Y)$ and taking the above expressions into account yields

$$0 = \frac{1}{4} - \alpha\beta + \frac{1}{2}\langle JU, JU \rangle^2 + 2\langle \nabla_U JU, \nabla_{JU} U \rangle - \langle \nabla_U U, \nabla_{JU} JU \rangle = \frac{3}{2},$$

which is a contradiction. Therefore, $J\xi_p$ cannot have non–trivial projection onto the three principal curvature spaces. □

$J\xi$ has non–trivial projection onto two eigenspaces

As usual, let ξ be a (local) unit normal vector field of M. We assume that $J\xi_p$ has non–trivial projection onto two principal curvature spaces, let us say, $T_\beta(p)$ and $T_\gamma(p)$. By continuity there exists a neighborhood $\mathcal{U}$ of p such that

$$J\xi = a\,X + b\,Y,$$

for some unit local vector fields $X \in \Gamma(T_\beta)$ and $Y \in \Gamma(T_\gamma)$ and everywhere non–zero functions $a, b : \mathcal{U} \to \mathbb{R}$.

Lemma 7.15. *There exists a local unit vector field $A \in \Gamma(T_\alpha)$ such that $\mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}\xi$ is a complex subbundle of TM. Moreover, by a suitable choice of orientation we can write*

$$JA = b\,X - a\,Y, \qquad JX = -b\,A - a\,\xi, \qquad JY = a\,A - b\,\xi.$$

In particular $\mathbb{R}X \oplus \mathbb{R}Y$ is a real subbundle of TM.

Proof. The principal curvature spaces T_β and T_γ are real by Corollary 7.5. Then we can write

$$JX = \langle JX, Y \rangle Y + U_1 + W - a\xi, \qquad JY = -\langle JX, Y \rangle X + U_2 + V - b\xi,$$

with $U_1, U_2 \in \Gamma(T_\alpha)$, $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$. Using the last expression we have

$$\begin{aligned} -\xi &= J^2\xi = J(a\,X + b\,Y) \\ &= a\big(\langle JX, Y \rangle Y + U_1 + W - a\xi\big) + b\big(- \langle JX, Y \rangle X + U_2 + V - b\xi\big) \\ &= (a\,U_1 + b\,U_2) - b\langle JX, Y \rangle X + a\langle JX, Y \rangle Y + b\,V + a\,W - (a^2 + b^2)\xi. \end{aligned}$$

Since $a^2 + b^2 = 1$ we get $a\,U_1 + b\,U_2 = V = W = 0$ and $\langle JX, Y \rangle = 0$.

The vectors U_1 and U_2 are both zero or both non–zero. If $U_1 = U_2 = 0$ we have $JX = -a\xi$ and $JY = -b\xi$, which is impossible. Hence we can choose a unit vector vector field $A \in \Gamma(\mathbb{R}U_1) = \Gamma(\mathbb{R}U_2) \subset \Gamma(T_\alpha)$. Since $JU_1 = aJ\xi - X \in \Gamma(\mathbb{R}X \oplus \mathbb{R}Y)$ we get $JA \in \Gamma(\mathbb{R}X \oplus \mathbb{R}Y)$. This shows that $\mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}\xi$ is a complex vector subbundle of TM.

The two unit vector fields $JA, J\xi \in \Gamma(\mathbb{R}X \oplus \mathbb{R}Y)$ are orthogonal and $J\xi = aX + bY$ by assumption. Then, by a suitable orientation of A we can write $JA = b\,X - a\,Y$ and the result follows. □

We will need the following covariant derivatives latter in this subsection.

Lemma 7.16. *With the notation of Lemma 7.15 we have*

$$\nabla_X X = \frac{3a\,b}{4(\alpha-\beta)}A, \qquad \nabla_Y Y = -\frac{3a\,b}{4(\alpha-\gamma)}A,$$

$$\nabla_X Y = -\left(\beta+\frac{3a^2}{4(\alpha-\beta)}\right)A, \qquad \nabla_Y X = \left(\gamma+\frac{3b^2}{4(\alpha-\gamma)}\right)A,$$

$$\nabla_X A = -\frac{3\,a\,b}{4(\alpha-\beta)}X+\left(\beta+\frac{3a^2}{4(\alpha-\beta)}\right)Y, \quad \nabla_Y A = -\left(\gamma+\frac{3b^2}{4(\alpha-\gamma)}\right)X+\frac{3\,a\,b}{4(\alpha-\gamma)}Y,$$

$$\nabla_A X = \frac{1}{\beta-\gamma}\left\{\left(\frac{3(\alpha-\gamma)}{4(\alpha-\beta)}-\frac{1}{4}\right)a^2+\frac{b^2}{2}+\beta(\alpha-\gamma)\right\}Y,$$

$$\nabla_A Y = -\frac{1}{\beta-\gamma}\left\{\left(\frac{3(\alpha-\gamma)}{4(\alpha-\beta)}-\frac{1}{4}\right)a^2+\frac{b^2}{2}+\beta(\alpha-\gamma)\right\}X,$$

$$\nabla_A A = 0.$$

Proof. Let $U \in \Gamma(T_\alpha \ominus \mathbb{R}A)$, $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$.

Since X and Y have unit length we have $\langle \nabla_X X, X\rangle = 0$ and $\langle \nabla_Y Y, Y\rangle = 0$. From Lemma 7.3 we easily get

$$\langle \nabla_X X, U\rangle = \langle \nabla_X X, Y\rangle = \langle \nabla_X X, W\rangle = 0, \qquad \langle \nabla_X X, A\rangle = \frac{3a\,b}{4(\alpha-\beta)},$$
$$\langle \nabla_Y Y, U\rangle = \langle \nabla_Y Y, X\rangle = \langle \nabla_Y Y, V\rangle = 0, \qquad \langle \nabla_Y Y, A\rangle = -\frac{3a\,b}{4(\alpha-\gamma)}.$$

We have $\langle V, \bar\nabla_X J\xi\rangle = \beta\langle V, JX\rangle = 0$. Using this, Lemma 7.3 and the expression for $J\xi$ we get

$$0 = X\langle V, J\xi\rangle = \langle \bar\nabla_X V, a\,X + b\,Y\rangle + \langle V, \bar\nabla_X J\xi\rangle = -a\langle \nabla_X X, V\rangle.$$

Hence, $\langle \nabla_X X, V\rangle = 0$. In a similar way we also obtain $\langle \nabla_Y Y, W\rangle = 0$. Therefore, the first two equations of Lemma 7.16 follow.

Since X and Y have unit length we have $\langle \nabla_X Y, Y\rangle = 0$ and $\langle \nabla_Y X, X\rangle = 0$. Using the expressions for $\nabla_X X$ and $\nabla_Y Y$ we immediately get $\langle \nabla_X Y, X\rangle = 0$ and $\langle \nabla_Y X, Y\rangle = 0$. Also, using Lemma 7.3 we obtain

$$\langle \nabla_X Y, V\rangle = \langle \nabla_Y X, W\rangle = 0.$$

As $\langle U, J\xi\rangle = 0$, taking derivatives, using the expressions for $J\xi$ and $\nabla_X X$ and the equality $\langle U, \bar\nabla_X J\xi\rangle = \beta\langle U, JX\rangle = 0$ we get

$$0 = X\langle U, J\xi\rangle = \langle \bar\nabla_X U, a\,X + b\,Y\rangle + \langle U, \bar\nabla_X J\xi\rangle = -b\langle \nabla_X Y, U\rangle.$$

Hence, $\langle \nabla_X Y, U\rangle = 0$ and in a similar way $\langle \nabla_Y X, U\rangle = 0$. Also, as $\langle W, \bar\nabla_X J\xi\rangle = \beta\langle W, JX\rangle = 0$, the formula for $\nabla_X X$ yields

$$0 = X\langle W, J\xi\rangle = \langle \bar\nabla_X W, a\,X + b\,Y\rangle + \langle W, \bar\nabla_X J\xi\rangle = -b\langle \nabla_X Y, W\rangle$$

and hence $\langle \nabla_X Y, W\rangle = 0$. In a similar way we obtain as well $\langle \nabla_Y X, V\rangle = 0$. Now, as $\langle JX, Y\rangle = 0$, using Lemma 7.15 and $\nabla_X X$ we get

$$\begin{aligned} 0 &= X\langle JX, Y\rangle = \langle \bar\nabla_X JX, Y\rangle + \langle JX, \bar\nabla_X Y\rangle \\ &= -\langle \bar\nabla_X X, aA - b\xi\rangle + \langle \bar\nabla_X Y, -bA - a\xi\rangle = -\frac{3a^2b}{4(\alpha-\beta)} - b\,\beta - b\langle \nabla_X Y, A\rangle. \end{aligned}$$

Hence, the above expression and the corresponding one for $0 = Y\langle JY, X\rangle$ yield

$$\langle \nabla_X Y, A\rangle = -\beta - \frac{3a^2}{4(\alpha-\beta)} \qquad \text{and} \qquad \langle \nabla_Y X, A\rangle = \gamma + \frac{3b^2}{4(\alpha-\gamma)}.$$

Therefore, the third and forth formulas of Lemma 7.16 follow.

Since A has constant length, $\langle \nabla_X A, A\rangle = \langle \nabla_Y A, A\rangle = 0$. Lemma 7.3 gives

$$\langle \nabla_X A, V\rangle = \langle \nabla_Y A, W\rangle = 0.$$

As $\langle JY, W\rangle = 0$, using Lemma 7.15 and the fact that $JW \in \Gamma((T_\alpha \ominus \mathbb{R}A) \oplus (T_\beta \ominus \mathbb{R}X))$ by Lemma 7.5, we obtain,

$$\begin{aligned} 0 &= X\langle JY, W\rangle = \langle \bar\nabla_X JY, W\rangle + \langle JY, \bar\nabla_X W\rangle \\ &= -\langle \nabla_X Y, JW\rangle + a\langle \nabla_X W, A\rangle + b\beta\langle X, W\rangle = -a\langle \nabla_X A, W\rangle. \end{aligned}$$

Hence, $\langle \nabla_X A, W\rangle = 0$. From $0 = X\langle JY, U\rangle$, $0 = Y\langle JX, V\rangle$ and $0 = Y\langle JX, U\rangle = 0$ we also get $\langle \nabla_X A, U\rangle = \langle \nabla_Y A, V\rangle = \langle \nabla_Y A, U\rangle = 0$. Altogether this means that $\nabla_X A = \langle \nabla_X A, X\rangle X + \langle \nabla_X A, Y\rangle Y$ and $\nabla_Y A = \langle \nabla_Y A, X\rangle X + \langle \nabla_Y A, Y\rangle Y$. Using the expressions obtained for $\nabla_X X$, $\nabla_Y Y$, $\nabla_X Y$ and $\nabla_Y X$ we get the formulas of the fifth and sixth equations of Lemma 7.16.

As X and Y have constant length, $\langle \nabla_A X, X\rangle = \langle \nabla_A Y, Y\rangle = 0$. Lemma 7.3 gives

$$\langle \nabla_A X, A\rangle = \langle \nabla_A X, U\rangle = \langle \nabla_A Y, A\rangle = \langle \nabla_A Y, U\rangle = 0.$$

Lemma 7.2 and the formula for $\nabla_X A$ gives

$$0 = -\bar R_{AX\xi W} = (\beta-\gamma)\langle \nabla_A X, W\rangle + (\gamma-\alpha)\langle \nabla_X A, W\rangle = (\beta-\gamma)\langle \nabla_A X, W\rangle.$$

Hence, $\langle \nabla_A X, W\rangle = 0$. Also, $0 = -\bar R_{AY\xi V} = (\gamma-\beta)\langle \nabla_A Y, V\rangle$ implies $\langle \nabla_A Y, V\rangle = 0$. Using this, the expression for $J\xi$ and $\langle W, \bar\nabla_A J\xi\rangle = \alpha\langle W, JA\rangle = 0$ we get

$$0 = A\langle W, J\xi\rangle = \langle \bar\nabla_A W, aX + bY\rangle + \langle W, \bar\nabla_A J\xi\rangle = -b\langle \nabla_A Y, W\rangle.$$

Hence $\langle \nabla_A Y, W\rangle = 0$. Analogously, $0 = A\langle V, J\xi\rangle$ yields $\langle \nabla_A X, V\rangle = 0$. Thus $\nabla_A X = \langle \nabla_A X, Y\rangle Y$ and $\nabla_A Y = -\langle \nabla_A X, Y\rangle X$. The latter inner product can be calculated by using the explicit expression of $\bar R$, Lemma 7.2 and $\nabla_X A$ as follows

$$\begin{aligned} -\frac{1}{4}(a^2 - 2b^2) &= -\bar R_{AX\xi Y} = (\beta-\gamma)\langle \nabla_A X, Y\rangle + (\gamma-\alpha)\langle \nabla_X A, Y\rangle \\ &= (\beta-\gamma)\langle \nabla_A X, Y\rangle + (\gamma-\alpha)\left(\beta + \frac{3a^2}{4(\alpha-\beta)}\right). \end{aligned}$$

Altogether this gives the expressions for $\nabla_A X$ and $\nabla_A Y$ in Lemma 7.16.

As A has constant length $\langle \nabla_A A, A\rangle = 0$. The previous calculations for $\nabla_A X$ and $\nabla_A Y$ show that $\langle \nabla_A A, X\rangle = \langle \nabla_A A, Y\rangle = 0$. Moreover, Lemma 7.3 yields $\langle \nabla_A A, V\rangle = \langle \nabla_A A, W\rangle = 0$. Using the expressions for $\nabla_A X$ and JX we get

$$\begin{aligned} 0 &= A\langle JX, U\rangle = \langle \bar{\nabla}_A JX, U\rangle + \langle JX, \bar{\nabla}_A U\rangle \\ &= -\langle \nabla_A X, JU\rangle + b\langle \nabla_A A, U\rangle + a\,\alpha\langle A, U\rangle = b\langle \nabla_A A, U\rangle. \end{aligned}$$

Hence, $\langle \nabla_A A, U\rangle = 0$ and the result follows. □

Corollary 7.17. *The integral curves of A are geodesics in M and the three vector fields A, X and Y span an autoparallel distribution on M.*

We will also need the following relation.

Corollary 7.18. *The principal curvatures of M and the functions a and b satisfy the equation*

$$\frac{3(\alpha-\gamma)}{4(\alpha-\beta)}\,a^2 + \frac{3(\alpha-\beta)}{4(\alpha-\gamma)}\,b^2 + (\alpha-\gamma)\beta + (\alpha-\beta)\gamma + \frac{1}{4} = 0.$$

Proof. From Lemma 7.2 we have

$$-\frac{1}{4} = -\frac{1}{4}(a^2+b^2) = -\bar{R}_{XY\xi A} = (\gamma-\alpha)\langle \nabla_X Y, A\rangle + (\alpha-\beta)\langle \nabla_Y X, A\rangle.$$

Plugging the corresponding expressions of $\langle \nabla_X Y, A\rangle$ and $\langle \nabla_Y X, A\rangle$ given by Lemma 7.16 we easily get the result. □

We are now ready to give the following relation among the eigenvalues. This will lead us to two different possibilities.

Lemma 7.19. *Let $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$. Then*

$$\Big(\frac{1}{4} - \alpha\beta\Big)\langle V, V\rangle = 0, \qquad \Big(\frac{1}{4} - \alpha\gamma\Big)\langle W, W\rangle = 0.$$

Proof. Our aim is to apply Lemma 7.7 to the pairs A, V and A, W. To do this, we first need the following intermediate result.

Claim 7.20. *For any $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$ we have $\langle \nabla_V A, \nabla_A V\rangle = \langle \nabla_W A, \nabla_A W\rangle = 0$.*

Lemma 7.3 implies

$$\langle \nabla_A V, A\rangle = \langle \nabla_A V, U\rangle = \langle \nabla_A W, A\rangle = \langle \nabla_A W, U\rangle = 0$$

and Lemma 7.16 gives

$$\langle \nabla_A V, X\rangle = \langle \nabla_A V, Y\rangle = \langle \nabla_A W, X\rangle = \langle \nabla_A W, Y\rangle = 0.$$

Hence we have $\nabla_A V, \nabla_A W \in \Gamma\big((T_\beta \ominus \mathbb{R}X) \oplus (T_\gamma \ominus \mathbb{R}Y)\big)$. On the other hand, Lemma 7.3 gives $\langle \nabla_V A, \tilde{V} \rangle = 0$ for any $\tilde{V} \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and $\langle \nabla_W A, \tilde{W} \rangle = 0$ for any $\tilde{W} \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$. Thus, to get Claim 7.20 it suffices to prove $\langle \nabla_V A, W \rangle = \langle \nabla_W A, V \rangle = 0$ for arbitrary $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$.

Using Lemma 7.15 and the Weingarten formula we get

$$\begin{aligned} 0 &= V\langle X, JW \rangle = \langle \bar{\nabla}_V X, JW \rangle + \langle X, \bar{\nabla}_V JW \rangle \\ &= \langle \nabla_V X, JW \rangle + b\langle A, \nabla_V W \rangle - a\beta \langle V, W \rangle = \langle \nabla_V X, JW \rangle - b\langle \nabla_V A, W \rangle. \end{aligned}$$

This, and similar equations for $0 = V\langle Y, JW \rangle$, $0 = W\langle Y, JV \rangle$ and $0 = W\langle X, JV \rangle$ yield

$$\begin{aligned} \langle \nabla_V A, W \rangle &= \frac{1}{b}\langle \nabla_V X, JW \rangle = -\frac{1}{a}\langle \nabla_V Y, JW \rangle, \\ \langle \nabla_W A, V \rangle &= \frac{1}{b}\langle \nabla_W X, JV \rangle = -\frac{1}{a}\langle \nabla_W Y, JV \rangle. \end{aligned} \tag{7.2}$$

Since T_γ is a real subbundle by Corollary 7.5, we may write $JW = \tilde{U} + \tilde{V}$ with $\tilde{U} \in \Gamma(T_\alpha \ominus \mathbb{R}A)$ and $\tilde{V} \in \Gamma(T_\beta \ominus \mathbb{R}X)$. Using Lemma 7.3 we get

$$\langle \nabla_V X, \tilde{U} \rangle = \frac{a}{2(\alpha - \beta)}\langle V, J\tilde{U} \rangle.$$

Also, since T_β is real by Corollary 7.5, $\langle \tilde{V}, \nabla_V J\xi \rangle = \langle \tilde{V}, JSV \rangle = \beta\langle \tilde{V}, JV \rangle = 0$. Then, the expression for $J\xi$ implies $0 = V\langle \tilde{V}, J\xi \rangle = -a\langle \nabla_V X, \tilde{V} \rangle$. Hence we get

$$\langle \nabla_V X, JW \rangle = \frac{a}{2(\alpha - \beta)}\langle V, J\tilde{U} \rangle.$$

In a similar way, Lemma 7.3 implies $\langle \nabla_V Y, \tilde{V} \rangle = 0$ and

$$0 = V\langle \tilde{U}, J\xi \rangle = -\beta\langle V, J\tilde{U} \rangle - \frac{a^2}{2(\alpha - \beta)}\langle V, J\tilde{U} \rangle - b\langle \nabla_V Y, \tilde{U} \rangle.$$

Thus

$$\langle \nabla_V Y, JW \rangle = -\frac{1}{b}\left(\beta + \frac{a^2}{2(\alpha - \beta)}\right)\langle V, J\tilde{U} \rangle.$$

Then (7.2) implies

$$\langle \nabla_V A, W \rangle = \frac{a}{2b(\alpha - \beta)}\langle V, J\tilde{U} \rangle = \frac{1}{a\,b}\left(\beta + \frac{a^2}{2(\alpha - \beta)}\right)\langle V, J\tilde{U} \rangle,$$

and a simple calculation shows that $\beta\langle V, J\tilde{U} \rangle = 0$.

In a similar way we may write $JV = \hat{U} + \hat{W}$ with $\hat{U} \in \Gamma(T_\alpha \ominus \mathbb{R}A)$ and $\hat{W} \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$ which eventually gives

$$\langle \nabla_W A, V \rangle = -\frac{b}{2a(\alpha - \gamma)}\langle W, J\hat{U} \rangle = -\frac{1}{a\,b}\left(\gamma + \frac{b^2}{2(\alpha - \gamma)}\right)\langle W, J\hat{U} \rangle.$$

As a consequence $\gamma\langle W, J\hat{U}\rangle = 0$.

Since either β or γ is non–zero it follows that $\langle V, J\tilde{U}\rangle = 0$ or $\langle W, J\hat{U}\rangle = 0$. Hence $\langle \nabla_V A, W\rangle = 0$ or $\langle \nabla_W A, V\rangle = 0$. But Lemma 7.2 shows

$$\begin{aligned} 0 &= -\bar{R}_{VW\xi A} = (\gamma-\alpha)\langle \nabla_V W, A\rangle + (\alpha-\beta)\langle \nabla_W V, A\rangle \\ &= (\alpha-\gamma)\langle \nabla_V A, W\rangle - (\alpha-\beta)\langle \nabla_W A, V\rangle. \end{aligned}$$

which implies $\langle \nabla_V A, W\rangle = \langle \nabla_W A, V\rangle = 0$ in both cases. This finishes the proof of Claim 7.20.

Applying Lemma 7.7 for A and $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and also for A and $W \in \Gamma(T_\gamma \ominus \mathbb{R}Y)$ and taking into account Lemmas 7.15, 7.16 and Claim 7.20, we get

$$\Big(\frac{1}{4} - \alpha\,\beta\Big)\langle V, V\rangle = 0 \qquad \text{and} \qquad \Big(\frac{1}{4} - \alpha\,\gamma\Big)\langle W, W\rangle = 0,$$

which is the statement of Lemma 7.19. □

This immediately implies

Corollary 7.21. *$m_\beta = 1$ or $m_\gamma = 1$.*

Proof. On the contrary, if $m_\beta, m_\gamma > 1$, both $T_\beta \ominus \mathbb{R}X$ and $T_\gamma \ominus \mathbb{R}Y$ are non–zero. Then Lemma 7.19 implies $1/4 - \alpha\beta = 1/4 - \alpha\gamma = 0$ and hence $\beta = \gamma$ which is impossible. □

According to the previous corollary we may assume $m_\gamma = 1$, that is, $T_\gamma = \mathbb{R}Y$. We distinguish two cases, $m_\beta > 1$ and $m_\beta = 1$.

Case 1: $m_\beta > 1$. Since $m_\beta > 1$ and T_β is a real subbundle of TM by Corollary 7.5 we have $J(T_\beta \ominus \mathbb{R}X) \subset T_\alpha \ominus \mathbb{R}A$. Also, by using Lemma 7.19 we get $4\,\alpha\,\beta = 1$.

Claim 7.22. *We have $\nabla_V Y = b\,\beta\,JV$ for any $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$.*

Since Y has unit length, $\langle \nabla_V Y, Y\rangle = 0$. Also, Lemma 7.3 implies $\langle \nabla_V Y, X\rangle = \langle \nabla_V Y, \tilde{V}\rangle = 0$ for any $\tilde{V} \in \Gamma(T_\beta \ominus \mathbb{R}X)$. Using the expression for $J\xi$ and Lemma 7.3 we get

$$0 = V\langle A, J\xi\rangle = \langle \bar{\nabla}_V A, aX + bY\rangle + \langle A, \bar{\nabla}_V J\xi\rangle = -b\langle \nabla_V Y, A\rangle$$

and for any $U \in \Gamma(T_\alpha \ominus (\mathbb{R}A \oplus \mathbb{R}JV))$,

$$\begin{aligned} 0 &= V\langle U, J\xi\rangle = \langle \bar{\nabla}_V U, aX + bY\rangle + \langle U, \bar{\nabla}_V J\xi\rangle \\ &= -\beta\langle JU, V\rangle - \frac{a^2}{2(\alpha-\beta)}\langle JU, V\rangle - b\langle \nabla_V Y, U\rangle = -b\langle \nabla_V Y, U\rangle. \end{aligned}$$

Finally using Lemmas 7.3 and 7.15 we get

$$0 = V\langle V, JY\rangle = \langle \bar{\nabla}_V V, aA - b\xi\rangle + \langle V, \bar{\nabla}_V JY\rangle = b\beta - \langle JV, \nabla_V Y\rangle.$$

Altogether this implies Claim 7.22.

Claim 7.23. *With the above assumptions and a suitable orientation of ξ we have $\alpha = \sqrt{3}/6$ and $\beta = \sqrt{3}/2$.*

Let $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ be a unit vector. Then $JV \in \Gamma(T_\alpha \ominus \mathbb{R}A)$. This, Lemma 7.3 and the expression for $J\xi$ imply

$$0 = V\langle JV, J\xi\rangle = \langle \bar{\nabla}_V JV, aX + bY\rangle + \langle JV, \bar{\nabla}_V J\xi\rangle = \frac{a^2}{2(\alpha-\beta)} - b\langle \nabla_V Y, JV\rangle + \beta.$$

Taking into account Claim 7.22 and the last expression we have

$$\langle \nabla_V Y, JV\rangle = \frac{1}{b}\left(\frac{a^2}{2(\alpha-\beta)} + \beta\right) = b\beta,$$

and using $a^2+b^2 = 1$ we obtain after a simple calculation $1+2\beta(\alpha-\beta) = 0$. This, together with $4\alpha\beta = 1$ gives the assertion of Claim 7.23.

Claim 7.24. *With the above assumptions, we have $\gamma \neq -\sqrt{3}/6$ and*

$$a^2 = \frac{10\left(\gamma + \frac{2\sqrt{3}}{15}\right)}{9\left(\gamma + \frac{\sqrt{3}}{6}\right)}, \qquad b^2 = -\frac{\gamma - \frac{\sqrt{3}}{6}}{9\left(\gamma + \frac{\sqrt{3}}{6}\right)}.$$

Inserting the values of α and β in the formula of Lemma 7.18 we get

$$\frac{3(2\gamma\sqrt{3}-1)}{8}a^2 + \frac{3\sqrt{3}}{2(6\gamma-\sqrt{3})}b^2 + \frac{1}{2} - \frac{5\gamma}{2\sqrt{3}} = 0.$$

The above equation and $a^2+b^2 = 1$ provide a linear system whose unknowns are a^2 and b^2. It is not hard to see that this system has solution only when $\gamma \neq -\sqrt{3}/6$ and in this case we obtain the expression given by Claim 7.24.

We now finish the prove of Proposition 7.8 (i). Using Lemma 7.2 and Claim 7.22 we obtain

$$-\frac{b}{4} = -\bar{R}_{YV\xi JV} = (\beta-\alpha)\langle \nabla_Y V, JV\rangle + (\alpha-\gamma)b\beta$$

and hence

$$\langle \nabla_Y V, JV\rangle = \frac{b\,(1+4\beta(\alpha-\gamma))}{4(\alpha-\beta)}.$$

Applying Lemma 7.7 to $V \in \Gamma(T_\beta \ominus \mathbb{R}X)$ and Y and taking into account the above expression, Corollary 7.4 and Claims 7.22, 7.23 and 7.24 we obtain

$$\begin{aligned} 0 &= \frac{1}{4} - \beta\gamma + 2\langle \nabla_V Y, \nabla_Y V\rangle + \frac{1}{4(\beta-\gamma)}\Big(-b\langle \nabla_V Y, JV\rangle + 2b\langle \nabla_Y V, JV\rangle\Big) \\ &= -\frac{2\sqrt{3}\,\gamma\left(\gamma - \frac{\sqrt{3}}{6}\right)}{3\left(\gamma + \frac{\sqrt{3}}{6}\right)}. \end{aligned}$$

Since $\gamma \neq \alpha$, we have $\gamma = 0$ and Proposition 7.8 (i) follows.

Case 2: $m_\beta = 1$. We assume $m_\alpha = 2n-3$, $m_\beta = m_\gamma = 1$. Thus $T_\beta = \mathbb{R}X$ and $T_\gamma = \mathbb{R}Y$. We easily determine a and b from these hypotheses.

Claim 7.25. *With the above notation and assumptions we have*

$$\frac{a^2}{4(\alpha-\beta)} + \frac{b^2}{4(\alpha-\gamma)} + \alpha = 0.$$

Hence, a and b are constant and

$$a^2 = -\frac{(\alpha-\beta)(1+4\alpha^2-4\alpha\gamma)}{\beta-\gamma} \qquad \textit{and} \qquad b^2 = \frac{(\alpha-\gamma)(1+4\alpha^2-4\alpha\beta)}{\beta-\gamma}.$$

Since $n \geq 3$ and $\mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}\xi$ is a complex subbundle of TM by Lemma 7.15, we have that $T_\alpha \ominus \mathbb{R}A$ is complex. Let $U \in \Gamma(T_\alpha \ominus \mathbb{R}A)$ be a unit vector field. Using the expression of $J\xi$, the fact that $JU \in \Gamma(T_\alpha \ominus \mathbb{R}A)$ and Lemma 7.3 we get

$$\begin{aligned} 0 &= U\langle JU, J\xi\rangle = \langle \bar\nabla_U JU, aX + bY\rangle + \langle JU, \bar\nabla_U J\xi\rangle \\ &= a\langle \nabla_U JU, X\rangle + b\langle \nabla_U JU, Y\rangle + \alpha\langle JU, JU\rangle = \frac{a^2}{4(\alpha-\beta)} + \frac{b^2}{4(\alpha-\gamma)} + \alpha. \end{aligned}$$

This equation together with $a^2+b^2=1$ yields a linear system of equations whose unknowns are a^2 and b^2. This system has a unique solution which is the one given by Claim 7.25.

Claim 7.26. *We have the following relation among the principal curvatures*

$$(\beta-\gamma)^2 - (\beta+\gamma-4\alpha)^2 = 1 - 4\alpha^2.$$

Plugging the values of a^2 and b^2 given by Claim 7.25 in the formula of Corollary 7.18 we get

$$-\frac{1}{2}\big(1 + 12\alpha^2 + 4\beta\gamma - 8\alpha\beta - 8\alpha\gamma\big) = 0,$$

which is equivalent to the formula of Claim 7.26.

Claim 7.27. *We have the following relation*

$$(\beta+\gamma)(1+4\alpha^2) - \alpha(1+4\beta^2+4\gamma^2) = 0.$$

Let $U \in \Gamma(T_\alpha \ominus \mathbb{R}A)$. Since X is a unit vector one gets $\langle \nabla_U X, X\rangle = 0$. Using Lemma 7.3 for any $V \in \Gamma(T_\alpha)$, we obtain

$$\langle \nabla_U X, V\rangle = -\frac{a}{4(\alpha-\beta)}\langle JU, V\rangle.$$

We have $\langle X, \bar\nabla_U J\xi\rangle = \alpha\langle X, JU\rangle = 0$. This and the expression for $J\xi$ yield

$$0 = U\langle X, J\xi\rangle = \langle \bar\nabla_U X, aX + bY\rangle + \langle X, \bar\nabla_U J\xi\rangle = b\langle \nabla_U X, Y\rangle.$$

Hence we have

$$\nabla_U X = -\frac{a}{4(\alpha-\beta)} JU.$$

This expression and the one for $\nabla_X X$ given in Lemma 7.16 when plugged in the formula of Lemma 7.7 for U and X yield

$$\begin{aligned} 0 &= \frac{1}{4} - \alpha\beta + 2\langle \nabla_U X, \nabla_X U\rangle - \langle \nabla_U U, \nabla_X X\rangle + \frac{a}{4(\alpha-\beta)}\Big\{2\langle \nabla_X U, JU\rangle - \langle \nabla_U X, JU\rangle\Big\} \\ &= \frac{1}{4} - \alpha\,\beta - \frac{3\,a\,b}{4(\alpha-\beta)}\langle \nabla_U U, A\rangle + \frac{a^2}{16(\alpha-\beta)^2} = 0. \end{aligned}$$

Proceeding in a similar way using Lemma 7.7 for U and Y we obtain

$$\frac{1}{4} - \alpha\,\gamma + \frac{3\,a\,b}{4(\alpha-\gamma)}\langle \nabla_U U, A\rangle + \frac{b^2}{16(\alpha-\gamma)^2} = 0.$$

Cancelling $\langle \nabla_U U, A\rangle$ in the last two equations and using the first equation of Claim 7.26 we get

$$\left(\frac{1}{4} - \alpha\beta\right)(\alpha-\beta) + \left(\frac{1}{4} - \alpha\gamma\right)(\alpha-\gamma) - \frac{\alpha}{4} = 0.$$

Easy calculations lead to the result of Claim 7.27.

We now finish the proof of Proposition 7.8.

If $\alpha = 0$, Claims 7.26 and 7.27 imply $\beta, \gamma \in \{-1/2, 1/2\}$. From now on we assume $\alpha \neq 0$. It is convenient to introduce the following notation. Let $x = \beta - \gamma$ and $y = \beta + \gamma - 4\alpha$. Then, the formulas of Claims 7.26 and 7.27 become

$$x^2 - y^2 = 1 - 4\alpha^2, \qquad x^2 + \left(y - \frac{1-12\alpha^2}{4\alpha}\right)^2 = \frac{1+16\alpha^4}{16\alpha^2}.$$

Obviously, these are the equations of a hyperbola and a circle. It is straightforward to calculate their common points, namely

$$(x,y) = \left(\pm\sqrt{1-3\alpha^2}, -\alpha\right) \qquad \text{and} \qquad (x,y) = \left(\pm\frac{1}{4\alpha}, \frac{1-8\alpha^2}{4\alpha}\right),$$

where the first possibility arises only if $3\alpha^2 \le 1$. Assume, without loss of generality, that $\beta < \gamma$. Since $\alpha \neq \beta, \gamma$, this eventually implies

$$\beta = \frac{1}{2}\left(3\alpha - \sqrt{1-3\alpha^2}\right) \qquad \text{and} \qquad \gamma = \frac{1}{2}\left(3\alpha + \sqrt{1-3\alpha^2}\right),$$

where $|\alpha| \le 1/\sqrt{3}$. If $|\alpha| = 1/2$ or $|\alpha| = 1/\sqrt{3}$ we easily see that the three principal curvatures cannot be different. Suppose $1/2 < |\alpha| < 1/\sqrt{3}$. Using the expression for β and γ we have just obtained, the first equation of Claim 7.25 becomes

$$\frac{a^2}{2\alpha\left(\alpha - \sqrt{1-3\alpha^2}\right)} + \frac{b^2}{2\alpha\left(\alpha + \sqrt{1-3\alpha^2}\right)} = 1,$$

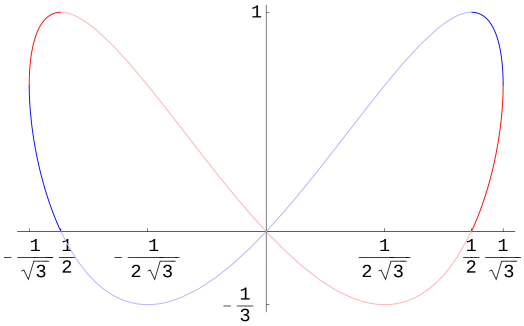

Figure 7.1: The axes of the ellipse $\frac{a^2}{2\alpha\left(\alpha-\sqrt{1-3\alpha^2}\right)}+\frac{b^2}{2\alpha\left(\alpha+\sqrt{1-3\alpha^2}\right)}=1$ as a function of α.

If $1/2<|\alpha|<1/\sqrt{3}$, it is just a matter of elementary calculus to show the inequalities $0<2\alpha\left(\alpha-\sqrt{1-3\alpha^2}\right)<1$ and $0<2\alpha\left(\alpha+\sqrt{1-3\alpha^2}\right)<1$ (see Figure 7.1, where we plot the above two functions as functions of α). This proves that the above equation is the equation of an ellipse centered at the origin with axes of length less than 1. Obviously, such an ellipse has no points of intersection with the circle $a^2+b^2=1$, which is a contradiction. Hence $|\alpha|<1/2$. This finishes the proof of Proposition 7.8.

7.2.2 Equidistant hypersurfaces and rigidity

In this subsection we finish the proof of Theorem 7.1. In order to achieve this, we study the equidistant hypersurfaces of a real hypersurface with constant principal curvatures. We need some facts and notation of Jacobi vector field theory. We refer to Section 4.1 where the main results and conventions are stated.

Let M be a real hypersurface with constant principal curvatures in $\mathbb{C}H^n$. Let ξ be a local unit normal vector field on M. Let c_p be the geodesic $c_p(t)=\exp_p(t\xi_p)$. Fix $r\in\mathbb{R}$, $r\neq 0$. As in Subsection 4.1 we define

$$\begin{array}{rccl}\Phi_r: & M & \longrightarrow & \mathbb{C}H^n \\ & p & \mapsto & \Phi_r(p)=\exp_p(r\xi_p).\end{array}$$

The vector field η_r along Φ_r is defined by $\eta_r(p)=c_p'(r)$. The Jacobi equation in $\mathbb{C}H^n$ reads

$$\zeta_v''-\frac{1}{4}\left(\zeta_v+3\langle\zeta_v,Jc_p'\rangle Jc_p'\right)=0.$$

We recall that $\zeta_v(r)=\Phi_{r*}(v)$ and $\zeta_v'(r)=\bar{\nabla}_v\eta_r$.

Lemma 7.28. *Denote by B_v the parallel translation of $v\in T_\lambda(p)$ along the geodesic c_p and by ζ_v the Jacobi vector field along c_p such that $\zeta_v(0)=v$ and $\zeta_v'(0)=Sv=\lambda v$. If $J\xi$ and v are not collinear then*

$$\zeta_v(r)=f_\lambda(r)B_v(r)+\langle v,J\xi\rangle g_\lambda(r)Jc_p'(r),$$

where

$$f_\lambda(r) = \cosh\frac{r}{2} + 2\lambda\sinh\frac{r}{2} \qquad \textit{and} \qquad g_\lambda(r) = \Big(\cosh\frac{r}{2} - 1\Big)\Big(1 + 2\cosh\frac{r}{2} + 2\lambda\sinh\frac{r}{2}\Big).$$

Now we assume that M has three constant principal curvatures. If M is a Hopf hypersurface, Theorem 6.5 asserts that M is holomorphically congruent to an open part of a tube around a totally geodesic $\mathbb{C}H^k$ for some $k \in \{0, \ldots, n-1\}$, to a tube around a totally geodesic $\mathbb{R}H^n$ or to a horosphere in $\mathbb{C}H^n$. We assume that M is not a Hopf hypersurface. Then M satisfies one of the two possibilities described in Proposition 7.8. We will see that in case (i) of Proposition 7.8 there exists a particular distance r at which the map Φ_r has constant rank $2n - m_\beta$, which means that the image of Φ_r forms locally a submanifold of codimension m_β. In case (ii) of Proposition 7.8 there exists a particular distance r at which the map Φ_r has constant rank $2n-1$ and the image is locally a minimal real hypersurface with constant principal curvatures. We then use the equation $\zeta_v'(r) = \bar\nabla_v \eta_r$ to obtain some information about the second fundamental form of these submanifolds. We continue using the notation introduced in Section 7.2.1.

Case (i): $m_\beta > 1$

We recall the situation of Proposition 7.8 (i). Throughout this section M is a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures α, β and γ. The Hopf vector field $J\xi$ of M has non–trivial projection onto T_β and T_γ. We write $J\xi = a\,X + b\,Y$, where $X \in \Gamma(T_\beta)$ and $Y \in \Gamma(T_\gamma)$ are unit vectors, $a, b > 0$ and $a^2 + b^2 = 1$. There exists a unit vector field $A \in \Gamma(T_\alpha)$ such that $JA = b\,X - a\,Y$. The subbundle $\mathbb{R}X \oplus \mathbb{R}Y$ is real and the subbundle $\mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}\xi$ is complex. We have $m_\alpha = 2n - m_\beta - 2$, $m_\beta > 1$ and $m_\gamma = 1$. The eigenvalues are $\alpha = \sqrt{3}/6$, $\beta = \sqrt{3}/2$ and $\gamma = 0$. Moreover $a = 2\sqrt{2}/3$, $b = 1/3$, the subbundle $T_\beta \ominus \mathbb{R}X$ is real and $J(T_\beta \ominus \mathbb{R}X) \subset T_\alpha \ominus \mathbb{R}A$.

Fix $r = -\log(2 + \sqrt{3})$. We will see that the map Φ_r is singular and that M has a focal manifold at distance r. Let $p \in M$ and $v \in T_pM$. As usual we denote by c_p the geodesic $c_p(t) = \exp_p(t\xi_p)$. We write $v = v_\alpha + v_\beta + v_\gamma$ with $v_\lambda \in T_\lambda(p)$ for all $\lambda \in \{\alpha, \beta, \gamma\}$. Since $\bar\nabla J = 0$ we have $Jc_p'(r) = aB_{X_p}(r) + bB_{Y_p}(r)$. Using Lemma 7.28 we get

$$\begin{aligned}
\Phi_{r*p}(v) &= \zeta_{v_\alpha}(r) + \zeta_{v_\beta}(r) + \zeta_{v_\gamma}(r) \\
&= \frac{\sqrt{6}}{3}B_{v_\alpha}(r) + \frac{1}{2}\langle v_\beta, J\xi_p\rangle Jc_p'(r) + \frac{\sqrt{6}}{2}B_{v_\gamma}(r) + \frac{4-\sqrt{6}}{2}\langle v_\gamma, J\xi_p\rangle Jc_p'(r) \\
&= \frac{\sqrt{6}}{3}B_{v_\alpha}(r) + \frac{1}{9}\left\{4\langle v_\beta, X_p\rangle + \left(4\sqrt{2} - 2\sqrt{3}\right)\langle v_\gamma, Y_p\rangle\right\} B_{X_p}(r) \\
&\quad + \frac{1}{9}\left\{\sqrt{2}\langle v_\beta, X_p\rangle + \left(4\sqrt{6} + 2\right)\langle v_\gamma, Y_p\rangle\right\} B_{Y_p}(r).
\end{aligned}$$

Straightforward calculations show that $\Phi_{r*}(v) = 0$ if and only if $v_\alpha = 0$ and $\langle v_\beta, X\rangle = \langle v_\gamma, Y\rangle = 0$. Consequently, $\ker\Phi_{r*p} = T_\beta(p) \ominus \mathbb{R}X_p$ and its dimension is $m_\beta - 1$. This implies that Φ_{r*} is singular and that the rank of Φ_r is constant. Then for every point in M

there exists an open neighborhood $\mathcal{V}$ such that $\mathcal{W} = \Phi_r(\mathcal{V})$ is an embedded submanifold of $\mathbb{C}H^n$ and $\Phi_r : \mathcal{V} \to \mathcal{W}$ is a submersion. Let $p \in \mathcal{V}$ and $q = \Phi_r(p) \in \mathcal{W}$. The above expression for the differential of Φ_r shows that the tangent space $T_q\mathcal{W}$ of $\mathcal{W}$, which is the image of $T_\alpha(p) \oplus \mathbb{R}X_p \oplus \mathbb{R}Y_p$ by the differential of Φ_r, coincides with the parallel translation along c_p of $T_\alpha(p) \oplus \mathbb{R}X_p \oplus \mathbb{R}Y_p$. Hence, the normal space $T_q^\perp\mathcal{W}$ is obtained by parallel translation of $\mathbb{R}\xi_p \oplus (T_\beta(p) \ominus \mathbb{R}X_p)$ along c_p from p to q. In particular, $\mathcal{W}$ has dimension $2n - m_\beta$ and $\mathcal{W}$ has a totally real normal vector bundle of dimension m_β.

Clearly, $\eta_r(p) = B_{\xi_p}(r)$ is a unit normal vector of $\mathcal{W}$ at q. The shape operator $S(r)$ of $\mathcal{W}$ in the direction of $\eta_r(p)$ is given by the equation $S(r)_{\eta_r(p)}\zeta_v(r) = (\zeta_v'(r))^\top$ for all $v \in T_\alpha(p) \oplus \mathbb{R}X_p \oplus \mathbb{R}Y_p$.

If $v \in T_\alpha(p)$, we have $\zeta_v(r) = (\sqrt{6}/3)B_v(r)$ and $\zeta_v'(r) = 0$ by Lemma 7.28. Hence,

$$S(r)_{\eta_r(p)}B_v(r) = 0 \quad \text{for all } v \in T_\alpha(p).$$

Also, Lemma 7.28 implies that

$$\begin{aligned}
\zeta_X(r) &= \frac{4}{9}B_X(r) + \frac{\sqrt{2}}{9}B_Y(r), & \zeta_X'(r) &= \frac{\sqrt{2}}{18}B_X(r) - \frac{2}{9}B_Y(r),\\
\zeta_Y(r) &= \frac{4\sqrt{2}-2\sqrt{3}}{9}B_X(r) - \frac{2+4\sqrt{6}}{9}B_Y(r), & \zeta_Y'(r) &= \frac{1-2\sqrt{6}}{9}B_X(r) - \frac{2\sqrt{2}+\sqrt{3}}{9}B_Y(r).
\end{aligned}$$

Therefore, $S(r)_{\eta_r(p)}$ leaves $\mathbb{R}B_{X_p}(r) \oplus \mathbb{R}B_{Y_p}(r)$ invariant and has the matrix representation

$$\frac{1}{18}\begin{pmatrix} -4\sqrt{2} & 7 \\ 7 & 4\sqrt{2} \end{pmatrix},$$

with respect to the basis $\{B_{X_p}(r), B_{Y_p}(r)\}$. We define

$$Z_p = JA_p = \frac{1}{3}X_p - \frac{2\sqrt{2}}{3}Y_p\,.$$

Using the above matrix representation we immediately get

$$\begin{aligned}
S(r)_{\eta_r(p)}J\eta_r(p) &= S(r)_{\eta_r(p)}\left(\frac{2\sqrt{2}}{3}B_{X_p}(r) + \frac{1}{3}B_{Y_p}(r)\right) = -\frac{1}{2}B_{Z_p}(r),\\
S(r)_{\eta_r(p)}B_{Z_p}(r) &= S(r)_{\eta_r(p)}\left(\frac{1}{3}B_{X_p}(r) - \frac{2\sqrt{2}}{3}B_{Y_p}(r)\right) = -\frac{1}{2}J\eta_r(p).
\end{aligned}$$

Since $J\big(T_q^\perp\mathcal{W} \ominus \mathbb{R}\eta_r(p)\big) \subset J\big(B_{T_\beta(p)\ominus\mathbb{R}X_p}(r)\big) \subset B_{T_\alpha(p)}(r)$, the fact that $S(r)_{\eta_r(p)|B_{T_\alpha(p)}(r)} = 0$ and the linearity of $S(r)_{\eta_r(p)}$ give

$$S(r)_{\eta_r(p)}J\tilde{\eta} = -\frac{1}{2}\langle \eta_r(p), \tilde{\eta}\rangle B_{Z_p}(r) \quad \text{for all } p \in \mathcal{V},\ \tilde{\eta} \in T_{\Phi_r(p)}^\perp\mathcal{W}.$$

Using the symmetry of the second fundamental form of $\mathcal{W}$, II_r, and the equation $\bar{\nabla}J = 0$ we get $\langle S(r)_{\tilde{\eta}} J\eta_r, U\rangle = \langle II_r(U, J\eta_r), \tilde{\eta}\rangle = \langle \bar{\nabla}_U J\eta_r, \tilde{\eta}\rangle = \langle \bar{\nabla}_U J\tilde{\eta}, \eta_r\rangle = \langle S(r)_{\eta_r} J\tilde{\eta}, U\rangle$ for any $U \in \Gamma(T\mathcal{W})$ and $\tilde{\eta} \in \Gamma(T^\perp\mathcal{W})$. Hence

$$S(r)_{\tilde{\eta}} J\eta_r(p) = S(r)_{\eta_r(p)} J\tilde{\eta},$$

for all $\tilde{\eta} \in T_q^\perp\mathcal{W}$. Now let χ be a curve in $\Phi_r^{-1}(\{q\}) \cap \mathcal{V}$ with $\chi(0) = p$. Since $\eta_r(p)$ and $\tilde{\eta}(\chi(t)) = \eta_r(\chi(t)) - \langle \eta_r(\chi(t)), \eta_r(p)\rangle\eta_r(p)$ are perpendicular, the linearity of $\eta \mapsto S(r)_\eta$ implies

$$\begin{aligned}
0 &= S(r)_{\eta_r(p)} J\tilde{\eta}(\chi(t)) \;=\; S(r)_{\tilde{\eta}(\chi(t))} J\eta_r(p) \\
&= S(r)_{\eta_r(\chi(t))} J\eta_r(p) - \langle \eta_r(\chi(t)), \eta_r(p) S(r)_{\eta_r(p)} J\eta_r(p) \\
&= -\frac{1}{2}\langle \eta_r(\chi(t)), \eta_r(p)\rangle B_{Z_{\chi(t)}}(r) + \frac{1}{2}\langle \eta_r(\chi(t)), \eta_r(p)\rangle B_{Z_p}(r) \\
&= -\frac{1}{2}\langle \eta_r(\chi(t)), \eta_r(p)\rangle \Big(B_{Z_{\chi(t)}}(r) - B_{Z_p}(r) \Big).
\end{aligned}$$

This shows that the map $\tilde{p} \mapsto B_{Z_{\tilde{p}}}(r)$ is of constant value z_q in the connected component $\mathcal{V}_0$ of $\Phi_r^1(\{q\}) \cap \mathcal{V}$ containing p.

For all $v \in T_\beta(p) \ominus \mathbb{R}X_p$ we have

$$\bar{\nabla}_v \eta_r = \zeta_v'(r) = \frac{\sqrt{2}}{2} B_v(r),$$

which implies that η_r is a local diffeomorphism from $\mathcal{V}_0$ onto the unit sphere in $T_q^\perp\mathcal{W}$. Thus $\eta_r(\mathcal{V}_0)$ is an open subset of the unit sphere in $T_q^\perp\mathcal{W}$. Since $S(r)_\eta$ depends analytically on $\eta \in T_q^\perp\mathcal{W}$, we conclude that

$$S(r)_\eta J\eta = -\frac{1}{2}\, z_q, \qquad S(r)_\eta\, z_q = -\frac{1}{2}\, J\eta, \qquad S(r)_\eta\, v = 0,$$

for all $\eta \in T_q^\perp\mathcal{W}$ and $v \in T_q\mathcal{W} \ominus \big(J(T_q^\perp\mathcal{W} \ominus \mathbb{R}\eta) \oplus \mathbb{R}z\big)$.

Therefore, the second fundamental form II_r of $\mathcal{W}$ at q is given by the trivial bilinear extension of $II_r(z, J\eta) = -(1/2)\eta$ for all $\eta \in T^\perp\mathcal{W}$. The construction of z shows that it depends smoothly on the point $q \in \mathcal{W}$. Hence, the second fundamental form of $\mathcal{W}$ has the form of that of Theorem 6.16. Then, $\mathcal{W}$ is holomorphically congruent to an open part of the ruled minimal submanifold W^{2n-m_β}. Altogether this means that M lies in a tube of radius $r = \log(2+\sqrt{3})$ around a ruled minimal submanifold holomorphically congruent to W^{2n-m_β}. We point out here that the unit normal vector ξ of M is outward pointing with respect to the focal submanifold. This finally implies that M is holomorphically congruent to an open part of the tube of radius $r = \log(2+\sqrt{3})$ around W^{2n-m_β}. This corresponds to case (c) of Theorem 7.1.

Case (ii): $m_\beta = 1$

We assume the notation of Proposition 7.8 (ii). Thus, M is a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures α, β and γ. The Hopf vector field $J\xi$ of M has non–trivial projection onto T_β and T_γ and $m_\beta = m_\gamma = 1$. We write $J\xi = a\,X + b\,Y$, where $X \in \Gamma(T_\beta)$ and $Y \in \Gamma(T_\gamma)$ are unit vectors, $a, b > 0$ and $a^2 + b^2 = 1$. There exists a unit vector field $A \in \Gamma(T_\alpha)$ such that $JA = b\,X - a\,Y$. The subbundle $T_\beta \oplus T_\gamma$ is real and the subbundle $\mathbb{R}A \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}\xi$ is complex. The eigenvalues satisfy

$$|\alpha| < \frac{1}{2}, \qquad \beta = \frac{1}{2}\left(3\alpha - \sqrt{1 - 3\alpha^2}\right), \qquad \gamma = \frac{1}{2}\left(3\alpha + \sqrt{1 - 3\alpha^2}\right),$$

and a and b are constant. In particular,

$$a^2 = -\frac{(\alpha - \beta)(1 + 4\alpha^2 - 4\alpha\gamma)}{\beta - \gamma}, \qquad b^2 = \frac{(\alpha - \gamma)(1 + 4\alpha^2 - 4\alpha\beta)}{\beta - \gamma}.$$

From now on we assume $\alpha \neq 0$. Otherwise we would have $\alpha = 0$, $\beta = -1/2$ and $\gamma = 1/2$ and the conditions of Theorem 6.8 would be satisfied for this hypersurface M. We will show that there exists a certain r such that the equidistant hypersurface of M at distance r is minimal and its geometry is that of W^{2n-1} in Theorem 6.8. First, as $|\alpha| < 1/2$ there exists $r \neq 0$ such that $2\alpha = -\tanh(r/2)$.

Let $v \in T_pM$. We write $v = v_\alpha + v_\beta + v_\gamma$, where $v_\lambda \in T_\lambda(p)$ for all $\lambda \in \{\alpha, \beta, \gamma\}$. Using Lemma 7.28 we get the differential of Φ_r as follows

$$\begin{aligned} \Phi_{r*}(v) &= \zeta_{v_\alpha}(r) + \zeta_{v_\beta}(r) + \zeta_{v_\gamma}(r) \\ &= f_\alpha(r) B_{v_\alpha}(r) + \Big(\langle v_\beta, X_p\rangle f_\beta(r) + \langle v_\beta, X_p\rangle a^2 g_\beta(r) + \langle v_\gamma, Y_p\rangle a\,b\,g_\gamma(r)\Big) B_{X_p}(r) \\ &\qquad + \Big(\langle v_\gamma, Y_p\rangle f_\gamma(r) + \langle v_\beta, X_p\rangle a\,b\,g_\beta(r) + \langle v_\gamma, Y_p\rangle b^2 g_\gamma(r)\Big) B_{Y_p}(r). \end{aligned}$$

Then, we have

$$\Phi_{r*}v = f_\alpha(r) B_v(r) \quad \text{for all } v \in T_\alpha(p) \qquad \text{and} \qquad \begin{pmatrix} \Phi_{r*}X_p \\ \Phi_{r*}Y_p \end{pmatrix} = D(r) \begin{pmatrix} B_{X_p}(r) \\ B_{Y_p}(r) \end{pmatrix}$$

where D is the endomorphism whose matrix representation is

$$D(t) = \begin{pmatrix} f_\beta(t) + a^2 g_\beta(t) & a\,b\,g_\beta(t) \\ a\,b\,g_\gamma(t) & f_\gamma(t) + b^2 g_\gamma(t) \end{pmatrix}.$$

We have $f_\alpha(r) = \cosh(r/2) + 2\alpha\sinh(r/2) = \operatorname{sech}(r/2) \neq 0$. Retrieving the expression of f_λ and g_λ from Lemma 7.28 and the known expressions of β, γ, a and b in terms of α we get after some straightforward calculations $\det D(r) = (\cosh(r/2) + 2\alpha\sinh(r/2))^3 = \operatorname{sech}^3(r/2) \neq 0$. Therefore, Φ_r has maximum rank $2n - 1$ everywhere. Hence, for every point in M there exists an open neighborhood $\mathcal{V}$ such that $\mathcal{W} = \Phi_r(\mathcal{V})$ is an embedded real hypersurface of $\mathbb{C}H^n$ and $\Phi_r : \mathcal{V} \to \mathcal{W}$ is a diffeomorphism. Let $p \in \mathcal{V}$ and $q = \Phi_r(p) \in \mathcal{W}$. Clearly, the tangent space $T_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $T_p\mathcal{V}$ along the geodesic c_p and $\eta_r(p)$ is a unit normal vector of $\mathcal{W}$ at q.

Let us denote by $S(r)$ the shape operator of $\mathcal{W}$ with respect to the unit normal $\eta_r(p) = c_p'(r)$, which is determined by the equation $S(r)\zeta_v(r) = \zeta_v'(r)$ for any $v \in TM$.

We easily get $f_\alpha'(r) = 0$. This implies $\zeta_v'(r) = 0$ for all $v \in T_\alpha(p)$. Hence

$$S(r)B_v(r) = 0 \quad \text{for all } v \in T_\alpha(p).$$

On the other hand, using the notation of Section 4.1 we have

$$\begin{pmatrix} S(r)B_{X_p}(r) \\ S(r)B_{Y_p}(r) \end{pmatrix} = D'(r)D(r) \begin{pmatrix} B_{X_p}(r) \\ B_{Y_p}(r) \end{pmatrix}.$$

A lengthly but straightforward calculation shows that $\det\bigl(D'(r)\bigr) = -(1/4)\operatorname{sech}^3(r/2)$ and $\bigl(\det D\bigr)'(r) = 0$. As a consequence we have

$$\det\bigl(D'(r)D(r)^{-1}\bigr) = \frac{\det D'(r)}{\det D(r)} = -\frac{1}{4} \quad \text{and} \quad \operatorname{tr}\bigl(D'(r)D(r)^{-1}\bigr) = -\frac{\bigl(\det D\bigr)'(r)}{\det D(r)} = 0.$$

This implies that the eigenvalues of $D'(r)D(r)^{-1}$ are $-1/2$ and $1/2$.

Altogether this means that $\mathcal{W}$ has three distinct principal curvatures 0, $-1/2$ and $1/2$ with corresponding multiplicities $2n-3$, 1 and 1. It follows from Theorem 6.8 that $\mathcal{W}$ is holomorphically congruent to an open part of the ruled real hypersurface W^{2n-1}. From this we eventually conclude that M is holomorphically congruent to an open part of an equidistant hypersurface to W^{2n-1}, where the distance r is given by the equation $2\alpha = -\tanh(r/2)$.

This finishes the proof of Theorem 7.1.

Open problems

We are interested in the following questions.

- Classification of real hypersurfaces with four or five distinct constant principal curvatures. This problem seems to be quite difficult. No examples of real hypersurfaces with constant principal curvatures in the complex hyperbolic space are known which are not open parts of homogeneous real hypersurfaces. Thus, one may expect examples (v) and (vi) of Theorem 6.4 to exhaust all the possibilities in this classification. However, there is no good analog of Lemmas 7.9 and 7.15 for four or five principal curvatures. This provokes that using our approach implies handling several different possibilities separately leading to long and tedious calculations.

- Are there any real hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures which are not an open part of a homogeneous real hypersurface? If there exists such an example it must have at least four distinct principal curvatures. Hence, the previous problem could be interesting if its proof led to a non–homogeneous example.

- The classification of real hypersurfaces in $\mathbb{C}H^2$ with 3 distinct constant principal curvatures remains open. We use the fact $n \geq 3$ in a few places and there is no straightforward generalization of our arguments to include the case $n = 2$.

See [103] for a survey on real hypersurfaces of complex projective and hyperbolic spaces where a wider list of open problems is given.

Resumo en galego

Ó estudiarmos as propiedades xeométricas dunha variedade semi–riemanniana, o punto de partida a miúdo provén da investigación de invariantes da estructura métrica. Entre tales invariantes, o tensor de curvatura é probablemente o máis natural. Segundo a opinión de R. Osserman [109],

> A noción de curvatura é un dos aspectos centrais da xeometría diferencial; pódese argumentar que é o central, distinguindo o núcleo xeométrico da materia daqueles aspectos que son analíticos, alxébricos ou topolóxicos. Nas palabras de M. Berger, a curvatura é o invariante riemanniano número un e o máis natural. Gauss e Riemann vírono instantaneamente.

A curvatura, non obstante, pode ser estudiada desde varios puntos de vista. En primeiro lugar, un problema esencial da xeometría diferencial é relacionar propiedades do tensor de curvatura coa xeometría subxacente da variedade. Outro problema importante é considerar diferentes tipos de obxectos naturalmente asociados ó tensor métrico e relacionar a curvatura da variedade coas propiedades destas construccións naturais.

Cando se trata con obxectos complicados como o tensor de curvatura, é interesante descompoñelo nos seus constituíntes elementais. A miúdo estas partes máis simples dan unha versión simplificada e unha visión máis profunda do problema. A Parte I desta memoria céntrase no estudio da curvatura desde un punto de vista alxébrico. No Capítulo 2 amosamos que o tensor de curvatura pode ser descomposto en termos dalgúns tensores de curvatura alxébricos máis simples. Isto é de especial importancia cando se consideran problemas nos que se pretende obter información xeométrica a partir de propiedades alxébricas de operadores asociados á curvatura. Entre todos estes operadores estamos especialmente interesados no operador de Jacobi, que codifica información xeométrica importante e que ten propiedades que influencian enormemente a xeometría subxacente da variedade. Así, entender o operador de Jacobi dunha variedade semi–riemanniana permítenos caracterizar a xeometría da variedade en moitos casos. O Capítulo 3 desta tese está adicado á investigación do operador de Jacobi en relación coa chamada conxectura de Osserman. Neste capítulo centrámonos no problema de Osserman en dimensión catro. O noso obxectivo primordial é amosar a existencia de métricas de Osserman que teñen operadores de Jacobi non nilpotentes e non diagonalizables. Isto contesta negativamente unha conxetura de non existencia de tales variedades. Ademais, dáse tamén unha descripción local destas métricas.

Tal e como se dixo anteriormente, outro problema interesante relacionado coa curvatura é esclarecer o noso entendemento dunha variedade por medio da investigación da relación existente entre a curvatura da propia variedade e as curvaturas de obxectos xeométricos naturalemente asociados á súa estructura métrica. Exemplos destas estructuras son as esferas xeodésicas, os discos xeodésicos e os tubos arredor de subvariedades significativas. A Parte II desta tese está adicada ó estudio dalgúns dos obxectos previamente mencionados. En particular investigamos os invariantes escalares da curvatura de esferas xeodésicas no Capítulo 4. Os invariantes escalares da curvatura teñen grande importancia e certas xeometrías poden ser caracterizadas en termos destas funcións. Consideramos a estes en relación coas esferas xeodésicas. Neste capítulo integramos os invariantes escalares da curvatura en esferas xeodésicas e discos obtendo os primeiros termos nos desenvolvementos en serie de potencias como función do radio. Isto dá lugar a algunhas caracterizacións de espacios homoxéneos dous puntos entre todas as variedades riemannianas con holonomía adaptada.

Inspirados pola construcción de discos xeodésicos en xeometría riemanniana, definimos as esferas celestes xeodésicas no contexto lorentziano. Resulta ser que esta familia de obxectos está adaptada á consideración de resultados de comparación de volume no contexto lorentziano, o cal sufría dunha falta de construccións análogas ás esferas xeodésicas e ós tubos en xeometría riemanniana. O Capítulo 5 adícase á investigación de propiedades de volume de esferas celestes xeodésicas así como ás súas curvaturas escalares totais. Isto permite caracterizar as variedades lorentzianas isotrópicas.

As esferas xeodésicas e os tubos son dalgún xeito os conxuntos de nivel da función de distancia riemanniana e por tanto están estreitamente vinculados á estructura métrica. Outros obxectos en variedades riemannianas que están relacionados coa estructura métrica son aquelas subvariedades invariantes baixo as isometrías da variedade ambiente. As órbitas de accións de cohomoxeneidade un son exemplos desta situación. Ademais, unha órbita principal dunha acción de cohomoxeneidade un é xeometricamente un tubo arredor dunha órbita singular desa acción. Isto involucra de novo a función distancia riemanniana e o operador de Jacobi que é a ferramenta principal para o cálculo da xeometría das esferas xeodésicas e dos tubos. A xeometría das órbitas das accións de cohomoxeneidade un é máis interesante desde o punto de vista extrínseco. Así, é a segunda forma fundamental a que estudiamos neste caso.

A Parte III deste traballo esta adicada á investigación de hipersuperficies reais con curvaturas principais constantes no espacio hiperbólico complexo. As órbitas de accións de cohomoxeneidade un son os principais candidatos para este tipo de hipersuperficies e son os únicos exemplos coñecidos ata o de agora. No Capítulo 6 estudiamos o operador de configuración das órbitas de accións de cohomoxeneidade un no espacio hiperbólico complexo. Empregando este estudio damos no Capítulo 7 unha clasificación completa das hipersuperficies reais con tres curvaturas principais constantes.

Parte I. Consecuencias xeométricas de propiedades alxébricas do tensor de curvatura

O espacio dos tensores de curvatura alxébricos dun espacio vectorial n–dimensional é un espacio vectorial $\mathcal{R}(V)$ de dimension $n^2(n^2-1)/12$, o que o fai moi difícil de manipular. Por tanto, a investigación centrouse en tentar atopar bases axeitadas ou conxuntos de xeneradores que permitisen simplificacións. Un exemplo típico é a base de Singer–Thorpe en dimensión catro.

Recentemente, o traballo de B. Fiedler [59] e P. Gikey [68] amosou a existencia de bos conxuntos de xeneradores de $\mathcal{R}(V)$ construidos a partir de formas bilineares simétricas e antisimétricas, o cal parece útil para entender algunhas condicións de curvatura. O noso método para atacar o problema, baseado no emprego do teorema de embebemento de Nash e na posibilidade de realizar xeométricamente calquera tensor de curvatura alxébrico, ten dúas vantaxes. A primeira é que nos permite obter unha estimación máis fina (aínda que non óptima) do número de xeneradores de $\mathcal{R}(V)$. A segunda é que amosa que cada tensor de curvatura alxébrico pode ser visto desde un punto de vista extrínseco como a segunda forma fundamental dun embebemento axeitado. Estas discusións son levadas a cabo no Capítulo 2.

Outro propósito desta parte é estudiar a influencia na xeometría da variedade de propiedades alxébricas de operadores naturais asociados á curvatura. De xeito máis preciso, adicamos a nosa atención á investigación do operador de Jacobi centrándonos na estructura de métricas de Osserman de dimensión catro. Recordamos que unha variedade semi–riemanniana se di de Osserman se os autovalores do operador de Jacobi son independentes da dirección e do punto base. Dado que as isometrías locais dun espacio isotrópico actúan transitivamente nos fibrados pseudo–esféricos unitarios, está claro que calquera espacio isotrópico é de Osserman. Non poden existir máis exemplos nos casos riemanniano (dimensión distinta de 16) nin lorentziano, pero existen métricas non simétricas e incluso non localmente homoxéneas en calquera signatura (p, q) con $p, q \geq 2$.

As métricas de Osserman de dimensión catro teñen particular interese. Primeiro, dimensión catro é a primeira dimensión non trivial considerada na investigación do problema de Osserman (nótese que calquera métrica de Osserman é Einstein e por tanto de curvatura seccional constante en dimensión 2 e 3), e ademais, catro é a dimensión máis pequena que soporta métricas non lorentzianas de signatura neutral, onde as primeiras métricas non simétricas de Osserman foron descubertas.

Debido ás identidades da curvatura, para calquera vector non nulo $x \in TM$ o operador de Jacobi é un operador autoadxunto en $x^\perp$, que ten unha métrica inducida lorentziana no caso de signatura $(2, 2)$. As métricas de Osserman con operador de Jacobi diagonalizable foron caracterizadas por N. Blažić, N. Bokan e Z. Rakić [21], quen tamén amosaron a non existencia en dimensión catro de métricas de Osserman con operadores de Jacobi que teñen autovalores complexos. A signatura lorentziana de $x^\perp$, soporta, non obstante, outras dúas posibilidades correspondentes a raíces dobles e triples do polinomio mínimo do operador de Jacobi. O feito de que todos os exemplos coñecidos nesas situación teñen operador de Jacobi nilpotente e que as métricas de Osserman simétricas en dimensión catro teñen operador de Jacobi diagonalizable ou nilpotente en dous pasos, motivaron a conxectura de que as métricas de Osserman que teñen operadores de Jacobi non diagonlizables deben ter operadores de Jacobi nilpotentes. O noso propósito no Capítulo 3 é responder á anterior conxectura de forma negativa amosando exemplos explícitos de métricas de Osserman con operadores de Jacobi que non son nin diagonalizables nin nilpotentes. Finalmente, unha descripción completa de tales métricas é dada na Sección 3.3.

Parte II. Invariantes da curvatura de esferas xeodésicas e esferas celestes xeodésicas

Para estudiar a xeometría dunha variedade de Riemann é a miúdo útil considerar obxectos naturalmente asociados á estructura métrica de M. Estes poden ser hipersuperficies especiais tales como esferas xeodésicas e tubos arredor de certas subvariedades, espacios fibrados con M como base ou familias de transformacións reflectindo as propiedades de simetría de M [128]. Nesta parte da tese centrámonos no estudio das esferas xeodésicas e da súa curvatura en relación coa curvatura da variedade ambiente. De feito, a existencia dunha relación entre a curvatura dunha variedade riemanniana e o volume das súas esferas xeodésicas levou a algúns autores a establecer a seguinte cuestión: "¿Ata que punto está a curvatura ou a xeometría dunha variedade riemanniana influenciada, ou incluso determinada, polas propiedades de certas familias de obxectos xeométricos naturalmente definidos sobre M?". Este problema semella bastante difícil de manipular con tanta xeneralidade. Sen embargo, cando un considera variedades cun alto grao de simetría (por exemplo os espacios homoxéneos dous puntos), estes obxectos xeométricos teñen boas propiedades e espérase obter caracterizacións de tales espacios por medio desas propiedades. Comparando unha variedade riemanniana cun espacio modelo tal como un espacio homoxéneo dous puntos obtemos unha idea da súa xeometría. Por tanto, entendendo a xeometría de espacios cun alto grao de simetría e por que as súas propiedades son características deles, conseguimos unha mellor visión da xeometría dunha variedade riemanniana.

Dado que as esferas xeodésicas son subvariedades compactas, ten siso calcular o seu volume. A. Gray e L. Vanhecke [83] calcularon os primeiros termos no desenvolvemento en serie de potencias do volume de esferas xeodésicas. Conxecturaron que o volume de esferas xeodésicas pode ser empregado para caracterizar a xeometría euclidiana. Máis especificamente, se cada esfera xeodésica dunha variedade riemanniana ten o mesmo volume ca unha esfera euclidiana do mesmo radio, entón a variedade é chá. Aínda que a resposta se sabe afirmativa en varios casos especiais, o problema segue aberto no caso xeral. Traballo ulterior con esferas xeodésicas involucrou a investigación das súas propiedades xeométricas e como estas influencian a xeometría da variedade ambiente. B.–Y. Chen e L. Vanhecke [33] estudiaron curvaturas intrínsecas e extrínsecas de esferas xeodésicas. Resultou que en moitos casos as propiedades de curvatura das esferas xeodésicas dan lugar a un mellor entendemento da xeometría cás propiedades de volume.

Estamos interesados nesta parte nos chamados invariantes escalares da curvatura. Á marxe da súa ubicuidade en xeometría riemanniana, especialmente cando se estudian esferas xeodésicas e obxectos relacionados, son de interese por eles mesmos. Véxase por exemplo [113], onde foi realizada unha caracterización de espacios homoxéneos empregando invariantes escalares da curvatura. O noso propósito no Capítulo 4 é investigar os invariantes da curvatura de esferas xeodésicas. Integrando os invariantes escalares da curvatura ó longo de cada esfera xeodésica dunha variedade riemanniana obtemos unha boa relación entre curvatura e propiedades de volume. A conxectura do volume de A. Gray e L. Vanhecke pode ser xeneralizada para estes novos obxectos. Vemos na Sección 4.2.3 que en certos casos os espacios homoxéneos dous puntos poden ser caracterizados mediante as integrais de invariantes escalares da curvatura en esferas xeodésicas. Enfatizamos que é suficiente un só invariante para dita caracterización. Véxase a Subsección 4.3.1 para exemplos de tales invariantes da curvatura.

Ademais de esferas xeodésicas, pódense considerar outros obxectos en xeometría riemanniana que están relacionados coa función distancia riemanniana: tubos arredor de subvariedades e discos. Os primeiros introdúcense no Capítulo 4 e son de interés na última parte desta tese. Os discos xeodésicos son a principal ocupación da Subsección 4.3.2. Foran previamente investigados por O. Kowalski e L. Vanhecke con especial atención ás súas propiedades de volume [93], [94], [95]. Nesta subsección estamos interesados na xeometría intrínseca dos bordes destes discos e centramos a nosa atención no estudio das súas curvaturas totales obtidas integrando a curvatura escalar e os invariantes escalares cuadráticos da curvatura nos bordes dos discos. O noso principal resultado é que os espacios homoxéneos dous puntos están caracterizados por algunhas das curvaturas totais dos bordes de discos xeodésicos entre as variedades riemannianas con holonomía adaptada.

Cando volvemos a nosa atención das variedades riemannianas cara ós espacios–tempo aparecen varias dificultades. Unha característica das variedades riemannianas é que teñen unha función distancia riemanniana que é continua e que induce unha topoloxía na variedade que coincide coa topoloxía de partida. Así, varios obxectos xeométricos tales como esferas xeodésicas poden ser definidos, polo menos localmente, por medio desta función. Estes obxectos son tamén variedades riemannianas. Teñen propiedades interesantes como compacidade e un comportamento aceptable con respecto doutras construccións. Cando se trata con variedades semi–riemannianas en xeral, non hai tal función "distancia semi–riemanniana". De feito, unha función tipo distancia só está definida para espacio–tempos, pero incluso neste caso as súas propiedades son completamente diferentes daquelas do contexto riemanniano [7]. Por exemplo, a función "distancia lorentziana" pode non ser continua ou limitada e os obxectos xeométricos definidos a partir dela teñen propiedades extrañas. Ademais, os conxuntos de nivel da función de distancia lorentziana con respecto dun punto dado non son compactos en xeral e aínda que algunhas propiedades destes conxuntos foron previamente investigadas, non parecen ser axeitados para a investigación de propiedades de volume.

No Capítulo 5 consideramos unha nova familia de obxectos en xeometría lorentziana, as chamadas esferas celestes xeodésicas. A grosso modo, son o conxunto de puntos acadados despois de viaxar unha distancia fixa, ó longo de xeodésicas radiais partindo dun punto, en direccións ortogonais a un vector temporal dado. En Relatividade, un vector temporal unitario representa un observador instantáneo e o subespacio vectorial do tanxente que é ortogonal a un observador instantáneo chámase o espacio de simultaneidade infinitesimal, é dicir, o universo newtoniano infinitesimal onde o observador percibe as partículas como partículas newtonianas relativas á súa posición de repouso. Entón, unha esfera celeste xeodésica non é máis cá imaxe mediante a aplicación exponencial dunha esfera celeste no espacio de simultaneidade infinitesimal.

Seguindo a idea de caracterizar espacios con alto grao de simetría por medio de propiedades de volume de obxectos xeométricos, levamos a cabo na Sección 5.2 o cálculo do volume de esferas celestes xeodésicas. Este depende do radio, do punto base e do observador instantáneo empregado para definila. Non obstante, nunha variedade lorentziana isotrópica esta medida só depende do radio. Vemos nesta sección que esta propiedade é característica das variedades localmente isotrópicas. Na Sección 5.2 discutimos resultados de comparación de volume e damos teoremas tipo Bishop–Günther e Gromov para estes obxectos. Finalmente na Sección 5.3 levamos a cabo a caracterización de variedades lorentzianas localmente isotrópicas empregando as integrais de invariantes escalares da curvatura en esferas celestes xeodésicas no espírito do Capítulo 4. Empregamos os resultados da Sección 4.2.3 para obter esta caracterización.

Nesta parte tratamos de explicitar o mínimo número de cálculos para facer o traballo máis fácil de ler. O autor implementou un paquete en *Mathematica* coas principais identidades do tensor de curvatura. Este paquete permite realizar cálculos involucrando invariantes escalares da curvatura e integración en esferas xeodésicas. Podemos obter tanto expresións explícitas en espacios homoxéneous dous puntos como desenvolvementos en serie de potencias en variedades riemannianas xerais.

Parte III. Hipersuperficies reais no espacio hiperbólico complexo

O obxectivo da xeometría de subvariedades é entender os invariantes xeométricos e clasificar as subvariedades a partir de datos xeométricos precisos. En xeometría riemanniana a estructura dunha variedade está codificada nas ecuacións de Gauss, Codazzi e Ricci. A situación simplifícase para hipersuperficies dado que a ecuación de Ricci é trivial e a segunda forma fundamental pode ser escrita en termos do operador de configuración. Os autovalores do operador de configuración, as chamadas curvaturas principais, son os obxectos xeométricos máis simples dunha hipersuperficie. Dous problemas básicos da xeometría de hipersuperficies son entender a xeometría de subvariedades para as que as curvaturas principais son constantes e clasificalas.

Empregando as ecuacións de Gauss e Codazzi, É. Cartan [28] provou que en espacios de curvatura constante unha hipersuperficie ten curvaturas principais constantes se e só se é isoparamétrica. A clasificación de superficies isoparamétricas ten unha longa historia e co paso dos anos moitas características sorprendentes foron descubertas. Ver [127] para un resumo. É. Cartan tamén provou en [28] que o número g de curvaturas principais distintas dunha superficie isoparamétrica no espacio hiperbólico real $\mathbb{R}H^n$ é 1 ou 2. Isto dá lugar a unha clasificación completa: esferas xeodésicas, horosferas, hiperplanos totalmente xeodésicos e as súas superficies equidistantes e tubos arredor de subespacios totalmente xeodésicos de dimensión maior ou igual ca un. Como consecuencia, todas as hipersuperficies no espacio hiperbólico real con curvaturas principais constantes son partes abertas de hipersuperficies homoxéneas.

Nesta parte tratamos o problema da clasificación de hipersuperficies reais con curvaturas principais constantes no espacio hiperbólico complexo. Describimos brevemente o estado do problema. Obviamente, calquera hipersuperficie real homoxénea ten curvaturas principais contantes. J. Berndt e H. Tamaru [16] derivaron recentemente a clasificación completa de hipersuperficies reais homoxéneas en $\mathbb{C}H^n$. O número g de curvaturas principais constantes de todas estas hipersuperficies homoxéneas é 2, 3, 4 ou 5. Non se coñecen exemplos de hipersuperficies reais con curvaturas principais constantes que non sexan un aberto dunha hipersuperficie homoxénea. Tampouco se sabe se para calquera hipersuperficie real con curvaturas principais constantes o número g debe ser necesariamente 2, 3, 4 ou 5.

No Capítulo 6 estudiamos en profundidade a xeometría das órbitas da clasificación dada por J. Berndt e H. Tamaru. Prestamos atención a aquelas accións que teñen órbitas singulares non totalmente xeodésicas, xa que estas consitúen novos exemplos só coñecidos recentemente. Interesámonos particularmente na existencia de distribucións nestas subvariedades que nos permitan describilas de xeito xeométrico. A Subsección 6.3.3 é un bo exemplo de tal estudio. De feito, nesta secciónséntanse as bases para a caracterización das órbitas singulares de todas as accións de cohomoxeneidade un descritas e demóstranse resultados de rixidez para elas nos Teoremas 6.8 e 6.16.

A partir da ecuación de Codazzi un pode facilmente deducir que o número de curvaturas constantes dunha hipersuperficie real de $\mathbb{C}H^n$ verifica $g > 1$ (ver Corolario 7.5). Séguese do traballo de S. Montiel [99] que toda hipersuperficie real con dúas curvaturas principais constantes en $\mathbb{C}H^n$, $n \geq 3$, é un aberto dunha esfera xeodésica, dunha horosfera, dun tubo arredor dun $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$ totalmente xeodésico ou dun tubo de radio $\log(2+\sqrt{3})$ arredor dun $\mathbb{R}H^n \subset \mathbb{C}H^n$ totalmente xeodésico. Para $n = 2$ o problema parece estar aberto. No Corolario 7.6 presentamos unha proba para a anterior clasificación que inclúe tamén este caso de dimensión baixa. Todas estas hipersuperficies son hipersuperficies homoxéneas de Hopf. Unha hipersuperficie de $\mathbb{C}H^n$ con campo de vectores normal unitario ξ dise de Hopf se $J\xi$ é un autovector do operador de configuración. J. Berndt obtivo en [10] a clasificación de todas as hipersuperficies de Hopf con curvaturas principais constantes en $\mathbb{C}H^n$. Calquera desas hipersuperfices é un aberto dunha horosfera, dun tubo arredor dun $\mathbb{C}H^k \subset \mathbb{C}H^n$ totalmente xeodésico para algún $k \in \{0, \ldots, n-1\}$ ou dun tubo arredor dun $\mathbb{R}H^n \subset \mathbb{C}H^n$ totalmente xeodésico. Todos estes tubos e horosferas son homoxémeos e $g \in \{2, 3\}$. Non obstante, non todas as hipersuperfices reais homoxéneas de $\mathbb{C}H^n$ son de Hopf. Ver o Capítulo 6 para unha dicussión deste tipo de exemplos.

No Capítulo 7 concluimos o estudio anterior dando unha clasificación das hipersuperficies reais en $\mathbb{C}H^n$ con tres curvaturas principais constantes distintas (Teorema 7.1). En particular, o noso resultado implica que calquera hipersuperficie con como máximo tres curvaturas principais constantes é homoxénea.

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