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Homological properties of transitive Lie algebroids via Sullivan models

(Propiedades homológicas de algebroides de Lie transitivos via modelos de Sullivan)

Jose Manuel Ribeiro Oliveira

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Tesis realizada en el Departamento de Geometria y Topologia de la Facultad de Matemáticas, bajo la dirección del profesor Aleksandr Mishchenko (Moscow State University), siendo tutor Jesús Antonio Álvarez López (de este Departamento), para obtener el grado de Doctor en Ciencias Matemáticas por la Universidad de Santiago de Compostela.

Autorización del director:	Autorización del tutor:	El autor:
Fdo.: A. Mishchenko	Fdo.: J.A. Álvarez López	Fdo.: J.M. Ribeiro Oliveira

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Homological properties of transitive Lie algebroids via Sullivan models

Abstract

D. Sullivan considered a new model for the underlying cochain complex of classical cohomologies with rational coefficients for arbitrary simplicial spaces which gives an isomorphism with classical rational cohomologies. This new model is determined by the Rham complex of all rational polynomial forms defined on the simplicial complex triangulating the space. Other cell-like constructions of cochain complexes which induce isomorphisms in cohomology with classical cohomologies had been already presented by H. Whitney. Recent ideas developed by K. Mackenzie and J. Kubarski concerning Lie algebroids are applied to a generalization of a cell-like construction for transitive Lie algebroids over combinatorial manifolds. Namely, given a compact smooth manifold M, smoothly triangulated by a simplicial complex K, and a transitive Lie algebroid $\mathcal{A}$ on M, we define a piecewise smooth form on $\mathcal{A}$ to be a family $\omega = (\omega_\Delta)_{\Delta\in K}$ of differential forms such that, for each simplex $\Delta \in K$, $\omega_\Delta \in \Omega^*(\mathcal{A}^{!!}_\Delta; \Delta)$ is a smooth form defined on the Lie algebroid $\mathcal{A}^{!!}_\Delta$, restriction of $\mathcal{A}$ to the simplex Δ, satisfying the compatibility condition under restrictions of the form ω_Δ to all faces of the simplex Δ, that is, if Δ' is a face of Δ, then $\lambda^*_{\Delta,\Delta'}(\omega_\Delta) = \omega_{\Delta'}$, in which $\lambda_{\Delta,\Delta'}$ denotes the canonical Lie algebroid morphism induced by the inclusion $\Delta' \hookrightarrow \Delta$. The set $\Omega^*(\mathcal{A}; K)$ of all piecewise smooth forms defined on $\mathcal{A}$ is a commutative cochain algebra. We define a map $\Omega^*(\mathcal{A}; M) \to \Omega^*(\mathcal{A}; K)$ which assigns, to each smooth form $\omega \in \Omega^*(\mathcal{A}; M)$, the piecewise smooth form $\xi = (\xi_\Delta)_{\Delta\in K} \in \Omega^*_{ps}(\mathcal{A}; K)$ defined by the condition $\xi_\Delta = \lambda^*_{M,\Delta}(\omega)$ for each simplex $\Delta \in K$. This map is a natural morphism of cochain algebras.

In this thesis, we prove that, for compact combinatorial manifolds, the cohomology of this construction is isomorphic to the Lie algebroid cohomology of $\mathcal{A}$. We apply this isomorphism in piecewise invariant cohomology of Lie algebroids and piecewise de Rham cohomology of locally trivial Lie groupoids.

Contents

Introduction

It is very well known that the integration map gives us an isomorphism between de Rham cohomology and the singular cohomology of a smooth manifold. This statement is the de Rham theorem. It was originally conjectured by Elie Cartan in 1928 and, on the beginning of years 30, Georges de Rham has proved the theorem. After de Rham, as cohomology theory developed, other proofs have been published. In particular, we mention the proof made by A. Weil [27] and by J. Dugundji [2].

The de Rham theorem is a result of great importance because it has been the main connecting link between analysis on manifolds and the topological properties of manifolds. In brief words, the homology spaces measure the number of holes of a manifold and its level of complexity. The de Rham theorem guarantees that the homology spaces of manifolds can be known by using differential forms and their analytical methods. The study of the homology spaces in terms of differential forms opened a way for studying the deeper structure of manifolds. Sullivan, in his paper "Infinitesimal computations in topology", says that within the world of topology there is more topological information in the de Rham algebra of the differential forms that simply the real cohomology. The de Rham theory quickly originated a deep development of the topology of manifolds. There are also a lot of mathematical situations in which the knowledge of differential forms has important consequences and consequently other mathematical theories were developed from the de Rham theorem. The Hopf invariant, the Massey product, the mapping degree and cohomology of compact Lie groups are some examples of the importance of the de Rham theorem. Sullivan and other mathematicians have implemented several strategies in the study of the de Rham algebra of all smooth forms. Among them, it is the theory of models. This theory consists in finding other graded algebras, inside the de Rham algebra of all smooth forms, such that the canonical inclusion induces an isomorphism in cohomology. From these developments, an important conclusion arose, which can be expressed in the following commutative diagram:

$$
\begin{array}{ccccc}
 & & H^*_{p.C^\infty}(M) & & \\
 & \nearrow & & \nwarrow & \\
H^*_{PL}(M) \otimes_{\mathbf{Q}} \mathbf{R} & & \Big\downarrow \int & & H^*_{dR}(M) \\
 & \underset{\cong}{\overset{\int}{\searrow}} & & \underset{\cong}{\overset{\int}{\swarrow}} & \\
 & & H^*(M, \mathbf{R}) & &
\end{array}
$$

This diagram incorporates a large amount of constructions and statements. The present work arose from efforts to extend those constructions to transitive Lie algebroids. Among those constructions, we are particularly interested in the one which says that de Rham cohomology of a smooth manifold, smoothly triangulated by a simplicial complex, is isomorphic to piecewise smooth cohomology of the simplicial complex. This isomorphism is given by restriction of smooth forms to all simplices. The study of this construction or other Sullivan's constructions in simplicial manifolds is based in the de Rham theorem for cells as well as extensions of smooth forms. Some difficulties rise from the use of de Rham theory in the study of Lie algebroid cohomology. Nevertheless, in spite of all difficulties that rise from de Rham theory of Lie algebroids, during the recent years, the cohomology theory of Lie algebroids has developed from a collection of great results with strong connections to many other parts of mathematics, in particular, to Chern-Weil theory. These refinements have reduced several obstructions in the development of our work.

The key ideas concerning class obstruction arising from non-abelian extensions of Lie algebroids have inspired Mishchenko and led him to conjecture that, given a transitive Lie algebroid on a combinatorial manifold, the morphism given by restriction, which takes smooth forms on the Lie algebroid into piecewise smooth forms on the same Lie algebroid, still remains an isomorphism in cohomology.

The aim of the present work is to prove Mishchenko's conjecture. For this purpose, we have used a structure called a complex of Lie algebroids. This structure commences by fixing a smooth triangulation of the base of a transitive Lie algebroid by a simplicial complex and taking the restriction of the Lie algebroid to all simplices of the triangulation. Since the Lie algebroid is transitive, the restriction of the Lie algebroid to each simplex always exists. When a complex of Lie algebroids is given, we define the notion of piecewise smooth form in a similar way to Whitney forms on a simplicial complex and the set of all piecewise smooth forms defined on a complex of Lie algebroids is naturally equipped with a differential, yielding a commutative differential graded algebra. Its cohomology is, by definition, the piecewise smooth cohomology of the Lie algebroid. Each smooth form defined on the Lie algebroid gives a piecewise smooth form defined on the corresponding complex of Lie algebroids by taking the restriction of the form to each simplex. This correspondence is a natural map from the usual algebra of the smooth forms of the Lie algebroid to the algebra of the piecewise smooth forms of the corresponding complex of Lie algebroids. Based on three crucial results, namely the triviality of a transitive Lie algebroid over a contractible smooth manifold (Mackenzie, Weinstein), the Künneth theorem for Lie algebroids (Kubarski) and the de Rham-Sullivan theorem for smooth manifolds, we show that this map is an isomorphism in cohomology.

We give an outline of this work. The manuscript is divided into 3 chapters and each chapter into several sections which are numerated.

The first chapter of this work is divided into three sections. The first section is a general discussion on restrictions of Lie algebroids to general submanifolds which may not be open submanifolds. The second section is a brief introduction to the algebra of smooth forms on a Lie algebroid and its cohomology. In the third section, we introduce the Lie algebroid of covariant differential operators and the Lie algebra bundle of inner and outer derivations. We recall the notions of couplings and operators extensions and state the main result concerning actions, which states that the additive group of the cohomology space in degree two of certain representation related to a coupling acts freely and transitively on the affine space of operator extensions.

In the second chapter, we define a complex of Lie algebroids to be a family of Lie algebroids defined on the simplices of a simplicial complex which is compatible with the restrictions to the faces. We define the notion of piecewise smooth form in a similar way to Whitney forms on a simplicial complex. We present some considerations related to the algebra of piecewise smooth forms. Some results concerning extensions of piecewise smooth forms and the Mayer-Vietoris sequence are presented. We generalize the notion of piecewise smooth cohomology to the situation in which we have, not only simplices of a simplicial complex, but also a finite collection of transverse submanifolds in an ambient space.

The third chapter is the main part of this work and is devoted to the proof of the Mishchenko's conjecture. We construct a morphism between Lie algebroid cohomology and piecewise smooth cohomology of a transitive Lie algebroid and prove that this morphism is an isomorphism. We apply this isomorphism in piecewise invariant cohomology of Lie algebroids and piecewise de Rham cohomology of locally trivial Lie groupoids.

Chapter 1

Preliminaries on Lie algebroids

The propose of this chapter is to give a brief revision of certain concepts and results on Lie algebroids, which will be used in the rest of this work. First, we shall recall some facts concerning to restrictions of Lie algebroids to open subsets and, more generally, to any submanifolds of the base. We shall introduce the cochain algebra of smooth forms on a Lie algebroid and respective Lie algebroid cohomology. Some facts concerning extensions of smooth forms in Lie algebroids, extensions of Lie algebroids and couplings are presented. The chapter finishes with a result on the triviality of transitive Lie algebroids over contractible smooth manifolds. Most theorems are simply stated without proofs. There are some exceptions and, in this case, some occasional indications for the proof are given. A reasonably complete description of definitions and results on Lie algebroids can be found in [7], [10], [11], [16], [17], [18] and [28].

1.1 Restriction of transitive Lie algebroids

Definition (Lie algebroid). Let M be a smooth manifold, possibly with boundary and corners, $\Upsilon(M)$ the Lie algebra of all smooth vector fields on M, and TM the tangent bundle to M. We recall that a Lie algebroid on base M is a vector bundle $\pi : \mathcal{A} \longrightarrow M$ on M equipped with a vector bundle morphism $\gamma : \mathcal{A} \longrightarrow TM$, called anchor of $\mathcal{A}$, and a structure of Lie algebra on the vector space $\Gamma(\mathcal{A})$ of sections of $\mathcal{A}$ such that the induced map $\gamma_\Gamma : \Gamma(\mathcal{A}) \longrightarrow \Upsilon(M)$ is a Lie algebra homomorphism and the action of the algebra $\mathcal{C}^\infty(M)$ on $\Gamma(\mathcal{A})$ satisfies the natural condition:

$$[\xi, f\eta] = f[\xi, \eta] + (\gamma_\Gamma(\xi) \cdot f)\eta$$

for each ξ, $\eta \in \Gamma(\mathcal{A})$ and $f \in \mathcal{C}^\infty(M)$. The Lie algebroid $\mathcal{A}$ is called transitive if the anchor γ is surjective. As usually, when there is no ambiguity, we drop the anchor map and the Lie bracket in the notation of the Lie algebroid but, when it is needed to emphasize them, we write $(\mathcal{A}, [\cdot, \cdot], \gamma)$ for denoting this structure.

Definition (Morphism of Lie algebroids). Let M and N two smooth manifolds and $(\mathcal{A}, [\cdot, \cdot], \gamma)$ and $(\mathcal{B}, [\cdot, \cdot], \delta)$ Lie algebroids over M e N respectively. A morphism of Lie algebroids from $(\mathcal{A}, [\cdot, \cdot], \gamma)$ to $(\mathcal{B}, [\cdot, \cdot], \delta)$ consists of a pair of mappings (ψ, φ), with $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\varphi : M \longrightarrow N$, such that (ψ, φ) is a vector bundle morphism satisfying the equality $\delta \circ \psi = T(\varphi) \circ \gamma$, in which $T(\varphi) : TM \longrightarrow TN$ means the tangential of φ, and preserving the Lie bracket condition for ψ-decompositions, that is, for each $\xi, \eta \in \Gamma(\mathcal{A})$ with decompositions

$$\psi \circ \xi = \sum_{i=1}^{m} a_i \otimes \xi_i \qquad \psi \circ \eta = \sum_{j=1}^{m} b_j \otimes \eta_j$$

ξ_i, $\eta_j \in \Gamma(\mathcal{B})$, then

$$\psi \circ [\xi, \eta]_{\mathcal{A}} = \sum_{i,j} a_i b_j \otimes [\xi_i, \eta_j]_{\mathcal{B}} + \sum_{j=1}^{m} (\gamma \circ \xi)(b_j) \otimes \eta_j - \sum_{i=1}^{m} (\gamma \circ \eta)(a_i) \otimes \xi_j$$

We notice that, when $M = N$, a simple characterization for a vector bundle morphism between two Lie algebroids on M to be a Lie algebroid morphism can seen in [7] or [10]. Namely, if $(\mathcal{A}, [\cdot, \cdot], \gamma)$ and $(\mathcal{B}, [\cdot, \cdot], \delta)$ are Lie algebroids over the same smooth manifold M, then a vector bundle morphism ψ from $\mathcal{A}$ to $\mathcal{B}$ is a Lie algebroid morphism if, and only if, $\gamma = \psi \circ \delta$ and the induced map $\psi_\Gamma : \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B})$ is a Lie algebra morphism.

We give some examples of Lie algebroids and algebraic constructions in Lie algebroids. Other examples of Lie algebroids are given in section 3.

Example 1 (Lie algebras). Any real finite dimensional Lie algebra $\mathfrak{g}$ over a one-point space $M = \{*\}$ (so $\mathcal{C}^\infty(M) = \mathbf{R}$) with anchor equal to zero is a totally intransitive Lie algebroid on M. Any Lie algebra morphism between two Lie algebras is a Lie algebroid morphism for this structure of Lie algebroid.

Example 2 (Tangent Lie algebroid). If M is a smooth manifold then TM is a Lie algebroid on M. The anchor map is the identity map of TM, and the Lie bracket is the usual Lie bracket of vector fields. This is called the tangent Lie algebroid of M. The anchor map $\gamma : A \longrightarrow TM$ is a Lie algebroid morphism from A to the tangent algebroid of M.

Let $\mathcal{F}$ be a regular foliation of M. The tangent Lie algebroid of $\mathcal{F}$ is, by definition, the vector subbundle of TM consisting of the tangent spaces to $\mathcal{F}$, with the usual Lie bracket of vector fields tangent to $\mathcal{F}$, and the inclusion map as the anchor. Conversely, if $\mathcal{A}$ is a Lie algebroid on M, whose its anchor map γ is injective, then, setting $E_x = \gamma_x(\mathcal{A}_x)$ for each $x \in M$, we obtain a vector subbundle E of TM defining a foliation of M, and the tangent Lie algebroid of this foliation is isomorphic to the Lie algebroid $\mathcal{A}$ (see [7]).

Example 3 (Trivial Lie algebroid). Let $\mathfrak{g}$ be a real finite dimensional Lie algebra and M a smooth manifold and consider the trivial vector bundle $M \times \mathfrak{g}$. On the Whitney sum

$$TM \oplus (M \times \mathfrak{g}) = TM \times_M (M \times \mathfrak{g})$$

we define an anchor map $\gamma : TM \oplus (M \times \mathfrak{g}) \longrightarrow TM$ by taking γ to be the projection of $TM \oplus (M \times \mathfrak{g})$ on TM and a Lie bracket on $\Gamma(TM \oplus (M \times \mathfrak{g}))$ by setting

$$[(X, u), (Y, v)] = ([X, Y]_{TM}, X(v) - Y(u) - [u, v])$$

for $X, Y \in TM$ and $u, v : M \longrightarrow \mathfrak{g}$ smooth maps. Then, $TM \oplus (M \times \mathfrak{g})$ is a transitive Lie algebroid on M and called the trivial Lie algebroid on M with structure algebra $\mathfrak{g}$.

Example 4 (Lie algebroid product). Let M and N be two smooth manifolds and $(\mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \gamma)$ and $(\mathcal{B}, [\cdot,\cdot]_{\mathcal{B}}, \widehat{\gamma})$ Lie algebroids over M and N respectively. The product of the Lie algebroids $(\mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \gamma)$ and $(\mathcal{B}, [\cdot,\cdot]_{\mathcal{B}}, \widehat{\gamma})$, denoted by $\mathcal{A} \times \mathcal{B}$, is the vector bundle product $\mathcal{A} \times \mathcal{B}$ over $M \times N$, in which the anchor is $\gamma \times \widehat{\gamma}$ and the Lie bracket is defined in the following way: for each $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2) \in \Gamma(\mathcal{A} \times \mathcal{B})$

$$[\xi, \eta]_{\mathcal{A}\times\mathcal{B}} = ([\xi, \eta]^1, [\xi, \eta]^2) \in \Gamma(\mathcal{A} \times \mathcal{B})$$

where

$$[\xi, \eta]^1_{(x,y)} = [\xi_1(-, y), \eta_1(-, y]_{\mathcal{A}}(x) + \widehat{\gamma}(\xi_{2_{(x,y)}})(\eta_1(x, -)) - \widehat{\gamma}(\eta_{2_{(x,y)}})(\xi_1(x.-))$$

and

$$[\xi, \eta]^2_{(x,y)} = [\xi_2(-, y), \eta_2(-, y]_{\mathcal{A}}(x) + \gamma(\xi_{1_{(x,y)}})(\eta_2(x, -)) - \gamma(\eta_{1_{(x,y)}})(\xi_2(x.-))$$

The projections $\pi_{\mathcal{A}} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{B}$ are morphisms of Lie algebroids.

Consider now a real finite dimensional Lie algebra $\mathfrak{g}$ and the tangent Lie algebroid TM. The Lie algebra $\mathfrak{g}$ is a Lie algebroid over a one-point space $N = \{*\}$. We can take the product of Lie algebroids $TM \times \mathfrak{g}$, which is defined over $M \simeq M \times N$. On the other hand, we can consider the trivial Lie algebroid $TM \oplus (M \times \mathfrak{g})$ over M and we easily see that the map $F : TM \times \mathfrak{g} \longrightarrow TM \oplus (M \times \mathfrak{g})$ given by $F(x, u, v) = (x, u, x, v)$ is a (strong) isomorphism of Lie algebroids. Henceforth, we will identify both Lie algebroids.

Example 5 (Lie algebroid of covariant differential operators). Let M be a smooth manifold and $\pi : E \longrightarrow M$ a vector bundle on M. For each $x \in M$, denote by $\mathcal{A}(E)_x$ the vector space of all linear maps $\psi : \Gamma(E) \longrightarrow E_x$ such that there exists a vector $u \in T_xM$ satisfying the equality

$$\psi(f\xi) = f(x)\psi(\xi) + (u \cdot f)_x \xi_x$$

for all $f \in C^\infty(M)$ and $\xi \in \Gamma(E)$. The vector u is unique and so we can define a map

$$\gamma : \bigsqcup_{x \in M} \mathcal{A}(E)_x \longrightarrow TM$$

We denote by $\mathcal{A}(E)$ the disjoint union $\bigsqcup_{x\in M}\mathcal{A}(E)_x$. We define a Lie bracket on the space of the sections of $\mathcal{A}(E)$ locally as follows. Fix a local trivialization $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ of the vector bundle E, in which U is an open subset of M and F is the fibre type of E. Let $\mathfrak{gl}(F)$ be the Lie algebra of F and, for each $\xi \in \Gamma(E)$, the map $\xi_\varphi : U \longrightarrow F$ defined by $\xi_\varphi(x) = \varphi_x(\xi_x)$, in which $\varphi_x : E_x \longrightarrow F$ is the linear map induced by φ. It can be seen in [7], [5], [8] that the map $\overline{\varphi} : TU \times \mathfrak{gl}(F) \longrightarrow \mathcal{A}(E)_U$ defined by

$$\overline{\varphi}(u,g)(\xi) = (\varphi_x)^{-1}(u \cdot \xi_\varphi + (g \circ \xi_\varphi(x))$$

is bijective. Hence, the Lie bracket and the anchor of the trivial Lie algebroid $TU \times \mathfrak{gl}(F)$ can be carried to $\mathcal{A}(E)_U$ and the space $\mathcal{A}(E)$ becomes a transitive Lie algebroid on M, which is denoted by $\mathcal{D}(E)$ and called the Lie algebroid of covariant differential operators on the space of sections of the vector bundle E (see [5], [7], [8], [10], [11], [16], [18]).

Example 6 (Lie algebra bundles). A Lie algebra bundle over a smooth manifold M is a vector bundle $\pi : K \longrightarrow M$ equipped with a section $[\cdot,\cdot]$ of the vector bundle $\bigwedge^2(K,K)$ such that, for each $x \in M$, $(K_x, [\cdot,\cdot]_x)$ is a Lie algebra and K admits an atlas

$$\{\psi_j : U_j \times \mathfrak{g} \longrightarrow \pi^{-1}(U)\} \qquad (\mathfrak{g} \text{ is a Lie algebra})$$

in which each ψ_{j_x} is a Lie algebra isomorphism. A Lie algebra bundle is a totally intransitive Lie algebroid (see [10]).

Let $\pi : E \longrightarrow M$ be a vector bundle on a smooth manifold M. Then, the vector bundle $\mathbf{End}(E) = L(E;E)$, whose fibres, at each point $x \in M$, are the vector spaces $L(E_x;E_x)$, is a Lie algebra bundle (see [10]).

Next example is not used in this work but we include it since it has been a crucial example on the development of integrability theory of Lie algebroids

Example 7 (Weinstein's transformation algebroid). Suppose that we have an action $\mu : \mathfrak{g} \longrightarrow \Upsilon(M)$ of a Lie algebra $\mathfrak{g}$ on a smooth manifold M. Then, we can associate to this action a Lie algebroid which is called the corresponding transformation algebroid and defined as follows. The vector bundle underlying this transformation algebroid is the trivial bundle $\mathfrak{g} \times M \longrightarrow M$. The anchor map $\gamma : \mathfrak{g} \times M \longrightarrow TM$ is defined by

$$\gamma(X, x) = \mu(X)(x)$$

for each $X \in \mathfrak{g}$ and $x \in M$, and the Lie bracket on the sections of $\mathfrak{g} \times M$, considered as maps from M to $\mathfrak{g}$, is defined as

$$[\xi, \eta](z) = [\xi(z), \eta(z)] + (\mu(\xi(z)))_z(\eta) - (\mu(\eta(z)))_z(\xi)$$

This algebroid is usually denoted by $\mathfrak{g} \ltimes M$.

We are going to handle restriction of Lie algebroids. We consider first the case in which the restriction is made to a open subset of the base. We are going to see that the restriction of a Lie algebroid to an open subset of base coincides with the restriction of the underlying vector bundle to that open subset. We note that, for restrictions to open subsets, we do not need the Lie algebroid to be transitive. We begin with the local property of the Lie bracket.

Proposition 1.1.1. Let M be a smooth manifold and $(\mathcal{A}, [\cdot, \cdot], \gamma)$ a Lie algebroid on M. Let U be an open subset of M and $X, Y \in \Gamma(\mathcal{A})$ such that Y vanishes on U. Then, the Lie bracket $[X, Y]$ vanishes on U.

Proof. Let $x_0 \in U$. Take the closed subset $M \backslash U$ and the open subset $M \backslash F$, where $F = \{x_0\}$. Obviously $M \backslash U \subseteq M \backslash F$ and then there exists a smooth function $g \in C^\infty(M)$ such that $g(M \backslash U) = \{1\}$ and **supp** $g \subseteq M \backslash F$. If $x \notin U$ then $g(x) = 1$ and so $g(x) Y_x = Y_x$. The restrictions of both sections to U also coincide and then $gY = Y$. Hence, we have

$$[X, Y](x_0) = [X, gY](x_0) = g(x_0)[X, Y](x_0) + (\gamma_\Gamma(X) \cdot g)(x_0) Y(x_0) = 0$$

because $g(x_0) = 0$ and $Y_{x_0} = 0$. Therefore, the Lie bracket $[X, Y]$ vanishes for all points of the open subset U. □

Although next proposition is not used in this section, we will note it here as an application of the local property of Lie bracket.

Proposition 1.1.2. Let M be a smooth manifold and $(\mathcal{A}, [\cdot,\cdot], \gamma)$ a Lie algebroid on M. For each $x \in M$, denote by $\mathfrak{g}_x$ the kernel of the linear map $\gamma_x : \mathcal{A}_x \longrightarrow T_xM$. Then, $\mathfrak{g}_x$ has a natural structure of Lie algebra defined by

$$[u, v] = [X, Y](x)$$

with $u, v \in \mathfrak{g}_x$ and $X, Y \in \Gamma\mathcal{A}$ such that $X_x = u$ and $Y_x = v$.

Proof. We are going to check that the definition is coherent. Let $\widehat{Y} \in \Gamma(\mathcal{A})$ be a section such that $\widehat{Y}_x = v$. Fix a local frame of the vector bundle $(\mathcal{A}, \pi, M)$, say us, $(s_1, \cdots, s_k)$ defined on an open subset U of M. Then, there are smooth functions $f_1, \cdots, f_k \in C^\infty(U)$ such that $Y_{/U} - \widehat{Y}_{/U} = \sum_j f_j s_j$. By using partition of unity, we can extend all maps f_j and sections s_j to all of M and we obtain $\widetilde{f}_1, \cdots, \widetilde{f}_k \in C^\infty(M)$ and $\widetilde{s}_1, \cdots, \widetilde{s}_k \in \Gamma(\mathcal{A})$ such that the restrictions of $\widetilde{f}_j$ and $\widetilde{s}_j$ to U coincide with f_j and s_j respectively. Denote by Z the section of $\mathcal{A}$ defined by $Z = \sum_j \widetilde{f}_j \widetilde{s}_j$. By the local property of Lie bracket and the third condition of the definition of Lie algebroid we have

$$[X, Y - \widehat{Y}](x) = [X, Z](x) = \sum_j \widetilde{f}_j(x)[X, \widetilde{s}_j](x) + (\gamma_\Gamma(X) \cdot \widetilde{f}_j)(x)\widetilde{s}_j(x) = 0$$

because $\widetilde{f}_j(x) = 0$ and $(\gamma \circ X)(x) = 0$. □

Let M be a smooth manifold and $(\mathcal{A}, [\cdot,\cdot]_\mathcal{A}, \gamma)$ a transitive Lie algebroid over M. Consider the vector bundle $\mathbf{Ker}\,\gamma$. The Lie bracket structure on $\Gamma(\mathcal{A})$ induces a bracket structure on $\Gamma(\mathbf{Ker}\,\gamma)$ and so $\mathbf{Ker}\,\gamma$ is a totally intransitive Lie algebroid on M, called adjoint Lie algebroid of $\mathcal{A}$. We notice that a totally intransitive Lie algebroid may not be a Lie algebra bundle. However, $\mathbf{Ker}\,\gamma$ is a Lie algebra bundle on M (see [7], [10]).

Proposition 1.1.3. Let $(\mathcal{A}, [\cdot,\cdot], \gamma)$ be a Lie algebroid on a smooth manifold M. Let U be an open subset of M and consider the vector bundle $\mathcal{A}_U$, restriction of $\mathcal{A}$ to U. Then, the Lie bracket

$$[\cdot,\cdot] : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{A})$$

restricts to a Lie bracket

$$[\cdot,\cdot]_{\mathcal{A}_U} : \Gamma(\mathcal{A}_U) \times \Gamma(\mathcal{A}_U) \longrightarrow \Gamma(\mathcal{A}_U)$$

Proof. We want define a Lie bracket on $\Gamma(\mathcal{A}_U)$. For that, fix two sections $X, Y \in \Gamma(\mathcal{A}_U)$ and $x \in U$. We can take two sections $\widetilde{X}$, $\widetilde{Y} \in \Gamma(\mathcal{A})$ and an open subset V of M such that $x \in V \subseteq \overline{V} \subseteq U$ and

$$X_{/V} = \widetilde{X}_{/V} \quad \text{and} \quad Y_{/V} = \widetilde{Y}_{/V}$$

We define

$$[X,Y]_{\mathcal{A}_U}(x) = [\widetilde{X},\widetilde{Y}](x)$$

This definition is coherent because, if we take other section $\widehat{X} \in \Gamma(A)$ and an open subset W of M such that $x \in W \subseteq \overline{W} \subseteq U$ and $X_W = \widehat{X}_W$, then, taking the difference $\widetilde{X} - \widehat{X}$ and applying the proposition 1.1.1 to the open subset $V \cap W$, we have that $[\widetilde{X}, Y](x) = [\widehat{X}, Y](x)$. Therefore, a bracket on $\Gamma(\mathcal{A}_U)$ is well defined and we easily can see that $[\cdot,\cdot]_{\mathcal{A}_U}$ satisfies the conditions of Lie bracket on $\Gamma(\mathcal{A}_U)$. $\square$

Next, if $(\mathcal{A}, [\cdot,\cdot], \gamma)$ is a Lie algebroid on a smooth manifold M and U an open subset of M, we want to define a structure of Lie algebroid in the vector bundle $\mathcal{A}_U$ defined on U. From previous proposition, we already have a Lie bracket on $\Gamma(\mathcal{A}_U)$. We only need to define an anchor map. Take then $b \in U$ and $u \in \mathcal{A}_b$. We have that $\gamma(u) \in (TM)_b$. Since U is an open subset of M, $(TU)_b = (TM)_b$ and so $\gamma(u) \in (TU)_b$. Hence, we may restrict the anchor $\gamma : \mathcal{A} \longrightarrow TM$ to a map $\gamma_U : \mathcal{A}_U \longrightarrow TU$. Obviously, γ_U is a vector bundle morphism. It remains to check that the morphism induced by γ_U on the sections of $\mathcal{A}_U$ is a Lie algebra morphism. Take $X, Y \in \Gamma(\mathcal{A}_U)$ and $b \in U$. Fix $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathcal{A})$ such that $\widetilde{X}_{/V} = X_{/V}$ and $\widetilde{Y}_{/V} = Y_{/V}$ where V is an open subset of M such that $b \in V \subseteq \overline{V} \subseteq U$. Firstly, we remark that $\gamma \circ \widetilde{X}$ and $\gamma \circ \widetilde{Y}$ are extensions of $\gamma_U \circ X$ and $\gamma_U \circ Y$ respectively. Consequently, we have that

$$[\gamma_U \circ X, \gamma_U \circ Y](b) = [\gamma \circ \widetilde{X}, \gamma \circ \widetilde{Y}](b)$$

and so,

$$(\gamma_{/U} \circ [X,Y])(b) = \gamma_{/U}([X,Y](b)) =$$
$$= \gamma([X,Y](b)) = \gamma([\widetilde{X},\widetilde{Y}](b)) =$$
$$= (\gamma \circ [\widetilde{X},\widetilde{Y}])(b)) = [\gamma \circ \widetilde{X}, \gamma \circ \widetilde{Y}](b) =$$
$$= [\gamma_U \circ X, \gamma_U \circ Y](b)$$

Also, for each $g \in C^\infty(U)$, $\widetilde{g} \in C^\infty(M)$ such that $\widetilde{g}_{/V} = g_{/V}$

$$[X, gY](b) = [\widetilde{X}, \widetilde{g}\widetilde{Y}](b) =$$
$$= \widetilde{g}(b)[\widetilde{X},\widetilde{Y}](b) + ((\gamma \circ \widetilde{X}) \cdot \widetilde{g})(b)\widetilde{Y}(b) =$$
$$= g(b)[X,Y](b) + ((\gamma_{/U} \circ X) \cdot g)(b)Y(b)$$

We have proved the following proposition.

Proposition 1.1.4. The vector bundle $\mathcal{A}_U$, with the structures above, becomes a Lie algebroid on U. This Lie algebroid will denoted by $(\mathcal{A}_U, [\cdot,\cdot]_{\mathcal{A}_U}, \gamma_{\mathcal{A}_U})$ or simply by $\mathcal{A}_U$. Moreover, if $(\widetilde{i}, i)$ denote the pair of smooth maps $i : U \longrightarrow M$ and $\widetilde{i} : A_U \longrightarrow A$ defined by the inclusions, $(\widetilde{i}, i)$ is a morphism of Lie algebroids, which is fibrewise injective.

We consider now the case in which the restriction is made, non necessarily to an open submanifold, but to a general submanifold of the base. In this case, the Lie algebroid restricted to a submanifold may not coincide with the restriction of the underlying vector bundle to that submanifold, but it will be a vector subbundle of the underlying vector bundle restricted to that submanifold, which is given by image inverse of Lie algebroids through the inclusion of submanifolds. The transitivity of Lie algebroids will be needed to show that inverse image always exists for any smooth embedding of manifolds. We begin by noting brief considerations on the construction of image inverse.

Let M and N be smooth manifolds and $\varphi : N \longrightarrow M$ a smooth map. Suppose that $(\mathcal{A}, [\cdot,\cdot], \gamma)$ is a transitive Lie algebroid on M. Let $\pi : \mathcal{A} \longrightarrow M$ denote the vector bundle underlying the Lie algebroid $\mathcal{A}$ and π_{TM} and π_{TN} the canonical projections of the tangent bundles TM and TN respectively. The anchor $\gamma : \mathcal{A} \longrightarrow TM$ defines $\mathcal{A}$ as a vector bundle on TM. In order to complete the following diagram

$$\begin{array}{ccc} & & \mathcal{A} \\ & & \downarrow \gamma \\ TN & \xrightarrow{T\varphi} & TM \end{array}$$

take the vector bundle $((T\varphi)^*\mathcal{A}, \widehat{\gamma}, TN)$, inverse image of the vector bundle $(\mathcal{A}, \gamma, TM)$ by the smooth map $T\varphi$, where $\widehat{\gamma}$ denotes the canonical projection $(T\varphi)^*\mathcal{A} \longrightarrow TN$. We notice that the vector bundle $(T\varphi)^*\mathcal{A}$ exists because γ is surjective. Obviously, $(T\varphi)^*\mathcal{A}$ is also a vector bundle over N, in which its projection is the composition of the projection $\widehat{\gamma}$ with the canonical projection π_{TN}. We obtain the following commutative diagram

$$\begin{array}{ccc} (T\varphi)^*\mathcal{A} & \longrightarrow & \mathcal{A} \\ \widehat{\gamma}\downarrow & & \downarrow\gamma \\ TN & \xrightarrow{T\varphi} & TM \\ \pi_N\downarrow & & \downarrow\pi_M \\ N & \xrightarrow{\varphi} & M \end{array} \qquad \pi : \mathcal{A} \longrightarrow M$$

Moreover, the vector bundle $\big((T\varphi)^*\mathcal{A}, \pi_{TN} \circ \widehat{\gamma}, N\big)$ is N-isomorphic to a vector subbundle of the Whitney sum $TN \oplus \varphi^*\mathcal{A}$, whose sections of this vector subbundle are the sections

$$s = (X, \xi) : N \longrightarrow TN \oplus \varphi^*\mathcal{A}$$

($X \in \Gamma(TN)$ and $\xi \in \Gamma(\varphi^*\mathcal{A})$) of the vector bundle $TN \oplus \varphi^*\mathcal{A}$ characterized by the equality $T(\varphi)(X) = \gamma(\Phi \circ \xi)$, where $\Phi : \varphi^*\mathcal{A} \longrightarrow \mathcal{A}$ stands for the canonical vector bundle morphism defined by $\Phi(x, u) = u$.

The notable fact is that the vector bundle $\big((T\varphi)^*\mathcal{A}, \pi_{TN} \circ \widehat{\gamma}, N\big)$ inherits a natural structure of transitive Lie algebroid on N. We note this structure of Lie algebroid on next proposition. The details of the proof can be found in ([7],[10]).

Proposition 1.1.5. Keeping the same hypothesis and notations as above, if $\mathcal{A}$ is a transitive Lie algebroid on M, then the vector bundle $\big((T\varphi)^*\mathcal{A}, \pi_{TN} \circ \widehat{\gamma}, N\big)$ carries a natural structure of transitive Lie algebroid on N, in which the canonical projection $\widehat{\gamma} : (T\varphi)^*\mathcal{A} \longrightarrow TN$ is the anchor map and the Lie bracket is defined in the following way: Fix a local frame $(s_1, \cdots, s_k)$ of the vector bundle $(\mathcal{A}, \pi, M)$ defined on an open subset U of M. Take now two sections (X, ξ), $(Y, \eta) \in \Gamma\big((T\varphi)^*\mathcal{A}\big)$, where $X, Y \in \Gamma(TN)$ and $\xi, \eta \in \Gamma(\varphi^*\mathcal{A})$. Then, on the open subset $V = \varphi^{-1}(U)$, we have the decompositions $\xi_{/V} = \sum_i f_i(\mathbf{s}_i \circ \varphi_{/V})$ and $\eta_{/V} = \sum_j g_j(\mathbf{s}_j \circ \varphi_{/V})$ with $f_i, g_j \in C^\infty(V)$. Define a Lie bracket by setting

$$[(X, \xi), (Y, \eta)]_{/V} =$$

$$= \Big([X, Y]_{/V}, \sum_{i,j} f_i g_j [\mathbf{s}_i, \mathbf{s}_j] \circ \varphi_{/V} + \sum_j (X \cdot g_j)(\mathbf{s}_j \circ \varphi_{/V}) - \sum_i (Y \cdot f_i)(\mathbf{s}_i \circ \varphi_{/V})\Big)$$

Thus, the pair of mappings $(\varphi^{!!}, \varphi)$, in which $\varphi^{!!} : (T\varphi)^*\mathcal{A} \longrightarrow \mathcal{A}$ is the smooth map defined by $\varphi^{!!}(X, a) = a$, is a morphism of Lie algebroids.

Definition (Inverse image of transitive Lie algebroid). Keeping the same hypothesis and notations as above, the vector bundle $\big((T\varphi)^*\mathcal{A}, \pi_{TN} \circ \widehat{\gamma}, N\big)$ equipped with this structure of Lie algebroid is called the inverse image Lie algebroid of $\mathcal{A}$ by the map φ and denoted by $(\varphi^{!!}\mathcal{A}, \widehat{\gamma}, [\cdot, \cdot]_{\varphi^{!!}\mathcal{A}})$. From now on, we write simply $\varphi^{!!}\mathcal{A}$ instead of $(\varphi^{!!}\mathcal{A}, \widehat{\gamma}, [\cdot, \cdot]_{\varphi^{!!}\mathcal{A}})$, dropping the anchor $\widehat{\gamma}$ and the Lie bracket $[\cdot, \cdot]_{\varphi^{!!}\mathcal{A}}$. The smooth map $\varphi^{!!}$ is called the induced map by φ and the Lie algebroid morphism $(\varphi^{!!}, \varphi)$ is called the canonical Lie algebroid morphism of an induced Lie algebroid.

Example. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The Lie algebra $\mathfrak{g}$ is a Lie algebroid over a one-point space $M = \{*\}$. Let N be a smooth manifold and $\varphi : N \longrightarrow M$ the constant map. Then, $(T\varphi)^*\mathfrak{g}$ is equal to $TN \oplus (TN \times \mathfrak{g})$ and we easily see that the anchor and the Lie bracket of the Lie algebroid $(T(\varphi))^*\mathfrak{g}$ coincides with the ones of the trivial Lie algebroid $TN \oplus (TN \times \mathfrak{g})$.

We define now restriction of a transitive Lie algebroid to a general submanifold of the base space.

Definition (Restriction of transitive Lie algebroids). Let M be a smooth manifold and $\varphi : N \hookrightarrow M$ an embedded submanifold, possibly with boundary and corners. Let $\mathcal{A}$ be a transitive Lie algebroid on M. The Lie algebroid $\varphi^{!!}\mathcal{A}$, constructed as inverse image of $\mathcal{A}$ by the map φ, is called the Lie algebroid restriction of $\mathcal{A}$ to the submanifold N and denoted by $\mathcal{A}_N^{!!}$.

Next three propositions are widely used in this work.

Proposition 1.1.6 (Transitivity of restriction of transitive Lie algebroids). Let M, N and P three smooth manifolds such that $\varphi : N \hookrightarrow M$ is an embedded submanifold and $\psi : P \hookrightarrow N$ is an embedded submanifold. Consider a transitive Lie algebroid $(\mathcal{A}, [\cdot,\cdot], \gamma)$ on M. Then,

$$(\mathcal{A}_N^{!!})_P^{!!} \simeq \mathcal{A}_P^{!!}$$

Proof.

$$\mathcal{A}_N^{!!} = \{(x,u,a) \in TN \times \mathcal{A} : (x,u) = \gamma(a)\}$$

and

$$(\mathcal{A}_N^{!!})_P^{!!} = \{((y,v),(x,u),a) \in TP \times (TN \times \mathcal{A}) : (y,v) = \gamma_{\mathcal{A}_N^{!!}}((x,u),a)\}$$

The map λ from $\mathcal{A}_P^{!!}$ onto $(\mathcal{A}_N^{!!})_P^{!!}$ given by $\lambda(y,v,a) = ((y,v),(y,v),a)$ is an isomorphism of Lie algebroids over P. □

Given a smooth manifold M and N a submanifold of M, the vector bundle TM_N, restriction of TM to a N, does not coincide in general with the tangent bundle TN. However, in the context restrictions of transitive Lie algebroids, the situation is different and much better. We note that in the next proposition. The proof is made directly from definition of image inverse of Lie algebroids.

Proposition 1.1.7 (Restriction of tangent Lie algebroids to submanifolds).

Let M be a smooth manifold and $\varphi : N \hookrightarrow M$ a submanifold of M. Consider the tangent Lie algebroids TM and TN. Then, $(TM)^{!!}_N = TN$.

Next proposition is concerning restrictions of trivial Lie algebroids.

Proposition 1.1.8 (Restriction of trivial Lie algebroids). Let $\mathfrak{g}$ be a real finite dimensional Lie algebra and M a smooth manifold. Consider the trivial Lie algebroid $TM \oplus (M \times \mathfrak{g})$ and let N be an embedded submanifold of M. Then,

$$(TM \oplus (M \times \mathfrak{g}))^{!!}_N = TN \oplus (N \times \mathfrak{g})$$

Proof. Let $\varphi : N \hookrightarrow M$ the inclusion. The Lie algebroid $(TM \oplus (M \times \mathfrak{g}))^{!!}_N$, seen as the Lie algebroid $\mathbf{Im}\, \varphi^{!!}$, is constituted by the elements $((x, u), (x, v)) \in TM \oplus (M \times \mathfrak{g})$ such that there exists $(\widetilde{x}, \widetilde{u}) \in TN$ satisfying the equality $T(\varphi)(\widetilde{x}, \widetilde{u}) = \gamma((x, u), (x, v))$. Then, $(x, u) = (\widetilde{x}, \widetilde{u}) \in TN$ and so we have that $(TM \oplus (M \times \mathfrak{g}))^{!!}_N = TN \oplus (N \times \mathfrak{g})$. $\square$

It is evident that, in the case in which U is an open subset of M, the Lie algebroid restriction $\mathcal{A}^{!!}_U$ has the natural structure of the transitive Lie algebroid given on the restricted vector bundle $\mathcal{A}_U$, that is, the map $\psi : \mathcal{A}_U \longrightarrow \mathcal{A}^{!!}_U$ defined by $\psi(a) = (\gamma(a), a)$ is an U-isomorphism of Lie algebroids. Therefore, the Lie algebra structure on the set of all sections of $\mathcal{A}^{!!}_U$ is defined by extending sections of $\mathcal{A}_U$ to sections of $\mathcal{A}$. We are going to show that, if M is a smooth manifold, $\varphi : N \hookrightarrow M$ an embedded submanifold of M, which is a closed subset of M in topological sense, and $(\mathcal{A}, [\cdot, \cdot], \gamma)$ a transitive Lie algebroid on M, the structure of Lie algebra on set of the sections of the Lie algebroid $\mathcal{A}^{!!}_N$ is also defined by natural extension of sections of $\mathcal{A}^{!!}_N$ to sections of $\mathcal{A}$. For that, we state a preparatory proposition in which the Lie algebroid $\mathcal{A}^{!!}_N$ can be seen as a vector subbundle of the restricted vector bundle $\mathcal{A}_N$.

Proposition 1.1.9. Let M be a smooth manifold and $\varphi : N \hookrightarrow M$ an embedded submanifold, possibly with boundary and corners. Let $(\mathcal{A}, [\cdot, \cdot], \gamma)$ be a transitive Lie algebroid on M and $\pi : \mathcal{A} \longrightarrow M$ the vector bundle underlying the Lie algebroid $\mathcal{A}$.

Denote by $\mathcal{A}_N$ the vector bundle restriction of $\mathcal{A}$ to N and $\varphi^{!!} : \varphi^{!!}\mathcal{A} \longrightarrow \mathcal{A}$ the canonical morphism defined by $\varphi^{!!}(X, a) = a$. Then, the following assertions holds.

a) The bundle $\mathbf{Im}\,\varphi^{!!}$ is a vector subbundle of $\mathcal{A}_N$ and $\varphi^{!!} : \varphi^{!!}\mathcal{A} \longrightarrow \mathbf{Im}\,\varphi^{!!}$ is an N-isomorphism of vector bundles. Its inverse is the map $(\varphi^{!!})^{-1}(a) = (\gamma(a), a)$.

b) $\mathbf{Im}\,\varphi^{!!} = \gamma^{-1}(TN)$ and so the anchor $\gamma : \mathcal{A} \longrightarrow TM$ restricts to $\varphi^{!!} \longrightarrow TN$.

c) If ξ, $\eta \in \Gamma(\mathcal{A})$ are sections such that $\xi_{/N}$ and $\eta_{/N} \in \Gamma(\mathbf{Im}\,\varphi^{!!})$ then $[\xi, \eta]_{/N}$ is a section of $\mathbf{Im}\,\varphi^{!!}$.

d) If ξ, $\eta \in \Gamma(\mathcal{A})$ such that $\xi_{/N} \in \Gamma(\mathbf{Im}\,\varphi^{!!})$ and $\eta_{/N} = 0$, then $[\xi, \eta]_{/N} = 0$.

Proof. a) Standard arguments.

b) Let $a \in \mathbf{Im}\,\varphi^{!!}$. Then, there exists $(x, u) \in TN$ such that

$$\gamma(a) = (\varphi(x), D\varphi_x(u)) = (x, u) \in TN$$

Hence, $\gamma(a) \in TN$.

c) Let $\xi, \eta \in \Gamma\mathcal{A}$ such that $\xi_{/N}$, $\eta_{/N} \in \Gamma(\mathbf{Im}\,\varphi^{!!})$. Then, there are vector fields $X, Y \in TN$ such that $(X, \xi_{/N})$, $(Y, \eta_{/N}) \in \Gamma(\mathcal{A}_N^{!!})$, and so, $\gamma \circ \xi_{/N} = X$ and $\gamma \circ \eta_{/N} = Y$. Then, we have $[(X, \xi_{/N}), (Y, \eta_{/N})] \in \Gamma(\mathcal{A}_N^{!!})$. We are going to calculate a local expression for this last Lie bracket. As done in proposition 1.1.5, fix a local frame $(s_1, \cdots, s_k)$ of the vector bundle $(\mathcal{A}, \pi, M)$ over an open subset U of M and set $V = U \cap N$. Writing $\xi_{/U} = \sum_i f_i \mathbf{s}_i$ and $\eta_{/U} = \sum_j g_j \mathbf{s}_j$, with $f_i, g_j \in C^\infty(U)$, we also can write $\xi_{/V} = \sum_i f_{i_{/V}} \mathbf{s}_{i/V}$ and $\eta_{/V} = \sum_j g_{j_{/V}} \mathbf{s}_{j/V}$, and so, $[(X, \xi_{/N}), (Y, \eta_{/N})]_{/V} = \big([X, Y]_{/V}, (*)\big)$ where $(*)$ means the following sum

$$(*) = \sum_{i,j} f_{i_{/V}} g_{j_{/V}} [\mathbf{s}_i, \mathbf{s}_j]_{/V} + \sum_j (X_{/V} \cdot g_{j_{/V}}) \mathbf{s}_{j_{/V}} \; - \sum_i (Y_{/V} \cdot f_{i_{/V}}) \mathbf{s}_{i_{/V}}$$

On other side,

$$[\xi, \eta]_{/U} = [\xi_{/U}, \eta_{/U}] = [\xi_{/U}, \sum_j g_j \mathbf{s}_j] =$$

$$= \sum_j g_j[\xi_{/U}, s_j] + \sum_j \big((\gamma \circ \xi_{/U}) \cdot g_j)s_j\big) =$$

$$= -\sum_j g_j[s_j, \xi_{/U}] + \sum_j \big((\gamma \circ \sum_i f_i s_i) \cdot g_j)s_j\big) =$$

$$= -\sum_j g_j[s_j, \xi_{/U}] + \sum_j \big((\sum_i (f_i(\gamma \circ s_i)) \cdot g_j)s_j\big) =$$

$$= -\sum_j g_j[s_j, \xi_{/U}] + \sum_j \big(\sum_i f_i(\gamma \circ s_i)\big) \cdot g_j)s_j\big) =$$

$$= \sum_{i,j} f_i g_j[s_i, s_j] - \sum_j g_j\big(\sum_i ((\gamma \circ s_j) \cdot f_i)s_i\big) + \sum_j \big(\sum_i f_i(\gamma \circ s_i)\big) \cdot g_j)s_j\big) =$$

$$= \sum_{i,j} f_i g_j[s_i, s_j] - \sum_j \big(\sum_i (g_j(\gamma \circ s_j) \cdot f_i)s_i\big) + \sum_j \big(\sum_i f_i(\gamma \circ s_i)\big) \cdot g_j)s_j\big) =$$

$$= \sum_{i,j} f_i g_j[s_i, s_j] - \sum_i \big(\sum_j (g_j(\gamma \circ s_j)) \cdot f_i)s_i\big) + \sum_j \big(\sum_i f_i(\gamma \circ s_i)\big) \cdot g_j)s_j\big) =$$

Since $X_{/V} = \gamma \circ \xi_{/V} = \gamma \circ (\sum_i f_{i/V} \mathbf{s}_{i/V}) = \sum_i f_{i/V}(\gamma \circ s_{i/N})$ and, analogously $Y_{/V} = \sum_j g_{j/V}(\gamma \circ s_{j/N})$ we conclude that

$$[\xi, \eta]_{/V} = \sum_{i,j} f_{i/V} g_{j/V}[s_i, s_j]_{/V} +$$

$$-\sum_i \big(\sum_j (g_{j/V}(\gamma \circ s_{j/V})) \cdot f_{i/V})s_{i/V}\big) + \sum_j \big(\sum_i f_{i/V}(\gamma \circ s_{i/V})\big) \cdot g_{j/V})s_{j/V}\big) =$$

$$= \sum_{i,j} f_{i/V} g_{j/V}[s_i, s_j]_{/V} - \sum_i \big(Y_{/V}) \cdot f_{i/V})s_{i/V}\big) + \sum_j \big(X_{/V} \cdot g_{j/V})s_{j/V}\big) = (*)$$

Therefore, $[(X, \xi_{/N}), (Y, \eta_{/N})]_{/V} = \big([X, Y]_{/V}, [\xi, \eta]_{/V}\big) \in \Gamma((\mathcal{A}^{!!}_N)_{/V})$ and then $[\xi, \eta]_{/V}$ is a local section of the vector bundle $\mathbf{Im}\,\varphi^{!!}$. Hence, $[\xi, \eta]$ is a section of the vector bundle $\mathbf{Im}\,\varphi^{!!}$.

d) From the proof of c) and by hypothesis, we have that $0 = \gamma \circ \eta_{/N} = Y$ and so $[X, Y] = 0$. From the equality $\eta_{/V} = \sum_j g_{j/V} \mathbf{s}_{j/V}$, we have that $g_{j/V} = 0$ for each j. Hence, on the sum denoted by $(*)$ in the proof of c), all summands are equal to zero. Then, $[\xi, \eta]_{/V} = 0$ and therefore, $[\xi, \eta]_{/N} = 0$ also. □

Proposition 1.1.10. Keeping the same hypothesis and notations as in previous proposition, if $\mathcal{A}$ is a transitive Lie algebroid on M and $\varphi : N \hookrightarrow M$ an embedded submanifold which is a closed subset of M in topological sense, then $\mathbf{Im}\ \varphi^{!!}$ carries a natural structure of transitive Lie algebroid on N, where the anchor of this structure is the restriction of γ to $\mathbf{Im}\ \varphi^{!!} \longrightarrow TN$ and the Lie bracket is defined by

$$[\xi, \eta] = [\widehat{\xi}, \widehat{\eta}]_{/N}$$

in which $\widehat{\xi}$ and $\widehat{\eta}$ are sections of $\mathcal{A}$ such that $\widehat{\xi}_{/N} = \xi$ and $\widehat{\eta}_{/N} = \eta$. Moreover, the canonical morphism $\varphi^{!!} : \varphi^{!!}\mathcal{A} \longrightarrow \mathbf{Im}\ \varphi^{!!}$ is an N-isomorphism of Lie algebroids whose its inverse is the map $\Psi : \mathbf{Im}\ \varphi^{!!} \longrightarrow \varphi^{!!}\mathcal{A}$ defined by $\Psi(a) = (\gamma(a), a)$.

Proof. Since N is closed subset of M, it is always possible to extends sections of $\mathbf{Im}\ \varphi^{!!}$ to a sections of $\mathcal{A}$. By using c) and d) of last proposition and similar arguments given on the definition of Lie bracket of a restriction to an open subset ([10]), we can define a Lie bracket on the sections of $\mathbf{Im}\ \varphi^{!!}$. □

Remark. The Lie algebroid $\mathbf{Im}\ \varphi^{!!}$ constructed on last proposition can be identified to the Lie algebroid $\mathcal{A}_N^{!!}$ and so, the Lie algebroid $\mathbf{Im}\ \varphi^{!!}$ is also called the Lie algebroid restriction of $\mathcal{A}$ to N. Henceforth, the Lie algebroid $\mathbf{Im}\ \varphi^{!!}$ will be denoted by $\mathcal{A}_N^{!!}$.

1.2 Smooth forms and cohomology

We shall recall briefly the cochain algebra of smooth forms on Lie algebroids and respective Lie algebroid cohomology. Some remarks on extensions of smooth forms are presented.

Let $\mathbf{R}_M$ denote the trivial bundle $M \times \mathbf{R}$ of base M. When $p = 0$, we set $\Lambda^0(\mathcal{A}^*; \mathbf{R}_M) = \mathbf{R}_M$. For each $p \geq 1$, we are going to denote $\Lambda^p(\mathcal{A}^*; \mathbf{R}_M)$ the vector bundle of the alternating p-linear maps from the vector bundle $\mathcal{A}$ to the vector bundle $\mathbf{R}_M$, that is, by definition

$$\Lambda^p(\mathcal{A}^*; \mathbf{R}_M) = \{(x, \xi) : \quad x \in M, \quad \xi \in \Lambda_{\mathbf{R}}(\mathcal{A}_x, \cdots, \mathcal{A}_x; \mathbf{R})\}$$

For $p = 1$ we have $\Lambda^1(\mathcal{A}^*; \mathbf{R}_M) = L(\mathcal{A}; \mathbf{R}_M)$.

The set $\Gamma\Lambda^p(\mathcal{A}^*; \mathbf{R}_M)$ of all sections of $\Lambda^p(\mathcal{A}^*; \mathbf{R}_M)$ is a $C^\infty(M)$-module for each $p \geq 0$. We notice that $\Gamma\Lambda^0(\mathcal{A}^*; \mathbf{R}_M) = C^\infty(M)$.

Definition. A smooth form of degree p on the Lie algebroid $\mathcal{A}$ is a section of $\Lambda^p(\mathcal{A}^*; \mathbf{R}_M)$. The set of all smooth forms of degree p on $\mathcal{A}$ will be denoted by $\Omega^p(\mathcal{A}; M)$. We may write $\Omega^p(\mathcal{A}; M) = \Gamma\Lambda^p(\mathcal{A}^*; \mathbf{R}_M)$. For $p = 0$, we have $\Omega^0(\mathcal{A}; M) = C^\infty(M)$. A smooth form of degree p on the Lie algebroid $\mathcal{A}$ can be seen as an element of the $C^\infty(M)$-module $\bigwedge^p(\Gamma(\mathcal{A}), C^\infty(M))$. We set

$$\Omega^*(\mathcal{A}; M) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{A}; M)$$

and the elements of $\Omega^*(\mathcal{A}; M)$ are called smooth forms on the Lie algebroid $\mathcal{A}$.

Clearly, the exterior product of alternated multi-linear maps induces an exterior product of smooth forms on $\mathcal{A}$, which is is associative and graded commutative (anti-commutative). Thus, this product makes $\Omega^*(\mathcal{A}; M)$ into a commutative graded algebra, in which the constant map $\mathbf{1} \in C^\infty(M)$ is the unit.

Remark 1. We notice that $\Omega^*(\mathcal{A}; M)$ vanishes for degrees $>$ rank of $\mathcal{A}$.

Remark 2. When $\mathcal{A}$ is the tangent Lie algebroid TM, the space $\Omega^*(\mathcal{A}; M)$ is the canonical space of the smooth forms on M.

Remark 3. When A is the Lie algebroid $\mathfrak{g}$ over a one-point set ($\mathfrak{g}$ a Lie algebra), the space $\Omega^*(\mathcal{A}; M)$ is the Chevalley-Eilenberg space.

Next, we recall the definition of exterior derivative. Let M be a smooth manifold and $\mathcal{A}$ a Lie algebroid on M, with anchor $\gamma : \mathcal{A} \longrightarrow TM$ and Lie bracket $[\cdot,\cdot]$ on $\Gamma(\mathcal{A})$. We first consider the algebra $\Omega^0(\mathcal{A}; M) = C^\infty(M)$. Let $f \in C^\infty(M)$. We can define the smooth 1-form

$$d(f) : M \longrightarrow \bigwedge^1(\Gamma(TM), C^\infty(M))$$

$$d(f)(X) = Df_x(X_x) = X \cdot f$$

for each $X \in \Upsilon(M)$. Hence, we define $d(f) \in \bigwedge^1(\Gamma(\mathcal{A}), C^\infty(M))$ by

$$d(f)(X) = (\gamma \circ X) \cdot f$$

for each $X \in \Gamma(\mathcal{A})$. Now, for each $p \geq 1$ we define

$$d^p : \Omega^p(\mathcal{A}; M) \longrightarrow \Omega^{p+1}(\mathcal{A}; M)$$

$$d^p\omega(X_1, X_2, \cdots, X_{p+1}) = \sum_{j=1}^{p+1}(-1)^{j+1}(\gamma \circ X_j) \cdot (\omega(X_1, \cdots, \widehat{X_j}, \cdots, X_{p+1})) +$$

$$+\sum_{i<k}(-1)^{i+k}\omega([X_i, X_k], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_k}, \cdots, X_{p+1})$$

for $\omega \in \Omega^p(\mathcal{A}; M)$ and $X_1, X_2, \cdots, X_{p+1} \in \Gamma(\mathcal{A})$.

The family of differential operators $d^* = (d^p)_{p\geq 0}$ defines, on the commutative graded algebra $\Omega^*(\mathcal{A}; M)$, a structure of differential graded algebra. Hence, $\Omega^*(\mathcal{A}; M)$ becomes a commutative cochain algebra, which is defined over $\mathbf{R}$.

Definition (Lie algebroid cohomology). Keeping the same hypothesis and notation as above, the Lie algebroid cohomology space of $\mathcal{A}$ is the cohomology space of the algebra $\Omega^*(\mathcal{A}; M)$ equipped with the structures defined above. This cohomology space is denoted by $H^*(\mathcal{A}; M)$.

In view of discuss of extensions of smooth forms in Lie algebroids, we recall the definition of inverse image of a smooth form. Let M and N two smooth manifolds and $\mathcal{A}$ and $\mathcal{B}$ Lie algebroids on M and N respectively. Let $\lambda = (F, f)$ be a morphism of Lie algebroids defined by the smooth maps $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $f : M \longrightarrow N$. If ω is a smooth form on $\mathcal{B}$ of degree p, we can consider a smooth form of degree p on $\mathcal{A}$, denoted by $\lambda^*\omega$, and defined by

$$\big(\lambda^*\omega\big)_x(v_1, v_2, \cdots, v_p) = \omega_{f(x)}(F(v_1), F(v_2), \cdots, F(v_p))$$

for each $x \in M$ and $v_1, v_2, \ldots v_p \in \mathcal{A}_x$. The form $\lambda^*\omega$ is called the pullback or inverse image of ω by the morphism λ. Thus, for each $p \geq 0$, there is a map

$$\lambda^{*p} : \Omega^p(\mathcal{B}; N) \longrightarrow \Omega^p(\mathcal{A}; M)$$

$$\omega \longrightarrow \lambda^*\omega$$

The family $\lambda^* = (\lambda^{*p})_{p\geq 0}$ is a morphism of cochain algebras. Therefore, we have a contravariant functor from the category of Lie algebroids to the category of cochain algebras. For details, see [4], [6] and [7].

Definition (Restriction of smooth forms). Let M be smooth manifold and $\mathcal{A}$ a Lie algebroid on M. Let $\varphi : N \hookrightarrow M$ be an embedded smooth manifold of M and consider the canonical morphism $\lambda = (\varphi^{!!}, \varphi)$ in which $\varphi^{!!} : \mathcal{A}_N^{!!} \longrightarrow \mathcal{A}$ is defined by $\varphi^{!!}(X, a) = a$. Consider the cochain algebras $\Omega^*(\mathcal{A}; M)$ and $\Omega^*(\mathcal{A}_N^{!!}; N)$ and the morphism

$$\lambda^* : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}_N^{!!}; N)$$

$$\omega \longrightarrow \lambda^*\omega$$

For each smooth form $\omega \in \Omega^p(\mathcal{A}; M)$, the form $\lambda^*(\omega) \in \Omega^p(\mathcal{A}_N^{!!}; N)$ is called restriction of ω to N and denoted by $\omega_N^{!!}$ or simply by ω_N, when there is no ambiguity with the restriction of ω to the restricted vector bundle $\mathcal{A}_N$. In subsequent sections, the smooth embedding $\varphi : N \hookrightarrow M$ will be often denoted by $\varphi_{M,N} : N \hookrightarrow M$ and the homomorphism $\lambda^* : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}_N^{!!}; N)$ by $\varphi_{M,N}^{\mathcal{A}}$. The homomorphism $\varphi_{M,N}^{\mathcal{A}}$ will be called the homomorphism of cochain algebras generated by the inclusion $\varphi_{M,N}$.

We will notice here some considerations on extension of smooth forms. We begin first with some remarks on extension of smooth forms on vector bundles. After those remarks, we then state a proposition concerned to extension of smooth forms on Lie algebroids. In the case of vector bundles, we are going to divide in two cases. The first case is very basic and is the one in which we have a vector bundle (E, π, M) defined over a smooth manifold M, N a submanifold of M which is closed subset of M, and we want to extend smooth forms belonging to the restricted vector bundle E_N. In this case, for each $x \in N$, the fibres of E_N are the same as the fibres of E. The second case is the one in which we have a vector subbundle (F, π', N) of a given vector bundle (E, π, M). In this case, the fibres of F are vector subspaces of the fibres of E and, for extension of smooth forms, we need to fix a Riemannian structure.

Remark 1. Let (E, π, M) be a vector bundle defined over a smooth manifold M of dimension n and N a submanifold of M which is closed subset of M in the topological sense. Let $\omega \in \Omega^p(E_N; N)$ be a smooth form on the restricted vector bundle E_N defined over N and suppose there exist an open subset U of M such that $N \subset U$ and a smooth form $\widetilde{\omega} \in \Omega^p(E_U; U)$ such that $\widetilde{\omega}_{/N} = \omega$. Then, the form ω extends to a smooth form $\xi \in \Omega^p(E; M)$.

Proof. We can consider a partition of unity $\varphi_1, \varphi_2 : [0,1] \longrightarrow M$ correspondent to the open covering of M made by $M \backslash N$ and U such that $\varphi_1(x) + \varphi_2(x) = 1 \quad \forall\, x \in M$, $\mathbf{supp}\, \varphi_1 \subset M \backslash N$ and $\mathbf{supp}\, \varphi_2 \subset U$. Define then $\xi \in \Omega^p(E; M)$ by

$$\xi_x = \begin{cases} \varphi_2(x)\ \widetilde{\omega}_x & \text{if } x \in U \\ \\ 0 \in \bigwedge^p(E_x; \mathbf{R}) & \text{if } x \in M \backslash \mathbf{supp}\, \varphi_2 \end{cases}$$

The form ξ is then a smooth extension of ω to the whole M. □

We can improve last remark and obtain the following.

Remark 2. Let (E, π, M) be a vector bundle defined over a smooth manifold M of dimension n and N a submanifold of M which is closed subset of M in the topological sense. Let $\omega \in \Omega^p(E_N; N)$ be a smooth form on the restricted vector bundle E_N defined over N and suppose there is an open subset U of M, with $N \subset U$, such that U is the domain of a chart of the vector bundle E

$$(\pi, \psi) : \pi^{-1}(U) \longrightarrow U \times \mathbf{R}^n$$

Then, the form ω extends to a smooth form $\xi \in \Omega^p(E_U; U)$.

Proof. Let

$$(\widehat{\pi}, \widehat{\psi}) : \widehat{\pi}^{-1}(U) \longrightarrow U \times \wedge^p(\mathbf{R}^n)$$

be the corresponding chart of the vector bundle $(\bigwedge^p E, \widehat{\pi}, M)$ and consider the smooth map $\theta = \widehat{\psi} \circ \omega : N \longrightarrow \bigwedge^p(\mathbf{R}^n)$. Let $\widetilde{\theta} : U \longrightarrow \bigwedge^p(\mathbf{R}^n)$ be a smooth extension of θ. Then, the form $\xi \in \Omega^p(E_U; M)$ defined by

$$\xi_x = (\widehat{\pi}, \widehat{\psi})^{-1}(x, \widetilde{\theta}(x))$$

is a smooth extension of ω to the open subset U. $\square$

We can again improve last remark and obtain the following.

Remark 3. Let (E, π, M) be a vector bundle defined over a smooth manifold M of dimension n and N a submanifold of M which is closed subset of M in the topological sense. Let $\omega \in \Omega^p(E_N; N)$ be a smooth form on the restricted vector bundle E_N defined over N. Then, the form ω extends to a smooth form $\xi \in \Omega^p(E; M)$.

Proof. For each $x \in N$, let U_x an open neighborhood of x in M where it is defined a chart of the vector bundle. We can apply last proposition to the closed submanifold $N \cap U_x$ and then there is a smooth form $\xi_x \in \Omega^p(E_{U_x}; U_x)$ such that $\xi_{x_{/N\cap U_x}} = \omega_{/N\cap U_x}$. We have $N \subset \bigcup_{x\in N} U_x$. Denote by U_* the open subset $U_* = M \setminus N$. Then $(U_x)_{x\in N\cup U_*}$ is an open covering of M. Let $(V_j)_{j\in J}$ be a locally finite refinement of the covering $(U_x)_{x\in N\cup U_*}$ and consider a partition of unity $(\varphi_j)_{j\in J}$ subordinated to the covering $(V_j)_{j\in J}$. Define now the index sets

$$J_* = \{j \in J : V_j \subset M \setminus N\} \quad \text{and} \quad J_0 = J \setminus J_*$$

We are going to check the following:

- $N \subset \bigcup_{j \in J_0} V_j$.
- For each $j \in J_0$, there exists $x \in N$ such that $V_j \subset U_x$.

For the first statement, if $x \in N$ then there is $j \in J$ with $x \in V_j$. If j could belong to J_*, we would conclude $x \in V_j \subset U_* = M \setminus N$ and so $x \notin N$. For the second statement, given $j \in J_0$, we have that $V_j \subset U_y$ for some $y \in N \cup U_*$. Since $j \notin J_*$ then $V_j \not\subset M \setminus N$ and so $V_j \subset U_x$ for some x which doesn't belong to U_*. Next, we can fix, for each $j \in J_0$, $x_j \in N$ with $V_j \subset U_{x_j}$. Denote the smooth form $\xi_{x_j/V_j} \in \Omega^p(E_{V_j}; V_j)$ by ξ_j. Finally, define $\xi \in \Omega^p(E; M)$ by

$$\xi = \sum_{j \in J0} \varphi_j \xi_j$$

where $\varphi_j \xi_j$ is defined by

$$\varphi_j \xi_j(x) = \begin{cases} \varphi_j(x)\ \xi(x) & \text{if } x \in V_j \\ \\ 0 \in \bigwedge^p(E_x; \mathbf{R}) & \text{if } x \in M \backslash \mathbf{supp}\ \varphi_j \end{cases}$$

We have that ξ is well defined since the covering is locally finite. Moreover, the form ξ is smooth extension of ω. $\square$

We consider now the case in which we have smooth forms defined on a vector subbundle of a given vector bundle. Let (E, π, M) be a vector bundle defined over a smooth manifold M. We recall that a vector bundle (F, π', N) is called vector subbundle of (E, π, M) if F is a submanifold of E, N is a submanifold of M, $\pi(F) \subset N$ and $\pi' = \pi_{/F} : F \longrightarrow N$ and, for each $x \in N$, F_x is a vector subspace of E_x. Fix Riemannian structure on the vector bundle (E, π, M). For each $x \in N$, we have $E_x = F_x \oplus F_x^{\perp}$ and denote $\psi_x : E_x \longrightarrow E_x$ the orthogonal projection over the fibre F_x. Then, the map $\psi : E_{/N} \longrightarrow F$ defined by ψ_x for each $x \in N$ is a N-morphism of vector bundles from $(E_{/N}, \pi, N)$ to (F, π', N). Using this discuss, we get immediately the following proposition.

Remark 4. Let (E, π, M) be a vector bundle on a smooth manifold M, N a submanifold of M and (F, π', N) a vector subbundle of (E, π, M). Let $\omega \in \Omega^p(F; N)$ be a smooth form. Then, the form ω extends to a smooth form $\xi \in \Omega^p(E_N; N)$.

Proof. Using the map ψ above, the form $\psi^*\omega$ is the required extension. □

We consider now the case of extensions smooth forms in Lie algebroids. We begin with two basic remarks.

Remark 5. Let M be a smooth manifold, U an open subset of M and $\varphi : U \longrightarrow M$ the inclusion map. Let $(\mathcal{A}, [\cdot,\cdot], \gamma)$ be a transitive Lie algebroid on M and consider the Lie algebroid $(\mathcal{A}_U, [\cdot,\cdot]_{\mathcal{A}_U}, \gamma_U)$ on U constructed in the proposition 1.1.4 as well the Lie algebroid $(\varphi^{!!}\mathcal{A}, [\cdot,\cdot]_{\varphi^!\mathcal{A}}, \widehat{\gamma})$ on U constructed in the proposition 1.1.5. Then, for each $p \geq 0$, $\Omega^p(\mathcal{A}_U; U)$ and $\Omega^p(\mathcal{A}^{!!}_U; U)$ are isomorphic. Namely, for the maps $\psi : \mathcal{A}_U \longrightarrow \mathcal{A}^{!!}_U$ defined by $\psi(a) = (\gamma(a), a)$ and $\widehat{\psi} : \varphi^{!!}\mathcal{A} \longrightarrow \mathcal{A}_U$ defined by $\widehat{\psi}((x, u), a) = a$, the map $\psi^{*p} : \Omega^p(\mathcal{A}^{!!}_U; U) \longrightarrow \Omega^p(\mathcal{A}_U; U)$ is the inverse of $(\widehat{\psi})^{*p} : \Omega^p(\mathcal{A}_U; U) \longrightarrow \Omega^p(\mathcal{A}^{!!}_U; U)$.

Remark 6. Let $\mathcal{A}$ be a transitive Lie algebroid on a smooth manifold M and $\varphi : N \hookrightarrow M$ an embedded submanifold such that N is a closed subset in M in the topological sense. Let $\varphi^{!!} : \varphi^!\mathcal{A} \longrightarrow \mathcal{A}$ be the canonical morphism defined by $\varphi^{!!}(X, a) = a$ and consider $\mathbf{Im}\, \varphi^{!!}$ equipped with the natural structure of transitive Lie algebroid on N given in the proposition 1.1.9. The map $\varphi^{!!}$ is a N-isomorphism of Lie algebroids between $\varphi^{!!}\mathcal{A}$ and $\mathbf{Im}\, \varphi^{!!}$. Hence, the map $\varphi^{!!}$ induces an isomorphism between $\Omega^p(\mathcal{A}^{!!}_N; N)$ and $\Omega^p(\mathbf{Im}\, \varphi^{!!}; N)$.

Let us notice now a proposition concerning extensions of smooth forms in Lie algebroids.

Proposition 1.2.1. Let M be a smooth manifold and $\varphi : N \hookrightarrow M$ an embedded submanifold such that N is a closed subset in M in the topological sense. Let $\mathcal{A}$ be a transitive Lie algebroid on M and consider the canonical morphism $\lambda = (\varphi^{!!}, \varphi)$ in which $\varphi^{!!} : \mathcal{A}^{!!}_N \longrightarrow \mathcal{A}$ is defined by $\varphi^{!!}(X, a) = a$. Consider the cochain algebras $\Omega^*(\mathcal{A}; M)$ and

$\Omega^*(\mathcal{A}_N^{!!}; N)$. Then, the cochain algebra morphism

$$\lambda^* : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}_N^{!!}; N)$$

$$\omega \longrightarrow \lambda^*\omega$$

is surjective.

Proof. Given a smooth form $\widetilde{\omega} \in \Omega^p(\mathcal{A}_N^{!!}; N)$, we can define a smooth form $\widehat{\omega} \in \Omega^p(\mathbf{Im}\,\varphi^{!!}; N)$ by $\widehat{\omega}(\xi_1, \ldots, \xi_p) = \widetilde{\omega}((\gamma \circ \xi_1, \xi_1), \ldots, (\gamma \circ \xi_1, \xi_p))$ where γ is the anchor of $\mathcal{A}$. We apply the remarks above to the form $\widehat{\omega}$ and the result follows since any smooth extension $\omega \in \Omega^*(\mathcal{A}; M)$ of $\widehat{\omega}$ satisfies $\lambda^*(\omega) = \widetilde{\omega}$. □

Previous propositions are used in the statement of the Mayer-Vietoris sequence in Lie algebroids. We recall it briefly. The details can be seen in [6]. We suppose that U and V are open subsets of M such that $M = U \cup V$ and consider the diagrams where all maps are the inclusions

These inclusions induce, by inverse image, the following cochain maps:

$$i^* : \Omega^*(\mathcal{A}_U; U) \longrightarrow \Omega^*(\mathcal{A}_{U\cap V}; U \cap V) \qquad j^* : \Omega^*(\mathcal{A}_V; V) \longrightarrow \Omega^*(\mathcal{A}_{U\cap V}; U \cap V)$$

$$k^* : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}_U; U) \qquad l^* : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}_V; V)$$

which are defined by $i^*(\omega) = \omega_{/U\cap V}$, $j^*(\omega) = \omega_{/U\cap V}$, $k^*(\omega) = \omega_{/U}$ and $l^*(\omega) = \omega_{/V}$. We denote by $\Omega^*(\mathcal{A}_U; U) \times \Omega^*(\mathcal{A}_V; V)$ the cochain complex product of $\Omega^*(\mathcal{A}_U; U)$ and $\Omega^*(\mathcal{A}_V; V)$ made from the cartesian product $\Omega^p(\mathcal{A}_U; U) \times \Omega^p(\mathcal{A}_V; V)$. Under these conditions, we have an short exact succession of cochain complexes

$$0 \longrightarrow \Omega^*(\mathcal{A}; M) \xrightarrow{\lambda^*} \Omega^*(\mathcal{A}_U; U) \times \Omega^*(\mathcal{A}_V; V) \xrightarrow{\delta^*} \Omega^*(\mathcal{A}_{U\cap V}; U \cap V) \longrightarrow 0$$

in which the maps

$$\lambda^p : \Omega^p(\mathcal{A}; M) \longrightarrow \Omega^p(\mathcal{A}_U; U) \times \Omega^p(\mathcal{A}_V;)$$

$$\delta : \Omega^p(\mathcal{A}_U; U) \times \Omega^p(\mathcal{A}_V; V) \longrightarrow \Omega^p(\mathcal{A}_{U\cap V}; U \cap V)$$

are defined by $\lambda^p(\omega) = (k^*\omega, l^*\omega) = (\omega_{/U}, \omega_{/V})$ and $\delta^p(\alpha, \beta) = j^*\beta - i^*\alpha = = \beta_{/U\cap V} - \alpha_{/U\cap V}$. Therefore, for the short exact succession mentioned above, it corresponds the long exact succession in cohomology

$$\cdots \longrightarrow H^{p-1}(\mathcal{A}_{/U\cap V}; U \cap V) \xrightarrow{\partial^{p-1}} H^p(\mathcal{A}; M)$$

$$H^p(\mathcal{A}; M) \xrightarrow{H^p(\lambda^*)} H^p(\Omega^*(\mathcal{A}_{/U}; U) \times \Omega^*(\mathcal{A}_{/V}; V))$$

$$H^p(\Omega^*(\mathcal{A}_{/U}; U) \times \Omega^*(\mathcal{A}_{/V}; V)) \xrightarrow{H^p(\delta^*)} H^p(\mathcal{A}_{/U\cap V}; U \cap V)$$

$$H^p(\mathcal{A}_{/U\cap V}; U \cap V) \xrightarrow{\partial^p} H^{p+1}(\mathcal{A}; M) \longrightarrow \cdots$$

in which ∂^* is the connecting homomorphism. We recall that the cohomology space $H^p(\Omega^*(\mathcal{A}_{/U}; U) \times \Omega^*(\mathcal{A}_{/V}; V))$ is a product of $H^p(\mathcal{A}_{/U}; U)$ and $H^p(\mathcal{A}_{/V}; V)$, with the projections naturally defined, and hence the space $H^p(\Omega^*(\mathcal{A}_{/U}; U) \times \Omega^*(\mathcal{A}_{/V}; V))$ is isomorphic to the cartesian product $H^p(\mathcal{A}_{/U}; U) \times H^p(\mathcal{A}_{/V}; V)$ by the isomorphism $([\xi], [\eta]) \longrightarrow [(\xi, \eta)]$.

1.3 Trivial Lie algebroids

The aim of this section is to state a result concerning the triviality of transitive Lie algebroids over contractible manifolds. This result is a direct consequence of a deep result on actions of certain cohomology space on the set of operators extensions of Lie algebroids by Lie algebra bundles. This deep result is due to Mackenzie (see [10]). In view of the statement of this result, some definitions and examples of Lie algebroids used in the study of this topic are given below. A complete description of these definitions and results can be found in Mackenzie's book "General Theory of Lie Groupoids and Lie Algebroids". The paper "Comparison of categorical characteristic classes of transitive Lie algebroid with Chern-Weil homomorphism" by Mishchenko and Xiaoyu contains an interesting summary of this study.

Example 1 (Lie algebroid of covariant derivatives). Let M be a smooth manifold and $\pi : E \longrightarrow M$ a vector bundle on M. Denote by $\Phi(E)$ the Lie groupoid on M made by all linear isomorphism $\xi : E_x \longrightarrow E_y$ for each $x, y \in M$ (see [10]). For each $n \geq 1$, let $\pi_L : L^n(E; E) \longrightarrow M$ be the vector bundle whose fibre at $z \in M$ is the vector space of all p-linear maps from $E_z \times \cdots \times E_z$ to E_z. The canonical action

$$\Phi(E) * L^n(E; E) \longrightarrow L^n(E; E)$$

of the Lie groupoid $\Phi(E)$ on the vector bundle $\pi_L : L^n(E; E) \longrightarrow M$ is defined by

$$\xi \cdot \varphi = \xi \circ \varphi \circ (\xi^{-1} \times \cdots \times \xi^{-1}) \in L^n(E_y; E_y)$$

where $x, y \in M$, $\xi : E_x \longrightarrow E_y$ is a linear isomorphism and $\varphi \in L^n(E_x; E_x)$. A section $\eta \in \Gamma(L^n(E; E))$ is stable for this action if, for all $x, y \in M$, there is a linear isomorphism $\xi : E_x \longrightarrow E_y$ such that $\xi \cdot \eta_x = \eta_y$. For a stable section $\eta \in \Gamma(L^n(E; E))$, the stabilizer subgroupoid of $\Phi(E)$ at η is defined by

$$\{\xi \in \Phi(E) : \quad \xi \cdot \eta(\alpha(\xi)) = \eta(\beta(\xi))\}$$

in which $\alpha : \Phi(E) \longrightarrow M$ and $\beta : \Phi(E) \longrightarrow M$ are the source and target projections of the Lie groupoid $\Phi(E)$. We notice that a section $\eta \in \Gamma(L^n(E; E))$ need not be stable. Nevertheless, for a Lie algebra bundle $\pi' : K \longrightarrow M$ on M with bracket $[\cdot, \cdot] \in \bigwedge^2(K, K)$, the bracket $[\cdot, \cdot]$ is a stable section for the action above restricted to the vector bundle $\bigwedge^2(K; K)$. Hence, the stabilizer subgroupoid of $\Phi(K)$ at $[\cdot, \cdot]$ is well defined and denoted by $\Phi_{Aut}(K)$ (see [10]). The Lie algebroid of $\Phi_{Aut}(K)$ is denoted by $\mathcal{D}_{Der}(K)$ and its elements are called covariant derivatives of K. The Lie algebroid $\mathcal{D}_{Der}(K)$ is transitive (see [10], [18]).

Example 2 (Adjoint Lie algebra bundle). Let M be a smooth manifold and $\pi : K \longrightarrow M$ a Lie algebra bundle with fibre type $\mathfrak{g}$. Consider the Lie subalgebra $Der(\mathfrak{g})$ of $\mathfrak{g}l(\mathfrak{g})$ made by the derivations of $\mathfrak{g}$. In [10], it can be seen that the Lie subalgebra $Der(\mathfrak{g})$ corresponds to a unique Lie algebra subbundle of the Lie algebra bundle $\mathbf{End}(K)$. This Lie algebra subbundle is denoted by $Der(K)$ and its elements are called derivations of K. Thus, the image $\mathbf{ad}(K)$ of K, by the Lie algebra bundle morphism $\mathbf{ad} : K \longrightarrow Der(K)$ defined by $\mathbf{ad}_x : K_x \longrightarrow Der(K_x)$ for each $x \in M$, is a Lie algebra subbundle of $Der(K)$, in which the fibre $\mathbf{ad}(K)_x$ is an ideal of $Der(K_x)$ (see [10]). Consequently, the Lie algebra bundle quotient $Der(K)/\mathbf{ad}(K)$ is well defined. The Lie algebra subbundle $\mathbf{ad}(K)$ is called the adjoint Lie algebra bundle of K. The Lie algebra bundle quotient $Der(K)/\mathbf{ad}(K)$ usually is denoted by $\mathbf{Out}\ (K)$.

Example 3 (Lie algebroid quotient). Let M be a smooth manifold and $\mathcal{A}$ a Lie algebroid over M, with anchor $\gamma : \mathcal{A} \longrightarrow TM$ and Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ on $\Gamma(\mathcal{A})$. Consider the Lie algebra bundle $\mathbf{Ker}\ \gamma$. An ideal of $\mathcal{A}$ is a Lie algebra subbundle K of $\mathbf{Ker}\ \gamma$ such that, for all sections $\xi \in \Gamma(\mathcal{A})$ and $\eta \in \Gamma K$, $[\xi,\eta]_{\mathcal{A}}$ is a section of K. In these conditions, one defines the Lie algebroid quotient of $\mathcal{A}$ by K as follows. Let $\overline{\mathcal{A}}$ be the vector bundle quotient $\mathcal{A}/K$ and $\overline{\gamma} : \overline{\mathcal{A}} \longrightarrow TM$ the map induced by γ. The Lie bracket in the space of the sections of $\overline{\mathcal{A}}$ is defined by

$$[\xi + \Gamma K, \eta + \Gamma K]_{\overline{\mathcal{A}}} = [\xi,\eta]_{\mathcal{A}} + \Gamma K$$

for each ξ, $\eta \in \Gamma(\mathcal{A})$. The Lie algebroid $\overline{\mathcal{A}}$ is transitive and usually denoted by $\mathcal{A}/K$ (see [10] or [11]).

A particular case of Lie algebroid quotient is the following. Let $\pi : K \longrightarrow M$ be a Lie algebra bundle on a smooth manifold M and consider the transitive Lie algebroid $\mathcal{D}_{Der}(K)$ of all covariant derivatives of a Lie algebra bundle K. The adjoint Lie algebra bundle $\mathbf{ad}(K)$ is an ideal of $\mathcal{D}_{Der}(K)$. Hence, we can consider the transitive Lie algebroid quotient $\mathcal{D}_{Der}(K)/\mathbf{ad}(K)$, which is denoted by $\mathbf{Out}\ \mathcal{D}(K)$. The sections of $\mathbf{Out}\ \mathcal{D}(K)$ are called outer derivations of K.

The Lie algebroid $\mathbf{Out}\ \mathcal{D}(K)$ of a Lie algebra bundle K plays a fundamental role in the development of cohomology theory of Lie algebroids. It is used to define couplings associated to extensions of Lie algebroids and consequently to define the affine space of operator extensions. The heart of this theory is that the additive group of the cohomology space in degree two of special representation induced by a coupling acts freely and transitively on the affine space of operator extensions (see proposition 1.3.1 below or [10], [11]). We briefly summary some definitions concerning couplings and representations, following [18]. A detailed study on those topics can be found in [10] or [11].

Representations. Let $\mathcal{A}$ be a Lie algebroid on a smooth manifold M, E a vector bundle on M and $\mathcal{D}(E)$ the Lie algebroid of covariant differential operators on $\Gamma(E)$. A representation of $\mathcal{A}$ on E is a Lie algebroid morphism $\rho : \mathcal{A} \longrightarrow \mathcal{D}(E)$.

On the graded algebra $\Omega^*(\mathcal{A}; M)$, an exterior derivative can be defined, in the same way as we have done in the second section, but taking $\rho(X_j)$ instead $\gamma \circ X_j$ (in our case, we used the trivial representation ρ' defined by $\rho'(X) = \gamma \circ X$, where γ is the anchor of $\mathcal{A}$). The cohomology space of this cochain algebra is denoted by $H(\mathcal{A}, \rho, E)$.

Couplings. Let $\mathcal{A}$ be a Lie algebroid on a smooth manifold M and K a Lie algebra bundle on M. A coupling of $\mathcal{A}$ with K is a morphism of Lie algebroids $\Xi : \mathcal{A} \longrightarrow \mathbf{Out}\ \mathcal{D}(K)$. In [10] or [11], it can be seen that a coupling induces a representation of $\mathcal{A}$, denoted by ρ^{Ξ}, on the Lie algebra bundle $\mathbf{Out}\ \mathcal{D}(ZK)$, in which ZK denotes the Lie algebra subbundle centre of K. The representation ρ^{Ξ} is called central representation of the coupling Ξ. A Lie derivation law covering the coupling Ξ is a vector bundle morphism $\nabla : \mathcal{A} \longrightarrow \mathcal{D}_{Der}(K)$ that preserves the anchor maps and satisfies the equality $\natural \circ \nabla = \Xi$, in which $\natural : \mathcal{D}_{Der}(K) \longrightarrow \mathbf{Out}\ \mathcal{D}(K) = \mathcal{D}_{Der}(K)/\mathbf{ad}(K)$ is the quotient map. Since the map $\natural$ is a surjective submersion, any coupling Ξ admits Lie derivation laws covering it. For transitive Lie algebroids, a Lie derivation law covering a couple Ξ can be obtained by taking the covariant derivative of a connection $\lambda : TM \longrightarrow \mathcal{A}$.

Obstruction. Let $\mathcal{A}$ be a Lie algebroid on a smooth manifold M, K a Lie algebra bundle on M and $\Xi : \mathcal{A} \longrightarrow \textbf{Out}\ \mathcal{D}(K)$ a coupling of $\mathcal{A}$ with K. For any Lie derivation law $\nabla : \mathcal{A} \longrightarrow \mathcal{D}_{Der}(K)$ covering the coupling Ξ, one defines the curvature $R_\nabla : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{D}_{Der}(K))$ of ∇ which is defined by

$$R_\nabla(\xi, \eta) = [\nabla(\xi), \nabla(\eta)] - \nabla([\xi, \eta])$$

for ξ, $\eta \in \Gamma(\mathcal{A})$. Let $\Lambda : \bigwedge^2(\mathcal{A}) \longrightarrow K$ be a lifting of R_∇ (see [18]). Then, the cyclic sum of

$$\nabla_\xi(\Lambda(\eta, \theta)) - \Lambda([\xi, \eta], \theta)$$

defines an element of $\mathbf{Z}^3(\mathcal{A}, \rho^\Xi, ZK)$. The notable fact is that, the cohomology class of this element is independent of the choice of ∇ and Λ, depending only on the coupling Ξ. The cohomology class of this element is called the obstruction class of the coupling Ξ, and is denoted by $Obs(\Xi)$ (see [10] or [18]).

Extensions of Lie algebroids. In view of the definition of set of equivalence classes of operator extensions, we recall that an exact sequence of Lie algebroids on a smooth manifold M is a sequence

$$\{0\} \longrightarrow \mathcal{A}' \xrightarrow{j} \mathcal{A} \xrightarrow{\lambda} \mathcal{A}'' \longrightarrow \{0\}$$

in which $\mathcal{A}'$, $\mathcal{A}$ and $\mathcal{A}''$ are Lie algebroids on M, j and λ are morphisms of Lie algebroids and the sequence is exact as a sequence of vector bundles. We are interested in sequences in which $\mathcal{A}'$ is a Lie algebra bundle. Given a Lie algebra bundle K, an extension of $\mathcal{A}$ by K is an exact sequence

$$\{0\} \longrightarrow K \xrightarrow{j} \mathcal{A}' \xrightarrow{\lambda} \mathcal{A} \longrightarrow \{0\}$$

of Lie algebroids over M. The most important extension for our work is the sequence

$$\{0\} \longrightarrow \mathbf{Ker}\,\gamma \xrightarrow{j} \mathcal{A} \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

of a transitive Lie algebroid $\mathcal{A}$.

Equivalent extensions. Let M be a smooth manifold, $\mathcal{A}$ a Lie algebroid on M and K a Lie algebra bundle on M. Two extensions

$$\{0\} \longrightarrow K \xrightarrow{j_1} \mathcal{A}_1 \xrightarrow{\lambda_1} \mathcal{A} \longrightarrow \{0\}$$

and

$$\{0\} \longrightarrow K \xrightarrow{j_2} \mathcal{A}_2 \xrightarrow{\lambda_2} \mathcal{A} \longrightarrow \{0\}$$

are equivalent if there is a Lie algebroid morphism $\varphi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that $\varphi \circ j_1 = j_2$ and $\lambda_2 \circ \varphi = \lambda_1$. In these conditions, φ is an isomorphism of Lie algebroids.

Transversals. Let M be a smooth manifold, $\mathcal{A}$ a Lie algebroid on M and K a Lie algebra bundle on M. Let

$$\{0\} \longrightarrow K \xrightarrow{j} \mathcal{A}' \xrightarrow{\lambda} \mathcal{A} \longrightarrow \{0\}$$

be an extension of $\mathcal{A}$ by K. A transversal in the extension is a vector bundle morphism $\chi : \mathcal{A} \longrightarrow \mathcal{A}'$ such that $\lambda \circ \chi = id_{\mathcal{A}}$. Since λ is a surjective submersion and a morphism of vector bundles, transversals always exist and they are anchor-preserving morphisms. Fix now any transversal $\chi : \mathcal{A} \longrightarrow \mathcal{A}'$ in the extension above. We can define a coupling of $\mathcal{A}$ with K as follows. Define the map $\nabla^{\chi} : \mathcal{A} \longrightarrow \mathcal{D}_{Der}(K)$ such that

$$j(\nabla^{\chi}(\xi)(Y)) = [\chi(\xi), j(Y)]$$

for each $\xi \in \Gamma(\mathcal{A})$ and $Y \in \Gamma(K)$. The map ∇^{χ} is a morphism of vector bundles preserving the anchor maps. The composition $\natural \circ \nabla^{\chi} : \mathcal{A} \longrightarrow \mathbf{Out}\,\mathcal{D}(K)$ is a coupling (see [10]). If $\zeta : \mathcal{A} \longrightarrow \mathcal{A}'$ is other transversal of the extension, then we obtain the same coupling since the equality $\natural \circ \nabla^{\chi} = \natural \circ \nabla^{\zeta}$ holds. The coupling of $\mathcal{A}$ with K constructed in this way for any transversal is called the coupling induced by the extension. The map ∇^{χ} is a Lie derivation law covering this coupling. Thus, in [10], it can be seen that, for any transversal in the extension and any Lie derivation law ∇ covering the coupling of $\mathcal{A}$ with K induced by the extension, there is other transversal $\zeta : \mathcal{A} \longrightarrow \mathcal{A}'$ such that $\nabla = \nabla^{\zeta}$.

Operator extensions. Let M be a smooth manifold, $\mathcal{A}$ a Lie algebroid on M and K a Lie algebra bundle on M. Let $\Xi : \mathcal{A} \longrightarrow \mathbf{Out}\ \mathcal{D}(K)$ be a coupling of $\mathcal{A}$ with K such that $Obs(\Xi) = 0 \in \mathbf{H}^3(\mathcal{A}, \rho^{\Xi}, ZK)$, in which ρ^{Ξ} denotes the central representation of the coupling. An operator extension of $\mathcal{A}$ with K is an extension

$$\{0\} \longrightarrow K \xrightarrow{j} \mathcal{A}' \xrightarrow{\lambda} \mathcal{A} \longrightarrow \{0\}$$

such that the coupling induced by this extension coincide with the coupling Ξ. The set of equivalence classes of operator extensions of $\mathcal{A}$ by K is denoted by $\mathcal{O}(\mathcal{A}, \Xi, K)$.

The result, due to Mackenzie ([10]), is that $\mathbf{H}^2(\mathcal{A}, \rho^{\Xi}, ZK)$ acts freely and transitively on the set $\mathcal{O}(\mathcal{A}, \Xi, K)$. We begin by defining this action.

Action on operation extensions. Let M be a smooth manifold, $\mathcal{A}$ a Lie algebroid on M and K a Lie algebra bundle on M. Let $\Xi : \mathcal{A} \longrightarrow \mathbf{Out}\ \mathcal{D}(K)$ be a coupling of $\mathcal{A}$ with K such that $Obs(\Xi) = 0 \in \mathbf{H}^3(\mathcal{A}, \rho^{\Xi}, ZK)$, in which ρ^{Ξ} denotes the central representation of the coupling. Consider an operator extension

$$\{0\} \longrightarrow K \xrightarrow{j} \mathcal{A}' \xrightarrow{\lambda} \mathcal{A} \longrightarrow \{0\}$$

Let $g \in \mathbf{Z}^2(\mathcal{A}; ZK)$. Then, the action of g on the extension yields the extension

$$\{0\} \longrightarrow K \xrightarrow{j} \mathcal{A}'_g \xrightarrow{\lambda} \mathcal{A} \longrightarrow \{0\}$$

in which $\mathcal{A}'_g = \mathcal{A}'$ as vector bundles, the maps j and λ are the same in both extensions, the anchors $\gamma' : \mathcal{A}' \longrightarrow TM$ and $\gamma'_g : \mathcal{A}'_g \longrightarrow TM$ are the same too and the Lie bracket $[\cdot,\cdot]_g$ on $\Gamma(\mathcal{A}'_g)$ is given by

$$[\xi, \eta]_g = [\xi, \eta] + (j \circ i \circ g)(\lambda(\xi), \lambda(\eta)$$

in which $i : ZK \longrightarrow K$ denotes the inclusion.

We can now state the main result concerning actions of operators extensions. All details of the (long) proof can be found in Mackenzie's book "General Theory of Lie Groupoids and Lie Algebroids" ([10]).

Proposition 1.3.1 (Mackenzie). Let $\mathcal{A}$ be a Lie algebroid on a smooth manifold M, K a Lie algebra bundle on M and $\Xi : \mathcal{A} \longrightarrow \textbf{Out}\ \mathcal{D}(K)$ a coupling of $\mathcal{A}$ with K. Denote by ρ^{Ξ} the central representation corresponding to the coupling Ξ. Suppose that the class obstruction of Ξ is the zero of $\mathbf{H}^3(\mathcal{A}, \rho^{\Xi}, ZK)$. Then, the additive group of $\mathbf{H}^2(\mathcal{A}, \rho^{\Xi}, ZK)$ acts freely and transitively on $\mathcal{O}(\mathcal{A}, \Xi, K)$.

There are several consequences from the previous result. For this work, the most important consequence is the triviality of a transitive Lie algebroid on a contractible manifold. This result is going to be applied in the proof of Mishchenko's theorem.

Proposition 1.3.2. Let $\mathcal{A}$ be a transitive Lie algebroid on a contractible smooth manifold M and $\gamma : \mathcal{A} \longrightarrow TM$ its anchor. Consider the Lie algebra bundle $K = \textbf{Ker}\ \gamma$ and denote by $\mathfrak{g}$ its fibre type. Then, $\mathcal{A}$ is isomorphic to the trivial Lie algebroid $TM \times \mathfrak{g}$, by a strong isomorphism of Lie algebroids.

Proof. Since $\mathcal{A}$ is transitive, we can fix a flat connection $a : TM \longrightarrow \mathcal{A}$ (see [11]). Let $\Xi : \mathcal{A} \longrightarrow \textbf{Out}\ \mathcal{D}(K)$ be the coupling defined by $\Xi = \natural \circ \nabla^a : \mathcal{A} \longrightarrow \textbf{Out}\ \mathcal{D}(K)$, in which $\natural$ is the quotient map $\natural : \mathfrak{D}_{Der}\mathfrak{g} \longrightarrow \textbf{Out}\ \mathcal{D}(K)$. Consider the extensions

$$\{0\} \longrightarrow K \xrightarrow{j} \mathcal{A} \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

and

$$\{0\} \longrightarrow K \xrightarrow{j} TM \times \mathfrak{g} \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

Since M is contractible then $\mathbf{H}^3(TM, \rho^{\Xi}, ZK) = \{0\}$ and, by transitivity of the action of the previous proposition, there exists $[g] \in \mathbf{H}^2(TM, \rho^{\Xi}, ZK)$ such that the extensions

$$\{0\} \longrightarrow \mathfrak{g} \xrightarrow{j} \mathcal{A}_g \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

and

$$\{0\} \longrightarrow \mathfrak{g} \xrightarrow{j} TM \times \mathfrak{g} \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

are equivalent. Since M is contractible, we can take $h \in \Omega^1(TM, \rho^{\Xi}, ZK)$ such that $dh = g$. We know that the extensions

$$\{0\} \longrightarrow \mathfrak{g} \xrightarrow{j} \mathcal{A}_{dh} \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

and

$$\{0\} \longrightarrow \mathfrak{g} \xrightarrow{j} \mathcal{A}_{g} \xrightarrow{\gamma} TM \longrightarrow \{0\}$$

are equivalent (see ([10]). The conclusion follows by transitivity. □

We finalize this section with other interesting proof of the proposition 1.3.2 made by A. Weinstein. Suppose that $\mathcal{A}$ is a transitive Lie algebroid on a contractible smooth manifold M. In the paper [3], they proved that every transitive Lie algebroid A over a 2-connected base M is integrable. Since M is contractible, by [10], we conclude that $\mathcal{A}$ is the Lie algebroid of the gauge groupoid of a principal bundle. This principal vector bundle is trivial because the base is contractible. Now, it is not difficult to see the Lie bracket in the set of sections of $\mathcal{A}$ is given by the derivative of functions by vector fields.

Chapter 2

Piecewise smooth cohomology

In this chapter, we deal with transitive Lie algebroids defined over simplices of a simplicial complex. We briefly discus a class of spaces for which the piecewise smooth cohomology spaces is defined, precisely the class of all complexes of Lie algebroids. Some facts concerning extension of piecewise smooth forms are presented. In what follows, all simplicial complexes considered are geometric and finite. Simplex means always closed simplex and each simplex can be represented as a convex body generated by its vertices. We shall denote the boundary of the simplex Δ by $\mathbf{bd}\,\Delta$. We shall write $\mathbf{s} \prec \Delta$, if $\mathbf{s}$ is a face of the simplex Δ. The notation $\varphi : \mathbf{s} \hookrightarrow \Delta$, where φ is the inclusion, will also be used when $\mathbf{s}$ is a face of Δ. Many of the definitions and properties stated through the entire chapter can be found, on level of cell spaces, in [1], [9], [22], [24] and [26].

2.1 Complex of Lie algebroids

Consider a simplicial complex K and two simplices Δ and Δ' of K such that Δ' is a face of Δ. Let $\mathcal{A}_\Delta$ be a transitive Lie algebroid on Δ and denote respectively by $\varphi_{\Delta,\Delta'} : \Delta' \hookrightarrow \Delta$

and $(\varphi_{\Delta,\Delta'})^{!!} : (\mathcal{A}_\Delta)^{!!}_{\Delta'} \longrightarrow \mathcal{A}_\Delta$ the inclusion and the induced maps. We recall that the Lie algebroid $(\mathcal{A}_\Delta)^{!!}_{\Delta'}$ denotes the restriction of $\mathcal{A}_\Delta$ to Δ', which is, by definition, the Lie algebroid $(\varphi_{\Delta,\Delta'})^{!!}\mathcal{A}_\Delta$, inverse image of $\mathcal{A}_\Delta$ by $\varphi_{\Delta,\Delta'}$. Since Δ' is a compact embedded submanifold of Δ, by proposition 1.1.10, the Lie algebroid $(\mathcal{A}_\Delta)^{!!}_{\Delta'}$ can be identified to the Lie algebroid $\mathbf{Im}\,(\varphi_{\Delta,\Delta'})^{!!}$. Hence, for each $x \in \Delta'$, the fibre $(\mathcal{A}_{\Delta'})_x$ is a vector subspace of the fibre $(\mathcal{A}_\Delta)_x$.

Definition (Complex of Lie algebroids). Let K be a simplicial complex. A complex of Lie algebroids on K is a family $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ such that, for each $\Delta \in K$, $\mathcal{A}_\Delta$ is a transitive Lie algebroid on Δ and $\mathcal{A}_{\Delta'} = (\mathcal{A}_\Delta)^{!!}_{\Delta'}$ for each face Δ' of Δ, that is, the Lie algebroid restriction of $\mathcal{A}_\Delta$ to Δ' is the Lie algebroid $\mathcal{A}_{\Delta'}$.

Alternatively, a complex of Lie algebroids on K means a family of transitive Lie algebroids defined on the simplices of K such that the structures of Lie algebroids induced on each intersection of two simplices coincide. We give now some examples of complexes of Lie algebroids.

Example 1 (Tangent complex). Let K be a simplicial complex. For each simplex $\Delta \in K$, consider the tangent Lie algebroid $T\Delta$ defined over Δ. If Δ' is a face of Δ then, by the proposition 1.1.7, $(T\Delta)^{!!}_{\Delta'} = T\Delta'$ and consequently we obtain a complex of Lie algebroids $\{T\Delta\}_{\Delta \in K}$, which is called the corresponding tangent complex on K.

Example 2 (Trivial complex). Let K be a simplicial complex and $\mathfrak{g}$ a real finite dimensional Lie algebra. For each simplex $\Delta \in K$ consider the transitive Lie algebroid $T\Delta \oplus (\Delta \times \mathfrak{g})$. If Δ' is a face of Δ then $(T\Delta \oplus (\Delta \times \mathfrak{g}))^{!!}_{\Delta'} = T\Delta' \oplus (\Delta' \times \mathfrak{g})$ by proposition 1.1.8. We conclude that the family $\{T\Delta \oplus (\Delta \times \mathfrak{g})\}_{\Delta \in K}$ is a complex of Lie algebroids. This complex is called the trivial complex on K.

Example 3 (Restriction of complexes of Lie algebroids). Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ a complex of Lie algebroids on K. Let L be a simplicial subcomplex of K. We can consider a new complex of Lie algebroids, defined over L and denoted by $\underline{\mathcal{A}}^L$, given by restriction of $\underline{\mathcal{A}}$ to the simplices of L, that is, $\underline{\mathcal{A}}^L = \{\mathcal{A}_\Delta\}_{\Delta \in L}$.

Example 4 (Complex corresponding a combinatorial manifold). Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Each simplex Δ of K is a compact embedded submanifold of M. We can consider the Lie algebroid restriction $\mathcal{A}^{!!}_\Delta$ on the compact submanifold Δ. By proposition 1.1.7, if Δ' is a face of Δ, then $(\mathcal{A}^{!!}_\Delta)^{!!}_{\Delta'} = \mathcal{A}^{!!}_{\Delta'}$ and so we obtain a complex of Lie algebroids on K which will be called the corresponding complex of $\mathcal{A}$ over the simplicial complex K and denoted by $\{\mathcal{A}^{!!}_\Delta\}_{\Delta \in K}$.

Notation. Let K, L be two simplicial complexes and $f : K \longrightarrow L$ a simplicial map. For each simplex Δ of K generated by the vertices $a_0, a_1, \ldots, a_p$, we denote by $f(\Delta)$ the simplex of L generated by $f(a_0), f(a_1), \ldots, f(a_p)$ with redundances removed.

Definition (Morphism of complexes of Lie algebroids). Let K, L be two simplicial complexes and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ and $\underline{\mathcal{B}} = \{\mathcal{B}_{\Delta'}\}_{\Delta' \in L}$ two complexes of Lie algebroids on K and L respectively. Let $f : K \longrightarrow L$ be a simplicial map and suppose that, for each $\Delta \in K$, a morphism of Lie algebroids $F_\Delta : \mathcal{A}_\Delta \longrightarrow \mathcal{B}_{f(\Delta)}$ over $f_{/\Delta} : \Delta \longrightarrow f(\Delta)$ is given. The family $\lambda = \big((F_\Delta)_{\Delta \in K}, f\big)$ is called a morphism of complex of Lie algebroids from $\underline{\mathcal{A}}$ to $\underline{\mathcal{B}}$ if this family is compatible with the restrictions, that is, if Δ and Δ' are two simplices of K, with Δ' face of Δ, the restriction $F_{\Delta_{/\Delta'}} : \mathcal{A}_{\Delta'} \longrightarrow \mathcal{B}_{f(\Delta')}$ coincides with the Lie algebroid morphism $F_{\Delta'}$. This is equivalent to say that the diagram

$$\begin{array}{ccc}
\mathcal{A}_{\Delta'} & \xrightarrow{F_{\Delta'}} & \mathcal{B}_{f(\Delta')} \\
\downarrow{\scriptstyle i_{\Delta',\Delta}} & & \downarrow{\scriptstyle i_{f(\Delta'),f(\Delta)}} \\
\mathcal{A}_\Delta & \xrightarrow{F_\Delta} & \mathcal{B}_{f(\Delta)}
\end{array}$$

is commutative, where $i_{\Delta',\Delta}$ and $i_{f(\Delta'),f(\Delta)}$ are the inclusions maps. Suppose that T is other simplicial complex and $g : L \longrightarrow T$ a simplicial map. Let $\underline{\mathcal{C}} = \{\mathcal{C}_{\Delta''}\}_{\Delta'' \in T}$ be a complex of Lie algebroids on T. For each $\Delta' \in L$, let $G_{\Delta'} : \mathcal{B}_{\Delta'} \longrightarrow \mathcal{C}_{g(\Delta')}$ be a morphism of Lie algebroids over $g_{/\Delta'} : \Delta' \longrightarrow g(\Delta')$ such that the family $\delta = \big((G_{\Delta'})_{\Delta' \in L}, g\big)$ is a morphism of complex of Lie algebroids from $\underline{\mathcal{B}}$ to $\underline{\mathcal{C}}$. Then, we define the composition $\delta \circ \lambda$ to be the family $\delta \circ \lambda = \big((G_{f(\Delta)} \circ F_\Delta)_{\Delta \in K}, g \circ f\big)$. It is routine to verify that the family $\delta \circ \lambda$ is a morphism of complexes of Lie algebroids from $\underline{\mathcal{A}}$ to $\underline{\mathcal{C}}$.

Example 1 (Identity morphism). Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ a complex of Lie algebroids on K. For each $\Delta \in K$, let $id_\Delta : \mathcal{A}_\Delta \longrightarrow \mathcal{A}_\Delta$ be the Lie algebroid morphism identity. Then, the family $\lambda = \big((id_\Delta)_{\Delta \in K}, id_K\big)$, where $id_K : K \longrightarrow K$ is the simplicial morphism identity, is a morphism of complex of Lie algebroids.

Example 2 (Inclusion morphism). Let K be a simplicial complex and suppose that $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ is a complex of Lie algebroids on K. Let L be a simplicial subcomplex of K and consider the complex of Lie algebroids $\underline{\mathcal{A}}^L = \{\mathcal{A}_\Delta\}_{\Delta \in L}$ given by restriction of $\underline{\mathcal{A}}$ to L (example 3, after the definition of complex of Lie algebroids). Consider the family $\lambda = \big((id_\Delta)_{\Delta \in L}, i_{L,K})\big)$, where $i_{L,K} : L \longrightarrow K$ is the simplicial inclusion and $id_\Delta : \mathcal{A}_\alpha \longrightarrow \mathcal{A}_\alpha$ is the Lie algebroid morphism identity. Then, the family λ is a morphism of complexes of Lie algebroids from $\underline{\mathcal{A}}^L$ to $\underline{\mathcal{A}}$.

Next proposition is obvious.

Proposition 2.1.1. The class of all complexes Lie algebroids and morphisms of complexes of Lie algebroids with the composition indicated as above is a category. This category is called the category of complexes of Lie algebroids.

2.2 Algebra of piecewise smooth forms

Let $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ be a complex of Lie algebroids on a simplicial complex K. For each simplex Δ of K, we denote, as done on previous sections, by $\big(\Omega^*(\mathcal{A}_\Delta; \Delta), d^*_\Delta\big)$ the cochain algebra of all smooth forms on $\mathcal{A}_\Delta$. Let Δ and Δ' be two simplices of K, with Δ' face of Δ, and $\varphi_{\Delta,\Delta'} : \Delta' \hookrightarrow \Delta$ the inclusion map. By definition of complex of Lie algebroids, we have that $(\mathcal{A}_\Delta)^{!!}_{\Delta'} = \mathcal{A}_{\Delta'}$. The homomorphism of cochain algebras generated by the inclusion $\varphi_{\Delta,\Delta'}$ is denoted by

$$\varphi^{\mathcal{A}_\Delta}_{\Delta,\Delta'} : \Omega^*(\mathcal{A}_\Delta; \Delta) \longrightarrow \Omega^*(\mathcal{A}_{\Delta'}; \Delta')$$

We give now the definition of piecewise smooth form. The idea of this definition is based in the Whitney book's [26] or in the Sullivan's paper [22].

Definition (Piecewise smooth form). Let K be a simplicial complex and suppose that $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ a complex of Lie algebroids on K. A piecewise smooth form of degree p ($p \geq 0$) on the complex of Lie algebroids $\underline{\mathcal{A}}$ is a family $\omega = (\omega_\Delta)_{\Delta\in K}$ such that the following conditions are satisfied.

- For each $\Delta \in K$, $\omega_\Delta \in \Omega^p(\mathcal{A}_\Delta; \Delta)$ is a smooth form of degree p on $\mathcal{A}_\Delta$.
- For each Δ, $\Delta' \in K$, with Δ' face of Δ,

$$\varphi^{\mathcal{A}_\Delta}_{\Delta,\Delta'}(\omega_\Delta) = \omega_{\Delta'}$$

Let $(\varphi_{\Delta,\Delta'})^{!!} : \mathcal{A}_{\Delta'} \longrightarrow \mathcal{A}_\Delta$ be the map induced by $\varphi_{\Delta,\Delta'}$. We recall that the spaces $\Omega^*(\mathcal{A}_{\Delta'}; \Delta')$ and $\Omega^*(\mathbf{Im}\,(\varphi_{\Delta,\Delta'})^{!!}; \Delta')$ are identified and that the fibre $(\mathcal{A}_{\Delta'})_x$ is a vector subspace of the fibre $(\mathcal{A}_\Delta)_x$. Hence, the second condition of the definition given above can be stated in the following form: for each $x \in \Delta'$ and vectors $u_1, \ldots, u_p \in (A_{\Delta'})_x$

$$\omega_{\Delta'}(x)(u_1, \cdots, u_p) = \omega_\Delta(x)(u_1, \cdots, u_p)$$

Thus, a piecewise smooth form is a collection of smooth forms, each one defined on a transitive Lie algebroid over a simplex of K, which are compatible under restriction to faces. The set of all piecewise smooth forms of degree p on the complex of Lie algebroids $\underline{\mathcal{A}}$ will be denoted by $\Omega^p_{\mathrm{ps}}(\underline{\mathcal{A}}; K)$ or simply $\Omega^p(\underline{\mathcal{A}}; K)$. We have then

$$\Omega^p_{\mathrm{ps}}(\underline{\mathcal{A}}; K) = \{(\omega_\Delta)_{\Delta\in K} : \omega_\Delta \in \Omega^p(\mathcal{A}_\Delta), \quad \Delta' \prec \Delta \implies (\omega_\Delta)^{!!}_{\Delta'} = \omega_{\Delta'}\}$$

Remark. When $p = 0$, a piecewise smooth form on $\underline{\mathcal{A}}$ of degree zero is a family $(\varphi_\Delta)_{\Delta\in K} \in \Omega^0_{ps}(\underline{\mathcal{A}}; K) \subset \prod_{\Delta\in K} C^\infty(\Delta)$ such that $\varphi_\Delta : \Delta \longrightarrow \mathbf{R}$ is smooth and the equality $\varphi_{\Delta'} = \varphi_{\Delta_{/\Delta'}}$ holds for each face Δ' of Δ. The compatibility condition of restrictions to faces gives a map $\varphi : |K| \longrightarrow \mathbf{R}$ which is continuous. The map φ may not be a differentiable map but it is a piecewise smooth function. The set of all maps $\varphi \in C(|K|; \mathbf{R})$ which are compatible with restrictions to the faces of $|K|$ and with smooth restrictions to the faces of $|K|$ is denoted by $C_{ps}(|K|; \mathbf{R})$. Obviously, $\Omega^0_{ps}(\underline{\mathcal{A}}; K)$ has a natural structure of algebra over $\mathbf{R}$ and is naturally identified to $C_{ps}(|K|; \mathbf{R})$.

Let $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ be a complex of Lie algebroids on a simplicial complex K. Since restrictions of smooth forms are compatible with sums and products, various operations on $\Omega^p_{ps}(\underline{\mathcal{A}}; K)$ can be defined by the corresponding operations on each simplex of K. Namely, if $\omega = (\omega_\Delta)_{\Delta\in K}$, $\eta = (\eta_\Delta)_{\Delta\in K} \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$ are two piecewise smooth forms of degree p on the complex of Lie algebroids $\underline{\mathcal{A}}$ and $f : |K| \longrightarrow \mathbf{R}$ a continuous map, we may define $\omega + \eta$, $f\omega$ and $\omega \wedge \eta$ to be

$$\omega + \eta = (\omega_\Delta + \eta_\Delta)_{\Delta\in K}$$

$$f\omega = (f_{/\Delta}\omega_\Delta)_{\Delta\in K}$$

$$\omega \wedge \eta = (\omega_\Delta \wedge \eta_\Delta)_{\Delta\in K}$$

The set $\Omega^p_{ps}(\underline{\mathcal{A}}; K)$, equipped with these operations, becomes a real vector subspace of $\prod_{\Delta\in K} \Omega^p(\mathcal{A}_\Delta; \Delta)$, for each natural $p \geq 0$. Thus, $\Omega^p_{ps}(\underline{\mathcal{A}}; K)$ is a module over the algebra $C_{ps}(|K|; \mathbf{R})$. When $p = 0$, $\Omega^0_{ps}(\underline{\mathcal{A}}; K) = C_{ps}(|K|; \mathbf{R})$ has a structure of an unitary associative algebra over $\mathbf{R}$. Moreover, the direct sum

$$\Omega^*_{ps}(\underline{\mathcal{A}}; K) = \bigoplus_{p\geq 0} \Omega^p_{ps}(\underline{\mathcal{A}}; K)$$

equipped with the exterior product defined by the corresponding exterior product on each algebra $\Omega^*_{ps}(\mathcal{A}_\Delta; \Delta) = \bigoplus_{p\geq 0} \Omega^p(\mathcal{A}_\Delta; \Delta)$, is a commutative graded algebra over $\mathbf{R}$.

In this section, we shall also include some basic functorial properties. We begin first with the definition of inverse image of a piecewise smooth form.

Definition (Inverse image of piecewise smooth forms). Let K, L be simplicial complexes and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ and $\underline{\mathcal{B}} = \{\mathcal{B}_{\Delta'}\}_{\Delta'\in L}$ complexes of Lie algebroids on K and L respectively. Let $f : K \longrightarrow L$ be a simplicial map and suppose that, for each $\Delta \in K$, a morphism of Lie algebroids $F_\Delta : \mathcal{A}_\Delta \longrightarrow \mathcal{B}_{f(\Delta)}$ over $f_{/\Delta} : \Delta \longrightarrow f(\Delta)$ is given such that the family $\lambda = \big((F_\Delta)_{\Delta\in K}, f\big)$ is a complex of Lie algebroids morphism. For each $\Delta \in K$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_\Delta & \xrightarrow{F_\Delta} & \mathcal{B}_{f(\Delta)} \\ \downarrow{\scriptstyle \pi_\Delta} & & \downarrow{\scriptstyle \pi_{f(\Delta)}} \\ \Delta & \xrightarrow{f_{/\Delta}} & f(\Delta) \end{array}$$

If $\omega = (\omega_{\Delta'})_{\Delta'\in L} \in \Omega^p_{ps}(\underline{\mathcal{B}}; L)$ is a piecewise smooth form, then, for each $\Delta \in K$, we can consider the smooth form $(F_\Delta, f_{/\Delta})^*(\omega_{f(\Delta)}) \in \Omega^p(\mathcal{A}_\Delta; \Delta)$. We define $\lambda^*\omega$ to be

$$\lambda^*\omega = \big((F_\Delta, f_{/\Delta})^*(\omega_{f(\Delta)})\big)_{\Delta\in K}$$

Proposition 2.2.1. On the same conditions above, the form $\lambda^*\omega$ is a piecewise smooth form of degree p defined on the complex of Lie algebroids $\underline{\mathcal{A}}$.

Proof. It remains to check the compatibility condition of the restriction to the faces. Let $\mathbf{s}$ be a face of Δ. Then $f(\mathbf{s})$ is also a face of the simplex $f(\Delta)$ and hence $(\omega_{f(\Delta)})_{/f(\mathbf{s})} = \omega_{f(\mathbf{s})}$. We also have that the equality

$$(F_\Delta, f_{/\Delta})^*\big(\omega_{f(\Delta)/_{f(\mathbf{s})}}\big) = \big((F_\Delta, f_{/\Delta})^*(\omega_{f(\Delta)})\big)_{/\mathbf{s}}$$

holds, and therefore

$$\big((\lambda^*\omega)_\Delta\big)_{/\mathbf{s}} = \big((F_\Delta, f_{/\Delta})^*(\omega_{f(\Delta)})\big)_{/\mathbf{s}} = (F_\Delta, f_{/\Delta})^*\big(\omega_{f(\Delta)/_{f(\mathbf{s})}}\big) = (F_\Delta, f_{/\Delta})^*(\omega_{f(\mathbf{s})})$$

Now, by the compatibility of $(F_\Delta, f_{/\Delta})$ to the restrictions, we have

$$(F_\Delta, f_{/\Delta})^*(\omega_{f(\mathbf{s})}) = (F_\mathbf{s}, f_{/\mathbf{s}})^*(\omega_{f(\mathbf{s})}) = (\lambda^*\omega)_{/\mathbf{s}}$$

and so $\big((\lambda^*\omega)_\Delta\big)_{/\mathbf{s}} = (\lambda^*\omega)_{/\mathbf{s}}$. $\square$

Proposition 2.2.2. Let $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$, $\underline{\mathcal{B}} = \{\mathcal{B}_{\Delta'}\}_{\Delta'\in L}$ and $\underline{\mathcal{C}} = \{\mathcal{C}_{\Delta''}\}_{\Delta''\in T}$ three complexes of Lie algebroids on the simplicial complexes K, L and T respectively. Let $f : K \longrightarrow L$ and $g : L \longrightarrow T$ simplicial maps and suppose that, for each $\Delta \in K$ and $\Delta' \in L$, morphisms of Lie algebroids $F_\Delta : \mathcal{A}_\Delta \longrightarrow \mathcal{B}_{f(\Delta)}$ over $f_{/\Delta} : \Delta \longrightarrow f(\Delta)$ and $G_{\Delta'} : \mathcal{B}_{\Delta'} \longrightarrow \mathcal{C}_{g(\Delta')}$ over $g_{/\Delta'} : \Delta' \longrightarrow g(\Delta')$ are given such that the families

$$\lambda = \big((F_\Delta)_{\Delta\in K}, f\big) : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{B}}$$

and

$$\widehat{\lambda} = \big((G_{\Delta'})_{\Delta'\in L}, g\big) : \underline{\mathcal{B}} \longrightarrow \underline{\mathcal{C}}$$

are morphisms of complexes of Lie algebroids. Then, the following properties hold:

a) $(\widehat{\lambda} \circ \lambda)^*\omega = \lambda^*(\widehat{\lambda}^*\omega)$.

b) $(id_{\underline{\mathcal{A}}})^*\zeta = \zeta$.

c) $\lambda^*(\xi + \eta) = \lambda^*\xi + \lambda^*\eta$ and $\lambda^*(\varphi\omega) = (\varphi \circ f)\lambda^*\omega$.

d) $\lambda^*(\xi \wedge \eta) = \lambda^*\xi \wedge \lambda^*\eta$.

for each $\zeta \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$, $\omega \in \Omega^p_{ps}(\underline{\mathcal{C}}; T)$, ξ, $\eta \in \Omega^p_{ps}(\underline{\mathcal{B}}; L)$ and $\varphi : |L| \longrightarrow \mathbf{R}$ continuous.

In order to obtain a complex of cochains, especially important is the analogues of exterior derivative. This operator also is obtained by the corresponding exterior derivative on each simplex. Namely, if $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ is a complex of Lie algebroids on a simplicial complex K, we can define the mapping

$$d^p : \Omega^p_{ps}(\underline{\mathcal{A}}; K) \longrightarrow \Omega^{p+1}_{ps}(\underline{\mathcal{A}}; K)$$

setting

$$d^p((\omega_\Delta)_{\Delta\in K}) = (d^p_\Delta \omega_\Delta)_{\Delta\in K}$$

for each $\omega = (\omega_\Delta)_{\Delta\in K} \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$. For $p = 0$, the algebra $\Omega^0_{ps}(\underline{\mathcal{A}}; K)$ is the vector space of all families $(\varphi_\Delta)_{\Delta\in K} \in \prod_{\Delta\in K} C^\infty(\Delta)$ such that $(\varphi_\Delta)_{\Delta\in K}$ is compatible with the restrictions to faces. So, the exterior derivative in degree zero is the usual derivative. We list below the main properties of the exterior derivative.

Proposition 2.2.3. Let $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ be a complex of Lie algebroids on a simplicial complex K and $\omega = (\omega_\Delta)_{\Delta\in K} \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$. Then, the followings properties hold:

- d^p is linear for any $p \geq 0$.
- $d^{p+1} \circ d^p = 0$ for any $p \geq 0$.
- For each $\xi = (\xi_\Delta)_{\Delta\in K} \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$ and $\eta = (\eta_\Delta)_{\Delta\in K} \in \Omega^q_{ps}(\underline{\mathcal{A}}; K)$,

$$d^{p+q}(\xi \wedge \eta) = (d^p\xi) \wedge \eta + (-1)^p \xi \wedge (d^q\eta)$$

Such as in the case of smooth forms on a Lie algebroid, the space $\Omega^*_{ps}(\underline{\mathcal{A}}; K)$, with the operations and differentiation above, becomes a commutative differential graded algebra, which is defined over $\mathbf{R}$.

Definition (Piecewise smooth cohomology). Keeping the same hypothesis and notation as above, the piecewise smooth cohomology space of $\underline{\mathcal{A}}$ is the cohomology space of the algebra $\Omega^p_*(\underline{\mathcal{A}}; K)$ equipped with the structures defined above. Its cohomology, $H(\Omega^*_{ps}(\underline{\mathcal{A}}; K))$, will be denoted by $H^*_{ps}(\underline{\mathcal{A}}; K)$ or simply by $H^*(\underline{\mathcal{A}}; K)$.

Proposition 2.2.4. Let $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$, $\underline{\mathcal{B}} = \{\mathcal{B}_{\Delta'}\}_{\Delta'\in L}$ two complexes of Lie algebroids on the simplicial complexes K e L respectively. Let $f : K \longrightarrow L$ be a simplicial map and suppose that, for each $\Delta \in K$, a morphism of Lie algebroids $F_\Delta : \mathcal{A}_\Delta \longrightarrow \mathcal{B}_{f(\Delta)}$ over $f_{/\Delta} : \Delta \longrightarrow f(\Delta)$ is given such that the family $\lambda = \big((F_\Delta)_{\Delta\in K}, f\big)$ is a complex of Lie algebroids morphism. Then, for each piecewise smooth form $\omega = (\omega_\Delta)_{\Delta\in K} \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$, the equality $d(\lambda^*\omega) = \lambda^*(d\omega)$ hold.

Proof. For each simplex $\Delta \in K$, the equality $d(\lambda^*\omega_\Delta) = \lambda^*(d\omega_\Delta)$ and so, the result follows. □

A particular case of inverse image of forms in the one in which considers the inclusion morphism of complex of Lie algebroids, given in example 2 following the definition of complex of Lie algebroids morphism. In this case, the inverse image of piecewise smooth forms is called restriction of piecewise smooth forms. We shall notice some basic results concerning restrictions of piecewise smooth forms.

Definition (Restriction of piecewise smooth forms). Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ a complex of Lie algebroids on K. Let L be a simplicial subcomplex of K and $\underline{\mathcal{A}}^L = \{\mathcal{A}_\Delta\}_{\Delta\in L}$ the complex of Lie algebroids given by restriction of $\underline{\mathcal{A}}$ to L. Consider the morphism of complexes of Lie algebroids $\lambda = \big((id_\Delta)_{\Delta\in L}, i_{L,K})\big)$, where $i_{L,K} : L \longrightarrow K$ is the simplicial inclusion and $id_\Delta : \mathcal{A}_\Delta \longrightarrow \mathcal{A}_\Delta$ is the identity. If $\omega = (\omega_\Delta)_{\Delta\in K} \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$ is a piecewise smooth form of degree p, we can define the restriction of ω to the subcomplex L, denoted by $\omega_{/L}$, to be

$$\omega_{/L} = \lambda^*\omega$$

Proposition 2.2.5. In the conditions of this definition, the form $\omega_{/L}$ is a piecewise smooth form on the complex of Lie algebroids $\underline{\mathcal{A}}^L$. Moreover, the equality

$$d(\omega_{/L}) = (d\omega)_{/L}$$

holds.

Proof. It is a consequence of propositions 2.2.1 and 2.2.4.

Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ a complex of Lie algebroids on K. Let L be a simplicial subcomplex of K and consider the complex of Lie algebroids $\underline{\mathcal{A}}^L = \{\mathcal{A}_\Delta\}_{\Delta\in L}$, restriction of $\underline{\mathcal{A}}$ to L. We obtain a new cochain complex, the cochain complex $\Omega^*_{ps}(\underline{\mathcal{A}}^L; L)$. For each $p \geq 0$, denote by

$$r^{p^K}_L : \Omega^p_{ps}\big(\underline{\mathcal{A}}; K\big) \longrightarrow \Omega^p_{ps}\big(\underline{\mathcal{A}}^L; L\big)$$

the map induced by restriction, that is, for each $\omega \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$,

$$r^{p^K}_L(\omega) = \omega_{/L}$$

In next proposition, we establish the main properties of this restriction map.

Proposition 2.2.6. Keeping the same hypothesis and notations as above, the following properties hold.

- For $p = 0$, $r_L^{0^K} : C_{ps}(|K|) \longrightarrow C_{ps}(|L|)$ is a homomorphism of algebras.
- For each $p \geq 0$, $r_K^{p^K} = id_{\Omega^p_{ps}(\underline{\mathcal{A}};K)}$.
- The map $r_L^{*^K} : \Omega^*_{ps}(\underline{\mathcal{A}}; K) \longrightarrow \Omega^p_{ps}(\underline{\mathcal{A}}^L; L)$ is a morphism of cochain algebras.
- If T is a simplicial subcomplex of L and $\underline{\mathcal{A}}^T = (\mathcal{A}_\alpha)_{\alpha\in T}$ then, for each $p \geq 0$, $r_T^{p^K} = r_T^{p^L} \circ r_L^{p^K}$ and so the diagram

$$\begin{array}{ccc} \Omega^*_{ps}(\underline{\mathcal{A}}; K) & \xrightarrow{\quad r_L^{*^K}\quad} & \Omega^p_{ps}(\underline{\mathcal{A}}^L; L) \\ & {\scriptstyle r_T^{*^K}} \searrow \quad \swarrow {\scriptstyle r_T^{*^L}} & \\ & \Omega^p_{ps}(\underline{\mathcal{A}}^T; T) & \end{array}$$

  is a commutative diagram of cochain algebras.

Proof. It is an immediate consequence of proposition 2.2.2. □

After proposition 2.3.4, we will see that the map $r_L^{p^K}$ is surjective.

2.3 Mayer-Vietoris sequence for piecewise smooth forms

Our propose now is to answer to the following question: let K_0 and K_1 be two simplicial subcomplexes of a simplicial complex K and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ a complex of Lie algebroids on K. We can consider the complexes of Lie algebroids $\underline{\mathcal{A}}^0 = \{\mathcal{A}_\Delta\}_{\Delta\in K_0}$, $\underline{\mathcal{A}}^1 = \{\mathcal{A}_\Delta\}_{\Delta\in K_1}$ and $\underline{\mathcal{A}}^{0,1} = \{\mathcal{A}_\Delta\}_{\Delta\in K_0\cap K_1}$. Naturally, our question is to know which relations hold between the cohomology of the complexes of cochains $\Omega^*_{ps}(\underline{\mathcal{A}};K)$, $\Omega^*_{ps}(\underline{\mathcal{A}^0};K_0)$, $\Omega^*_{ps}(\underline{\mathcal{A}^1};K_1)$ and $\Omega^*_{ps}(\underline{\mathcal{A}^{0,1}};K_0\cap K_1)$. This question is answered by the Mayer-Vietoris sequence. We start by stating the extension lemma for piecewise smooth forms (see [19], [22]). Once this is done, we establish the result concerned to the Mayer-Vietoris sequence.

Proposition 2.3.1 (Extension lemma for piecewise smooth forms - particular case). Let Δ_k denote the canonical k-simplex in $\mathbf{R}^\infty$ having the vertices

$$e_0 = (0,0,\dots,0,\dots)$$

$$e_1 = (1,0,\dots,0,\dots)$$

$$\dots$$

$$e_k = (0,0,\dots,1,\dots)$$

(e_j is the vector with 1 in the jth coordinate and zeros elsewhere). Let $\mathcal{A}$ be a transitive Lie algebroid on Δ_k and consider the complexes of Lie algebroids $\underline{\mathcal{A}^{\Delta_k}} = \{\mathcal{A}^{!!}_\alpha\}_{\alpha\in\Delta_k}$ and $\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}} = \{\mathcal{A}^{!!}_\alpha\}_{\alpha\in\mathbf{bd}\ \Delta_k}$ given by restriction of $\mathcal{A}$ to the correspondent simplicial complexes Δ_k and $\mathbf{bd}\ \Delta_k$ respectively. Let $\xi\in\Omega^p_{ps}\big(\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}};\mathbf{bd}\ \Delta_k\big)$ be a piecewise smooth form of degree p defined on $\mathbf{bd}\ \Delta_k$. Then, there is a piecewise smooth form $\omega\in\Omega^p_{ps}\big(\underline{\mathcal{A}^{\Delta_k}};\Delta_k\big)$ of degree p defined on Δ_k such that $\omega_{/\mathbf{bd}\ \Delta_k}=\xi$.

Proof. We are going to divide the proof in three parts.

Part 1. Let α be a face of dimension $k-1$ of Δ_k, say us, α is the face spanned by the vertices $e_0,\dots,e_{j-1},e_{j+1},\dots,e_k$. The face α consists of all points $x\in\mathbf{R}^\infty$ such that

$$x = t_0e_0+\cdots+t_{j-1}e_{j-1}+0e_j+t_{j+1}e_{j+1}+\cdots+t_ke_k$$

with $\sum_i t_i = 1$ and $t_i\geq 0$. Let e_j be the opposite vertex to the face α and U the complement of this vertex. U is an open subset in Δ_k. For each

$$x = t_0e_0+\cdots+t_{j-1}e_{j-1}+t_je_j+t_{j+1}e_{j+1}+\cdots+t_ke_k\in U$$

we have that $t_j \neq 1$ and the element $\frac{t_0}{1-t_j}e_0 + \cdots + \frac{t_{j-1}}{1-t_j}e_{j-1} + \frac{t_{j+1}}{1-t_j}e_{j+1} \cdots + \frac{t_k}{1-t_j}e_k$ belongs to α. So, we may define a map

$$\varphi : U \longrightarrow \alpha$$

by

$$\varphi(t_0e_0 + \cdots + t_{j-1}e_{j-1} + t_je_j + t_{j+1}e_{j+1} + \cdots + t_ke_k) =$$
$$= \frac{t_0}{1-t_j}e_0 + \cdots + \frac{t_{j-1}}{1-t_j}e_{j-1} + \frac{t_{j+1}}{1-t_j}e_{j+1} \cdots + \frac{t_k}{1-t_j}e_k$$

Obviously φ is smooth map. Thus, φ is a retraction. Since Δ_k is contractible, $\mathcal{A}$ is a trivial Lie algebroid and then we can find a map $\psi : \mathcal{A}_U \longrightarrow \mathcal{A}_\alpha$ such that $\lambda = (\psi, \varphi)$ is a morphism of Lie algebroids and, for each $x \in \alpha$, $\psi_x : \mathcal{A}_x \longrightarrow \mathcal{A}_x$ is the identity map. Consider now a smooth form $\omega \in \Omega^p(\mathcal{A}^{!!}_\alpha; \alpha)$. Take the form $\lambda^*\omega$. This form is smooth and belongs to $\Omega^p(\mathcal{A}_U; U)$. By extension lemmas, the form $\lambda^*\omega$ damps out smoothly to a smooth form $\widetilde{\omega} \in \Omega^p(\mathcal{A}; \Delta_k)$. By taking the restriction of $\widetilde{\omega}$ to each face of the simplex Δ_k, we obtain a piecewise smooth form $\widetilde{\omega} \in \Omega^p_{ps}(\underline{\mathcal{A}^{\Delta_k}}; \Delta_k)$, which is a piecewise smooth extension of ω.

Part 2. The piecewise smooth form $\widetilde{\omega}$ obtained in the first part has the following property: for each face β of α,

$$\omega_{/\beta} = 0 \quad \Longrightarrow \quad \widetilde{\omega}_{/\beta * e_j} = 0$$

where $\beta * e_j$ is the join of β and the vertex e_j. This happens because $\widetilde{\omega}_{e_j}$ is obviously equal to zero and, for each $x \in \beta * e_j$ with $x \neq e_j$, $\varphi(x)$ lives in β.

Part 3. Let $\alpha_0, \ldots, \alpha_k$ be the $k+1$ faces of dimension $k-1$ of Δ_k and let

$$\xi = (\xi_\alpha)_{\alpha \in \, \mathbf{bd}\Delta_k} = (\xi_{\alpha_0}, \xi_{\alpha_1}, \ldots, \xi_{\alpha_k}) \in \Omega^p_{ps}(\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}}; \mathbf{bd}\ \Delta_k)$$

be a piecewise smooth form of degree p defined over $\mathbf{bd}\Delta_k$. By the part 1, the smooth form $\xi_{\alpha_0} \in \Omega^p(\mathcal{A}^{!!}_{\alpha_0}; \alpha_0)$ can be extended to a smooth form $\widetilde{\xi_0} \in \Omega^p(\mathcal{A}; \Delta_k)$ defined on Δ_k and the form $\widetilde{\xi_0}$ defines, by restriction to each face of Δ_k, a piecewise smooth form $\widehat{\xi_0} \in \Omega^p_{ps}(\underline{\mathcal{A}^{\Delta_k}}; \Delta_k)$ defined on Δ_k. Let $\xi_1 = \xi - (\widehat{\xi_0}_{/\mathbf{bd}\Delta_k}) \in \Omega^p_{ps}(\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}}; \mathbf{bd}\ \Delta_k)$. The form ξ_1 vanishes on α_0. Repeating the same process for the face α_1 by using the form ξ_1, the smooth form $\xi_{1/\alpha_1} \in \Omega^p(\mathcal{A}^{!!}_{\alpha_1}; \alpha_1)$ extends to a smooth form $\widetilde{\xi_1} \in \Omega^p(\mathcal{A}; \Delta_k)$ defined on Δ_k, and then we obtain a piecewise smooth form $\widehat{\xi_1} \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\Delta_k}}; \Delta_k\big)$ defined on Δ_k by restrictions to its faces. Since the faces α_0 and α_1 have a common vertex, we have that $\widehat{\xi_1}_{/\alpha_0} = 0$ by the part 2 above. Let $\xi_2 = \xi - (\widehat{\xi_0} + \widehat{\xi_1})_{/\mathbf{bd}\Delta_k} \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}}; \mathbf{bd}\ \Delta_k\big)$. Then, ξ_2 is a piecewise smooth form defined on $\mathbf{bd}\Delta_k$, $\xi_{2/\alpha_0} = 0$ and $\xi_{2/\alpha_1} = \xi_{1/\alpha_1} - \widetilde{\xi}_{1/\alpha_1} = 0$. Hence $\xi_{2/\alpha_0\cup\alpha_1} = 0$. Therefore, we construct inductively a finite sequence $\xi_j \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}}; \mathbf{bd}\ \Delta_k\big)$ and $\widehat{\xi_j} \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\Delta_k}}; \Delta_k\big)$, with $j = 1, \ldots, k+1$, such that

$$\xi_j = \xi - (\widehat{\xi_0} + \widehat{\xi_1} + \cdots + \widehat{\xi_{j-1}})_{/\partial\Delta}$$

and

$$\xi_{j/\alpha_0\cup\alpha_1\cup\cdots\cup\alpha_{j-1}} = 0$$

for each $j \in \{1, \ldots, k+1\}$. Then, setting

$$\omega = \widehat{\xi_0} + \widehat{\xi_1} + \cdots + \widehat{\xi_l}$$

we have that $\omega \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\Delta_k}}; \Delta_k\big)$ is a piecewise smooth form defined on Δ_k such that $\omega_{/\mathbf{bd}\Delta_k} = \xi$ and so the result is proved. □

Proposition 2.3.2 (**Extension lemma for piecewise smooth forms - general case**). Let Δ be any simplex of dimension k. Let $\mathcal{A}$ be a transitive Lie algebroid on Δ and consider the complexes of Lie algebroids $\underline{\mathcal{A}^{\Delta_k}} = \{\mathcal{A}^{!!}_\alpha\}_{\alpha\in\Delta_k}$ and $\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}} = \{\mathcal{A}^{!!}_\alpha\}_{\alpha\in\mathbf{bd}\ \Delta_k}$ given by restriction of $\mathcal{A}$ to the correspondent simplicial complexes Δ_k and $\mathbf{bd}\ \Delta_k$ respectively. Let $\xi \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\mathbf{bd}\Delta_k}}; \mathbf{bd}\ \Delta_k\big)$ be a piecewise smooth form of degree p defined on $\mathbf{bd}\ \Delta_k$. Then, there is a piecewise smooth form $\omega \in \Omega^p_{ps}\big(\underline{\mathcal{A}^{\Delta_k}}; \Delta_k\big)$ of degree p defined on Δ_k such that $\omega_{/\mathbf{bd}\ \Delta_k} = \xi$.

Proof. There is an affine isomorphism φ from the simplex Δ_k onto the simplex Δ which maps the boundary $\mathbf{bd}\ \Delta_k$ onto the boundary $\mathbf{bd}\ \Delta$. Consider the transitive Lie algebroid $\varphi^{!!}(\mathcal{A})$ on Δ_k. Then, $\varphi^{!!}(\mathcal{A})$ is isomorphic (non strong isomorphism of Lie algebroids) to the Lie algebroid $\mathcal{A}$. Take the inverse image of the form ξ, apply the previous proposition, take the direct image and we have the required extension. □

On last propositions, we began with a piecewise smooth form defined on whole boundary. However, we can improve slightly last propositions and establish a result concerning extension of piecewise smooth forms when the form is defined, not on all $(k-1)-$dimensional faces, but just on some $(k-1)-$dimensional faces of Δ. We note this improvement on next proposition.

Proposition 2.3.3. Let Δ be any simplex of dimension k and $\mathcal{A}$ a transitive Lie algebroid on the simplex Δ. Consider the complex of Lie algebroids $\underline{\mathcal{A}^{\Delta}} = \{\mathcal{A}^{!!}_{\alpha}\}_{\alpha\in\Delta}$ given by restriction of $\mathcal{A}$ to the correspondent simplicial complex Δ. Suppose that α_0, $\ldots$, α_k are the $k+1$ faces of dimension $k-1$ of simplex Δ and that we have r smooth forms $\xi_{j_1} \in \Omega^p(\mathcal{A}^{!!}_{\alpha_{j_1}}; \alpha_{j_1})$, $\ldots$, $\xi_{j_r} \in \Omega^p(\mathcal{A}^{!!}_{\alpha_{j_r}}; \alpha_{j_r})$ with $\{j_1, \ldots, j_r\} \subset \{1, \ldots, k\}$ such that, for each j_i, j_e with $\alpha_{j_i} \cap \alpha_{j_e}$ non empty, the forms ξ_{j_i} and ξ_{j_e} agree on the intersection $\alpha_{j_i} \cap \alpha_{j_e}$. Then, there is a piecewise smooth form $\omega \in \Omega^p_{ps}\left(\underline{\mathcal{A}^{\Delta}}; \Delta\right)$ such that $\omega_{/\alpha_{j_i}} = \xi_{j_i}$ for $i = 1, \ldots, r$.

Proof. For each vertex which doesn't belong to any face $\alpha_{j_1}, \ldots, \alpha_{j_r}$, we take the smooth form zero on this vertex and then we have a family of smooth forms, each one defined on each vertex of Δ. Now, for each two vertices defining a face of Δ of dimension 1 different of any $1-$dimensional face of α_{j_i}, with $i = 1, \ldots, r$, we apply the previous proposition and we obtain a piecewise smooth form defined on the skeleton of Δ of dimension 1. This piecewise smooth form is an extension of each smooth forms given on the $1-$dimensional faces of α_{j_i} $(i = 1, \ldots, r)$ by restriction of the forms ξ_{j_r}. We repeat the same argument for dimension 2. This process will end on dimension k and the form obtained is a piecewise smooth form defined on Δ, which is an extension of the forms ξ_{j_i} $(i = 1, \ldots, r)$. $\square$

From last propositions, we easily obtain the next general lemma on extensions.

Proposition 2.3.4 (**Extension lemma**). Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_{\alpha}\}_{\alpha\in K}$ a complex of Lie algebroids on K. Let L be a simplicial subcomplex of K and consider the subcomplex of Lie algebroids $\{\mathcal{A}_{\alpha}\}_{\alpha\in L}$ defined on L. Then, any piecewise smooth form of degree p defined on L can be piecewise smoothly extended to a piecewise smooth form of degree p defined on the whole K.

We conclude from this proposition that the map $r_L^{p^K} : \Omega^p_{ps}\big(\underline{\mathcal{A}}; K\big) \longrightarrow \Omega^p_{ps}\big(\underline{\mathcal{A}}^L; L\big)$ from proposition 2.2.6 is surjective.

Proposition 2.3.5. Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\alpha\}_{\alpha\in K}$ a complex of Lie algebroids on K. Let K_0 and K_1 be two simplicial subcomplexes of K such that $K = K_0 \cup K_1$ and set $L = K_0 \cap K_1$. Consider the complexes of Lie algebroids $\underline{\mathcal{A}^0} = \{\mathcal{A}_\alpha\}_{\alpha\in K_0}$, $\underline{\mathcal{A}^1} = \{\mathcal{A}_\alpha\}_{\alpha\in K_1}$ and $\underline{\mathcal{A}^{0,1}} = \{\mathcal{A}_\alpha\}_{\alpha\in L}$ given by restriction of $\mathcal{A}$ to the simplicial subcomplexes K_0, K_1 and L. Then, it holds a exact short sequence of cochain complexes

$$\{0\} \longrightarrow \Omega^*_{ps}(\underline{\mathcal{A}}; K) \xrightarrow{\lambda^*} \Omega^*_{ps}(\underline{\mathcal{A}^0}; K_0) \oplus \Omega^*_{ps}(\underline{\mathcal{A}^1}; K_1) \xrightarrow{\mu^*} \Omega^*_{ps}(\underline{\mathcal{A}^{0,1}}; L) \longrightarrow \{0\}$$

in which the linear maps

$$\lambda^p : \Omega^*_{ps}(\underline{\mathcal{A}}; K) \longrightarrow \Omega^*_{ps}(\underline{\mathcal{A}^0}; K_0) \oplus \Omega^*_{ps}(\underline{\mathcal{A}^1}; K_1)$$

$$\mu^p : \Omega^*_{ps}(\underline{\mathcal{A}^0}; K_0) \oplus \Omega^*_{ps}(\underline{\mathcal{A}^1}; K_1) \longrightarrow \Omega^*_{ps}(\underline{\mathcal{A}^{0,1}}; L)$$

are defined by $\lambda^p(\omega) = (\omega_{/K_0}, \omega_{/K_1})$ and $\mu^p(\xi,\eta) = \eta_{/L} - \xi_{/L}$.

Proof. As in the case of smooth forms on a transitive Lie algebroid over a smooth manifold, the exterior derivative commutes with the restrictions to a simplicial subcomplexes (proposition 2.2.5) and, since $d^p(\xi,\eta) = (d^p(\xi), d^p(\eta))$, one deduces immediately that λ^* and μ^* are effectively cochain complex morphisms. Obviously, the linear map λ^p is injective. Since, for each piecewise smooth form $\omega \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$, the forms $\omega_{/K_0}$ and $\omega_{/K_1}$ have the same restriction ω_L to L, we conclude that $\mu^p \circ \lambda^p = 0$, and hence the image of the linear map λ^p is contained in the kernel of the linear map μ^p. Reciprocally, if $\mu^p(\xi,\eta) = 0$, we have $\xi_\alpha = \eta_\alpha$, for each $\alpha \in L$, and this equality allows to define a piecewise smooth form $\omega \in \Omega^p_{ps}(\underline{\mathcal{A}}; K)$ by the condition $\omega_\alpha = \xi_\alpha$, for each $\alpha \in K_0$, and $\omega_\alpha = \eta_\alpha$, for each $\alpha \in K_1$. We have then $\lambda^p(\omega) = \mu^p(\xi,\eta)$. We want now to prove that μ^p is surjective.

Let $\gamma \in \Omega^p_{ps}(\underline{\mathcal{A}^{0,1}}; L)$ be a piecewise smooth form and consider the piecewise smooth form $-\frac{1}{2}\gamma \in \Omega^p_{ps}(\underline{\mathcal{A}^{0,1}}; L)$. By the extension lemma, we can consider a piecewise smooth form $\alpha \in \Omega^p_{ps}(\underline{\mathcal{A}^0}; K_0)$ such that $\alpha_{/L} = -\frac{1}{2}\gamma$. Analogously, we can consider a piecewise smooth form $\beta \in \Omega^p_{ps}(\underline{\mathcal{A}^1}; K_1)$ such that $\beta_{/L} = \frac{1}{2}\gamma$. We have then that $\mu^p(\alpha, \beta) = \gamma$. $\square$

By applying the zig-zag lemma to the sequence above, we obtain the long exact sequence in cohomology

$$H^{p-1}_{ps}(\underline{\mathcal{A}^{0,1}}; L) \xrightarrow{\partial^{p-1}} H^p_{ps}(\underline{\mathcal{A}}; K) \xrightarrow{H^p(\lambda^*)} H^p_{ps}(\underline{\mathcal{A}^0}; K_0) \oplus H^p_{ps}(\underline{\mathcal{A}^1}; K_1)$$

$$H^p_{ps}(\underline{\mathcal{A}^0}; K_0) \oplus H^p_{ps}(\underline{\mathcal{A}^1}; K_1) \xrightarrow{H^p(\mu^*)} H^p_{ps}(\underline{\mathcal{A}^{0,1}}; L) \xrightarrow{\partial^p} H^{p+1}_{ps}(\underline{\mathcal{A}}; K)$$

which is the Mayer-Vietoris sequence for piecewise smooth cohomology.

2.4 Generalization of piecewise smooth context

It should be remarked that, in previous sections, specific properties of the simplices were not required neither in the formulation of the piecewise smooth context nor in the statement of some properties. Indeed, these notions and properties can be extended to more general spaces. Moreover, for the proof of the Mishchenko's theorem given in next section, we are going to need a slightly modification of the concept of piecewise smooth cohomology given in the previous section. We shall notice, in this section, a general notion of piecewise smooth cohomology to other spaces which may not be simplicial complexes. A sheaf of the piecewise smooth forms on a complex of Lie algebroids will be constructed. As remarked in previous section, all simplicial complexes considered are geometric and finite and simplex means always closed simplex.

Definition. Let $\underline{K} = \{N_1, \ldots, N_s\}$ be a finite collection of submanifolds in an ambient space such that, for any $j_1, \ldots, j_e \in \{1, \ldots, s\}$, the intersection $N_{j_1} \cap \cdots \cap N_{j_e}$ is a submanifold. A complex of Lie algebroids on $\underline{K}$ is a family $\underline{\mathcal{A}} = \{\mathcal{A}_j\}_{j \in J}$ such that the following conditions hold.

- For each $j \in \{1, \ldots, s\}$, $\mathcal{A}_j$ is a transitive Lie algebroid on N_j.
- For each $i, j \in \{1, \ldots, s\}$, one has $(\mathcal{A}_j)^{!!}_{N_j \cap N_i} = (\mathcal{A}_i)^{!!}_{N_j \cap N_i}$.

It is obvious that, by transitivity of restrictions of Lie algebroids, for each subset $\widetilde{J}$ of $\{1, \ldots, s\}$ and any partition $\{\{j_1, \ldots, j_r\}, \{i_1, \ldots, i_t\}\}$ of $\widetilde{J}$, we have $(\mathcal{A}^{!!}_{N_r})^{!!}_{N_t} = (\mathcal{A}^{!!}_{N_t})^{!!}_{N_r}$, where $N_r = N_{j_1} \cap \cdots \cap N_{j_r}$ and $N_t = N_{i_1} \cap \cdots \cap N_{i_t}$

Definition. Let $\underline{K} = \{N_1, \ldots, N_s\}$ be a finite collection of submanifolds in an ambient space such that, for any $j_1, \ldots, j_e \in \{1, \ldots, s\}$, the intersection $N_{j_1} \cap \cdots \cap N_{j_e}$ is a submanifolds. Assume that a complex of Lie algebroids $\underline{\mathcal{A}} = \{\mathcal{A}_j\}_{j \in J}$ on $\underline{K}$ is given. A piecewise smooth form of degree p ($p \geq 0$) on $\underline{\mathcal{A}}$ is a family $\omega = (\omega_1 \ldots, \omega_s)$ such that, for each $j \in \{1, \ldots, s\}$,

- For each $j \in \{1, \ldots, s\}$, $\omega_j \in \Omega^p(\mathcal{A}_j; N_j)$ is a smooth form on $\mathcal{A}_j$.
- For each $i, j \in \{1, \ldots, s\}$, one has

$$\varphi^{\mathcal{A}_j}_{N_j \cap N_i, N_j}(\omega_j) = \varphi^{\mathcal{A}_i}_{N_j \cap N_i, N_i}(\omega_i)$$

where

$$\varphi^{\mathcal{A}_j}_{N_j \cap N_i, N_j} : \Omega^*(\mathcal{A}_j; N_j) \longrightarrow \Omega^*((\mathcal{A}_j)^{!!}_{N_i \cap N_j}; N_i \cap N_j)$$
$$\varphi^{\mathcal{A}_i}_{N_j \cap N_i, N_i} : \Omega^*(\mathcal{A}_i; N_i) \longrightarrow \Omega^*((\mathcal{A}_i)^{!!}_{N_i \cap N_j}; N_i \cap N_j)$$

denote the homomorphisms of cochain algebras generated by the inclusions maps $\varphi_{N_j \cap N_i, N_j} : N_i \cap N_j \longrightarrow N_j$ and $\varphi_{N_j \cap N_i, N_i} : N_i \cap N_j \longrightarrow N_i$ respectively.

The family of all such forms obtained in this way will be denoted by $\Omega^p_{ps}(\underline{\mathcal{A}}; \underline{K})$. This set is a real vector subspace of the product vector space $\Omega^p(A_1; N_1) \times \cdots \times \Omega^p(A_s; N_s)$. A wedge product and an exterior derivative can be defined on $\Omega^*_{ps}(\underline{\mathcal{A}};\underline{K}) = \bigoplus_{p\geq 0}\Omega^p_{ps}(\underline{\mathcal{A}};\underline{K})$ by the corresponding operations on each algebra $\Omega^*_{ps}(\mathcal{A}_j;N_j) = \bigoplus_{p\geq 0}\Omega^p(\mathcal{A}_j;N_j)$, giving to $\Omega^p_*(\underline{\mathcal{A}};\underline{K})$ a structure of cochain algebra defined over $\mathbf{R}$. The cohomology space of this algebra will be denoted by $H^*_{ps}(\underline{\mathcal{A}};\underline{K})$.

We notice that piecewise smooth cohomology of a complex of Lie algebroids defined on a simplicial complex is a particular case of this generalization. The reason for this generalization is that, as mentioned at the introduction of this section, we are going to deal with a complex of piecewise smooth forms that may not be defined over the family of all closed simplices of a simplicial complex. To illustrate this idea, let us briefly look at some cases of this construction. A first example of this generalization is take a simplicial complex and to fix our attention on an open star of one its vertex. In this case, we have the open star smoothly triangulated by a non-complete simplicial complex since each simplex of this triangulation does not contain the face opposite to the vertex. The family of submanifolds made by those simplices without the faces opposite to the vertex satisfies the conditions required in our definition of complex of Lie algebroids given at the beginning of this section. Any transitive Lie algebroid over the open star gives, by restriction, a complex of Lie algebroids. Another illustrative example consists of taking the family defined by intersections of open stars with any open subset of the polytope of a simplicial complex. The first example is obviously a particular example of this second case. The construction of a complex of Lie algebroids can be done in similar way. These two examples will be used in the proof of the main theorem of next section. Our third example extends the second one and and consists of taking intersections of generalized stars with open subsets of the polytope. This third example is not quite different of previous examples. Nevertheless, it enhances the construction of the sheaf of the piecewise smooth forms on a complex of Lie algebroids. We provide, in next section, a description of this third example as well of the corresponding sheaf of piecewise smooth forms.

2.5 Sheaves of piecewise smooth forms

In this section, we describe a corresponding sheaf of piecewise smooth forms, which we can define on a special complex of Lie algebroids defined by using regular open subsets. Definitions and main properties of regular open subsets can be seen in [24]. The idea of construction of the sheaf of the piecewise smooth forms on a complex of Lie algebroids comes from [9] or [1].

Definition (Generalized star). Let K be a simplicial complex and a a point of the polyhedron $|K|$. The generalized star of a, denoted also by $\mathbf{St}\, a$, is the union of the interiors of all simplices of K such that a belongs to those simplices.

Remark. When the point a is a vertex of K, it is obvious that the generalized star of a is the same as the star of a.

Remark 2. For each $a \in K$, there is a unique simplex Δ_a of K such that a belongs to the interior of Δ_a (see [24]).

Proposition 2.5.1. Let K be a simplicial complex and a a point of the polytope $|K|$. Denote by Δ_a the unique simplex of K such that a belongs to the interior of Δ_a. Then, the generalized star of a coincide with the star $\mathbf{St}\, \Delta_a$. Consequently, the generalized star of a is an open subset of the polyhedron $|K|$.

Proof. If a is one of the vertices of K, then $\Delta_a = \{a\}$ and the result is proved. Suppose now that a is different of any vertex of K. Then a belongs to the interior of Δ_a. We shall see first that $\mathbf{St}\, a \subset \mathbf{St}\, \Delta_a$. Let Δ be a simplex of K such that $a \in \Delta$. Since a is different of any vertex of K, it follows that $a \in \overset{\circ}{s}$, for some face s of Δ. But $a \in \overset{\circ}{\Delta}_a$ and so $\overset{\circ}{s} = \overset{\circ}{\Delta}_a$. Hence $s = \Delta_a$ and therefore Δ_a is a face of Δ. We conclude then $\overset{\circ}{\Delta} \subset \mathbf{St}\, \Delta_a$. Now, let $\widetilde{\Delta}$ be a simplex of K such that Δ_a is a face of $\widetilde{\Delta}$. Then, $a \in \widetilde{\Delta}$ and so $\overset{\circ}{\widetilde{\Delta}} \subset \mathbf{St}\, \Delta_a$. The other inclusion is obvious. The second part of the proposition is immediate. □

The collection of the stars of all vertices of a simplicial complex K form an open covering of its polytope $|K|$ and the base obtained from this covering is a contractible base for the topology of $|K|$. Below, we are going to deal with other coverings which are not obtained from its stars of its vertices, but from regular open subsets. We recall that a regular open subset of $|K|$ is a star of some simplex of K (see [24]). Next definition is to generalize this notion in order to obtain other coverings of $|K|$ which fit better for the sheaf of piecewise smooth forms on a complex of Lie algebroids.

Definition (Regular open subset). Let K be a simplicial complex and $|K|$ its polytope. Let $a \in |K|$ and U an open subset of $|K|$ with $a \in U$. The open subset U is called regular open neighborhood of a if U is the intersection of an open neighborhood of a in $|K|$ with the generalized star of a. Given any open subset V of $|K|$, V is called a regular open subset of $|K|$, if there exists a point $a \in |K|$ such that V is a regular open neighborhood of the point a.

Remark. Obviously, a star of some simplex of a simplicial complex is a regular open subset of its polytope.

Proposition 2.5.2. Let K be a simplicial complex and $|K|$ its polytope. For each $a \in |K|$, the set of all regular open neighborhoods of a is a fundamental system of neighborhoods of a and the set of all regular open subsets of $|K|$ is a base for the topology of the space $|K|$.

Proof. Standard arguments.

The proposition 2.2.6 still remains true in the piecewise context obtained by using the set of all generalized regular open subsets of the polytope of a simplicial complex. We notice those facts below, beginning first to describe a special construction of a complex of Lie algebroids based in regular open subsets.

Derived complex corresponding to regular open subsets. Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta\in K}$ a complex of Lie algebroids on K. Let U be a regular open subset of $|K|$ and consider $a \in |K|$ such that $U = Z \cap \mathbf{St}\, a$, in which Z is an open neighborhood of a in $|K|$. Consider the unique simplex Δ_a of K such that a belongs to the interior of Δ_a. For each simplex $\Delta \in K$ such that Δ_a is a face of Δ, denote by Δ_U the set $\Delta_U = U \cap \Delta$. We have that Δ_U is an open submanifold of Δ. If Δ' is other simplex of K such that Δ_a is a face of Δ', then Δ_a is a face of $\Delta \cap \Delta'$ and the intersection $U \cap (\Delta \cap \Delta)$ is a submanifold. The collection $\underline{K}^U$, made by the manifolds $\Delta_U = U \cap \Delta$ such that Δ_a is a face of Δ, satisfies the conditions required for the definition of complex of Lie algebroids given at the beginning of this section. The Lie algebroid $\mathcal{A}_\Delta$ is transitive and so we can take the Lie algebroid restriction $(\mathcal{A}_\Delta)^{!!}_{\Delta_U}$ to Δ_U. Therefore, we can consider the family

$$\mathfrak{A}_U = \{(\mathcal{A}_\Delta)^{!!}_{\Delta_U} : \Delta \in K, \quad \Delta_a \prec \Delta\}$$

We claim that the family $\mathfrak{A}_U$ is a complex of Lie algebroids defined over the set of manifolds $\underline{K}^U$.

Before proving this statement, we are going to check that the triangulation obtained in U does not depend on the point a chosen, that is, if $Z \cap \mathbf{St}\, a = \widetilde{Z} \cap \mathbf{St}\, b$, then $\mathbf{St}\, a = \mathbf{St}\, b$. To see this, denote by Δ_a and Δ_b the unique simplices of K which contain a and b in its interior respectively. Then, $\mathbf{St}\, a = \mathbf{St}\, \Delta_a$ and $\mathbf{St}\, b = \mathbf{St}\, \Delta_b$. Since $b \in V \cap \mathbf{St}\, a$, there exists a simplex $\Delta' \in K$ such that Δ_a is a face of Δ' and b belongs to the interior of Δ'. Hence, $\Delta_b = \Delta'$ by uniqueness of Δ_b, and so Δ_a is a face of Δ_b. Analogously, we conclude can that Δ_b is a face of Δ_a and so it holds that $\Delta_b = \Delta_a$.

It remains to check that the family $\mathfrak{A}_U$ is indeed a complex of Lie algebroids. For that, fix two simplices Δ and Δ' of K such that Δ_a is a common face of Δ and Δ'. Let $s = \Delta \cap \Delta'$. We have

$$\left((\mathcal{A}_\Delta)^{!!}_{\Delta_U}\right)^{!!}_{U\cap s} = (\mathcal{A}_\Delta)^{!!}_{U\cap s} = \left((\mathcal{A}_\Delta)^{!!}_{s}\right)^{!!}_{U\cap s} = (\mathcal{A}_s)^{!!}_{U\cap s}$$

and analogously

$$\left((\mathcal{A}_{\Delta'})^{!!}_{\Delta'_U}\right)^{!!}_{U\cap s} = (\mathcal{A}_s)^{!!}_{U\cap s}$$

Hence, the family $\mathfrak{A}_U$ is a complex of Lie algebroids over the set of manifolds

$$\underline{K}^U = \{U \cap \Delta : \Delta \in K, \Delta_a \prec \Delta\}$$

The complex $\mathfrak{A}_U$ is called the derived complex of the complex $\underline{\mathcal{A}}$ corresponding to the regular open subset U.

The cochain algebra of the piecewise smooth forms on the derived complex of a complex of Lie algebroids will be denoted simply by $\Omega^*_{ps}(\mathfrak{A}_U)$, dropping the letter that represents the family of submanifolds which the derived complex of Lie algebroids is defined on.

Keeping the same hypothesis and notations as above, let U and V two regular open subsets of $|K|$ such that $V \subset U$. Let a and $b \in |K|$ such that U and V are regular open neighborhoods of a and b respectively. Denote by Δ_a and Δ_b the unique simplices of K which contain a and b in its interior respectively. Since $b \in U$, there exists a simplex $\Delta \in K$ such that Δ_a is a face of Δ and the point b belongs to the interior of Δ. Hence, $\Delta_b = \Delta$ and so Δ_a is a face of Δ_b. If Δ' is a simplex of K such that Δ_b is a face Δ', then Δ_a is a face of Δ' and, consequently, every element of $\underline{K}^V = \{\Delta_V : \Delta \in K, \Delta_b \prec \Delta\}$ is a submanifold of the respective element of $\underline{K}^U = \{\Delta_U : \Delta \in K, \Delta_a \prec \Delta\}$. Let $\omega = (\omega_{\Delta_U})_{\Delta_U \in \underline{K}^U} \in \Omega^*_{ps}(\mathfrak{A}_U)$ be a piecewise sooth form on the complex of Lie algebroids $\mathfrak{A}_U$. For each simplex $\Delta \in K$ such that Δ_b is a face of Δ, we have that $\big((\mathcal{A}_\Delta)^{!!}_{\Delta_U}\big)^{!!}_{\Delta_V} = (\mathcal{A}_\Delta)^{!!}_{\Delta_V}$ and we can restrict the smooth form $\omega_{\Delta_U} \in \Omega^*\big((\mathcal{A}_\Delta)^{!!}_{\Delta_U}; \Delta_U\big)$ to the submanifold Δ_V, obtaining the smooth form $\omega_{\Delta_V} = (\omega_{\Delta_U})^{!!}_{\Delta_V} \in \Omega^*\big((\mathcal{A}_\Delta)^{!!}_{\Delta_V}; \Delta_V\big)$. Therefore, we obtain the differential form $(\omega_{\Delta_V})_{\Delta_V \in \underline{K}^V}$. Similar arguments given in the proof of the proposition 3.2 can be used to show that the form $(\omega_{\Delta_V})_{\Delta_V \in \underline{K}^V}$ is a piecewise smooth form and so it belongs to $\in \Omega^*_{ps}(\mathfrak{A}_V)$. As done in the definition following proposition 3.4, the piecewise smooth form $(\omega_{\Delta_V})_{\Delta_V \in \underline{K}^V}$ is denoted by $\omega_{/V}$ or simply by ω_V.

Proposition 2.5.3. Let K be a simplicial complex and $\underline{\mathcal{A}} = \{\mathcal{A}_\Delta\}_{\Delta \in K}$ a complex of Lie algebroids on K. Let U and V be two regular open subsets of $|K|$ such that $U \subset V$ and consider the derived complexes of Lie algebroids $\mathfrak{A}_U$ and $\mathfrak{A}_V$ corresponding to U and

V respectively. For each $p \geq 0$, denote by

$$r^{p^K}_{L} : \Omega^p_{ps}(\mathfrak{A}_U) \longrightarrow \Omega^p_{ps}(\mathfrak{A}_V)$$

the map induced by restriction, that is, for each $\omega \in \Omega^p_{ps}(\mathfrak{A}_U)$,

$$r^{p^U}_{V}(\omega) = \omega_{/V}$$

- For each $p \geq 0$, $r^{p^U}_{U} = id_{\Omega^p_{ps}(\mathfrak{A}_U)}$.
- $r^{*^U}_{V} : \Omega^*_{ps}(\mathfrak{A}_U) \longrightarrow \Omega^*_{ps}(\mathfrak{A}_V)$ is a morphism of graded algebras.
- If W is other generalized regular open subset of $|K|$ with $W \subset V$ and $\mathfrak{A}_W$ is the derived complex of Lie algebroids corresponding to W, then the diagram below is a commutative diagram of cochain complexes

$$\begin{array}{ccc} \Omega^*_{ps}(\mathfrak{A}_U) & \xrightarrow{\; r^{*^U}_{V} \;} & \Omega^*_{ps}(\mathfrak{A}_V) \\ & {\scriptstyle r^{*^U}_{W}} \searrow \quad \swarrow {\scriptstyle r^{*^V}_{W}} & \\ & \Omega^*_{ps}(\mathfrak{A}_W) & \end{array}$$

Consequently, for each $p \geq 0$, the correspondence which associates, to each regular open subset U of $|K|$ the real vector space $\Omega^p_{ps}(\mathfrak{A}_U)$ of the piecewise smooth forms defined on U and, to each pair of regular open subsets U and V of $|K|$ with $V \subset U$ the homomorphism $r^{p^U}_{V}$, is a presheaf, which is called the presheaf of the piecewise smooth forms of degree p of the complex $\underline{\mathcal{A}}$.

Proof. Standard arguments.

The last proposition leads us to the following definition (see [9]).

Definition. Let K be a simplicial complex and $\mathfrak{A} = (\mathcal{A}_\alpha)_{\alpha \in K}$ a sheaf of Lie algebroids on K. For each $p \geq 0$, the sheaf of the piecewise smooth forms of degree p on the sheaf of Lie algebroids $\mathfrak{A}$ is the sheaf constructed canonically from the presheaf of the piecewise smooth forms of degree p on $(\mathcal{A}_\alpha)_{\alpha \in K}$.

Proposition 2.5.4. Let K be a simplicial complex and $\mathfrak{A} = (\mathcal{A}_\alpha)_{\alpha \in K}$ a sheaf of Lie algebroids on K. Then the sheaf $\mathfrak{S}$ of the piecewise smooth forms of degree p on the sheaf of Lie algebroids $\mathfrak{A}$ is fine.

Proof. Let $\mathfrak{U} = \{U_j\}_{j \in J}$ be a locally finite open covering of $|K|$ by regular open subsets of $|K|$. Since the set of all regular open subsets of $|K|$ is a base for the topology of $|K|$, we can assume that each open subset $U \in \mathfrak{U}$ is a regular open subset. If $\{\varphi_j\}_{j \in J}$ is piecewise smooth partition of unity subordinated to the covering $\mathfrak{U}$, the homomorphisms of presheaves $h_j : \Omega^p_{ps}(\mathfrak{A}_U) \longrightarrow \Omega^p_{ps}(\mathfrak{A}_U)$ defined by $h_j(\omega) = \varphi_{j/U}\omega$ for each $\omega \in \Omega^p_{ps}(\mathfrak{A}_U)$ induce homomorphisms from $\mathfrak{A}^p$ to $\mathfrak{A}^p$ satisfying the conditions which characterize the definition of fine sheaf. Therefore, the result is proved if we find a piecewise smooth partition of unity subordinated to the covering $\mathfrak{U}$. By lemma shrinking, there is an open covering $\mathfrak{V} = \{V_j\}_{j \in J}$ such that, for each $j \in J$, $\overline{V_j} \subset U_j$. Let $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$ such that $\overline{V} \subset U$. Consider $a \in |K|$ such that U is a regular open neighborhood of a in $|K|$. For each simplex $\Delta \in K$ such that Δ_a is a face of Δ, consider the closed subset $\overline{V} \cap \Delta$ of Δ. Take the union of all $\overline{V} \cap \Delta$ such that Δ_a is a face of Δ and denote that union by W. Since $|K|$ is compact, the topology of $|K|$ coincide with the topology induced from the Euclidian space. We have that W a closed subset of the Euclidian space. The open star $\mathbf{St}\,\Delta_a$ is open in $|K|$ and so there is an open subset Z of the Euclidian space such that $\mathbf{St}\,\Delta_a = Z \cap |K|$. The closed subset is contained in the open subset Z. Hence, we can fix a smooth function $\varphi : Z \longrightarrow \mathbf{R}$ such that φ does not vanish on W. By restriction to each submanifold $\Delta_U = U \cap \Delta$, we have a piecewise smooth function on U which does not vanish on each $\overline{V} \cap \Delta$. Take the sum of these functions and consider the quotient of each function by the sum. This defines a partition of unity made by piecewise smooth functions.

Chapter 3

Main theorems

In this chapter, we state the main results of our work. The chapter is divided in three sections. The first section is devoted to the proof of Mishchenko's conjecture, which sates that Lie algebroid cohomology and piecewise smooth cohomology of a transitive Lie algebroid over a combinatorial manifold are isomorphic. The remaining sections outline two applications of Mishchenko's theorem to invariant cohomology of transitive Lie algebroids and to piecewise de Rham cohomology of locally trivial Lie groupoids. In this chapter, all simplicial complexes are finite and consequently all smooth manifolds considered in this chapter are compact manifolds.

3.1 Mishchenko's theorem

The central purpose of this section is to give a relationship between de Lie algebroid cohomology and piecewise smooth cohomology of transitive Lie algebroids. Mishchenko conjectured that the Lie algebroid cohomology and the piecewise smooth cohomology of a transitive Lie algebroid over a combinatorial compact manifold are isomorphic and this isomorphism is induced by restriction of smooth forms to the simplices. We prove that this conjecture holds for any transitive Lie algebroid over a combinatorial compact manifold and this theorem will be called Mishchenko's theorem.

Let M be a smooth manifold, smoothly triangulated by a simplicial complex K. Suppose that $\mathcal{A}$ is a transitive Lie algebroid on M. We can consider the corresponding complex of Lie algebroids $\underline{\mathcal{A}} = \{\mathcal{A}^{!!}_{\Delta}\}_{\Delta \in K}$ obtained by restriction of $\mathcal{A}$ to the simplices of K (example 4 given after the definition of complex of Lie algebroids, section 2.1 of the second chapter). We recall that $\Omega^*(\mathcal{A}; M)$ denotes the cochain algebra of all smooth forms on the Lie algebroid $\mathcal{A}$. The cochain algebra of all piecewise smooth forms on the corresponding complex $\{\mathcal{A}_{\Delta}\}_{\Delta \in K}$ is denoted by $\Omega^*_{ps}(\mathcal{A}; M)$ or $\Omega^*_{ps}(\mathcal{A}; K)$. Given a simplex Δ of K, consider the canonical Lie algebroid morphism $\lambda = (\varphi_{M,\Delta}, \varphi^{!!}_{M,\Delta})$ in which $\varphi_{M,\Delta} : \Delta \longrightarrow M$ and $\varphi^{!!}_{M,\Delta} : \mathcal{A}^{!!}_{\Delta} \longrightarrow \mathcal{A}$ are respectively the inclusion map and the corresponding induced map. The morphism of cochain algebras generated by the map $\varphi_{M,\Delta}$ is

$$\varphi^{\mathcal{A}}_{M,\Delta} : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}^{!!}_{\Delta}; \Delta)$$

which is defined by $\varphi^{\mathcal{A}}_{M,\Delta}(\omega) = \lambda^*(\omega)$. The form $\varphi^{\mathcal{A}}_{M,\Delta}(\omega)$ is simply denote by ω_{Δ} (definition given in the section 1.2 of the first chapter). Moreover, any smooth form $\omega \in \Omega^p(\mathcal{A}; M)$ of degree p determines, on the corresponding complex $\underline{\mathcal{A}} = \{\mathcal{A}^{!!}_{\Delta}\}_{\Delta \in K}$, a piecewise smooth form of degree p defined by the restriction of ω to each simplex of K, that is,

$$\Omega^{\mathbf{p}}(\mathcal{A}; M) \ni \omega \longrightarrow (\omega_{\Delta})_{\Delta \in K} \in \Omega^{\mathbf{p}}_{ps}(\mathcal{A}; K)$$

Hence, we have a linear map

$$\Psi^p : \Omega^p(\mathcal{A}; M) \longrightarrow \Omega^p_{\mathrm{ps}}(\mathcal{A}; K)$$

defined by

$$\omega \longrightarrow (\omega_{\Delta})_{\Delta \in K}$$

Since the exterior derivative commutes with restrictions to any submanifold of M, the family $\Psi = (\Psi^p)_{p \geq 0}$ defines a cochain algebra morphism from $\Omega^*(\mathcal{A}; M)$ to $\Omega^*_{ps}(\mathcal{A}; K)$.

Mishchenko conjectured that the map Ψ induces an isomorphism in cohomology. The rest of this section is devoted to prove his conjecture, which we have called Mishchenko's theorem.

Main theorem - Mishchenko's theorem. Ψ induces an isomorphism in cohomology.

The connection between Lie algebroid and piecewise smooth cohomology of a transitive Lie algebroid on a combinatorial compact manifold will be made through the main theorem above. There are some facts we will need for the proof of this theorem. We will need the Mayer-Vietoris sequences for regular open subsets in the smooth and piecewise smooth cases, the triviality of Lie algebroids over contractible manifolds and a few background on simplicial manifolds as well the de Rham-Sullivan theorem for combinatorial manifolds.

We emphasize that, in the second chapter, it was given an example in which piecewise smooth forms may not be defined over a collection of closed simplices, but over a collection of submanifolds obtained by the intersection of closed simplices with a regular open subset. We restrict now our attention to this generalization of the piecewise smooth setting and note some notations. Once these ideas are established, we shall then turn towards to the statement and the proof of the Mishchenko's theorem.

Definitions. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and suppose that $\mathcal{A}$ is a transitive Lie algebroid on M. As done before, for each two submanifolds X and Z of M such that $\varphi_{Z,X} : X \hookrightarrow Z$ is an embedded submanifold, the map

$$\varphi_{Z,X}^{\mathcal{A}_Z^{!!}} : \Omega^*(\mathcal{A}_Z^{!!}; Z) \longrightarrow \Omega^*(\mathcal{A}_X^{!!}; X)$$

$$\varphi_{Z,X}^{\mathcal{A}_Z^{!!}}(\omega) = \lambda^*(\omega)$$

denotes the cochain algebra morphism generated by the inclusion $\varphi_{Z,X} : X \hookrightarrow Z$, in which $\lambda = (\varphi_{Z,X}, \varphi_{Z,X}^{!!})$ is the canonical Lie algebroid morphism defined by the inclusion $\varphi_{Z,X} : X \hookrightarrow Z$ and the induced smooth map $\varphi_{Z,X}^{!!} : \mathcal{A}_X^{!!} \longrightarrow \mathcal{A}_Z^{!!}$.

- **Definition 1**. Let $\mathbf{s}$ be a simplex of K and consider the regular open subset $U = \mathbf{St}(\mathbf{s})$ of M. For each simplex $\Delta \in K$ such that $\mathbf{s}$ a face of Δ, we denote by Δ_U the intersection $\Delta_U = \Delta \cap U$. The set Δ_U is an embedded submanifold of M and it holds the equality

$$U = \bigcup_{\Delta \in K,\ \mathbf{s} \prec \Delta} \Delta \cap U$$

We can consider the family $\underline{K}^U$ of all submanifolds $\Delta_U = \Delta \cap U$ such that Δ is a simplex of K provide that $\mathbf{s}$ is a face of Δ, that is,

$$\underline{K}^U = \{U \cap \Delta : \Delta \in K, \mathbf{s} \prec \Delta\}$$

It is obvious that, for any simplices $\Delta^1, \cdots, \Delta^q$ of K such that $\mathbf{s}$ is face of each Δ^j $(j = 1, \cdots, q)$ the intersection $\Delta^1_U \cap \cdots \cap \Delta^q_U$ is a submanifold and belongs to $\underline{K}^U$. For each $\Delta_U \in \underline{K}^U$, the Lie algebroid restriction $\mathcal{A}^{!!}_{\Delta_U}$ exists, since Δ_U is an open subset in the compact submanifold Δ. Thus, by transitivity of restrictions, one has

$$(\mathcal{A}^{!!}_{\Delta_U})^{!!}_{\Delta \cap \Delta' \cap U} = (\mathcal{A}^{!!}_{\Delta'_U})^{!!}_{\Delta \cap \Delta' \cap U}$$

for each simplices Δ and Δ' of K such that $\mathbf{s}$ is a face of both Δ and Δ'. Hence, the family $\{\mathcal{A}^{!!}_{\Delta_U}\}_{\Delta_U \in \underline{K}^U}$ is a complex of Lie algebroids on $\underline{K}^U$. We can apply the construction made in the section 2.4 of the second chapter to obtain a graded vector space of piecewise smooth forms over the complex of Lie algebroids $\{\mathcal{A}^{!!}_{\Delta_U}\}_{\Delta_U \in \underline{K}^U}$. This graded vector space of piecewise smooth forms will be denoted by $\Omega^*_{ps}(\mathcal{A}^{!!}_U; U)$. We have then that a piecewise smooth form $\omega \in \Omega^p_{ps}(\mathcal{A}^{!!}_U; U)$ of degree p is a family

$$\omega = (\omega_\Delta)_{\Delta_U \in \underline{K}^U} \in \prod_{\Delta_U \in \underline{K}^U} \Omega^p(\mathcal{A}^{!!}_{\Delta_U}; \Delta_U)$$

such that, if Δ and Δ' are simplices of K with $\mathbf{s} \prec \Delta' \prec \Delta$, one has

$$\varphi^{\mathcal{A}_{\Delta_U}}_{\Delta,\Delta'}(\omega_\Delta) = \omega_{\Delta'}$$

or simply $(\omega_\Delta)_{/\Delta'} = \omega_{\Delta'}$.

- **Definition 2**. More generally, suppose that $\mathbf{s}_1, \ldots, \mathbf{s}_e$ are simplices of K and set $U_1 = \mathbf{St}\,(\mathbf{s}_1), \ldots, U_e = \mathbf{St}\,(\mathbf{s}_e)$ and $U = U_1 \cup \cdots \cup U_e$. As done on the previous definition, we denote by Δ_U the intersection $\Delta \cap U$, for each simplex Δ of K such that $\mathbf{s}_j$ is a face of Δ for some $j \in \{1, \ldots, e\}$. Let $\underline{K}^U$ be the family of all submanifolds $\Delta_U = \Delta \cap U$ such that Δ is a simplex of K provide that $\mathbf{s}_j$ is a face of Δ for some $j \in \{1, \ldots, e\}$. Similar arguments given in the previous definition lead us to consider the complex of Lie algebroids $\{\mathcal{A}^{!!}_{\Delta_U}\}_{\Delta_U \in \underline{K}^U}$ defined on $\underline{K}^U$. A piecewise smooth form of degree p over the complex of Lie algebroids $\{\mathcal{A}^{!!}_{\Delta_U}\}_{\Delta_U \in \underline{K}^U}$ is a family

$$\omega = (\omega_\Delta)_{\Delta_U \in \underline{K}^U} \in \prod_{\Delta_U \in \underline{K}^U} \Omega^p(\mathcal{A}^{!!}_{\Delta_U}; \Delta_U)$$

such that, if Δ and Δ' are simplices of K with $\mathbf{s}_j \prec \Delta' \prec \Delta$ for some $j \in \{1, \ldots, e\}$, one has

$$\varphi^{\mathcal{A}_{\Delta_U}}_{\Delta,\Delta'}(\omega_\Delta) = \omega_{\Delta'}$$

The graded vector space of all piecewise smooth forms over the complex of Lie algebroids $\{\mathcal{A}^{!!}_{\Delta_U}\}_{\Delta_U \in \underline{K}^U}$ will be denoted by $\Omega^*_{ps}(\mathcal{A}^{!!}_U; U)$.

We notice that, for each simplex $\Delta \in K$ with $\mathbf{s} \prec \Delta$, the smooth form ω_Δ on the definition above is a smooth form defined on the submanifold $\Delta_U = U \cap \Delta$ and it is neither a smooth form defined on the smooth submanifold Δ nor a restriction of a smooth form defined on Δ.

Keeping the same hypothesis and notations of the previous definitions, a wedge product and a differential can be defined on $\Omega^*_{ps}(\mathcal{A}^{!!}_U; U)$ by the corresponding operations on each cochain algebra $\Omega^*(\mathcal{A}^{!!}_{\Delta_U}; \Delta_U)$, giving to $\Omega^*_{ps}(\mathcal{A}^{!!}_U; U)$ a structure of cochain algebra defined over $\mathbf{R}$. The cohomology space of this complex is denoted by $\mathbf{H}^*_{ps}(\mathcal{A}^{!!}_{\Delta_U}; U)$.

We establish and prove the main results for the proof of the Mishchenko's theorem. We begin by checking that the map Ψ is natural.

Proposition 3.1.1. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K and $\mathcal{A}$ a transitive Lie algebroid on M. Then, the map

$$\Psi^* : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*_{\mathrm{ps}}(\mathcal{A}; K)$$

given by $\Psi(\omega) = (\omega_\Delta)_{\Delta \in K}$ is natural.

Proof. Let M and N be two smooth manifolds, smoothly triangulated the the simplicial complexes K and L respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be transitive Lie algebroids on M and on N respectively and (F, f) a morphism of Lie algebroids from $\mathcal{A}$ into $\mathcal{B}$. We shall see that the following diagram

$$\begin{array}{ccc} H^p(\mathcal{B}; N) & \stackrel{(F,f)^*}{\longrightarrow} & H^p(\mathcal{A}; M) \\ \big\downarrow \Psi & & \big\downarrow \Psi \\ H^p_{ps}(\mathcal{B}; L) & \stackrel{(F,f)^*}{\longrightarrow} & H^p_{\mathrm{ps}}(\mathcal{A}; K) \end{array}$$

commutes. Let $\omega \in \Omega^p(\mathcal{B}; N)$ a smooth form. For each simplex $\Delta \in K$, the equality $(F^*\omega)_{/\Delta} = F^*(\omega_{/f(\Delta)})$ holds, and therefore, $\Psi(F^*\omega) = F^*(\omega_{/f(\Delta)})_{\Delta \in K}$. On the other side, $\Psi(\omega) = (\omega_{\Delta'})_{\Delta' \in L}$ and so, by definition of inverse image of differential form for the piecewise case, we have

$$F^*(\Psi(\omega)) = F^*\big((\omega_{\Delta'})_{\Delta' \in L}\big) = (F^*\omega_{/f(\Delta)})_{\Delta \in K}$$

From here, the commutativity of the diagram above can be readily derived. □

Next proposition is concerning the Mishchenko's theorem for trivial Lie algebroids. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Take a simplex $\mathbf{s}$ of K and let $U = \mathbf{St}(\mathbf{s})$. Assume that $\mathfrak{g}$ is a real Lie algebra and consider the trivial Lie algebroid $\mathcal{A} = TU \oplus (M \times \mathfrak{g})$ on U, which is identified to the Lie algebroid $\mathcal{A} = TU \times \mathfrak{g}$ by a strong homomorphism of Lie algebroids over U. For each $p \geq 0$, one has $\Omega^p(\mathfrak{g}) = \bigwedge^p \mathfrak{g}$. Consider the Lie algebroids morphisms

$$\gamma : TU \times \mathfrak{g} \longrightarrow TU$$

and

$$\pi : TU \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

For each simplex $\Delta \in K$ such that $\mathbf{s}$ is a face of Δ, we denote $\Delta_U = U \cap \Delta$ as done before. Consider the Lie algebroids morphisms

$$\gamma_{\Delta_U} : T\Delta_U \times \mathfrak{g} \longrightarrow T\Delta_U$$

and

$$\pi_{\Delta_U} : T\Delta_U \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

given by the projections on the first and second factors respectively.

Proposition 3.1.2. Keeping the same hypothesis and notations as above, the morphism

$$\Psi : \Omega^*(\mathcal{A}; U) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U)$$

$$\omega \longrightarrow (\omega_\Delta)_{\Delta_U \subset U}$$

induces an isomorphism in cohomology.

Proof. By the Künneth theorem [6], we have that

$$\mathrm{H}(\mathcal{A}; U) \simeq \mathrm{H}_{\mathrm{dR}}(U) \otimes \mathrm{H}(\mathfrak{g})$$

Next, we wish to check that

$$\mathrm{H}_{ps}(\mathcal{A}; U) \simeq \mathrm{H}_{ps}(U) \otimes \mathrm{H}(\mathfrak{g})$$

For that, we are going to divide the proof in three parts.

Part 1. We are going to check that

$$\Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g}) \simeq \Omega^*_{ps}(\mathcal{A}; U)$$

Let $\xi = (\xi_\Delta)_{\Delta_U \subset U} \in \Omega^*_{ps}(U)$ and $\eta \in \Omega^*(\mathfrak{g})$. If Δ' and Δ are two simplices of K such that $s \prec \Delta' \prec \Delta$, denote by $(\varphi^{T\Delta_U \times \mathfrak{g}}_{\Delta,\Delta'})^{!!}$ and $(\varphi^{T\Delta_U}_{\Delta,\Delta'})^{!!}$ the canonical map induced from the diagrams

$$\begin{array}{ccc} & & T\Delta_U \times \mathfrak{g} \\ & & \big\downarrow \gamma \\ \Delta'_U & \xrightarrow{\varphi_{\Delta,\Delta'}} & \Delta_U \end{array} \qquad\qquad \begin{array}{ccc} & & T\Delta_U \\ & & \big\downarrow \gamma \\ \Delta'_U & \xrightarrow{\varphi_{\Delta,\Delta'}} & \Delta_U \end{array}$$

It is obvious that

$$\gamma_{\Delta_U} \circ (\varphi^{T\Delta_U \times \mathfrak{g}}_{\Delta,\Delta'})^{!!} = (\varphi^{T\Delta_U}_{\Delta,\Delta'})^{!!} \circ \gamma_{\Delta'_U} \qquad \text{and} \qquad \pi_{\Delta_U} \circ (\varphi^{T\Delta_U \times \mathfrak{g}}_{\Delta,\Delta'})^{!!} = \pi_{\Delta'_U}$$

so the equalities

$$(\gamma^*_{\Delta_U} \xi_\Delta)_{\Delta'_U} = \gamma^*_{\Delta'_U} \xi_{\Delta'}$$

and

$$(\pi^*_{\Delta_U} \eta)_{\Delta'_U} = \pi^*_{\Delta'_U} \eta$$

hold. These equalities show that the differential form

$$(\gamma^*_{\Delta_U} \xi_\Delta \wedge \pi^*_{\Delta_U} \eta)_{\Delta_U \subset U}$$

belongs to $\Omega^*_{ps}(\mathcal{A}; U)$. Hence, we can consider a map

$$k : \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g}) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U)$$

such that

$$k(\xi \otimes \eta) = (\gamma^*_{\Delta_U} \xi_\Delta \wedge \pi^*_{\Delta_U} \eta)_{\Delta_U \subset U}$$

where $\xi = (\xi_\Delta)_{\Delta_U \subset U}$. This map is well defined. Now, we will see that the map k is an isomorphism of differential graded algebras. Obviously, the map k is a morphism of graded algebras. For each $\Delta \in K$ such that $s \prec \Delta$, let

$$k_\Delta : \Omega^*(\Delta_U) \otimes \Omega(\mathfrak{g}) \longrightarrow \Omega^*(T\Delta_U \times \mathfrak{g})$$

be the Künneth isomorphism (see theorem [6]?. We have that,

$$(k(\xi \otimes \eta))_{\Delta_U} = \gamma^*_{\Delta_U} \xi_\Delta \wedge \pi^*_{\Delta_U} \eta = k_\Delta(\xi_\Delta \otimes \eta)$$

Therefore, if $\omega = \sum \xi \otimes \eta \in \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g})$ and $k(\omega) = 0$, then $k(\omega)_{\Delta_U} = 0$ and so

$$0 = \big(k(\sum \xi \otimes \eta)\big)_{\Delta_U} = k_\Delta(\sum(\xi_\Delta \otimes \eta))$$

Hence $\omega = \sum(\xi_\Delta \otimes \eta) = 0$ and, with this, we have checked that k is injective. Take now $\lambda = (\lambda_\Delta)_{\Delta_U \subset U} \in \Omega^*_{ps}(\mathcal{A}; U)$. We want to find $\omega \in \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g})$ such that $k(\omega) = \lambda$. Since k_Δ is surjective, we can consider smooth forms $\xi_{j_\Delta} \in \Omega^*(\Delta_U)$ and $\eta \in \Omega^*(\mathfrak{g})$ such that

$$k_\Delta(\sum_j(\xi_{j_\Delta} \otimes \eta)) = \lambda_{\Delta_U}$$

Take then the form $\omega_\Delta = \sum_j(\xi_{j_\Delta} \otimes \eta)$. If Δ' and Δ are simplices of K with $s \prec \Delta' \prec \Delta$, we have the equalities

$$k_{\Delta'}\big(\sum_j(\xi_{j_\Delta})_{\Delta'_U} \otimes \eta\big) = \sum_j k_{\Delta'}((\xi_{j_\Delta})_{\Delta'_U} \otimes \eta\big) = \sum_j \big(\gamma^*_{\Delta'_U}(\xi_{j_\Delta})_{\Delta'_U} \wedge \pi^*_{\Delta'}\eta\big) = (*)$$

and

$$\begin{aligned} k_{\Delta'}\big(\sum_j(\xi_{j_{\Delta'}} \otimes \eta)\big) &= \lambda_{\Delta'} = (\lambda_\Delta)_{/\Delta'_U} = \big(k_\Delta(\sum_j(\xi_{j_\Delta} \otimes \eta)\big)_{/\Delta'_U} = \\ &= \big(\sum_j(\gamma^*_{\Delta_U}(\xi_{j_\Delta}) \wedge \pi^*_\Delta\eta)\big)_{/\Delta'_U} = \sum_j(\gamma^*_{\Delta_U}(\xi_{j_\Delta}) \wedge \pi^*_\Delta\eta)\big)_{/\Delta'_U} = \\ &= \sum_j(\gamma^*_{\Delta_U}(\xi_{{j_\Delta}_{/\Delta'_U}}) \wedge \pi^*_{\Delta'}\eta)) = \sum_j \big(\gamma^*_{\Delta'_U}(\xi_{j_\Delta})_{\Delta'_U} \wedge \pi^*_{\Delta'}\eta\big) = (*) \end{aligned}$$

Hence,

$$k_{\Delta'}\big(\sum_j(\xi_{j_\Delta})_{\Delta'_U} \otimes \eta\big) = k_{\Delta'}\big(\sum_j(\xi_{j_{\Delta'}} \otimes \eta)\big)$$

and, since $k_{\Delta'}$ is bijective, $\sum_j(\xi_{j_\Delta})_{\Delta'_U} \otimes \eta = \sum_j(\xi_{j_{\Delta'}} \otimes \eta)$. Therefore, we can conclude that $\xi_{j_\Delta}/_{\Delta'_U} = \xi_{j_{\Delta'}}$. Then, the form $\omega = (\omega_\Delta)_{\Delta_U \subset U}$ where, for each $\Delta_U \subset U$, $\omega_\Delta = \sum_j(\xi_{j_\Delta} \otimes \eta)$ belongs to $\Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g})$. Obviously $k(\omega) = \lambda$ and then it is checked that k is an isomorphism of graded algebras.

Part 2. In next part we are going to check that k commutes with differential, being then proved that k is an isomorphism of differential graded algebras. For each $\Delta \in K$ such that $s \prec \Delta$, denoting the differentials on the complexes $\Omega^*_{ps}(\mathcal{A};U)$ and $\Omega^*_{ps}(U)$ by $d^{\mathcal{A}}_{ps}$ and d^{U}_{ps} respectively, we have

$$(d^{\mathcal{A}}_{ps} \circ k)(\xi \otimes \eta) = d^{\mathcal{A}}_{ps}(\gamma^*\xi \wedge \pi^*\eta) =$$

$$= d^{\mathcal{A}}_{ps}(\gamma^*\xi) \wedge \pi^*\eta + (-1)^{deg\xi}\gamma^*\xi \wedge d^{\mathcal{A}}_{ps}(\pi^*\eta) =$$

$$= \gamma^*(d^{U}_{ps}\xi) \wedge \pi^*\eta + (-1)^{deg\omega}\gamma^*\xi \wedge \pi^*(d_{\mathfrak{g}}\eta) =$$

$$= k((d^{U}_{ps}\xi) \otimes \eta) + (-1)^{deg\xi}k(\xi \otimes d_{\mathfrak{g}}\eta) = k \circ \delta(\xi \otimes \eta)$$

Part 3. The isomorphism k above induces an isomorphism in cohomology. By applying the Künneth theorem, we obtain

$$H^*_{ps}(\mathcal{A};U) \simeq H^*(\Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g})) \simeq H^*_{ps}(U) \otimes H^*(\mathfrak{g})$$

Now, we shall see that Ψ induces an isomorphism in cohomology. Take the diagram

$$\begin{array}{ccc} \Omega^*(U) \otimes \Omega^*(\mathfrak{g}) & \xrightarrow{\lambda} & \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g}) \\ \downarrow k & & \downarrow k_{ps} \\ \Omega^*(\mathcal{A};U) & \xrightarrow{\Psi} & \Omega^*_{ps}(\mathcal{A};U) \end{array}$$

where $k_{ps} = k : \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g}) \longrightarrow \Omega^*_{ps}(\mathcal{A};U)$ is the isomorphism defined above, k is the Künneth isomorphism ([6]) and $\lambda = \Phi \otimes \mathbf{Id}$ in which Φ is the restriction map given on the Rham-Sullivan theorem for smooth manifolds. Obviously, the diagram is commutative and, by the de Rham-Sullivan theorem, Φ is an isomorphism in cohomology. Therefore, in cohomology, we have the commutative diagram

$$\begin{array}{ccc} H^*_{dR}(U) \otimes H^*(\mathfrak{g}) & \xrightarrow{H(\lambda)} & H^*_{ps}(U) \otimes H^*(\mathfrak{g}) \\ \simeq \downarrow & & \simeq \downarrow \\ H^*(\mathcal{A};U) & \xrightarrow{H(\Psi)} & H^*_{ps}(\mathcal{A};U) \end{array}$$

Hence, $H(\Psi)$ is an isomorphism and the result is proved. $\square$

Next, we want to show that Ψ induces an isomorphism in cohomology, not only for the trivial Lie algebroid defined over a regular open subset but for any arbitrary transitive Lie algebroid over a regular open subset. For that, we state first a basic result needed for the statement. This result is a basic consequence of the functor homology.

Proposition 3.1.3. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, $\mathbf{s}$ a simplex of K and $U = \mathbf{St}\,(\mathbf{s})$. Let $\mathcal{A}$ and $\mathcal{B}$ be two transitive Lie algebroids on M and suppose there is an isomorphism of Lie algebroids between them. Then, the cohomology spaces $H_{ps}(\mathcal{A}; U)$ and $H_{ps}(\mathcal{B}; U)$ are isomorphic.

Proposition 3.1.4. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, $\mathbf{s}$ a simplex of K and $U = \mathbf{St}\,(\mathbf{s})$. Let $\mathcal{A}$ be a transitive Lie algebroid on U. Then, the morphism

$$\Psi : \Omega^*(\mathcal{A}; U) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U)$$

$$\omega \longrightarrow (\omega_\Delta)_{\Delta_U \subset U}$$

induces an isomorphism in cohomology.

Proof. Since U is contractible, $\mathcal{A}$ is isomorphic to the trivial Lie algebroid $\mathcal{B} = TU \times \mathfrak{g}$ on U, in which $\mathfrak{g}$ is the fibre type of $K = \mathbf{Ker}\ \gamma$. We conclude the result by the commutativity of the diagram

$$\begin{array}{ccc} \Omega^p(\mathcal{A}; U) & \longrightarrow & \Omega^p_{ps}(\mathcal{A}; U) \\ \big\downarrow & & \big\downarrow \\ \Omega^p(TU \times \mathfrak{g}) & \xrightarrow{\Psi} & \Omega^p_{ps}(TU \times \mathfrak{g}) \end{array}$$

and applying the propositions 3.1.2 and 3.1.3. □

Proposition 3.1.5. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Let $\mathbf{s}_1, \ldots, \mathbf{s}_k$ be simplices of the simplicial complex K and consider the regular open subsets $U_j = \mathbf{St}\,(\mathbf{s}_j)$. For $l \in \{1, \ldots, k\}$ fixed, consider the open subsets $U = U_1 \cup \cdots \cup U_l$ and $V = U_{l+1} \cup \cdots \cup U_k$ of M and assume that $M = U \cup V$. Denote by $\underline{K}^U$ the set of all submanifolds $\Delta_U = U \cap \Delta$ such that $\Delta \in K$ and $\mathbf{s}_j$ is a face of Δ for some $j \in \{1, \ldots, l\}$, $\underline{K}^V$ the set of all submanifolds $\Delta_V = V \cap \Delta$ such that $\Delta \in K$ and $\mathbf{s}_i$ is a face of Δ for some $i \in \{l+1, \ldots, k\}$, and $\underline{K}^{U\cap V}$ the set of all submanifolds $\Delta_{U\cap V} = (U \cap V) \cap \Delta$ such that $\Delta \in K$ and $\mathbf{s}_j$ and $\mathbf{s}_i$ are faces of Δ for some $j \in \{1, \ldots, l\}$ and $i \in \{l+1, \ldots, k\}$. Then, we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \Omega^p(\mathcal{A}; M) & \xrightarrow{\lambda} & \Omega^p(\mathcal{A}_U; U) \oplus \Omega^p(\mathcal{A}_V; V) & \xrightarrow{\mu} & \Omega^p(\mathcal{A}_{U\cap V}; U \cap V) & \longrightarrow & \{0\} \\
 & & \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi & & \\
\{0\} & \longrightarrow & \Omega^p_{ps}(\mathcal{A}; K) & \xrightarrow{\delta} & \Omega^p_{ps}(\mathcal{A}_U; U) \oplus \Omega^p_{ps}(\mathcal{A}_V; V) & \xrightarrow{\pi} & \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U \cap V) & \longrightarrow & \{0\}
\end{array}$$

in which the maps λ and μ are the canonical maps given by the restriction and the difference and the maps

$$\delta : \Omega^p_{ps}(\mathcal{A}; K) \longrightarrow \Omega^p_{ps}(\mathcal{A}_U; U) \times \Omega^p_{ps}(\mathcal{A}_V; V)$$

$$\pi : \Omega^p_{ps}(\mathcal{A}_U; U) \times \Omega^p_{ps}(\mathcal{A}_V; V) \longrightarrow \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U \cap V)$$

are defined by

$$\delta\big((\omega_\Delta)_{\Delta\in K}\big) = \big((\omega_{\Delta_U})_{\Delta_U \in \underline{K}^U}, (\omega_{\Delta_V})_{\Delta_V \in \underline{K}^V}\big)$$

and

$$\pi\big((\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}, (\eta_{\Delta_V})_{\Delta_V \in \underline{K}^V}\big) = \big((\eta_{\Delta_{U\cap V}} - \xi_{\Delta_{U\cap V}})_{\Delta_{U\cap V} \in \underline{K}^{U\cap V}}\big)$$

Proof. In the section 1.2 of the first chapter, it was stated that the first arrow is exact. Regarding the second arrow, the proof is similar to the proof of proposition 2.3.5, section 2.3 of the second chapter. We shall only check that the map π is surjective. Since the set $\{U, V\}$ is an open covering of M we can fix two smooth maps $\varphi, \psi : M \longrightarrow [0,1]$ such that **supp** $\varphi \subset U$, **supp** $\psi \subset V$ and $\varphi(x) + \psi(x) = 1 \;\; \forall\, x \in M$. Let

$$(\gamma_{\Delta_{U\cap V}})_{\Delta_{U\cap V} \in \underline{K}^{U\cap V}} \in \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U \cap V)$$

be a piecewise smooth form. We shall define a differential form

$$(\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U} \in \Omega^p_{ps}(\mathcal{A}_U; U)$$

as follows. For each $\Delta_U \in \underline{K}^U$, set

$$\xi_{\Delta_U}(x) = \begin{cases} -\psi(x)\ \gamma_{\Delta_U}(x) & \text{if } x \in \Delta_U \cap V \\ \\ 0_x \in (\mathcal{A}_{\Delta_U})_x & \text{if } x \in \Delta_U \cap (M \setminus \mathbf{supp}\ \psi) \end{cases}$$

The sets $\Delta_U \cap V$ and $\Delta_U \cap (M \setminus \mathbf{supp}\ \psi)$ are open in Δ_U with union equal to Δ_U. Obviously, the restrictions of ξ_{Δ_U} to $\Delta_U \cap V$ and to $\Delta_U \cap (M \setminus \mathbf{supp}\ \psi)$ are smooth. Therefore, we conclude that $\xi_{\Delta_U} \in \Omega^p(\mathcal{A}_{\Delta_U})$. In order to obtain a piecewise smooth form belonging to $\Omega^p_{ps}(\mathcal{A}_U)$ it remains to check that $(\xi_{\Delta_U})_{\Delta_U \subset U}$ is compatible with restrictions to faces. Let Δ and Δ' be two simplices of K such that $s_j \prec \Delta \prec \Delta'$ for some $j \in \{1, \ldots, e\}$. Then, one has $\Delta_U \cap V \subset \Delta'_U \cap V \subset U \cap V$ and, since γ is piecewise smooth, we have $\gamma_{\Delta_U}(x) = (\gamma_{\Delta'_U})_{/\Delta_U}(x)$ for each $x \in \Delta_U$. Hence, if $x \in \Delta_U \cap V$,

$$\xi_{\Delta_U}(x) = -\psi(x)\ \gamma_{\Delta_U}(x) = -\psi(x)(\gamma_{\Delta'_U})_{/\Delta_U}(x) = (\xi_{\Delta'_U})_{/\Delta_U}(x)$$

If $x \in \Delta_U \cap (M \setminus \mathbf{supp}\ \psi)$ we have that $\xi_{\Delta_U}(x) = (\xi_{\Delta'_U})_{\Delta_U}(x) = 0$. Hence, the differential form $(\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}$ is a piecewise smooth form belonging to $\Omega^p_{ps}(\mathcal{A}_U; U)$. Analogously, we define a piecewise smooth form $(\eta_{\Delta_V})_{\Delta_V \in \underline{K}^V} \in \Omega^p_{ps}(\mathcal{A}_V; V)$ by

$$\eta_{\Delta_V}(x) = \begin{cases} -\varphi(x)\ \gamma_{\Delta_V}(x) & \text{if } x \in \Delta_V \cap U \\ \\ 0_x \in (\mathcal{A}_{\Delta_V})_x & \text{if } x \in \Delta_V \cap (M \setminus \mathbf{supp}\ \varphi) \end{cases}$$

and we have that, for each $x \in \Delta_{U\cap V} \in \underline{K}^{U\cap V}$,

$$\eta_{\Delta_{U\cap V}}(x) - \xi_{\Delta_{U\cap V}}(x) = \gamma_{\Delta_{U\cap V}}(x)$$

and so

$$(\eta_{\Delta_{U\cap V}} - \xi_{\Delta_{U\cap V}})_{\Delta_{U\cap V} \in \underline{K}^{U\cap V}} = (\gamma_{\Delta_{U\cap V}})_{\Delta_{U\cap V} \in \underline{K}^{U\cap V}}$$

Hence, the result is proved. □

Proof of the Mishchenko's theorem. We will prove the result by induction on the number of vertices of the simplicial complex K. Suppose then that $v_0, \ldots, v_N$ is the family of all vertices of K. If K has only one vertex, the result is trivial. Suppose we have established the result for all $l < N$. We know that $M = \bigcup_{j=0}^{j=N} \mathbf{St}\, v_j$. Taking the open subsets $U = \bigcup_{j=0}^{N-1} \mathbf{St}\, v_j$ and $V = \mathbf{St}\, v_N$ of M, we have that

$$U \cap V = \Big(\bigcup_{j=0}^{N-1} \mathbf{St}\,(v_j)\Big) \cap \mathbf{St}\,(v_N) = \bigcup_{j=0}^{N-1} \big(\mathbf{St}\,(v_j) \cap \mathbf{St}\,(v_N)\big) = \bigcup_{j=0}^{N-1} \mathbf{St}\,[v_j, v_N]$$

where $\big[v_j, v_N\big]$ denotes the closed simplex generated by the vertices v_j and v_N. By last proposition, we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \Omega^p(\mathcal{A}; M) & \xrightarrow{\lambda} & \Omega^p(\mathcal{A}_U; U) \oplus \Omega^p(\mathcal{A}_V; V) & \xrightarrow{\mu} & \Omega^p(\mathcal{A}_{U\cap V}; U\cap V) & \longrightarrow & \{0\} \\
 & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} & & \\
\{0\} & \longrightarrow & \Omega^p_{ps}(\mathcal{A}; K) & \xrightarrow{\delta} & \Omega^p_{ps}(\mathcal{A}_U; U) \oplus \Omega^p_{ps}(\mathcal{A}_V; V) & \xrightarrow{\pi} & \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U\cap V) & \longrightarrow & \{0\}
\end{array}$$

The map Ψ on the right side is quasi-isomorphism by induction. The map Ψ on the middle is quasi-isomorphism by induction and the proposition 3.1.4. By the Steenrod lemma, the map Ψ on the left side is also a quasi-isomorphism. □

From Mishchenko's theorem we easily conclude that the piecewise smooth cohomology of a combinatorial compact manifold does not depend on the triangulation used, that is, for any simplicial division of the simplicial complex, the piecewise smooth cohomology spaces of both combinatorial manifolds remains isomorphic. Precisely, this statement is our next proposition.

Proposition 3.1.6. Let M be a smooth manifold smoothly triangulated by a simplicial complex K and $\mathcal{A}$ a transitive Lie algebroid on M. Let L be other simplicial complex and assume that L a subdivision of K. Then, the piecewise smooth cohomology of the complex $\{\mathcal{A}^{!!}_{\Delta}\}_{\Delta\in K}$ is isomorphic to the one of the complex $\{\mathcal{A}^{!!}_{\Delta}\}_{\Delta\in L}$. Thus, the morphism from $\Omega^p_{ps}(\mathcal{A}; K)$ to $\Omega^p_{ps}(A; L)$ which induces that isomorphism in cohomology is given by restriction of forms.

Proof. The result follows from the commutativity of the following diagram

$$\begin{array}{ccc} & \Omega^p(\mathcal{A};M) & \\ {}^{\Psi}\swarrow & & \searrow^{\Psi} \\ \Omega^p_{ps}(\mathcal{A};K) & \xrightarrow{\Phi} & \Omega^p_{ps}(\mathcal{A};L) \end{array}$$

where Φ is also given by restriction. $\square$

3.2 Piecewise invariant cohomology

In this section, we shall note a consequence of the Mishchenko's theorem in piecewise invariant cohomology of transitive Lie algebroids equipped with an action of a Lie group. We recall basic definitions and the main result regarding invariant cohomology, following the paper [4] by Kubarski. As in the previous section all simplicial complexes are finite. We begin by stating a general result concerning natural transformations between functors.

For next proposition, consider the category $\mathfrak{C}$ of all transitive Lie algebroids over combinatorial compact manifolds and the category $\mathfrak{D}$ of all cochain algebras. Suppose that F and G are two functors from $\mathfrak{C}$ to $\mathfrak{D}$. Let t be a natural transformation between the functors F and G. For each transitive Lie algebroid $\mathcal{A}$ over a combinatorial compact manifold, denote by $t_{\mathcal{A}} : F(\mathcal{A}) \longrightarrow G(\mathcal{A})$ the corresponding cochain algebra morphism. Our next proposition is the following.

Proposition 3.2.1. Keeping the same hypothesis and notations as above, suppose yet that the following conditions hold.

- For each finite dimensional real Lie algebra $\mathfrak{g}$ and each contractible combinatorial compact manifold M,

$$H(F(TM \times \mathfrak{g})) \simeq H(G(TM \times \mathfrak{g}))$$

- For each transitive Lie algebroid $\mathcal{A}$ on a combinatorial compact manifold M, if U and V are regular open subsets in M and $t_{\mathcal{A}_U}$, $t_{\mathcal{A}_V}$ and $t_{\mathcal{A}_{U\cap V}}$ induce isomorphism in cohomology, then $t_{\mathcal{A}_{U\cup V}}$ also induces isomorphism in cohomology.

Then, under these conditions, $t_{\mathcal{A}}$ induces an isomorphism in cohomology for all transitive Lie algebroids $\mathcal{A}$ over a combinatorial compact manifold M.

Proof. The proof is essentially the same as the proof of Mishchenko's theorem. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Let $U = \mathbf{St}\,\Delta$, for some simplex $\Delta \in K$. As done in the proof of the proposition 3.1.4, $\mathcal{A}_U$ is isomorphic, by a Lie algebroid isomorphism, to the trivial Lie algebroid $TU \times \mathfrak{g}$ on U, in which $\mathfrak{g}$ is the fibre type of $K = \mathbf{Ker}\,\gamma$. Hence, by the first hypothesis and properties of the functor homology, we have

$$H(F(\mathcal{A}_U)) \simeq H(F(TU \times \mathfrak{g})) \simeq$$

$$\simeq H(G(TU \times \mathfrak{g})) \simeq H(G(\mathcal{A}_U))$$

and so $t_{\mathcal{A}_U}$ induces an isomorphism in cohomology. By using the open covering of M made of the stars of all vertices of K, the arguments given in the proof of the Mishchenko's theorem are valid mutatis-mutandis in this case and therefore $t_{\mathcal{A}} : F(\mathcal{A}) \longrightarrow G(\mathcal{A})$ is a quasi-isomorphism. □

We shall now introduce the notion of invariant cohomology of Lie algebroids over combinatorial manifolds. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Denote by $\gamma : \mathcal{A} \longrightarrow TM$ the anchor of $\mathcal{A}$ and by $\pi : \mathcal{A} \longrightarrow M$ the projection of the underlying vector bundle of $\mathcal{A}$. Assume that G is a Lie group. A left action of G on $\mathcal{A}$ is a pair (T, t) of maps

$$T : G \times \mathcal{A} \longrightarrow \mathcal{A} \qquad t : G \times M \longrightarrow M$$

satisfying the followings conditions.

- The maps T and t are smooth actions.

- π is an equivariant map.

- For each $g \in G$, the left translation $T_g : \mathcal{A} \longrightarrow \mathcal{A}$ is a morphism of Lie algebroids.

- For each $g \in G$, the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T_g} & \mathcal{A} \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle\gamma}\downarrow \\ TM & \xrightarrow{T(t_g)} & TM \\ {\scriptstyle\pi_N}\downarrow & & {\scriptstyle\pi_M}\downarrow \\ M & \xrightarrow{t_g} & M \end{array}$$

  commutes.

- For each Δ of K, the simplex Δ is stable for the action t.

Keeping the same hypothesis and notations as above, for each $\Delta \in K$ and $g \in G$, we also have that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{A}^{!!}_{\Delta} & \xrightarrow{T_{g_\Delta}} & \mathcal{A}^{!!}_{\Delta} \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle\gamma}\downarrow \\ T\Delta & \xrightarrow{T(t_{g_\Delta})} & T\Delta \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{t_{g_\Delta}} & \Delta \end{array}$$

Moreover, if $\underline{\mathcal{A}} = \{\mathcal{A}^{!!}_{\Delta}\}_{\Delta \in K}$ is the corresponding complex obtained by restriction of $\mathcal{A}$ to all simplices of K, then, for each $g \in G$, the family $\lambda_g = ((T_{g_\Delta})_{\Delta \in K}, id_K)$ is a morphism of complexes of Lie algebroids from $\underline{\mathcal{A}}$ to $\underline{\mathcal{A}}$.

Definition (Piecewise invariant form). Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Assume that G is a Lie group and (T, t) is a left action of G on $\mathcal{A}$. Consider the corresponding complex $\underline{\mathcal{A}} = \{\mathcal{A}^{!!}_{\Delta}\}_{\Delta \in K}$ obtained by restriction of $\mathcal{A}$ to all simplices of K and the complex of Lie algebroids morphism $\lambda_g = ((T_{g_\Delta})_{\Delta\in K}, id_K)$. Let $\omega = (\omega_\Delta)_{\Delta\in K} \in \Omega^*_{ps}(\underline{\mathcal{A}}; K)$ be a piecewise smooth form. The form ω is called piecewise invariant if, for each $g \in G$,

$$\lambda_g^*(\omega) = \omega$$

Keeping the same hypothesis and notations from the definition above, the space of all piecewise invariant forms on $\mathcal{A}$ is denoted by $\Omega^*_{I_{ps}}(\mathcal{A}; K)$ or $\Omega^*_{I_{ps}}(\mathcal{A}; M)$. Since $d(\lambda_g^*\omega) = \lambda_g^*(d\omega)$ (by proposition 2.2.4), the space $\Omega^*_{I_{ps}}(\mathcal{A}; K)$ is stable under the exterior differential and so it is a complex in which the differential is the restriction of the exterior derivative to each simplex of K. The cohomology space of the differential complex $\Omega^*_{I_{ps}}(\mathcal{A}; K)$ will be denoted by $H_{I_{ps}}(\mathcal{A}; K)$.

We shall see that the restriction map induces an isomorphism is cohomology.

Proposition 3.2.2. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Assume that G is a compact Lie group and (T, t) is a left action of G on $\mathcal{A}$. Consider the map

$$\Phi : \Omega_I(\mathcal{A}; M) \longrightarrow \Omega_{I_{ps}}(\mathcal{A}; K)$$

$$\Phi(\omega) = (\omega_\Delta)_{\Delta\in K}$$

Then, the map Φ induces an isomorphism in cohomology.

Proof. Let d and d_{ps} denote the differential of the complexes $\Omega^*(\mathcal{A}; M)$ and $\Omega^*_{I_{ps}}(\mathcal{A}; K)$ respectively and Ψ the restriction map given on the Mishchenko's theorem. We are going to apply the proposition 3.2.1. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra, M a contractible smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A} = TM \times \mathfrak{g}$ the trivial Lie algebroid on M. We want to check that

$$H^*_I(\mathcal{A}; M) \simeq H^*_{I_{ps}}(\mathcal{A}; K)$$

The steps used in the proof of the proposition 3.1.2 are valid here provide we check that, if $\xi = (\xi_\Delta)_{\Delta\in K} \in \Omega^*_{I_{ps}}(\mathcal{A}; M)$ is a piecewise invariant form, then $(\gamma_\Delta^*\xi_\Delta)_{\Delta\in K}$ is a piecewise invariant form, which is true since γ commutes with T_{g_*}. For the second condition of the proposition 3.2.1, we cam mimic the proof of the proposition 3.1.5, provided that we fix a partition unity made by invariant functions to prove the sequence

$$\{0\} \longrightarrow \Omega^p_{Ips}(\mathcal{A};K) \xrightarrow{\delta} \Omega^p_{Ips}(\mathcal{A}_U;U) \oplus \Omega^p_{Ips}(\mathcal{A}_V;V) \xrightarrow{\pi} \Omega^p_{Ips}(\mathcal{A}_{U\cap V};U\cap V) \longrightarrow \{0\}$$

is exact. Since G is compact, for each $f \in C^\infty(M)$, the function $f_I \in C^\infty(M)$ given by $f_I(x) = \int_G f \circ t_g(x)$ is invariant. Hence, the result is proved. □

For next proposition, suppose we have we a transitive Lie algebroid $\mathcal{A}$ over a smooth manifold M and an action (T,t) of a compact Lie group G on $\mathcal{A}$. Suppose that there is a Lie algebroid morphism $\widehat{T} : TG \times \mathcal{A} \longrightarrow \mathcal{A}$ such that $\widehat{T}_{/G\times\mathcal{A}} = T$. Let

$$i : \Omega^*_I(\mathcal{A};M) \longrightarrow \Omega^*(\mathcal{A};M)$$

be the inclusion map. We recall that Kubarski showed that, if the inclusion i induces a monomorphism in cohomology and, if G is also connected, the inclusion i induces an isomorphism in cohomology (see [4]). Assume now that the manifold M is smoothly triangulated by a simplicial complex K and consider the inclusion map

$$i_{ps} : \Omega^*_{Ips}(\mathcal{A};M) \longrightarrow \Omega^*_{ps}(\mathcal{A};M)$$

Next result is the following.

Proposition 3.2.3. Keeping the same hypothesis and notations as above, the inclusion i_{ps} induces a monomorphism in cohomology. If G is compact and connected, then the inclusion i_{ps} induces an isomorphism in cohomology.

Proof. It is clear that the following diagram

$$\begin{array}{ccc} \Omega^*_I(\mathcal{A};M) & \xrightarrow{i} & \Omega^*(\mathcal{A};M) \\ \Phi \downarrow & & \downarrow \Psi \\ \Omega^*_{I_{ps}}(\mathcal{A};M) & \xrightarrow{i_{ps}} & \Omega^*_{ps}(\mathcal{A};M) \end{array}$$

is commutative, in which Φ is the restriction map given in the proposition 3.2.2 and Ψ is the restriction map given in the Mishchenko's theorem. So, it is also commutative the following diagram

$$\begin{array}{ccc} H_I^*(\mathcal{A};M) & \xrightarrow{H(i)} & H^*(\mathcal{A};M) \\ {\scriptstyle H(\Phi)}\downarrow & & \downarrow{\scriptstyle H(\Psi)} \\ H_{I_{ps}}^*(\mathcal{A};M) & \xrightarrow{H(i_{ps})} & H^*(\mathcal{A};M) \end{array}$$

The maps $H(i)$ and $H(\Psi)$ are isomorphisms by Kubarski's and Mishchenko's theorems respectively and so $H(i_{ps})$ is an isomorphism. □

3.3 Piecewise de Rham cohomology of Lie groupoids

In this section, we establish a basic relationship between de Rham cohomology of left invariants forms on a Lie groupoid and Lie algebroid cohomology of its Lie algebroid. Constructions involving differential forms in Lie groupoids are quite analogous to constructions in Lie groups. An illustrative example of this analogy, is the algebra of all invariant forms on a Lie groupoid. As in the case of Lie groups, the algebra of all invariant forms on a Lie groupoid is isomorphic to the algebra of all smooth forms on its Lie algebroid. We shall state this property below. Since we are working in a piecewise smooth context, we begin by summarizing some considerations on restrictions of Lie groupoids.

We recall the construction of the Lie algebroid of a Lie groupoid. Let M be a smooth manifold and G a Lie groupoid on M with source projection $\alpha : G \longrightarrow M$ and target projection $\beta : G \longrightarrow M$. Denote by $1 : M \longrightarrow G$ the object inclusion map of G and $G_x = \alpha^{-1}(x)$ the α-fibre of G in x, for each $x \in M$. The Lie algebroid of G is

$$(\mathcal{A}(G), [\cdot,\cdot], \gamma)$$

in which

- $\mathcal{A}(G) = \bigsqcup_{x \in M} T_{1_x} G_x$ (disjoint union).

- The anchor $\gamma : \mathcal{A}(G) \longrightarrow TM$ is defined by $\gamma(a) = D\beta_{1_x}(a)$.

- For each ξ and $\eta \in \Gamma(\mathcal{A}(G))$, the Lie bracket is defined by

$$[\xi, \eta] = [\xi', \eta']_G$$

  in which ξ' and η' denote the unique α-right-invariant vector fields on G such that $\xi'_{1_x} = \xi_x$ and $\eta'_{1_x} = \eta_x$, $\forall x \in M$.

Definition (Locally trivial Lie groupoid). Let M be a smooth manifold and G a Lie groupoid on M with source projection $\alpha : G \longrightarrow M$ and target projection $\beta : G \longrightarrow M$. The anchor of G is the map $(\beta, \alpha) : G \longrightarrow M \times M$. The Lie groupoid G is called locally trivial if its anchor $(\beta, \alpha) : G \longrightarrow M \times M$ is a surjective submersion.

In [10], it can be seen the following proposition.

Proposition 3.3.1. Let M be a smooth manifold and G a Lie groupoid on M. Let $\mathcal{A}(G)$ be the Lie algebroid of G. Then, if G is locally trivial, $\mathcal{A}(G)$ is transitive Lie algebroid.

Definition (Submanifold transversal to a Lie groupoid). Let M be a smooth manifold and G a Lie groupoid on M with source projection $\alpha : G \longrightarrow M$ and target projection $\beta : G \longrightarrow M$. A submanifold $\varphi : N \hookrightarrow M$ is called transversal to G if the anchor $(\beta, \alpha) : G \longrightarrow M \times M$ and the smooth map $\varphi \times \varphi : N \times N \longrightarrow M \times M$ are transversal.

In [10], it can be seen the following proposition.

Proposition 3.3.2 (Restrictions of Lie groupoids). Let M be a smooth manifold and G a Lie groupoid on M with source projection $\alpha : G \longrightarrow M$ and target projection

$\beta : G \longrightarrow M$. Let $\varphi : N \hookrightarrow M$ be a submanifold. Denote by G_N^N the set $\alpha^{-1}(N) \cap \beta^{-1}(N)$. Then, if N is transversal to G, the set G_N^N is a Lie subgroupoid of G with base N and G_N^N is called the Lie groupoid restriction of G to N.

The main proposition for our piecewise context is the following. The proof is a direct consequence of the definition of Lie algebroid of a Lie groupoid.

Proposition 3.3.2 (Restrictions of Lie groupoids to simplices). Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and G a Lie groupoid on M. For each simplex Δ of K, one has

$$\mathcal{A}(G_\Delta^\Delta) = \mathcal{A}(G)_\Delta^{!!}$$

We recall now the definition of de Rham cohomology of left invariants forms of a Lie groupoid. Let M be a smooth manifold and G a Lie groupoid on M with source projection α and target projection β. For each $p \geq 0$, let $\Omega_\alpha^p(G; M)$ be the $C^\infty(G)$-module of the smooth sections of the vector bundle of all alternating p-linear maps from the vector bundle $\bigsqcup_{g \in G} TG_{\alpha(g)}$ (disjoint union) to the trivial vector bundle $\mathbf{R}_M$. A smooth α-form of degree p on the Lie groupoid G is, by definition, an element of $\Omega_\alpha^p(G; M)$. Thus, a smooth α-form $\omega \in \Omega_\alpha^p(G; M)$ is a family defined on G such that, for each $g \in G$, one has

$$\omega_g \in \Lambda^p\big(\bigsqcup_{g \in G} T_g^* G_{\alpha(g)}; \mathbf{R}\big)$$

The usual exterior derivative along the α-fibres is defined by

$$(d_\alpha^p \omega)(X_1, X_2, \cdots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} X_j \cdot (\omega(X_1, \cdots, \widehat{X_j}, \cdots, X_{p+1})) +$$

$$+ \sum_{i<k} (-1)^{i+k} \omega([X_i, X_k], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_k}, \cdots, X_{p+1})$$

in which $\omega \in \Omega_\alpha^p(G; M)$ and X_1, X_2, $\cdots$, X_{p+1} are smooth vector α-fields on G. The complex $(\Omega_\alpha^*(G; M), d_\alpha^*)$ is a commutative cochain algebra defined on $\mathbf{R}$. The set $\Omega_{\alpha,L}^p(G; M)$

consisting of all α-forms on G which are invariant under the groupoid left translations is a subcomplex of $(\Omega^*_\alpha(G;M), d^*_\alpha)$. Its cohomology is denoted by $H^*_{\alpha,L}(G;M)$.

Denote by $1 : M \longrightarrow G$ the object inclusion map of G. There is an isomorphism

$$\psi : \Omega^*_{\alpha,L}(G : M) \longrightarrow \Omega^*(\mathcal{A}(G); M)$$

of cochain algebras defined by $\psi(\omega)_x = \omega_{1_x}$. Consequently, we have the following proposition.

Proposition 3.3.3. Keeping the same hypothesis and notations as above, we have

$$H^*_{\alpha,L}(G;M) \simeq H^*(\mathcal{A}(G);M)$$

Let us to introduce the notion of piecewise smooth cohomology of Lie groupoids. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and G a locally trivial Lie groupoid on M with source projection α and target projection β. For each simplex $\Delta \in K$, let G^Δ_Δ be the Lie groupoid restriction of G to Δ and $\mathcal{A}(G^\Delta_\Delta)$ its Lie algebroid. Since G is locally trivial its Lie algebroid $\mathcal{A}(G)$ is transitive and we know that $\mathcal{A}(G^\Delta_\Delta) \simeq \mathcal{A}^{!!}_\Delta$. Similarly to piecewise smooth forms on Lie algebroids, we give now the notion of piecewise smooth form on G.

Definition (Piecewise smooth form). Keeping the same hypothesis and notations as above, a piecewise smooth α-form of degree p ($p \geq 0$) on G is a family $\omega = (\omega_\Delta)_{\Delta\in K}$ such that the following conditions are satisfied.

- For each $\Delta \in K$, $\omega_\Delta \in \Omega^p_{\alpha,L}(G^\Delta_\Delta;\Delta)$ is a α-smooth form of degree p on G^Δ_Δ.
- If Δ and Δ' are two simplices of K such that $\Delta' \prec \Delta$, one has $(\omega_\Delta)_{/\Delta'} = \omega_{\Delta'}$.

The $C^\infty(G)$-module of all piecewise α-smooth forms of degree p on G is denoted by $\Omega^p_{\alpha,L,ps}(G;M)$ or $\Omega^p_{\alpha,L,ps}(G;K)$. As done in previous sections, a wedge product and an exterior derivative can be defined on $\Omega^*_{\alpha,L,ps}(G;K)$ by the corresponding operations on each submanifold G^Δ_Δ, giving to $\Omega^*_{\alpha,L,ps}(G;K)$ a structure of cochain algebra defined over $\mathbf{R}$. The cohomology space of this complex is denoted by $H^*_{\alpha,L,ps}(G;M)$ or $H^*_{\alpha,L,ps}(G;K)$.

Our aim is to relate the cohomology space $H^*_{\alpha,L,ps}(G;K)$ of G to the cohomology space $H^*_{ps}(\mathcal{A}(G);K)$ of its Lie algebroid $\mathcal{A}(G)$. For that, we have to consider a map ϕ from the complex $\Omega^*_{\alpha,L,ps}(G;K)$ to the complex $\Omega^*_{ps}(\mathcal{A}(G);K)$. In order to obtain such map ϕ, we recall that, for each simplex Δ of K, we have an isomorphism

$$\psi_\Delta : \Omega^p_{\alpha,L}(G^\Delta_\Delta;\Delta) \longrightarrow \Omega^p(\mathcal{A}(G^\Delta_\Delta);\Delta)$$

given by $\psi(\omega)_x = \omega_{1_x}$. Consider now a piecewise smooth α-form

$$\omega = (\omega_\Delta)_{\Delta\in K} \in \Omega^p_{\alpha,L,ps}(G;K)$$

For each simplex $\Delta \in K$, take the smooth form $\xi_\Delta = \psi_\Delta(\omega_\Delta) \in \Omega^p(\mathcal{A}(G^\Delta_\Delta);\Delta)$. Next proposition says that this process gives a piecewise smooth form on $\mathcal{A}(G)$.

Proposition 3.3.4. Keeping the same hypothesis and notations as above, if Δ and Δ' are two simplices of K such that Δ' is a face of Δ, then $(\xi_\Delta)^{!!}_{\Delta'} = \xi_{\Delta'}$ and so $\xi = (\xi_\Delta)_{\Delta\in K}$ is a piecewise smooth on $\mathcal{A}(G)$. Consequently, the map

$$\Phi : \Omega^*_{\alpha,L,ps}(G;K) \longrightarrow \Omega^*_{ps}(\mathcal{A}(G);K)$$

defined by $\Phi((\omega_\Delta)_{\Delta\in K}) = (\psi_\Delta(\omega_\Delta))_{\Delta\in K}$ is an isomorphism of cochain algebras.

We can state now the main proposition of this section. Let r_G be the restriction map

$$r_G : \Omega^*_{\alpha,L}(G;M) \longrightarrow \Omega^*_{\alpha,L,ps}(G;K)$$

defined $r_G(\omega) = (\omega_{/\Delta})_{\Delta\in K}$. Our proposition is the following.

Proposition 3.3.5. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and G a locally trivial Lie groupoid on M with source projection α

and target projection β. Then, the map $r_G : \Omega^*_{\alpha,L}(G;M) \longrightarrow \Omega^*_{\alpha,L,ps}(G;K)$ induces an isomorphism in cohomology. Consequently, the Rham cohomology of G is isomorphic to the piecewise Rham cohomology of G.

Proof. The diagram

$$\begin{array}{ccc} \Omega^*_{\alpha,L}(G;M) & \xrightarrow{iso} & \Omega^*(\mathcal{A};M) \\ {\scriptstyle r_G}\downarrow & & \downarrow{\scriptstyle r_{\mathcal{A}}} \\ \Omega^*_{\alpha,L,ps}(G;K) & \xrightarrow{iso} & \Omega^*_{ps}(\mathcal{A};K) \end{array}$$

is commutative, where $r_{\mathcal{A}}$ is the restriction map given at the Mishchenko's theorem. We apply the Mishchenko's theorem in cohomology and the proof is done. □

Our last proposition says that the piecewise de Rham cohomology of a locally trivial Lie groupoid over a combinatorial manifold doesn't depend on the triangulation.

Proposition 3.3.6. Let M be a smooth manifold smoothly triangulated by a simplicial complex K and L other simplicial complex which a subdivision of K. Let G be a locally trivial Lie groupoid on M. Then, the piecewise de Rham cohomology of G obtained by the triangulation corresponding K is isomorphic to the the piecewise de Rham cohomology of G obtained by the y the triangulation corresponding to L. Thus, this isomorphism is induced by the restriction map.

Proof. Denote by $\phi : \Omega^*_{\alpha,L,ps}(G;K) \longrightarrow \Omega^*_{\alpha,L,ps}(G;L)$ the map given by restriction. The diagram

$$\begin{array}{ccccc} & & \Omega^p_{\alpha,L}(G;M) & & \\ & \swarrow & & \searrow & \\ \Omega^*_{\alpha,L,ps}(G;K) & & \xrightarrow{\phi} & & \Omega^*_{\alpha,L,ps}(G;L) \\ \downarrow & & & & \downarrow \\ \Omega^p_{ps}(\mathcal{A};K) & & \longrightarrow & & \Omega^p_{ps}(\mathcal{A};L) \end{array}$$

is commutative. By propositions 3.1.6, 3.3.4 and 3.3.5, the maps non labeled are isomorphisms in cohomology and so the map ϕ also is isomorphism in cohomology. $\square$

Conclusion

H. Whitney started the study of cohomologies of cell-like spaces by taking different notions of differential form. Roughly speaking, Whitney used notions of forms such as piecewise smooth forms, elementary forms, polyhedral forms and flat forms. The relationship of these constructions is described in the Whitney's book "Geometric Integration theory" and it is the genesis of the use of differential forms to solve the commutative cochain problem. Similar constructions were found out by Sullivan in the study of the rational homotopy type of a space. Namely, Sullivan considered the algebra of the polynomial forms on a cell space and proved that this algebra is quasi-isomorphic to the classic algebra of smooth forms. The algebra of the polynomial forms originated the theory of models, which has revealed crucial in the development of homotopy and formality theories for cell spaces. Our work was written in the effort to understand those constructions for transitive Lie algebroids over combinatorial manifolds. Our first aim was to study piecewise smooth cohomology of Lie algebroids on combinatorial manifolds. Some methods used in the study of those constructions on cell-like spaces had to be changed in order to be applied to Lie algebroids, especially because we do not have the notion of cell in Lie algebroids. The notion of cell structure was changed to obtain what we called complex of Lie algebroids and to define a complex of piecewise differential forms and consequently the notion of piecewise smooth cohomology of Lie algebroids. We have seen that the restriction map induces an isomorphism in cohomology between piecewise smooth and Lie algebroid cohomology. This result is based in the Rham-Sullivan theorem well as in some results on non-abelian extensions of Lie algebroids, which lead us to the triviality of transitive Lie algebroids over contractible manifolds. It is not known whether other cell constructions can be developed for Lie algebroids. For transitive Lie algebroids with commutative kernel, the transition functions are flat and this is a nice hypothesis for developing other kind of forms in Lie algebroids, especially polynomial forms in Lie algebroids, giving us an alternative way to study formality of Lie algebroids.

Resumen

El teorema de de Rham es un resultado de gran importancia, ya que ha sido el principal vínculo de unión entre el análisis en variedades y la propiedades topológicas de las variedades. En breves palabras, la homología espacios mide el número de agujeros de una variedad y su nivel de complejidad. El teorema de de Rham garantiza que los espacios de homología de variedades se pueden expresar mediante el uso de formas diferenciales y sus métodos analíticos. El estudio de los espacios de homología en términos de formas diferenciales abrió un camino para el estudio de la estructura más profunda de variedades. Sullivan, en su artículo “ Cálculos infinitesimales en topología ”, dice que dentro del mundo de la topología hay más información topológica en el álgebra de Rham de formas diferenciales que simplemente la cohomología real. La teoría de Rham rápidamente originó un profundo desarrollo de la topología de variedades. Hay también una gran cantidad de situaciones matemáticas en las que el conocimiento de formas diferenciales tiene consecuencias importantes y por consiguiente otras teorías matemáticas se desarrollaron a partir del teorema de de Rham. El invariante de Hopf, el producto de Massey, el grado de una aplicación y la cohomología de grupos de Lie compactos son algunos ejemplos de la importancia del teorema de de Rham. Sullivan y otros matemáticos han implementado varias estrategias en el estudio del álgebra de Rham de todas las formas diferenciables. Entre ellos, está la teoría de modelos. Esta teoría consiste en encontrar otras álgebras graduadas, dentro del álgebra de de Rham de todas las formas defirenciables, de tal manera que la inclusión canónica induce un isomorfismo en cohomología. A partir de estos desarrollos, una conclusión importante surgió, que puede expresarse en el siguiente diagrama conmutativo:

$$\begin{array}{ccccc} & & H^*_{p.C^\infty}(M) & & \\ & \nearrow & \Big\downarrow \int & \nwarrow & \\ H^*_{PL}(M)\otimes_{\mathbf{Q}}\mathbf{R} & & & & H^*_{dR}(M) \\ & \searrow^{\int}_{\cong} & & \swarrow^{\int}_{\cong} & \\ & & H^*(M,\mathbf{R}) & & \end{array}$$

Este esquema incorpora una gran cantidad de construcciones y enunciados. El presente trabajo surge de los esfuerzos por extender esas construcciones a algebroides de Lie transitivos. Entre esas construcciones, estamos particularmente interesados en la que dice que la cohomología de de Rham de una variedad diferenciable, triangulada diferenciablemente por un complejo simplicial, es isomorfo a la cohomología diferenciable por partes del complejo simplicial. Este isomorfismo viene dado por la restricción de formas diferenciables a todos los símplices. El estudio de esta construcción o de otras construcciones de Sullivan en variedades simpliciales se basa en el teorema de de Rham para células, así como extensiones de las formas diferenciables. Algunas dificultades surgen de la utilización de la teoría de de Rham en el estudio de la cohomología de algebroide de Lie. Sin embargo, a pesar de todas las dificultades que surgen de la teoría de de Rham de algebroides de Lie, durante los últimos años, la teoría de cohomología de algebroides de Lie se ha desarrollado a partir de una colección de grandes resultados con fuertes conexiones con muchas otras partes de las matemáticas, en particular, con la teoría de Chern-Weil. Estas mejoras han reducido varios obstáculos en el desarrollo de nuestro trabajo.

Las ideas clave relativas a la clase de obstrucción derivada de las extensiones no abelianas de algebroides de Lie han inspirado Mishchenko y le llevó a conjeturar que, dado un algebroide de Lie transitivo en una variedad combinatórica, el morfismo dado por la restricción, que lleva formas diferenciables del algebroide de Lie en formas diferenciables a trozos en el mismo algebroide de Lie, sigue siendo un isomorfismo en cohomología.

El objetivo del presente trabajo es demostrar la conjetura de Mishchenko. Para este propósito, se ha utilizado una estructura llamada complejo de algebroides de Lie. Esta estructura se inicia fijando una triangulación diferenciable de la base de un algebroide de Lie transitivo por un complejo simplicial y tomando la restricción del algebroide de Lie a todos los símplices de la triangulación. Como el algebroide de Lie es transitivo, siempre existe la restricción del algebroide de Lie a cada símplice. Dado un complejo de algebroides de Lie, se define la noción de forma diferenciable a trozos de manera similar a las formas de Whitney en un complejo simplicial y el conjunto de todas las formas diferenciables a trozos definidas en un complejo de algebroides de Lie es, naturalmente, equipado con una diferencial, produciendo un álgebra diferencial graduada conmutativa. Su cohomología es, por definición, la cohomología diferenciable a trozos del algebroide de Lie. Cada forma diferenciable definida en el algebroide Lie da una forma diferenciable a trozos definida en el correspondiente complejo de algebroides de Lie tomando la restricción de la forma a cada símplice. Esta correspondencia es una aplicación natural del álgebra usual de las formas suaves del algebroide Lie al álgebra de las formas diferenciables a trozos del complejo correspondiente de algebroides de Lie. Basándose en tres resultados importantes, a saber, la trivialidad de un algebroide de Lie transitivo sobre una variadad diferenciable contráctil (Mackenzie, Weinstein), el teorema de Künneth para algebroides de Lie (Kubarski) y el teorema de Rham-Sullivan para variedades diferenciables, mostramos que esta aplicación es un isomorfismo en cohomología.

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