2024 IEEE INTERNATIONAL WORKSHOP ON MACHINE LEARNING FOR SIGNAL PROCESSING, SEPT. 22–25, 2024, LONDON, UK

A $\mathbb{T}$-PROPER KARHUNEN-LOÈVE EXPANSION AND ITS APPLICATION TO THE PROBLEM OF SIMULATION

R. M. Fernández-Alcalá⋆*, J. Navarro-Moreno*⋆*, J. D. Jiménez-López*⋆*, C. Cheong-Took*†*, J. C. Ruiz-Molina*⋆

⋆ Department of Statistics and Operations Research, University of Jaén, Spain.
† Department of Electronic Engineering, Royal Holloway, University of London, UK.

ABSTRACT

The paper addresses the Karhunen-Loéve series representation in the tessarine domain. Based on augmented statistics, a tessarine widely linear Karhunen-Loève expansion is defined. Then, the impact of $\mathbb{T}$-properness on this representation is analyzed, leading to a $\mathbb{T}$-proper Karhunen-Loève expansion that means a dimensionality reduction. Furthermore, this series representation serves as a versatile simulation tool, valid for both stationary and non-stationary, Gaussian and non-Gaussian random signals. Finally, the applicability of the simulation technique proposed is examined numerically.

Index Terms— Karhunen-Loève expansion, simulation techniques, tessarine algebra, $\mathbb{T}$-properness.

1. INTRODUCTION

In recent decades, the field of neural networks has grown exponentially, becoming a key tool in a wide range of applications, from image processing to complex decision making (see, e.g., [1] and references therein). However, despite their ability to learn complex patterns, they are often faced with unpredictable and variable environments. The ability to adapt to unpredictable data and model the associated uncertainty is essential for their effective application in fields such as finance, biology, and telecommunications, among others. In this context, stochastic process simulation techniques are an essential component for enhancing the robustness and effectiveness of neural networks. Not only do these techniques enable neural networks to learn from deterministic data, but they also enable them to handle the variability inherent in real-world processes, thereby enhancing their adaptability and performance [2, 3, 4, 5].

The Karhunen-Loéve (KL) expansion has been satisfactorily used as a simulation tool due to its versatility and efficiency in representing random signals. KL expansion decomposes the stochastic signal into a series of orthogonal functions and random variables coefficients, with the advantage that it is optimal in terms of minimizing the total mean square error resulting from the truncation of the series. An essential feature is its applicability to address stationary and non-stationary, Gaussian and non-Gaussian signals, providing a flexible framework for simulating a wide range of phenomena. This problem has been broadly studied in the real field (see, e.g., [6, 7, 8]), and extended to the complex [9] and quaternion [10] domain.

Interestingly, not only do quaternions offer a suitable architecture for processing signals describing the physical phenomena of dimensions three and four, but they can also, under certain signal properness conditions, lead to a dimensionality reduction of the signals to be processed. Specifically, in the context of quaternion signal processing, three main categories are defined based on the cancellation of complementary functions [11]: $\mathbb{Q}$-proper (when all the complementary functions are zero), $\mathbb{C}^\eta$-proper (characterized by the vanishing of two complementary functions), and improper (when none of the complementary functions are cancelled). In general, in the improper case, the optimal linear processing is the quaternion widely linear processing (QWL), which amounts to operating on a four dimensional vector formed by the quaternion signal and its three involutions [12]. Conversely, in the $\mathbb{Q}$-proper scenario, the most effective approach is the quaternion strictly linear (QSL) processing, which operates solely on the signal itself, resulting in a dimensionality reduction to a quarter. In the $\mathbb{C}^\eta$-proper case, the appropriate processing is the semi-widely linear (QSWL) processing, which means to operate on a two-dimensional vector formed by the signal and its conjugate, thereby reducing the dimensionality by half when compared to QWL processing [13]. Following this approach, a QWL KL expansion was proposed in [14], and its repercussions in the $\mathbb{C}^\eta$-proper scenario were subsequently analyzed [10], giving rise to a QSWL KL expansion that results in a reduction of the computational burden.

Nevertheless, recent research suggests that quaternions may not always be the best choice for processing 3D or 4D signals, and there has been a growing interest in exploring new hypercomplex algebras that, under certain property conditions, lead to a dimensionality reduction [18]. Notably, the tessarine algebra has gained special attention. Unlike quaternions, tessarines constitute a commutative algebra, which

This work is part of the I+D+i project PID2021-124486NB-I00, funded by MICIU/AEI/10.13039/501100011033/ and ERDF/EU.

facilitates the extension of numerous existing techniques developed in the real and complex domains to this new field [15], [16]. Moreover, analogous to the $\mathbb{Q}$ and $\mathbb{C}^{\eta}$ properness in the quaternion domain, [17] and [18] have introduced two novel properness concepts in the tessarine domain, labeled $\mathbb{T}$-(or $\mathbb{T}_1$-) and $\mathbb{T}_2$- properness, respectively, and they have established the basis of the corresponding tessarine strictly linear (TSL) and the tessarine semi-widely linear (TSWL) processing, which mean a reduction in dimensionality with respect to the tessarine widely linear processing (TWL).

The purpose of this paper is twofold: Firstly, a TWL KL series expansion is introduced and analyzed under $\mathbb{T}$-properness conditions. As a direct consequence, a $\mathbb{T}$-proper KL series expansion is derived on the basis of the second-order information of the signal itself, which lead to a reduction in the computational load when compared to the TWL KL series expansion. Secondly, the $\mathbb{T}$-proper KL expansion is applied to the simulation problem, demonstrating its efficacy in addressing the simulation of $\mathbb{T}$-proper random signals.

The rest of the paper unfolds as follows: Section II introduces the reader to the basic notation and concepts essential for understanding tessarine algebra. In Section III, a TWL KL series expansion is derived and the implications of $\mathbb{T}$-properness on this representation are examined. Then, a simulation procedure based on the novel $\mathbb{T}$-proper KL expansion is proposed. Section IV presents a numerical example to illustrate the benefits of the method presented here in a specific context and Section V includes the conclusions of this paper.

2. PRELIMINARIES

The notation used in this paper is fairly standard. Specifically, scalar quantities, column vectors and matrices are denoted using lowercase lightface, lowercase boldface, and uppercase boldface letters, respectively. Additionally, the following symbols are employed:

Symbols	Description
$\mathbb{R}$	set of real numbers
$\mathbb{T}$	set of tessarine numbers
$\mathbb{T}^p$	set of all p-dimensional tessarine vectors
$*$	tessarine conjugate
T	transpose
H	Hermitian transpose
$E[\cdot]$	expectation operator
$\mathrm{diag}(\cdot)$	diagonal (block diagonal) matrix
$\mathbf{0}_{n\times m}$	$n \times m$ zero matrix
$\mathbf{I}_{n\times n}$	$n \times n$ identity matrix
δ_{ij}	Kronecker delta function
$\otimes$	Kronecker product

Moreover, throughout this paper, all random variables are assumed to have zero-mean.

2.1. Review of the tessarine processing

In this section, the key concepts and properties within the tessarine domain are revisited.

Let L be a compact interval of $\mathbb{R}$, and let $\{\mathbf{x}(t) \in \mathbb{T}^p, t \in L\}$ be a p-dimensional tessarine random signal, that can be defined component-wise as [15]:

$$\mathbf{x}(t) = \mathbf{a}(t) + \mathbf{b}(t)\mathrm{i} + \mathbf{c}(t)\mathrm{j} + \mathbf{d}(t)\mathrm{k}, \quad t \in L$$

where $\mathbf{a}(t)$, $\mathbf{b}(t)$, $\mathbf{c}(t)$, and $\mathbf{d}(t)$ are real-valued second-order stochastic signals, and $(\mathrm{i}, \mathrm{j}, \mathrm{k})$ are the imaginary units that satisfy the following multiplication rules: $\mathrm{i}^2 = \mathrm{k}^2 = -1$, $\mathrm{j}^2 = 1$, $\mathrm{ij} = \mathrm{k}$, $\mathrm{jk} = \mathrm{i}$, $\mathrm{ki} = -\mathrm{j}$.

For any two tessarine random signals $\mathbf{x}(t) \in \mathbb{T}^p$ and $\mathbf{y}(t) \in \mathbb{T}^q$, the corresponding *pseudo* autocorrelation and *pseudo* cross-correlation functions are denoted by $\mathbf{\Gamma}_{\mathbf{x}}(t,s) = E[\mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(s)]$ and $\mathbf{\Gamma}_{\mathbf{xy}}(t,s) = E[\mathbf{x}(t)\mathbf{y}^{\mathrm{H}}(s)]$, respectively.

Note that, a complete description of the second-order statistical properties of $\mathbf{x}(t) \in \mathbb{T}^p$ is obtained from the augmented vector [17]:

$$\bar{\mathbf{x}}(t) = \left[\mathbf{x}^{\mathrm{T}}(t), \mathbf{x}^{*^{\mathrm{T}}}(t), \mathbf{x}^{\mathrm{i}^{\mathrm{T}}}(t), \mathbf{x}^{\mathrm{k}^{\mathrm{T}}}(t)\right]^{T} \quad t \in L$$

where $\mathbf{x}^*(t) = \mathbf{a}(t) - \mathbf{b}(t)\mathrm{i} + \mathbf{c}(t)\mathrm{j} - \mathbf{d}(t)\mathrm{k}$, $\mathbf{x}^{\mathrm{i}}(t) = \mathbf{a}(t) + \mathbf{b}(t)\mathrm{i} - \mathbf{c}(t)\mathrm{j} - \mathbf{d}(t)\mathrm{k}$, and $\mathbf{x}^{\mathrm{k}}(t) = \mathbf{a}(t) - \mathbf{b}(t)\mathrm{i} - \mathbf{c}(t)\mathrm{j} + \mathbf{d}(t)\mathrm{k}$, and whose *pseudo* autocorrelation function is, for $t, s \in L$,

$$\mathbf{\Gamma}_{\bar{\mathbf{x}}}(t,s) = E[\bar{\mathbf{x}}(t)\bar{\mathbf{x}}^{\mathrm{H}}(s)] = \begin{pmatrix} \mathbf{\Gamma}_{\mathbf{x}}(t,s) & \mathbf{\Gamma}_{\mathbf{xx}^*}(t,s) & \mathbf{\Gamma}_{\mathbf{xx}^{\mathrm{i}}}(t,s) & \mathbf{\Gamma}_{\mathbf{xx}^{\mathrm{k}}}(t,s) \\ \mathbf{\Gamma}^*_{\mathbf{xx}^*}(t,s) & \mathbf{\Gamma}^*_{\mathbf{x}}(t,s) & \mathbf{\Gamma}^*_{\mathbf{xx}^{\mathrm{k}}}(t,s) & \mathbf{\Gamma}^*_{\mathbf{xx}^{\mathrm{i}}}(t,s) \\ \mathbf{\Gamma}^{\mathrm{i}}_{\mathbf{xx}^{\mathrm{i}}}(t,s) & \mathbf{\Gamma}^{\mathrm{i}}_{\mathbf{xx}^{\mathrm{k}}}(t,s) & \mathbf{\Gamma}^{\mathrm{i}}_{\mathbf{x}}(t,s) & \mathbf{\Gamma}^{\mathrm{i}}_{\mathbf{xx}^*}(t,s) \\ \mathbf{\Gamma}^{\mathrm{k}}_{\mathbf{xx}^{\mathrm{k}}}(t,s) & \mathbf{\Gamma}^{\mathrm{k}}_{\mathbf{xx}^{\mathrm{i}}}(t,s) & \mathbf{\Gamma}^{\mathrm{k}}_{\mathbf{xx}^*}(t,s) & \mathbf{\Gamma}^{\mathrm{k}}_{\mathbf{x}}(t,s) \end{pmatrix}. \quad (1)$$

The augmented vector $\bar{\mathbf{x}}(t)$ is related to the real vector $\mathbf{x}_r(t) = [\mathbf{a}^{\mathrm{T}}(t), \mathbf{b}^{\mathrm{T}}(t), \mathbf{c}^{\mathrm{T}}(t), \mathbf{d}^{\mathrm{T}}(t)]^{\mathrm{T}}$ as follows:

$$\bar{\mathbf{x}}(t) = 2\boldsymbol{\mathcal{T}}_p\mathbf{x}_r(t), \quad t \in L \quad (2)$$

where $\boldsymbol{\mathcal{T}}_p = \frac{1}{2}\mathcal{B} \otimes \mathbf{I}_{p\times p}$, with

$$\mathcal{B} = \begin{pmatrix} 1 & \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & -\mathrm{i} & \mathrm{j} & -\mathrm{k} \\ 1 & \mathrm{i} & -\mathrm{j} & -\mathrm{k} \\ 1 & -\mathrm{i} & -\mathrm{j} & \mathrm{k} \end{pmatrix},$$

and $\boldsymbol{\mathcal{T}}_p^H\boldsymbol{\mathcal{T}}_p = \mathbf{I}_{4p\times 4p}$.

Moreover, working with hypercomplex signals has the advantage of potentially reducing the dimensionality of the problem. This reduction is achieved by considering specific signal properness characteristics that rely on the vanishing of the *pseudo* correlation functions in (1). An interesting case is the $\mathbb{T}$-properness introduced in [17] as an extension of the $\mathbb{Q}$-properness from the quaternion domain to the tessarine field. Essentially, $\mathbb{T}$-properness entails a dimensionality reduction to quater compared to optimal or widely linear processing. This concept is detailed in the following definition.

Definition 1 *Given the tessarine random signals* $\mathbf{x}(t) \in \mathbb{T}^p$, $\mathbf{y}(t) \in \mathbb{T}^q$, *it is said that* $\mathbf{x}(t)$ *is* $\mathbb{T}$*-proper if, and only if,* $\mathbf{\Gamma}_{\mathbf{x}\mathbf{x}^\nu}(t,s)$*=0, for* $\nu = *, \mathrm{i}, \mathrm{k}$, *and* $\forall t, s \in L$. *Moreover,* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are said to be cross* $\mathbb{T}$*-proper if, and only if,* $\mathbf{\Gamma}_{\mathbf{x}\mathbf{y}^\nu}(t,s)$*=0, for* $\nu = *, \mathrm{i}, \mathrm{k}$ *(similarly,* $\nu = \mathrm{i}, \mathrm{k}$*), and* $\forall t, s \in L$. *Finally,* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are said to be jointly* $\mathbb{T}$*-proper, if, and only if, they are* $\mathbb{T}$*-proper and cross* $\mathbb{T}$*-proper.*

The following result established in [17] gives a characterization of $\mathbb{T}$-properness.

Proposition 1 *A tessarine random signal* $\mathbf{x}(t)$ *is* $\mathbb{T}$*-proper, if and only if, the following identities are satisfied:*

$$\begin{aligned} \mathbf{\Gamma}_{\mathbf{a}}(t,s) = \mathbf{\Gamma}_{\mathbf{b}}(t,s) = \mathbf{\Gamma}_{\mathbf{c}}(t,s) = \mathbf{\Gamma}_{\mathbf{d}}(t,s) \\ -\mathbf{\Gamma}_{\mathbf{ab}}(t,s) = \mathbf{\Gamma}_{\mathbf{ba}}(t,s) = -\mathbf{\Gamma}_{\mathbf{cd}}(t,s) = \mathbf{\Gamma}_{\mathbf{dc}}(t,s) \\ \mathbf{\Gamma}_{\mathbf{ac}}(t,s) = \mathbf{\Gamma}_{\mathbf{bd}}(t,s) = \mathbf{\Gamma}_{\mathbf{ca}}(t,s) = \mathbf{\Gamma}_{\mathbf{db}}(t,s) \\ -\mathbf{\Gamma}_{\mathbf{ad}}(t,s) = \mathbf{\Gamma}_{\mathbf{bc}}(t,s) = -\mathbf{\Gamma}_{\mathbf{cb}}(t,s) = \mathbf{\Gamma}_{\mathbf{da}}(t,s) \end{aligned}$$

Further details about the tessarine processing can be found in [17]–[18] and references therein.

3. $\mathbb{T}$-PROPER KARHUNEN-LOÈVE EXPANSION

In this section, the KL series expansion is analyzed in the tessarine domain. Firstly, a KL series representation for the augmented signal $\bar{\mathbf{x}}(t)$ is introduced and then, the repercussion of $\mathbb{T}$-properness on this representation is investigated.

Consider a tessarine random signal $\{\mathbf{x}(t), t \in L\}$, where L is a compact interval of $\mathbb{R}$. Assume that $\mathbf{\Gamma}_{\mathbf{x}_r}(t,s)$ is continuous. The following result can be obtained by similar reasoning to that used in the quaternion case developed in [14].

Theorem 1 *The augmented vector* $\bar{\mathbf{x}}(t) \in \mathbb{T}^{4p}$ *allows for the following TWL KL expansion:*

$$\bar{\mathbf{x}}(t) = \sum_{i=1}^{\infty} \boldsymbol{\theta}_i(t) \varsigma_i, \tag{3}$$

where the series converges in quadratic mean (q.m.) uniformly for $t \in L$. *Additionally,* $\varsigma_i = \int_L \boldsymbol{\theta}_i^H(t)\bar{\mathbf{x}}(t)dt$ *(in q.m.) are real-valued random variables with* $E[\varsigma_i\varsigma_j] = l_i\delta_{ij}$*, where* l_i *and* $\boldsymbol{\theta}_i(t)$ *are the eigenvalues and associated eigenfunctions of* $\mathbf{\Gamma}_{\bar{\mathbf{x}}}(t,s)$*, given by*

$$\int_L \mathbf{\Gamma}_{\bar{\mathbf{x}}}(t,s)\boldsymbol{\theta}_i(s)ds = l_i\boldsymbol{\theta}_i(t).$$

From Theorem 1, a TWL KL representation for the tessarine random signal $\mathbf{x}(t) \in \mathbb{T}^p$ can be devised. This representation is optimal in the sense that the total mean square error resulting from the truncation of the series is minimized.

Our interest now is to investigate the impact of $\mathbb{T}$-properness on this representation. In this regard, it is noteworthy that, under $\mathbb{T}$-properness conditions, the augmented *pseudo* autocorrelation function of $\bar{\mathbf{x}}(t)$ adopts the form

$$\mathbf{\Gamma}_{\bar{\mathbf{x}}}(t,s) = \mathrm{diag}\left(\mathbf{\Gamma}_{\mathbf{x}}(t,s), \mathbf{\Gamma}^*_{\mathbf{x}}(t,s), \mathbf{\Gamma}^{\mathrm{i}}_{\mathbf{x}}(t,s), \mathbf{\Gamma}^{\mathrm{k}}_{\mathbf{x}}(t,s)\right).$$

This specific structure, in turn, leads to eigenfunctions and eigenvalues of a particular form, as shown in the Lemma 1.

Lemma 1 *Let* $\mathbf{x}(t) \in \mathbb{T}^p$ *be* $\mathbb{T}$*-proper. If* $\mathbf{\Gamma}_{\mathbf{x}}(t,s)$ *has eigenvalues of the form* $\tau_i = \tau_{i_1} + \mathrm{j}\tau_{i_2}$ *with associated eigenfunctions* $\mathfrak{f}_i(t)$*, then* $\mathbf{\Gamma}_{\bar{\mathbf{x}}}(t,s)$ *has eigenvalues* τ_i, τ_i, τ_i^{i}, τ_i^{k}*, and the corresponding eigenfunctions are of the form*

$$\boldsymbol{\phi}_i(t) = [\mathfrak{f}_i^T(t), \mathbf{0}_p^T, \mathbf{0}_p^T, \mathbf{0}_p^T]^T, \quad \boldsymbol{\psi}_i(t) = [\mathbf{0}_p^T, \mathfrak{f}_i^{*^T}(t), \mathbf{0}_p^T, \mathbf{0}_p^T]^T$$
$$\boldsymbol{\kappa}_i(t) = [\mathbf{0}_p^T, \mathbf{0}_p^T, \mathfrak{f}_i^{\mathrm{i}^T}(t), \mathbf{0}_p^T]^T, \quad \boldsymbol{\varphi}_i(t) = [\mathbf{0}_p^T, \mathbf{0}_p^T, \mathbf{0}_p^T, \mathfrak{f}_i^{\mathrm{k}^T}(t)]^T$$

Furthermore, the tessarine random variables $\chi_i = \int_L \mathfrak{f}_i^H(t)\mathbf{x}(t)dt$ *(in q.m.) are* $\mathbb{T}$*-proper, where* $E[\chi_i\chi_j^*] = \tau_i\delta_{ij}$ *and* $E[\chi_i\chi_j^\nu] = 0$*, for* $\nu = \mathrm{i}, \mathrm{k}$, $\forall\, i, j$.

Note that unlike (3), where the random coefficients are real-valued, χ_i are tessarine random variables.

From Lemma 1, the following KL expansion under $\mathbb{T}$-properness conditions can be devised for $\mathbf{x}(t) \in \mathbb{T}^p$.

Theorem 2 *Under conditions of Lemma 1,* $\mathbf{x}(t)$ *allows for the* $\mathbb{T}$*-proper KL expansion*

$$\mathbf{x}(t) = \sum_{i=1}^{\infty} \mathfrak{f}_i(t)\chi_i \tag{4}$$

where the series converges in q.m. uniformly in $t \in L$.

Remark 1 *The series expansion* (4) *means a computational saving in representing* $\mathbf{x}(t)$ *compared to the TWL KL series expansion since, under* $\mathbb{T}$*-properness conditions, only the information contained in* $\mathbf{\Gamma}_{\mathbf{x}}(t,s)$ *needs consideration, reducing the dimensionality of the problem by a quarter.*

3.1. Application to the simulation of $\mathbb{T}_k$-proper signals

Series representation (4) can be used as a simulation tool to obtain numerical realizations of $\mathbb{T}$-proper tessarine signals. The implementation of this simulation technique involves the following steps:

1. Determine the *pseudo* autocorrelation function $\mathbf{\Gamma}_{\mathbf{x}}(t,s)$[1].
2. Obtain the eigenvalues τ_i and eigenfunctions $\mathfrak{f}_i(t)$ corresponding to $\mathbf{\Gamma}_{\mathbf{x}}(t,s)$ by solving the equation
$$\int_L \mathbf{\Gamma}_{\mathbf{x}}(t,s)\mathfrak{f}_i(s)ds = l_i\mathfrak{f}_i(t).$$
3. Truncate series expansion (4) at a finite number n of terms. It is noteworthy that the choice of this value n has a significant impact on the accuracy of the simulated process as well as the computation

[1] In certain real-world applications, this function is known at first. Actually, it could be inferred from experimental data or formulated from mathematical models.

load involved. A suitable approach to give an optimal truncation level and avoid unnecessary computation, is to select the minimum value of n such that $\sum_{i=1}^{n}(\tau_{i_1}+\tau_{i_2}) \geq 0.95 \sum_{i=1}^{\infty}(\tau_{i_1}+\tau_{i_2})$.

4. Use series expansion (4) truncated at a finite number n of terms to generate approximate sample functions of the signal $\mathbf{x}(t)$. In order to generate the values of the tessarine $\mathbb{T}$-proper vector $[\chi_1,\dots,\chi_n]^T$, we generate n uncorrelated real random vectors χ_{i_r}, for $i=1,\dots,n$, each with *pseudo* autocorrelation function

$$\mathbf{\Gamma}_{\chi_{i_r}} = \frac{1}{4}\begin{pmatrix} \tau_{i_1} & 0 & \tau_{i_2} & 0 \\ 0 & \tau_{i_1} & 0 & \tau_{i_2} \\ \tau_{i_2} & 0 & \tau_{i_1} & 0 \\ 0 & \tau_{i_2} & 0 & \tau_{i_1} \end{pmatrix}, \quad i=1,\dots,n.$$

Then, these values correspond to the real vectors of χ_i.

Remark 2 *The derivation of the $\mathbb{T}-proper$ KL expansion is based solely on the assumption that the pseudo autocorrelation function $\mathbf{\Gamma}_{\mathbf{x}}(t,s)$ is known. However, a particular pseudo autocorrelation function does not specify a unique random signal; in fact, it characterizes a broader class of signals with shared second-order properties. To employ these series expansions effectively as a simulation tool, additional information about the distribution of the signal is necessary.*

4. NUMERICAL EXAMPLE

In order to analyze the effectiveness of the simulation technique proposed here, let us consider a tessarine random signal $\{x(t), t\in[0,1]\}$ with the *pseudo* autocorrelation function

$$\gamma_x(t,s) = E[x(t)x^*(s)] = \sum_{i=1}^{2}\tau_i f_i(t) f_i^*(s),$$

with $\tau_1 = 0.04 + \mathrm{j}0.004$, $\tau_2 = 0.02 - \mathrm{j}0.008$ and

$$f_i(t) = \frac{\sqrt{2}}{2}\Big(\cos\big((2i-1)\pi t\big) + \mathrm{i}\sin\big((2i-1)\pi t\big) + \mathrm{j}\cos\big(2i\pi t\big) + \mathrm{k}\sin\big(2i\pi t\big)\Big), \qquad i=1,2$$

By using Monte Carlo simulation, 20000 trajectories for the signal $x(t)$ have been generated. They are denoted as $\{x(t,i)\}_{i=1}^{20000}$. From these trajectories, the *pseudo* autocorrelation and *pseudo* cross-correlation functions $\gamma_x(t,s)$ and $\gamma_{xx^\nu}(t,s) = E[x(t)x^{\nu^*}(s)]$, for $\nu = *, \mathrm{i}, \mathrm{k}$, have been simulated as follows:

$$\begin{aligned} \hat{\gamma}_x(t,s) &= \frac{1}{20000}\sum_{i=1}^{20000}\sum_{j=1}^{20000} x(t,i)x^*(t,j) \\ \hat{\gamma}_{xx^\nu}(t,s) &= \frac{1}{20000}\sum_{i=1}^{20000}\sum_{j=1}^{20000} x(t,i)x^{\nu^*}(t,j). \end{aligned} \tag{5}$$

The four components of the true *pseudo* autocorrelation function $\gamma_x(t,s)$ are depicted in Fig. 1 a)–d), respectively. Similarly, the four components of simulated *pseudo* autocorrelation function $\hat{\gamma}_x(t,s)$ are shown in Fig. 1 e)–h), respectively. As observed, the simulated components closely match with their corresponding true counterparts.

Moreover, the real part of the squared modulus of $\hat{\gamma}_{xx^\nu}(t,s)$, for $\nu = *, \mathrm{i}, \mathrm{k}$, is represented in Fig. 2 a)–c), respectively. As expected, these values are all closely approaching zero.

5. CONCLUSIONS

The KL series representation has been successfully extended to the tessarine domain. Interestingly, this expansion enables dimensionality reduction in $\mathbb{T}$-proper scenarios. In [17] and [18], statistical tests for experimentally determining whether a signal is $\mathbb{T}_k$-proper, $k=1,2$, have been provided. The applicability of this $\mathbb{T}$-proper KL expansion has been examined in the simulating random signals, where its computational advantages have been exploited under $\mathbb{T}$-properness conditions. Future research will explore more general hypercomplex structures, such as the generalized Segre's quaternions or beta quaternions, offering the possibility to determine the best proper conditions for each specific problem, if they exist.

6. REFERENCES

[1] C. Hua, X. Cao, Q. Xu, B. Liao, and S. Li, "Dynamic neural network models for time-varying problem solving: A survey on model structures," in IEEE Access, vol. 11, pp. 65991–66008, 2023, doi: 10.1109/ACCESS.2023.3290046.

[2] M. Soleymani Shishvan, and J. Benndorf, "The effect of geological uncertainty on achieving short-term targets: A quantitative approach using stochastic process simulation," J. South. Afr. Inst. Min. Metall. , vol. 116, pp. 259–264., 2016, doi: 10.17159/2411-9717/2016/v116n3a7.

[3] M. Camargo, M. Dumas, and O. González-Rojas, "Learning Accurate LSTM models of business processes," in Business Process Management: 17th International Conference BPM 2019, Vienna (Austria), Sept. 1–6, 2019, Proceedings, Springer-Verlag, 2019, pp. 286–302. doi: 10.1007/978-3-030-26619-6_19.

[4] G.T. Nwaila, H.E. Frimmel, S. E. Zhang, J. E. Bourdeau, L.C.K. Tolmay, R. J. Durrheim, and Y. Ghorbani, "The minerals industry in the era of digital transition: An energy-efficient and environmentally conscious approach," Resources Policy, vol. 78, 102851, 2022.

[5] R. Galić, Radoslav, E. Čajić, Z. Stojanovic, and D. Galić, "Stochastic Methods in Artificial Intelligence,"

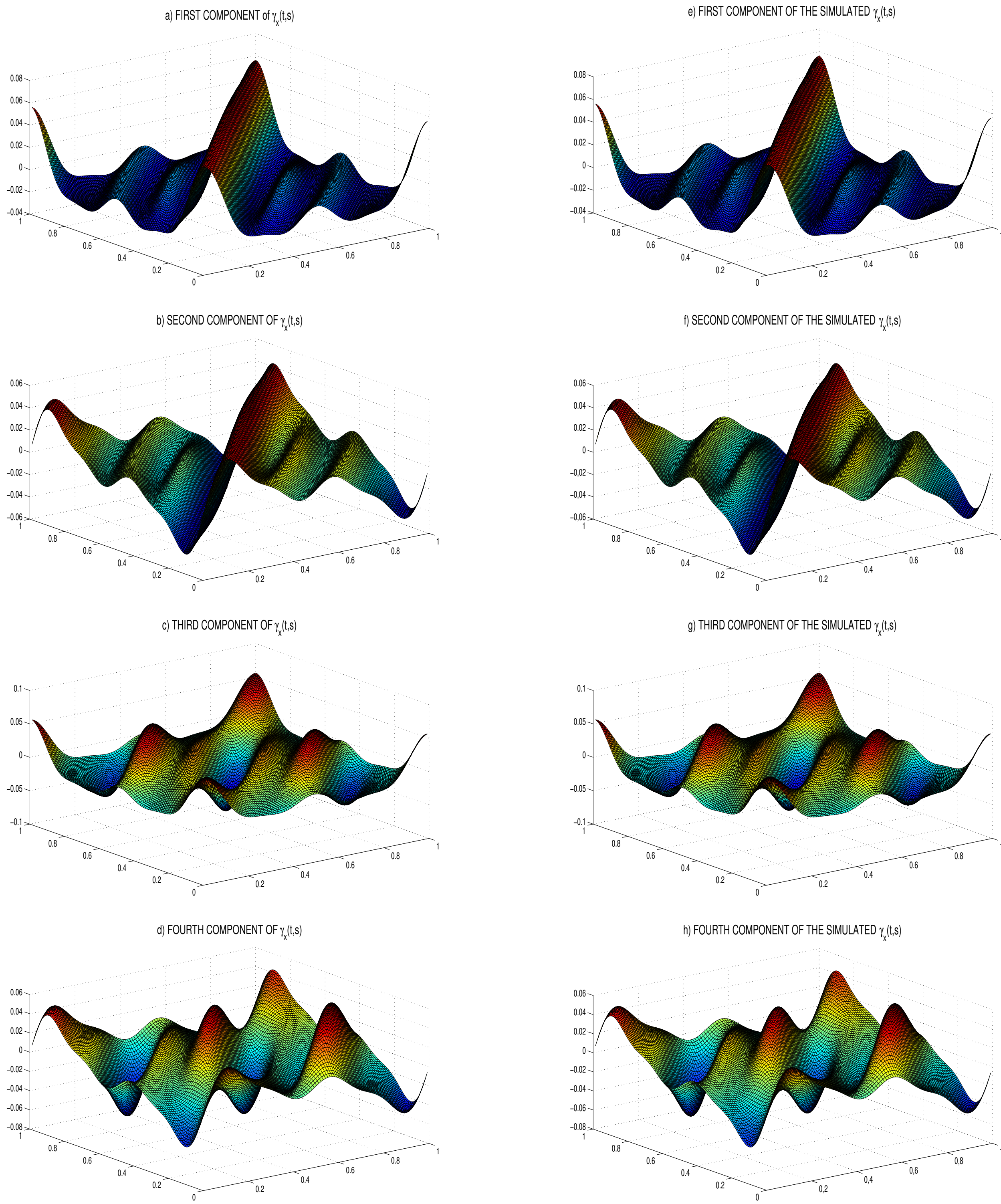

Fig. 1. a) First component of $\gamma_x(t,s)$. b) Second component of $\gamma_x(t,s)$. c) Third component of $\gamma_x(t,s)$. d) Fourth component of $\gamma_x(t,s)$. e) First component of $\hat{\gamma}_x(t,s)$. f) Second component of $\hat{\gamma}_x(t,s)$. g) Third component of $\hat{\gamma}_x(t,s)$. h) Fourth component of $\hat{\gamma}_x(t,s)$.

in Symposium Mathematics and Applications, vol. XIII, 2023, doi:10.21203/rs.3.rs-3597781/v1.

[6] K.K. Phoon, H.W. Huang, and S.T. Quek, "Comparison between Karhunen-Loève and wavelet expansions for simulation of Gaussian processes," Computers and Structures, vol. 82, pp. 985–991, 2004.

[7] K.K. Phoon, H.W. Huang, and S.T. Quek, "Simulation of strongly non-Gaussian processes using Karhunen-Loève expansion," Probabilistic Engineering Mechanics, vol. 20, pp. 188–198, 2005.

[8] L. Weihong, and J. Anjia, "A quasi-Monte Carlo method based on the Karhunen-Loève expansion for groundwater flow simulation, " Simulation, vol. 97 (10), pp.677–685, 2021, doi 10.1177/0037549720906006.

[9] A. Oya, J. Navarro-Moreno, and J.C. Ruiz-Molina, "Widely linear simulation of continuous-time complex-valued random signals," IEEE Signal Process. Letters, vol. 18 (9), pp. 513–516, 2011.

[10] J. Navarro-Moreno, R. M. Fernández-Alcalá, and J. C. Ruiz-Molina, "Semi-widely simulation and estimation of continuous-time undefined control sequence $\mathbb{C}$-proper quaternion random signals," IEEE Trans. . Signal Process., vol. 63 (18), pp. 4999–5012, 2015, doi: 10.1109/TSP.2015.2448521.

[11] J. Vía, D. Ramírez, and I. Santamaría,"Properness and widely linear processing of quaternion random vectors," IEEE Trans. Inform. Theory, vol. 56 (7), pp.3502–3515, 2010.

[12] C. Cheong Took, and D.P. Mandic, "Augmented second-order statistics of quaternion random signals," Signal Process., vol. 91 (2), pp.214–224, 2011.

[13] T. Nitta, M. Kobayashi, and D.P. Mandic, "Hypercomplex widely linear estimation through the lens of underpinning geometry, " IEEE Trans. Signal Process. , vol.67 (15), pp.3985–3994, 2019.

[14] J. Navarro-Moreno, R. M. Fernández-Alcalá, and J. C. Ruiz-Molina, "A quaternion widely linear series expansion and its applications," IEEE Signal Process. Lett. , vol. 19 (12), pp.868–871, 2012.

[15] F. Ortolani, M. Scarpiniti, D. Comminiello, and A. Uncini, "On 4-Dimensional hypercomplex algebras in adaptive signal processing," *In Neural Advances in Processing Nonlinear Dynamic Signals*, pp. 131–140, 2017.

[16] F. Ortolani, Hypercomplex Adaptive Filtering, PhD Thesis, University di Roma, 2017.

[17] J. Navarro-Moreno, R.M. Fernández-Alcalá, J. D. Jiménez-López, and J.C. Ruiz-Molina, "Tessarine signal processing under the $\mathbb{T}$-properness condition," J. Franklin Inst. , vol.357 (14), pp. 10100–10126, 2020.

[18] J. Navarro-Moreno, and J. C. Ruiz-Molina, " Wide-sense Markov signals on the tessarine domain. A study under properness conditions," Signal Process. , vol. 183, 108022, 2021.

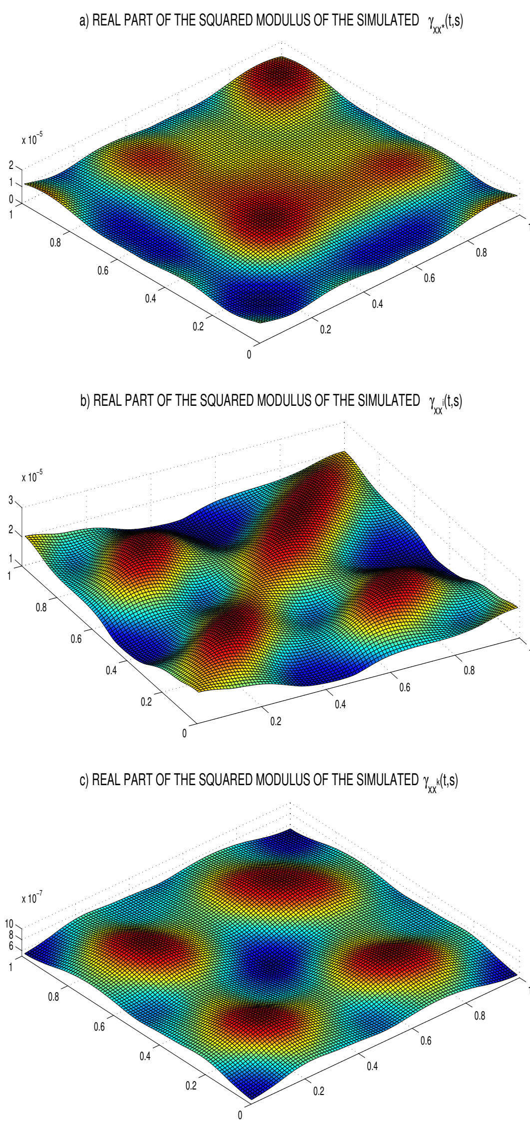

Fig. 2. Real part of the squared modulus of $\hat{\gamma}_{xx^{\nu}}(t,s)$, for $\nu = *, \mathrm{i}, \mathrm{k}$.