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Second Kind Chebyshev Wavelet Analysis of Abel's Integral Equations

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Abstract

This paper presents two approximations of the solution functions of Abel's integral equations belonging to classes $H^{\alpha}[0,1)$, $H^{\phi}[0,1)$ by $(\lambda^{k+1}-1, M)^{th}$ partial sums of their second kind Chebyshev wavelet expansion in the interval $[0,1)$, for $\lambda > 1$. These approximations are $E^{(1)}_{\lambda^{k+1}-1,M}(f)$, $E^{(2)}_{\lambda^{k+1}-1,M}(f)$. Chebyshev wavelets of the second kind were used to solve the Abel's integral equations. The Chebyshev wavelet of the second kind leads to a solution that is almost identical to their exact solution. This research paper's accomplishment in wavelet analysis is noteworthy.

Keywords: $H^{\alpha}[0,1)$ class, $H^{\phi}[0,1)$ class, Chebyshev wavelet of the second kind and Abel's integral equations.
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1 Introduction

In recent years, wavelets have become widely utilized across diverse fields such as biology, mathematics, and engineering. These are primarily used for

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signal analysis tasks such as time-frequency analysis and waveform segmentation, provide quick results for straightforward performance. Additionally, they utilize efficient organization of numerical data to compute solutions to integral and differential equations effectively. This capability presents a significant advantage in various scientific and engineering applications, where accurate and efficient computation is paramount. This research study investigates the properties associated with Hölder's classes $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$, presenting an innovative concept in the fields of mathematical sciences and physics. Numerous esteemed researchers have contributed to the understanding of functions within Hölder's class with order α $(0 < \alpha \leq 1)$. Notable among them are Chui [1], Debnath [2], Kumar et al. ([4], [9]) Lal and Yadav ([7], [8]), Lepik [10], Meyer [11], Pandey et al. ([3], [12], [13], [14], [15], [16]), Ray and Sahu [17] and Sripathy et al. [18], whose extensive studies have paved the way for advancements in function approximation theory and related fields. Their work provides a solid foundation upon which this study builds, extending the understanding of Hölder's classes and their applications. This paper explores solving integral equations by employing the Chebyshev wavelet of the second kind within the interval [0,1), enhancing efficiency and broadening the toolkit for mathematical and computational scientists. For instance, the method for solving Abel's integral equations of the form:

$$y(t)\gamma(t) = f(t) + \int_0^t \frac{y(x)}{\sqrt{t-x}}dx \tag{1}$$

has been introduced using second kind Chebyshev wavelet.

This research paper aims to achieve several objectives. Firstly, it seeks to define the Chebyshev wavelet of the second kind within the interval [0,1), providing a clear understanding of its properties and characteristics. Secondly, it aims to define Hölder's classes $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$, portraying their significance in mathematical analysis. Thirdly, the paper aims to assess the approximation of function f within these classes, exploring the effectiveness of wavelet-based techniques in function approximation. Additionally, the procedure for solving Abel's integral equations utilizing the Chebyshev wavelet of the second kind have been discussed, offering insights into the practical application of wavelet methods in solving integral equations. Lastly, the paper intends to compare the exact solutions of Abel's integral equations with the approximate solutions obtained using various types of wavelets, including the Legendre wavelet and both kinds of Chebyshev wavelets, shedding light on the comparative efficacy of different wavelet approaches.

The subsequent sections of this paper are structured as follows: Section 2 outlines the definitions and properties of Hölder's class, the Chebyshev wavelet of the second kind, and their orthonormality. Section 3 examines the approximation of solution functions of Abel's integral equations using Chebyshev wavelets of the second kind. Theorems are presented in Section 4, with their proofs in Section 5. Section 6 presents the algorithm for solving Abel's integral equations using the Chebyshev wavelet of the second kind within the interval [0,1). Section 7 demonstrates the solutions of Abel's integral equations using this wavelet. Finally, the conclusions of this research paper are summarized in Section 8.

2 Definitions and Preliminaries

2.1 Function of Hölder's class

A function $f \in H^{\alpha}[0,1), 0 < \alpha \leq 1$ if

$$f(x+t) - f(x) = O(|t|^{\alpha}),\ \forall\, x+t,\ x,\ t \in [0,1) \text{ (Titchmarsh[19])}.$$

2.2 Function of Hölder's class $H^{\phi}[0,1)$

A function $f \in H^{\phi}[0,1), 0 < \alpha \leq 1$ if

$$f(x+t) - f(x) = O(\phi(|t|)),\ \forall\, x+t,\ x,\ t \in [0,1),$$

where $\phi(t)$ is a positive monotonic increasing function of t such that $\phi(|t|) \to 0$ as $t \to 0$. If $\phi(t) = |t|^{\alpha}$ then $H^{\phi}[0,1)$ class reduces to $H^{\alpha}[0,1)$ class.

2.3 Chebyshev wavelet of the second kind

Chebyshev wavelet of the second kind is denoted by $\psi_{n,m}^{(\lambda)}$ and defined by

$$\psi_{n,m}^{(\lambda)}(t) = \begin{cases} \frac{2}{\sqrt{\pi}} \lambda^{\frac{k+1}{2}} U_m(2\lambda^{k+1}t - 2n - 1), & \text{if } t \in [\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}}); \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $n = 0, 1, 2, ..., \lambda^{k+1} - 1, m = 0, 1, 2, ..., M-1$ and k, λ are positive integer. $U_m(t)$ are the second kind Chebyshev polynomial of degree m which are orthogonal with respect to weight function $\omega(t) = \sqrt{1-t^2}$ on [-1,1] and satisfy $U_0(t) = 1, U_1(t) = 2t$ and $U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t)$, $m \in \{1, 2, 3, ...\}$.

2.4 Orthonormality of the second kind Chebyshev wavelets

Proposition 2.1. $\{\psi_{n,m}^{(\lambda)}(t),\ n = 0, 1, 2, ..., \lambda^{k+1} - 1,\ m = 0, 1, 2, ..., M-1\}$ *forms an orthonormal set. i.e.*

$$\langle \psi_{n,m}^{(\lambda)}, \psi_{n',m'}^{(\lambda)} \rangle_{\omega_{n,k}} = \begin{cases} 1, & \textit{if } n = n' \textit{ and } m = m'; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. For $m = 0$ and $n = n'$,

$$\begin{aligned}
\langle \psi^{(\lambda)}_{n,0}, \psi^{(\lambda)}_{n,0} \rangle_{\omega_{n,k}} &= \int_0^1 \psi^{(\lambda)}_{n,0}(t)\overline{\psi^{(\lambda)}_{n,0}(t)}\omega_{n,k}(t)dt \\
&= \int_0^1 (\psi^{(\lambda)}_{n,0}(t))^2 \omega_{n,k}(t)dt \\
&= \frac{4}{\pi}\int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \lambda^{k+1}\omega_{n,k}(t)dt \\
&= \frac{2}{\pi}\int_0^{\pi} \sin^2\theta d\theta = 1,\ 2\lambda^{k+1}t - 2n + 1 = \cos\theta.
\end{aligned}$$

For $m \neq 0$,

1. For $m = m'$ and $n = n'$,

$$\begin{aligned}
\langle \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n,m} \rangle_{\omega_{n,k}} &= \int_0^1 \psi^{(\lambda)}_{n,m}(t)\overline{\psi^{(\lambda)}_{n,m}(t)}\omega_{n,k}(t)dt \\
&= \int_0^1 (\psi^{(\lambda)}_{n,m}(t))^2 \omega_{n,k}(t)dt \\
&= \frac{4}{\pi}\int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \lambda^{k+1}U_m^2(2\lambda^{k+1}t - 2n + 1)\omega_{n,k}(t)dt \\
&= \frac{2}{\pi}\int_0^{\pi} U_m^2(\cos\theta)\sin^2\theta d\theta,\ 2\lambda^{k+1}t - 2n + 1 = \cos\theta \\
&= \frac{2}{\pi}\int_0^{\pi} \sin^2(m+1)\theta d\theta = 1.
\end{aligned}$$

2. For $n \neq n'$,

$$\langle \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n',m} \rangle_{\omega_{n,k}} = \int_0^1 \psi^{(\lambda)}_{n,m}(t)\overline{\psi^{(\lambda)}_{n',m}(t)}\omega_{n,k}(t)dt.$$

Since, $\psi^{(\lambda)}_{n,m}$ is defined in $[\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}}) \subset [0,1)$ and $\psi^{(\lambda)}_{n',m}$ is defined in $[\frac{n'}{\lambda^{k+1}}, \frac{n'+1}{\lambda^{k+1}}) \subset [0,1)$, therefore if $n \neq n'$, then the intervals $[\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}})$ and $[\frac{n'}{\lambda^{k+1}}, \frac{n'+1}{\lambda^{k+1}})$ are disjoint i.e.

$$\left[\frac{n}{\lambda^{k+1}}, \frac{n+1}{\lambda^{k+1}}\right) \cap \left[\frac{n'}{\lambda^{k+1}}, \frac{n'+1}{\lambda^{k+1}}\right) = \phi.$$

Therefore $\langle \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n',m} \rangle_{\omega_{n,k}} = 0$ if $n \neq n' \,\forall\, m, m'$.

3. For $m \neq m'$,

$$\begin{aligned}\langle \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n,m'} \rangle_{\omega_{n,k}} &= \int_0^1 \psi^{(\lambda)}_{n,m}(t)\overline{\psi^{(\lambda)}_{n,m'}(t)}\omega_{n,k}(t)dt \\ &= 0, \quad \text{by the orthogonality of Chebyshev polynomials.}\end{aligned}$$

□

3 Approximation of functions and Multiresolution analysis

This section contains the approximations of the solution functions of the Abel's integral equations and Multiresolution analysis by the second kind Chebyshev wavelet.

3.1 Approximation of functions

Since $\{\psi^{(\lambda)}_{m,n}\}$ forms an orthonormal basis for $L^2[0,1)$, therefore the solution function $f \in L^2[0,1)$ of the Abel's integral equations can be expressed into Chebyshev wavelet of second kind as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\lambda)}_{m,n}(t), \;\; c_{n,m} = \langle f, \psi^{(\lambda)}_{m,n}\rangle_{\omega_{n,k}}. \tag{3}$$

The $(\lambda^{k+1}-1, M)^{\text{th}}$ partial sum $(S_{\lambda^{k+1}-1,M}f)(t)$ of wavelet series (3) is given by

$$(S_{\lambda^{k+1}-1,M}f)(t) = \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=0}^{M-1} c_{n,m}\psi^{(\lambda)}_{m,n}(t) = C^T\psi^{(\lambda)}(t), \tag{4}$$

where $C = [c_{0,0}, c_{0,1}, ..., c_{0,M-1}, c_{1,0}, ..., c_{1,M-1}, ..., c_{\lambda^{k+1}-1,0}, ..., c_{\lambda^{k+1}-1,M-1}]^T$ and $\psi^{(\lambda)}(t) = [\psi^{(\lambda)}_{0,0}(t), \psi^{(\lambda)}_{0,1}(t), ..., \psi^{(\lambda)}_{0,M-1}(t), \psi^{(\lambda)}_{1,0}(t), ..., \psi^{(\lambda)}_{1,M-1}(t), ..., \psi^{(\lambda)}_{\lambda^{k+1}-1,0}(t), ..., \psi^{(\lambda)}_{\lambda^{k+1}-1,M-1}(t)]^T$.

The second kind Chebyshev wavelet approximation $E_{\lambda^{k+1}-1,M}(f)$ of f by $(\lambda^{k+1}-1, M)^{\text{th}}$ partial sum $(S_{\lambda^{k+1}-1,M}f)$ of the wavelet series (3) is defined by

$$E_{\lambda^{k+1}-1,M}(f) = \min_{(S_{\lambda^{k+1}-1,M}f)} ||f - (S_{\lambda^{k+1}-1,M}f)||_2 \;\; \text{(Lal and Priya [5])}. \tag{5}$$

$E_{\lambda^{k+1}-1,M}(f)$ is said to be best approximation of the function f if $E_{\lambda^{k+1}-1,M}(f) \to 0$ as $k \to \infty$, $M \to \infty$ (Zygmund [20]).

3.2 Multiresolution analysis

The sequence of subspaces $\{V_n^{(\lambda)}\}_{n=0}^{\infty}$ of $L^2[0,1)$ is defined by

$$V_n^{(\lambda)} = \bigoplus_{l=1}^{n-1} W_l^{(\lambda)} = W_1^{(\lambda)} \oplus W_2^{(\lambda)} \oplus ... \oplus W_{n-1}^{(\lambda)};$$

where $W_n^{(\lambda)} = \underset{L^2[0,1)}{\text{clos}} \langle \psi_{n,m}^{(\lambda)} : m \in \mathbb{N} \cup \{0\} \rangle, \ n = 0, 1, 2, ..., \lambda^{k+1} - 1,$

then $\{V_n^{(\lambda)}\}_{n=0}^{\infty}$ is a multiresolution analysis of $L^2[0,1)$ (Chui [1]).

4 Theorems

The following theorems have been proved in this section:

4.1 Theorem 4.1

If a function $f \in H^\alpha[0,1)$ and its second kind Chebyshev wavelet expansion be

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(\lambda)}(t)$$

having $(\lambda^{k+1} - 1, M)^{\text{th}}$ partial sums

$$(S_{\lambda^{k+1}-1,M} f)(t) = \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(\lambda)}(t),$$

then the second kind Chebyshev wavelet approximation of f is given by

$$E_{\lambda^{k+1}-1,M}^{(1)}(f) = \min ||f - \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(\lambda)}(t)||_2 = O\left(\frac{1}{\lambda^{(k+1)\alpha}\sqrt{M}} \right), \ M > 1.$$

4.2 Theorem 4.2

If a function $f \in H^\phi[0,1)$ class such that $\phi(|t|) \to 0$ as $t \to 0$ then the second kind Chebyshev wavelet approximation of f by $(S_{\lambda^{k+1}-1,M} f)$ satisfies for $M > 1$

$$E_{\lambda^{k+1}-1,M}^{(2)}(f) = \min ||f - \sum_{n=0}^{\lambda^{k+1}-1} \sum_{m=0}^{M-1} c'_{n,m} \psi_{n,m}^{(\lambda)}(t)||_2 = O\left(\frac{1}{\lambda^{k+1}\sqrt{M}} \phi\left(\frac{1}{\lambda^{k+1}} \right) \right).$$

5 Proof of Theorems

5.1 Proof of Theorem 4.1

$$
\begin{aligned}
f(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t). \\
c_{n,m} &= \langle f, \psi_{n,m}^{(\lambda)}\rangle_{\omega_{n,k}} \\
&= \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} f(t)\psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt \\
&= \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \left(f(t) - f\left(\frac{n}{\lambda^{k+1}}\right)\right)\psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt \\
&+ f\left(\frac{n}{\lambda^{k+1}}\right)\int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt \\
&= \int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \left(f(t) - f\left(\frac{n}{\lambda^{k+1}}\right)\right)\psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt \\
|c_{n,m}| &\leq N\left(\frac{1}{\lambda^{k+1}}\right)^{\alpha}\left|\int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}} \psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt\right|,\ f \in H^{\alpha}[0,1) \\
&= N\left(\frac{1}{\lambda^{k+1}}\right)^{\alpha}\left|\int_{0}^{\pi} \frac{2}{\sqrt{\pi}}\lambda^{\frac{k+1}{2}}U_m(\cos\theta)\omega(\cos\theta)\frac{\sin\theta}{2\lambda^{k+1}}d\theta\right| \\
&= \frac{N}{\pi\lambda^{(k+1)(\alpha+\frac{1}{2})}}\left|\int_{0}^{\pi}\sin(m+1)\theta\sin\theta d\theta\right|,\ U_m(\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta} \\
&= \frac{N}{2\pi\lambda^{(k+1)(\alpha+\frac{1}{2})}}\left|\int_{0}^{\pi}\cos m\theta - \cos(m+2)\theta d\theta\right| \\
&\leq \frac{N}{2\pi\lambda^{(k+1)(\alpha+\frac{1}{2})}}\left(\left|\int_{0}^{\pi}\frac{1}{m}\frac{d}{d\theta}(\sin m\theta)d\theta\right| + \left|\int_{0}^{\pi}\frac{1}{m+2}\frac{d}{d\theta}(\sin(m+2)\theta)d\theta\right|\right) \\
&\leq \frac{N}{2\pi\lambda^{(k+1)(\alpha+\frac{1}{2})}}\left(\frac{1}{m}\left(\max_{0\leq\theta\leq\pi}(\sin m\theta)\right) + \frac{1}{m+2}\left(\max_{0\leq\theta\leq\pi}(\sin(m+2)\theta)\right)\right) \\
&= \frac{N}{\pi\lambda^{(k+1)(\alpha+\frac{1}{2})}(m+2)}. \qquad (6)
\end{aligned}
$$

Now, $\quad f(t) - (S_{\lambda^{k+1}-1,M}f)(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t) - \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)$

$$
= \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t).
$$

Then, $\quad (E_{\lambda^{k+1}-1,M}^{(1)}(f))^2 = ||f(t) - (S_{\lambda^{k+1}-1,M}f)(t)||_2^2$

$$
\begin{aligned}
&= \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=M}^{\infty}|c_{n,m}|^2\\
&\le \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=M}^{\infty}\left(\frac{N}{\pi\lambda^{(k+1)(\alpha+\frac{1}{2})}(m+2)}\right)^2\\
&= \frac{N^2}{\pi^2\lambda^{2(k+1)\alpha}}\sum_{m=M}^{\infty}\frac{1}{(m+2)^2}\\
&\le \frac{N^2}{\pi^2\lambda^{2(k+1)\alpha}}\left(\frac{1}{(M+2)^2}+\int_M^\infty\frac{dm}{(m+2)^2}\right)\\
&\le \frac{N^2}{\pi^2\lambda^{2(k+1)\alpha}}\left(\frac{1}{M^2}+\frac{1}{M}\right)\le\frac{2N^2}{\pi^2\lambda^{2(k+1)\alpha}M}.
\end{aligned}
$$

$$
\text{Therefore, } E^{(1)}_{\lambda^{k+1}-1,M}(f) \le \frac{\sqrt{2}N}{\pi\lambda^{(k+1)\alpha}\sqrt{M}} = O\left(\frac{1}{\lambda^{(k+1)\alpha}\sqrt{M}}\right),\ M>1.
$$

This completes the proof of the Theorem 4.1.

5.2 Proof of Theorem 4.2

Following the proof of Theorem 4.1 and for $f\in H^{\phi}[0,1)$ class

$$
\begin{aligned}
|c'_{n,m}| &\le N\phi\left(\frac{1}{\lambda^{k+1}}\right)\left|\int_{\frac{n}{\lambda^{k+1}}}^{\frac{n+1}{\lambda^{k+1}}}\psi^{(\lambda)}_{n,m}(t)\omega_{n,k}(t)dt\right|\\
&= N\phi\left(\frac{1}{\lambda^{k+1}}\right)\left|\int_0^\pi\frac{2}{\sqrt{\pi}}\lambda^{\frac{k+1}{2}}U_m(\cos\theta)\omega(\cos\theta)\frac{\sin\theta}{2\lambda^{k+1}}d\theta\right|\\
&= \frac{N}{\pi\lambda^{\frac{k+1}{2}}}\phi\left(\frac{1}{\lambda^{k+1}}\right)\left|\int_0^\pi\sin(m+1)\theta\sin\theta d\theta\right|\\
&= \frac{N}{2\pi\lambda^{\frac{k+1}{2}}}\phi\left(\frac{1}{\lambda^{k+1}}\right)\left|\int_0^\pi\cos m\theta-\cos(m+2)\theta d\theta\right|\\
&= \frac{N}{\pi\lambda^{\frac{k+1}{2}}(m+2)}\phi\left(\frac{1}{\lambda^{k+1}}\right). \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
(E^{(2)}_{\lambda^{k+1}-1,M}(f))^2 &= \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=M}^{\infty}|c'_{n,m}|^2\\
&\le \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=M}^{\infty}\left(\frac{N}{\pi\lambda^{\frac{k+1}{2}}(m+2)}\phi\left(\frac{1}{\lambda^{k+1}}\right)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{N^2}{\pi^2\lambda^{k+1}}\phi^2\left(\frac{1}{\lambda^{k+1}}\right)\sum_{m=M}^{\infty}\frac{1}{(m+2)^2} \\
&\leq \frac{N^2}{\pi^2\lambda^{k+1}}\phi^2\left(\frac{1}{\lambda^{k+1}}\right)\left(\frac{1}{M^2}+\frac{1}{M}\right) \\
&\leq \frac{2N^2}{\pi^2\lambda^{k+1}M}\phi^2\left(\frac{1}{\lambda^{k+1}}\right).
\end{aligned}
$$

$$
\text{Therefore, } E^{(2)}_{\lambda^{k+1}-1,M}(f) = O\left(\frac{1}{\lambda^{k+1}\sqrt{M}}\phi\left(\frac{1}{\lambda^{k+1}}\right)\right),\ M>1.
$$

This completes the proof of the Theorem 4.2.

6 Method for solving Abel's integral equations by second kind Chebyshev wavelet

Consider the Abel's integral equation (1). Let us consider the wavelet series expansion,

$$
y(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\lambda)}_{n,m}(t) \tag{8}
$$

and $(\lambda^{k+1}-1, M)^{th}$ partial sum of series (8) which is the approximate solution of (1) i.e.

$$
y(t) = \sum_{n=0}^{\lambda^{k+1}-1}\sum_{m=0}^{M-1} c_{n,m}\psi^{(\lambda)}_{n,m}(t) = C^T\psi^{(\lambda)}(t). \tag{9}
$$

In Eqn. (9), C^T contains $\lambda^{k+1}M$ unknown coefficients $c_{n,m}$. For determining the values of $\lambda^{k+1}M$ unknown coefficients $c_{n,m}$, taking suitable collocation points near $t_i = \frac{i-1}{\lambda^{k+1}M}, i = 1, ..., \lambda^{k+1}M$, the $\lambda^{k+1}M$ system of algebraic equations are obtained. By solving these $\lambda^{k+1}M$ system of algebraic equations, the values of $c_{n,m}$ have been obtained. Substituting these values in Eqn. (9), the second kind Chebyshev wavelet solution of the integral equation (1) is obtained.

7 Results and Discussion

In this section, the second kind Chebyshev wavelet method has been used to find the approximate solution of Abel's integral equations. The exact solutions of Abel's integral equations are compared with the Legendre wavelet, first kind, and second kind Chebyshev wavelet solutions (Sharma and Lal [6]). This is illustrated in the following examples:

7.1 Example 1

Consider the Abel's integral equation of the first kind

$$\int_0^t \frac{y(x)}{\sqrt{t-x}}dx = \frac{2\sqrt{t}(8t^2+10t+15)}{15} \tag{10}$$

It is obtained by taking $\gamma(t) = 0$, $f(t) = -\frac{2\sqrt{t}(8t^2+10t+15)}{15}$ in Abel's integral equation (1). The exact solution of Eqn. (10) is $y(t) = t^2 + t + 1$.
The six basis function of the second kind Chebyshev wavelet for $k = 0$, $M = 3, \lambda = 2$ are as follows:

$$\left.\begin{array}{l} \text{If} \quad 0 \le t < \frac{1}{2}, \\ \psi_{0,0}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}} \\ \psi_{0,1}^{(2)}(t) = 4\sqrt{\frac{2}{\pi}}(4t-1) \\ \psi_{0,2}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1) \end{array}\right\} \left.\begin{array}{l} \text{If} \quad \frac{1}{2} \le t < 1, \\ \psi_{1,0}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}} \\ \psi_{1,1}^{(2)}(t) = 4\sqrt{\frac{2}{\pi}}(4t-3) \\ \psi_{1,2}^{(2)}(t) = 2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1) \end{array}\right\}$$

By using the algorithm of the second kind Chebyshev wavelet method described in Section 6, the first kind Abel's integral equation has been solved. The approximate solution $y(t)$ will be

$$\begin{aligned} y(t) &= c_{0,0}\psi_{0,0}^{(2)}(t) + c_{0,1}\psi_{0,1}^{(2)}(t) + c_{0,2}\psi_{0,2}^{(2)}(t),\ t \in \left[0, \frac{1}{2}\right) \\ &= 2\sqrt{\frac{2}{\pi}}c_{0,0} + 4\sqrt{\frac{2}{\pi}}(4t-1)c_{0,1} + 2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1)c_{0,2}. \quad (11) \\ y(t) &= c_{1,0}\psi_{1,0}^{(2)}(t) + c_{1,1}\psi_{1,1}^{(2)}(t) + c_{1,2}\psi_{1,2}^{(2)}(t),\ t \in \left[\frac{1}{2}, 1\right) \\ &= 2\sqrt{\frac{2}{\pi}}c_{1,0} + 4\sqrt{\frac{2}{\pi}}(4t-3)c_{1,1} + 2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1)c_{1,2}. \quad (12) \end{aligned}$$

For values of unknowns $c_{0,0}$, $c_{0,1}$, $c_{0,2}$, $c_{1,0}$, $c_{1,1}$, and $c_{1,2}$, we collocate equation Eqn. (10) at $t = 0.15, 0.3, 0.45, 0.6, 0.75,$ and 0.9, six system of algebraic equations are obtained. Solving these systems of algebraic equations, the values of the unknowns are obtained as follows:

$$\begin{aligned} &c_{0,0} = 0.832278919311074,\ c_{0,1} = 0.117498200373328, \\ &c_{0,2} = 0.009791516697778,\ c_{1,0} = 1.458935987968824, \\ &c_{1,1} = 0.195830333955546,\ c_{1,2} = 0.009791516697777. \end{aligned} \tag{13}$$

Putting the values of $c_{0,0}$, $c_{0,1}$, $c_{0,2}$, $c_{1,0}$, $c_{1,1}$, and $c_{1,2}$ from Eqn. (13) into Eqns. (11) and (12).

$$\begin{aligned} y(t) &= 0.832278919311074\left(2\sqrt{\frac{2}{\pi}}\right) + 0.117498200373328\left(4\sqrt{\frac{2}{\pi}}(4t-1)\right) \\ &+ 0.009791516697778\left(2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1)\right),\ t \in \left[0, \frac{1}{2}\right), \\ y(t) &= 1.458935987968824\left(2\sqrt{\frac{2}{\pi}}\right) + 0.195830333955546\left(4\sqrt{\frac{2}{\pi}}(4t-3)\right) \\ &+ 0.009791516697777\left(2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1)\right),\ t \in \left[\frac{1}{2}, 1\right). \end{aligned}$$

The approximate solution of the first kind Abel's integral equation (10) obtained by the second kind Chebyshev wavelet method (SKCWM) for different values of t in the interval [0,1) has been obtained. The comparison of exact solutions (ES) of Abel's integral equation with the Legendre wavelet method (LWM), first kind Chebyshev wavelet method (FKCWM), and second kind Chebyshev wavelet method are given in Table 1.

Table (1)

t	ES	LWM	FKCWM	SKCWM
0.1	1.11	0.929532564	1.0719999995	1.10999999999999
0.2	1.24	1.115745774	1.1546666661	1.23999999999999
0.3	1.39	1.309633163	1.2479999994	1.38999999999999
0.4	1.56	1.511194733	1.3519999994	1.56000000000000
0.5	1.75	1.720430483	1.4666666667	1.75000000000000
0.6	1.96	1.937340412	1.5919999994	1.96000000000000
0.7	2.19	2.161924522	1.7279999994	2.18999999999999
0.8	2.44	2.394182812	1.8746666661	2.44000000000000
0.9	2.71	2.634115282	2.0319999994	2.70999999999999

Table(1): Comparison between exact and approximate solutions of Eqn. (10).

The graphs of the exact solution and approximate solution of the Abel's integral equation (10) are shown in Figure 1.

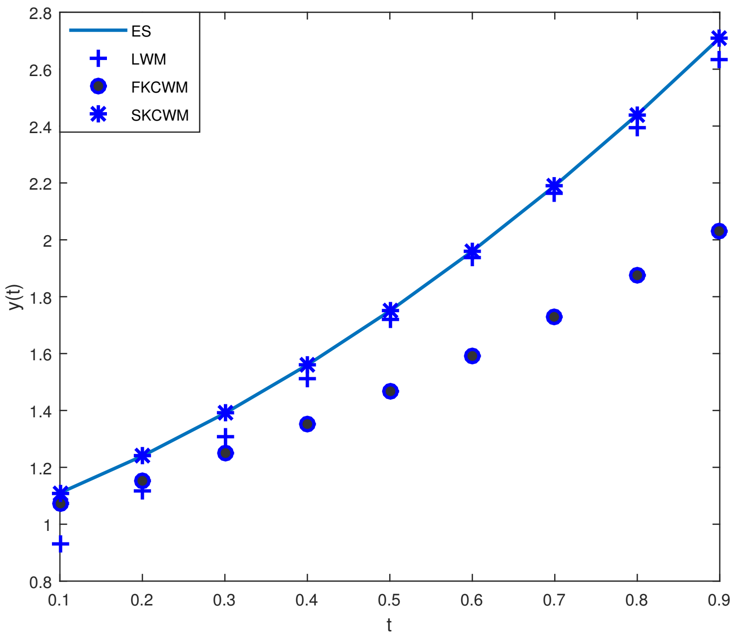

Fig.(1): The graphs of exact and approximate solutions of Eqn. (10).

7.2 Example 2

Consider the Abel's integral equation of the second kind

$$y(t) = 4t^{\frac{3}{2}} - \int_0^t \frac{y(x)}{\sqrt{t-x}}dx \tag{14}$$

It is obtained by taking $\gamma(t) = -1$, $f(t) = -4t^{\frac{3}{2}}$ in the Abel's integral equation (1). The exact solution of Eqn. (14) is $y(t) = 3t + \frac{3}{\pi}(1 - 2\sqrt{t} - erfc(\sqrt{\pi t}))$. Following the procedure of example (1), the approximate solution $y(t)$ will be

$$\begin{aligned} y(t) &= 0.203127378663919\left(2\sqrt{\frac{2}{\pi}}\right) + 0.1245710999898258\left(4\sqrt{\frac{2}{\pi}}(4t-1)\right) \\ &+ 0.007508790407423\left(2\sqrt{\frac{2}{\pi}}(4(4t-1)^2-1)\right),\ t \in \left[0, \frac{1}{2}\right), \\ y(t) &= 0.788695399256314\left(2\sqrt{\frac{2}{\pi}}\right) + 0.159580510841338\left(4\sqrt{\frac{2}{\pi}}(4t-3)\right) \\ &+ 0.002685161812766\left(2\sqrt{\frac{2}{\pi}}(4(4t-3)^2-1)\right),\ t \in \left[\frac{1}{2}, 1\right). \end{aligned}$$

The exact solution and approximate solutions of the second kind Abel's integral equation (14) are shown in Table 2.

Table (2)

t	ES	LWM	FKCWM	SKCWM
0.1	0.0914468358	0.0296708741	0.0772954668400	0.09087255103875016
0.2	0.2313248892	0.1868894237	0.1890183222312	0.23456458416817604
0.3	0.3928020550	0.3529379159	0.3135382389570	0.39359395601475531
0.4	0.5682530617	0.5278163507	0.4467872378730	0.56796066657848785
0.5	0.7538699265	0.7115247282	0.5858830996710	0.76212315618636044
0.6	0.9473720681	0.9040630482	0.7291945119710	0.95487673780814230
0.7	1.1472485450	1.1054313110	0.8762823974830	1.15311498926330412
0.8	1.3524303100	1.3156295160	1.0263605224509	1.35683791055184616
0.9	1.5621248430	1.5346576640	1.1786727537049	1.56604550167376791

Table(2): Comparison between exact and approximate solutions of Eqn. (14).

The graphs of the exact solution and approximate solution of the second kind Abel's integral equation (14) are shown in Figure 2.

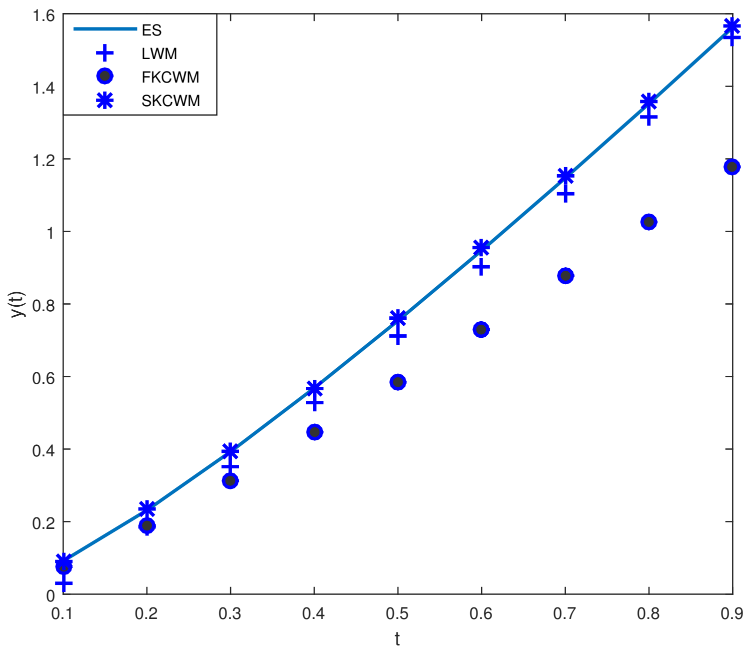

Fig.(2): The graphs of exact and approximate solutions of Eqn. (14).

7.3 Absolute Error

The absolute error in the approximate solution of the Abel's integral equations (10) and (14) are shown in Tables 3. This shows that absolute error is negligible by the second kind Chebyshev wavelet method as compared to Legendre wavelet method, and first kind Chebyshev wavelet method.

Table (3)

	Absolute error in solution of Example 1			Absolute error in solution of Example 2		
t	LWM	FKCWM	SKCWM	LWM	FKCWM	SKCWM
0.1	0.18046	0.03800	0.222×10^{-15}	0.06177	0.01415	0.00057
0.2	0.12425	0.08533	0.222×10^{-15}	0.04443	0.04230	0.00323
0.3	0.08036	0.14200	0	0.03986	0.07926	0.00079
0.4	0.04880	0.20800	0	0.04043	0.12146	0.00029
0.6	0.02956	0.28333	0	0.04234	0.16798	0.00825
0.6	0.02265	0.36800	0	0.04330	0.21817	0.00750
0.7	0.02807	0.46200	0	0.04181	0.27096	0.00586
0.8	0.04581	0.56533	0	0.03680	0.32606	0.00441
0.9	0.07588	0.67800	0	0.02746	0.38345	0.00392

Table(3): Absolute error in approximate solutions of Eqns. (10) and (14).

8 Conclusions

1. The second kind Chebyshev wavelet approximations of Theorem 4.1 and 4.2 are given by $E^{(1)}_{\lambda^{k+1}-1,M}(f) = O\left(\frac{1}{\lambda^{(k+1)\alpha}\sqrt{M}}\right) \to 0$ as $k \to \infty$ and $M \to \infty$;
and $E^{(2)}_{\lambda^{k+1}-1,M}(f) = O\left(\frac{1}{\lambda^{k+1}\sqrt{M}}\phi\left(\frac{1}{\lambda^{k+1}}\right)\right) \to 0$ as $k \to \infty$ and $M \to \infty$.
Therefore, estimators $E^{(1)}_{\lambda^{k+1}-1,M}(f)$ and $E^{(2)}_{\lambda^{k+1}-1,M}(f)$ of the solution function of the Abel's integral equations are best possible in wavelet analysis.

2. From the tables 1, 2, 3 and figures 1, 2, it is observed that the solutions of the Abel's integral equations by the second kind Chebyshev wavelet method coincide with their exact solution as compare to Legendre and first kind Chebyshev wavelet method. This shows the validity and applicability of the proposed algorithm.

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References

[1] Chui, C.K.: An Introduction to Wavelets (Wavelet Analysis and its Applications), Vol.1, Academic Press Cambridge (1992).

[2] Debnath, L.: Wavelet Transforms and Their Applications, Birkhäuser, Boston (2002).

[3] Kumar, S., et al.: Numerical solutions for the reaction-diffusion, diffusion-wave, and Cattaneo equations using a new operational matrix for the Caputo-Fabrizio derivative. Mathematical Methods in the Applied Sciences, 43(15), pp.8595-8607, (2020).

[4] Kumar, S.: Linear and non-linear wavelet approximations of functions of Lipschitz class and related classes using the Haar wavelet series, Journal of Ramanujan Society of Mathematics and Mathematical Sciences, Vol. 10, pp. 161-176, (2023).

[5] Lal, S., Kumari, P.; Approximation of a function f of Generalized Lipschitz Class by its Extended Legendre Wavelet Series, Int. J. Comput. Math., 4:147, (2018).

[6] Lal, S., Sharma, R.; Approximation of function belonging to generalized Hölder's class by first and second kind Chebyshev wavelets and their applications in the solutions of Abel's integral equations, Arabian Journal of Mathematics, Vol. 10, pp. 157-174, (2020).

[7] Lal, S.; Yadav, H.C.: Approximation of functions belonging to Hölder's class and solution of Lane-Emden differential equation using Gegenbauer wavelets. Filomat, Vol. 37(12), pp. 4029-4045, (2023).

[8] Lal, S.; Yadav, H.C.: Approximation in Hölder's class and solution of Bessel's differential equations by extended Haar wavelet. Poincare J. Anal. Appl., Vol. 10(1), pp. 29-53, (2023).

[9] Lal, S. et al.: Error bounds of a function related to generalized Lipschitz class via the pseudo-Chebyshev wavelet and its applications in the approximation of functions, Carpathian Math. Publ., Vol. 14 (1), 29-48, (2022).

[10] Lepik, $\ddot{U}$; Application of Haar wavelet transform to solving integral and differential equations, Proc. Estonian Acad. Sci. Phys. Math., Vol.56, 28-46, (2007).

[11] Meyer, Y.: Wavelets their post and their future, Meyer, Y., Roques, S. (eds.); Progress in Wavelet Analysis and Applications (Toulouse,1992), pp. 9-18. Frontiers, Gif-sur-Yvette, (1993).

[12] Pandey, P.; Aguilar, J.F.C.: On solution of a class of nonlinear variable order fractional reaction–diffusion equation with Mittag–Leffler kernel, Numerical Methods for Partial Differential Equations, Vol. 37(2), pp. 998-1011, (2021).

[13] Pandey, P. et al.: Two-dimensional nonlinear time fractional reaction–diffusion equation in application to sub-diffusion process of the multicomponent fluid in porous media, Meccanica, Vol. 56, pp. 95-115, (2021).

[14] Pandey, P.; Singh, J.: An efficient computational approach for nonlinear variable order fuzzy fractional partial differential equations, Computational and Applied Mathematics, Vol. 41, pp. 1-21, (2022).

[15] Pandey, P. et al.: Approximate analytical solution of two-dimensional space-time fractional diffusion equation, Mathematical Methods in the Applied Sciences, Vol. 43(12), pp. 7194-7207, (2020).

[16] Pandey, P. et al.: An efficient technique for solving the space-time fractional reaction-diffusion equation in porous media. Chinese Journal of Physics, 68, pp.483-492, (2020).

[17] Ray, S., Sahu, P.K.; Numerical Methods for Solving Fredholm Integral Equations of Second Kind, Hindawi Publishing Corporation, Article ID 426916, (2013).

[18] Sripathy, B., Vijayaraju, P., Hariharan, G.; Chebyshev wavelet based approximation method to some non-linear differential equations arisingin engineering, Int. J. Math. Anal., 9(20), 993-1010, (2015).

[19] Titchmarsh, E.C.; The Theory of functions, 2nd edn. Oxford University Press, Oxford, (1939).

[20] Zygmund A.; Trigonometric Series, Vol. 1, Cambridge University Press, Cambridge, (1959).