CONTRIBUTIONS TO THE MATHEMATICAL STUDY OF SOME PROBLEMS IN MAGNETOHYDRODYNAMICS AND INDUCTION HEATING

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PhD Dissertation

October 13, 2008

Agradecementos

Ante todo, quero agradecer aos meus dous directores, Alfredo Bermúdez e Rafael Muñoz, toda a axuda que me prestaron para poder chegar a finalizar este traballo. A Alfredo téñolle que agradecer a confianza depositada en min dende o primeiro momento, introducíndome no mundo da Matemática Aplicada e da simulación numérica. Del partiron a maioría das ideas que foron aparecendo no desenvolvemento desta tese. A Rafa quérolle agradecer a aprendizaxe (e o repaso) durante os últimos anos de tantas ferramentas matemáticas que non tería comprendido sen a súa axuda. Ademais diso, a súa dedicación e paciencia para revisar cada nova versión do texto foron imprescindibles para mellorar o resultado final.

Esta memoria é o resultado da colaboración entre o Departamento de Matemática Aplicada e a empresa interesada na simulación do forno de inducción. Quixera agradecer a dita empresa o seu financiamento e a posibilidade de utilizar para esta memoria simulacións correspondentes a casos reais. Tamén quero agradecer a Javier B. a súa decidida aposta pola investigación, axudando a introducir aos recén licenciados en temas punteiros de investigación. Quixera lembrar tamén ao resto de membros da empresa que nos recibiron nas visitas á fábrica: Antonio, Alejandro, Álvaro e Jose Manuel, co que mantiven tantas charlas por teléfono intentando unir a simulación numérica co mundo real.

Unha parte desta tese foi realizada durante dúas estadías no Departamento de Matemáticas da Universidade de Trento. Para min foi un pracer poder traballar durante ese tempo cos profesores Alberto Valli e Ana Alonso. Á parte de aprender deles gran cantidade de cousas, disfrutei moito coa súa maneira tan distendida (pero rigorosa) de traballar.

Tamén foron imprescindibles para a realización deste traballo as melloras introducidas no programa durante a estadía no Departamento de Análisis Matemático da Universidad de Málaga, baixo a supervisión dos profesores Carlos Parés e Manolo Castro. Quixera agradecer a eles dous e a todo o seu grupo a boa acollida que me deron e o bo ambiente de traballo que conseguiron crear. En particular, quero lembrar aos bolseiros Alberto e Juan Antonio, porque o pasei realmente ben con eles dous.

Non podo esquecer nos agradecementos a toda a xente do Departamento de Matemática Aplicada coa que fun coincidindo durante este tempo. Dende os primeiros anos, escribindo o DEA e facendo simulacións con Fluent, ata os últimos anos impartindo docencia, e polo medio unha chea de burocracia e incidencias informáticas. Non podería ter feito tantas cousas de estar eu só. Quero agradecer particularmente o apoio e a paciencia de todo o grupo de electromagnetismo: Dolores, Fran, Mari Carmen e sobre todo Pili, que me resolveu un bo número de dúbidas e problemas durante estes anos.

E por suposto, quérolle dar as grazas a todos os bolseiros que foron pasando durante estes anos (seis!) pola facultade, e cos que fun compartindo a aula de bolseiros, as manías, os cortes de luz, os doces de cada viaxe, as ceas por calquera motivo (calquera vale), e en xeral unha chea de momentos de agobio e de relax. Polo tempo e as cousas que pasei con eles, quero lembrar aquí a Cris (la mamma), Tere, Andrés e ás novas xeracións, Marta e Laura, que ademais foron as únicas que pediron aparecer nos agradecementos. Dentro de nada xa sodes as veteranas, así que a comportarse.

Por último, quixera lembrar á miña familia pola paciencia que me tiveron estes anos, nos que non souben explicar ben o que facía no traballo (agora o vedes), e tamén pola gran axuda que foi telos sempre por aí. Isto non tería saído ben sen vós. Moitísimas grazas.

Este traballo foi financiado polo antigo Ministerio de Educación y Ciencia a través dunha beca FPI con referencia BES-2004-3283.

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Introduction

Induction furnaces are widely used by metallurgical industry in several processes involving different materials, such as metal hardening, casting or melting. Basically, these furnaces consist of an electrically conducting inductor and a workpiece to be heated. An alternating current traversing the conductor generates induction currents in the workpiece, which is then heated due to Ohmic losses.

An efficient design of the induction furnace must take into account several parameters that can affect its performance. These parameters include the geometrical properties, such as the size and shape of the workpiece or its distance to the coil, the thermal and electromagnetic properties of the materials in the furnace, and the properties of the alternating electric current supplied to the coil. Numerical simulation is a very helpful tool to understand the influence of these parameters in the behaviour of the furnace. It is also useful to test modifications on the furnace avoiding the cost of real experiments.

The motivation of this work is the implementation of a computer code to simulate an induction melting furnace. To this end an adequate mathematical model must be introduced and mathematically analyzed. Due to the physical processes occurring in the furnace this model must take into account thermal, electromagnetic and hydrodynamic phenomena. This leads to a coupled mathematical model consisting of a system of partial differential equations with several non-linearities. As we will see, the coupling and the nonlinear terms introduce some difficulties, not only in the mathematical analysis of the equations but also in its numerical solving.

The outline of the thesis is the following:

In *Chapter 1* we give a description of the induction furnace we want to simulate and explain the involved physical phenomena. In this chapter we also introduce the mathematical models describing the physics of the problem. However, these models will be modified in other chapters, depending on whether we are interested in their analytical study or in their numerical approximation.

Chapter 2 is devoted to the analytical study of two mathematical models concerning a stationary problem in magnetohydrodynamics, including Joule effect and viscous heating. The first one uses the Boussinesq approximation and extends the results of [72] by including quadratic terms in the heat sources. The second model is developed following an idea of [42] and permits us to prove the existence of solution to the problem under less restrictive conditions.

In *Chapter 3* we come back to the mathematical model of the furnace which is used for the numerical simulation. Since the thermal and hydrodynamic submodels are very well known, we focus our interest in the formulation of the electromagnetic model. The problem is formulated in an axisymmetrical setting, hence substituting the helical coil by torus-shaped rings. In order to use either the voltage or the current intensity as the given data, we introduce an hybrid formulation in terms of the magnetic vector potential with the current voltages acting as Lagrange multipliers. Finally, we give an integro-differential formulation of the electromagnetic problem with a mixed boundary element/finite element discretization in view.

In the first section of *Chapter 4* we explain the development of the computer code for the numerical simulation of the furnace. The thermal and hydrodynamic problems are discretized using Lagrange-Galerkin methods, in order to treat the convection terms of the equations, whereas the electromagnetic problem is solved using a mixed boundary element/finite element method. We also present the iterative algorithm designed to deal with the nonlinearities of the problem. In the second section we present the results of some simulations. The first one is a simulation of a thermal-hydrodynamic problem with phase change, which has been used for validation of the code. The second is a simulation of a real industrial furnace designed for induction melting and stirring.

Finally, in *Chapter 5* we present a different formulation of the electromagnetic model. This formulation is intended to be used in full three-dimensional simulations of the furnace, thus maintaining the helical geometry of the coil. The problem is written in terms of the electric field in the conductor, and the magnetic field in the insulators. Either the current intensity or the voltage can be given as data, in the latter case with the intensities acting as Lagrange multipliers. After the analysis of the continuous formulation we present its finite element discretization and numerical analysis. Then, two different problems are solved using a computer code written in MATLAB.

The thesis also includes three appendices. In *Appendix A* we introduce some functional spaces and notations that appear throughout the thesis. We also present some known results that will be used in the mathematical analysis of some problems. In *Appendix B* we review the concept of solution by transposition in the sense given by Stampacchia in [94], as it is going to be used in *Chapter 2*. Finally, *Appendix C* is devoted to recall the cylindrical coordinates system, in order to make easier the reading of *Chapter 3*.

Chapter 1

Motivation. The physical problem.

1.1 Motivation.

Induction heating is a non-contact method by which conducting materials subjected to an alternating electromagnetic field are heated. This method has been widely used in the last years in many areas of metallurgical and semiconductor industry, such as metal hardening ([34, 95]), casting ([17]), melting([35, 39, 88]) and crystal growing ([37, 66]).

An induction heating system consists of one or several inductors supplied with alternating electrical current and a conductive workpiece to be heated. The alternating current traversing the inductor generates eddy currents in the workpiece, and through the Ohmic losses the workpiece is heated (see Figure 1.1).

Figure 1.1: A simple induction heating device.

Different furnaces can be designed depending on its application. In our case we want to simulate the behaviour of a coreless induction furnace designed for melting and stirring. A simple sketch of this furnace is presented in Figure 1.2. It consists of an inductor coil and a workpiece, which is formed by the crucible and the load within:

- The inductor is a helical copper coil, connected to a power supply and carrying an alternating electrical current. Since Ohmic losses also take place in the coil, it is water-cooled to avoid overheating.

- The load is the material we want to melt, which is usually an electrically conducting metal.

- The crucible is usually a refractory material designed to contain the material to melt and to resist very high temperatures.

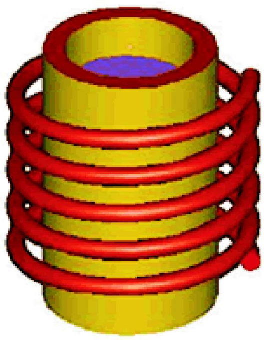

Figure 1.2: Sketch of the induction furnace.

It is well known that, due to the skin effect, the Ohmic losses are concentrated in the external part of the workpiece, and this concentration is more important at high frequencies. In induction heating it is crucial to control the distribution of Ohmic losses, since they could cause very high temperatures in the crucible and the load which could harm the crucible, thus reducing its lifetime. Moreover, the frequency and intensity of the alternating current also affects the stirring of the molten bath. Since the stirring will determine some of the properties of the final product, it is convenient to know accurately the influence of these parameters to achieve the desired stirring.

To control the stirring of the bath and the temperature profile in the furnace, it is possible to adjust the power supplied to the furnace and the frequency of the alternating current traversing the coil. Moreover, we are also interested in understanding the influence on the furnace performance of certain geometrical parameters such as the crucible thickness or its distance to the coil, or physical parameters, such as the thermal and electrical conductivity of the refractory layer. Numerical simulation can be a good tool for this purpose, since it permits to introduce these changes in the simulation avoiding the high cost of experimentation with real processes. Many papers have been published concerning the numerical simulation of induction heating devices, from some pioneering articles published in the early eighties (see [68] and references therein) to more recent works dealing with different coupled problems, such as the thermoelectrical problem appearing in induction heating ([35, 88]), the magnetohydrodynamic problem related to induction stirring ([78]) and also a thermal-magneto-hydrodynamic problem ([57, 64]) but not fully coupled because material properties are not supposed to depend on temperature. Some other related works include mechanical effects in the workpiece ([12, 61]). A more extensive bibliographic review can be found in [69].

1.2 The mathematical model.

In this section we introduce the mathematical model for the behaviour of an induction furnace. The physical problem involves thermal, electromagnetic and hydrodynamic phenomena in the molten region of the workpiece. Thus, the three submodels and the couplings among them must be considered to achieve a realistic simulation of the furnace.

The electromagnetic submodel consists of Maxwell equations, and is described in Section 1.2.1. In these equations velocity appears in Ohm's law, and some electromagnetic properties may depend on temperature.

The thermal submodel is presented in Section 1.2.2. We use a formulation in terms of enthalpy, to take into account the change of state during melting. The thermal model is coupled with the electromagnetic one, since the Joule effect is one of the source terms in the heat transfer equation. Moreover, velocity is needed if we want to consider convective heat transfer.

The hydrodynamic submodel is described in Section 1.2.3. It consists of the compressible Navier-Stokes equations, including buoyancy forces and Lorentz forces, which couple the hydrodynamic model with the two other submodels. Furthermore, the physical properties of the fluid can depend on temperature.

Throughout this chapter we do not mention either the domain of our problem or the boundary conditions. These conditions will be made precise in the two following chapters and may change depending on whether we are interested in mathematical analysis or in numerical simulation.

1.2.1 Electromagnetic submodel.

To model the electric current traversing the coil and the eddy currents inducted in the workpiece by this current, we introduce an electromagnetic model which is described by Maxwell equations

$$\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} - \mathbf{curl}\,\boldsymbol{\mathcal{H}} = -\boldsymbol{\mathcal{J}}, \tag{1.1}$$

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} = \mathbf{0}, \tag{1.2}$$

$$\operatorname{div} \boldsymbol{\mathcal{B}} = 0, \tag{1.3}$$

$$\operatorname{div} \boldsymbol{\mathcal{D}} = q, \tag{1.4}$$

where $\boldsymbol{\mathcal{D}}$ denotes the electric displacement, $\boldsymbol{\mathcal{H}}$ is the magnetic field, $\boldsymbol{\mathcal{J}}$ is the current density, $\boldsymbol{\mathcal{B}}$ is the magnetic induction, $\boldsymbol{\mathcal{E}}$ is the electric field and q is the electric charge density.

These equations must be completed with some constitutive laws,

$$\boldsymbol{\mathcal{B}} = \mu \boldsymbol{\mathcal{H}}, \tag{1.5}$$

$$\boldsymbol{\mathcal{D}} = \varepsilon \boldsymbol{\mathcal{E}}, \tag{1.6}$$

and (Ohm's law),

$$\boldsymbol{\mathcal{J}} = \sigma(\boldsymbol{\mathcal{E}} + \mathbf{u} \times \boldsymbol{\mathcal{B}}), \tag{1.7}$$

where μ is the magnetic permeability, ε denotes the electric permittivity, σ is the electrical conductivity and $\mathbf{u}$ is the velocity field. We will only consider isotropic materials, in which case μ, ε and σ are bounded scalar functions of the spatial variable $\mathbf{x}$, and we will allow them to depend on temperature T. Moreover, both ε and μ are strictly positive while σ is assumed to be strictly positive in conductors and to be null in dielectrics.

In many physical applications, such as those involving molten metals, the size of the term involving the displacement current, $\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t}$, is negligible with respect to the size of the other two terms appearing in Maxwell-Ampère law (1.1) (see [77, p. 8]). By neglecting this term and leaving aside equation (1.4) that will only be useful to compute the electric charge q, we obtain the so-called eddy currents model

$$\mathbf{curl}\,\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}, \tag{1.8}$$
$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} = \mathbf{0}, \tag{1.9}$$
$$\operatorname{div} \boldsymbol{\mathcal{B}} = 0, \tag{1.10}$$

that must be completed with equations (1.5) and (1.7).

In magnetohydrodynamics (MHD), it is very useful to consider magnetic induction, $\boldsymbol{\mathcal{B}}$, as the main unknown and to compute all the other electromagnetic quantities by using Maxwell equations and Ohm's law. To do this, we assume that there is only one conductor, *i.e.* $\sigma > 0$, and that magnetic permeability is constant. We replace $\boldsymbol{\mathcal{J}}$ in (1.1) by its value given in (1.7) and, by considering the constitutive law (1.5) we obtain

$$\boldsymbol{\mathcal{E}} = \frac{1}{\mu\sigma}\mathbf{curl}\,\boldsymbol{\mathcal{B}} - \mathbf{u} \times \boldsymbol{\mathcal{B}}. \tag{1.11}$$

Now, using this new expression for $\boldsymbol{\mathcal{E}}$ in Faraday's law (1.2), we obtain a formulation of the electromagnetic problem in terms of the magnetic induction $\boldsymbol{\mathcal{B}}$ which is usually called the induction equation:

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathbf{curl}\left(\frac{1}{\mu\sigma}\mathbf{curl}\,\boldsymbol{\mathcal{B}}\right) - \mathbf{curl}\,(\mathbf{u} \times \boldsymbol{\mathcal{B}}) = \mathbf{0}. \tag{1.12}$$

The quasi-static time-harmonic model.

Next we are going to introduce some simplifications in the electromagnetic submodel, to attain the equations that will be used for the numerical simulation.

The term $\mathbf{curl}\,(\mathbf{u} \times \boldsymbol{\mathcal{B}})$ in equation (1.12) represents the influence of the fluid motion on the magnetic field. This term is important in certain applications related to MHD, such as astrophysics and plasma confinement. However, when working with molten metals in a laboratory scale this term is negligible in comparison with the two other terms in the induction equation [77, p. 30]. This leads us to neglect velocity in Ohm's law that simply becomes

$$\boldsymbol{\mathcal{J}} = \sigma\boldsymbol{\mathcal{E}}\,. \tag{1.13}$$

Moreover, since the induction furnace works with an alternating current, we can consider that all fields vary harmonically with time. Hence they all have the form

$$\boldsymbol{\mathcal{F}}(\mathbf{x}, t) = \mathrm{Re}\,[\mathrm{e}^{\mathrm{i}\omega t}\,\mathbf{F}(\mathbf{x})], \tag{1.14}$$

where $\mathbf{F}$ is the complex amplitude corresponding to the $\boldsymbol{\mathcal{F}}$ field and ω is the angular frequency, $\omega = 2\pi f$, f being the frequency of the alternating current.

Under these assumptions we arrive at the so-called eddy-currents model in the frequency domain

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{1.15}$$
$$\mathrm{i}\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{1.16}$$
$$\mathrm{div}\,\mathbf{B} = 0, \tag{1.17}$$

with the constitutive laws

$$\mathbf{B} = \mu\mathbf{H}, \tag{1.18}$$
$$\mathbf{J} = \sigma\mathbf{E}. \tag{1.19}$$

1.2.2 Thermal submodel.

The induction furnace we are considering is used to melt and stir a metal. To model the behaviour of the molten metal, we will use the convective heat transfer equation. Moreover, since we are interested in how the material melts and solidifies, the standard heat equation is modified by introducing an enthalpy formulation allowing us to consider changes of state.

We consider the convective heat transfer equation in terms of the enthalpy

$$\frac{\partial e}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,e - \mathrm{div}\,(k\mathbf{grad}\,T) = \frac{|\boldsymbol{\mathcal{J}}|^2}{\sigma} + \frac{\eta}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2 + \psi, \tag{1.20}$$

e being the enthalpy density, T the temperature and k the thermal conductivity which is supposed to be a scalar function depending on position $\mathbf{x}$ and temperature T. The first term in the right-hand side represents the Joule effect whereas the second one accounts for viscous heating, η being the dynamic viscosity and $\mathbf{u}$ the velocity field. The third term, ψ, is some given heat source as for instance radiative heat transfer.

The enthalpy density e can be expressed in terms of the temperature by means of a multi-valued function:

$$e(\mathbf{x}, t) \in \mathcal{H}(\mathbf{x}, T(\mathbf{x}, t)), \tag{1.21}$$

where $\mathcal{H}(\mathbf{x}, T)$ is given by

$$\mathcal{H}(\mathbf{x}, T) = \begin{cases} \displaystyle\int_0^T \rho(\mathbf{x}, s)\, c(\mathbf{x}, s)\, ds, \quad T < T_S(\mathbf{x}), \\ \displaystyle\left[\int_0^T \rho(\mathbf{x}, s)\, c(\mathbf{x}, s)\, ds, \int_0^T \rho(\mathbf{x}, s)\, c(\mathbf{x}, s)\, ds + \rho(\mathbf{x}, T_S)\, L(\mathbf{x})\right], \; T = T_S(\mathbf{x}), \\ \displaystyle\int_0^T \rho(\mathbf{x}, s)\, c(\mathbf{x}, s)\, ds + \rho(\mathbf{x}, T_S)\, L(\mathbf{x}), \quad T > T_S(\mathbf{x}), \end{cases} \tag{1.22}$$

L being the latent heat, *i.e.*, the heat per unit mass necessary to achieve the change of state at temperature T_S, ρ being the mass density and c the specific heat.

1.2.3 Hydrodynamic submodel.

In a melting induction furnace, the movement of the molten region strongly influences the distribution of the temperature, due to convective heat transfer. Furthermore, in certain industrial applications related to MHD, such as electromagnetic stirring, the objective is to control the fluid motion via the electromagnetic forces, to attain certain qualities or patterns in the way the material solidifies. Thus, a hydrodynamic submodel is crucial to understand the behaviour of the metal in the furnace.

The movement of the fluid is governed by the motion equation, which reads as follows

$$\rho(\mathbf{x},T)\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{grad}\,\mathbf{u})\mathbf{u}\right)-\mathrm{div}\,(l(D))+\mathbf{grad}\,p \quad = \quad \mathbf{f}_0+\boldsymbol{\mathcal{J}}\times\boldsymbol{\mathcal{B}}+\rho\mathbf{g}\,, \tag{1.23}$$

where $\mathbf{u}$ is the velocity field, p is the pressure and ρ denotes mass density, which is supposed to be a function of the spatial coordinate $\mathbf{x}$ and the temperature T. The term $\boldsymbol{\mathcal{J}}\times\boldsymbol{\mathcal{B}}$ represents the Lorentz force, $\rho\mathbf{g}$ represents the buoyancy forces, and $\mathbf{f}_0$ is some other given force, e.g., a Coriolis force. Moreover, $l(D)$ denotes the viscous part of Cauchy stress tensor. In Newtonian fluids this is given by

$$l(D)=2\eta(\mathbf{x},T)D+\xi(\mathbf{x},T)tr(D)I\,, \tag{1.24}$$

where $D\equiv D(\mathbf{u})$ is the symmetric part of Cauchy stress tensor, and the physical parameters η and ξ are the dynamic viscosity and the second viscosity coefficient, respectively.

Boussinesq approximation.

A usual simplification of thermal-hydrodynamic processes is the Boussinesq approximation. It consists on assuming that the variations of the thermodynamical coefficients, such as dynamic viscosity, specific heat and thermal conductivity, are negligible so they can be taken as constant in equations (1.23) and (1.20). Mass density ρ is supposed to be constant except for the buoyancy force term where the density is assumed to depend linearly on temperature. More precisely, rewriting the heat equation in terms of the temperature, the Boussinesq approximation, in our particular case, consists of the equations

$$\rho_0 c_0\left(\frac{\partial T}{\partial t}+\mathbf{u}\cdot\mathbf{grad}\,T\right)-k_0\Delta T \quad = \quad \frac{|\boldsymbol{\mathcal{J}}|^2}{\sigma}+\frac{\eta_0}{2}\left|\mathbf{grad}\,\mathbf{u}+\mathbf{grad}\,\mathbf{u}^t\right|^2+\psi, \tag{1.25}$$

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{grad}\,\mathbf{u})\mathbf{u}\right)-\eta_0\Delta\mathbf{u}+\mathbf{grad}\,p' \quad = \quad \boldsymbol{\mathcal{J}}\times\boldsymbol{\mathcal{B}}-\rho_0\beta_0(T-T_r)\mathbf{g}, \tag{1.26}$$

$$\mathrm{div}\,\mathbf{u} \quad = \quad 0, \tag{1.27}$$

where T_r is a constant reference temperature, ρ_0, η_0, k_0 and c_0 denote the physical properties at the reference temperature T_r, and β_0 is the coefficient of thermal expansion at temperature T_r. Moreover, p' represents the corrected pressure, which is given by $p'=p-\rho_0\mathbf{g}\cdot\mathbf{x}$.

1.2.4 Coupled models. The system of MHD equations.

Once we have introduced the equations for the three submodels, we are in a position to join them together, and present the system of MHD equations. As usual in MHD, we write the electromagnetic problem by means of the induction equation. Moreover, to avoid the use of other electromagnetic properties, we write the Joule effect in the thermal equation in terms of $\boldsymbol{\mathcal{B}}$, by using (1.8) and (1.5):

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathbf{curl}\left(\frac{1}{\mu\sigma}\mathbf{curl}\,\boldsymbol{\mathcal{B}}\right) - \mathbf{curl}\,(\mathbf{u}\times\boldsymbol{\mathcal{B}}) = \mathbf{0}, \tag{1.28}$$

$$\operatorname{div}\boldsymbol{\mathcal{B}} = 0, \tag{1.29}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{grad}\,\mathbf{u})\mathbf{u}\right) - \operatorname{div}\left(2\eta D(\mathbf{u})\right) + \mathbf{grad}\,p' = \boldsymbol{\mathcal{J}}\times\boldsymbol{\mathcal{B}} + \rho\mathbf{g}, \tag{1.30}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.31}$$

$$\frac{\partial e}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,e - \operatorname{div}\left(k\,\mathbf{grad}\,T\right) = \frac{1}{\sigma\mu^2}|\mathbf{curl}\,\boldsymbol{\mathcal{B}}|^2 + \frac{\eta}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2 + \psi, \tag{1.32}$$

$$e(\mathbf{x},t) \in \mathcal{H}(\mathbf{x}, T(\mathbf{x},t))\,. \tag{1.33}$$

In steady state, by writing the heat equation in terms of the temperature and considering the Boussinesq approximation, the MHD system becomes

$$\mathbf{curl}\left(\frac{1}{\mu\sigma}\mathbf{curl}\,\boldsymbol{\mathcal{B}}\right) - \mathbf{curl}\,(\mathbf{u}\times\boldsymbol{\mathcal{B}}) = \mathbf{0}, \tag{1.34}$$

$$\operatorname{div}\boldsymbol{\mathcal{B}} = 0, \tag{1.35}$$

$$\rho_0(\mathbf{grad}\,\mathbf{u})\mathbf{u} - \eta_0\Delta\mathbf{u} + \mathbf{grad}\,p' = \boldsymbol{\mathcal{J}}\times\boldsymbol{\mathcal{B}} - \rho_0\beta_0(T-T_r)\mathbf{g}, \tag{1.36}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.37}$$

$$-k_0\Delta T + \rho_0 c_0\mathbf{u}\cdot\mathbf{grad}\,T = \frac{1}{\sigma\mu^2}|\mathbf{curl}\,\boldsymbol{\mathcal{B}}|^2 + \frac{\eta_0}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2 + \psi. \tag{1.38}$$

We notice that this system of equations corresponds to a steady problem for the thermal and the hydrodynamic models, and a magnetostatic problem as the electromagnetic model. Thus, when using this model, induction effects cannot be taken into account as we are neglecting the term $\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}$.

When working with alternating currents, it is also possible to write a stationary version of the problem taking into account the induction effects. To do so, one assumes that the electromagnetic fields are harmonic and have the form (1.14), and that the influence of the fluid motion on the magnetic field is negligible. Since the electromagnetic fields vary in a time scale much smaller than the one for the variation of temperature and velocity, it is possible to achieve a steady state for these two fields, whereas the electromagnetic fields remain harmonic. In this case, the electromagnetic model is written in its quasi-static time-harmonic version, given by equations (1.15)-(1.19). The thermal and hydrodynamic problems are given by equations (1.36)-(1.38), but the Joule effect term, $\frac{1}{\sigma\mu^2}|\mathbf{curl}\,\boldsymbol{\mathcal{B}}|^2$, and the Lorentz force $\boldsymbol{\mathcal{J}}\times\boldsymbol{\mathcal{B}}$ are replaced by their mean values on a cycle (see Remark 3.6), namely,

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{1}{\sigma\mu^2}|\mathbf{curl}\,\boldsymbol{\mathcal{B}}|^2\,,$$

and

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \boldsymbol{\mathcal{J}} \times \boldsymbol{\mathcal{B}}\,.$$

In fact, this time harmonic approximation is used in Chapters 3 and 4 for the numerical simulation of the induction furnace. The mathematical analysis of the steady MHD system of equations given by (1.34)-(1.38) is carried out in Chapter 2. Our decision to analyze this problem, instead of the one used for the numerical simulation, is that they share many of their features and difficulties, such as the Joule effect and other coupling terms, and at the same time it avoids the change of state, that would lead to a varying domain in the equations of the hydrodynamic model.

Chapter 2

Analysis of two stationary MHD systems of equations.

This chapter is devoted to the analysis of two systems of stationary partial differential equations in magnetohydrodynamics. In Section 2.1 we present a brief bibliographic review of some works dealing with problems similar to ours. In Section 2.2 we analyze the stationary system of equations given by (1.34)-(1.38), which has been obtained by considering the Boussinesq approximation. For this model we prove the existence of solution under some smallness conditions on the boundary and source data. Section 2.3 is devoted to the study of a quite similar model that does not come from Boussinesq approximation. Instead of considering a linear approximation of the buoyancy force term, as it is done in the Boussinesq approximation, we maintain its original expression assuming some conditions on the density function. In this case it is possible to find an *a priori* bound for the solution, and also to prove the existence of solution independently of the data size. Finally, in Section 2.4 we present some results of uniqueness for both models. For the first one we can only prove the uniqueness of solution in a small closed ball, under some hypotheses of small data. For the second one we present a criterion of small data which guarantees the global uniqueness of solution.

2.1 Bibliographic review.

In this section we present a brief review of several articles concerning the mathematical study of some problems related to ours. The analytical study of a fully coupled thermal-magneto-hydrodynamic problem is rather unusual, and not many works can be found in the subject. Thus, we will also mention other interesting papers dealing with some of the couplings.

Articles on the thermoelectrical problem.

In the study of the thermoelectrical coupling the main difficult is the treatment of the Joule effect in the heat equation, due to its quadratic nature. This coupling can be found in several articles devoted to the study of the so-called *thermistor problem*, in which the electromagnetic phenomena are described by the equation of electrostatics, usually considering that the electrical conductivity is temperature dependent. The study of the stationary thermistor problem is carried out by Howison, Rodrigues and Shillor in [62] (see also references therein). A deeper review on this subject can be found in the PhD thesis of Pena [85].

Another problem, very related to the thermistor, is the *in situ vitrification problem*. The stationary formulation is studied by Gariepy, Shillor and Xu in [47], whereas in [96] Xu studies the coupling of the transient heat equation, including phase change, with the equation of electrostatics, considering temperature dependent properties.

Concerning induction heating the stationary problem is treated by Bermúdez and Muñoz in [18] by considering cylindrical symmetry and seeking a solution in certain weighted Sobolev spaces. The transient thermoelectrical problem including Joule effect is studied by Bossavit and Rodrigues in [28]. In that paper, both thermal and electrical conductivity are assumed to be inhomogeneous and temperature dependent. In [19] we find a similar problem, but in this case with an enthalpy formulation of the thermal problem, taking the enthalpy as a multivalued function of temperature, and assuming different dependencies (on spatial coordinates, on temperature or both) for each physical property.

Finally, we also mention among these works the one of Hömberg [61], which considers a transient model which couples a thermoelectrical problem with a mechanical one. The model includes quadratic heat sources arising from Joule heating.

Articles on the thermal-hydrodynamic problem.

The study of the thermal-hydrodynamic coupling has been considered by many authors in a vast number of papers. We just mention here some papers dealing with the Boussinesq approximation that we have found interesting.

The stationary version of the Boussinesq approximation is treated by Boland and Layton in [26]. The authors prove the existence of solution and its uniqueness under small data, using similar techniques to the ones appearing in the study of Navier-Stokes equations. Moreover, they prove there is an open and dense set of sources such that the number of solutions is finite. The paper also includes the numerical analysis of a finite element method for the discretization of the problem.

Concerning the transient model, existence and uniqueness results for different kinds of solution related to semigroups theory are presented by Hishida in [60] and Kagei in [63]. The analysis of a generalized version of Boussinesq approximation can be found in the work by Díaz and Galiano [41]. The authors prove the existence of weak solution, which is seen to be unique in the two-dimensional case.

We also mention here the work of Díaz, Rakotoson and Schmidt [42], which treats a particular case of a transient thermal-hydrodynamic problem. The authors assume all physical properties are constant, except for density in the buoyancy force term, which is considered to be strictly positive and non-increasing. The authors state an existence result, but the proof is left for a forthcoming paper.

Articles on the magnetohydrodynamic problem.

Concerning the magnetohydrodynamic problem, without considering the heat transfer equation, Gunzburger, Meir and Peterson [55] prove the existence and uniqueness of solution to the steady state equations under smallness conditions on the data. The paper also includes the numerical analysis of a finite element method. The result is improved by Alekseev in [2] for the case of a tangential velocity boundary condition, in which case the smallness of the data is no longer needed.

Meir and Schmidt in [73] deal with a magnetohydrodynamic problem where the electromagnetic phenomena are considered in the whole space. They propose two different mixed formulations: the first one in terms of velocity and magnetic induction, and the second one in terms of velocity and current density, which leads to an integro-differential formulation of the problem.

Several works on the subject have been published by Gerbeau and coauthors. The study of the time-dependent problem can be seen in [50]. A further review on this subject can also be found in this book. In [48] they propose a transient model with density dependent parameters. In [49] they couple the transient hydrodynamic equations with a steady electromagnetic model, and prove existence and uniqueness for small data and small time interval.

Finally, Rappaz and Touzani analyze in [89] a particular problem, coupling the steady incompressible Navier-Stokes equations with the time harmonic eddy currents model in a particular two-dimensional setting. The numerical analysis of a numerical method for solving the problem is done in [90].

Articles on the thermal-magneto-hydrodynamic problem.

As we said before, not many papers are devoted to the mathematical analysis of a fully coupled thermal-magneto-hydrodynamic problem.

Cimatti in [38] treats the stationary two-dimensional case. He assumes that velocity and magnetic field have the same direction, which permits him to reduce the problem to a much simpler formulation.

In [97] Xu and Shillor study a coupled thermal-magneto-hydrodynamic problem with phase change. In that paper the thermal and hydrodynamic models are written in its transient version, and the electromagnetic model is described by the equation of electrostatics with temperature dependent properties. Lorentz's force is neglected in this model.

In [72] we find the closest problem to the one we will study below. In that paper Meir treats the stationary problem (1.34)-(1.38), but neglecting viscous heating and Joule heating. Existence of solution is proved under certain restrictions on the boundary data, and uniqueness is proved under more stringent conditions. The paper also includes the numerical analysis of a finite element method for the discretization of the equations.

Finally, Meir and Schmidt propose in [74] the coupling of the mixed velocity-current density formulation (presented in [73]) with the heat transfer equation, but neglecting viscous and Joule heating. The authors prove the existence and uniqueness of solution under different smallness conditions of the source and boundary data.

2.2 Steady MHD equations using the Boussinesq approximation.

The first problem we analyze arises by considering that the domain is constituted by only one fluid and that, in system (1.34)-(1.38), all physical properties are independent of temperature except density in the buoyancy force which is treated using Boussinesq approximation. In this section we present an existence result, which extends the one proved by Meir in [72] by including L^1 sources in the heat equation, corresponding to viscous and Joule heating.

2.2.1 Equations and non-dimensionalization.

Let $\Omega \subset \mathbb{R}^3$ be either a bounded simply connected domain of class $\mathcal{C}^{1,1}$ or a Lipschitz polyhedron. Considering that all physical properties are constant in system (1.34)-(1.38) (unless density in the buoyancy force term, as it is given by Boussinesq approximation), we obtain the following system of equations which holds in Ω:

$$\frac{1}{\mu\sigma}\mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}) - \mathbf{curl}\,(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{2.1}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{2.2}$$

$$-\eta_0\Delta\mathbf{u} + \rho_0(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p' - \frac{1}{\mu}(\mathbf{curl}\,\mathbf{B})\times\mathbf{B} = \mathbf{f}_0 - \rho_0\beta_0(T - T_r)\mathbf{g}, \tag{2.3}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2.4}$$

$$-k_0\Delta T + \rho_0 c_{p0}\mathbf{u}\cdot\mathbf{grad}\,T = \frac{1}{\sigma\mu^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{\eta_0}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2 + \psi. \tag{2.5}$$

The notation is the one used throughout Chapter 1, except for the magnetic induction $\boldsymbol{\mathcal{B}}$ which is now denoted by $\mathbf{B}$. We notice, however, that in this chapter $\mathbf{B}$ is not the complex magnitude appearing in equation (1.14). From this point we will drop the subscript 0 for the physical properties at the reference temperature.

Let us now introduce some non-dimensional quantities,

$$\begin{aligned}
&\text{Hartmann number} && H_a = \mathcal{B}\mathcal{L}\left(\frac{\sigma}{\eta}\right)^{1/2},\\
&\text{interaction parameter} && N = \sigma\mathcal{B}^2\frac{\mathcal{L}}{\rho u},\\
&\text{Reynolds number} && R_e = \frac{\rho u\mathcal{L}}{\eta},\\
&\text{magnetic Reynolds number} && R_m = \mu\sigma u\mathcal{L},\\
&\text{Prandtl number} && P_r = \frac{\eta c_p}{k},\\
&\text{Grashof number} && G_r = \frac{\beta g\Delta\mathcal{T}\mathcal{L}^3}{\nu^2},\\
&\text{Eckert number} && E_c = \frac{u^2}{c_p\Delta\mathcal{T}}.
\end{aligned}$$

where $\mathcal{B}$, u, $\mathcal{L}$ and $\Delta\mathcal{T}$ are typical values of magnetic induction, velocity, length and temperature difference, respectively. Moreover, $\nu = \rho/\eta$ is the kinematic viscosity and $g = |\mathbf{g}|$ is the magnitude of gravity acceleration. See further details in [92], [72] and references therein.

We replace the temperature T by the temperature difference with respect to a reference temperature T_r. From now on this difference will be also denoted by T. Then, we normalize equations as follows: the magnetic induction $\mathbf{B}$ by $\mathcal{B}$, the velocity $\mathbf{u}$ by u, the (corrected) pressure p by $\sigma u \mathcal{B}^2 \mathcal{L}$, the body force $\mathbf{f}_0$ by $\sigma u \mathcal{B}^2$, the temperature difference T by $\Delta\mathcal{T}$, and the heat source ψ by $\rho c_p u \Delta\mathcal{T}/\mathcal{L}$ (see [72] and references therein). After this normalization we arrive at the following non-dimensionalized system of equations, which holds in Ω:

$$\frac{1}{R_m}\mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}) - \mathbf{curl}\,(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{2.6}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{2.7}$$

$$-\frac{1}{H_a^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p - \frac{1}{R_m}(\mathbf{curl}\,\mathbf{B})\times\mathbf{B} = \mathbf{f}_0 - \frac{G_r}{NR_e^2}\frac{\mathbf{g}}{g}T, \tag{2.8}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2.9}$$

$$-\frac{1}{P_r R_e}\Delta T + \mathbf{u}\cdot\mathbf{grad}\,T = \frac{E_c}{R_e}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2\right] + \psi. \tag{2.10}$$

We denote by $\partial\Omega$ the boundary of Ω, and by $\mathbf{n}$ the unit outward-pointing normal vector to $\partial\Omega$. For the hydrodynamic model we impose a Dirichlet boundary condition,

$$\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d\,, \tag{2.11}$$

where $\mathbf{u}_d$ is a given vector function on $\partial\Omega$.

For the electromagnetic model we impose the boundary conditions

$$(\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l, \tag{2.12}$$

$$\left[\left(\frac{1}{R_m}(\mathbf{curl}\,\mathbf{B}) - (\mathbf{u}\times\mathbf{B})\right)\times\mathbf{n}\right]_{|\partial\Omega} = \mathbf{k}, \tag{2.13}$$

where the second equation arises from a condition of the form $\mathbf{E}\times\mathbf{n} = \mathbf{k}$, after an appropriated non-dimensionalization. We notice that l and $\mathbf{k}$ must satisfy some compatibility conditions that will be detailed below. For the temperature, we also impose a Dirichlet boundary condition:

$$T_{|\partial\Omega} = T_d. \tag{2.14}$$

This problem, without the Joule heating and viscous heating terms, has been treated in [72]. The reason to neglect the Joule effect is the difficulty of its mathematical analysis, due to its quadratic nature. However, in many real applications, the Joule heating is the main heat source, so its mathematical study cannot be avoided. The treatment of the above system of equations with quadratic source terms is the main contribution to the theoretical part of this work.

2.2.2 Function spaces.

In Appendix A we introduce most of the spaces and theoretical results that will be used throughout this work. Nevertheless, some particular spaces and results will be only used in this chapter so we introduce them in this section.

First we recall the definition of the space $\mathbf{X}(\Omega) := \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$, which is equipped with the norm $\|\mathbf{D}\|_{\mathbf{X}} := (\|\mathbf{D}\|_0^2 + \|\mathbf{curl}\,\mathbf{D}\|_0^2 + \|\mathrm{div}\,\mathbf{D}\|_0^2)^{1/2}$, and its closed subspace $\mathbf{X}_0(\Omega) := \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}_0(\mathrm{div};\Omega)$ for which the seminorm $|\mathbf{D}|_{\mathbf{X}} := (\|\mathbf{curl}\,\mathbf{D}\|_0^2 + \|\mathrm{div}\,\mathbf{D}\|_0^2)^{1/2}$ is an equivalent norm to $\|\cdot\|_{\mathbf{X}}$. We also introduce the space

$$\mathbf{Y}(\Omega) \quad := \quad \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{L}^3(\Omega)\,,$$

endowed with the norm

$$\|\mathbf{D}\|_{\mathbf{Y}} \quad := \quad \left(\|\mathbf{curl}\,\mathbf{D}\|_0^2 + \|\mathrm{div}\,\mathbf{D}\|_0^2\right)^{1/2} + \|\mathbf{D}\|_{\mathbf{L}^3}\ .$$

For the coupled magnetohydrodynamic problem we need the product spaces

$$\begin{aligned}
\mathcal{W}(\Omega) &:= \mathbf{H}^1(\Omega) \times \mathbf{X}(\Omega),\\
\mathcal{Y}(\Omega) &:= \mathbf{H}^1(\Omega) \times \mathbf{Y}(\Omega),\\
\mathcal{W}_0(\Omega) &:= \mathbf{H}_0^1(\Omega) \times \mathbf{X}_0(\Omega),\\
\mathcal{Z}(\Omega) &:= \mathbf{Z}(\Omega) \times \mathbf{Y}(\Omega),\\
\mathcal{Z}_0(\Omega) &:= \mathbf{Z}_0(\Omega) \times \mathbf{X}_0(\Omega),
\end{aligned}$$

equipped with the usual product norms

$$\begin{aligned}
\|(\mathbf{w},\mathbf{D})\|_{\mathcal{W}} &:= \left(\|\mathbf{w}\|_1^2 + \|\mathbf{D}\|_{\mathbf{X}}^2\right)^{1/2}\ ,\\
\|(\mathbf{w},\mathbf{D})\|_{\mathcal{Y}} &:= \left(\|\mathbf{w}\|_1^2 + \|\mathbf{D}\|_{\mathbf{Y}}^2\right)^{1/2}\ .
\end{aligned}$$

Due to inequalities (A.8) and (A.10) the expression

$$|(\mathbf{v},\mathbf{C})|_{\mathcal{W}} := \left(|\mathbf{v}|_1^2 + |\mathbf{C}|_{\mathbf{X}}^2\right)^{1/2}\ ,$$

defines a norm in $\mathcal{W}_0(\Omega)$, equivalent to the product norm $\|\cdot\|_{\mathcal{W}}$.

Besides the spaces defined above and in Appendix A, we will also need three results to ensure that our functions have a proper regularity. These results involve the spaces $H^\delta(\partial\Omega)$ and $W^{s,p}(\Omega)$, with δ, $s \in \mathbb{R}$. The definition of these spaces is classical, and can be found for instance in [54].

The first result is taken from [4, Th. 4.4]:

Theorem 2.1. *Let Ω be a Lipschitz polyhedron. Then, for each $\delta \in (0,1/2)$ the space*

$$\{\mathbf{C} \in \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega) : \mathbf{C}\cdot\mathbf{n}_{|\partial\Omega} \in H^\delta(\partial\Omega)\} = \{\mathbf{C} \in \mathbf{X}(\Omega) : \mathbf{C}\cdot\mathbf{n}_{|\partial\Omega} \in H^\delta(\partial\Omega)\}$$

is continuously imbedded in $\mathbf{H}^{1/2+\varepsilon}(\Omega)$ for some $\varepsilon \in (0,1/2)$.

The proof of this theorem can be found in the aforementioned paper. The next result that we need can be found, for instance, in [54, Th. 1.4.3.2]:

Theorem 2.2. *Let $r > s \geq 0$ and assume that Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary. Then $H^r(\Omega)$ is compactly imbedded in $H^s(\Omega)$.*

Finally, we will also use the Sobolev imbeddings (see equation (1,4,4,5) and Theorem 1.4.3.1 in [54])

Theorem 2.3. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ *with a Lipschitz boundary, then*

$$W^{s,p}(\Omega) \subset W^{t,q}(\Omega)\,,$$

for $t \leq s$ *and* $q \geq p$ *such that* $s - N/p = t - N/q$.

These three theorems allow us to affirm that, for any $\delta \in (0, 1/2)$

$$\{\mathbf{C} \in \mathbf{X}(\Omega) : \mathbf{C} \cdot \mathbf{n}_{|\partial\Omega} \in H^{\delta}(\partial\Omega)\} \subset \mathbf{H}^{1/2+\varepsilon}(\Omega) \subset\subset \mathbf{H}^{1/2}(\Omega) \subset \mathbf{L}^3(\Omega)\,,$$

where $\subset\subset$ denotes a compact imbedding, and there exists a constant κ, depending on δ and Ω, such that

$$\|\mathbf{D}\|_{\mathbf{L}^3} \leq \kappa(\|\mathbf{D}\|_{\mathbf{X}} + \|\mathbf{D} \cdot \mathbf{n}\|_{\delta,\partial\Omega}) \quad \forall\, \mathbf{D} \in \{\mathbf{C} \in \mathbf{X}(\Omega) : \mathbf{C} \cdot \mathbf{n}_{|\partial\Omega} \in H^{\delta}(\partial\Omega)\}\,, \tag{2.15}$$

where $\|\centerdot\|_{\delta,\partial\Omega}$ denotes the norm of $H^{\delta}(\partial\Omega)$.

As an immediate consequence of this result we have that $\mathbf{X}_0(\Omega)$ is compactly imbedded in $\mathbf{L}^3(\Omega)$, and

$$\|\mathbf{D}\|_{\mathbf{L}^3} \leq \kappa\,\|\mathbf{D}\|_{\mathbf{X}} \quad \forall\, \mathbf{D} \in \mathbf{X}_0(\Omega)\,. \tag{2.16}$$

Finally, we remark that in this chapter we will make use of the tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$ as presented in Section A.1.1, along with the spaces defined there.

2.2.3 Compatibility and regularity conditions for source and boundary data.

In this subsection we specify the precise compatibility and regularity conditions for boundary conditions and given sources, in order to obtain a weak formulation of the problem. First, for the Navier-Stokes equations we assume

$$\mathbf{f}_0 \in \mathbf{H}^{-1}(\Omega), \tag{2.17}$$

$$\mathbf{u}_d \in \mathbf{H}^{1/2}(\partial\Omega) \quad \text{with} \quad \int_{\partial\Omega} \mathbf{u}_d \cdot \mathbf{n}\, d\mathbf{x} = 0, \tag{2.18}$$

the compatibility condition for the boundary data being needed because the velocity field is divergence-free.

Next, for the electromagnetic data, we have the following conditions

$$l \in H^{\delta}(\partial\Omega) \quad \text{with} \quad \int_{\partial\Omega} l\, d\mathbf{x} = 0 \quad \text{and } 0 < \delta < 1/2\,, \tag{2.19}$$

$$\mathbf{k} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma}, \partial\Omega) \quad \text{and } \mathrm{div}_{\Gamma}\, \mathbf{k} = 0\,. \tag{2.20}$$

The compatibility condition for l is a consequence of $\mathbf{B}$ being a divergence-free field. Furthermore, we impose $0 < \delta < 1/2$ in order to obtain a magnetic induction field $\mathbf{B} \in \mathbf{L}^3(\Omega)$. The first condition of (2.20) is a direct consequence of the boundary condition (2.13): if we define the non-dimensionalized electric field $\mathbf{E} := \frac{1}{R_m}\mathbf{curl}\,\mathbf{B} - \mathbf{u}\times\mathbf{B}$, since we shall require $\mathbf{u}\in\mathbf{H}^1(\Omega)\subset\mathbf{L}^6(\Omega)$ and $\mathbf{B}\in\mathbf{Y}(\Omega)$, we will have that $\mathbf{E}\in\mathbf{L}^2(\Omega)$. Moreover, from (2.6) we also know that $\mathbf{curl}\,\mathbf{E}=\mathbf{0}$, so $\mathbf{E}\in\mathbf{H}(\mathbf{curl};\Omega)$ and its tangential trace $\mathbf{E}\times\mathbf{n}=\gamma_\tau(\mathbf{E})\in\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$. The second one is a consequence of $\mathbf{E}$ being an irrotational field: for any $\varphi\in H^{3/2}(\partial\Omega)$ we consider any extension $\widetilde{\varphi}\in H^2(\Omega)$ such that $\widetilde{\varphi}_{|\partial\Omega}=\varphi$. From the definitions of the tangential divergence and of the tangential gradient we have

$$\langle \mathrm{div}_\Gamma\,\mathbf{k},\varphi\rangle_{\partial\Omega} = -\,\langle \mathbf{k},\mathbf{grad}_\Gamma\,\widetilde{\varphi}\rangle_{\partial\Omega} = -\,\langle \mathbf{k},\pi_\tau(\mathbf{grad}\,\widetilde{\varphi})\rangle_{\partial\Omega} = -\,\langle \gamma_\tau(\mathbf{E}),\pi_\tau(\mathbf{grad}\,\widetilde{\varphi})\rangle_{\partial\Omega}\ ,$$

and from Green's formula (A.17) we conclude that $\mathrm{div}_\Gamma\,\mathbf{k}=0$, because $\mathbf{curl}\,\mathbf{E}=\mathbf{0}$.

Remark 2.1. *If* $\mathbf{k}\in\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$ *and* $\mathrm{div}_\Gamma\,\mathbf{k}=0$*, it also holds that*

$$\langle \mathbf{k},\pi_\tau(\mathbf{grad}\,\psi)\rangle_{\partial\Omega}=0\quad\forall\,\psi\in H^1(\Omega)\,.$$

To prove this result we first use the definition of the surface divergence, which says that the equality holds for any $\varphi\in H^2(\Omega)$*. Since* $H^2(\Omega)$ *is dense in* $H^1(\Omega)$ *there exists a sequence* $\{\varphi_n\}\subset H^2(\Omega)$ *such that* $\varphi_n\to\psi$ *in* $H^1(\Omega)$*. Since the curl of any gradient is null, it holds that* $\mathbf{grad}\,\varphi_n\to\mathbf{grad}\,\psi$ *in* $\mathbf{H}(\mathbf{curl};\Omega)$*. As a consequence* $\pi_\tau(\mathbf{grad}\,\varphi_n)\to\pi_\tau(\mathbf{grad}\,\psi)$ *in* $\mathbf{H}_{\perp}^{-1/2}(\mathbf{curl}_\Gamma,\partial\Omega)$ *and the equality is true for any* $\psi\in H^1(\Omega)$*.*

For the heat transfer equation, the heat source and the boundary data must satisfy

$$\psi\in L^1(\Omega),\quad T_d\in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega). \tag{2.21}$$

In what follows we are going to consider the thermal-magnetohydrodynamic problem (2.6)-(2.14) along with the conditions (2.17)-(2.21).

To prove the existence of a solution for our coupled problem, we will first study two subproblems separately: a pure magnetohydrodynamic problem, where the unknowns are $\mathbf{u}$ and $\mathbf{B}$, and the temperature is supposed to be known, and a thermal problem where the only unknown is the temperature T. Moreover, during the study of the magnetohydrodynamic problem we will also introduce a linearized problem, that will further help us. We will show the existence of a unique solution for each subproblem and introduce bounds for these solutions in the spaces where they are defined.

2.2.4 Magnetohydrodynamic problem.

This subsection is devoted to the study of the pure magnetohydrodynamic problem, where the temperature is supposed to be known. We begin by introducing the weak formulation of the problem and proving that it is equivalent to the strong formulation as partial differential equations (understood in the sense of distributions), along with boundary conditions. Then we prove some properties of the forms appearing in the weak formulation and finish analyzing a linearized version of the problem that will help us to study the coupled problem.

Weak formulation.

Before presenting the weak formulation of problem (2.6)-(2.14), we introduce some forms that will allow us to simplify the notation. Let $a_0 : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, $a_1 : \mathbf{X}(\Omega) \times \mathbf{X}(\Omega) \to \mathbb{R}$, $c_0 : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, $c_1 : \mathbf{X}(\Omega) \times \mathbf{Y}(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, $b : \mathbf{H}^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$, be given by

$$\begin{aligned}
a_0(\mathbf{u},\mathbf{v}) &:= \frac{1}{H_a^2}\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\,\mathrm{d}\mathbf{x}\,,\\
a_1(\mathbf{B},\mathbf{C}) &:= \frac{1}{R_m^2}\int_\Omega [(\mathbf{curl}\,\mathbf{B})\cdot(\mathbf{curl}\,\mathbf{C}) + (\operatorname{div}\mathbf{B})(\operatorname{div}\mathbf{C})]\,\mathrm{d}\mathbf{x},\\
c_0(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \frac{1}{N}\int_\Omega (\mathbf{grad}\,\mathbf{v})\mathbf{u}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}\,,\\
c_1(\mathbf{B},\mathbf{C},\mathbf{u}) &:= \frac{1}{R_m}\int_\Omega (\mathbf{curl}\,\mathbf{B})\times\mathbf{C}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,,\\
b(\mathbf{u},p) &:= -\int_\Omega p(\operatorname{div}\mathbf{u})\,\mathrm{d}\mathbf{x},
\end{aligned}$$

and let $F : \mathbf{H}_0^1(\Omega) \times \mathbf{X}_0(\Omega) \to \mathbb{R}$ be given by

$$F((\mathbf{v},\mathbf{C})) \;:=\; \langle \mathbf{f}_0,\mathbf{v}\rangle_\Omega + \frac{1}{R_m}\langle \mathbf{k},\pi_\tau(\mathbf{C})\rangle_{\partial\Omega}\,.$$

We remark that all the forms are well defined. Firstly, the integrals and duality pairings of the forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$, and $b(\cdot,\cdot)$ are trivially well defined. The integral appearing in the trilinear form $c_0(\cdot,\cdot,\cdot)$ also makes sense due to the Sobolev imbedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$. Finally, the integral in the trilinear form $c_1(\cdot,\cdot,\cdot)$ is also well defined due to the previous Sobolev imbedding and because $\mathbf{Y}(\Omega) \subset \mathbf{L}^3(\Omega)$. Besides from being well defined, all the forms can be proved to be continuous in a standard way.

We will also make use of the mapping $G : L^{6/5}(\Omega) \to \mathbf{H}^{-1}(\Omega)$, defined as

$$G(T) := \frac{G_r}{NR_e^2}\frac{\mathbf{g}}{g}T\,.$$

For any $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ we have

$$\langle G(T),\mathbf{v}\rangle_\Omega = \frac{G_r}{NR_e^2}\int_\Omega T\frac{\mathbf{g}}{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = (G(T),\mathbf{v})_\Omega\,,$$

and the integral is meaningful because of the Sobolev imbedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$.

We notice that, in the heat equation, the sources belong to $L^1(\Omega)$. For the Poisson's equation with source in $L^1(\Omega)$ and homogeneous Dirichlet boundary conditions, a result presented in [94] states the regularity of the solution in the space $W_0^{1,q}(\Omega)$ with $q < N/(N-1) = 3/2$. Thus, for the heat equation we will require $T \in W^{1,q}(\Omega)$ with $q < 3/2$. In particular, when treating the coupled problem we will work with $T \in W^{1,6/5}(\Omega)$.

It will also be useful to define the bilinear form $a : \mathcal{W}(\Omega) \times \mathcal{W}(\Omega) \to \mathbb{R}$ given by

$$a((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})) := a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{B},\mathbf{C})\,,$$

which is obviously well defined and continuous. We can now introduce the weak formulation of the magnetohydrodynamic problem.

Given $\mathbf{f}_0$, $\mathbf{u}_d$, l *and* $\mathbf{k}$ *satisfying (2.17)-(2.20), and* $T \in L^{6/5}(\Omega)$ *find*

$$(\mathbf{u},\mathbf{B}) \in \mathcal{Z}(\Omega)\,, \tag{2.22}$$

satisfying

$$\begin{aligned} a((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) - c_1(\mathbf{B},\mathbf{B},\mathbf{v}) + c_1(\mathbf{C},\mathbf{B},\mathbf{u}) \\ = F((\mathbf{v},\mathbf{C})) - (G(T),\mathbf{v})_\Omega \quad \forall (\mathbf{v},\mathbf{C}) \in \mathcal{Z}_0(\Omega), \end{aligned} \tag{2.23}$$

$$\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d, \quad (\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l, \tag{2.24}$$

We now proceed to show that, for any given $T \in L^{6/5}(\Omega)$, any pair $(\mathbf{u},\mathbf{B})$ satisfying (2.22)-(2.24) is also a solution of equations (2.6)-(2.9) along with boundary conditions (2.11)-(2.13). To do that we will make use of the following lemma.

Lemma 2.4. *Let* $\mathbf{B} \in \mathbf{H}(\mathrm{div};\Omega)$ *with* $(\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l$, l *satisfying (2.19). Then there exists a unique scalar function* $\chi \in H^1(\Omega) \cap L^2_0(\Omega)$ *such that*

$$\left\{ \begin{aligned} \Delta\chi &= \mathrm{div}\,\mathbf{B}\,, \\ (\mathbf{grad}\,\chi\cdot\mathbf{n})_{|\partial\Omega} &= 0\,. \end{aligned} \right. \tag{2.25}$$

As a consequence, $\mathbf{grad}\,\chi \in \mathbf{X}_0(\Omega)$.

Proof. Since $\mathbf{B} \in \mathbf{H}(\mathrm{div};\Omega)$, we know that $\mathrm{div}\,\mathbf{B} \in L^2(\Omega)$. Moreover, using Gauss' theorem and the compatibility condition (2.19) we get $\mathrm{div}\,\mathbf{B} \in L^2_0(\Omega)$. Thus, there exists $\chi \in H^1(\Omega) \cap L^2_0(\Omega)$ being the unique solution of Neumann problem (2.25) in $L^2_0(\Omega)$ (see [53, Prop. 1.2]). Furthermore, since $\mathrm{div}\,(\mathbf{grad}\,\chi) = \mathrm{div}\,\mathbf{B}$ and $\mathbf{curl}\,(\mathbf{grad}\,\chi) = \mathbf{0}$, and due to the Neumann boundary condition, $\mathbf{grad}\,\chi \in \mathbf{X}_0(\Omega)$. □

Proposition 2.5. *Given* $T \in L^{6/5}(\Omega)$, *if* $(\mathbf{u},\mathbf{B})$ *is a pair satisfying (2.22)-(2.24) then there exists a unique* $p \in L^2_0(\Omega)$ *such that* $((\mathbf{u},\mathbf{B}),p)$ *satisfy (2.6)-(2.9) and boundary conditions (2.11)-(2.13).*

Proof. Taking $\mathbf{C} = \mathbf{0}$ in (2.23) we get

$$\begin{aligned} \frac{1}{H_a^2}\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\,\mathrm{d}\mathbf{x} + \frac{1}{N}\int_\Omega (\mathbf{grad}\,\mathbf{u})\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \frac{1}{R_m}\int_\Omega (\mathbf{curl}\,\mathbf{B}) \times \mathbf{B}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \\ = \langle \mathbf{f}_0,\mathbf{v}\rangle_\Omega - \frac{G_r}{NR_e^2}\int_\Omega T\frac{\mathbf{g}}{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{Z}_0(\Omega)\,. \end{aligned} \tag{2.26}$$

The first task is to construct the pressure p. To do that let us introduce the linear form $\mathbf{l} \in \mathbf{H}^{-1}(\Omega)$ defined as

$$\langle \mathbf{l}, \mathbf{v} \rangle_\Omega := \frac{1}{H_a^2} \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{N} \int_\Omega (\mathbf{grad}\,\mathbf{u})\mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} - \frac{1}{R_m} \int_\Omega (\mathbf{curl}\,\mathbf{B}) \times \mathbf{B} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \\ -\langle \mathbf{f}_0, \mathbf{v} \rangle_\Omega + \frac{G_r}{N R_e^2} \int_\Omega T \frac{\mathbf{g}}{g} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,.$$

Since $\mathbf{f}_0$, $G(T) \in \mathbf{H}^{-1}(\Omega)$ and the forms $a_0(\cdot,\cdot)$, $c_0(\cdot,\cdot,\cdot)$ and $c_1(\cdot,\cdot,\cdot)$ are continuous, we have $\mathbf{l} \in \mathbf{H}^{-1}(\Omega)$. Now, as $\langle \mathbf{l}, \mathbf{v} \rangle_\Omega = 0 \quad \forall \mathbf{v} \in \mathbf{Z}_0(\Omega)$, we know (see [53, Lemma 2.1]) there exists a unique $p \in L_0^2(\Omega)$ such that

$$\mathbf{l} = -\mathbf{grad}\, p \quad \text{in } \mathbf{H}^{-1}(\Omega)\,,$$

which is equivalent to say that

$$\langle \mathbf{l}, \mathbf{v} \rangle_\Omega = -\langle \mathbf{grad}\, p, \mathbf{v} \rangle_\Omega = \int_\Omega p(\operatorname{div} \mathbf{v})\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,.$$

Thus we also have

$$\frac{1}{H_a^2} \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\, \mathrm{d}\mathbf{x} + \frac{1}{N} \int_\Omega (\mathbf{grad}\,\mathbf{u})\mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} - \frac{1}{R_m} \int_\Omega (\mathbf{curl}\,\mathbf{B}) \times \mathbf{B} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \\ - \int_\Omega p(\operatorname{div} \mathbf{v})\, \mathrm{d}\mathbf{x} = \langle \mathbf{f}_0, \mathbf{v} \rangle_\Omega - \frac{G_r}{N R_e^2} \int_\Omega T \frac{\mathbf{g}}{g} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,.$$

Since the equation is valid for any test function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, it can also be written as

$$-\frac{1}{H_a^2} \langle \Delta \mathbf{u}, \mathbf{v} \rangle_\Omega + \frac{1}{N} \int_\Omega (\mathbf{grad}\,\mathbf{u})\mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} - \frac{1}{R_m} \int_\Omega (\mathbf{curl}\,\mathbf{B}) \times \mathbf{B} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} + \langle \mathbf{grad}\, p, \mathbf{v} \rangle_\Omega \\ = \langle \mathbf{f}_0, \mathbf{v} \rangle_\Omega - \frac{G_r}{N R_e^2} \int_\Omega T \frac{\mathbf{g}}{g} \cdot \mathbf{v}\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,.$$

Hence equation (2.8) is satisfied in $\mathbf{H}^{-1}(\Omega)$.

Now setting $\mathbf{v} = \mathbf{0}$ in (2.23) we have

$$\frac{1}{R_m^2} \int_\Omega [(\mathbf{curl}\,\mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C}) + (\operatorname{div} \mathbf{B})(\operatorname{div} \mathbf{C})]\, \mathrm{d}\mathbf{x} + \frac{1}{R_m} \int_\Omega (\mathbf{curl}\,\mathbf{C}) \times \mathbf{B} \cdot \mathbf{u}\, \mathrm{d}\mathbf{x} \\ = \frac{1}{R_m} \langle \mathbf{k}, \pi_\tau(\mathbf{C}) \rangle_{\partial\Omega} \quad \forall \mathbf{C} \in \mathbf{X}_0(\Omega)\,. \tag{2.27}$$

As $\mathbf{B} \in \mathbf{Y}(\Omega) \subset \mathbf{H}(\operatorname{div};\Omega)$ and $(\mathbf{B} \cdot \mathbf{n})_{|\partial\Omega} = l$ we are in the hypotheses of Lemma 2.4, so we can take $\mathbf{C} = \mathbf{grad}\,\chi$ to obtain

$$\frac{1}{R_m^2} \int_\Omega (\operatorname{div} \mathbf{B})(\operatorname{div} \mathbf{B})\, \mathrm{d}\mathbf{x} = \frac{1}{R_m^2} \int_\Omega (\operatorname{div} \mathbf{B})(\operatorname{div} \mathbf{grad}\,\chi)\, \mathrm{d}\mathbf{x} = \frac{1}{R_m} \langle \mathbf{k}, \mathbf{grad}\,\chi_{|\partial\Omega} \rangle_{\partial\Omega}\,, \tag{2.28}$$

and since $\mathbf{k} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \partial\Omega)$ and $\operatorname{div}_\Gamma \mathbf{k} = 0$, from Remark 2.1 we can affirm that

$$\int_\Omega (\operatorname{div} \mathbf{B})^2\, \mathrm{d}\mathbf{x} = 0\,,$$

so that

$$\operatorname{div} \mathbf{B} = 0 \quad \text{a.e. in } \Omega\,, \tag{2.29}$$

and equation (2.7) holds. Incorporating (2.29) into (2.27) and using that $(\mathbf{curl}\,\mathbf{C}) \times \mathbf{B} \cdot \mathbf{u} = -(\mathbf{u} \times \mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C})$ we get

$$\begin{aligned} \frac{1}{R_m^2} \int_\Omega (\mathbf{curl}\,\mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C})\,\mathrm{d}\mathbf{x} - \frac{1}{R_m} \int_\Omega (\mathbf{u} \times \mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C})\,\mathrm{d}\mathbf{x} \\ = \frac{1}{R_m} \langle \mathbf{k}, \pi_\tau(\mathbf{C}) \rangle_{\partial\Omega} \quad \forall\, \mathbf{C} \in \mathbf{X}_0(\Omega)\,. \end{aligned} \tag{2.30}$$

Since $\mathbf{H}_0^1(\Omega) \subset \mathbf{X}_0(\Omega)$ we also have

$$\frac{1}{R_m^2} \langle \mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}), \mathbf{C} \rangle_\Omega - \frac{1}{R_m} \langle \mathbf{curl}\,(\mathbf{u} \times \mathbf{B}), \mathbf{C} \rangle_\Omega = 0 \quad \forall\, \mathbf{C} \in \mathbf{H}_0^1(\Omega)\,,$$

that is

$$\frac{1}{R_m^2} \mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}) - \frac{1}{R_m} \mathbf{curl}\,(\mathbf{u} \times \mathbf{B}) = \mathbf{0} \quad \text{in } \mathbf{H}^{-1}(\Omega)\,, \tag{2.31}$$

and equation (2.6) holds. Incorporating this result into (2.30), we obtain

$$\begin{aligned} \frac{1}{R_m^2} \int_\Omega (\mathbf{curl}\,\mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C})\,\mathrm{d}\mathbf{x} - \frac{1}{R_m} \int_\Omega (\mathbf{u} \times \mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C})\,\mathrm{d}\mathbf{x} \\ - \int_\Omega \left(\frac{1}{R_m^2} \mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}) - \frac{1}{R_m} \mathbf{curl}\,(\mathbf{u} \times \mathbf{B}) \right) \cdot \mathbf{C}\,\mathrm{d}\mathbf{x} = \frac{1}{R_m} \langle \mathbf{k}, \pi_\tau(\mathbf{C}) \rangle_{\partial\Omega} \quad \forall\, \mathbf{C} \in \mathbf{X}_0(\Omega)\,. \end{aligned}$$

If we define the electric field by $\mathbf{E} := \frac{1}{R_m} \mathbf{curl}\,\mathbf{B} - \mathbf{u} \times \mathbf{B}$ we know that $\mathbf{E} \in \mathbf{L}^2(\Omega)$ and, from equation (2.31), we know that $\mathbf{curl}\,\mathbf{E} = \mathbf{0}$, so $\mathbf{E} \in \mathbf{H}(\mathbf{curl}; \Omega)$ and $\mathbf{E} \times \mathbf{n}_{|\partial\Omega} = \gamma_\tau(\mathbf{E}) \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \partial\Omega)$. Using Green's formula (A.17), from the previous equation we get

$$\frac{1}{R_m} \langle \gamma_\tau(\mathbf{E}), \pi_\tau(\mathbf{C}) \rangle_{\partial\Omega} = \frac{1}{R_m} \langle \mathbf{k}, \pi_\tau(\mathbf{C}) \rangle_{\partial\Omega} \quad \forall\, \mathbf{C} \in \mathbf{X}_0(\Omega)\,, \tag{2.32}$$

and from Lemma A.1 it follows that $\mathbf{k} = \gamma_\tau(\mathbf{E})$ in $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \partial\Omega)$.

Finally, the equation of conservation of mass (2.9) and the boundary conditions (2.11) and (2.12) are trivially satisfied. □

Remark 2.2. *The inverse result can also be proved. Assume that we have* $T \in L^{6/5}(\Omega)$, $(\mathbf{u}, \mathbf{B}) \in \mathcal{Y}(\Omega)$ *and* $p \in L_0^2(\Omega)$ *satisfying (2.6)-(2.9), along with boundary conditions (2.11)-(2.13) and compatibility conditions (2.17)-(2.20). Since equation (2.8) is valid in a distributional sense and all its terms belong to* $\mathbf{H}^{-1}(\Omega)$, *it is also valid in* $\mathbf{H}^{-1}(\Omega)$, *so multiplying by a test function* $\mathbf{v} \in \mathbf{Z}_0(\Omega)$ *we obtain (2.26). Moreover, multiplying equation (2.6) by* $\mathbf{C} \in \mathbf{X}_0(\Omega)$, *and using Green's formula (A.17), boundary condition (2.13) and equation (2.7) we get (2.27). Summing up the two equations we obtain (2.23). Finally, since* $(\mathbf{u}, \mathbf{B}) \in \mathcal{Y}(\Omega)$ *and* $\mathbf{u}$ *satisfies (2.9), then* $(\mathbf{u}, \mathbf{B}) \in \mathcal{Z}(\Omega)$.

Reduction to homogeneous boundary conditions.

What we will do now is to split the unknowns into two parts, the first one satisfying the inhomogeneous boundary conditions, and the second one satisfying homogeneous boundary conditions.

If the domain Ω is a Lipschitz polyhedron, supposing that the boundary conditions $\mathbf{u}_d$ and l satisfy equations (2.18) and (2.19), respectively, there exist extensions $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and $\mathbf{B}_0 \in \mathbf{Y}_0(\Omega)$ satisfying

$$\mathbf{u}_{0|\partial\Omega} = \mathbf{u}_d \quad \text{with} \quad \operatorname{div}\mathbf{u}_0 = 0 \ \text{ and } \ \|\mathbf{u}_0\|_1 \le \Lambda_1 \|\mathbf{u}_d\|_{1/2,\partial\Omega}\,, \tag{2.33}$$

$$(\mathbf{B}_0 \cdot \mathbf{n})_{|\partial\Omega} = l \quad \text{with} \quad \operatorname{div}\mathbf{B}_0 = 0,\ \mathbf{curl}\,\mathbf{B}_0 = \mathbf{0} \ \text{ and } \ \|\mathbf{B}_0\|_{\mathbf{L}^3} \le \Lambda_2 \|l\|_{\delta,\partial\Omega}\,, \tag{2.34}$$

where Λ_1 and Λ_2 are two constants that depend on Ω.

The construction of $\mathbf{u}_0$ is well known and can be found, for instance, in [55] or [53]. The construction of $\mathbf{B}_0$ is the same as in [55] and is based on the solution of Neumann problem

$$\begin{cases} -\Delta\chi = 0\,, \\ (\mathbf{grad}\,\chi \cdot \mathbf{n})_{|\partial\Omega} = l\,. \end{cases}$$

Due to the compatibility condition (2.19) this problem has a unique solution $\chi \in H^1(\Omega) \cap L^2_0(\Omega)$. Taking $\mathbf{B}_0 = \mathbf{grad}\,\chi$ it is clear that $\operatorname{div}\mathbf{B}_0 = 0$ and $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, hence $\mathbf{B}_0 \in \mathbf{X}(\Omega)$. Moreover, $(\mathbf{B}_0 \cdot \mathbf{n})_{|\partial\Omega} = l \in H^\delta(\Omega)$ and $\|\mathbf{B}_0\|_{\mathbf{X}} = \|\mathbf{B}_0\|_0 \le \kappa_1 \|l\|_{-1/2,\partial\Omega}$. As a consequence of Theorems 2.1, 2.2 and 2.3 we know that $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ and taking into account that $\mathbf{B}_0$ is irrotational and divergence-free we have $\|\mathbf{B}_0\|_{\mathbf{Y}} = \|\mathbf{B}_0\|_{\mathbf{L}^3} \le \kappa(\|\mathbf{B}_0\|_{\mathbf{X}} + \|\mathbf{B}_0 \cdot \mathbf{n}\|_{\delta,\partial\Omega}) \le \kappa(\kappa_1 \|l\|_{-1/2,\partial\Omega} + \|l\|_{\delta,\partial\Omega}) \le \kappa(\kappa_1\kappa_2 + 1)\|l\|_{\delta,\partial\Omega} = \Lambda_2 \|l\|_{\delta,\partial\Omega}$ where κ is the constant introduced in (2.15) and κ_2 is the constant of the imbedding $H^\delta(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$. Both κ and κ_2 depend on δ and Ω, and κ_1 depends on Ω.

Remark 2.3. *In the case of Ω being of class $\mathcal{C}^{1,1}$ we can require the compatibility condition $l \in H^{1/2}(\partial\Omega)$. The construction of $\mathbf{B}_0$ is analogous to that presented before, but in this case the field χ is known to be in $H^2(\Omega)$. Therefore, $\mathbf{B}_0 \in \mathbf{H}^1(\Omega)$ and we can find an estimate of the form $\|\mathbf{B}_0\|_1 \le \hat{\Lambda}_2 \|l\|_{1/2,\partial\Omega}$.*

Once we have constructed these fields we can split the unknowns in the following way: $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$, with $\hat{\mathbf{u}} \in \mathbf{Z}_0(\Omega)$ and $\mathbf{B} = \hat{\mathbf{B}} + \mathbf{B}_0$ with $\hat{\mathbf{B}} \in \mathbf{X}_0(\Omega)$. Using these splittings we can rewrite problem (2.22)-(2.24) as follows:

Given $\mathbf{u}_0 \in \mathbf{Z}(\Omega)$, $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ satisfying (2.33)-(2.34), and $T \in L^{6/5}(\Omega)$ find

$$(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{2.35}$$

such that

$$\begin{aligned} &a((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\mathbf{v}, \mathbf{C})) + c_0(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) + c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \mathbf{v}) + c_0(\mathbf{u}_0, \hat{\mathbf{u}}, \mathbf{v}) \\ &-c_1(\hat{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) + c_1(\mathbf{C}, \hat{\mathbf{B}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \hat{\mathbf{B}}, \mathbf{u}_0) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}) \\ &= F((\mathbf{v}, \mathbf{C})) - (G(T), \mathbf{v})_\Omega - a_0(\mathbf{u}_0, \mathbf{v}) - c_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) - c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}_0) \quad \forall\,(\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega)\,. \end{aligned} \tag{2.36}$$

We notice that, since $\mathbf{X}_0(\Omega) \subset \mathbf{Y}(\Omega)$ all the terms concerning the trilinear form $c_1(\cdot,\cdot,\cdot)$ make sense. It is easily seen that $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ is a solution of (2.35)-(2.36) if and only if $(\mathbf{u}, \mathbf{B}) = (\hat{\mathbf{u}}, \hat{\mathbf{B}}) + (\mathbf{u}_0, \mathbf{B}_0)$ is a solution of problem (2.22)-(2.24).

Continuity and coerciveness properties.

Next we state some results about the properties of the forms appearing in our problem, which will be needed to prove the existence result.

Lemma 2.6. *The linear form $F(\cdot)$, the bilinear forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ and the trilinear forms $c_0(\cdot,\cdot,\cdot)$ and $c_1(\cdot,\cdot,\cdot)$ are continuous in the spaces where they have been defined.*

Proof. Most of the inequalities can be proved in a standard way as it is done in [2] and [55], but we reproduce them here in detail because the continuity constants will be important in the forthcoming results. The following inequalities are straightly obtained from the definitions of the forms:

$$|a_0(\mathbf{u},\mathbf{v})| \leq \frac{1}{H_a^2} |\mathbf{u}|_1 |\mathbf{v}|_1 \leq \frac{1}{H_a^2} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 ,$$

which holds for all $\mathbf{u}$, $\mathbf{v} \in \mathbf{H}^1(\Omega)$;

$$|a_1(\mathbf{B},\mathbf{C})| \leq \frac{1}{R_m^2} (\|\mathbf{curl}\,\mathbf{B}\|_0 \|\mathbf{curl}\,\mathbf{C}\|_0 + \|\mathrm{div}\,\mathbf{B}\|_0 \|\mathrm{div}\,\mathbf{C}\|_0) \leq \frac{1}{R_m^2} |\mathbf{B}|_{\mathbf{X}} |\mathbf{C}|_{\mathbf{X}} \leq \frac{1}{R_m^2} \|\mathbf{B}\|_{\mathbf{X}} \|\mathbf{C}\|_{\mathbf{X}} ,$$

which holds for all $\mathbf{B}$, $\mathbf{C} \in \mathbf{X}(\Omega)$;

$$|c_0(\mathbf{u},\mathbf{v},\mathbf{w})| \leq \frac{1}{N} \|\mathbf{u}\|_{\mathbf{L}^4} |\mathbf{v}|_1 \|\mathbf{w}\|_{\mathbf{L}^4} \leq \frac{\gamma_4}{N} \|\mathbf{u}\|_{\mathbf{L}^4} |\mathbf{v}|_1 \|\mathbf{w}\|_1 \leq \frac{\gamma_4^2}{N} \|\mathbf{u}\|_1 |\mathbf{v}|_1 \|\mathbf{w}\|_1 \leq \frac{\gamma_4^2}{N} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 ,$$

which holds for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega)$, with γ_4 being the constant of the Sobolev imbedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^4(\Omega)$;

$$\begin{aligned}|c_1(\mathbf{B},\mathbf{C},\mathbf{u})| &\leq \frac{1}{R_m} \|\mathbf{curl}\,\mathbf{B}\|_0 \|\mathbf{C}\|_{\mathbf{L}^3} \|\mathbf{u}\|_{\mathbf{L}^6} \leq \frac{1}{R_m} |\mathbf{B}|_{\mathbf{X}} \|\mathbf{C}\|_{\mathbf{L}^3} \|\mathbf{u}\|_{\mathbf{L}^6} \leq \frac{\gamma_6}{R_m} |\mathbf{B}|_{\mathbf{X}} \|\mathbf{C}\|_{\mathbf{L}^3} \|\mathbf{u}\|_1 \\ &\leq \frac{\gamma_6}{R_m} \|\mathbf{B}\|_{\mathbf{X}} \|\mathbf{C}\|_{\mathbf{L}^3} \|\mathbf{u}\|_1 \leq \frac{\gamma_6}{R_m} \|\mathbf{B}\|_{\mathbf{X}} \|\mathbf{C}\|_{\mathbf{Y}} \|\mathbf{u}\|_1 ,\end{aligned}$$

which holds for all $\mathbf{B} \in \mathbf{X}(\Omega)$, $\mathbf{C} \in \mathbf{Y}(\Omega)$ and $\mathbf{u} \in \mathbf{H}^1(\Omega)$;

$$\begin{aligned}|F((\mathbf{v},\mathbf{C}))| &\leq \|\mathbf{f}_0\|_{-1} \|\mathbf{v}\|_1 + \frac{1}{R_m} \|\mathbf{k}\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)} \|\pi_\tau(\mathbf{C})\|_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega)} \\ &\leq \|\mathbf{f}_0\|_{-1} \|\mathbf{v}\|_1 + \frac{1}{R_m} \|\mathbf{k}\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)} \|\pi_\tau\|_{\mathcal{L}(\mathbf{H}(\mathbf{curl};\Omega),\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega))} \|\mathbf{C}\|_{\mathbf{X}} \\ &\leq C_0 \|\mathbf{f}_0\|_{-1} |\mathbf{v}|_1 + \frac{C_1}{R_m} \|\mathbf{k}\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)} \|\pi_\tau\|_{\mathcal{L}(\mathbf{H}(\mathbf{curl};\Omega),\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega))} |\mathbf{C}|_{\mathbf{X}} \\ &\leq \left(C_0^2 \|\mathbf{f}_0\|_{-1}^2 + \frac{C_1^2}{R_m^2} \|\mathbf{k}\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)} \|\pi_\tau\|^2_{\mathcal{L}(\mathbf{H}(\mathbf{curl};\Omega),\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega))} \right)^{1/2} |(\mathbf{v},\mathbf{C})|_{\mathcal{W}} ,\end{aligned}$$

which holds for all $(\mathbf{v},\mathbf{C}) \in \mathcal{W}_0(\Omega)$, with C_0 and C_1 the constants appearing in the inequalities (A.8) and (A.10), respectively.

□

Remark 2.4. *In the case where Ω is $\mathcal{C}^{1,1}$ the boundary data $\mathbf{k}$ belongs to the space $\mathbf{H}_T^{-1/2}(\partial\Omega)$ and the continuity constant for the linear form $F(\boldsymbol{.})$ can be written in the following form:*

$$\begin{aligned}|F((\mathbf{v},\mathbf{C}))| &\leq \|\mathbf{f}_0\|_{-1}\,\|\mathbf{v}\|_1 + \frac{1}{R_m}\,\|\mathbf{k}\|_{\mathbf{H}_T^{-1/2}(\partial\Omega)}\,\|\mathbf{C}\|_1\\ &\leq \left(C_0^2\,\|\mathbf{f}_0\|_{-1}^2 + \frac{C_2^2}{R_m^2}\,\|\mathbf{k}\|^2_{\mathbf{H}_T^{-1/2}(\partial\Omega)}\right)^{1/2} |(\mathbf{v},\mathbf{C})|_{\mathcal{W}}\,,\end{aligned}$$

where C_2 is the constant appearing in (A.11).

From the continuity of $a_0(\boldsymbol{.},\boldsymbol{.})$ and $a_1(\boldsymbol{.},\boldsymbol{.})$ it can be easily proved that the bilinear form $a(\boldsymbol{.},\boldsymbol{.})$ is also continuous, and

$$|a((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C}))| \leq \lambda_a\,\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}\,\|(\mathbf{v},\mathbf{C})\|_{\mathcal{W}} \quad \forall(\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})\in\mathcal{W}(\Omega)\,,$$

with $\lambda_a := \max\{1/H_a^2, 1/R_m^2\}$. Finally, the following result will also be helpful:

$$|(G(T),\mathbf{v})_\Omega| \leq \frac{G_r}{NR_e^2}\,\|T\|_{L^{6/5}}\,\|\mathbf{v}\|_{\mathbf{L}^6} \leq \frac{G_r\gamma_6}{NR_e^2}\,\|T\|_{L^{6/5}}\,\|\mathbf{v}\|_1 \leq C_0\frac{G_r\gamma_6}{NR_e^2}\,\|T\|_{L^{6/5}}\,|\mathbf{v}|_1\,,$$

which holds for all $T\in L^{6/5}(\Omega)$ and $\mathbf{v}\in\mathbf{H}_0^1(\Omega)$.

For the sake of simplicity, in what follows we will use the notations $\lambda_{a0} := 1/H_a^2$, $\lambda_{a1} := 1/R_m^2$, $\lambda_{c0} := 1/N$, $\lambda_{c1} := 1/R_m$ and $\lambda_G := C_0G_r\gamma_6/(NR_e^2)$. Furthermore, we shall denote $\lambda_F := \left(C_0^2\,\|\mathbf{f}_0\|_{-1}^2 + (C_1^2/R_m^2)\,\|\mathbf{k}\|^2_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)}\,\|\pi_\tau\|^2_{\mathcal{L}(\mathbf{H}(\mathbf{curl};\Omega),\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega))}\right)^{1/2}$ in the case where Ω is a Lipschitz polyhedron and $\lambda_F := \left(C_0^2\,\|\mathbf{f}_0\|_{-1}^2 + (C_2^2/R_m^2)\,\|\mathbf{k}\|^2_{\mathbf{H}_T^{-1/2}(\partial\Omega)}\right)^{1/2}$ in the case where Ω is a smooth domain.

Lemma 2.7. *The bilinear forms $a_0(\boldsymbol{.},\boldsymbol{.})$ and $a_1(\boldsymbol{.},\boldsymbol{.})$ are coercive on $\mathbf{H}_0^1(\Omega)$ and $\mathbf{X}_0(\Omega)$, respectively. As a consequence, the bilinear form $a(\boldsymbol{.},\boldsymbol{.})$ is coercive on $\mathcal{W}_0(\Omega)$.*

Proof. For any $\hat{\mathbf{u}}\in\mathbf{H}_0^1(\Omega)$ we have

$$a_0(\hat{\mathbf{u}},\hat{\mathbf{u}}) = \frac{1}{H_a^2}\int_\Omega |\mathbf{grad}\,\hat{\mathbf{u}}|^2\,\mathrm{d}\mathbf{x} \geq \frac{1}{H_a^2}\,\|\mathbf{grad}\,\hat{\mathbf{u}}\|_0^2 = \frac{1}{H_a^2}|\hat{\mathbf{u}}|_1^2\,,$$

and, for any $\hat{\mathbf{B}}\in\mathbf{X}_0(\Omega)$,

$$a_1(\hat{\mathbf{B}},\hat{\mathbf{B}}) = \frac{1}{R_m^2}\int_\Omega (|\mathbf{curl}\,\hat{\mathbf{B}}|^2 + |\mathrm{div}\,\hat{\mathbf{B}}|^2)\,\mathrm{d}\mathbf{x} \geq \frac{1}{R_m^2}\left(\|\mathbf{curl}\,\hat{\mathbf{B}}\|_0^2 + \|\mathrm{div}\,\hat{\mathbf{B}}\|_0^2\right) = \frac{1}{R_m^2}\,|\hat{\mathbf{B}}|_{\mathbf{X}}^2\,.$$

Therefore, for any $(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathcal{W}_0(\Omega)$ we have:

$$a((\hat{\mathbf{u}},\hat{\mathbf{B}}),(\hat{\mathbf{u}},\hat{\mathbf{B}})) \geq \frac{1}{H_a^2}|\hat{\mathbf{u}}|_1^2 + \frac{1}{R_m^2}\,|\hat{\mathbf{B}}|_{\mathbf{X}}^2 \geq \min\left\{\frac{1}{H_a^2},\frac{1}{R_m^2}\right\}|(\hat{\mathbf{u}},\hat{\mathbf{B}})|_{\mathcal{W}}^2\,.$$

□

Analogously to what we did with the continuity constants, we denote the coerciveness constants by $\alpha_{a0} := 1/H_a^2$, $\alpha_{a1} := 1/R_m^2$ and $\alpha_a := \min\left\{1/H_a^2, 1/R_m^2\right\}$.

Lemma 2.8. *Let $\mathbf{u} \in \mathbf{Z}(\Omega)$, $\mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ and assume that one of the functions $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belongs to $\mathbf{H}_0^1(\Omega)$. Then the trilinear form $c_0(\cdot,\cdot,\cdot)$ is antisymmetric with respect to its second and third arguments, i.e.,*

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -c_0(\mathbf{u}, \mathbf{w}, \mathbf{v}). \tag{2.37}$$

Proof. The result is well known and a proof can be found in [55]. It relies on the fact that $\mathbf{u}$ is divergence-free and uses a Green's formula to state the result.

□

Remark 2.5. *Under the assumptions of the previous lemma and, in particular, for any $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, we also have*

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0. \tag{2.38}$$

Linearized MHD problem.

We introduce now a linearized version of the MHD problem that will be helpful to prove the existence of solution for the coupled problem via a fixed point theorem. The linearized problem reads:

Given $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$, $T \in L^{6/5}(\Omega)$ and $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}(\Omega)$ with $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, find

$$(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{2.39}$$

such that

$$\begin{aligned} a((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\mathbf{v}, \mathbf{C})) + c_0(\hat{\mathbf{w}}, \hat{\mathbf{u}}, \mathbf{v}) + c_0(\mathbf{u}_0, \hat{\mathbf{u}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) \\ +c_1(\mathbf{C}, \hat{\mathbf{D}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}) = F((\mathbf{v}, \mathbf{C})) - (G(T), \mathbf{v})_\Omega - a_0(\mathbf{u}_0, \mathbf{v}) \\ -c_0(\hat{\mathbf{w}}, \mathbf{u}_0, \mathbf{v}) - c_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) - c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}_0) - c_1(\mathbf{C}, \hat{\mathbf{D}}, \mathbf{u}_0) \quad \forall\, (\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega)\,. \end{aligned} \tag{2.40}$$

Proposition 2.9. *There exists a unique solution $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ to problem (2.39)-(2.40). Moreover,*

$$\begin{aligned} |(\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \leq \frac{1}{\alpha_a} \Big(\lambda_F + \lambda_G \|T\|_{L^{6/5}} + \lambda_{a0} |\mathbf{u}_0|_1 + \lambda_{c0} \|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \lambda_{c1} \|\mathbf{B}_0\|_{\mathbf{L}^3} \|\mathbf{u}_0\|_{\mathbf{L}^6} \\ + \max\left\{ \tilde{\lambda}_{c0} \|\mathbf{u}_0\|_{\mathbf{L}^4}, \tilde{\lambda}_{c1} \|\mathbf{u}_0\|_{\mathbf{L}^6} \right\} |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \Big), \end{aligned}$$

where α_a, λ_F, λ_G, λ_{a0}, λ_{c0} and λ_{c1} are the constants introduced above, $\tilde{\lambda}_{c0} := \gamma_4 C_0/N$ and $\tilde{\lambda}_{c1} := \kappa C_1/R_m$.

Proof. The result is a consequence of the continuity, coerciveness and antisymmetry properties proved before. Since $(\mathbf{u}_0, \mathbf{B}_0), (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ and T are given, we can define the bilinear form

$$\begin{aligned} \tilde{a}((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C})) := a((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C})) + c_0(\hat{\mathbf{w}}, \mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}_0, \mathbf{u}, \mathbf{v}) \\ -c_1(\mathbf{B}, \hat{\mathbf{D}}, \mathbf{v}) - c_1(\mathbf{B}, \mathbf{B}_0, \mathbf{v}) + c_1(\mathbf{C}, \hat{\mathbf{D}}, \mathbf{u}) + c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}), \end{aligned}$$

and the linear form

$$\tilde{F}((\mathbf{v},\mathbf{C})) := F((\mathbf{v},\mathbf{C})) - (G(T),\mathbf{v})_\Omega - a_0(\mathbf{u}_0,\mathbf{v})$$
$$-c_0(\hat{\mathbf{w}},\mathbf{u}_0,\mathbf{v}) - c_0(\mathbf{u}_0,\mathbf{u}_0,\mathbf{v}) - c_1(\mathbf{C},\mathbf{B}_0,\mathbf{u}_0) - c_1(\mathbf{C},\hat{\mathbf{D}},\mathbf{u}_0),$$

which act on $\mathcal{Z}_0(\Omega)\times\mathcal{Z}_0(\Omega)$ and on $\mathcal{Z}_0(\Omega)$, respectively.

Due to the continuity results proved in Lemma 2.6, it is clear that both $\tilde{a}$ and $\tilde{F}$ are continuous. Moreover, since we are in the hypotheses of Lemma 2.8, and due to the coerciveness of $a(\cdot,\cdot)$ proved in Lemma 2.7, the bilinear form $\tilde{a}(\cdot,\cdot)$ is coercive. Hence, reminding the antisymmetry property of $c_0(\cdot,\cdot,\cdot)$, the result is obtained by applying Lax-Milgram lemma. The bound for $|(\hat{\mathbf{u}},\hat{\mathbf{B}})|_{\mathcal{W}}$ is obtained from the continuity and coerciveness results mentioned before, and taking into account that

$$
\begin{aligned}
|-c_0(\hat{\mathbf{w}},\mathbf{u}_0,\hat{\mathbf{u}}) - c_1(\hat{\mathbf{B}},\hat{\mathbf{D}},\mathbf{u}_0)| &= |c_0(\hat{\mathbf{w}},\hat{\mathbf{u}},\mathbf{u}_0) - c_1(\hat{\mathbf{B}},\hat{\mathbf{D}},\mathbf{u}_0)|\\
= \left|\frac{1}{N}\int_\Omega (\mathbf{grad}\,\hat{\mathbf{u}})\hat{\mathbf{w}}\cdot\mathbf{u}_0\,\mathrm{d}\mathbf{x} \right.&\left. - \frac{1}{R_m}\int_\Omega (\mathbf{curl}\,\hat{\mathbf{B}})\times\hat{\mathbf{D}}\cdot\mathbf{u}_0\,\mathrm{d}\mathbf{x}\right|\\
&\le \frac{1}{N}\,\|\hat{\mathbf{w}}\|_{\mathbf{L}^4}\,|\hat{\mathbf{u}}|_1\,\|\mathbf{u}_0\|_{\mathbf{L}^4} + \frac{1}{R_m}\,|\hat{\mathbf{B}}|_{\mathbf{X}}\,\|\hat{\mathbf{D}}\|_{\mathbf{L}^3}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\\
&\le \frac{\gamma_4 C_0}{N}\,|\hat{\mathbf{w}}|_1\,|\hat{\mathbf{u}}|_1\,\|\mathbf{u}_0\|_{\mathbf{L}^4} + \frac{\kappa C_1}{R_m}\,|\hat{\mathbf{B}}|_{\mathbf{X}}\,|\hat{\mathbf{D}}|_{\mathbf{X}}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\\
\le \max\left\{\frac{\gamma_4 C_0}{N}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}\,,\frac{\kappa C_1}{R_m}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}&\,|(\hat{\mathbf{w}},\hat{\mathbf{D}})|_{\mathcal{W}}\,|(\hat{\mathbf{u}},\hat{\mathbf{B}})|_{\mathcal{W}}\,. \qquad (2.41)
\end{aligned}
$$

□

Definition 2.1. *Let us define the mapping*

$$
\begin{aligned}
\mathcal{G}_1 : \mathcal{Z}_0(\Omega)\times L^{6/5}(\Omega) &\longrightarrow \mathcal{Z}_0(\Omega)\\
((\hat{\mathbf{w}},\hat{\mathbf{D}}),T) &\longmapsto \mathcal{G}_1((\hat{\mathbf{w}},\hat{\mathbf{D}}),T) = (\hat{\mathbf{u}},\hat{\mathbf{B}}),
\end{aligned}
\qquad (2.42)
$$

where $(\hat{\mathbf{u}},\hat{\mathbf{B}})$ is the solution of the MHD linearized problem (2.39)-(2.40) for a given pair $(\hat{\mathbf{w}},\hat{\mathbf{D}})\in\mathcal{Z}_0(\Omega)$ and a given temperature $T\in L^{6/5}(\Omega)$.

Lemma 2.10. *Let $(\hat{\mathbf{w}}_n,\hat{\mathbf{D}}_n)\rightharpoonup(\hat{\mathbf{w}},\hat{\mathbf{D}})$ weakly in $\mathcal{Z}_0(\Omega)$ and $T_n\to T$ strongly in $L^{6/5}(\Omega)$. Then $\mathcal{G}_1((\hat{\mathbf{w}}_n,\hat{\mathbf{D}}_n),T_n)\to\mathcal{G}_1((\hat{\mathbf{w}},\hat{\mathbf{D}}),T)$ strongly in $\mathcal{Z}_0(\Omega)$.*

Proof. Let us denote $(\hat{\mathbf{u}}_n,\hat{\mathbf{B}}_n)=\mathcal{G}_1((\hat{\mathbf{w}}_n,\hat{\mathbf{D}}_n),T_n)$ and $(\hat{\mathbf{u}},\hat{\mathbf{B}})=\mathcal{G}_1((\hat{\mathbf{w}},\hat{\mathbf{D}}),T)$. Writing equation (2.40) of the MHD linearized problem for both solutions, and then subtracting the two equations we have

$$
\begin{aligned}
&a((\hat{\mathbf{u}}_n,\hat{\mathbf{B}}_n)-(\hat{\mathbf{u}},\hat{\mathbf{B}}),(\mathbf{v},\mathbf{C})) + c_0(\hat{\mathbf{w}}_n,\hat{\mathbf{u}}_n,\mathbf{v}) - c_0(\hat{\mathbf{w}},\hat{\mathbf{u}},\mathbf{v}) + c_0(\mathbf{u}_0,\hat{\mathbf{u}}_n-\hat{\mathbf{u}},\mathbf{v}) - c_1(\hat{\mathbf{B}}_n,\hat{\mathbf{D}}_n,\mathbf{v})\\
&\quad + c_1(\hat{\mathbf{B}},\hat{\mathbf{D}},\mathbf{v}) - c_1(\hat{\mathbf{B}}_n-\hat{\mathbf{B}},\mathbf{B}_0,\mathbf{v}) + c_1(\mathbf{C},\hat{\mathbf{D}}_n,\hat{\mathbf{u}}_n) - c_1(\mathbf{C},\hat{\mathbf{D}},\hat{\mathbf{u}}) + c_1(\mathbf{C},\mathbf{B}_0,\hat{\mathbf{u}}_n-\hat{\mathbf{u}})\\
&\quad = -(G(T_n-T),\mathbf{v})_\Omega - c_0(\hat{\mathbf{w}}_n-\hat{\mathbf{w}},\mathbf{u}_0,\mathbf{v}) - c_1(\mathbf{C},\hat{\mathbf{D}}_n-\hat{\mathbf{D}},\mathbf{u}_0)\quad\forall(\mathbf{v},\mathbf{C})\in\mathcal{Z}_0(\Omega)\,.
\end{aligned}
$$

Now, by adding and subtracting the terms $c_0(\hat{\mathbf{w}}_n, \hat{\mathbf{u}}, \mathbf{v})$, $c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}_n, \mathbf{v})$ and $c_1(\mathbf{C}, \hat{\mathbf{D}}_n, \hat{\mathbf{u}})$ the equation reads as follows

$$\begin{aligned} a((\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\mathbf{v}, \mathbf{C})) + c_0(\hat{\mathbf{w}}_n, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}, \mathbf{v}) + c_0(\hat{\mathbf{w}}_n - \hat{\mathbf{w}}, \hat{\mathbf{u}}, \mathbf{v}) \\ + c_0(\mathbf{u}_0, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \hat{\mathbf{D}}_n, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) \\ + c_1(\mathbf{C}, \hat{\mathbf{D}}_n, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) + c_1(\mathbf{C}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) \\ = -(G(T_n - T), \mathbf{v})_\Omega - c_0(\hat{\mathbf{w}}_n - \hat{\mathbf{w}}, \mathbf{u}_0, \mathbf{v}) - c_1(\mathbf{C}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \mathbf{u}_0) \quad \forall (\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega)\,. \end{aligned}$$

If we choose as test function $(\mathbf{v}, \mathbf{C}) = (\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})$, and take into account the antisymmetry result stated in Lemma 2.8, we get

$$\begin{aligned} a((\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})) + c_0(\hat{\mathbf{w}}_n - \hat{\mathbf{w}}, \hat{\mathbf{u}}, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) \\ - c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) + c_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \hat{\mathbf{u}}) \\ = -(G(T_n - T), \hat{\mathbf{u}}_n - \hat{\mathbf{u}})_\Omega - c_0(\hat{\mathbf{w}}_n - \hat{\mathbf{w}}, \mathbf{u}_0, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) - c_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \mathbf{u}_0)\,. \end{aligned}$$

Then, using the coerciveness of $a(\cdot, \cdot)$, the continuity results for $c_0(\cdot, \cdot, \cdot)$, $c_1(\cdot, \cdot, \cdot)$ and the antisymmetry property for $c_0(\cdot, \cdot, \cdot)$ we obtain

$$\begin{aligned} \alpha_a |(\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \le \lambda_G \|T_n - T\|_{L^{6/5}} + \lambda_{c0} \|\hat{\mathbf{w}}_n - \hat{\mathbf{w}}\|_{\mathbf{L}^4} \|\hat{\mathbf{u}}\|_{\mathbf{L}^4} + \lambda_{c0} \|\hat{\mathbf{w}}_n - \hat{\mathbf{w}}\|_{\mathbf{L}^4} \|\mathbf{u}_0\|_{\mathbf{L}^4} \\ + \gamma_6 \lambda_{c1} C_0 \, |\hat{\mathbf{B}}|_{\mathbf{X}} \|\hat{\mathbf{D}}_n - \hat{\mathbf{D}}\|_{\mathbf{L}^3} + \lambda_{c1} \|\hat{\mathbf{D}}_n - \hat{\mathbf{D}}\|_{\mathbf{L}^3} \|\hat{\mathbf{u}}\|_{\mathbf{L}^6} + \lambda_{c1} \|\hat{\mathbf{D}}_n - \hat{\mathbf{D}}\|_{\mathbf{L}^3} \|\mathbf{u}_0\|_{\mathbf{L}^6}\,. \end{aligned}$$

Due to the compact imbeddings $\mathbf{H}^1(\Omega) \subset\subset \mathbf{L}^4(\Omega)$ and $\mathbf{X}_0(\Omega) \subset\subset \mathbf{L}^3(\Omega)$, already introduced in Section 2.2.2, and since $(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ in $\mathcal{Z}_0(\Omega) = \mathbf{Z}_0(\Omega) \times \mathbf{X}_0(\Omega)$, we know that $\hat{\mathbf{w}}_n \to \hat{\mathbf{w}}$ and $\hat{\mathbf{D}}_n \to \hat{\mathbf{D}}$ strongly in $\mathbf{L}^4(\Omega)$ and $\mathbf{L}^3(\Omega)$, respectively. Thus, considering also the strong convergence $T_n \to T$ in $L^{6/5}(\Omega)$, we obtain from the previous inequality $|(\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \to 0$, and due to the equivalence of this seminorm to the usual norm in $\mathcal{W}_0(\Omega)$ the result follows. □

We have proved the existence and uniqueness of the solution to the linearized MHD problem. Moreover, we have proved, in the preceding lemma, that the mapping $\mathcal{G}_1$ is sequentially continuous from $\mathbf{Z}_0(\Omega)$ – weak × $L^{6/5}(\Omega)$ – strong into $\mathbf{Z}_0(\Omega)$ – strong. This property will be necessary to prove the existence of solution for the coupled problem. In the next subsection we analyze the thermal subproblem.

2.2.5 The thermal problem.

As we mentioned before, the main difficulty in the thermal problem are the quadratic source terms, which belong to $L^1(\Omega)$. For the treatment of this problem, we will make use of the concept of solution by transposition, as studied by Stampacchia in [94]. For the ease of reading we will write some of the results from that paper in the general case $\Omega \subset \mathbb{R}^N$, with $N \ge 3$, but we recall that in our problem we are considering $N = 3$. Moreover, throughout this section we will seek a temperature $T \in W^{1,q}(\Omega)$, with $q < N/(N-1)$, even if for the coupled problem we always work with $T \in W^{1,6/5}(\Omega)$.

First of all, let us remind the equations of the thermal problem.

Given $(\mathbf{u},\mathbf{B}) \in \mathcal{Z}(\Omega)$*, and* T_d *and* ψ *satisfying (2.21) find* T *such that*

$$\frac{1}{P_r R_e}\Delta T + \mathbf{u}\cdot\mathbf{grad}\,T = \frac{E_c}{R_e}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2\right] + \psi\,, \tag{2.43}$$

$$T_{|\partial\Omega} = T_d\,. \tag{2.44}$$

Since $\mathbf{u} \in \mathbf{Z}(\Omega)$ and $\mathbf{B} \in \mathbf{X}(\Omega)$, from the definition of these spaces we know that $\mathbf{grad}\,\mathbf{u} \in L^2(\Omega)^{3\times 3}$ and $\mathbf{curl}\,\mathbf{B} \in \mathbf{L}^2(\Omega)$ so that all the source terms in equation (2.43) belong to $L^1(\Omega)$. Let us consider the following problem, for any given source in $L^1(\Omega)$:

Given $\mathbf{u} \in \mathbf{Z}(\Omega)$*,* $f \in L^1(\Omega)$ *and* $T_d \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$*, find* T *satisfying*

$$\begin{cases} -\dfrac{1}{P_r R_e}\Delta T + \mathbf{u}\cdot\mathbf{grad}\,T = f, \\ T_{|\partial\Omega} = T_d\,. \end{cases} \tag{2.45}$$

In our case f will be the sum of Joule heating, viscous heating and the given heat source ψ. We notice that we have not made precise the functional space to which T belongs. We will seek our solution T in $W^{1,q}(\Omega)$, $1 \le q < N/(N-1)$, as it is done in [94], but since in that paper the problem was treated with homogeneous boundary conditions, we will first split the problem into two ones: the first one with homogeneous boundary conditions and sources in $L^1(\Omega)$, and the second one with non-homogeneous boundary conditions and null sources, namely

$$T = T_1 + T_2, \tag{2.46}$$

where T_1 satisfies

$$\begin{cases} -\dfrac{1}{P_r R_e}\Delta T_1 + \mathbf{u}\cdot\mathbf{grad}\,T_1 = f, \\ T_{1|\partial\Omega} = 0, \end{cases} \tag{2.47}$$

and T_2 is solution to

$$\begin{cases} T_2 \in H^1(\Omega), \\ -\dfrac{1}{P_r R_e}\Delta T_2 + \mathbf{u}\cdot\mathbf{grad}\,T_2 = 0, \\ T_{2|\partial\Omega} = T_d\,. \end{cases} \tag{2.48}$$

The analysis of problem (2.47) is based on the theory of solution by transposition (see [94]). Instead, the mathematical analysis of problem (2.48) is standard. We will begin by presenting the results for this second problem, and then we will show the results for problem (2.47).

Analysis of the thermal subproblem with inhomogeneous boundary conditions.

Let us introduce the forms $e : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ and $d : \mathbf{H}^1(\Omega)\times H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$, defined as

$$e(T,z) := \frac{1}{P_r R_e}\int_\Omega \mathbf{grad}\,T\cdot\mathbf{grad}\,z\,\mathrm{d}\mathbf{x},$$

$$d(\mathbf{u},T,z) := \int_\Omega \mathbf{u}\cdot\mathbf{grad}\,T\,z\,\mathrm{d}\mathbf{x},$$

where the integral appearing in $d(\cdot,\cdot,\cdot)$ is well defined due to the imbeddings $\mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$ and $H^1(\Omega) \subset L^6(\Omega)$. These two forms satisfy the following properties.

Lemma 2.11. *The bilinear form $e(\cdot,\cdot)$ and the trilinear form $d(\cdot,\cdot,\cdot)$ are continuous. Moreover*

$$|e(T,z)| \leq \frac{1}{P_r R_e} \|\mathbf{grad}\, T\|_0 \|\mathbf{grad}\, z\|_0 = \frac{1}{P_r R_e} |T|_1 |z|_1 \leq \frac{1}{P_r R_e} \|T\|_1 \|z\|_1 ,$$
$$|d(\mathbf{u},T,z)| \leq \|\mathbf{u}\|_{\mathbf{L}^4} \|\mathbf{grad}\, T\|_0 \|z\|_{L^4} \leq \gamma_4^2 \|\mathbf{u}\|_1 |T|_1 \|z\|_1 \leq \gamma_4^2 \|\mathbf{u}\|_1 \|T\|_1 \|z\|_1 ,$$

and

$$|d(\mathbf{u},T,z)| \leq \|\mathbf{u}\|_{\mathbf{L}^6} \|\mathbf{grad}\, T\|_0 \|z\|_{L^3} \leq meas(\Omega)^{1/3} \|\mathbf{u}\|_{\mathbf{L}^6} |T|_1 \|z\|_{L^\infty} .$$

Lemma 2.12. *The bilinear form $e(\cdot,\cdot)$ is coercive on $H_0^1(\Omega)$. More precisely,*

$$e(T,T) = \frac{1}{P_r R_e} \|\mathbf{grad}\, T\|_0^2 = \frac{1}{P_r R_e} |T|_1^2 .$$

Lemma 2.13. *Let $\mathbf{u} \in \mathbf{Z}(\Omega)$ and assume that either $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ or one of the functions T or z belongs to $H_0^1(\Omega)$. Then the trilinear form $d(\cdot,\cdot,\cdot)$ is antisymmetric with respect to its last two arguments, i.e.*

$$d(\mathbf{u},T,z) = -d(\mathbf{u},z,T) .$$

As a consequence, under the same hypotheses we have

$$d(\mathbf{u},T,T) = 0 .$$

The proof of these three lemmas is straightforward from the definition of the forms. In order to simplify the notation we denote $\lambda_e := 1/(P_r R_e)$, $\lambda_d := \gamma_4^2$, $\tilde{\lambda}_d := meas(\Omega)^{1/3}$ and $\alpha_e := 1/(P_r R_e)$.

Now, as we did in the MHD problem, we construct a field that satisfies the inhomogeneous boundary condition and that will allow us to rewrite the problem with homogeneous boundary conditions. Since $T_d \in H^{1/2}(\partial\Omega)$, it is known that there exists $T_0 \in H^1(\Omega)$ such that

$$T_{0|\partial\Omega} = T_d \quad \text{with} \quad \|T_0\|_1 \leq \Lambda_3 \|T_d\|_{1/2,\partial\Omega} , \tag{2.49}$$

with Λ_3 a constant depending on the domain Ω.

We can now take $T_2 = \hat{T}_2 + T_0$ and write problem (2.48) in the form

$$\begin{cases} \hat{T}_2 \in H_0^1(\Omega), \\ e(\hat{T}_2, z) + d(\mathbf{u}, \hat{T}_2, z) = -e(T_0, z) - d(\mathbf{u}, T_0, z) \quad \forall z \in H_0^1(\Omega). \end{cases} \tag{2.50}$$

With the three lemmas introduced above we can prove the existence of a unique solution to this problem, in the following:

Proposition 2.14. *There exists a unique solution $\hat{T}_2$ to problem (2.50). Moreover, the following inequality holds*

$$|\hat{T}_2|_1 = \|\mathbf{grad}\, \hat{T}_2\|_0 \leq \frac{1}{\alpha_e} (\lambda_e + \lambda_d \|\mathbf{u}\|_1) \|T_0\|_1 . \tag{2.51}$$

Proof. The result is a consequence of the continuity of $d(\cdot,\cdot,\cdot)$ and $e(\cdot,\cdot)$, the inequalities appearing in Lemma 2.11, the coerciveness of $e(\cdot,\cdot)$ proved in Lemma 2.12, and the antisymmetry property for $d(\cdot,\cdot,\cdot)$ stated in Lemma 2.13. With all these properties, the result is obtained just by applying Lax-Milgram lemma. □

As an immediate consequence of the previous proposition we obtain that there exists a unique solution of problem (2.48). Moreover, due to Poincaré inequality, we have the following estimate

$$\|T_2\|_1 \leq \left[\frac{C_0}{\alpha_e}(\lambda_e + \lambda_d \|\mathbf{u}\|_1) + 1\right] \|T_0\|_1 \,, \tag{2.52}$$

where C_0 is the constant appearing in inequality (A.7).

Another bound for T_2 can be stated when the given boundary data T_d is essentially bounded.

Proposition 2.15. *If $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and $T_{0|\partial\Omega} = T_d$, the solution T_2 to problem (2.48) satisfies*

$$\|T_2\|_{L^\infty(\Omega)} \leq \|T_d\|_{L^\infty(\partial\Omega)} \,. \tag{2.53}$$

Proof. The result is a consequence of Theorems 3.6 and 3.7 in [94]. □

Definition 2.2. *Let us now introduce the following mapping:*

$$\begin{array}{rcl} G_D : \mathbf{Z}(\Omega) & \longrightarrow & H^1(\Omega) \\ \mathbf{u} & \longmapsto & G_D(\mathbf{u}) := T_2 = \hat{T}_2 + T_0 \,, \end{array} \tag{2.54}$$

T_2 and $\hat{T}_2$ being the solutions of problems (2.48) and (2.50), respectively, for the velocity field $\mathbf{u}$.

Lemma 2.16. *The mapping G_D is continuous.*

Proof. Let $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{Z}(\Omega)$, we denote $\hat{T}_{2,n} = G_D(\mathbf{u}_n) - T_0$ and $\hat{T}_2 = G_D(\mathbf{u}) - T_0$. We can write the equation of problem (2.50) for $\mathbf{u}_n$, $\hat{T}_{2,n}$ and for $\mathbf{u}$, $\hat{T}_2$. Subtracting the two equations, and adding and subtracting the term $d(\mathbf{u}, \hat{T}_{2,n}, z)$ we get,

$$e(\hat{T}_{2,n} - \hat{T}_2, z) + d(\mathbf{u}_n - \mathbf{u}, \hat{T}_{2,n}, z) + d(\mathbf{u}, \hat{T}_{2,n} - \hat{T}_2, z) = -d(\mathbf{u}_n - \mathbf{u}, T_0, z).$$

Taking $z = \hat{T}_{2,n} - \hat{T}_2$ as test function, and reminding the coerciveness property for $e(\cdot,\cdot)$, and the continuity and antisymmetry properties for $d(\cdot,\cdot,\cdot)$, we have

$$\alpha_e \left|\hat{T}_{2,n} - \hat{T}_2\right|_1 \leq \lambda_d \left(\|\mathbf{u}_n - \mathbf{u}\|_1 \|\hat{T}_{2,n}\|_1 + \|\mathbf{u}_n - \mathbf{u}\|_1 \|T_0\|_1\right) .$$

Since $\hat{T}_{2,n}$ is bounded in $H^1(\Omega)$ and $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{H}^1(\Omega)$, the result holds. □

Analysis of the thermal subproblem with L^1 source and homogeneous boundary conditions.

For the mathematical analysis of (2.47), we shall make use of some of the results proved in [94], which are summarized in Appendix B. First, let us write the problem in a more convenient way to apply the theory of solution by transposition. Given the velocity $\mathbf{u} \in \mathbf{Z}(\Omega) \cap \mathbf{L}^N(\Omega)$ we define the operator $L_{\mathbf{u}}$ by

$$L_{\mathbf{u}} T = -\frac{1}{P_r R_e} \Delta T + \mathbf{u} \cdot \mathbf{grad}\, T, \tag{2.55}$$

which is of the form (B.1), with $A(\mathbf{x}) = -\frac{1}{P_r R_e} I$, I being the identity matrix, $\mathbf{b}(\mathbf{x}) = \mathbf{u}$, $\mathbf{d}(\mathbf{x}) = \mathbf{0}$ and $c(\mathbf{x}) = 0$. Thus, the assumptions (B.2)-(B.5) hold true.

Using this notation, the thermal subproblem with L^1 source consists on finding T_1 such that

$$\begin{cases} L_{\mathbf{u}} T_1 = f, \\ T_{1|\partial\Omega} = 0. \end{cases} \tag{2.56}$$

We will prove the existence of a unique solution to this problem by transposition. First, we notice that $L_{\mathbf{u}}$, when considered as an operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, is associated to the bilinear form

$$a_{\mathbf{u}}(T, z) := \frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, T \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, T\, z \,\mathrm{d}\mathbf{x} = e(T, z) + d(\mathbf{u}, T, z). \tag{2.57}$$

We can also define $L^*_{\mathbf{u}} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ the adjoint operator of $L_{\mathbf{u}}$, which is given by

$$L^*_{\mathbf{u}} T = -\frac{1}{P_r R_e} \Delta T - \mathrm{div}\, (\mathbf{u} T) = -\frac{1}{P_r R_e} \Delta T - \mathbf{u} \cdot \mathbf{grad}\, T, \tag{2.58}$$

where the equality holds due to $\mathbf{u}$ being divergence free. In view of this last equation, it is clear that $L^*_{\mathbf{u}} = L_{-\mathbf{u}}$. Moreover, the operator $L^*_{\mathbf{u}}$ is associated to the bilinear form $a^*(u, v) := a_{\mathbf{u}}(v, u)$. As it is explained in Appendix B, we can construct the Green operator $G^* = G_{-\mathbf{u}} : H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)$ such that, for any $g \in H^{-1}(\Omega)$, $G_{-\mathbf{u}} g = w$, where w is the unique solution to the problem

$$\begin{cases} w \in H_0^1(\Omega), \\ a_{\mathbf{u}}(v, w) = \langle g, v \rangle_{H^{-1}, H_0^1} \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{2.59}$$

In order to define the solution to problem (2.56) by transposition, we must check that our operator $L_{\mathbf{u}}$ satisfies conditions (B.13) and (B.14). Since $c = 0$ and $\mathrm{div}\, \mathbf{b} = \mathrm{div}\, \mathbf{u} = 0$, condition (B.13) is clearly satisfied. Moreover, for a given velocity $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the coerciveness of the bilinear operator $a_{\mathbf{u}}(\cdot, \cdot)$ is a consequence of the coerciveness of $e(\cdot, \cdot)$ on $H_0^1(\Omega)$, proved in Lemma 2.7, and the antisymmetry property of $d(\cdot, \cdot, \cdot)$ stated in Lemma 2.8.

The solution to problem (2.56) by transposition is defined as $T_1 = G^t_{-\mathbf{u}} f$, where $G^t_{-\mathbf{u}}$ is the adjoint operator of $G_{-\mathbf{u}}$, and it is the unique solution of

$$\begin{cases} T_1 \in W_0^{1,q}(\Omega), \quad 1 < q < \frac{N}{N-1}, \\ \displaystyle\int_\Omega T_1 \varphi \,\mathrm{d}\mathbf{x} = \int_\Omega f(G_{-\mathbf{u}} \varphi) \,\mathrm{d}\mathbf{x} \quad \forall \varphi \in \mathcal{D}(\Omega)\,. \end{cases} \tag{2.60}$$

This problem is equivalent to

$$\left\{\begin{array}{l} T_1 \in W_0^{1,q}(\Omega), \quad 1 < q < \frac{N}{N-1}, \\ \displaystyle\int_\Omega T_1(L_{-\mathbf{u}}\psi)\,\mathrm{d}\mathbf{x} = \int_\Omega f\psi\,\mathrm{d}\mathbf{x} \quad \forall\psi \in H_0^1(\Omega)\cap L^\infty(\Omega) \text{ such that } L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega). \end{array}\right. \tag{2.61}$$

What we must do next is to find a bound for the solution T_1 of problem (2.56), independent of the velocity $\mathbf{u}$.

Lemma 2.17. *Assume $N \geq 3$. Given $\mathbf{u} \in \mathbf{Z}(\Omega) \cap \mathbf{L}^N(\Omega)$, $f_i \in L^p(\Omega)$, $i = 1, \ldots, N$, $p > N$ and the operator $L_{-\mathbf{u}}$ defined analogously to (2.55), then for the solution of the problem*

$$\left\{\begin{array}{l} w \in H_0^1(\Omega), \\ \displaystyle L_{-\mathbf{u}}w = -\sum_{i=1}^N \frac{\partial f_i}{\partial x_i} \quad \text{in } \mathcal{D}'(\Omega), \end{array}\right. \tag{2.62}$$

the following inequality holds

$$\|w\|_{L^\infty} \leq K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{p}} \left(\sum_{i=1}^N \|f_i\|_{L^p}^2\right)^{1/2},$$

with $K_1 = K_1(N,p)$ a constant independent of the velocity $\mathbf{u}$ and the domain Ω.

Proof. The proof follows most of the steps of the proof of Theorem 4.2 in [94]. First we recall that the problem is equivalent to

$$\left\{\begin{array}{l} w \in H_0^1(\Omega), \\ \displaystyle\frac{1}{P_r R_e}\int_\Omega \mathbf{grad}\,w \cdot \mathbf{grad}\,z\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{grad}\,w\,z\,\mathrm{d}\mathbf{x} = \sum_{i=1}^N \int_\Omega f_i \frac{\partial z}{\partial x_i}\,\mathrm{d}\mathbf{x} \quad \forall z \in H_0^1(\Omega). \end{array}\right. \tag{2.63}$$

For any $k \geq 0$, we take $z = (w-k)_+$ as test function, which is defined by

$$(w-k)_+(\mathbf{x}) = \left\{\begin{array}{ll} w(\mathbf{x}) - k & \text{if } w(\mathbf{x}) > k, \\ 0 & \text{if } w(\mathbf{x}) \leq k, \end{array}\right.$$

It is well known that $z \in H_0^1(\Omega)$. Moreover, its partial derivatives are given by

$$\frac{\partial z}{\partial x_i} = \left\{\begin{array}{ll} \dfrac{\partial w}{\partial x_i} & \text{if } w(\mathbf{x}) > k, \\ 0 & \text{if } w(\mathbf{x}) \leq k. \end{array}\right.$$

If we denote by $A(k) = \{\mathbf{x} \in \Omega : w(\mathbf{x}) > k\}$, it is clear that for our choice of z we have

$$\frac{1}{P_r R_e}\int_{A(k)} \mathbf{grad}\,w \cdot \mathbf{grad}\,z\,d\mathbf{x} - \int_{A(k)} \mathbf{u}\cdot\mathbf{grad}\,w\,z\,d\mathbf{x} = \sum_{i=1}^N \int_{A(k)} f_i \frac{\partial z}{\partial x_i}\,d\mathbf{x},$$

then

$$\frac{1}{P_rR_e}\int_\Omega \mathbf{grad}\, z\cdot\mathbf{grad}\, z\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{grad}\, z\, z\, d\mathbf{x}$$
$$= \frac{1}{P_rR_e}\int_{A(k)} \mathbf{grad}\, z\cdot\mathbf{grad}\, z\, d\mathbf{x} - \int_{A(k)} \mathbf{u}\cdot\mathbf{grad}\, z\, z\, d\mathbf{x} = \sum_{i=1}^N \int_{A(k)} f_i \frac{\partial z}{\partial x_i}\, d\mathbf{x}.$$

Since $\operatorname{div}\mathbf{u} = 0$, using the antisymmetry property already proved for $d(\cdot,\cdot,\cdot)$ and the Cauchy-Schwarz inequality, we obtain

$$\alpha_e \|\mathbf{grad}\, z\|_0^2 = \sum_{i=1}^N \int_{A(k)} f_i \frac{\partial z}{\partial x_i}\, d\mathbf{x} \leq \left(\int_{A(k)} \sum_{i=1}^N f_i^2(\mathbf{x})\, d\mathbf{x}\right)^{1/2} \|\mathbf{grad}\, z\|_0\,,$$

and hence

$$\|\mathbf{grad}\, z\|_0^2 \leq \frac{1}{\alpha_e^2}\int_{A(k)} \sum_{i=1}^N f_i^2\, d\mathbf{x}.$$

We will make use of the following Sobolev inequality which states that $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, with $1/2^* = 1/2 - 1/N$:

$$\|z\|_{L^{2^*}} \leq S\left(\sum_{i=1}^N \left\|\frac{\partial z}{\partial x_i}\right\|_0^2\right)^{1/2} = S\,\|\mathbf{grad}\, z\|_0\,, \quad \forall z \in H_0^1(\Omega), \tag{2.64}$$

with $S = S(N)$ a certain constant independent of Ω (see [29, Ch. IX]). Then, by applying this result and Hölder inequality, we get

$$\left(\int_{A(k)} (w(\mathbf{x}) - k)^{2^*}\, d\mathbf{x}\right)^{2/2^*} = \|z\|_{L^{2^*}}^2 \leq S^2\,\|\mathbf{grad}\, z\|_0^2 \leq \frac{S^2}{\alpha_e^2}\int_{A(k)} \sum_{i=1}^N f_i^2\, d\mathbf{x}$$
$$\leq \frac{S^2}{\alpha_e^2}\sum_{i=1}^N \|f_i\|_{L^p}^2\, meas(A(k))^{1-2/p}. \tag{2.65}$$

Let $h > k$, then $A(h) = \{\mathbf{x} \in \Omega : w(\mathbf{x}) > h\} \subset A(k)$ and it holds

$$\int_{A(k)} (w(\mathbf{x}) - k)^{2^*}\, d\mathbf{x} \geq \int_{A(h)} (w(\mathbf{x}) - k)^{2^*}\, d\mathbf{x} \geq \int_{A(h)} (h-k)^{2^*}\, d\mathbf{x} \geq (h-k)^{2^*} meas(A(h))\,.$$

Using this inequality and (2.65) we obtain

$$(h-k)^2 meas(A(h))^{2/2^*} \leq \frac{S^2}{\alpha_e^2}\sum_{i=1}^N \|f_i\|_{L^p}^2\, meas(A(k))^{1-2/p},$$

or equivalently

$$meas(A(h)) \leq \frac{1}{(h-k)^{2^*}}\left(\frac{S}{\alpha_e}\right)^{2^*}\left(\sum_{i=1}^N \|f_i\|_{L^p}^2\right)^{2^*/2} meas(A(k))^{\beta},$$

with $\beta = (2^*/2)(1-2/p) = (1-2/p)/(1-2/N)$, and since $N < p$ we have that $\beta > 1$.

If we apply Lemma B.1 (see Appendix B) with

$$\begin{aligned} \varphi(h) &= meas(A(h)), \\ C &= \left(\frac{S}{\alpha_e}\right)^{2^*} \left(\sum_{i=1}^{N} \|f_i\|_{L^p}^2\right)^{2^*/2}, \\ \alpha &= 2^*, \\ \beta &= \frac{2^*}{2}\left(1-\frac{2}{p}\right) > 1, \\ k_0 &= 0, \end{aligned}$$

we get

$$\varphi(d) = meas(A(d)) = 0,$$

with

$$d^\alpha = C(\varphi(0))^{\beta-1} 2^{\alpha\beta/(\beta-1)}.$$

Thus, we have obtained $meas(A(d)) = meas(\{\mathbf{x} \in \Omega : w(\mathbf{x}) > d\}) = 0$, which implies

$$\|w\|_{L^\infty} \le d.$$

Since the constants will play an important role in the existence result for the coupled problem, we will give a more precise bound for d. We must take into account that

$$\varphi(0) = meas(A(0)) \le meas(\Omega),$$
$$\beta - 1 = \frac{1-2/p}{1-2/N} - 1 = \frac{2/N-2/p}{1-2/N} = \frac{1/N-1/p}{1/2^*},$$
$$\frac{\beta}{\beta-1} = \frac{1/2-1/p}{1/2-1/N}\,\frac{1/2-1/N}{1/N-1/p} = \frac{1/2-1/p}{1/N-1/p}.$$

Using these results and the definitions of α, β, φ we obtain the inequality

$$\|w\|_{L^\infty} \le d = C^{1/\alpha}(\varphi(0))^{(\beta-1)/\alpha} 2^{\beta/(\beta-1)} \le meas(\Omega)^{\frac{1}{N}-\frac{1}{p}} 2^{\frac{1/2-1/p}{1/N-1/p}} \frac{S}{\alpha_e}\left(\sum_{i=1}^{N}\|f_i\|_{L^p}^2\right)^{1/2},$$

and the result holds with $K_1 = \frac{S}{\alpha_e} 2^{\frac{1/2-1/p}{1/N-1/p}}$. □

Proposition 2.18. *Given $f \in L^1(\Omega)$, $\mathbf{u} \in \mathbf{Z}(\Omega) \cap \mathbf{L}^N(\Omega)$ and $L_{\mathbf{u}}$ the operator defined in (2.55), then the solution to problem (2.61) satisfies*

$$\|T_1\|_{1,q} \le K_2 \|f\|_{L^1}, \tag{2.66}$$

with $K_2 \equiv K_2(q)$ a constant independent of the velocity $\mathbf{u}$ and of the right-hand side f.

Proof. The idea of the proof is to use the previous Lemma to find a bound for the norm of the Green operator $G_{-\mathbf{u}}$, and then, by a transposition argument, to obtain a bound for its adjoint operator $G^t_{-\mathbf{u}}$.

To obtain the bound for $G_{-\mathbf{u}}$ we first remind that for $F \in W^{-1,q'}(\Omega) \subset H^{-1}(\Omega)$, with $1 < q < N/(N-1)$, $w = G_{-\mathbf{u}}F$ is the unique solution to problem

$$\begin{cases} w \in H^1_0(\Omega), \\ L_{-\mathbf{u}} w = F \ \text{ in } \mathcal{D}'(\Omega), \end{cases} \tag{2.67}$$

or equivalently

$$\begin{cases} w \in H^1_0(\Omega), \\ \dfrac{1}{P_r R_e} \displaystyle\int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, w\, z \,\mathrm{d}\mathbf{x} = \langle F, z \rangle_{H^{-1},H^1_0} \quad \forall z \in H^1_0(\Omega). \end{cases} \tag{2.68}$$

Since $F \in W^{-1,q'}(\Omega)$ and Ω is bounded, we know that (see [29, Prop. IX.20])

$$F = -\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}, \quad f_i \in L^{q'}(\Omega)\,,$$

with

$$\max_{1 \le i \le N} \|f_i\|_{L^{q'}} = |F|_{-1,q'}\ , \tag{2.69}$$

so equation (2.68) can be rewritten in the form

$$\frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, w \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, w\, z \,\mathrm{d}\mathbf{x} = \sum_{i=1}^N \int_\Omega f_i \frac{\partial z}{\partial x_i} \,\mathrm{d}\mathbf{x} \quad \forall z \in H^1_0(\Omega). \tag{2.70}$$

Since $q < N/(N-1)$ we know that $q' > N$. Then, from Lemma 2.17 and taking into account the equality (2.69) we have

$$\begin{aligned} \|w\|_{L^\infty} = \|G_{-\mathbf{u}}F\|_{L^\infty} &\le K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}} \left(\sum_{i=1}^N \|f_i\|^2_{L^{q'}} \right)^{1/2} \\ &\le K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}} \sqrt{N}\, |F|_{-1,q'}\,, \end{aligned}$$

with a constant independent of the velocity $\mathbf{u}$, provided $\operatorname{div} \mathbf{u} = 0$. Thus,

$$\|G_{-\mathbf{u}}\|_{\mathcal{L}(W^{-1,q'}(\Omega), L^\infty(\Omega))} \le K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}} \sqrt{N}.$$

Next step is to estimate, from the bound for the norm of $G_{-\mathbf{u}}$, a bound for $\left\|G^t_{-\mathbf{u}}\right\|_{\mathcal{L}(L^1(\Omega), W^{1,q}_0(\Omega))}$. If we consider in $W^{1,q}_0(\Omega)$ the norm $|\cdot|_{1,q}$, and in its dual space $W^{-1,q'}(\Omega)$ the corresponding induced norm $|\cdot|_{-1,q'}$, we obtain

$$\left\|G^t_{-\mathbf{u}}\right\|_{\mathcal{L}(L^\infty(\Omega)', W^{1,q}_0(\Omega))} = \|G_{-\mathbf{u}}\|_{\mathcal{L}(W^{-1,q'}(\Omega), L^\infty(\Omega))}\ ,$$

because $W_0^{1,q}(\Omega)$ is reflexive.

As $L^1(\Omega)$ is isometrically imbedded in $L^\infty(\Omega)'$, we also have

$$\left\|G^t_{-\mathbf{u}}\right\|_{\mathcal{L}(L^1(\Omega),W_0^{1,q}(\Omega))} \leq \|G_{-\mathbf{u}}\|_{\mathcal{L}(W^{-1,q'}(\Omega),L^\infty(\Omega))} \leq K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}}\sqrt{N}\,,$$

so, if T_1 is solution to problem (2.60), we have

$$\|T_1\|_{1,q} \leq C(q)|T_1|_{1,q} \leq C(q)\left\|G^t_{-\mathbf{u}}\right\|_{\mathcal{L}(L^1(\Omega),W_0^{1,q}(\Omega))}\|f\|_{L^1} \leq C(q)K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}}\sqrt{N}\,\|f\|_{L^1}\ ,$$

with $C(q)$ the constant appearing in (A.3), and the result is true with

$$K_2(q) = C(q)K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}}\sqrt{N},$$

a constant dependent on Ω, q and N. □

Now we are going to prove the continuity of the operator $G^t_{-\mathbf{u}}$ with respect to the velocity $\mathbf{u}$ and the source f. To do that let us introduce the mapping

$$\begin{aligned} \widetilde{G} : \big(\mathbf{Z}(\Omega)\cap\mathbf{L}^N(\Omega)\big)\times L^1(\Omega) &\longrightarrow W_0^{1,q}(\Omega) \\ (\mathbf{u},f) &\longmapsto \widetilde{G}(\mathbf{u},f) := G^t_{-\mathbf{u}}f\,, \end{aligned} \tag{2.71}$$

where $G^t_{-\mathbf{u}}f$ is the solution by transposition to problem (2.56). We will first prove the continuity of the mapping with respect to the velocity, for a fixed source $f \in L^2(\Omega)$, and then prove the general result of continuity with respect to the velocity and to the source in $L^1(\Omega)$.

Lemma 2.19. *Given $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{Z}(\Omega)\cap\mathbf{L}^N(\Omega)$ and $g \in L^2(\Omega)$, it holds that $\widetilde{G}(\mathbf{u}_n,g) \to \widetilde{G}(\mathbf{u},g)$ strongly in $W^{1,q}(\Omega)$.*

Proof. Since $g \in L^2(\Omega)$, the solution by transposition $\widetilde{G}(\mathbf{u},g)$ coincides with the weak solution in $H_0^1(\Omega)$ (see Appendix B). The convergence in $H_0^1(\Omega)$ is easily proved, with the arguments already used in Lemma 2.16. Since $q < N/(N-1) < 2$, we have $H_0^1(\Omega) \subset W_0^{1,q}(\Omega)$, which implies the strong convergence in $W^{1,q}(\Omega)$. □

Proposition 2.20. *The mapping $\widetilde{G}$ is continuous on $\big(\mathbf{Z}(\Omega)\cap\mathbf{L}^N(\Omega)\big)\times L^1(\Omega)$.*

Proof. Let $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{Z}(\Omega)\cap\mathbf{L}^N(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$. We have to prove that $\widetilde{G}(\mathbf{u}_n,f_n) \to \widetilde{G}(\mathbf{u},f)$ converges strongly in $W_0^{1,q}(\Omega)$. The proof will consist in using several triangular inequalities.

First, we have

$$\widetilde{G}(\mathbf{u}_n,f_n) - \widetilde{G}(\mathbf{u},f) = \widetilde{G}(\mathbf{u}_n,f_n) - \widetilde{G}(\mathbf{u}_n,f) + \widetilde{G}(\mathbf{u}_n,f) - \widetilde{G}(\mathbf{u},f)\,, \tag{2.72}$$

and using Proposition 2.18 there exists a constant C, independent of $\mathbf{u}_n$, such that

$$\|\widetilde{G}(\mathbf{u}_n,f_n) - \widetilde{G}(\mathbf{u}_n,f)\|_{1,q} \leq C\,\|f_n - f\|_{L^1}\ , \tag{2.73}$$

which tends to zero, due to the convergence of f_n to f in $L^1(\Omega)$.

Now, for any $k \geq 0$ we introduce the truncation function

$$\tau_k(x) = \begin{cases} k, & \text{if } x > k, \\ x, & \text{if } -k \leq x \leq k, \\ -k, & \text{if } x < -k. \end{cases} \tag{2.74}$$

Composing f with τ_k, we obtain the truncated function $\tau_k f$. We can use this truncated function to prove the convergence to zero of the remaining term, in the following way:

$$\widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}, f) = \widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}_n, \tau_k f) + \widetilde{G}(\mathbf{u}_n, \tau_k f) - \widetilde{G}(\mathbf{u}, \tau_k f) + \widetilde{G}(\mathbf{u}, \tau_k f) - \widetilde{G}(\mathbf{u}, f),$$

and using triangular inequalities and Proposition 2.18 we obtain

$$\begin{aligned} \|\widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}, f)\|_{1,q} &\leq \|\widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}_n, \tau_k f)\|_{1,q} + \|\widetilde{G}(\mathbf{u}_n, \tau_k f) - \widetilde{G}(\mathbf{u}, \tau_k f)\|_{1,q} \\ &+ \|\widetilde{G}(\mathbf{u}, \tau_k f) - \widetilde{G}(\mathbf{u}, f)\|_{1,q} \leq 2C\|f - \tau_k f\|_{L^1} + \|\widetilde{G}(\mathbf{u}_n, \tau_k f) - \widetilde{G}(\mathbf{u}, \tau_k f)\|_{1,q}. \end{aligned}$$

The first term converges to zero as $k \to \infty$ by applying the Lebesgue dominated convergence theorem, whereas for fixed $k \geq 0$ the second one converges to zero as $n \to \infty$ due to the result proved in Lemma 2.19, as $\tau_k f \in L^2(\Omega)$. Joining these two convergence results with (2.72) and (2.73), the desired result follows. □

2.2.6 Coupled problem.

In order to prove the existence of a solution to our coupled problem via a fixed point theorem, a mapping from $\mathcal{Z}_0(\Omega)$ into itself will be introduced, and then we will prove the existence of a fixed point for that mapping. To do that, we first introduce the two following mappings:

$$\begin{aligned} \mathcal{G}_2 : \mathbf{Z}_0(\Omega) \times L^1(\Omega) &\longrightarrow W^{1,6/5}(\Omega) \\ (\hat{\mathbf{w}}, f) &\longmapsto \mathcal{G}_2(\hat{\mathbf{w}}, f) := \widetilde{G}(\mathbf{w}, f) + G_D(\mathbf{w}), \end{aligned} \tag{2.75}$$

where $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{u}_0$ and mappings $\widetilde{G}$ and G_D have been introduced in the previous section;

$$\begin{aligned} \mathcal{G}_3 : \mathcal{Z}_0(\Omega) &\to L^1(\Omega) \\ (\hat{\mathbf{w}}, \hat{\mathbf{D}}) &\mapsto \mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \frac{E_c}{R_e}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\hat{\mathbf{D}}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{w} + \mathbf{grad}\,\mathbf{w}^t\right|^2\right] + \psi, \end{aligned} \tag{2.76}$$

where $\mathbf{w}$ is defined as above and $\psi \in L^1(\Omega)$. Thus, the application maps any $(\hat{\mathbf{w}}, \hat{\mathbf{D}})$ to its correspondent heat source in the heat equation.

To find a solution of our problem it suffices to find a fixed point of the mapping

$$\begin{aligned} \mathcal{G} : \mathcal{Z}_0(\Omega) &\longrightarrow \mathcal{Z}_0(\Omega) \\ (\hat{\mathbf{w}}, \hat{\mathbf{D}}) &\longmapsto \mathcal{G}((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \mathcal{G}_1\left((\hat{\mathbf{w}}, \hat{\mathbf{D}}), \mathcal{G}_2\left(\hat{\mathbf{w}}, \mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))\right)\right), \end{aligned} \tag{2.77}$$

recalling that $\mathcal{G}_1$ is the mapping introduced in Definition 2.1 at the end of Section 2.2.4.

Lemma 2.21. *If* $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$*, with* $\mathbf{curl}\, \mathbf{B}_0 = \mathbf{0}$*,* $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ *and* $\psi \in L^1(\Omega)$*, under the assumptions*

$$\begin{aligned} \|\mathbf{u}_0\|_{\mathbf{L}^4} &< \frac{\alpha_a}{\tilde{\lambda}_{c0}} \,, \\ \|\mathbf{u}_0\|_{\mathbf{L}^6} &< \frac{\alpha_a}{\tilde{\lambda}_{c1}} \,, \\ \mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) &\le \frac{(k_1 - \alpha_a)^2}{4\lambda_G K_2 k_f}, \end{aligned}$$

there exists a constant $R > 0$ *such that if* $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \le R$ *then* $|\mathcal{G}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \le R$*.*

In the assumptions, $\mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi)$ *depends on the boundary and source data and it is given by the expression*

$$\begin{aligned} \mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) = \lambda_F + \lambda_G \Big(8K_2 k_{f2} \, |\mathbf{u}_0|_1^2 + K_2 \, \|\psi\|_{L^1} + meas(\Omega)^{5/6} \, \|T_d\|_{L^\infty(\partial\Omega)}\Big) \\ + \lambda_{a0} \, |\mathbf{u}_0|_1 + \lambda_{c0} \, \|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \lambda_{c1} \, \|\mathbf{B}_0\|_{\mathbf{L}^3} \, \|\mathbf{u}_0\|_{\mathbf{L}^6} \;. \end{aligned}$$

The constants have the expressions $k_{f1} = E_c H_a^2/(R_e R_m^2)$*,* $k_{f2} = E_c/(2R_e)$*,* $k_f = \max\{k_{f1}, 8k_{f2}\}$ *and*

$$k_1 = \max\left\{\tilde{\lambda}_{c0} \, \|\mathbf{u}_0\|_{\mathbf{L}^4} \,, \tilde{\lambda}_{c1} \, \|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}.$$

The remaining constants have been already introduced in Lemmas 2.6 and 2.7, and in Propositions 2.9 and 2.18.

Proof. Let $R > 0$ be a real number and $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ such that $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \le R$. According to the definition of k_{f1}, k_{f2} and k_f, we have

$$\begin{aligned} &\|\mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))\|_{L^1} \\ &\le k_{f1} \| |\mathbf{curl}\, \hat{\mathbf{D}}|^2 \|_{L^1} + k_{f2} \| |\mathbf{grad}\, \hat{\mathbf{w}} + \mathbf{grad}\, \hat{\mathbf{w}}^t + \mathbf{grad}\, \mathbf{u}_0 + \mathbf{grad}\, \mathbf{u}_0^t|^2 \|_{L^1} + \|\psi\|_{L^1} \\ &\le k_{f1} \| |\mathbf{curl}\, \hat{\mathbf{D}}|^2 \|_{L^1} + 8k_{f2} \| |\mathbf{grad}\, \hat{\mathbf{w}}|^2 + |\mathbf{grad}\, \mathbf{u}_0|^2 \|_{L^1} + \|\psi\|_{L^1} \\ &= k_{f1} \|\mathbf{curl}\, \hat{\mathbf{D}}\|_0^2 + 8k_{f2} (\|\mathbf{grad}\, \hat{\mathbf{w}}\|_0^2 + \|\mathbf{grad}\, \mathbf{u}_0\|_0^2) + \|\psi\|_{L^1} \\ &\le k_f |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}}^2 + 8k_{f2} \, |\mathbf{u}_0|_1^2 + \|\psi\|_{L^1} \;. \end{aligned} \tag{2.78}$$

Next, if we denote $f = \mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))$, then

$$\begin{aligned} \|\mathcal{G}_2(\hat{\mathbf{w}}, f)\|_{L^{6/5}} = \|\widetilde{G}(\mathbf{w}, f) + G_D(\mathbf{w})\|_{L^{6/5}} \le \|\widetilde{G}(\mathbf{w}, f)\|_{L^{6/5}} + \|G_D(\mathbf{w})\|_{L^{6/5}} \\ \le K_2 \, \|f\|_{L^1} + meas(\Omega)^{5/6} \, \|T_d\|_{L^\infty(\partial\Omega)} \;, \end{aligned} \tag{2.79}$$

with $K_2 \equiv K_2(6/5) = C(6/5)\sqrt{3} K_1 meas(\Omega)^{1/6}$, as given in Proposition 2.18. The value $meas(\Omega)^{5/6}$ appears as a consequence of Proposition 2.15.

For the third step, if we denote $T = \mathcal{G}_2(\hat{\mathbf{w}}, f)$ and recall Proposition 2.9, we get

$$\begin{aligned} \alpha_a |\mathcal{G}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T)|_{\mathcal{W}} \le \lambda_F + \lambda_G \, \|T\|_{L^{6/5}} + \lambda_{a0} \, |\mathbf{u}_0|_1 + \lambda_{c0} \, \|\mathbf{u}_0\|_{\mathbf{L}^4}^2 \\ + \lambda_{c1} \, \|\mathbf{B}_0\|_{\mathbf{L}^3} \, \|\mathbf{u}_0\|_{\mathbf{L}^6} + k_1 |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \;, \end{aligned} \tag{2.80}$$

with $k_1 := \max\{\tilde{\lambda}_{c0}\left\|\mathbf{u}_0\right\|_{\mathbf{L}^4}, \tilde{\lambda}_{c1}\left\|\mathbf{u}_0\right\|_{\mathbf{L}^6}\}$.

Joining the three inequalities we obtain

$$\begin{aligned}
\alpha_a|\mathcal{G}((\hat{\mathbf{w}},\hat{\mathbf{D}}))|_{\mathcal{W}} &\leq \lambda_F + \lambda_G\left\|T\right\|_{L^{6/5}} + \lambda_{a0}\left|\mathbf{u}_0\right|_1 \\
&\quad +\lambda_{c0}\left\|\mathbf{u}_0\right\|^2_{\mathbf{L}^4} + \lambda_{c1}\left\|\mathbf{B}_0\right\|_{\mathbf{L}^3}\left\|\mathbf{u}_0\right\|_{\mathbf{L}^6} + k_1|(\hat{\mathbf{w}},\hat{\mathbf{D}})|_{\mathcal{W}} \\
&\leq \lambda_F + \lambda_G(K_2\|f\|_{L^1} + meas(\Omega)^{5/6}\|T_d\|_{L^\infty(\partial\Omega)}) + \lambda_{a0}\left|\mathbf{u}_0\right|_1 \\
&\quad +\lambda_{c0}\left\|\mathbf{u}_0\right\|^2_{\mathbf{L}^4} + \lambda_{c1}\left\|\mathbf{B}_0\right\|_{\mathbf{L}^3}\left\|\mathbf{u}_0\right\|_{\mathbf{L}^6} + k_1|(\hat{\mathbf{w}},\hat{\mathbf{D}})|_{\mathcal{W}} \\
&\leq \lambda_F + \lambda_G\left[K_2(k_f|(\hat{\mathbf{w}},\hat{\mathbf{D}})|^2_{\mathcal{W}} + 8k_{f2}\left|\mathbf{u}_0\right|^2_1 + \|\psi\|_{L^1}) + meas(\Omega)^{5/6}\|T_d\|_{L^\infty(\partial\Omega)}\right] \\
&\quad +\lambda_{a0}\left|\mathbf{u}_0\right|_1 + \lambda_{c0}\left\|\mathbf{u}_0\right\|^2_{\mathbf{L}^4} + \lambda_{c1}\left\|\mathbf{B}_0\right\|_{\mathbf{L}^3}\left\|\mathbf{u}_0\right\|_{\mathbf{L}^6} + k_1|(\hat{\mathbf{w}},\hat{\mathbf{D}})|_{\mathcal{W}} \\
&= k_0 + k_1|(\hat{\mathbf{w}},\hat{\mathbf{D}})|_{\mathcal{W}} + k_2|(\hat{\mathbf{w}},\hat{\mathbf{D}})|^2_{\mathcal{W}}\,, \qquad (2.81)
\end{aligned}$$

where we have defined the constants

$$\begin{aligned}
k_0 = \lambda_F + \lambda_G\Big(K_2(8k_{f2}\left|\mathbf{u}_0\right|^2_1 + \|\psi\|_{L^1}) + meas(\Omega)^{5/6}\left\|T_d\right\|_{L^\infty(\partial\Omega)}\Big) \\
+\lambda_{a0}\left|\mathbf{u}_0\right|_1 + \lambda_{c0}\left\|\mathbf{u}_0\right\|^2_{\mathbf{L}^4} + \lambda_{c1}\left\|\mathbf{B}_0\right\|_{\mathbf{L}^3}\left\|\mathbf{u}_0\right\|_{\mathbf{L}^6}
\end{aligned}$$

and $k_2 = \lambda_G K_2 k_f$. We notice that $k_0 = \mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi)$, the expression appearing in the statement of the theorem. We also notice that k_0 and k_1 depend on the given source and boundary data, whereas k_2 is independent of them. Moreover, we know that $k_2 > 0$ and $k_0, k_1 \geq 0$.

We have to prove that there exists a certain constant $R > 0$ such that $|(\hat{\mathbf{w}},\hat{\mathbf{D}})|_{\mathcal{W}} \leq R$ implies $|\mathcal{G}((\hat{\mathbf{w}},\hat{\mathbf{D}}))|_{\mathcal{W}} \leq R$. In view of the last inequality, it is enough to prove that there exists a constant $R > 0$ such that

$$k_0 + (k_1 - \alpha_a)R + k_2R^2 \leq 0. \qquad (2.82)$$

The case $k_0 = 0$ is trivial. Indeed, by the definition of k_0 and k_1, $k_0 = 0$ implies that k_1 must be equal to zero. Thus, the roots of the quadratic equation are $R_- = 0$ and $R_+ = \alpha_a/k_2$, and the result holds for any $R \in (0, \alpha_a/k_2)$.

Let us now consider the case $k_0 > 0$. The roots of the corresponding quadratic equation are given by

$$R_\pm = \frac{\alpha_a - k_1 \pm \sqrt{(k_1-\alpha_a)^2 - 4k_0k_2}}{2k_2}, \qquad (2.83)$$

and since $R_+R_- = k_0/k_2 > 0$, the two roots are either both positive, or both negative or complex. In order to avoid complex roots, which prevent us from finding such an R, we must require $\Delta = (k_1 - \alpha_a)^2 - 4k_0k_2 \geq 0$. Assuming this, we also know that

$$R_- \leq \frac{\alpha_a - k_1}{2k_2} \leq R_+, \qquad (2.84)$$

so, in order to find positive roots we must require $\alpha_a - k_1 > 0$ too. Summarizing, we are able to find a real constant R such that $\mathcal{G}(\overline{B(0,R)}) \subset \overline{B(0,R)}$ by requiring

$$k_1 < \alpha_a, \qquad k_0 \leq \frac{(k_1-\alpha_a)^2}{4k_2}, \qquad (2.85)$$

and, recalling the values for constants k_0, k_1 and k_2, this is equivalent to

$$\|\mathbf{u}_0\|_{\mathbf{L}^4} < \frac{\alpha_a}{\tilde{\lambda}_{c0}},$$

$$\|\mathbf{u}_0\|_{\mathbf{L}^6} < \frac{\alpha_a}{\tilde{\lambda}_{c1}},$$

$$\lambda_F + \lambda_G \left(8K_2 k_{f2} |\mathbf{u}_0|_1^2 + K_2 \|\psi\|_{L^1} + meas(\Omega)^{5/6} \|T_d\|_{L^\infty(\partial\Omega)}\right) + \lambda_{a0} |\mathbf{u}_0|_1$$
$$+\lambda_{c0} \|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \lambda_{c1} \|\mathbf{B}_0\|_{\mathbf{L}^3} \|\mathbf{u}_0\|_{\mathbf{L}^6} \leq \frac{(k_1 - \alpha_a)^2}{4\lambda_G K_2 k_f}.$$

□

Remark 2.6. *The existence of solution for the case $k_0 = 0$ is trivial. Indeed, if $k_0 = 0$ all the source terms are null, except maybe the lifting $\mathbf{B}_0$. Therefore, a fixed point of the mapping $\mathcal{G}$ is the trivial solution, $(\hat{\mathbf{u}}, \hat{\mathbf{B}}) = (\mathbf{0}, \mathbf{0})$, and a solution to problem (2.6)-(2.14) is given by $\mathbf{u} = \mathbf{0}$, $\mathbf{B} = \mathbf{B}_0$ and $T = 0$.*

Now we are in a position to prove the main result of this section:

Theorem 2.22. *Under the hypotheses of Lemma 2.21 the mapping $\mathcal{G}$ has at least one fixed point.*

Proof. The result is a consequence of applying Schauder fixed point theorem to the mapping $\mathcal{G}$. Under the given hypotheses we have already proved in Lemma 2.21 that there exists a constant $R > 0$ such that $\mathcal{G}$ maps the ball $\overline{B(0,R)} \subset \mathcal{Z}_0(\Omega)$ into itself. Moreover, the mapping $\mathcal{G}$ is continuous, because of the continuity of the mappings $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. In order to apply the Schauder fixed point theorem, we must prove the compactness of $\mathcal{G}$, *i.e.*, that $\mathcal{G}$ maps bounded sets into precompact sets.

Let $B \subset \mathcal{Z}_0(\Omega)$ be a bounded set. We must prove that for any sequence $\{(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n)\} \subset B$, $\{\mathcal{G}((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n))\}$ has a convergent subsequence. Since $\mathcal{Z}_0(\Omega)$ is a reflexive Banach space, there exists a subsequence still denoted by $\{(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n)\}$ such that $(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ weakly in $\mathcal{Z}_0(\Omega)$. Due to the definition of $\mathcal{G}_3$, the sequence $f_n = \mathcal{G}_3((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n))$ is bounded in $L^1(\Omega)$. Moreover, due to Proposition 2.18 we also have that $T_n = \mathcal{G}_2(\hat{\mathbf{w}}_n, f_n)$ is bounded in $W^{1,6/5}(\Omega)$. Hence, there is a subsequence that we still denote by T_n such that $T_n \to T$ strongly in $L^{6/5}(\Omega)$. Finally, due to the result proved in Lemma 2.10, it holds that $\mathcal{G}_1((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n), T_n) = (\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) \to \mathcal{G}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T) = (\hat{\mathbf{u}}, \hat{\mathbf{B}})$ strongly in $\mathcal{Z}_0(\Omega)$. Thus $\mathcal{G}(B)$ is relatively compact which ends the proof. □

The previous theorem proves the existence of a solution for the coupled thermal-magneto-hydrodynamic problem. Moreover, we know that the temperature T satisfies (see (B.21))

$$T \in \bigcap_{1<q<3/2} W^{1,q}(\Omega). \tag{2.86}$$

2.3 Steady MHD equations without using the Boussinesq approximation.

In the previous section we have considered the stationary MHD equations using the Boussinesq approximation. Under this approximation we were constrained to impose very strict conditions on the boundary data to ensure the existence of a solution to our problem (see Lemma 2.21). As a first step, in order to study a more complicated model and following some of the ideas appearing in [42], we propose a different mathematical model for which, instead of using Boussinesq approximation, we assume that the density appearing in the buoyancy force is a function of temperature satisfying certain properties. For this density function we are able to obtain an *a priori* bound for the solution of the model, which will lead to prove the existence of solution under less severe constraints on the data.

2.3.1 Mathematical model.

For the mathematical model we consider the steady MHD equations along with the heat transfer equation. Following some ideas appeared in [42], instead of the Boussinesq approximation we consider that density is constant in the left-hand side terms and that it is a function of temperature in the buoyancy force, $\rho = \hat{\rho}(T)$. Moreover, we assume that function $\hat{\rho} : (0, +\infty) \to (0, +\infty)$ is strictly positive, continuous and non-increasing. Notice that these assumptions do not hold for the Boussinesq approximation.

The equations of the model read as follows:

$$\frac{1}{\mu\sigma}\mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}) - \mathbf{curl}\,(\mathbf{u} \times \mathbf{B}) = \mathbf{0}, \tag{2.87}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{2.88}$$

$$-\eta\Delta\mathbf{u} + \rho(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p - \frac{1}{\mu}(\mathbf{curl}\,\mathbf{B}) \times \mathbf{B} = \mathbf{f}_0 + \hat{\rho}(T)\mathbf{g}, \tag{2.89}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{2.90}$$

$$-k\Delta T + \rho c_p \mathbf{u} \cdot \mathbf{grad}\,T = \frac{1}{\sigma\mu^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{\eta}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2 + \psi. \tag{2.91}$$

We can now introduce the dimensionless form of the equations. As we did in Section 2.2.1 let us denote by $\mathcal{B}$, u and $\mathcal{L}$ the typical values of magnetic induction, velocity and length, respectively. Let us also denote by $\mathcal{T}$ the typical value of temperature. We normalize the equations as follows: the magnetic induction $\mathbf{B}$ by $\mathcal{B}$, the velocity $\mathbf{u}$ by u, the pressure p by $\sigma u \mathcal{B}^2 \mathcal{L}$, the body force $\mathbf{f}_0$ by $\sigma u \mathcal{B}^2$, the temperature T by $\mathcal{T}$, the heat source ψ by $\rho c_p u \mathcal{T}/\mathcal{L}$ and the density function $\tilde{\rho}$ by $\sigma u \mathcal{B}^2$. After this normalization, the buoyancy term is expressed in the form $\hat{\varrho}(\hat{T}) = \frac{1}{\sigma u \mathcal{B}^2}\hat{\rho}(T) = \frac{1}{\sigma u \mathcal{B}^2}\hat{\rho}(\mathcal{T}\hat{T})$. Now, maintaining the same notation for the normalized fields we arrive at the following non-dimensionalized system of equations, which holds in Ω:

$$\frac{1}{R_m}\mathbf{curl}\,(\mathbf{curl}\,\mathbf{B}) - \mathbf{curl}\,(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{2.92}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{2.93}$$

$$-\frac{1}{H_a^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p - \frac{1}{R_m}(\mathbf{curl}\,\mathbf{B})\times\mathbf{B} = \mathbf{f}_0 + \hat{\rho}(T)\mathbf{g}, \tag{2.94}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2.95}$$

$$-\frac{1}{P_r R_e}\Delta T + \mathbf{u}\cdot\mathbf{grad}\,T = \frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2\right] + \psi. \tag{2.96}$$

This system of equations is completed with the same boundary conditions we introduced in Section 2.2.1, which are given by

$$\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d\,, \tag{2.97}$$

$$(\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l\,, \tag{2.98}$$

$$\left[\left(\frac{1}{R_m}(\mathbf{curl}\,\mathbf{B}) - (\mathbf{u}\times\mathbf{B})\right)\times\mathbf{n}\right]_{|\partial\Omega} = \mathbf{k}\,, \tag{2.99}$$

$$T_{|\partial\Omega} = T_d\,. \tag{2.100}$$

The compatibility and regularity conditions for the given data are the same as those previously introduced in Section 2.2.3, but we also assume the heat source ψ to be non-negative and the temperature on the boundary to be strictly positive, *i.e.*

$$\psi \in L^1(\Omega), \qquad \psi(\mathbf{x}) \geq 0 \quad \forall \mathbf{x}\in\Omega\,, \tag{2.101}$$

$$T_d \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega), \quad T_d(\mathbf{x}) \geq T_{\min} > 0 \quad \forall\mathbf{x}\in\partial\Omega\,. \tag{2.102}$$

Moreover, as we mentioned above, the density response function $\hat{\rho} : (0,+\infty) \to (0,+\infty)$ is requested to be strictly positive, continuous and non-increasing. Obviously, the same properties are satisfied by the non-dimensionalized response function, also denoted by $\hat{\rho}$.

Remark 2.7. *In order to obtain the non-dimensionalized system, it is more usual to work with the temperature difference with respect to a reference temperature rather than with the temperature itself, as we did in Section 2.2.1. This would lead to express the sources in the heat equation in terms of the Eckert number, as in (2.10), but at the cost of working with a density response function $\hat{\rho}$ defined on an interval different from $(0,+\infty)$. From the mathematical point of view the results of existence and uniqueness are analogous for both non-dimensionalizations.*

Remark 2.8. *If we denote by $\Delta\mathcal{T}$ a typical value for the temperature difference with respect to the reference temperature, the constant appearing in the sources of the heat equation can be expressed as*

$$\frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}} = \frac{E_c}{R_e}\frac{\Delta\mathcal{T}}{\mathcal{T}}\,.$$

Moreover, in the case of a perfect gas, this expression is known to be equal to $(\gamma-1)M^2/R_e$ (see equation (12.25) in [92]).

2.3.2 An *a priori* bound for the solutions.

In the equations considering the Boussinesq approximation it was not possible to obtain an *a priori* bound for the solution, due to the buoyancy force term appearing in the right-hand side of the Navier-Stokes equations. Now, since Joule effect and viscous heating are non-negative, due to condition (2.101) and as a consequence of Theorem 3.7 in [94], we know that any temperature T solution to (2.96) and (2.100) satisfies

$$\operatorname{ess}\inf_{\mathbf{x}\in\Omega} T(\mathbf{x}) \geq \operatorname{ess}\inf_{\mathbf{x}\in\partial\Omega} T_d(\mathbf{x}) = T_{\min} > 0\,. \tag{2.103}$$

Hence, since $\hat{\rho}$ is continuous and non-increasing, we have

$$0 < \hat{\rho}(T(\mathbf{x})) \leq \hat{\rho}(T_{\min}) = \hat{\rho}_{\max}\,, \tag{2.104}$$

and the product $\hat{\rho}(T)\mathbf{g}$ satisfies

$$\|\hat{\rho}(T)\mathbf{g}\|_{-1} \leq g\,\|\hat{\rho}(T)\|_{L^\infty}\, meas(\Omega)^{1/2} \leq g\hat{\rho}_{\max} meas(\Omega)^{1/2}\,,$$

where $g = |\mathbf{g}|$ is the modulus of gravity acceleration.

Reasoning as in Proposition 2.5 it can be seen that $(\mathbf{u},\mathbf{B}) \in \mathcal{Z}(\Omega)$ is a solution to (2.92)-(2.95) along with boundary conditions (2.97)-(2.99) if and only if it is a solution to the problem:

Given $\mathbf{f}_0$, $\mathbf{u}_d$, l *and* $\mathbf{k}$ *satisfying (2.17)-(2.20) and* $T : \Omega \to \mathbb{R}$ *a measurable function such that* $T(\mathbf{x}) \geq T_{\min} > 0$ *a.e. in* Ω, *find*

$$(\mathbf{u},\mathbf{B}) \in \mathcal{Z}(\Omega)\,, \tag{2.105}$$

satisfying

$$a((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{C})) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) - c_1(\mathbf{B},\mathbf{B},\mathbf{v}) + c_1(\mathbf{C},\mathbf{B},\mathbf{u})$$
$$= F((\mathbf{v},\mathbf{C})) + \langle\hat{\rho}(T)\mathbf{g},\mathbf{v}\rangle_\Omega \quad \forall(\mathbf{v},\mathbf{C}) \in \mathcal{Z}_0(\Omega), \tag{2.106}$$
$$\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d, \quad (\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l\,. \tag{2.107}$$

Using the splittings $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$, $\mathbf{B} = \hat{\mathbf{B}} + \mathbf{B}_0$ the problem can be reduced to homogeneous boundary conditions, and taking into account that $\mathbf{B}_0$ is irrotational it can be rewritten in the form:

Given $\mathbf{u}_0 \in \mathbf{Z}(\Omega)$, $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ *satisfying (2.33)-(2.34), and* $T : \Omega \to \mathbb{R}$ *a measurable function such that* $T(\mathbf{x}) \geq T_{\min} > 0$ *a.e. in* Ω, *find*

$$(\hat{\mathbf{u}},\hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{2.108}$$

such that

$$a((\hat{\mathbf{u}},\hat{\mathbf{B}}),(\mathbf{v},\mathbf{C})) + c_0(\hat{\mathbf{u}},\hat{\mathbf{u}},\mathbf{v}) + c_0(\hat{\mathbf{u}},\mathbf{u}_0,\mathbf{v}) + c_0(\mathbf{u}_0,\hat{\mathbf{u}},\mathbf{v})$$
$$-c_1(\hat{\mathbf{B}},\hat{\mathbf{B}},\mathbf{v}) - c_1(\hat{\mathbf{B}},\mathbf{B}_0,\mathbf{v}) + c_1(\mathbf{C},\hat{\mathbf{B}},\hat{\mathbf{u}}) + c_1(\mathbf{C},\hat{\mathbf{B}},\mathbf{u}_0) + c_1(\mathbf{C},\mathbf{B}_0,\hat{\mathbf{u}}) = F((\mathbf{v},\mathbf{C}))$$
$$+\langle\hat{\rho}(T)\mathbf{g},\mathbf{v}\rangle_\Omega - a_0(\mathbf{u}_0,\mathbf{v}) - c_0(\mathbf{u}_0,\mathbf{u}_0,\mathbf{v}) - c_1(\mathbf{C},\mathbf{B}_0,\mathbf{u}_0) \quad \forall\,(\mathbf{v},\mathbf{C}) \in \mathcal{Z}_0(\Omega)\,. \tag{2.109}$$

The next proposition gives us an *a priori* bound for any solution of the MHD problem.

Proposition 2.23. *If* $\mathbf{u}_0 \in \mathbf{Z}(\Omega)$ *is a lifting of the boundary condition* $\mathbf{u}_d$ *satisfying*

$$\|\mathbf{u}_0\|_{\mathbf{L}^4} < \frac{\alpha_a}{\tilde{\lambda}_{c0}}, \qquad \|\mathbf{u}_0\|_{\mathbf{L}^6} < \frac{\alpha_a}{\tilde{\lambda}_{c1}}, \tag{2.110}$$

with $\tilde{\lambda}_{c0}$ *and* $\tilde{\lambda}_{c1}$ *defined as in Proposition 2.9, then, for any solution* $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ *to problem (2.108)-(2.109) it holds*

$$|(\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \leq \frac{k_0}{\alpha_a - k_1}, \tag{2.111}$$

with $k_0 := \left(\lambda_F + \lambda_{\hat{G}} + \lambda_{a0}\,|\mathbf{u}_0|_1 + \lambda_{c0}\,\|\mathbf{u}_0\|^2_{\mathbf{L}^4} + \lambda_{c1}\,\|\mathbf{B}_0\|_{\mathbf{L}^3}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right)$, $\lambda_{\hat{G}} := Sg\hat{\rho}_{\max} meas(\Omega)^{5/6}$, *and* $k_1 := \max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}, \tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}$.

Proof. Taking $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ as a test function in (2.109), by the antisymmetry property of $c_0(\cdot,\cdot,\cdot)$ we have

$$\begin{aligned} a((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\hat{\mathbf{u}}, \hat{\mathbf{B}})) + c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \hat{\mathbf{u}}) + c_1(\hat{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{u}_0) \\ = F((\hat{\mathbf{u}}, \hat{\mathbf{B}})) + \langle \hat{\rho}(T)\mathbf{g}, \hat{\mathbf{u}}\rangle_\Omega - a_0(\mathbf{u}_0, \hat{\mathbf{u}}) - c_0(\mathbf{u}_0, \mathbf{u}_0, \hat{\mathbf{u}}) - c_1(\hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{u}_0)\,. \end{aligned}$$

By the continuity results of Lemma 2.6 and reasoning as in (2.41) we obtain

$$\begin{aligned} |c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \hat{\mathbf{u}}) + c_1(\hat{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{u}_0)| &\leq \max\left\{\frac{\gamma_4 C_0}{N}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}, \frac{\kappa C_1}{R_m}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\} |(\hat{\mathbf{u}}, \hat{\mathbf{B}})|^2_{\mathcal{W}}, \\ &= \max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}, \tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}. \end{aligned}$$

Then, due to the coerciveness result for $a(\cdot,\cdot)$ proved in Lemma 2.7 we get

$$\begin{aligned} |a((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\hat{\mathbf{u}}, \hat{\mathbf{B}})) + c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \hat{\mathbf{u}}) + c_1(\hat{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{u}_0)| \\ \geq |a((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\hat{\mathbf{u}}, \hat{\mathbf{B}}))| - |c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \hat{\mathbf{u}}) + c_1(\hat{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{u}_0)| \\ \geq \left(\alpha_a - \max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}, \tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}\right) |(\hat{\mathbf{u}}, \hat{\mathbf{B}})|^2_{\mathcal{W}} = (\alpha_a - k_1)|(\hat{\mathbf{u}}, \hat{\mathbf{B}})|^2_{\mathcal{W}}\,. \end{aligned} \tag{2.112}$$

As a consequence of (2.110) we know that $\alpha_a - k_1 > 0$, which gives us a coerciveness result for the left-hand side of equation (2.112). Besides, we have

$$|\langle \hat{\rho}(T)\mathbf{g}, \hat{\mathbf{u}}\rangle_\Omega| = \left|\int_\Omega \hat{\rho}(T)\mathbf{g}\cdot\hat{\mathbf{u}}\,\mathrm{d}\mathbf{x}\right| \leq g\,\|\hat{\rho}(T)\|_{L^{6/5}}\,\|\hat{\mathbf{u}}\|_{\mathbf{L}^6} \leq Sg\hat{\rho}_{\max} meas(\Omega)^{5/6}\,|\hat{\mathbf{u}}|_1\,.$$

Hence, using this inequality, the coerciveness result given by (2.112) and the estimates of Proposition 2.6, we obtain

$$|(\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \leq \frac{1}{\alpha_a - k_1}\left(\lambda_F + \lambda_{\hat{G}} + \lambda_{a0}\,|\mathbf{u}_0|_1 + \lambda_{c0}\,\|\mathbf{u}_0\|^2_{\mathbf{L}^4} + \lambda_{c1}\,\|\mathbf{B}_0\|_{\mathbf{L}^3}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right) = \frac{k_0}{\alpha_a - k_1}\,.$$

□

From this *a priori* bound for the term $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$, and using Propositions 2.15 and 2.18, it is also possible to find an *a priori* estimate for the temperature $T \in W^{1,q}(\Omega)$, with $q < 3/2$, and in particular for $q = 6/5$.

2.3.3 Linearized version of the MHD problem.

As in the previous model, to prove the existence of solution of the coupled problem it is convenient to analyze the thermal and the MHD submodels uncoupled. The analysis of the thermal problem is identical to that presented in Section 2.2.5, thus we will now focus on the analysis of the MHD submodel. We are going to introduce the linearized version of the MHD problem. Then we will prove the existence of a solution to this linearized problem, and find an estimate independent of the temperature.

First of all, and due to the lower bound given in (2.103), it is convenient to introduce the set

$$L^{6/5}_{\min}(\Omega) := \{\theta \in L^{6/5}(\Omega) : \theta(\mathbf{x}) \geq T_{\min} \text{ a.e. in } \Omega\},$$

which is a closed convex subset of $L^{6/5}(\Omega)$.

As we mentioned before, the difference of the current model and the one analyzed in Section 2.2 is the use of Boussinesq approximation, which leads to different buoyancy force terms in the right-hand side of the Navier-Stokes equations. To simplify the notation let us introduce the mapping

$$\begin{aligned} \hat{G} : L^{6/5}_{\min}(\Omega) &\longrightarrow \mathbf{H}^{-1}(\Omega) \\ T &\longmapsto \hat{G}(T) := \hat{\rho}(T)\mathbf{g}\,. \end{aligned}$$

Mapping $\hat{G}$ is continuous as an immediate consequence of the following lemma:

Lemma 2.24. *The operator*

$$\begin{aligned} L^{6/5}_{\min}(\Omega) &\longrightarrow L^p(\Omega) \\ T &\longmapsto \hat{\rho}(T) \end{aligned}$$

is continuous for any $1 \leq p < +\infty$.

Proof. The proof is analogous to that of Nemytskii's theorem, which can be found, for instance, in [93, Th. II.3.2]. □

We can now introduce the linearized version of the MHD problem, which differs from the one presented in 2.2.4 only in the buoyancy term. This linearized version of the problem reads:

Given $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$, $T \in L^{6/5}_{\min}(\Omega)$ *and* $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}(\Omega)$, *with* $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, *find*

$$(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{2.113}$$

such that

$$\begin{aligned} a((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\mathbf{v}, \mathbf{C})) + c_0(\hat{\mathbf{w}}, \hat{\mathbf{u}}, \mathbf{v}) + c_0(\mathbf{u}_0, \hat{\mathbf{u}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) \\ + c_1(\mathbf{C}, \hat{\mathbf{D}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}) = F((\mathbf{v}, \mathbf{C})) + \langle \hat{G}(T), \mathbf{v}\rangle_\Omega - a_0(\mathbf{u}_0, \mathbf{v}) \\ - c_0(\hat{\mathbf{w}}, \mathbf{u}_0, \mathbf{v}) - c_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) - c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}_0) - c_1(\mathbf{C}, \hat{\mathbf{D}}, \mathbf{u}_0) \quad \forall\,(\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega)\,. \end{aligned} \tag{2.114}$$

In the following proposition we prove the existence of a unique solution to this problem, and give an estimate for this solution independent of the temperature field T.

Proposition 2.25. *There exists a unique solution* $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ *to problem (2.113)-(2.114). Moreover,*

$$|(\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \leq \frac{1}{\alpha_a} \Big(\lambda_F + \lambda_{\hat{G}} + \lambda_{a0} |\mathbf{u}_0|_1 + \lambda_{c0} \|\mathbf{u}_0\|^2_{\mathbf{L}^4} + \lambda_{c1} \|\mathbf{B}_0\|_{\mathbf{L}^3} \|\mathbf{u}_0\|_{\mathbf{L}^6} + \max\left\{ \tilde{\lambda}_{c0} \|\mathbf{u}_0\|_{\mathbf{L}^4}, \tilde{\lambda}_{c1} \|\mathbf{u}_0\|_{\mathbf{L}^6} \right\} |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \Big), \tag{2.115}$$

where α_a, λ_F, λ_{a0}, λ_{c0}, λ_{c1}, $\tilde{\lambda}_{c0}$ *and* $\tilde{\lambda}_{c1}$ *are the same constants appearing in Proposition 2.9, and* $\lambda_{\hat{G}}$ *the constant appearing in Proposition 2.23.*

Proof. The proof is analogous to that of Proposition 2.9. □

Definition 2.3. *Let us define the mapping*

$$\begin{array}{rcl} \hat{\mathcal{G}}_1 : \mathcal{Z}_0(\Omega) \times L^{6/5}_{\min}(\Omega) & \longrightarrow & \mathcal{Z}_0(\Omega) \\ ((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T) & \longmapsto & \hat{\mathcal{G}}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T) = (\hat{\mathbf{u}}, \hat{\mathbf{B}}), \end{array} \tag{2.116}$$

where $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ *is the solution of the MHD linearized problem (2.113)-(2.114) for a given pair* $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ *and a given temperature* $T \in L^{6/5}_{\min}(\Omega)$.

Lemma 2.26. *Let* $(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ *weakly in* $\mathcal{Z}_0(\Omega)$ *and* $\{T_n\} \subset L^{6/5}_{\min}(\Omega)$ *such that* $T_n \to T$ *strongly in* $L^{6/5}(\Omega)$. *Then* $T \in L^{6/5}_{\min}(\Omega)$ *and* $\hat{\mathcal{G}}_1((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n), T_n) \to \hat{\mathcal{G}}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T)$ *strongly in* $\mathcal{Z}_0(\Omega)$.

Proof. The fact that $T \in L^{6/5}_{\min}(\Omega)$ is clear, because $L^{6/5}_{\min}(\Omega)$ is a closed set. The rest of the proof is analogous to that of Lemma 2.10, but substituting the term $(G(T_n - T), \mathbf{v})_\Omega$ by $\langle \hat{G}(T_n) - \hat{G}(T), \mathbf{v} \rangle_\Omega = \int_\Omega (\hat{\rho}(T_n) - \hat{\rho}(T)) \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x}$. The result is obtained as a consequence of the continuity of the operator considered in Lemma 2.24. □

2.3.4 Coupled problem.

As we already mentioned, the thermal subproblem is identical to the one analyzed in Section 2.2.5. Since we are dealing with positive heat sources, it is convenient to introduce the set

$$L^1_+(\Omega) := \{ f \in L^1(\Omega) : f(\mathbf{x}) \geq 0 \text{ a.e. in } \Omega \},$$

which is a closed convex subset of $L^1(\Omega)$. Analogously to what we did in (2.75), we introduce the mapping $\hat{\mathcal{G}}_2$, which maps a velocity $\hat{\mathbf{w}}$ and a heat source f into the corresponding solution of the heat equation. Moreover, due to the result presented in (2.103), we know that for a positive f this solution belongs to $L^{6/5}_{\min}(\Omega)$, thus the mapping is defined as

$$\begin{array}{rcl} \hat{\mathcal{G}}_2 : \mathbf{Z}_0(\Omega) \times L^1_+(\Omega) & \longrightarrow & W^{1,6/5}(\Omega) \cap L^{6/5}_{\min}(\Omega) \\ (\hat{\mathbf{w}}, f) & \longmapsto & \hat{\mathcal{G}}_2(\hat{\mathbf{w}}, f) := \widetilde{G}(\mathbf{w}, f) + G_D(\mathbf{w}), \end{array} \tag{2.117}$$

where $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{u}_0$ and the mappings $\widetilde{G}$ and G_D have been introduced in Section 2.2.5. We will also make use of the mapping $\hat{\mathcal{G}}_3$, which maps any pair $(\hat{\mathbf{w}}, \hat{\mathbf{D}})$ to its correspondent heat source in the heat equation. It is given by

$$\begin{aligned}\hat{\mathcal{G}}_3 : \mathcal{Z}_0(\Omega) &\rightarrow L^1_+(\Omega)\\ (\hat{\mathbf{w}}, \hat{\mathbf{D}}) &\mapsto \hat{\mathcal{G}}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\hat{\mathbf{D}}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{w} + \mathbf{grad}\,\mathbf{w}^t\right|^2\right] + \psi, \quad (2.118)\end{aligned}$$

where $\mathbf{w}$ is defined as above and ψ satisfies (2.101).

As in the previous case, we will prove the existence of a solution to our problem via a fixed point result. It suffices to prove the existence of a fixed point for the mapping

$$\begin{aligned}\hat{\mathcal{G}} : \mathcal{Z}_0(\Omega) &\longrightarrow \mathcal{Z}_0(\Omega)\\ (\hat{\mathbf{w}}, \hat{\mathbf{D}}) &\longmapsto \hat{\mathcal{G}}((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \hat{\mathcal{G}}_1\left((\hat{\mathbf{w}}, \hat{\mathbf{D}}), \hat{\mathcal{G}}_2\left(\hat{\mathbf{w}}, \hat{\mathcal{G}}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))\right)\right). \quad (2.119)\end{aligned}$$

Lemma 2.27. *If $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$, with $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, ψ satisfies (2.101) and T_d satisfies (2.102), under the assumptions*

$$\|\mathbf{u}_0\|_{\mathbf{L}^4} < \frac{\alpha_a}{\tilde{\lambda}_{c0}}, \quad (2.120)$$

$$\|\mathbf{u}_0\|_{\mathbf{L}^6} < \frac{\alpha_a}{\tilde{\lambda}_{c1}}, \quad (2.121)$$

there exists a constant $R > 0$ such that if $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \le R$ then $|\hat{\mathcal{G}}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \le R$. The constants α_a, $\tilde{\lambda}_{c0}$ and $\tilde{\lambda}_{c1}$ have been introduced in Lemma 2.7 and Proposition 2.9.

Proof. Let $R > 0$ and $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ be such that $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})| \le R$. From the result of Proposition 2.25 we have that

$$|\hat{\mathcal{G}}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \le \frac{1}{\alpha_a}(k_0 + k_1|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}}) \le \frac{1}{\alpha_a}(k_0 + k_1 R),$$

where the constants k_0 and k_1 have the same expression that in Proposition 2.23. In order to obtain $|\hat{\mathcal{G}}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \le R$ we must require $k_0 + k_1 R \le \alpha_a R$, which is equivalent to $R \ge k_0/(\alpha_a - k_1)$. Since $k_0 > 0$ and $R > 0$, the result holds whenever $\alpha_a - k_1 > 0$ which is equivalent to conditions (2.120) and (2.121). □

Using the result of the previous lemma we can prove the existence of a solution to the coupled problem:

Theorem 2.28. *Under the assumptions of Lemma 2.27 the mapping $\hat{\mathcal{G}}$ has at least one fixed point.*

Proof. The proof is analogous to that of Theorem 2.22, using Lemmas 2.27 and 2.26 instead of Lemmas 2.21 and 2.10. □

Remark 2.9. *We notice that conditions (2.120) and (2.121) have already appeared in Proposition 2.23. Moreover, let us take*

$$R = R_0 = \frac{k_0}{\alpha_a - k_1}.$$

Under the mentioned conditions and from the results of Proposition 2.23 and Lemma 2.27, we know that any solution $(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega)$ of the problem belongs to $\overline{B(0, R_0)}$, the closed ball centered at the origin with radius R_0, and that there exists at least one solution in the mentioned ball.

2.3.5 Existence of solution without assuming smallness of the data.

The existence results of the two coupled problems analyzed above, given in Theorems 2.22 and 2.28, rely on some smallness of the given data. Following the ideas of [2] it is possible to prove that, for a tangential boundary condition $\mathbf{u}_d$, it is always possible to construct a lifting $\mathbf{u}_0$ satisfying (2.120) and (2.121), which leads to an existence result for the problem independently of the size of the given data. The proof makes use of some definitions and results appearing in [46] that we reproduce below.

Definition 2.4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary. For $1 < q < \infty$ we define the space*

$$\mathbf{H}^q(\mathrm{div};\Omega) := \{\mathbf{u} \in \mathbf{L}^q(\Omega) : \mathrm{div}\,\mathbf{u} \in L^q(\Omega)\} \ .$$

We notice that the space $\mathbf{H}^q(\mathrm{div};\Omega)$ is denoted in [46] by $\widetilde{H}_q(\Omega)$.

The following theorem can be found in [46, Th. III.2.1].

Theorem 2.29. *Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^N$, $N \geq 2$. Let $\mathbf{H}^q_0(\mathrm{div};\Omega)$ denote the completion of $(\mathcal{D}(\Omega))^N$ in the norm $\|\cdot\|_{\mathbf{H}^q(\mathrm{div};\Omega)}$. Then*

$$\mathbf{H}^q_0(\mathrm{div};\Omega) = \{\mathbf{u} \in \mathbf{H}^q(\mathrm{div};\Omega) : \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\} \ .$$

The following result is a consequence of Theorem III.3.3 in [46].

Theorem 2.30. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary. Then, given*

$$\mathbf{g} \in \mathbf{H}^q_0(\mathrm{div};\Omega) \cap \mathbf{L}^s(\Omega), \quad 1 < \mathrm{q} < +\infty\,, 1 < \mathrm{s} < +\infty\,,$$

there exists at least one $\mathbf{v} \in (W^{1,q}_0(\Omega))^N \cap \mathbf{L}^s(\Omega)$ such that

$$\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{g}\,, \tag{2.122}$$
$$\|\mathbf{v}\|_{1,q} \leq \beta_q \|\mathrm{div}\,\mathbf{g}\|_{L^q} \ , \tag{2.123}$$
$$\|\mathbf{v}\|_{\mathbf{L}^s} \leq \kappa_s \|\mathbf{g}\|_{\mathbf{L}^s} \ . \tag{2.124}$$

Proof. According to [46, Th. III.3.3], for each $\mathbf{g} \in \mathbf{H}^q_0(\mathrm{div};\Omega)$, with $1 < q < +\infty$, it is possible to construct $\mathbf{v} \in (W^{1,q}_0(\Omega))^N$ satisfying (2.122), (2.123) and also

$$\|\mathbf{v}\|_{\mathbf{L}^q} \leq \kappa_q \|\mathbf{g}\|_{\mathbf{L}^q} \ , \tag{2.125}$$

Moreover, it can be seen that the construction procedure in [46] is independent of q, and it holds that $\mathbf{v} = \mathcal{R}\mathbf{g}$, where $\mathcal{R}$ is a linear and continuous operator, *i.e.* $\mathcal{R} \in \mathcal{L}(\mathbf{H}^q_0(\mathrm{div};\Omega), (\mathrm{W}^{1,q}_0(\Omega))^{\mathrm{N}})$. Therefore, the result is clear for $s \leq q$.

For $q < s$, let us consider the restriction of $\mathcal{R}$ to $(\mathcal{D}(\Omega))^3$. Thanks to (2.125), this operator is continuous from $(\mathcal{D}(\Omega))^3$, endowed with the norm $\|\cdot\|_{\mathbf{L}^q}$, into $\mathbf{L}^q(\Omega)$. Now we can define, in a unique way, a linear and continuous operator $T_q \in \mathcal{L}(\mathbf{L}^q(\Omega), \mathbf{L}^q(\Omega))$ such that $T_q\mathbf{g} = \mathcal{R}\mathbf{g}$ for all

$\mathbf{g} \in (\mathcal{D}(\Omega))^3$, and by the definition of $\mathbf{H}_0^q(\text{div};\Omega)$ (see Theorem 2.29), we have $T_q\mathbf{g} = \mathcal{R}\mathbf{g}$ for all $\mathbf{g} \in \mathbf{H}_0^q(\text{div};\Omega)$.

In an analogous way, we define the operator $T_s \in \mathcal{L}(\mathbf{L}^s(\Omega), \mathbf{L}^s(\Omega))$. Since $q < s$ we can say that $T_q, T_s \in \mathcal{L}(\mathbf{L}^s(\Omega), \mathbf{L}^q(\Omega))$. By the way these operators have been constructed, we know that $T_s\mathbf{g} = T_q\mathbf{g}$ for all $\mathbf{g} \in (\mathcal{D}(\Omega))^3$, and as a consequence $T_s\mathbf{g} = T_q\mathbf{g}$ for all $\mathbf{g} \in \mathbf{L}^s(\Omega)$.

Joining the two previous results, we find that $T_s\mathbf{g} = T_q\mathbf{g} = \mathcal{R}\mathbf{g}$ for all $\mathbf{g} \in \mathbf{H}_0^q(\text{div};\Omega) \cap \mathbf{L}^s(\Omega)$, which implies

$$\|\mathcal{R}\mathbf{g}\|_{\mathbf{L}^s(\Omega)} = \|T_s\mathbf{g}\|_{\mathbf{L}^s(\Omega)} \leq \kappa_s \|\mathbf{g}\|_{\mathbf{L}^s(\Omega)} ,$$

and the result is proved. □

Lemma 2.31. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary, and let $\mathbf{u}_d \in \mathbf{H}_T^{1/2}(\partial\Omega)$. The two following results hold:*

(i) Let $1 < p < 6$. Then, for every number $\varepsilon > 0$ there exists a vector $\mathbf{u}_\varepsilon \in \mathbf{H}_T^1(\Omega)$ such that

$$\operatorname{div}\mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega\,, \tag{2.126}$$
$$\mathbf{u}_\varepsilon = \mathbf{u}_d \quad \text{on } \partial\Omega\,, \tag{2.127}$$
$$\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^p} \leq C(p)\varepsilon \|\mathbf{u}_d\|_{1/2,\partial\Omega} , \tag{2.128}$$
$$\|\mathbf{u}_\varepsilon\|_1 \leq C_\varepsilon \|\mathbf{u}_d\|_{1/2,\partial\Omega} , \tag{2.129}$$

where the constant $C(p)$ depends on p and the domain Ω, and the constant C_ε depends on ε and Ω.

(ii) For every number $\varepsilon > 0$ there exists a vector $\mathbf{u}_\varepsilon \in \mathbf{H}_T^1(\Omega)$ satisfying (2.126), (2.127) and

$$\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^6} \leq \varepsilon\,. \tag{2.130}$$

Proof. (i) The proof is analogous to the one of Lemma 2.2 in [2], but in that paper it is done for $p = 4$. It is reproduced here because some steps are also used in the proof of (ii).

Let us denote by $\mathbf{u}_0$ the standard extension of $\mathbf{u}_d$ to Ω satisfying $\|\mathbf{u}_0\|_1 \leq \Lambda_1 \|\mathbf{u}_d\|_{1/2,\partial\Omega}$, with Λ_1 independent of $\mathbf{u}_d$. For each real number $\varepsilon_0 > 0$ we introduce the truncation function $\theta_{\varepsilon_0} \in \mathcal{C}^1(\overline{\Omega})$ defined as in [46, Lemma III.6.2] (see also [53, Lemma IV.2.4]), satisfying the following conditions: $|\theta_{\varepsilon_0}(\mathbf{x})| \leq 1$ in $\overline{\Omega}$, $\theta_{\varepsilon_0}(\mathbf{x}) = 1$ in a neighborhood of $\partial\Omega$ and $\theta_{\varepsilon_0}(\mathbf{x}) = 0$ for $\text{dist}(\mathbf{x}, \partial\Omega) \geq 2\delta(\varepsilon_0)$, with $\delta(\varepsilon_0) = e^{-1/\varepsilon_0}$.

Setting $\mathbf{w}_{\varepsilon_0} = \theta_{\varepsilon_0}\mathbf{u}_0$ it is obvious that $\mathbf{w}_{\varepsilon_0} \in \mathbf{H}_T^1(\Omega)$, $\mathbf{w}_{\varepsilon_0} = \mathbf{u}_d$ on $\partial\Omega$ and $\|\mathbf{w}_{\varepsilon_0}\|_1 \leq C'_{\varepsilon_0} \|\mathbf{u}_d\|_{1/2,\partial\Omega}$, with C'_{ε_0} a constant depending on ε_0 and Ω. Moreover, using Hölder inequality we obtain

$$\|\mathbf{w}_{\varepsilon_0}\|_{\mathbf{L}^p} \leq \|\mathbf{u}_0\|_{\mathbf{L}^6} \|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \leq \gamma_6 \|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \|\mathbf{u}_0\|_1 \leq \gamma_6\Lambda_1 \|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \|\mathbf{u}_d\|_{1/2,\partial\Omega} , \tag{2.131}$$

with γ_6 the constant of the imbedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$, and $q = 6p/(6-p)$.

Since $\mathbf{w}_{\varepsilon_0} \in \mathbf{H}^1_T(\Omega)$, we can use Theorem 2.30 with $q = 2$ and $s = p$, hence for $\mathbf{w}_{\varepsilon_0}$ we can construct a vector $\mathbf{v}_{\varepsilon_0} \in \mathbf{H}^1_0(\Omega)$ such that $\operatorname{div} \mathbf{v}_{\varepsilon_0} = \operatorname{div} \mathbf{w}_{\varepsilon_0}$ and

$$\|\mathbf{v}_{\varepsilon_0}\|_{\mathbf{L}^p} \leq \kappa_p \|\mathbf{w}_{\varepsilon_0}\|_{\mathbf{L}^p} \leq \kappa_p \gamma_6 \Lambda_1 \|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \|\mathbf{u}_d\|_{1/2,\partial\Omega} \, ,$$
$$\|\mathbf{v}_{\varepsilon_0}\|_1 \leq \beta_2 \|\operatorname{div} \mathbf{w}_{\varepsilon_0}\|_0 \leq \sqrt{N}\beta_2 \|\mathbf{w}_{\varepsilon_0}\|_1 \leq \sqrt{N}\beta_2 C'_{\varepsilon_0} \|\mathbf{u}_d\|_{1/2,\partial\Omega} \, .$$

Now, setting $\mathbf{u}_\varepsilon = \mathbf{w}_{\varepsilon_0} - \mathbf{v}_{\varepsilon_0}$ it is clear that $\mathbf{u}_\varepsilon \in \mathbf{H}^1_T(\Omega)$, $\mathbf{u}_\varepsilon = \mathbf{u}_d$ on $\partial\Omega$, and we have the estimates

$$\|\mathbf{u}_\varepsilon\|_1 \leq (1 + \beta_2 C'_{\varepsilon_0}) \|\mathbf{u}_d\|_{1/2,\partial\Omega} \, ,$$
$$\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^p} \leq (1 + \kappa_p)\gamma_6 \Lambda_1 \|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \|\mathbf{u}_d\|_{1/2,\partial\Omega} \, .$$

From the definition of θ_{ε_0} it is clear that $\|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \to 0$ as $\varepsilon_0 \to 0$. Therefore, choosing ε_0 such that $\|\theta_{\varepsilon_0}\|_{\mathbf{L}^q} \leq \varepsilon$, the result follows with $C(p) = (1 + \kappa_p)\gamma_6\Lambda_1$ and $C_\varepsilon = (1 + \sqrt{N}\beta_2 C'_{\varepsilon_0})$.

(ii) Let $\varepsilon_0 > 0$ and construct $\mathbf{w}_{\varepsilon_0} = \theta_{\varepsilon_0}\mathbf{u}_0$ as we did before. Due to the properties of θ_{ε_0} we know that $\mathbf{w}_{\varepsilon_0}(\mathbf{x}) \to 0$ as $\varepsilon_0 \to 0$ a.e. in Ω, and $|\mathbf{w}_{\varepsilon_0}(\mathbf{x})| \leq |\mathbf{u}_0(\mathbf{x})|$ a.e. in Ω. As a consequence of Lebesgue dominated convergence theorem, for any $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that

$$\|\mathbf{w}_{\varepsilon_1}\|_{\mathbf{L}^6} \leq \frac{\varepsilon}{1 + \kappa_6} \, . \tag{2.132}$$

Once again, since $\mathbf{w}_{\varepsilon_1} \subset \mathbf{H}^1_T(\Omega)$ we can use Theorem 2.30 with $q = 2$ and $s = 6$ to construct a vector $\mathbf{v}_{\varepsilon_1} \in \mathbf{H}^1_0(\Omega)$ such that $\operatorname{div} \mathbf{v}_{\varepsilon_1} = \operatorname{div} \mathbf{w}_{\varepsilon_1}$ and

$$\|\mathbf{v}_{\varepsilon_1}\|_{\mathbf{L}^6} \leq \kappa_6 \|\mathbf{w}_{\varepsilon_1}\|_{\mathbf{L}^6} \, .$$

Setting $\mathbf{u}_\varepsilon = \mathbf{w}_{\varepsilon_1} - \mathbf{v}_{\varepsilon_1}$ it is clear that $\mathbf{u}_\varepsilon \in \mathbf{H}^1_T(\Omega)$, $\mathbf{u}_\varepsilon = \mathbf{u}_d$ on $\partial\Omega$ and, from the two previous inequalities, we have the estimate

$$\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^6} \leq (1 + \kappa_6) \|\mathbf{w}_{\varepsilon_1}\|_{\mathbf{L}^6} \leq \varepsilon \, .$$

□

Remark 2.10. *We notice that in point (ii) of the previous lemma it is not possible to obtain an estimate of the form (2.129). This is because for* $p = 6$ *we cannot use Hölder inequality to obtain an estimate of the form (2.131), and we just have the estimate (2.132) instead. As a consequence, the choice of* ε_1 *depends not only on* ε *but also on the boundary data* $\mathbf{u}_d$ *and this fact makes it impossible to obtain something like (2.129), because the constant would also depend on* $\mathbf{u}_d$.

The previous Lemma allows us to prove the existence of solution, for a tangential velocity boundary condition, without assuming smallness of the data.

Theorem 2.32. *Let* $\mathbf{f}_0$, l, $\mathbf{k}$, ψ *and* T_d *satisfying (2.17), (2.19), (2.20), (2.101) and (2.102), respectively. Let* $\hat{\rho} : (0, +\infty) \to (0, +\infty)$ *be continuous and non-increasing and* $\mathbf{u}_d \in \mathbf{H}^{1/2}_T(\partial\Omega)$. *Then there exists at least one solution* $((\mathbf{u}, \mathbf{B}), T)$ *to problem (2.92)-(2.100).*

Proof. It is well known that for any $\mathbf{u} \in \mathbf{L}^4(\Omega)$ we have $\|\mathbf{u}\|_{\mathbf{L}^4} \leq meas(\Omega)^{1/12} \|\mathbf{u}\|_{\mathbf{L}^6}$. Since $\mathbf{u}_d \in \mathbf{H}_T^{1/2}(\partial\Omega)$ we can apply Lemma 2.31 with $\varepsilon = \min\left\{meas(\Omega)^{-1/12}\alpha_a/\tilde{\lambda}_{c0}\,,\, \alpha_a/\tilde{\lambda}_{c1}\right\}$ to construct a field $\mathbf{u}_0 \in \mathbf{H}_T^1(\Omega)$ such that $\mathbf{u}_{0|\partial\Omega} = \mathbf{u}_d$ and satisfying

$$\|\mathbf{u}_0\|_{\mathbf{L}^4} < \frac{\alpha_a}{\tilde{\lambda}_{c0}}\,, \quad \|\mathbf{u}_0\|_{\mathbf{L}^6} < \frac{\alpha_a}{\tilde{\lambda}_{c1}}\,.$$

The result follows from Theorem 2.28 and the fact that any fixed point of $\hat{\mathcal{G}}$ is also a solution to problem (2.92)-(2.100). □

2.4 Results of uniqueness.

In the previous sections we have proved existence results for the two models presented in the chapter. We are now going to prove uniqueness results under more severe restrictions on the given data and on the domain. In particular, in the sequel we will assume that Ω is a bounded domain of class $\mathcal{C}^{1,1}$.

2.4.1 Equivalence of the solution by transposition and the weak solution.

The technique we will use to prove the uniqueness under smallness of the data will require some results of Lipschitz continuity on bounded sets for the mappings appearing in the definition of $\mathcal{G}$ (see eqn. (2.77)). In particular we will make use of the Lipschitz continuity on bounded sets of the mapping $\tilde{G}$ defined in (2.71) with respect to $\mathbf{u}$ and f. In order to prove this result it is not convenient to write the thermal problem with L^1 sources in the form of (2.61), because the test functions depend on the velocity $\mathbf{u}$. Instead, we will rewrite the problem in a weak formulation, using a theorem presented in [51] to show that both formulations are equivalent. This theorem requires a smooth domain, therefore from this point we are assuming that Ω is bounded and of class $\mathcal{C}^{1,1}$. The result is proved for a general temperature $T \in W^{1,q}(\Omega)$ with $q < N/(N-1) = 3/2$, but later on we will only require $T \in W^{1,6/5}(\Omega)$.

First of all, let us recall the definition of the bilinear form $e : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, which is given by

$$e(T, z) := \frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, T \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x}\,.$$

Given a vector field $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $6/5 \leq q < 3/2$, we introduce the bilinear forms $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $a_q : W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \to \mathbb{R}$ defined as

$$a(T, z) := \frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, T \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, T\, z \,\mathrm{d}\mathbf{x}\,,$$
$$a_q(T, z) := \frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, T \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, T\, z \,\mathrm{d}\mathbf{x}\,.$$

We notice that these two bilinear forms only differ in the spaces on which they are defined. Moreover, it holds that $a(T, z) = a_q(T, z)\ \forall T \in H^1(\Omega)\ \forall z \in W^{1,q'}(\Omega)$.

Finally, we remind the definition of the operator $L_{-\mathbf{u}} : H_0^1(\Omega) \to H^{-1}(\Omega)$ and analogously define the operator $L_{\mathbf{0}} : H_0^1(\Omega) \to H^{-1}(\Omega)$,

$$L_{-\mathbf{u}}T := -\frac{1}{P_r R_e}\Delta T - \mathbf{u}\cdot\mathbf{grad}\, T\,,$$
$$L_{\mathbf{0}}T := -\frac{1}{P_r R_e}\Delta T\,.$$

The following lemma is an immediate consequence of [51, Th. 9.15]:

Lemma 2.33. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{1,1}$ and $g \in L^p(\Omega)$, with $p \geq (2^*)' = 6/5$. Then, the unique solution to the problem*

$$\left\{\begin{array}{l} \psi \in H_0^1(\Omega)\,, \\ e(v,\psi) = \displaystyle\int_\Omega gv\, d\mathbf{x} \quad \forall v \in H_0^1(\Omega) \end{array}\right. \tag{2.133}$$

satisfies $\psi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Lemma 2.34. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^{1,1}$ and $\mathbf{u} \in \mathbf{Z}(\Omega)$. If $\psi \in H_0^1(\Omega)$ is such that $L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega)$, then $\psi \in \mathcal{C}^1(\overline{\Omega})$.*

Proof. Let us introduce the function g defined as

$$g = L_{-\mathbf{u}}\psi = -\frac{1}{P_r R_e}\Delta\psi - \mathbf{u}\cdot\mathbf{grad}\,\psi\,, \tag{2.134}$$

which is clearly in $\mathcal{D}(\Omega)$. With this definition, ψ is solution to the problem

$$\left\{\begin{array}{l} \psi \in H_0^1(\Omega)\,, \\ L_{\mathbf{0}}\psi = g + \mathbf{u}\cdot\mathbf{grad}\,\psi\,. \end{array}\right.$$

Denoting $\widetilde{g} = g + \mathbf{u}\cdot\mathbf{grad}\,\psi$, since $\mathbf{u} \in \mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$ and $\mathbf{grad}\,\psi \in \mathbf{L}^2(\Omega)$ we know that $\widetilde{g} \in L^{3/2}(\Omega)$ and, due to the previous lemma, $\psi \in W^{2,3/2}(\Omega) \cap W_0^{1,3/2}(\Omega)$. Now, from Theorem 2.3 we know that $W^{2,3/2}(\Omega) \subset W^{1,3}(\Omega)$ hence $\mathbf{grad}\,\psi \in \mathbf{L}^3(\Omega)$. Reasoning as before, $\widetilde{g} \in L^2(\Omega)$. Moreover, from the previous lemma we get $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, and due to Sobolev imbeddings we get $\psi \in W^{1,6}(\Omega)$. Repeating the previous steps we get $\widetilde{g} \in L^3(\Omega)$ and $\psi \in W^{2,3}(\Omega) \subset W^{1,r}(\Omega)$ for any $r \in [1,+\infty)$. Repeating the process once more with any $r > 6$ we obtain $\psi \in \mathcal{C}^1(\overline{\Omega})$, which is the desired result. □

Now we can prove that any solution by transposition is in fact a solution for the weak formulation.

Lemma 2.35. *Let $\Omega \subset \mathbb{R}^3$ a bounded domain of class $\mathcal{C}^{1,1}$, $\mathbf{u} \in \mathbf{Z}(\Omega)$ and $f \in L^1(\Omega)$. Then, T is solution to*

$$\left\{\begin{array}{l} T \in W_0^{1,q}(\Omega), \quad 1 < q < 3/2, \\ \displaystyle\int_\Omega T(L_{-\mathbf{u}}\psi)\, d\mathbf{x} = \int_\Omega f\psi\, d\mathbf{x} \quad \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega) \;\; \textit{such that } L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega)\,, \end{array}\right. \tag{2.135}$$

if and only if it is solution to

$$\left\{\begin{array}{l} T \in \bigcap_{1\leq q<3/2} W_0^{1,q}(\Omega), \\ \dfrac{1}{P_r R_e}\displaystyle\int_\Omega \mathbf{grad}\, T\cdot \mathbf{grad}\,\varphi\, d\mathbf{x} + \int_\Omega \mathbf{u}\cdot\mathbf{grad}\, T\,\varphi\, d\mathbf{x} = \int_\Omega f\varphi\, d\mathbf{x} \quad \forall\varphi\in\mathcal{D}(\Omega)\,. \end{array}\right. \tag{2.136}$$

Proof. Since the solution to problem (2.135) is independent of q, we know that $T \in \bigcap_{1\leq q<3/2} W_0^{1,q}(\Omega)$, and in particular $T \in W_0^{1,6/5}(\Omega)$. As a consequence, we can apply Proposition B.3 with $q = 6/5$ to obtain that

$$\left\{\begin{array}{l} T \in W_0^{1,q}(\Omega), \\ a_q(T,\varphi) = \displaystyle\int_\Omega f\varphi\,\mathrm{d}\mathbf{x} \quad \forall\varphi\in\mathcal{D}(\Omega)\,, \end{array}\right. \tag{2.137}$$

and as a consequence, T also satisfies (2.136).

The other implication is a consequence of the previous lemma. Let T be a solution of (2.136), and $\psi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ be such that $L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega)$. From the previous lemma we know that $\psi \in \mathcal{C}^1(\overline{\Omega})$ and, in particular, $\psi \in W^{1,q'}(\Omega)$ $\forall q < 3/2$. Then, for any $q \in [6/5, 3/2)$ we have

$$\int_\Omega (L_{-\mathbf{u}}\psi)\varphi\,\mathrm{d}\mathbf{x} = \langle L_{-\mathbf{u}}\psi, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = a(\varphi,\psi) = a_q(\varphi,\psi) \quad \forall\varphi\in\mathcal{D}(\Omega)\,. \tag{2.138}$$

Now, since $\mathcal{D}(\Omega)$ is dense in $W_0^{1,q}(\Omega)$, $L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega) \subset L^\infty(\Omega)$, $\psi \in W^{1,q'}(\Omega)$ and $a_q(\cdot,\cdot)$ is a bilinear and continuous form, the following Green's formula holds:

$$\int_\Omega \varphi(L_{-\mathbf{u}}\psi)\,\mathrm{d}\mathbf{x} = a_q(\varphi,\psi) \quad \forall\varphi\in W_0^{1,q}(\Omega)\,.$$

In particular, taking $\varphi = T$, since T is solution to (2.136) we have

$$\int_\Omega T(L_{-\mathbf{u}}\psi)\,\mathrm{d}\mathbf{x} = a_q(T,\psi) = \int_\Omega f\psi\,\mathrm{d}\mathbf{x}\,.$$

As the result is valid for any arbitrary ψ, we have proved that T is solution to (2.135). □

Once we have proved the equivalence of both formulations, we can prove the Lipschitz continuity result using the weak formulation.

Lemma 2.36. *Let $f \in L^1(\Omega)$ and $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbf{Z}(\Omega)$. Let us consider the mapping $\tilde{G}$ defined in (2.71). Then the following estimate holds*

$$\|\tilde{G}(\mathbf{u}_1,f) - \tilde{G}(\mathbf{u}_2,f)\|_{1,q} \leq K_3\,|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1\,\|\tilde{G}(\mathbf{u}_2,f)\|_{1,6/5}\,, \tag{2.139}$$

with $K_3 \equiv K_3(q)$ a constant independent of the velocities $\mathbf{u}_1$, $\mathbf{u}_2$ and of the source term f.

Proof. Let us denote $\tilde{T}_i = \tilde{G}(\mathbf{u}_i, f)$ for $i = 1, 2$. Lemma 2.35 states that the fields $\tilde{T}_i = \tilde{G}(\mathbf{u}_i, f)$ are in fact solutions to

$$\begin{cases} \tilde{T}_i \in \bigcap_{1\le q<3/2} W_0^{1,q}(\Omega), \\ \dfrac{1}{P_r R_e}\displaystyle\int_\Omega \mathbf{grad}\,\tilde{T}_i \cdot \mathbf{grad}\,\varphi \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_i \cdot \mathbf{grad}\,\tilde{T}_i\,\varphi \,\mathrm{d}\mathbf{x} = \int_\Omega f\varphi \,\mathrm{d}\mathbf{x} \quad \forall\varphi \in \mathcal{D}(\Omega)\,. \end{cases} \tag{2.140}$$

Subtracting the two problems we get

$$\begin{cases} \tilde{T}_1 - \tilde{T}_2 \in \bigcap_{1\le q<3/2} W_0^{1,q}(\Omega), \\ \dfrac{1}{P_r R_e}\displaystyle\int_\Omega \mathbf{grad}\,(\tilde{T}_1 - \tilde{T}_2) \cdot \mathbf{grad}\,\varphi \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_1 \cdot \mathbf{grad}\,(\tilde{T}_1 - \tilde{T}_2)\,\varphi \,\mathrm{d}\mathbf{x} \\ \qquad = -\displaystyle\int_\Omega (\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\tilde{T}_2\,\varphi \,\mathrm{d}\mathbf{x} \quad \forall\varphi \in \mathcal{D}(\Omega)\,. \end{cases} \tag{2.141}$$

Since $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{Z}(\Omega) \subset \mathbf{L}^6(\Omega)$ and $\mathbf{grad}\,\tilde{T}_2 \in \mathbf{L}^q(\Omega)$ for every $1 \le q < 3/2$, and in particular for $q = 6/5$, we have $(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\tilde{T}_2 \in L^1(\Omega)$. Moreover, $\mathbf{u}_1 \in \mathbf{Z}(\Omega)$ and as a consequence of Lemma 2.35 we know that

$$\tilde{T}_1 - \tilde{T}_2 = \tilde{G}(\mathbf{u}_1, -(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\tilde{T}_2)\,.$$

Hence, we can apply Proposition 2.18 to obtain

$$\begin{aligned} \|\tilde{T}_1 - \tilde{T}_2\|_{1,q} \le K_2(q)\|(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\tilde{T}_2\|_{L^1} &\le K_2(q)\,\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6}\,\|\mathbf{grad}\,\tilde{T}_2\|_{\mathbf{L}^{6/5}} \\ &\le SK_2(q)\,|\mathbf{u}_1 - \mathbf{u}_2|_1\,\|\tilde{T}_2\|_{1,6/5}\,, \end{aligned}$$

where S and $K_2(q) = C(q)K_1 meas(\Omega)^{\frac{1}{N}-\frac{1}{q'}}\sqrt{N}$ are the constants appearing in (2.64) and in Proposition 2.18, respectively. The result holds with $K_3 = SK_2(q)$. □

2.4.2 Uniqueness result for the model with Boussinesq approximation.

We want to prove the uniqueness of solution in a certain closed ball $\overline{B(0,R)}$ contained in the space $\mathcal{Z}_0(\Omega)$, with $R > 0$. The idea is to prove that, if $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)$, $(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \overline{B(0,R)}$ are two fixed points of mapping $\mathcal{G}$ defined in (2.77), then there exists a constant $L < 1$ such that

$$|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}} = |\mathcal{G}((\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)) - \mathcal{G}((\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2))|_{\mathcal{W}} \le L|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}}\,. \tag{2.142}$$

In order to obtain the previous inequality, we must remind some of the estimates obtained for the mappings appearing in the definition of $\mathcal{G}$. Moreover, we must also prove some results of Lipschitz continuity for those mappings.

First of all, we recall that for $\mathcal{G}_3$ defined in (2.76), we have the estimate in (2.78),

$$\|\mathcal{G}_3((\hat{\mathbf{u}}, \hat{\mathbf{B}}))\|_{L^1} \le k_f|(\hat{\mathbf{u}}, \hat{\mathbf{B}})|^2_{\mathcal{W}} + 8k_{f2}\,|\mathbf{u}_0|_1^2 + \|\psi\|_{L^1}\,.$$

Moreover, from its definition, the mapping $\mathcal{G}_3$ can also be written in the form

$$\mathcal{G}_3((\hat{\mathbf{u}},\hat{\mathbf{B}})) = \frac{E_c}{R_e}\left(b((\hat{\mathbf{u}},\hat{\mathbf{B}}),(\hat{\mathbf{u}},\hat{\mathbf{B}})) + d(\hat{\mathbf{u}}) + \frac{1}{2}|\mathbf{grad}\,\mathbf{u}_0 + \mathbf{grad}\,\mathbf{u}_0^t|^2\right) + \psi\,,$$

with the bilinear map $b : \mathcal{Z}_0(\Omega)\times\mathcal{Z}_0(\Omega) \to L^1(\Omega)$ and the linear map $d : \mathbf{Z}_0(\Omega) \to L^1(\Omega)$ respectively defined as

$$\begin{aligned} b((\hat{\mathbf{u}},\hat{\mathbf{B}}),(\hat{\mathbf{w}},\hat{\mathbf{D}})) &:= \frac{H_a^2}{R_m^2}\mathbf{curl}\,\hat{\mathbf{B}}\cdot\mathbf{curl}\,\hat{\mathbf{D}} + \frac{1}{2}(\mathbf{grad}\,\hat{\mathbf{u}} + \mathbf{grad}\,\hat{\mathbf{u}}^t) : (\mathbf{grad}\,\hat{\mathbf{w}} + \mathbf{grad}\,\hat{\mathbf{w}}^t)\,,\\ d(\hat{\mathbf{u}}) &:= (\mathbf{grad}\,\hat{\mathbf{u}} + \mathbf{grad}\,\hat{\mathbf{u}}^t) : (\mathbf{grad}\,\mathbf{u}_0 + \mathbf{grad}\,\mathbf{u}_0^t)\,. \end{aligned}$$

Therefore, for $\mathcal{G}_3$ we have the following Lipschitz continuity

$$\begin{aligned} &\|\mathcal{G}_3((\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)) - \mathcal{G}_3((\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2))\|_{L^1}\\ &= \left\|\frac{E_c}{R_e}\left(b((\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1),(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)) + d(\hat{\mathbf{u}}_1) - b((\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2),(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)) - d(\hat{\mathbf{u}}_2)\right)\right\|_{L^1}\\ &= \frac{E_c}{R_e}\left\|b((\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1),(\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2)) + b((\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2),(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)) + d(\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2)\right\|_{L^1}\\ &\le \frac{E_c}{R_e}\left(\max\left\{\frac{H_a^2}{R_m^2},2\right\}\left(|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)|_{\mathcal{W}} + |(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\right)|(\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2)|_{\mathcal{W}}\right.\\ &\qquad\left. +4|\mathbf{u}_0|_1|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1\right)\,. \end{aligned}$$

For the analysis of the mapping $\mathcal{G}_2$, we recall that it has been defined in (2.75). For the mapping G_D we have the bound given in Proposition 2.15, namely,

$$\|G_D(\mathbf{u})\|_{L^\infty} \le \|T_d\|_{L^\infty(\partial\Omega)}\,. \tag{2.143}$$

To obtain a Lipschitz constant for G_D we denote $T_{D,i} = G_D(\mathbf{u}_i)$, with $\mathbf{u}_i = \mathbf{u}_0 + \hat{\mathbf{u}}_i$ for $i = 1,2$. Reasoning as in Lemma 2.16 we obtain

$$e(T_{D,1}-T_{D,2},z) + d(\mathbf{u}_2,T_{D,1}-T_{D,2},z) = -d(\mathbf{u}_1-\mathbf{u}_2,T_{D,1},z) \quad \forall z\in H_0^1(\Omega)\,.$$

Since $T_{D,1}-T_{D,2}\in H_0^1(\Omega)$ we can take it as test function and using the antisymmetry property of Lemma 2.13 we find that

$$e(T_{D,1}-T_{D,2},T_{D,1}-T_{D,2}) = d(\mathbf{u}_1-\mathbf{u}_2,T_{D,1}-T_{D,2},T_{D,1})\,.$$

Then, the coerciveness of $e(\cdot,\cdot)$, the continuity of $d(\cdot,\cdot,\cdot)$ and the estimate (2.143) yield

$$|T_{D,1}-T_{D,2}|_1 \le \frac{\tilde{\lambda}_d}{\alpha_e}\|T_{D,1}\|_{L^\infty}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{L}^6} \le \frac{\tilde{\lambda}_d}{\alpha_e}\|T_d\|_{L^\infty(\partial\Omega)}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{L}^6}\,, \tag{2.144}$$

with $\tilde{\lambda}_d = meas(\Omega)^{1/3}$ as it was given in Lemma 2.11.

Next step is the study of $\tilde{G}$. We recall that in Proposition 2.18 we have already obtained the estimate

$$\|\tilde{G}(\mathbf{u},f)\|_{1,6/5} \le K_2\|f\|_{L^1}\,, \tag{2.145}$$

with $K_2 = C(6/5)K_1 meas(\Omega)^{1/6}\sqrt{3}$ a constant independent of the velocity $\mathbf{u}$. To prove a result of Lipschitz continuity for $\tilde{G}$ we first assume that $f_i \in L^1(\Omega)$, for $i = 1, 2$. Then we have

$$\|\tilde{G}(\mathbf{u}_1, f_1) - \tilde{G}(\mathbf{u}_2, f_2)\|_{1,6/5} \le \|\tilde{G}(\mathbf{u}_1, f_1 - f_2)\|_{1,6/5} + \|\tilde{G}(\mathbf{u}_1, f_2) - \tilde{G}(\mathbf{u}_2, f_2)\|_{1,6/5} . \tag{2.146}$$

For the first term we can use Proposition 2.18 to obtain

$$\|\tilde{G}(\mathbf{u}_1, f_1 - f_2)\|_{1,6/5} \le K_2 \|f_1 - f_2\|_{L^1} , \tag{2.147}$$

with K_2 the same constant appearing above, which is independent of the velocity $\mathbf{u}_1$. For the second term we use Lemma 2.36 and get

$$\|\tilde{G}(\mathbf{u}_1, f_2) - \tilde{G}(\mathbf{u}_2, f_2)\|_{1,6!5} \le K_3 |\mathbf{u}_1 - \mathbf{u}_2|_1 \|\tilde{G}(\mathbf{u}_2, f_2)\|_{1,6/5} , \tag{2.148}$$

with $K_3 = SK_2$.

Now let us assume that $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)$, $(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \mathcal{Z}_0(\Omega)$ are two fixed points of the mapping $\mathcal{G}$ and such that $|(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)|_{\mathcal{W}} \le R$, $i = 1, 2$. Then, denoting $f_i = \mathcal{G}_3(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)$ and $T_i = \mathcal{G}_2(\hat{\mathbf{u}}_i, f_i)$, we have

$$(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i) = \mathcal{G}((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)) = \mathcal{G}_1((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i), T_i) .$$

Since $(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)$, $i = 1, 2$, are the solutions to the corresponding linearized MHD problems, subtracting the equations of the two problems and reasoning as in Lemma 2.10 we arrive at

$$\begin{aligned} a((\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2), (\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)) + c_0(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) \\ -c_1(\hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) + c_1(\hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{u}}_2) \\ = -(G(T_1 - T_2), \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)_\Omega - c_0(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \mathbf{u}_0, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) - c_1(\hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \mathbf{u}_0) . \end{aligned}$$

For any $(\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\hat{\mathbf{v}}, \hat{\mathbf{C}}), (\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ we have

$$\begin{aligned} & |c_0(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) - c_1(\hat{\mathbf{C}}, \hat{\mathbf{B}}, \hat{\mathbf{w}}) + c_1(\hat{\mathbf{D}}, \hat{\mathbf{B}}, \hat{\mathbf{v}})| \\ & \le \lambda_{c0} \|\hat{\mathbf{u}}\|_{\mathbf{L}^4} \|\hat{\mathbf{v}}\|_{\mathbf{L}^4} |\hat{\mathbf{w}}|_1 + \lambda_{c1} |\hat{\mathbf{C}}|_{\mathbf{X}} \|\hat{\mathbf{B}}\|_{\mathbf{L}^3} \|\hat{\mathbf{w}}\|_{\mathbf{L}^6} + \lambda_{c1} |\hat{\mathbf{D}}|_{\mathbf{X}} \|\hat{\mathbf{B}}\|_{\mathbf{L}^3} \|\hat{\mathbf{v}}\|_{\mathbf{L}^6} \\ & \le \lambda_{c0} C_0^2 \gamma_4^2 |\hat{\mathbf{u}}|_1 |\hat{\mathbf{v}}|_1 |\hat{\mathbf{w}}|_1 + \lambda_{c1} C_0 \gamma_6 C_1 \kappa |\hat{\mathbf{B}}|_{\mathbf{X}} \left(|\hat{\mathbf{C}}|_{\mathbf{X}} |\hat{\mathbf{w}}|_1 + |\hat{\mathbf{D}}|_{\mathbf{X}} |\hat{\mathbf{v}}|_1 \right) \\ & \le \lambda_{c0} C_0^2 \gamma_4^2 |\hat{\mathbf{u}}|_1 |\hat{\mathbf{v}}|_1 |\hat{\mathbf{w}}|_1 + \lambda_{c1} C_0 \gamma_6 C_1 \kappa |\hat{\mathbf{B}}|_{\mathbf{X}} |(\hat{\mathbf{v}}, \hat{\mathbf{C}})|_{\mathcal{W}} |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \\ & \le \max\left\{\lambda_{c0} C_0^2 \gamma_4^2, \lambda_{c1} C_0 \gamma_6 C_1 \kappa\right\} |(\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \left(|\hat{\mathbf{v}}|_1^2 |\hat{\mathbf{w}}|_1^2 + |(\hat{\mathbf{v}}, \hat{\mathbf{C}})|_{\mathcal{W}}^2 |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}}^2 \right)^{1/2} \\ & \le \sqrt{2} \max\left\{\lambda_{c0} C_0^2 \gamma_4^2, \lambda_{c1} C_0 \gamma_6 C_1 \kappa\right\} |(\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} |(\hat{\mathbf{v}}, \hat{\mathbf{C}})|_{\mathcal{W}} |(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} . \end{aligned} \tag{2.149}$$

Using this last result, along with the inequality (2.41) and the coerciveness of the form $a(\cdot, \cdot)$, we obtain

$$\begin{aligned} \alpha_a |(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}} \le \lambda_G \|T_1 - T_2\|_{L^{6/5}} + \max\left\{ \tilde{\lambda}_{c0} \|\mathbf{u}_0\|_{\mathbf{L}^4} , \tilde{\lambda}_{c1} \|\mathbf{u}_0\|_{\mathbf{L}^6} \right\} |(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}} \\ + \sqrt{2} \max\left\{\lambda_{c0} C_0^2 \gamma_4^2, \lambda_{c1} C_0 \gamma_6 C_1 \kappa\right\} |(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}} |(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}} . \end{aligned}$$

Then, using the estimates for $\mathcal{G}_2$ we get

$$\alpha_a|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}} \leq \lambda_G\Big(meas(\Omega)^{2/3}S\,|T_{D,1}-T_{D,2}|_1+K_2\,\|f_1-f_2\|_{L^1}$$
$$+SK_2\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1\,\|\tilde{G}(\mathbf{u}_2,f_2)\|_{1,6/5}\Big)+\max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4},\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$\leq \lambda_G\big(meas(\Omega)^{2/3}S^2\frac{\tilde{\lambda}_d}{\alpha_e}\,\|T_d\|_{L^\infty(\partial\Omega)}\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1+K_2\,\|f_1-f_2\|_{L^1}$$
$$+SK_2^2\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1\,\|f_2\|_{L^1}\big)+\max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4},\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\,.$$

Finally, using the Lipschitz continuity result and the estimate obtained for $\mathcal{G}_3$ we have

$$\alpha_a|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}} \leq \lambda_G\Big[meas(\Omega)^{2/3}S^2\frac{\tilde{\lambda}_d}{\alpha_e}\,\|T_d\|_{L^\infty(\partial\Omega)}\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1$$
$$+K_2\frac{E_c}{R_e}\left(\max\left\{\frac{H_a^2}{R_m^2},2\right\}\Big(|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)|_{\mathcal{W}}+|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\Big)+4\,|\mathbf{u}_0|_1\right)|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+SK_2^2(k_f|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}^2+8k_{f2}\,|\mathbf{u}_0|_1^2+\|\psi\|_{L^1})\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1\Big]$$
$$+\max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4},\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\,.$$

As we have assumed that $|(\hat{\mathbf{u}}_i,\hat{\mathbf{B}}_i)|_{\mathcal{W}} \leq R$ for $i=1,2$, it is possible to find a constant $L<1$ such that (2.142) is satisfied if

$$\tilde{k}_2R^2+\tilde{k}_1R+(\tilde{k}_0-\alpha_a)<0\,, \tag{2.150}$$

with

$$\tilde{k}_2=\lambda_GSK_2^2k_f\,, \tag{2.151}$$
$$\tilde{k}_1=2\lambda_GK_2\frac{E_c}{R_e}\max\left\{\frac{H_a^2}{R_m^2},2\right\}+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}, \tag{2.152}$$
$$\tilde{k}_0=\lambda_G\bigg(meas(\Omega)^{2/3}S^2\frac{\tilde{\lambda}_d}{\alpha_e}\,\|T_d\|_{L^\infty(\partial\Omega)}+4K_2\frac{E_c}{R_e}\,|\mathbf{u}_0|_1$$
$$+SK_2^2(8k_{f2}\,|\mathbf{u}_0|_1^2+\|\psi\|_{L^1})\bigg)+\max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4},\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}. \tag{2.153}$$

We are seeking a closed ball of radius R to ensure the uniqueness but, at the same time, we must have a result of existence in that ball. We first consider the case $k_0>0$, where k_0 is the constant appearing in Lemma 2.21. We recall that in this case, in order to affirm the existence of solution we assume conditions (2.85) are satisfied, and there exists a solution to our problem in the closed ball $\overline{B(0,R_-)}$, with R_- the radius given in (2.84). We know that

$$R_-^2 = \frac{-k_0 + (\alpha_a - k_1)R_-}{k_2},$$

with k_1 and k_2 the two other constants appearing in the proof of Lemma 2.21. Substituting R by R_- in equation (2.150) we obtain that R_- should satisfy

$$\left(\tilde{k}_1 + \frac{\tilde{k}_2}{k_2}(\alpha_a - k_1)\right) R_- + \tilde{k}_0 - \alpha_a - \frac{\tilde{k}_2}{k_2}k_0 < 0\,. \tag{2.154}$$

Now, on one hand we know that

$$R_- = \frac{(\alpha_a - k_1) - \sqrt{(\alpha_a - k_1)^2 - 4k_0k_2}}{2k_2} = \frac{2k_0}{(\alpha_a - k_1) + \sqrt{(\alpha_a - k_1)^2 - 4k_0k_2}} \le \frac{2k_0}{(\alpha_a - k_1)},$$

and on the other hand, since $R_-R_+ = k_0/k_2 > 0$ and $0 < R_- \le R_+$, we have

$$R_- \le \sqrt{\frac{k_0}{k_2}}\,.$$

Using these two inequalities we obtain that (2.154) holds if

$$\tilde{k}_1\sqrt{\frac{k_0}{k_2}} + \frac{\tilde{k}_2}{k_2}2k_0 + \tilde{k}_0 - \alpha_a - \frac{\tilde{k}_2}{k_2}k_0 < 0\,. \tag{2.155}$$

Hence, we can prove that there exists a unique solution assuming certain smallness of the data:

Theorem 2.37. *If* $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$, *with* $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, $T_d \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$ *and* $\psi \in L^1(\Omega)$, *under the assumptions*

$$\|\mathbf{u}_0\|_{\mathbf{L}^4} < \frac{\alpha_a}{\tilde{\lambda}_{c0}}\,, \tag{2.156}$$

$$\|\mathbf{u}_0\|_{\mathbf{L}^6} < \frac{\alpha_a}{\tilde{\lambda}_{c1}}\,, \tag{2.157}$$

$$0 < \mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) \le \frac{(k_1 - \alpha_a)^2}{4\lambda_G K_2 k_f}, \tag{2.158}$$

and

$$\mathcal{E}_u(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) < \alpha_a\,, \tag{2.159}$$

there exists a constant $R > 0$ *such that there is only one fixed point of mapping* $\mathcal{G}$ *in the ball* $\overline{B(0,R)}$.

Both $\mathcal{E}_e$ *and* $\mathcal{E}_u$ *depend only on the boundary and source data. The first one is defined as in Lemma 2.21, whereas the second one has the expression*

$$\mathcal{E}_u(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) = \tilde{k}_1\sqrt{\frac{k_0}{k_2}} + \frac{\tilde{k}_2}{k_2}k_0 + \tilde{k}_0\,,$$

with k_0 *and* k_2 *the constants appearing in Lemma 2.21, and* $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ *the constants defined in (2.151)-(2.153).*

Proof. The existence has been proved in Theorem 2.22 under the hypotheses (2.156)-(2.158); in particular, for a ball of radius

$$R_- = \frac{(\alpha_a - k_1) - \sqrt{(\alpha_a - k_1)^2 - 4k_0k_2}}{2k_2}.$$

Under the assumption (2.159) we have proved that R_- satisfies

$$\tilde{k}_2 R_-^2 + \tilde{k}_1 R_- + (\tilde{k}_0 - \alpha_a) < 0\,,$$

and this implies that there exists a constant $L < 1$ such that, if $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1), (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \overline{B(0, R_-)}$ in the space $\mathcal{Z}_0(\Omega)$ are two fixed points of the mapping $\mathcal{G}$, then

$$|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}} = |\mathcal{G}((\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)) - \mathcal{G}((\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2))|_{\mathcal{W}} \leq L|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)|_{\mathcal{W}}\,,$$

and as a consequence $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) = (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)$. □

Summarizing, we have proved in the previous theorem that, assuming smallness of the boundary and source data of the coupled problem and for a certain lifting $(\mathbf{u}_0, \mathbf{B}_0)$ of the boundary data, there exists a unique solution to the original MHD problem in a neighborhood of the lifting.

Remark 2.11. *Concerning the case* $k_0 = 0$ *we have already seen in Remark 2.6 that the trivial solution is a fixed point of mapping* $\mathcal{G}$*. We can prove that this solution is unique in a certain ball. First of all, we notice that, due to their definitions,* $k_0 = 0$ *yields* $\tilde{k}_0 = 0$ *and the inequality (2.150) becomes*

$$\tilde{k}_2 R^2 + \tilde{k}_1 R - \alpha_a < 0\,.$$

Since $\tilde{k}_2\alpha_a > 0$*, the corresponding quadratic equation has one real negative and one real positive roots. In particular, the inequality is satisfied for any value* $R \in [0, \tilde{R}_+)$*, with*

$$\tilde{R}_+ = \frac{-\tilde{k}_1 + \sqrt{\tilde{k}_1^2 + 4\tilde{k}_2\alpha_a}}{2\tilde{k}_2}.$$

As a consequence, we know that the trivial solution is unique in the ball $B(0, \tilde{R}_+)$ *of* $\mathcal{Z}_0(\Omega)$*.*

2.4.3 Uniqueness result for the second model.

In the previous section we have proved a uniqueness result for the MHD model under the Boussinesq approximation. Now we are going to prove a uniqueness result for the model introduced in Section 2.3. Once again we are going to prove, under smallness of the data, that there exists a closed ball $\overline{B(0, R)}$ such that there is only one fixed point of the mapping $\hat{\mathcal{G}}$, defined in (2.119). The idea is to obtain some results of Lipschitz continuity on bounded sets for the mappings appearing in the definition of $\hat{\mathcal{G}}$.

In order to prove the Lipschitz continuity result, we must first assume that the response function $\hat{\rho}$ is Lipschitz continuous in the sense that

$$|\hat{\rho}(\theta_1) - \hat{\rho}(\theta_2)| \leq \Lambda_\rho |\theta_1 - \theta_2| \qquad \forall \theta_1, \theta_2 \in [T_{\min}, +\infty)\,. \tag{2.160}$$

Now, we have to analyze one by one the mappings appearing in the definition of $\hat{\mathcal{G}}$. First of all, for the mapping $\hat{\mathcal{G}}_3$ we have

$$\|\hat{\mathcal{G}}_3((\hat{\mathbf{u}},\hat{\mathbf{B}}))\|_{L^1} \leq k_f|(\hat{\mathbf{u}},\hat{\mathbf{B}})|^2_{\mathcal{W}} + 8k_{f2}\,|\mathbf{u}_0|^2_1 + \|\psi\|_{L^1}\ ,$$

with $k_f = \max\{k_{f1}, 8k_{f2}\}$, $k_{f1} = u^2H_a^2/(c_p\mathcal{T}R_eR_m^2)$, $k_{f2} = u^2/(2c_p\mathcal{T}R_e)$, and also

$$\begin{aligned}&\|\mathcal{G}_3((\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)) - \mathcal{G}_3((\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2))\|_{L^1}\\ \leq \frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}&\left(\max\left\{\frac{H_a^2}{R_m^2},2\right\}\left(|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)|_{\mathcal{W}} + |(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\right)|(\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2)|_{\mathcal{W}}\right.\\ &\left.+4|\mathbf{u}_0|_1|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1\right).\end{aligned}$$

The definition of mapping $\hat{\mathcal{G}}_2$ is identical to that of mapping $\mathcal{G}_2$, hence the same estimates proved in the previous section for this mapping, or for the mappings G_D and $\tilde{G}$, remain valid. However, the result given in (2.144) can be improved in the sense that, instead of the norm $\|T_d\|_{L^\infty(\partial\Omega)}$, it is the difference between the maximum and the minimum temperatures on the boundary what is involved. To obtain the result, we first observe that the solution $G_D(\mathbf{u})$ indeed depends on the boundary condition T_d. Therefore, introducing a slight abuse of notation, we will refer to $G_D(\mathbf{u})$ as $G_D(\mathbf{u}, T_d)$. Since problem (2.48) is linear, we know that $G_D(\mathbf{u}, T_d) = G_D(\mathbf{u}, T_d - c) + G_D(\mathbf{u}, c) = G_D(\mathbf{u}, T_d - c) + c$ for every constant $c \in \mathbb{R}$. In particular, taking $c = T_{\min}$ the value defined in (2.102), we get

$$G_D(\mathbf{u}_1, T_d) - G_D(\mathbf{u}_2, T_d) = G_D(\mathbf{u}_1, T_d - T_{\min}) - G_D(\mathbf{u}_2, T_d - T_{\min})\,,$$

and the result obtained in (2.144) is transformed into

$$|T_{D,1} - T_{D,2}|_1 \leq \frac{\tilde{\lambda}_d}{\alpha_e}\,\|T_d - T_{\min}\|_{L^\infty(\partial\Omega)}\,\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6}\ . \tag{2.161}$$

Finally, let us assume that $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)\,, (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \mathcal{Z}_0(\Omega)$ are two fixed points of the mapping $\hat{\mathcal{G}}$ and such that $|(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)|_{\mathcal{W}} \leq R$, $i = 1, 2$. We denote $f_i = \hat{\mathcal{G}}_3((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i))$ and $T_i = \hat{\mathcal{G}}_2(\hat{\mathbf{u}}_i, f_i)$. With this notation we know that

$$(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i) = \hat{\mathcal{G}}((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)) = \hat{\mathcal{G}}_1((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i), T_i)\,,$$

which means that the fields $(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)$ are solution to their respective linearized MHD problems. Reasoning as in Lemma 2.10 (see also Lemma 2.26), we obtain

$$\begin{aligned}&a((\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2),(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)) + c_0(\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2,\hat{\mathbf{u}}_2,\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2)\\ &\qquad -c_1(\hat{\mathbf{B}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2,\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2) + c_1(\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2,\hat{\mathbf{u}}_2)\\ &= -\langle\hat{G}(T_1)-\hat{G}(T_2),\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2\rangle_\Omega - c_0(\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2,\mathbf{u}_0,\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2) - c_1(\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2,\hat{\mathbf{B}}_1-\hat{\mathbf{B}}_2,\mathbf{u}_0)\,,\end{aligned}$$

with $\langle\hat{G}(T_1) - \hat{G}(T_2), \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2\rangle_\Omega = \int_\Omega (\hat{\rho}(T_1) - \hat{\rho}(T_2))\mathbf{g}\cdot(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\,\mathrm{d}\mathbf{x}$. For this term, we know that

$$\left|\int_\Omega (\hat{\rho}(T_1) - \hat{\rho}(T_2))\mathbf{g}\cdot(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\,\mathrm{d}\mathbf{x}\right| \leq Sg\,\|\hat{\rho}(T_1) - \hat{\rho}(T_2)\|_{L^{6/5}}\,|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1\ ,$$

being g the modulus of the gravity acceleration, and S the constant appearing in (2.64).

Using this inequality along with (2.41), (2.149), the coerciveness of $a(\cdot,\cdot)$ and the Lipschitz continuity of $\hat{\rho}$ stated in (2.160) we get

$$\alpha_a|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}} \leq \max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}\,,\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+Sg\Lambda_\rho\,\|T_1-T_2\|_{L^{6/5}}+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\,.$$

Then, using the estimates for $\hat{\mathcal{G}}_2$ and $\hat{\mathcal{G}}_3$ and reasoning as in the previous section, we obtain

$$\alpha_a|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}} \leq Sg\Lambda_\rho\Bigg[meas(\Omega)^{2/3}S^2\frac{\tilde{\lambda}_d}{\alpha_e}\,\|T_d-T_{\min}\|_{L^\infty(\partial\Omega)}\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1$$
$$+K_2\frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}\left(\max\left\{\frac{H_a^2}{R_m^2},2\right\}\left(|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)|_{\mathcal{W}}+|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\right)+4\,|\mathbf{u}_0|_1\right)|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+SK_2^2(k_f|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}^2+8k_{f2}\,|\mathbf{u}_0|_1^2+\|\psi\|_{L^1})\,|\hat{\mathbf{u}}_1-\hat{\mathbf{u}}_2|_1\Bigg]$$
$$+\max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}\,,\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}$$
$$+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}|(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\,.$$

As we did in the previous section and since we have assumed that the fixed points $(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)$, $(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)$ belong to the closed ball $\overline{B(0,R)}$ in $\mathcal{Z}_0(\Omega)$, we can prove that there exists a constant $L<1$ such that

$$|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}=|\hat{\mathcal{G}}((\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1))-\hat{\mathcal{G}}((\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2))|_{\mathcal{W}}\leq L|(\hat{\mathbf{u}}_1,\hat{\mathbf{B}}_1)-(\hat{\mathbf{u}}_2,\hat{\mathbf{B}}_2)|_{\mathcal{W}}\,, \tag{2.162}$$

whenever R satisfies the inequality

$$\tilde{k}_2R^2+\tilde{k}_1R+(\tilde{k}_0-\alpha_a)<0\,, \tag{2.163}$$

with

$$\tilde{k}_2 = Sg\Lambda_\rho SK_2^2k_f\,, \tag{2.164}$$
$$\tilde{k}_1 = 2Sg\Lambda_\rho K_2\frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}\max\left\{\frac{H_a^2}{R_m^2},2\right\}+\sqrt{2}\max\left\{\lambda_{c0}C_0^2\gamma_4^2,\lambda_{c1}C_0\gamma_6C_1\kappa\right\}, \tag{2.165}$$
$$\begin{aligned}\tilde{k}_0 &= Sg\Lambda_\rho\Bigg(meas(\Omega)^{2/3}S^2\frac{\tilde{\lambda}_d}{\alpha_e}\,\|T_d-T_{\min}\|_{L^\infty(\partial\Omega)}+4K_2\frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}\,|\mathbf{u}_0|_1\\ &+ SK_2^2(8k_{f2}\,|\mathbf{u}_0|_1^2+\|\psi\|_{L^1})\Bigg)+\max\left\{\tilde{\lambda}_{c0}\,\|\mathbf{u}_0\|_{\mathbf{L}^4}\,,\tilde{\lambda}_{c1}\,\|\mathbf{u}_0\|_{\mathbf{L}^6}\right\}.\end{aligned} \tag{2.166}$$

As we have already noticed in Remark 2.9, under the conditions $\|\mathbf{u}_0\|_{\mathbf{L}^4}<\alpha_a/\tilde{\lambda}_{c0}$ and $\|\mathbf{u}_0\|_{\mathbf{L}^6}<\alpha_a/\tilde{\lambda}_{c1}$, and setting $R_0=k_0/(\alpha_a-k_1)$, we know that every solution $(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathcal{Z}_0(\Omega)$ belongs to the closed ball $\overline{B(0,R_0)}$ and there exists at least one solution in that ball. Thus, to prove the uniqueness of solution in the mentioned ball we must give conditions to ensure that R_0 satisfies (2.163).

Theorem 2.38. *Let* $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$*, with* $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$*, and let* $\psi \in L^1(\Omega)$ *and* $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\Omega)$ *satisfying (2.101) and (2.102), respectively. Under the assumptions*

$$\|\mathbf{u}_0\|_{\mathbf{L}^4} < \frac{\alpha_a}{\tilde{\lambda}_{c0}}\,, \tag{2.167}$$

$$\|\mathbf{u}_0\|_{\mathbf{L}^6} < \frac{\alpha_a}{\tilde{\lambda}_{c1}}\,, \tag{2.168}$$

$$\tilde{k}_2 k_0^2 + \tilde{k}_1(\alpha_a - k_1)k_0 + (\tilde{k}_0 - \alpha_a)(\alpha_a - k_1)^2 < 0\,, \tag{2.169}$$

there exists a unique fixed point of the mapping $\hat{\mathcal{G}} : \mathcal{Z}_0(\Omega) \to \mathcal{Z}_0(\Omega)$*. Moreover, it belongs to the closed ball* $\overline{B(0, R_0)}$*, with* $R_0 = k_0/(\alpha_a - k_1)$*.*

The constants k_1 *and* k_0 *have been introduced in Proposition 2.23, and* $\tilde{k}_2$*,* $\tilde{k}_1$*,* $\tilde{k}_0$ *are given by (2.164)-(2.166).*

Proof. From Proposition 2.23 and Theorem 2.28 we know that, under the assumptions (2.167)-(2.168), there exists at least one fixed point for the mapping $\hat{\mathcal{G}}$ and every fixed point belongs to the closed ball $\overline{B(0, R_0)}$. Moreover, from the assumption (2.169) we easily deduce that R_0 fulfills (2.163). Thus, there exists a constant $L < 1$ such that (2.162) is satisfied for every couple of fixed points $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)$, $(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \overline{B(0, R_0)}$. As a consequence, there is only one fixed point for the mapping $\hat{\mathcal{G}}$ in that ball. □

The assumptions (2.167)-(2.168) clearly impose a condition of smallness on the lifting $\mathbf{u}_0$, and as we have seen before, they are equivalent to the condition $\alpha_a - k_1 > 0$. Concerning the assumption (2.169), and recalling the definition of constants k_0 and $\tilde{k}_0$, we notice that this condition first imposes smallness of the source data $\mathbf{f}, \mathbf{k}$ and ψ, and of the liftings $\mathbf{u}_0$ and $\mathbf{B}_0$. It also requires a small difference between the minimum and maximum temperature on the boundary, and a condition of smallness on the maximum density $\hat{\rho}_{\max}$. If we assume that the density function $\hat{\rho}$ tends to zero as the temperature tends to infinity, then these two conditions can be fulfilled at the same time, and therefore the condition (2.169) can be also satisfied.

Remark 2.12. *In Section 2.3.1 we have assumed that* $\hat{\rho}$ *is continuous, strictly positive and non-increasing. Hence, there exists a limit of* $\hat{\rho}$ *as the temperature tends to infinity, but we cannot assure that this limit is equal to zero. Anyway, the equations of the model can be modified to obtain a result analogous to Theorem 2.38. Let us denote by* $\hat{\rho}_\infty$ *the previous limit. Summing up and subtracting the term* $\hat{\rho}_\infty \mathbf{g}$ *in equation (2.94), denoting* $\breve{\rho} = \hat{\rho} - \hat{\rho}_\infty$ *and introducing a modified pressure* $p' = p - \hat{\rho}_\infty \mathbf{g} \cdot \mathbf{x}$*, as it is done in the Boussinesq approximation, we arrive at the equation*

$$-\frac{1}{H_a^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p' - \frac{1}{R_m}(\mathbf{curl}\,\mathbf{B}) \times \mathbf{B} = \mathbf{f}_0 + \breve{\rho}(T)\mathbf{g},$$

with $\breve{\rho}$ *a function satisfying the same properties that* $\hat{\rho}$*, and its limit is zero as temperature tends to infinity.*

Remark 2.13. *In Section 2.3.5 we proved that it is always possible to construct a lifting* $\mathbf{u}_0$ *satisfying (2.167)-(2.168). We cannot affirm the same about condition (2.169) because, due to the definitions of* k_0 *and* $\tilde{k}_0$*, this condition requires at the same time smallness of* $|\mathbf{u}_0|_1$ *and* $\|\mathbf{u}_0\|_{\mathbf{L}^6}$*.*

According to Theorem 1.5.1.10 in [54] we know that there exists a constant K such that, for any lifting $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ of the boundary data $\mathbf{u}_d$, it holds

$$\|\mathbf{u}_d\|_{\mathbf{L}^2(\partial\Omega)} \leq K \left(\varepsilon^{1/2} |\mathbf{u}_0|_1^2 + \varepsilon^{-1/2} \|\mathbf{u}_0\|_0^2 \right) \qquad \forall \varepsilon \in (0,1) \, .$$

Using Lemma 2.31 we can construct $\mathbf{u}_0(\varepsilon)$ satisfying $\|\mathbf{u}_0(\varepsilon)\|_{\mathbf{L}^6} \leq \varepsilon$, but taking the same ε in the previous inequality, and by using Hölder inequality, it is easily seen that $|\mathbf{u}_0(\varepsilon)|_1$ tends to infinity as ε tends to zero.

Chapter 3

Mathematical model for the induction furnace.

The mathematical results presented in the previous chapter correspond to a general MHD model. However, that model is not the most convenient in order to carry out a numerical simulation of the induction furnace. First, in a realistic setting the domain must be split into several subdomains, which correspond to different materials in the furnace. Moreover, the hydrodynamic model is only taken into account in the molten part and, in order to consider the melting of the region, the heat equation should be rewritten in terms of the enthalpy.

This chapter is devoted to present the mathematical model that will be used to simulate the induction furnace, considering some simplifications from the full MHD equations, but different from those introduced in the previous chapter. The model is written in an axisymmetrical setting, which simplifies the numerical simulation.

3.1 Statement of the problem.

We first recall the geometry description of the induction furnace given in Chapter 1. The induction furnace consists of a helical coil surrounding a cylindrical crucible. The electrically conducting crucible contains the material to be melted, and is surrounded by refractory and insulating materials to avoid heat losses. An alternating low frequency current traversing the coil produces an oscillating magnetic field, which generates eddy currents in the conducting materials within the workpiece. These currents, due to the Joule effect, produce heat in the conducting crucible in such a way that the metal is also heated until it melts.

In order to state the problem in an axisymmetric setting, the helical induction coil has to be replaced by m rings having toroidal geometry. Let Ω_0 be the radial section of the conducting parts of the workpiece and $\Omega_1, \Omega_2, \ldots, \Omega_m$ be the radial sections of the turns of the coil, which are assumed to be simply connected. Moreover, we denote by Ω the radial section of the set of

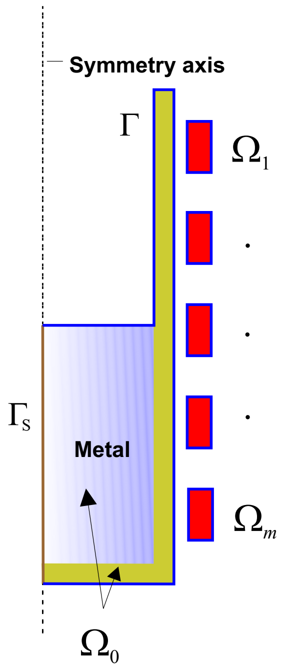

Figure 3.1: Radial section of inductors and workpiece.

conductors, *i.e.*, inductors and conducting materials in the workpiece, given by (see Figure 3.1),

$$\Omega = \bigcup_{k=0}^{m} \Omega_k \,.$$

Finally, we denote by Ω^c the complementary set of Ω in the half-plane $\{(r,z) \in \mathbb{R}^2 : r \geq 0\}$, and notice that this is an unbounded set corresponding to the radial section of all the insulating regions of the workpiece and the air surrounding the whole device.

Let $\Delta \subset \mathbb{R}^3$ be the bounded open set generated by the rotation around the $z-$axis of Ω and Δ^c the complementary set of $\overline{\Delta}$ in $\mathbb{R}^3$, which corresponds to the set generated by rotation of Ω^c around the $z-$axis. Analogously, we denote by Δ_k, $k = 0, \ldots, m$ the subset of $\mathbb{R}^3$ generated by the rotation of Ω_k, $k = 0, \ldots, m$, respectively, around the $z-$axis (see Figure 3.2). In particular, Δ_0 is assumed to be simply connected. Moreover, from the way we have constructed $\Delta_k, k = 1, \ldots, m$, we know that the spaces of Neumann harmonic functions $\boldsymbol{\mathcal{H}}_\sigma(\Delta_k)$, introduced in (A.19), have dimension equal to one.

We denote by Σ the boundary of Δ and by Γ its intersection with the half-plane $\{(r,z) \in \mathbb{R}^2 : r > 0\}$. We notice that $\Sigma = \cup_{k=0}^{m} \Sigma_k$, where Σ_k denotes the boundary of Δ_k. Moreover, we assume that the boundary of Ω is the union of Γ and Γ_{s}, the latter being a subset of the symmetry axis (see Figure 3.1).

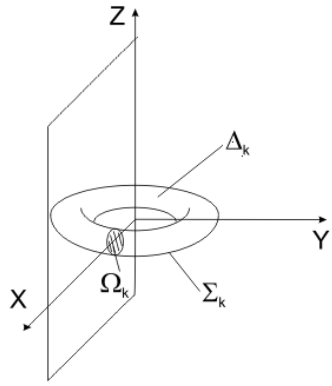

Figure 3.2: Sketch of a toroidal turn Δ_k.

3.2 The electromagnetic model.

Since the furnace works with alternating currents, we will use the time harmonic eddy-currents model, which was already introduced in Chapter 1. The equations of the model are the following

$$\begin{aligned} \mathbf{curl}\,\mathbf{H} &= \mathbf{J} \quad \text{in } \mathbb{R}^3\,, & (3.1)\\ \mathrm{i}\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} &= \mathbf{0} \quad \text{in } \mathbb{R}^3\,, & (3.2)\\ \operatorname{div}\mathbf{B} &= 0 \quad \text{in } \mathbb{R}^3\,, & (3.3) \end{aligned}$$

which are completed with the constitutive laws

$$\mathbf{B} = \mu\mathbf{H} \quad \text{in } \mathbb{R}^3\,, \tag{3.4}$$

$$\mathbf{J} = \begin{cases} \sigma\mathbf{E} & \text{in } \Delta,\\ \mathbf{0} & \text{in } \Delta^c. \end{cases} \tag{3.5}$$

Since the equations hold in $\mathbb{R}^3$, we also require for the fields a certain behaviour at infinity

$$\mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \text{uniformly for } |\mathbf{x}| \to \infty, \tag{3.6}$$

$$\mathbf{H}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \text{uniformly for } |\mathbf{x}| \to \infty. \tag{3.7}$$

Moreover, equations (3.1) and (3.5) force us to impose the following compatibility conditions

$$\operatorname{div}\mathbf{J} = 0 \ \text{ in } \Delta, \qquad \mathbf{J}\cdot\mathbf{n} = 0 \ \text{ on } \Sigma\,, \tag{3.8}$$

where $\mathbf{n}$ is a unit normal vector to Σ outward from Δ.

We also assume that the current intensities traversing each coil section $\mathbf{I} = (I_1, \ldots, I_m)$ are given, and we add to the model the following constraints

$$\int_{\Omega_k} \mathbf{J}\cdot\boldsymbol{\nu} = I_k, \ \ k = 1, \ldots, m, \tag{3.9}$$

where $\boldsymbol{\nu}$ denotes a unit normal vector to the sections Ω_k.

Remark 3.1. *In order to add the constraints given in (3.9) to the model, we will have to relax some of the equations (3.1)-(3.7). This need is due to the fact that problem (3.1)-(3.9) does not admit solution unless all the prescribed intensities I_k are null. Indeed, multiplying the conjugate of (3.2) by $\mathbf{H}$, integrating in $\mathbb{R}^3$ and using (3.6)-(3.7) we have*

$$\int_{\mathbb{R}^3} \mathbf{curl}\,\bar{\mathbf{E}} \cdot \mathbf{H} + \mathrm{i}\omega \int_{\mathbb{R}^3} \mu \bar{\mathbf{H}} \cdot \mathbf{H} = \int_{\mathbb{R}^3} \bar{\mathbf{E}} \cdot \mathbf{curl}\,\mathbf{H} + \mathrm{i}\omega \int_{\mathbb{R}^3} \mu \bar{\mathbf{H}} \cdot \mathbf{H} = 0\,.$$

Now equations (3.1) and (3.5) allow us to replace $\mathbf{E}$ by $\sigma^{-1}\mathbf{curl}\,\mathbf{H}$ in Δ, and recalling that $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ in Δ^c we obtain

$$\int_{\Delta} \frac{1}{\sigma} \mathbf{curl}\,\bar{\mathbf{H}} \cdot \mathbf{curl}\,\mathbf{H} + \mathrm{i}\omega \int_{\mathbb{R}^3} \mu \bar{\mathbf{H}} \cdot \mathbf{H} = 0\,,$$

hence $\mathbf{H} = \mathbf{0}$ in $\mathbb{R}^3$ and $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ in Δ, which implies $\mathbf{E}_{|\Delta} = \mathbf{0}$ and $\mathbf{J} = \mathbf{0}$. As a consequence the vector $\mathbf{I}$ also vanishes.

To overcome this difficulty, throughout this chapter we are going to relax equation (3.2) in the sense that it will be imposed in conductor and insulator separately, i.e.,

$$\mathrm{i}\omega \mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \textit{in } \Delta, \tag{3.10}$$
$$\mathrm{i}\omega \mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \textit{in } \Delta^c. \tag{3.11}$$

We are going to introduce a formulation in terms of a magnetic vector potential. To do that, we will also need to introduce a suitable scalar potential, which will be a function of the space of Neumann harmonic fields $\boldsymbol{\mathcal{H}}_\sigma(\Delta)$. Firstly, from equation (3.3) we can affirm that there exists a magnetic vector potential $\mathbf{A}$ such that

$$\mathbf{B} = \mathbf{curl}\,\mathbf{A}, \tag{3.12}$$

so equation (3.10) can be rewritten in the form

$$\mathrm{i}\omega \mathbf{curl}\,\mathbf{A} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \Delta. \tag{3.13}$$

Now, since $\mathbf{curl}\,(\mathrm{i}\omega\mathbf{A} + \mathbf{E}) = \mathbf{0}$ we can use the decomposition (A.23) considering each connected component of Δ separately, so that

$$(\mathrm{i}\omega\mathbf{A} + \mathbf{E})_{|\Omega_k} = -\mathbf{grad}\,\psi_k - V_k \boldsymbol{\varrho}_k \quad \text{in } \Delta_k\,, \quad k = 0, \ldots, m\,, \tag{3.14}$$

where $\psi_k \in H^1(\Delta_k)$ and $\boldsymbol{\varrho}_k$ is the basis function of the space $\boldsymbol{\mathcal{H}}_\sigma(\Delta_k)$, as defined in (A.19). From a physical point of view, the complex numbers V_k can be interpreted as voltage drops (see, for instance, [59]). In our particular case, since the workpiece Δ_0 is simply connected we know that $\boldsymbol{\varrho}_0 = \mathbf{0}$.

Taking into account that $\mathbf{H} = \mu^{-1}\mathbf{curl}\,\mathbf{A}$ and considering equations (3.1) and (3.5) we obtain

$$\mathrm{i}\omega\sigma\mathbf{A} + \mathbf{curl}\,(\frac{1}{\mu}\mathbf{curl}\,\mathbf{A}) = -\sigma(\sum_{k=0}^{m} \mathbf{grad}\,\psi_k + \sum_{k=1}^{m} V_k \boldsymbol{\varrho}_k). \tag{3.15}$$

Notice, however, that the vector potential $\mathbf{A}$ is not unique because it can be altered by any gradient. To ensure the uniqueness of $\mathbf{A}$ we also require some gauge conditions.

In the conductor region Δ we set $\operatorname{div}(\sigma\mathbf{A}) = 0$ and the boundary condition $\sigma\mathbf{A}\cdot\mathbf{n} = 0$ on Σ. From these gauge conditions, and from the compatibility conditions (3.8) we can infer that

$$\operatorname{div}(\mathrm{i}\omega\sigma\mathbf{A} + \sigma\mathbf{E}) = 0\,, \quad (\mathrm{i}\omega\sigma\mathbf{A} + \sigma\mathbf{E})\cdot\mathbf{n} = 0 \text{ on } \Sigma\,, \tag{3.16}$$

then recalling (3.13) and the definition of $\boldsymbol{\mathcal{H}}_\sigma(\Delta_k)$ we conclude that, in each connected component $(\mathrm{i}\omega\mathbf{A} + \mathbf{E})_{|\Delta_k} \in \boldsymbol{\mathcal{H}}_\sigma(\Delta_k)$, which implies that functions ψ_k in (3.14) are null and equation (3.15) can be rewritten as

$$\mathrm{i}\omega\sigma\mathbf{A} + \mathbf{curl}\,(\frac{1}{\mu}\mathbf{curl}\,\mathbf{A}) = -\sigma\sum_{k=1}^{m} V_k\boldsymbol{\varrho}_k \quad \text{in } \Delta\,. \tag{3.17}$$

In the air region Δ^c we impose the gauge conditions

$$\operatorname{div}\mathbf{A} = 0 \quad \text{in } \Delta^c \quad \text{and} \quad \int_{\Sigma_j}\mathbf{A}\cdot\mathbf{n} = 0, \quad j = 0,\ldots,m\,. \tag{3.18}$$

We notice that $\mathbf{A} \in \mathbf{H}(\operatorname{div};\Delta^c)$, but we are not going to assume that $\mathbf{A} \in \mathbf{H}(\operatorname{div};\mathbb{R}^3)$ so the normal trace $\mathbf{A}\cdot\mathbf{n}$ on Σ may be discontinuous and hence the integral conditions in (3.18) are not redundant.

Next, we obtain a weak formulation of equations (3.17) and (3.18) assigning the voltage drops V_k. We will come back later to the assignment of the current intensities given by conditions (3.9).

3.2.1 Weak formulation.

In order to propose a weak formulation of the previous problem we introduce some functional spaces and sets. Let $\Omega_e \subset \mathbb{R}^3$ be an open set, the complement of which is bounded in $\mathbb{R}^3$. We define the Beppo-Levi space

$$W^{1,-1}(\Omega_e) := \left\{\phi : \frac{\phi(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}} \in L^2(\Omega_e),\ \mathbf{grad}\,\phi \in \mathbf{L}^2(\Omega_e)\right\},$$

and its vectorial counterpart

$$\mathbf{W}^{1,-1}(\Omega_e) := \left\{\boldsymbol{\Phi} : \frac{\boldsymbol{\Phi}(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}} \in \mathbf{L}^2(\Omega_e),\ \mathbf{grad}\,\boldsymbol{\Phi} \in (L^2(\Omega_e))^{3\times 3}\right\}.$$

In our particular setting, we will also make use of the Beppo-Levi space

$$\boldsymbol{\mathcal{X}} = \left\{\mathbf{G} : \frac{\mathbf{G}(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}} \in \mathbf{L}^2(\mathbb{R}^3),\ \mathbf{curl}\,\mathbf{G} \in \mathbf{L}^2(\mathbb{R}^3)\right\},$$

and its subset

$$\boldsymbol{\mathcal{Y}} = \left\{\mathbf{G} \in \boldsymbol{\mathcal{X}} : \operatorname{div}\mathbf{G} = 0 \text{ in } \Delta^c,\ \int_{\Sigma_j}\mathbf{G}\cdot\mathbf{n} = 0,\ j = 0,\ldots,m\right\}.$$

We assume, in a first step, that the complex numbers V_k are given for $k = 1, \ldots, m$ and try to find the vector potential $\mathbf{A}$. After that, we will show how to solve the problem by giving the intensity vector $\mathbf{I} = (I_1, \ldots, I_m)$ as data, and with the voltages V_k playing the role of Lagrange multipliers.

Multiplying equation (3.17) by the complex conjugate of a test function $\mathbf{G}$, integrating in $\mathbb{R}^3$ and using a Green's formula we can easily obtain the following weak formulation:

Problem PV.- Given $\mathbf{V} = (V_1, \ldots, V_m) \in \mathbb{C}^m$, find $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$ such that

$$\mathrm{i}\omega \int_{\mathbb{R}^3} \sigma \mathbf{A} \cdot \bar{\mathbf{G}} + \int_{\mathbb{R}^3} \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \bar{\mathbf{G}} = -\sum_{k=1}^{m} V_k \int_{\Delta_k} \sigma \boldsymbol{\varrho}_k \cdot \bar{\mathbf{G}}, \quad \forall \mathbf{G} \in \boldsymbol{\mathcal{Y}}. \tag{3.19}$$

Remark 3.2. *In order to impose the current intensities across the inductors and to avoid relaxing the Faraday's law as we did in previous sections, the authors of [9] propose to modify the Ohm's law as follows*

$$\mathbf{J} = \sigma \mathbf{E} - \sigma \sum_{k=1}^{m} V_k \boldsymbol{\varrho}_k \quad \text{in } \Delta. \tag{3.20}$$

Notice that, in this way, the current density is divided into two parts: $\sigma \mathbf{E}$ *and a source term which is distributed in the coils* Δ_k*. Then, we notice that, by using the Faraday's law in* $\mathbb{R}^3$*, equation (3.14) reads*

$$\mathrm{i}\omega \mathbf{A} + \mathbf{E} = -\mathbf{grad}\, \Phi \quad \text{in } \mathbb{R}^3\,. \tag{3.21}$$

Imposing the same gauge conditions and arguing as in Section 3.2, we can arrive at the same weak problem **PV***.*

Theorem 3.1. *Problem* **PV** *has a unique solution.*

Proof. The proof is essentially similar to the one of Theorem 2.1 in [58] (see also [10]). We reproduce it here for the sake of completeness and because some steps will also be used in the proof of another result below. From now on, C denotes a generic positive constant.

The main task of the proof is to show the coerciveness of the sesquilinear form $a(\cdot, \cdot)$, defined as

$$a(\mathbf{A}, \mathbf{G}) := \mathrm{i}\omega \int_{\mathbb{R}^3} \sigma \mathbf{A} \cdot \bar{\mathbf{G}} + \int_{\mathbb{R}^3} \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \bar{\mathbf{G}}\,, \tag{3.22}$$

on the subspace

$$\widetilde{\boldsymbol{\mathcal{Y}}} := \{ \mathbf{G} \in \boldsymbol{\mathcal{X}} : \operatorname{div} \mathbf{G} = 0 \ \text{ in } \Delta^c \}\,. \tag{3.23}$$

To do this we take $\mathbf{G} \in \widetilde{\boldsymbol{\mathcal{Y}}}$ and consider its restriction to the conductors $\mathbf{G}_{|\Delta} \in \mathbf{H}(\mathbf{curl}\,; \Delta)$. According to [58, Th. 3.2] we can construct $\widetilde{\mathbf{G}} \in \mathbf{H}(\mathbf{curl}\,; \mathbb{R}^3)$ an extension of $\mathbf{G}_{|\Delta}$ to $\mathbb{R}^3$ satisfying

$$\|\widetilde{\mathbf{G}}\|_{\mathbf{H}(\mathbf{curl}\,;\Delta^c)} \leq C\, \|\mathbf{G} \times \mathbf{n}\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Sigma)}\,.$$

Moreover, using the same technique described in [79, Th. 5.4.2] this extension can be divergence-free. Defining $\mathbf{w} := \mathbf{G} - \widetilde{\mathbf{G}} \in \widetilde{\boldsymbol{\mathcal{Y}}}$ we notice that $\mathbf{w}$ has vanishing tangential components on the interface Σ. Next, we set $\mathbf{u}^c := \mathbf{curl}\,\mathbf{w}_{|\Delta^c}$, and from its definition and equation (A.16) we know that

$$\mathbf{u}^c \in \{\mathbf{\Phi} \in \mathbf{H}(\mathrm{div}\,;\Delta^c) : \mathrm{div}\,\mathbf{\Phi} = 0,\ \mathbf{\Phi}\cdot\mathbf{n} = 0 \text{ on } \Sigma\}\,. \tag{3.24}$$

Denoting by $\mathbf{u}$ its extension by zero into the set of conductors Δ, we obtain $\mathbf{u} \in \mathbf{H}(\mathrm{div}\,;\mathbb{R}^3)$ with $\mathrm{div}\,\mathbf{u} = 0$. According to [52, Th. 2.5] we can find a unique vector potential $\mathbf{\Psi} \in \mathbf{W}^{1,-1}(\mathbb{R}^3)$ such that $\mathbf{curl}\,\mathbf{\Psi} = \mathbf{u}$, $\mathrm{div}\,\mathbf{\Psi} = 0$ and

$$\|\mathbf{\Psi}\|_{\mathbf{W}^{1,-1}(\mathbb{R}^3)} \le C\,\|\mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^3)}\,. \tag{3.25}$$

By definition, $\mathbf{curl}\,\mathbf{\Psi} = \mathbf{0}$ in Δ, and from (A.23) we obtain the orthogonal decomposition

$$\mathbf{\Psi}_{|\Delta} = \mathbf{grad}\,\phi + \boldsymbol{\varrho}, \quad \phi \in H^1(\Delta)/\mathbb{C},\ \boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}(\Delta)\,, \tag{3.26}$$

where $\boldsymbol{\mathcal{H}}(\Delta)$ is the finite dimensional space of Neumann harmonic vector-fields introduced in (A.19), taking $\boldsymbol{\varepsilon}$ as the identity matrix. Now we solve the exterior problem

$$\mathrm{div}\,(\mathbf{grad}\,\lambda) = 0 \quad \text{in } \Delta^c, \quad \lambda = \phi \quad \text{on } \Sigma\,,$$

in the Beppo-Levi space $W^{1,-1}(\Delta^c)$ and observe that

$$\|\lambda\|_{W^{1,-1}(\Delta^c)} \le C\,\left\|\phi_{|\Sigma}\right\|_{1/2,\Sigma} \le C\,\|\phi\|_{H^1(\Delta)} \le C\,\|\mathbf{\Psi}\|_{\mathbf{H}^1(\Delta)} \le C\,\|\mathbf{curl}\,\mathbf{w}\|_{\mathbf{L}^2(\mathbb{R}^3)}\,. \tag{3.27}$$

Now, defining $\boldsymbol{\zeta} := \mathbf{\Psi} - \mathbf{grad}\,\lambda - \mathbf{w}$ we have

$$\mathbf{curl}\,\boldsymbol{\zeta} = \mathbf{0} \quad \text{in } \Delta^c, \quad \mathrm{div}\,\boldsymbol{\zeta} = 0 \quad \text{in } \Delta^c\,. \tag{3.28}$$

Moreover, the tangential components of $\boldsymbol{\zeta}$ on Σ agree with those of $\boldsymbol{\varrho} \in \boldsymbol{\mathcal{H}}(\Delta)$. Along with (3.28) this implies that $\boldsymbol{\zeta}$ belongs to the following space of harmonic vector fields in Δ^c

$$\boldsymbol{\mathcal{N}}(\Delta^c) := \left\{\mathbf{G} \in \mathbf{L}^2(\Delta^c) : \mathbf{curl}\,\mathbf{G} = \mathbf{0},\ \mathrm{div}\,\mathbf{G} = 0,\ \mathbf{G}\times\mathbf{n} = \boldsymbol{\mu}\times\mathbf{n} \text{ on } \Sigma,\ \text{for some } \boldsymbol{\mu} \in \boldsymbol{\mathcal{H}}(\Delta)\right\}\,.$$

This space can be easily seen to be finite dimensional, because the space of Dirichlet harmonic fields in Δ^c, already introduced in (A.24) and defined as

$$\boldsymbol{\mathcal{D}}(\Delta^c) := \left\{\mathbf{G} \in \mathbf{L}^2(\Delta^c) :\ \mathbf{curl}\,\mathbf{G} = \mathbf{0},\ \mathrm{div}\,\mathbf{G} = 0 \text{ in } \Delta^c \text{ and } \mathbf{G}\times\mathbf{n} = \mathbf{0} \text{ on } \Sigma\right\}\,,$$

and the space of Neumann harmonic fields in Δ, $\boldsymbol{\mathcal{H}}(\Delta)$, are also finite dimensional.

Summing up, we get in Δ^c

$$\mathbf{G} = \widetilde{\mathbf{G}} + \mathbf{w} = \mathbf{q} - \boldsymbol{\zeta}, \quad \mathbf{q} := \widetilde{\mathbf{G}} + \mathbf{\Psi} - \mathbf{grad}\,\lambda\,,$$

and using the estimates (3.25) and (3.27), the construction of the extension $\widetilde{\mathbf{G}}$ and the continuity of the tangential trace operator, stated in Section A.1.1, we get

$$\begin{aligned}\left\|\frac{\mathbf{q}(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}}\right\|_{\mathbf{L}^2(\Delta^c)} &\le C\left(\|\mathbf{\Psi}\|_{\mathbf{W}^{1,-1}(\Delta^c)} + \|\widetilde{\mathbf{G}}\|_{\mathbf{L}^2(\Delta^c)} + \|\lambda\|_{W^{1,-1}(\Delta^c)}\right)\\ \le C\left(\|\mathbf{curl}\,\mathbf{G}\|_{\mathbf{L}^2(\Delta^c)} + \|\widetilde{\mathbf{G}}\|_{\mathbf{H}(\mathbf{curl}\,;\Delta^c)}\right) &\le C\left(\|\mathbf{curl}\,\mathbf{G}\|_{\mathbf{L}^2(\Delta^c)} + \|\mathbf{G}\|_{\mathbf{H}(\mathbf{curl}\,;\Delta)}\right)\,.\end{aligned}$$

We conclude that, for a constant $c > 0$ depending on the physical properties μ, σ and on the conducting subdomain Δ, it holds

$$|a(\mathbf{G},\mathbf{G})| \geq c \left(\left\| \frac{(\mathbf{G}+\boldsymbol{\zeta})(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}} \right\|^2_{\mathbf{L}^2(\Delta^c)} + \|\mathbf{G}\|^2_{\mathbf{L}^2(\Delta)} + \|\mathbf{curl}\,\mathbf{G}\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \right) .$$

Recalling that the norm on $\widetilde{\boldsymbol{\mathcal{Y}}}$ agrees with the norm on $\boldsymbol{\mathcal{X}}$ and that $\boldsymbol{\zeta}$ belongs to a finite dimensional space, the bilinear form $a(\cdot,\cdot)$ is seen to be $\widetilde{\boldsymbol{\mathcal{Y}}}$-elliptic modulo a compact perturbation, or coercive in the sense of [71, Th. 2.34].

According to Theorem 2.34 in [71], existence of solutions of the weak problem follows from uniqueness. In order to prove the uniqueness we are going to see that if the right hand side of Problem **PV** is equal to zero, then the solution is $\mathbf{A} = \mathbf{0}$. Indeed, if $\mathbf{V} = \mathbf{0}$, by taking $\mathbf{G} = \mathbf{A}$ as the test function in (3.19) we get $a(\mathbf{A},\mathbf{A}) = 0$, hence $\mathbf{curl}\,\mathbf{A} = \mathbf{0}$ in $\mathbb{R}^3$ and $\mathbf{A}_{|\Delta} = \mathbf{0}$. The latter yields $\mathbf{A}\times\mathbf{n} = \mathbf{0}$ on Σ. Moreover, since $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$, we also have $\operatorname{div}\mathbf{A} = 0$ in Δ^c, and as a consequence $\mathbf{A} \in \boldsymbol{\mathcal{D}}(\Delta^c)$.

Finally, from the definition of $\boldsymbol{\mathcal{Y}}$ we also know that

$$\int_{\Sigma_j} \mathbf{A}\cdot\mathbf{n} = 0, \quad j = 0,\dots,m\,,$$

and as $\operatorname{div}\mathbf{A} = 0$ in Δ^c, the result presented in (A.27) allows us to affirm that the vector field $\mathbf{A}$ is orthogonal to the space $\boldsymbol{\mathcal{D}}(\Delta^c)$. Since $\mathbf{A} \in \boldsymbol{\mathcal{D}}(\Delta^c)$, we conclude that $\mathbf{A}_{|\Delta^c} = \mathbf{0}$. □

The next Proposition is a straightforward adaptation to unbounded domains of results included in Section 2 of [8]:

Proposition 3.2. *Let* $\mathbf{A}$ *be the unique solution of problem* **PV**. *Then equation (3.19) also holds for any* $\mathbf{G} \in \boldsymbol{\mathcal{X}}$.

Theorem 3.3. *Given* $\mathbf{V} = (V_1,\dots,V_m) \in \mathbb{C}^m$, *let* $\mathbf{A}$ *be the corresponding solution to Problem* **PV**. *Let us define* $\mathbf{B} := \mathbf{curl}\,\mathbf{A}$, $\mathbf{H} := \mu^{-1}\mathbf{B}$, $\mathbf{E} := -\mathrm{i}\omega\mathbf{A} - \sum_{k=1}^m V_k\boldsymbol{\varrho}_k$ *in* Δ, $\mathbf{J}_{|\Delta} := \sigma\mathbf{E}$ *and* $\mathbf{J}_{|\Delta^c} := \mathbf{0}$. *Then the following equalities hold true:*

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J} \quad \text{in } \mathbb{R}^3, \tag{3.29}$$

$$\mathrm{i}\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \Delta, \tag{3.30}$$

$$\operatorname{div}\mathbf{B} = 0 \quad \text{in } \mathbb{R}^3, \tag{3.31}$$

$$\mathbf{J}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma, \tag{3.32}$$

and the gauge conditions for $\mathbf{A}$

$$\operatorname{div}(\sigma\mathbf{A}) = 0 \quad \text{in } \Delta, \qquad \sigma\mathbf{A}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma\,, \tag{3.33}$$

are also satisfied.

Proof. We notice that, from the previous Proposition we are allowed to take as test function $\mathbf{G}$ in (3.19) any smooth function with compact support in $\mathbb{R}^3$. By doing so, we obtain,

$$\mathrm{i}\omega\sigma\mathbf{A} + \mathbf{curl}\,(\frac{1}{\mu}\mathbf{curl}\,\mathbf{A}) + \sigma\sum_{k=1}^{m} V_k\boldsymbol{\varrho}_k = \mathbf{0} \quad \text{in } \mathbb{R}^3\,, \tag{3.34}$$

in the sense of distributions. Hence, in particular, $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{A})$ belongs to $\mathbf{L}^2(\mathbb{R}^3)$. Then, if we define the magnetic induction $\mathbf{B}$ in $\mathbb{R}^3$ as

$$\mathbf{B} := \mathbf{curl}\,\mathbf{A},$$

and the electric field by

$$\mathbf{E} := -\mathrm{i}\omega\mathbf{A} - \sum_{k=1}^{m} V_k\boldsymbol{\varrho}_k \quad \text{in } \Delta, \tag{3.35}$$

taking the curl operator in (3.35) we obtain that (3.30) is satisfied, because $\mathbf{curl}\,\boldsymbol{\varrho}_k = \mathbf{0}$. Moreover, from the definition of $\mathbf{B}$ it is clear that (3.31) holds.

Then, if we define the magnetic field as $\mathbf{H} := \mu^{-1}\mathbf{B}$ and the current density as $\mathbf{J}_{|\Delta} := \sigma\mathbf{E}$ and $\mathbf{J}_{|\Delta^c} := \mathbf{0}$, from equations (3.34) and (3.35) we clearly obtain (3.29).

Moreover, since $\mathbf{curl}\,\mathbf{H} = \mathbf{J}$ in $\mathbf{L}^2(\mathbb{R}^3)$, we know that $\operatorname{div}\mathbf{J} = 0$ in $\mathbb{R}^3$. As a consequence $\mathbf{J} \in \mathbf{H}(\operatorname{div};\mathbb{R}^3)$ and the normal trace $\mathbf{J}\cdot\mathbf{n}$ is well defined on the interface Σ. Since $\mathbf{J}$ vanishes in Δ^c we have $\mathbf{J}\cdot\mathbf{n} = 0$ on Σ so equation (3.32) is also satisfied.

The first gauge condition for $\mathbf{A}$ in the conductors is easily obtained taking the divergence operator in equation (3.34) and reminding that $\operatorname{div}(\sigma\boldsymbol{\varrho}_k) = 0$. The second gauge condition is a consequence of equation (3.35), and the fact that $\sigma\boldsymbol{\varrho}_k\cdot\mathbf{n} = 0$ and $\sigma\mathbf{E}\cdot\mathbf{n} = \mathbf{J}\cdot\mathbf{n} = 0$ on the interface Σ. Finally, the gauge conditions in the insulator are trivially satisfied as $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$. □

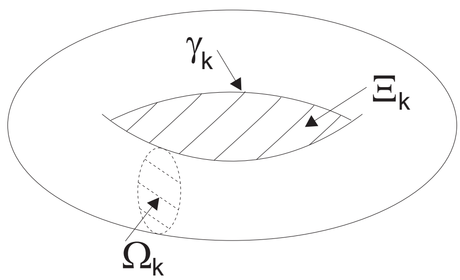

Figure 3.3: Cutting surfaces and loops for a torus.

Remark 3.3. *If we define curl-free extensions of the functions $\boldsymbol{\varrho}_k$ to Δ^c, denoted by $\widetilde{\boldsymbol{\varrho}}_k$, and set $\widetilde{\mathbf{E}} := -\mathrm{i}\omega\mathbf{A} - \sum_{k=1}^{m} V_k\widetilde{\boldsymbol{\varrho}}_k$, then the Faraday's law also holds in this set. However, one can easily show that there is no such extension belonging to the space $\boldsymbol{\mathcal{X}}$ (see [59]). We notice that the radial sections Ω_k play the role of the surfaces Σ_k in Section A.2, and that the non-bounding cycles γ_k introduced in that section can be chosen as $\gamma_k = \partial\Xi_k$, where Ξ_k is an orientable two-dimensional* *surface in* Δ^c *and* $\int_{\gamma_k} \boldsymbol{\varrho}_k d\gamma = 1$ *(see Figure 3.3). Hence using equations (3.2), (3.12) and the definition of functions* $\boldsymbol{\varrho}_k$ *we get*

$$0 = \int_{\Xi_k} \mathbf{curl}\,(\widetilde{\mathbf{E}} + \mathrm{i}\omega\mathbf{A}) = \int_{\gamma_k} (\widetilde{\mathbf{E}} + \mathrm{i}\omega\mathbf{A}) d\gamma = -\int_{\gamma_k} V_k \widetilde{\boldsymbol{\varrho}}_k d\gamma = -V_k\,.$$

The conclusion is that one cannot define $\widetilde{\mathbf{E}}$ *in* $\boldsymbol{\mathcal{X}}$ *so as to satisfy the eddy current model in the whole space together with the conditions prescribing non-null intensities in the rings* Δ_k.

Remark 3.4. *Under the modified Ohm's law approach, the electric field* $\mathbf{E}$ *and the current density* $\mathbf{J}$ *in Theorem 3.3 have to be redefined by*

$$\mathbf{E} = -\mathrm{i}\omega\mathbf{A} \quad \text{in } \mathbb{R}^3, \tag{3.36}$$

$$\mathbf{J} = \sigma\mathbf{E} - \sigma\sum_{k=1}^{m} V_k \boldsymbol{\varrho}_k \quad \text{in } \Delta, \tag{3.37}$$

and (3.30) holds in $\mathbb{R}^3$.

3.2.2 Imposing the current intensities in a weak sense.

We recall that we are interested in finding a solution of the eddy current problem satisfying the conditions for the intensities given in (3.9). To attain this goal, we start by writing these conditions in a weak sense.

Firstly, we remind the definition of functions $\boldsymbol{\varrho}_k := \widetilde{\mathbf{grad}}\,\eta_k$, where η_k is the solution of problem (A.20) in $\Delta_k \setminus \Omega_k$. Since the current density $\mathbf{J} = \sigma\mathbf{E}$ satisfies $\operatorname{div}\mathbf{J} = 0$ in Δ and $\mathbf{J}\cdot\mathbf{n} = 0$ on Σ, we have

$$\int_{\Delta_k} \mathbf{J}\cdot\boldsymbol{\varrho}_k = \int_{\Delta_k\setminus\Omega_k} \mathbf{J}\cdot\widetilde{\mathbf{grad}}\,\eta_k = -\int_{\Delta_k\setminus\Omega_k} \operatorname{div}\mathbf{J}\,\eta_k + \int_{\Sigma_k} \mathbf{J}\cdot\mathbf{n}\,\eta_k + \int_{\Omega_k} [\eta_k]\,\mathbf{J}\cdot\boldsymbol{\nu}$$
$$= \int_{\Omega_k} \mathbf{J}\cdot\boldsymbol{\nu} = I_k, \tag{3.38}$$

for $k = 1, \dots, m$. Thus, we can impose the current intensities as follows:

$$\sum_{k=1}^{m} \bar{W}_k \int_{\Delta_k} \sigma\mathbf{E}\cdot\boldsymbol{\varrho}_k = \sum_{k=1}^{m} I_k \bar{W}_k, \quad \forall\,\mathbf{W} = (W_1, \dots, W_m) \in \mathbb{C}^m,$$

and taking into account (3.35), we obtain the following weak form of constraint (3.9) which is well defined for any vector function $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$:

$$-\sum_{k=1}^{m} \bar{W}_k \int_{\Delta_k} \mathrm{i}\omega\sigma\boldsymbol{\varrho}_k\cdot\mathbf{A} - \sum_{k=1}^{m} \bar{W}_k \int_{\Delta_k} \sigma V_k |\boldsymbol{\varrho}_k|^2 = \sum_{k=1}^{m} I_k \bar{W}_k \qquad \forall\,\mathbf{W} \in \mathbb{C}^m. \tag{3.39}$$

Therefore, given the vector field of intensities $\mathbf{I} = (I_1, \dots, I_m)$, we are led to solve the following mixed problem:

Problem MPI.- Given $\mathbf{I} = (I_1, \ldots, I_m) \in \mathbb{C}^m$, find $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$ and $\mathbf{V} \in \mathbb{C}^m$, such that:

$$\mathrm{i}\omega \int_{\mathbb{R}^3} \sigma \mathbf{A} \cdot \bar{\mathbf{G}} + \int_{\mathbb{R}^3} \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \bar{\mathbf{G}} + \sum_{k=1}^{m} V_k \int_{\Delta_k} \sigma \boldsymbol{\varrho}_k \cdot \bar{\mathbf{G}} = 0 \quad \forall\, \mathbf{G} \in \boldsymbol{\mathcal{Y}},$$

$$\sum_{k=1}^{m} \bar{W}_k \int_{\Delta_k} \mathrm{i}\omega\sigma \boldsymbol{\varrho}_k \cdot \mathbf{A} + \sum_{k=1}^{m} \bar{W}_k \int_{\Delta_k} \sigma V_k |\boldsymbol{\varrho}_k|^2 = -\sum_{k=1}^{m} I_k \bar{W}_k \quad \forall\, \mathbf{W} \in \mathbb{C}^m.$$

From the solution $(\mathbf{A}, \mathbf{V})$, the vector fields $\mathbf{H}$, $\mathbf{E}$ and $\mathbf{B}$ defined as in Theorem 3.3 would be the solution of the full eddy current model (3.1)-(3.9), except that the Faraday's law (3.2) does not hold on the interface Σ separating the conducting and the dielectric domains.

Notice moreover, that the complex vector of potentials, $\mathbf{V}$, can be interpreted as a Lagrange multiplier introduced to impose the current intensities in a weak sense.

Remark 3.5. *Considering the modified Ohm's law approach, equation (3.38) also holds for* $\mathbf{J} = \sigma\mathbf{E} - \sigma \sum_{k=1}^{m} V_k \boldsymbol{\varrho}_k$ *and taking into account that* $\mathbf{E} = -\mathrm{i}\omega\mathbf{A}$*, equation (3.39) is easily obtained. Thus, by modifying the Ohm's law, we also obtain the mixed problem* **MPI**.

3.2.3 Analysis of the mixed problem.

An important feature of the mixed problem **MPI** is that the second equation allows us to obtain the components of vector $\mathbf{V}$ in terms of $\mathbf{I}$ and $\mathbf{A}$. Then, by replacing $\mathbf{V}$ in the first equation we can obtain a weak problem with $\mathbf{A}$ being the only unknown, which can be analyzed in a classical setting. To attain this goal, we start by introducing some notation.

Firstly, we remind the definition of the scalar product in $\mathbf{L}^2(\Delta)$:

$$(\mathbf{F}, \mathbf{G})_\sigma = \int_\Delta \sigma \mathbf{F} \cdot \bar{\mathbf{G}},$$

and its corresponding induced norm denoted by $\|\cdot\|_{0,\sigma}$. We recall that $\{\boldsymbol{\varrho}_k,\ k = 1, \ldots, m\}$ is a basis of the space $\boldsymbol{\mathcal{H}}_\sigma(\Delta)$ which is orthogonal for the scalar product $(\cdot,\cdot)_\sigma$. In what follows we will consider $\{\mathbf{a}_k,\ k = 1, \ldots, m\}$, an orthonormal basis of $\boldsymbol{\mathcal{H}}_\sigma(\Delta)$ equipped with the norm $\|\cdot\|_{0,\sigma}$, given by

$$\mathbf{a}_k = \frac{\boldsymbol{\varrho}_k}{\|\boldsymbol{\varrho}_k\|_{0,\sigma}}.$$

Given a vector field $\mathbf{F} \in \mathbf{L}^2(\Delta)$, let us denote by $\boldsymbol{\mathcal{P}}(\mathbf{F})$ its projection onto $\boldsymbol{\mathcal{H}}_\sigma(\Delta)$ defined by

$$\boldsymbol{\mathcal{P}}(\mathbf{F}) = \sum_{k=1}^{m} (\mathbf{F}, \mathbf{a}_k)_\sigma\, \mathbf{a}_k.$$

By using this notation, from the second equation of the mixed problem **MPI**, the components of $\mathbf{V}$ can be written as:

$$V_k = -\frac{I_k}{\|\boldsymbol{\varrho}_k\|_{0,\sigma}^2} - \mathrm{i}\omega \frac{(\mathbf{A}, \mathbf{a}_k)_\sigma}{\|\boldsymbol{\varrho}_k\|_{0,\sigma}}, \qquad k = 1, \ldots, m. \tag{3.40}$$

By replacing this expression in the first equation of the mixed problem and taking into account that $\sigma = 0$ in Δ^c, we have:

$$\mathrm{i}\omega \int_{\Delta} \sigma \mathbf{A} \cdot \bar{\mathbf{G}} + \int_{\mathbb{R}^3} \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \bar{\mathbf{G}} - \mathrm{i}\omega \sum_{k=1}^{m} (\mathbf{A}, \mathbf{a}_k)_\sigma \int_{\Delta_k} \sigma \mathbf{a}_k \cdot \bar{\mathbf{G}}$$
$$= \sum_{k=1}^{m} \frac{I_k}{\|\boldsymbol{\varrho}_k\|_{0,\sigma}} \int_{\Delta_k} \sigma \mathbf{a}_k \cdot \bar{\mathbf{G}}, \qquad \forall\, \mathbf{G} \in \boldsymbol{\mathcal{Y}}.$$

Thus, the mixed problem **MPI** is equivalent to the following one:

Problem PI.- Given $\mathbf{I} = (I_1, \ldots, I_m) \in \mathbb{C}^m$, find $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$ satisfying:

$$\mathrm{i}\omega \int_{\Delta_0} \sigma \mathbf{A} \cdot \bar{\mathbf{G}} + \int_{\mathbb{R}^3} \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \bar{\mathbf{G}} + \mathrm{i}\omega \sum_{k=1}^{m} \int_{\Delta_k} (\mathbf{A} - \boldsymbol{\mathcal{P}}(\mathbf{A})) \cdot \sigma \bar{\mathbf{G}}$$
$$= \sum_{k=1}^{m} \frac{I_k}{\|\boldsymbol{\varrho}_k\|_{0,\sigma}} \int_{\Delta_k} \sigma \mathbf{a}_k \cdot \bar{\mathbf{G}}, \qquad \forall\, \mathbf{G} \in \boldsymbol{\mathcal{Y}}. \tag{3.41}$$

Theorem 3.4. *Problem* **PI** *has a unique solution.*

Proof. We follow the same technique already used in the proof of Theorem 3.1. Firstly, we introduce the sesquilinear form

$$a(\mathbf{A}, \mathbf{G}) = \mathrm{i}\omega \sum_{k=0}^{m} \int_{\Delta_k} (\mathbf{A} - \boldsymbol{\mathcal{P}}(\mathbf{A})) \cdot \sigma \bar{\mathbf{G}} + \int_{\mathbb{R}^3} \frac{1}{\mu} \mathbf{curl}\, \mathbf{A} \cdot \mathbf{curl}\, \bar{\mathbf{G}},$$

taking into account that $\boldsymbol{\mathcal{P}}(\mathbf{A}) = \mathbf{0}$ in Δ_0.

Since $\boldsymbol{\mathcal{P}}(\mathbf{A})$ is the projection of $\mathbf{A}$ onto $\boldsymbol{\mathcal{H}}_\sigma(\Delta)$, from the $\sigma-$orthogonal decomposition of $\mathbf{L}^2(\Delta)$ we know that $(\mathbf{A} - \boldsymbol{\mathcal{P}}(\mathbf{A}), \boldsymbol{\mathcal{P}}(\mathbf{G}))_\sigma = 0$ for all $\mathbf{G} \in \boldsymbol{\mathcal{Y}}$. Using this, and reasoning as in the proof of Theorem 3.1, we obtain

$$|a(\mathbf{A}, \mathbf{A})| + \frac{\omega}{\sqrt{2}} \|\boldsymbol{\mathcal{P}}(\mathbf{A})\|_{0,\sigma}^2 \geq C \left(\|\mathbf{A}\|_{0,\sigma}^2 + \|\mathbf{curl}\, \mathbf{A}\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \left\| \frac{(\mathbf{A} + \boldsymbol{\zeta})(\mathbf{x})}{\sqrt{1 + |\mathbf{x}|^2}} \right\|_{\mathbf{L}^2(\Delta^c)}^2 \right) \tag{3.42}$$

where $\boldsymbol{\zeta}$ belongs to a finite dimensional space of harmonic fields in Δ^c.

Taking into account that $\boldsymbol{\mathcal{P}}(\mathbf{A})$ and $\boldsymbol{\zeta}$ belong to finite dimensional spaces, we deduce that $a(\cdot, \cdot)$ is $\boldsymbol{\mathcal{Y}}$-coercive, *i.e.*, $\boldsymbol{\mathcal{Y}}$-elliptic modulo a compact perturbation (see again [58]). Then, existence of solution of the weak problem follows from uniqueness.

In order to prove the uniqueness the reasoning is similar to that of Theorem 3.1. Let us suppose that $\mathbf{I} = \mathbf{0}$, and take $\mathbf{G} = \mathbf{A}$ as the test function in (3.41). From the definition of $a(\cdot, \cdot)$ we deduce that $\mathbf{curl}\, \mathbf{A} = \mathbf{0}$ in $\mathbb{R}^3$ and also that $\mathbf{A}_{|\Delta} = \boldsymbol{\mathcal{P}}(\mathbf{A})$, hence $\mathbf{A} \in \boldsymbol{\mathcal{H}}_\sigma(\Delta)$. The former yields,

$$\mathbf{A} = -\mathbf{grad}\, \phi \quad \text{in } \mathbb{R}^3,$$

with $\phi \in W^{1,-1}(\mathbb{R}^3)$, and in particular

$$\mathbf{A}_{|\Delta} = -\mathbf{grad}\,\phi, \qquad \text{with } \phi \in H^1(\Delta).$$

But then $\mathbf{A}_{|\Delta} = \mathbf{0}$ because $\mathbf{A}$ belongs to $\boldsymbol{\mathcal{H}}_\sigma(\Delta)$ which is $\mathbf{L}^2(\Delta;\Omega)$-orthogonal to the gradients of functions in $H^1(\Delta)$ as stated in Section A.2. As a consequence $\mathbf{A}\times\mathbf{n} = \mathbf{0}$ on Σ. Moreover, since $\mathbf{A} \in \boldsymbol{\mathcal{Y}}$ we get $\mathbf{A} \in \boldsymbol{\mathcal{D}}(\Delta^c)$, the space of Dirichlet harmonic fields in Δ^c.

Finally, from the definition of $\boldsymbol{\mathcal{Y}}$ we know that $\operatorname{div}\mathbf{A} = 0$ in Δ^c, and also

$$\int_{\Sigma_j} \mathbf{A}\cdot\mathbf{n} = 0, \quad j = 0,\ldots,m\,.$$

The orthogonality result presented in (A.27) allows us to affirm that the vector field $\mathbf{A}$ is orthogonal to the space $\boldsymbol{\mathcal{D}}(\Delta^c)$. Then, as $\mathbf{A} \in \boldsymbol{\mathcal{D}}(\Delta^c)$, we conclude that $\mathbf{A}_{|\Delta^c} = \mathbf{0}$. □

3.2.4 An axisymmetric BEM/FEM formulation of problems PV and MPI.

In the previous section we have used the equivalence between problems **MPI** and **PI** to analyze the mixed problem. However, for the numerical solution it is advisable to discretize problem **MPI** since the term involving $\boldsymbol{\mathcal{P}}(\mathbf{A})$ in problem **PI** leads to a full matrix.

In order to solve the problems **PV** and **MPI** by using a hybrid boundary elements/finite elements method (in the sequel BEM/FEM), we are going to write the equations of these problems in another form involving only the values of the magnetic vector potential $\mathbf{A}$ in Δ and on its boundary Σ. To attain this goal we first notice that the field $\mu^{-1}\mathbf{curl}\,\mathbf{A}$, which is the intensity of the magnetic field, belongs to $\boldsymbol{\mathcal{X}}$, and then its tangential trace $(\mu^{-1}\mathbf{curl}\,\mathbf{A})\times\mathbf{n}$ is continuous across Σ. Besides

$$\mathbf{curl}\left(\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\right) = \mathbf{curl}\,\mathbf{H} = \mathbf{0} \quad \text{in } \Delta^c, \tag{3.43}$$

where μ_0 denotes the vacuum magnetic permeability. Then, by using a Green's formula in Δ^c, we have,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{\mu}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}} = \int_\Delta \frac{1}{\mu}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}} + \int_{\Delta^c} \frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}} \\
&= \int_\Delta \frac{1}{\mu}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}} + \int_{\Delta^c} \mathbf{curl}\left(\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\right)\cdot\bar{\mathbf{G}} \\
&- \int_\Sigma \left(\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\right)\times\mathbf{n}\cdot\bar{\mathbf{G}} = \int_\Delta \frac{1}{\mu}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}} - \int_\Sigma \left(\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\right)\times\mathbf{n}\cdot\bar{\mathbf{G}} \quad \forall\mathbf{G}\in\boldsymbol{\mathcal{Y}}.
\end{aligned}$$

Thus, the equation of problem **PV** can be formally written as

$$\begin{aligned}
&\mathrm{i}\omega\int_\Delta \sigma\mathbf{A}\cdot\bar{\mathbf{G}} + \int_\Delta \frac{1}{\mu}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}} - \int_\Sigma \left(\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\right)\times\mathbf{n}\cdot\bar{\mathbf{G}} \\
&= -\sum_{k=1}^m V_k \int_{\Delta_k} \sigma\boldsymbol{\varrho}_k\cdot\bar{\mathbf{G}} = 0 \quad \forall\mathbf{G}\in\boldsymbol{\mathcal{Y}},
\end{aligned} \tag{3.44}$$

and analogously, the first equation of problem **MPI** can be formally written as

$$\begin{aligned}&\mathrm{i}\omega\int_{\Delta}\sigma\mathbf{A}\cdot\bar{\mathbf{G}}+\int_{\Delta}\frac{1}{\mu}\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\bar{\mathbf{G}}-\int_{\Sigma}\left(\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}\right)\times\mathbf{n}\cdot\bar{\mathbf{G}}\\&+\sum_{k=1}^{m}V_k\int_{\Delta_k}\sigma\boldsymbol{\varrho}_k\cdot\bar{\mathbf{G}}=0\quad\forall\mathbf{G}\in\boldsymbol{\mathcal{Y}}.\end{aligned}\tag{3.45}$$

We notice that the value of $\mu_0^{-1}(\mathbf{curl}\,\mathbf{A})\times\mathbf{n}$ on Σ can be determined by solving an exterior problem in Δ^c. In [58] the author analyzes a BEM/FEM eddy current formulation in terms of the electric field involving the same boundary term. However, in this work we are more interested in obtaining a numerical simulation of the induction furnace in a reasonable computational time, and therefore we will focus in the analysis of the problem in an axisymmetrical domain. In order to do so, we consider a cylindrical coordinate system (r,θ,z) with the $z-$axis coinciding with the symmetry axis of the device. Hereafter we denote by $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ the local unit vectors in the corresponding coordinate directions. In Appendix C the reader can find the expression for these vectors, along with the expression of several differential operators in cylindrical coordinates.

Now, cylindrical symmetry leads us to consider that no field depends on the angular variable θ. We further assume that the current density field has non-zero component only in the tangential direction $\mathbf{e}_\theta$, namely,

$$\mathbf{J}=J_\theta(r,z)\,\mathbf{e}_\theta.$$

From this condition, equations (3.1) and (3.2), and taking into account the expression of the curl operator in cylindrical coordinates, we know that

$$\mathbf{B}=B_r(r,z)\mathbf{e}_r+B_z(r,z)\mathbf{e}_z.\tag{3.46}$$

We also notice that, since the conductors $\Delta_k,\ k=1,\dots,m$ are tori generated by rotation around the z-axis, the functions $\boldsymbol{\varrho}_k$ have the form

$$\boldsymbol{\varrho}_k=\frac{1}{2\pi r}\mathbf{e}_\theta\quad\text{in }\Delta_k,\ k=1,\dots,m.\tag{3.47}$$

Now, due to the assumed conditions on $\mathbf{J}$, the expressions for $\mathbf{B}$ and $\boldsymbol{\varrho}_k$, and recalling equations (3.2), (3.5) and (3.12), only the θ-component of the magnetic vector potential, hereafter denoted by A_θ, does not vanish, *i.e.*,

$$\mathbf{A}=A_\theta(r,z)\mathbf{e}_\theta.\tag{3.48}$$

Note that this $\mathbf{A}$ automatically satisfies (3.18), because in an axisymmetric geometry $\mathbf{n}=n_r\mathbf{e}_r+n_z\mathbf{e}_z$. Moreover, taking into account the expression for the curl operator in cylindrical coordinates we have

$$\mathbf{curl}\,\mathbf{A}=-\frac{\partial A_\theta}{\partial z}\mathbf{e}_r+\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}\mathbf{e}_z.\tag{3.49}$$

Now let $\mathbf{G} = \psi(r,z)\mathbf{e}_\theta$ be a test function. Thus, taking into account the cylindrical symmetry and the expression for the functions $\boldsymbol{\varrho}_k$, the axisymmetric version of problem **PV** writes formally as follows:

Problem APV.- Given $\mathbf{V} = (V_1, \ldots, V_m) \in \mathbb{C}^m$, find $A_\theta : \Omega \longrightarrow \mathbb{C}$ satisfying,

$$\mathrm{i}\omega \int_\Omega \sigma A_\theta \cdot \bar{\psi} r\, drdz + \int_\Omega \frac{1}{\mu r} \frac{\partial (rA_\theta)}{\partial r} \frac{\partial (r\bar{\psi})}{\partial r}\, drdz + \int_\Omega \frac{1}{\mu} \frac{\partial A_\theta}{\partial z} \frac{\partial \bar{\psi}}{\partial z}\, r\, drdz \\ - \int_\Gamma \frac{1}{\mu_0} \frac{\partial (rA_\theta)}{\partial \mathbf{n}} \bar{\psi}\, d\gamma = -\frac{1}{2\pi} \sum_{k=1}^m V_k \int_{\Omega_k} \sigma \bar{\psi}\, drdz \quad \forall \psi\,, \tag{3.50}$$

and the axisymmetric version of problem **MPI** is formally written:

Problem AMPI.- Given $\mathbf{I} = (I_1, \ldots, I_m) \in \mathbb{C}^m$, find $A_\theta : \Omega \longrightarrow \mathbb{C}$ and $\mathbf{V} \in \mathbb{C}^m$, satisfying,

$$\mathrm{i}\omega \int_\Omega \sigma A_\theta \cdot \bar{\psi} r\, drdz + \int_\Omega \frac{1}{\mu r} \frac{\partial (rA_\theta)}{\partial r} \frac{\partial (r\bar{\psi})}{\partial r}\, drdz + \int_\Omega \frac{1}{\mu} \frac{\partial A_\theta}{\partial z} \frac{\partial \bar{\psi}}{\partial z}\, r\, drdz \\ - \int_\Gamma \frac{1}{\mu_0} \frac{\partial (rA_\theta)}{\partial \mathbf{n}} \bar{\psi}\, d\gamma + \frac{1}{2\pi} \sum_{k=1}^m V_k \int_{\Omega_k} \sigma \bar{\psi}\, drdz = 0 \quad \forall \psi\,, \tag{3.51}$$

$$\frac{1}{2\pi} \sum_{k=1}^m \bar{W}_k \int_{\Omega_k} \sigma A_\theta\, drdz + \frac{1}{4\pi^2 \mathrm{i}\omega} \sum_{k=1}^m \bar{W}_k \int_{\Omega_k} \sigma \frac{V_k}{r}\, drdz = -\frac{1}{2\pi \mathrm{i}\omega} \sum_{k=1}^m I_k \bar{W}_k \quad \forall \mathbf{W} \in \mathbb{C}^m\,. \tag{3.52}$$

In order to apply a hybrid BEM/FEM for the numerical solution, the next step is to transform the integral $\int_\Gamma \mu_0^{-1} \partial(rA_\theta)/\partial\mathbf{n}\, \bar{\psi}\, d\gamma$ by using the single-double layer potentials. To do that we will make use of some properties of A_θ in Δ^c and on the interface Σ.

(i) Since $\mathbf{A} \in \boldsymbol{\mathcal{X}}$ we can ensure the continuity of the tangential component of $\mathbf{A}$ on the interface Σ, *i.e.*,

$$[\mathbf{A} \times \mathbf{n}] = \mathbf{0} \quad \text{on } \Sigma\,, \tag{3.53}$$

where $[\cdot]$ denotes the jump of the function into the brackets and $\mathbf{n}$ is the unit normal vector to Σ pointing to Δ^c. Moreover, in the axisymmetrical case the normal vector has always the form $\mathbf{n} = n_r \mathbf{e}_r + n_z \mathbf{e}_z$, and from (3.48) we can infer the continuity of A_θ.

As it was mentioned before, $\mu^{-1}\mathbf{curl}\,\mathbf{A}$ belongs to $\boldsymbol{\mathcal{X}}$, so its tangential trace is continuous across Σ, namely

$$\left[\frac{1}{\mu}(\mathbf{curl}\,\mathbf{A}) \times \mathbf{n}\right] = \mathbf{0} \quad \text{on } \Sigma\,, \tag{3.54}$$

and from the expression of the curl operator in cylindrical coordinates we get

$$\left[\frac{1}{\mu r} \frac{\partial (rA_\theta)}{\partial \mathbf{n}}\right] = 0 \quad \text{on } \Sigma. \tag{3.55}$$

(ii) At infinity, Biot-Savart law implies the following expression for A_θ

$$A_\theta = O\left(\frac{1}{r^2 + z^2}\right) \quad \text{as } (r^2 + z^2)^{1/2} \to \infty\,, \tag{3.56}$$

and as a consequence we also have

$$A_\theta = O\left(\frac{1}{(r^2+z^2)^{1/2}}\right) \quad \text{as } (r^2+z^2)^{1/2} \to \infty\,. \tag{3.57}$$

(iii) Since μ_0 is a constant, we can deduce from equation (3.43) and the first equation in (3.18) that

$$\Delta \mathbf{A} = \mathbf{0} \quad \text{in } \Delta^c\,. \tag{3.58}$$

Moreover, from equation (3.48) and considering the expression in cylindrical coordinates of the Laplacian operator for a vector field, we know that

$$\Delta A_\theta - \frac{A_\theta}{r^2} = 0 \quad \text{in } \Delta^c\,, \tag{3.59}$$

and from the expression of the Laplacian operator for a scalar field, we obtain

$$\Delta(A_\theta \cos\theta) = 0 \quad \text{in } \Delta^c\,. \tag{3.60}$$

If we define $A_2(\mathbf{x}) = A_\theta(r,z)\cos\theta$, from (3.60) and (3.57) we know that A_2 can be expressed by using a single-double layer representation formula on the interface Σ (see e.g. [79, p.111]),

$$\frac{A_2(\mathbf{x})}{2} = \int_\Sigma G_{\mathbf{n}}(\mathbf{x},\mathbf{y})A_2(\mathbf{y})\,d\Gamma_{\mathbf{y}} - \int_\Sigma G(\mathbf{x},\mathbf{y})\frac{\partial A_2}{\partial \mathbf{n}}(\mathbf{y})\,d\Gamma_{\mathbf{y}} \quad \text{on } \Sigma, \tag{3.61}$$

where G denotes the so-called fundamental solution of Laplace's equation in $\mathbb{R}^3$ (see [67, p.68]) and $G_{\mathbf{n}}$ its normal derivative, namely

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,, \tag{3.62}$$

$$G_{\mathbf{n}}(\mathbf{x},\mathbf{y}) = \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|^3}\,(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_{\mathbf{y}}\,, \tag{3.63}$$

$\mathbf{n}_{\mathbf{y}}$ being the outward unit normal vector to Σ at point $\mathbf{y}$ and $d\Gamma_{\mathbf{y}}$ the differential surface element.

The axisymmetry of the problem leads to take in equation (3.61) the point $\mathbf{x} \in \Gamma$, the radial section of the boundary Σ. However, we note that since the integrals are computed on Σ, we are constrained to take $\mathbf{y} \in \Sigma$. Thus we have $\mathbf{x} = (r,0,z) \in \Gamma$ and $\mathbf{y} = (\tilde r\cos\tilde\theta, \tilde r\sin\tilde\theta, \tilde z) \in \Sigma$ expressed in cylindrical coordinates, therefore $\mathbf{n}_{\mathbf{y}} = (\tilde n_r\cos\tilde\theta, \tilde n_r\sin\tilde\theta, \tilde n_z)$. Straightforward computations yield

$$|\mathbf{x}-\mathbf{y}| = \sqrt{d^2 - 2\,r\,\tilde r(1+\cos\tilde\theta)}, \tag{3.64}$$

$$(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_{\mathbf{y}} = r\,\tilde n_r\cos\tilde\theta - \tilde r\tilde n_r + \tilde n_z(z-\tilde z), \tag{3.65}$$

where $d^2 = (r+\tilde r)^2 + (z-\tilde z)^2$. Taking into account that

$$\frac{\partial A_2}{\partial \mathbf{n}}(\mathbf{y}) = \left(\frac{\partial A_\theta}{\partial r}(\tilde r,\tilde z)\,\tilde n_r + \frac{\partial A_\theta}{\partial z}(\tilde r,\tilde z)\,\tilde n_z\right)\cos\tilde\theta, \tag{3.66}$$

it is convenient to define the new variables

$$A_\theta^{'} = r A_\theta \tag{3.67}$$

$$\lambda(r,z) = \frac{\partial A_\theta}{\partial r} n_r + \frac{\partial A_\theta}{\partial z} n_z, \tag{3.68}$$

$$\lambda^{'}(r,z) = \frac{\partial A_\theta^{'}}{\partial r} n_r + \frac{\partial A_\theta^{'}}{\partial z} n_z. \tag{3.69}$$

Note that

$$\lambda^{'}(r,z) = A_\theta(r,z)\, n_r + r\,\lambda(r,z), \tag{3.70}$$

and from equations (3.53) and (3.55) we deduce that $\lambda^{'}$ is continuous on Γ. Then (3.61) yields the following integral equation relating A_θ and λ:

$$\begin{aligned} A_\theta(r,z) = {} & \frac{1}{2\pi}\int_\Gamma A_\theta(\tilde r,\tilde z)\, r\,\tilde n_r\,\omega_{2,3/2}(r,z,\tilde r,\tilde z)\tilde r\, d\gamma_{(\tilde r,\tilde z)} \\ & + \frac{1}{2\pi}\int_\Gamma A_\theta(\tilde r,\tilde z)\,(\tilde n_z\,(z-\tilde z) - \tilde r\,\tilde n_r)\,\omega_{1,3/2}(r,z,\tilde r,\tilde z)\,\tilde r\, d\gamma_{(\tilde r,\tilde z)} \\ & - \frac{1}{2\pi}\int_\Gamma \lambda(\tilde r,\tilde z)\,\omega_{1,1/2}(r,z,\tilde r,\tilde z)\,\tilde r\, d\gamma_{(\tilde r,\tilde z)}, \end{aligned} \tag{3.71}$$

where we have used the notation

$$\omega_{p,\alpha}(r,z,\tilde r,\tilde z) = \int_0^{2\pi} \frac{\cos^p\tilde\theta}{[d^2 - 2\,r\,\tilde r(1+\cos\tilde\theta)]^\alpha}\, d\tilde\theta, \quad p\in\{1,2\},\ \alpha\in\{1/2,3/2\}. \tag{3.72}$$

Equation (3.71) can be rewritten in a more condensed form as

$$\frac{A_\theta^{'}(r,z)}{r} = (\mathcal{G}_n\,A_\theta^{'})(r,z) - (\mathcal{G}\,\lambda')(r,z), \tag{3.73}$$

where

$$(\mathcal{G}\,\xi)(r,z) = \frac{1}{2\pi}\int_\Gamma \xi(\tilde r,\tilde z)\,\omega_{1,1/2}(r,z,\tilde r,\tilde z)\, d\gamma_{(\tilde r,\tilde z)}\,, \tag{3.74}$$

$$\begin{aligned} (\mathcal{G}_n\,\upsilon)(r,z) = {} & \frac{1}{2\pi}\int_\Gamma \upsilon(\tilde r,\tilde z)\, r\,\tilde n_r\,\omega_{2,3/2}(r,z,\tilde r,\tilde z)\, d\gamma_{(\tilde r,\tilde z)} \\ & + \frac{1}{2\pi}\int_\Gamma \upsilon(\tilde r,\tilde z)\,(\tilde n_z\,(z-\tilde z) - \tilde r\,\tilde n_r)\,\omega_{1,3/2}(r,z,\tilde r,\tilde z)\, d\gamma_{(\tilde r,\tilde z)} \\ & + \frac{1}{2\pi}\int_\Gamma \upsilon(\tilde r,\tilde z)\,\frac{\tilde n_r}{\tilde r}\omega_{1,1/2}(r,z,\tilde r,\tilde z)\, d\gamma_{(\tilde r,\tilde z)}. \end{aligned} \tag{3.75}$$

Multiplying (3.73) by the conjugate of a test function ζ and integrating in Γ we get the weak formulation

$$\int_\Gamma \frac{1}{\mu r} A_\theta^{'}(r,z)\bar\zeta(r,z) d\gamma_{(r,z)} = \int_\Gamma \frac{1}{\mu}(\mathcal{G}_n\,A_\theta^{'})(r,z)\,\bar\zeta(r,z)\, d\gamma_{(r,z)} - \int_\Gamma \frac{1}{\mu}(\mathcal{G}\,\lambda^{'})(r,z)\,\bar\zeta(r,z)\, d\gamma_{(r,z)}. \tag{3.76}$$

We can now write the axisymmetric version of the two electromagnetic problems in the bounded domain Ω:

Problem WEPV.- Given $\mathbf{V} = (V_1, \ldots, V_m) \in \mathbb{C}^m$, find $A_\theta^{'} : \Omega \longrightarrow \mathbb{C}$ and $\lambda^{'} : \Gamma \longrightarrow \mathbb{C}$ such that

$$\begin{aligned}
\mathrm{i}\omega \int_\Omega \frac{\sigma}{r}\; A_\theta^{'}\bar{\psi}'\, drdz + \int_\Omega \frac{1}{\mu r}\mathbf{grad}\, A_\theta^{'} \cdot \mathbf{grad}\, \bar{\psi}'\, drdz - \int_\Gamma \frac{1}{\mu r}\, \lambda^{'}\bar{\psi}'\, d\gamma \\
= -\frac{1}{2\pi}\sum_{k=1}^m V_k \int_{\Omega_k} \frac{\sigma}{r}\bar{\psi}'\, drdz, \quad \forall\, \psi^{'}, \\
\int_\Gamma \frac{1}{\mu r}\; A_\theta^{'}\bar{\zeta}\, d\gamma - \int_\Gamma \frac{1}{\mu}(\mathcal{G}_n A_\theta^{'})\bar{\zeta}\, d\gamma + \int_\Gamma \frac{1}{\mu}(\mathcal{G}\lambda^{'})\bar{\zeta}\, d\gamma \;\; = 0\,, \quad \forall \zeta.
\end{aligned}$$

Problem WEPI.- Given $\mathbf{I} = (I_1, \ldots, I_m) \in \mathbb{C}^m$, find $A_\theta^{'} : \Omega \longrightarrow \mathbb{C}$, $\mathbf{V} \in \mathbb{C}^m$ and $\lambda^{'} : \Gamma \longrightarrow \mathbb{C}$ such that

$$\begin{aligned}
\mathrm{i}\omega \int_\Omega \frac{\sigma}{r}\; A_\theta^{'}\bar{\psi}'\, drdz + \int_\Omega \frac{1}{\mu r}\mathbf{grad}\, A_\theta^{'} \cdot \mathbf{grad}\, \bar{\psi}'\, drdz - \int_\Gamma \frac{1}{\mu r}\, \lambda^{'}\bar{\psi}'\, d\gamma \\
+\frac{1}{2\pi}\sum_{k=1}^m V_k \int_{\Omega_k} \frac{\sigma}{r}\bar{\psi}'\, drdz = 0, \quad \forall\, \psi^{'}, \\
\frac{1}{2\pi}\sum_{k=1}^m \bar{W}_k \int_{\Omega_k} \frac{\sigma}{r} A_\theta^{'}\, drdz + \frac{1}{4\pi^2 \mathrm{i}\omega}\sum_{k=1}^m \bar{W}_k \int_{\Omega_k} \sigma \frac{V_k}{r}\, drdz \\
= -\frac{1}{2\pi \mathrm{i}\omega}\sum_{k=1}^m I_k \bar{W}_k, \quad \forall\, \mathbf{W} \in \mathbb{C}^m, \\
\int_\Gamma \frac{1}{\mu r}\; A_\theta^{'}\bar{\zeta}\, d\gamma - \int_\Gamma \frac{1}{\mu}(\mathcal{G}_n A_\theta^{'})\bar{\zeta}\, d\gamma + \int_\Gamma \frac{1}{\mu}(\mathcal{G}\lambda^{'})\bar{\zeta}\, d\gamma \;\; = 0\,, \quad \forall \zeta.
\end{aligned}$$

For the sake of simplicity in writing we introduce the following notations

$$\begin{aligned}
a(\varphi, \psi) &:= \mathrm{i}\omega \int_\Omega \frac{\sigma}{r}\; \varphi\bar{\psi}\, drdz, + \int_\Omega \frac{1}{\mu r}\mathbf{grad}\, \varphi \cdot \mathbf{grad}\, \bar{\psi}\, drdz, \\
b(\zeta, \psi) &:= -\int_\Gamma \frac{1}{\mu r}\; \zeta\bar{\psi}\, d\gamma, \\
c(\varphi, \zeta) &:= -\int_\Gamma \frac{1}{\mu}(\mathcal{G}_n\varphi)\bar{\zeta}(r, z)\, d\gamma, \\
d(\xi, \zeta) &:= \int_\Gamma \frac{1}{\mu}(\mathcal{G}\xi)\bar{\zeta}(r, z)\, d\gamma, \\
g(\varphi, \mathbf{W}) &:= \frac{1}{2\pi}\sum_{k=1}^m \bar{W}_k \int_{\Omega_k} \frac{\sigma}{r}\varphi\, drdz, \\
p(\mathbf{V}, \mathbf{W}) &:= \frac{1}{4\pi^2\mathrm{i}\omega}\sum_{k=1}^m \bar{W}_k \int_{\Omega_k} \sigma\frac{V_k}{r}\, drdz, \\
l(\mathbf{W}) &:= -\frac{1}{2\pi\mathrm{i}\omega}\sum_{k=1}^m I_k\bar{W}_k,
\end{aligned}$$

so that the two previous problems can be written in the analogous form:

Problem WEPV.- Given $\mathbf{V} = (V_1, \ldots, V_m) \in \mathbb{C}^m$, find $A_\theta^{'} : \Omega \longrightarrow \mathbb{C}$ and $\lambda^{'} : \Gamma \longrightarrow \mathbb{C}$ such that

$$\begin{aligned} a(A_\theta', \psi') + b(\lambda', \psi') &= -\overline{g(\psi', \mathbf{V})} \quad \forall \psi' , \\ -\overline{b(\zeta, A_\theta')} + c(A_\theta', \zeta) + d(\lambda', \zeta) &= 0 \quad \forall \zeta . \end{aligned}$$

Problem WEPI.- Given $\mathbf{I} = (I_1, \ldots, I_m) \in \mathbb{C}^m$, find $A_\theta^{'} : \Omega \longrightarrow \mathbb{C}$, $\mathbf{V} \in \mathbb{C}^m$ and $\lambda^{'} : \Gamma \longrightarrow \mathbb{C}$ such that

$$\begin{aligned} a(A_\theta', \psi') + \overline{g(\psi', \mathbf{V})} + b(\lambda', \psi') &= 0 \quad \forall \psi' , \\ g(A_\theta', \mathbf{W}) + p(\mathbf{V}, \mathbf{W}) &= l(\mathbf{W}) \quad \forall \mathbf{W} \in \mathbb{C}^m , \\ -\overline{b(\zeta, A_\theta')} + c(A_\theta', \zeta) + d(\lambda', \zeta) &= 0 \quad \forall \zeta . \end{aligned}$$

In Chapter 4 we present the way to approximate problems **WEPV** and **WEPI** by a BEM-FEM.

3.3 The thermal model.

The electromagnetic model must be coupled with the heat equation to study the thermal effects of the electromagnetic fields in the workpiece. The mathematical analysis of the thermal equation with phase change is beyond the scope of this work. We will focus on introducing the equations of the model in an axisymmetric setting, paying attention to the terms which couple the thermal problem with the electromagnetic one.

The computational domain for the thermal model in the axisymmetric setting is a radial section of the whole furnace, that we shall denote by $\Omega_T := \Omega_0 \cup \Omega_1 \cup \ldots \Omega_m \cup \Omega_{m+1}$, where Ω_{m+1} denotes the radial section of the dielectric parts of the induction furnace. We notice that this domain includes the radial section of the windings of the coil (see Figure 3.4), which are water-cooled. Since the metal is introduced in solid state and then melted, we shall use the transient heat transfer equation with change of phase, that is written in terms of the enthalpy. Furthermore, since the molten metal is subject to electromagnetic and buoyancy forces, we also need to consider convective heat transfer. Let us suppose that we already know the velocity field $\mathbf{u}$ which is null in the solid part of the workpiece. Then the equation for energy conservation is

$$\left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \mathbf{grad}\, e \right) - \operatorname{div} \left(k(\mathbf{x}, T)\, \mathbf{grad}\, T \right) = \frac{|\mathbf{J}|^2}{2\sigma(\mathbf{x}, T)} \quad \text{in } \Omega_T, \tag{3.77}$$

where e is the enthalpy, T is the temperature and k is the thermal conductivity, depending on temperature as well. Hereafter, we also assume that other material properties, such as the electrical conductivity σ and the magnetic permeability μ may also depend on temperature. The term $|\mathbf{J}|^2/(2\sigma)$ on the right-hand side of (3.77) represents the heat released by the electric current due to the Joule effect. It is obtained by solving the electromagnetic problem introduced in Section 3.2.

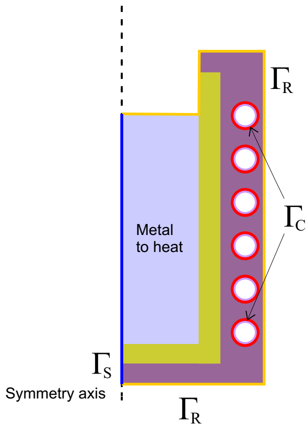

Figure 3.4: Computational domain for the thermal problem.

Remark 3.6. *Notice that the time scale for the variation of the electromagnetic field is much smaller than the one for the variation of temperature. Indeed, the physical parameters used in a typical industrial situation give a time scale for temperature of the order of 1 second, whereas the alternating current is at a frequency of several kHz, which means that the time scale for the electromagnetic fields is of the order of* 10^{-5} *seconds. Thus, we may consider the time harmonic eddy current model to compute the electromagnetic field, and then the heat source is determined by taking the mean value on a cycle (see [39]), namely*

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \boldsymbol{\mathcal{J}}(\mathbf{x},t)\cdot\boldsymbol{\mathcal{E}}(\mathbf{x},t)\,dt\,. \tag{3.78}$$

Since $\boldsymbol{\mathcal{J}}$ *and* $\boldsymbol{\mathcal{E}}$ *are harmonic fields of the form (1.14), we have*

$$\boldsymbol{\mathcal{J}}(\mathbf{x},t)\cdot\boldsymbol{\mathcal{E}}(\mathbf{x},t) = \mathrm{Re}\,[e^{\mathrm{i}\omega t}\mathbf{J}(\mathbf{x})]\cdot\mathrm{Re}\,[e^{\mathrm{i}\omega t}\mathbf{E}(\mathbf{x})] =$$
$$(\cos(\omega t)\,\mathrm{Re}\,[\mathbf{J}(\mathbf{x})] - \sin(\omega t)\,\mathrm{Im}\,[\mathbf{J}(\mathbf{x})])\cdot(\cos(\omega t)\,\mathrm{Re}\,[\mathbf{E}(\mathbf{x})] - \sin(\omega t)\,\mathrm{Im}\,[\mathbf{E}(\mathbf{x})])\,,$$

Since

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t)\,dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t)\,dt = \frac{1}{2}\,,$$

and

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos(\omega t)\sin(\omega t)\,dt = 0\,,$$

by considering Ohm's law (3.5) we get

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \boldsymbol{\mathcal{J}}(\mathbf{x},t)\cdot\boldsymbol{\mathcal{E}}(\mathbf{x},t)\,dt = \frac{\mathrm{Re}\,[\mathbf{J}(\mathbf{x})]\cdot\mathrm{Re}\,[\mathbf{E}(\mathbf{x})] + \mathrm{Im}\,[\mathbf{J}(\mathbf{x})]\cdot\mathrm{Im}\,[\mathbf{E}(\mathbf{x})]}{2} = \frac{|\mathbf{J}(\mathbf{x})|^2}{2\sigma}\,. \tag{3.79}$$

The enthalpy density $e(\mathbf{x},T)$ can be expressed as a function of temperature similar to that used for Stefan problems (see [44]),

$$e(\mathbf{x},t) \in \mathcal{H}(\mathbf{x},T(\mathbf{x},t))\,. \tag{3.80}$$

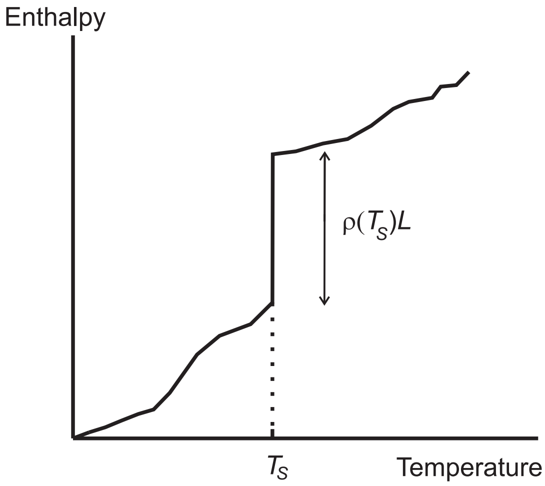

Figure 3.5: Typical graph of the enthalpy.

If the change of state takes place at a constant temperature T_S, and if we denote by L the latent heat, *i.e.*, the heat per unit mass necessary to achieve the change of state at temperature T_S, $\mathcal{H}(\mathbf{x},T)$ is represented as the following multi-valued function:

$$\mathcal{H}(\mathbf{x},T)=\begin{cases}\displaystyle\int_0^T \rho(\mathbf{x},s)c(\mathbf{x},s)\,ds\,, \quad T<T_S(\mathbf{x})\\ \displaystyle\left[\int_0^T \rho(\mathbf{x},s)c(\mathbf{x},s)\,ds\,,\int_0^T \rho(\mathbf{x},s)c(\mathbf{x},s)\,ds+\rho(\mathbf{x},T_S)L(\mathbf{x})\right],\ T=T_S(\mathbf{x})\,,\\ \displaystyle\int_0^T \rho(\mathbf{x},s)c(\mathbf{x},s)\,ds+\rho(\mathbf{x},T_S)L(\mathbf{x})\,,\quad T>T_S(\mathbf{x})\,,\end{cases} \tag{3.81}$$

ρ denoting the mass density and c the specific heat. Both of them are supposed to depend on the temperature T. In Figure 3.5 one can see the graph of a typical function for the enthalpy.

We assume cylindrical symmetry so that T does not depend on the angular coordinate θ. Using the expressions of the divergence and gradient operators in cylindrical coordinates, equation (3.77) becomes

$$\left(\frac{\partial e}{\partial t}+\mathbf{u}\cdot\mathbf{grad}\,e\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(rk(r,z,T)\frac{\partial T}{\partial r}\right)-\frac{\partial}{\partial z}\left(k(r,z,T)\frac{\partial T}{\partial z}\right)=\frac{|J_\theta|^2}{2\sigma(r,z,T)}\quad \text{in } \Omega_T. \tag{3.82}$$

Note that, from equations (3.5), (3.35) and the expression for functions $\boldsymbol{\varrho}_k$, we can infer that

$$J_\theta=-\mathrm{i}\omega\sigma A_\theta \qquad \text{in } \Omega_0\,, \tag{3.83}$$

$$J_\theta=-\mathrm{i}\omega\sigma A_\theta-\frac{V_k}{2\pi r} \qquad \text{in } \Omega_k\,,\quad k=1,\ldots,m\,, \tag{3.84}$$

$$J_\theta=0 \qquad \text{in } \Omega_{m+1}\,, \tag{3.85}$$

and the heat source is easily computed from the solution of problem **WEPI**.

3.3.1 Thermal boundary conditions.

Equation (3.82) must be completed with suitable boundary conditions on Γ_T, the boundary of Ω_T. We shall denote by Γ_{S} the intersection of the symmetry axis with Γ_T, by Γ_{R} the part of the boundary in contact with the air, and by Γ_{C} the internal boundary of the coil, which is in contact with the cooling water (see Figure 3.4). In Γ_{C} we consider a convection condition

$$k(\mathbf{x},T)\frac{\partial T}{\partial \mathbf{n}} = \alpha(T_w - T) \quad \text{on } \Gamma_{\mathrm{C}}, \tag{3.86}$$

α being the coefficient of convective heat transfer and T_w the temperature of the cooling water. In the boundary Γ_R we impose the radiation-convection condition

$$k(\mathbf{x},T)\frac{\partial T}{\partial \mathbf{n}} = \alpha(T_c - T) + \gamma(T_r^4 - T^4) \quad \text{on } \Gamma_{\mathrm{R}}, \tag{3.87}$$

where T_c and T_r are the external convection and radiation temperature, respectively, and γ is the product of emissivity by Stefan-Boltzmann constant (5.669e-8 $\mathrm{W/m^2K^4}$). Finally, on the axis we set the symmetry condition

$$k(\mathbf{x},T)\frac{\partial T}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_{\mathrm{S}}. \tag{3.88}$$

In Chapter 4 we present an appropriate discretization of the thermal model. Moreover, we also describe in that section the algorithms proposed to deal with the non-linearities of the problem.

3.4 The hydrodynamic model.

As mentioned before, in order to achieve a realistic simulation of the overall process occurring in the furnace, convective heat transfer must be taken into account. The hydrodynamic domain is the molten region of the metal, which varies as the metal melts or solidifies, making our hydrodynamic domain time dependent.

Let $\Omega_l(t)$ be the radial section of the molten metal at time t. We assume that the fluid motion is governed by the incompressible Navier-Stokes equations:

$$\rho(\mathbf{x},T)\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{grad}\,\mathbf{u})\mathbf{u}\right) - \operatorname{div}\left(2\eta(\mathbf{x},T)D(\mathbf{u})\right) + \mathbf{grad}\,p = \mathbf{f} \qquad \text{in } \Omega_l(t), \tag{3.89}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega_l(t), \tag{3.90}$$

where ρ denotes the density, $\mathbf{u}$ is the velocity field, η is the dynamic viscosity, p is the pressure and $D(\mathbf{u})$ denotes the symmetric part of $\mathbf{grad}\,\mathbf{u}$, namely

$$D(\mathbf{u}) = \frac{\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t}{2}.$$

We remark that both density and viscosity are material properties which depend on temperature, *i.e.*, $\rho = \rho(\mathbf{x},T)$ and $\eta = \eta(\mathbf{x},T)$. Moreover, the molten region at the time instant t must be computed from the temperature profile, so the solution of the thermal problem is needed to solve the hydrodynamic problem. We will see below how the domain $\Omega_l(t)$ is determined to carry out the numerical simulation.

The right-hand side term $\mathbf{f}$ contains the forces supported by the fluid due to natural convection (buoyancy forces, $\mathbf{f}_g$) and those due to the electromagnetic field (Lorentz forces, $\mathbf{f}_l$):

$$\mathbf{f} = \mathbf{f}_g(\mathbf{x}, T) + \mathbf{f}_l(\mathbf{x}) \,, \tag{3.91}$$

where the buoyancy forces are given by

$$\mathbf{f}_g = \rho(\mathbf{x}, T)\mathbf{g} \,, \tag{3.92}$$

$\mathbf{g}$ being the acceleration of gravity. As it was done in the thermal model, for the electromagnetic forces we actually consider the mean value on a cycle, namely

$$\mathbf{f}_l = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \boldsymbol{\mathcal{J}}(\mathbf{x}, t) \times \boldsymbol{\mathcal{B}}(\mathbf{x}, t)\, dt, \tag{3.93}$$

where $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{B}}$ are the current density and the magnetic induction fields, respectively. Arguing as in Remark 3.6 (see also [89]) it is easily seen that

$$\mathbf{f}_l = \frac{\mathrm{Re}[\mathbf{J}(\mathbf{x})] \times \mathrm{Re}[\mathbf{B}(\mathbf{x})] + \mathrm{Im}[\mathbf{J}(\mathbf{x})] \times \mathrm{Im}[\mathbf{B}(\mathbf{x})]}{2} \,. \tag{3.94}$$

Recalling that $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$, $\mathbf{J} = -\mathrm{i}\omega\sigma A_\theta \mathbf{e}_\theta - \sum_{k=1}^m \sigma V_k \boldsymbol{\varrho}_k$ and the harmonic functions $\boldsymbol{\varrho}_k$ are null in the workpiece, taking into account the expression in cylindrical coordinates of the $\mathbf{curl}$ operator we obtain

$$\mathrm{Re}[\mathbf{J}(\mathbf{x})] \times \mathrm{Re}[\mathbf{B}(\mathbf{x})] = \omega\sigma \mathrm{Im}[A_\theta] \left(\mathrm{Re}\left[\frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r}\right] \mathbf{e}_r + \mathrm{Re}\left[\frac{\partial A_\theta}{\partial z}\right] \mathbf{e}_z \right) ,$$
$$\mathrm{Im}[\mathbf{J}(\mathbf{x})] \times \mathrm{Im}[\mathbf{B}(\mathbf{x})] = -\omega\sigma \mathrm{Re}[A_\theta] \left(\mathrm{Im}\left[\frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r}\right] \mathbf{e}_r + \mathrm{Im}\left[\frac{\partial A_\theta}{\partial z}\right] \mathbf{e}_z \right) .$$

Thus, summing up the two equations, Lorentz force can be expressed in terms of A_θ as

$$\mathbf{f}_l = \frac{\omega\sigma}{2} \left(\mathrm{Im}[A_\theta] \left(\mathrm{Re}\left[\frac{\partial A_\theta}{\partial r}\right] \mathbf{e}_r + \mathrm{Re}\left[\frac{\partial A_\theta}{\partial z}\right] \mathbf{e}_z \right) - \mathrm{Re}[A_\theta] \left(\mathrm{Im}\left[\frac{\partial A_\theta}{\partial r}\right] \mathbf{e}_r + \mathrm{Im}\left[\frac{\partial A_\theta}{\partial z}\right] \mathbf{e}_z \right) \right) ,$$
$$= \frac{\omega\sigma}{2} \Big(\mathrm{Im}[A_\theta]\, \mathrm{Re}[\mathbf{grad}\, A_\theta] - \mathrm{Re}[A_\theta]\, \mathrm{Im}[\mathbf{grad}\, A_\theta] \Big) = \frac{\omega\sigma}{2} \mathrm{Im}[A_\theta \mathbf{grad}\, \overline{A}_\theta] \,.$$

Equations (3.89)-(3.90) must be completed with suitable boundary conditions. Let us denote by $\Gamma_{\mathrm{s}}(t)$ the intersection of the symmetry axis with the boundary of $\Omega_l(t)$, and by $\Gamma_d(t)$ and $\Gamma_n(t)$ the parts of the boundary next to a solid region (solid metal or crucible) or next to the air, respectively (see Figure 3.6). We notice that since the computational domain is time dependent, so are the boundary regions. In the solid parts we consider a non-slip boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_d(t) \,, \tag{3.95}$$

whereas in the symmetry axis and in the upper boundary we demand

$$S\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_n(t) \,, \tag{3.96}$$
$$S\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{s}}(t) \,, \tag{3.97}$$

where S denotes the Cauchy stress tensor, $S = -pI + 2\eta D(\mathbf{u})$, I being the identity tensor.

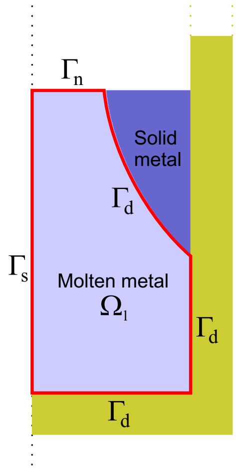

Figure 3.6: Computational domain for the hydrodynamic problem.

Boussinesq approximation.

As the range of temperatures in the molten region is not very large, we can use the Boussinesq approximation to model the fluid motion. Briefly, this approximation consists in considering that the variations of density and viscosity are negligible, except for the mass density in the buoyancy force $\mathbf{f}_g$ appearing in the momentum equation. In this term it is assumed that density is a linear function of temperature, of the form $\rho(T) = \rho_0[1 - \beta_0(T - T_0)]$, where T_0 denotes the reference temperature, ρ_0 is the mass density at the reference temperature and β_0 is the coefficient of thermal expansion, which is assumed to be constant. With this approximation, the buoyancy force is written

$$\mathbf{f}_g = \rho_0[1 - \beta_0(T - T_0)]\mathbf{g}\,. \tag{3.98}$$

If we introduce the reference pressure $p_0(\mathbf{x}) = \rho_0\mathbf{g}\cdot\mathbf{x} + c$, for some constant c, and define the modified pressure $p' = p - p_0$, our equations under the Boussinesq approximation read

$$\rho_0\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{grad}\,\mathbf{u})\mathbf{u}\right) - \operatorname{div}(2\eta_0 D(\mathbf{u})) + \mathbf{grad}\,p' = -\rho_0\beta_0(T - T_0)\mathbf{g} + \mathbf{f}_l \quad \text{in } \Omega_l(t)\,, \tag{3.99}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_l(t)\,, \tag{3.100}$$

where η_0 denotes the dynamic viscosity at the reference temperature.

The heat equation in the molten region is also modified in a similar form, considering that the physical properties in the molten region are constant, to obtain

$$\rho_0 c_0\left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,T\right) - \operatorname{div}(k_0\mathbf{grad}\,T) = \frac{|\mathbf{J}|^2}{2\sigma(T)}, \quad \text{in } \Omega_l(t) \tag{3.101}$$

where c_0 and k_0 denote the specific heat and the thermal conductivity at the reference temperature, respectively. We note that this equation is valid in the molten region of the metal, but in the rest of the domain the thermal equation remains non-linear.

Remark 3.7. *Equation (3.101) is obtained by considering that the enthalpy density in the molten part is given by*

$$e(T) = e_S + \int_{T_S}^{T} \rho_0 c_0 \, ds = e_S + \rho_0 c_0 (T - T_S)\,,$$

with $e_S = \int_0^{T_S} \rho(T)c(T)\,ds + \rho(T_S)L$. *Substituting this expression for the enthalpy in (3.77) one easily gets equation (3.101), which holds only in the molten region. However, even if the heat equation in the molten region can be written in a linear form, it is more convenient to maintain the equation in terms of enthalpy, in order to have the same equation in the whole domain.*

3.4.1 An algebraic turbulence model: Smagorinsky's model.

We recall that the Reynolds number is a dimensionless quantity which gives the ratio of inertial to viscosity forces. It is expressed as

$$Re = \frac{\rho U L}{\eta}, \tag{3.102}$$

where U and L represent a characteristic velocity and a characteristic length (here taken as the inner radius of the crucible), respectively. When this number goes beyond a threshold, the flow becomes turbulent and then it is impossible to model its behaviour using the Navier-Stokes equations due to the big mesh that the computational domain would require. To deal with turbulent flows, all the fields are decomposed into a mean part and an oscillating part which takes into account the small variations due to turbulent flow. By rewriting the Navier-Stokes equations using the decomposed fields and filtering the equations (see [76]) we arrive at the Reynolds-averaged Navier-Stokes equations:

$$\rho_0 \left(\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\mathbf{grad}\,\hat{\mathbf{u}})\hat{\mathbf{u}} \right) - \mathrm{div}\,(2\eta_0 D(\hat{\mathbf{u}})) - \rho_0 \mathrm{div}\,\widehat{\mathbf{u}' \otimes \mathbf{u}'} + \mathbf{grad}\,\hat{p} = \hat{\mathbf{f}} \quad \text{in } \Omega_l(t), \tag{3.103}$$

$$\mathrm{div}\,\hat{\mathbf{u}} = 0 \quad \text{in } \Omega_l(t), \tag{3.104}$$

where $\hat{\mathbf{u}}$ denotes the mean velocity, $\hat{p}$ the mean pressure, $\mathbf{u}'$ the oscillating part of the velocity field and $\otimes$ the tensor product. Hereafter, the symbol $\hat{}$ denotes the mean value of a variable or an expression. The term $R = \rho_0(\widehat{\mathbf{u}' \otimes \mathbf{u}'})$ is called the Reynolds tensor and represents the contribution of the turbulent part to the mean flow.

Analogously, the averaged heat equation is written as

$$\rho_0 c_0 \left(\frac{\partial \hat{T}}{\partial t} + \hat{\mathbf{u}} \cdot \mathbf{grad}\,\hat{T} \right) + \rho_0 c_0 \mathrm{div}\,(\widehat{T'\mathbf{u}'}) - \mathrm{div}\,(k_0 \mathbf{grad}\,\hat{T}) = \frac{\widehat{|\mathbf{J}|^2}}{2\sigma}, \tag{3.105}$$

being $\hat{T}$ the mean temperature and T' its oscillating part. The tensor $\rho_0 c_0 \widehat{T'\mathbf{u}'}$ takes into account the contribution of the turbulent flow to the mean temperature profile.

The Boussinesq assumption consists in taking these two tensors as

$$-\rho_0 \widehat{\mathbf{u}' \otimes \mathbf{u}'} = -\frac{1}{3}\mathrm{tr}(R)I + 2\eta_t D(\hat{\mathbf{u}}), \tag{3.106}$$

$$\rho_0 c_0 \widehat{T'\mathbf{u}'} = -k_t \mathbf{grad}\,\hat{T}, \tag{3.107}$$

where η_t is the turbulent viscosity, k_t is the turbulent thermal conductivity and $\mathrm{tr}(\cdot)$ denotes the trace operator. Using these assumptions we can now rewrite equations (3.103) and (3.105) as

$$\rho_0\left(\frac{\partial\hat{\mathbf{u}}}{\partial t}+(\mathbf{grad}\,\hat{\mathbf{u}})\hat{\mathbf{u}}\right)-\mathrm{div}\,(2\eta_{eff}D(\hat{\mathbf{u}}))+\mathbf{grad}\,\hat{p}^*=\hat{\mathbf{f}}, \tag{3.108}$$

$$\rho_0 c_0\left(\frac{\partial\hat{T}}{\partial t}+\hat{\mathbf{u}}\cdot\mathbf{grad}\,\hat{T}\right)-\mathrm{div}\,(k_{eff}\mathbf{grad}\,\hat{T})=\frac{\widehat{|\mathbf{J}|^2}}{2\sigma}, \tag{3.109}$$

where $p^* = p' - \frac{1}{3}\mathrm{tr}(R)$ and η_{eff} is the effective viscosity, which is given by $\eta_{eff} = \eta_0 + \eta_t$. Analogously, k_{eff} represents the effective thermal conductivity, given by $k_{eff} = k_0 + k_t$. Different models are obtained depending on the way in which the turbulent viscosity η_t and the turbulent conductivity k_t are computed. An efficient and easy to implement model is the one proposed by Smagorinsky (see e.g. [76]), which consists in considering

$$\eta_t = \rho_0\, C\, h^2\, |D(\hat{\mathbf{u}})|, \quad C \cong 0.01, \quad k_t = c_0\frac{\eta_t}{Pr_t} \tag{3.110}$$

where $h(\mathbf{x})$ is the mesh size of the numerical method around point $\mathbf{x}$ and Pr_t is the turbulent Prandtl number which is taken equal to 0.9 (see [76]).

Chapter 4

Model discretization and numerical results.

4.1 Discretization and implementation.

The mathematical model presented in the previous chapter is rather complicated and its numerical discretization and implementation are far from being straightforward due to the presence of several non-linearities, different domains for each problem and the coupling of the three submodels. Moreover, since a real process in the furnace takes place in several hours, the numerical simulation must be done for several hours too, thus the problem becomes very large in time. In this chapter we present the discretization techniques used to approximate the solution of the mathematical model considered in the previous chapter, paying special attention to some issues about the implementation which can help to reduce the computational time.

We notice that in the liquid region of the domain we are always considering equations (3.108) and (3.109). Nevertheless, since most of the techniques presented here are independent of the turbulence model, to simplify the notation we will refer to equations (3.99) and (3.82) instead.

4.1.1 Time discretization.

As it was said in Remark 3.6, the variations of the electromagnetic fields take place in a time scale very small compared to the variations of temperature, which allows us to consider the time harmonic model for the electromagnetic problem and perform a time discretization just of the thermal and hydrodynamic problems. However, since electrical conductivity depends on temperature, the electromagnetic problem must be solved at each time step.

To obtain a suitable time discretization of the problem one must consider the convective terms appearing in the thermal and hydrodynamic models. In order to do that, we first rewrite equations

(3.82) and (3.99) in terms of the material time derivative, obtaining

$$\dot{e} - \frac{1}{r}\frac{\partial}{\partial r}\left(rk(r,z,T)\frac{\partial T}{\partial r}\right) - \frac{\partial}{\partial z}\left(k(r,z,T)\frac{\partial T}{\partial z}\right) = \frac{|J_\theta|^2}{2\sigma(r,z,T)} \quad \text{in } \Omega_T\,, \tag{4.1}$$

$$\dot{\mathbf{u}} - \operatorname{div}(2\eta_0 D(\mathbf{u})) + \mathbf{grad}\, p' = -\rho_0\beta_0(T-T_0)\mathbf{g} + \mathbf{f}_l \quad \text{in } \Omega_l(t)\,, \tag{4.2}$$

where $\dot{e} = \partial e/\partial t + \mathbf{u}\cdot\mathbf{grad}\, e$, and $\dot{\mathbf{u}} = \partial \mathbf{u}/\partial t + (\mathbf{grad}\,\mathbf{u})\mathbf{u}$. To approximate the material time derivative we use the method of characteristics (see, for instance [87]). We will only explain the discretization for the material time derivative of the enthalpy, since the same holds for the material time derivative of the velocity $\mathbf{u}$.

Given a velocity field $\mathbf{u}$ we define the characteristic curve going through point $\mathbf{x}$ at time t as the solution of the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{X}(\mathbf{x},t;\tau) = \mathbf{u}(\mathbf{X}(\mathbf{x},t;\tau),\tau)\,, \\ \mathbf{X}(\mathbf{x},t;t) = \mathbf{x}\,, \end{cases} \tag{4.3}$$

so $\mathbf{X}(\mathbf{x},t;\tau)$ is the trajectory of the material point being at position $\mathbf{x}$ at time t. The material time derivative of e can also be written as

$$\dot{e}(\mathbf{x},t) = \frac{d}{d\tau}[e(\mathbf{X}(\mathbf{x},t;\tau),\tau)]_{|\tau=t}. \tag{4.4}$$

We consider a time interval $[0,t_f]$ and a discretization time step $\Delta t = t_f/N$, to obtain a uniform partition of the interval $\Pi = \{t^n = n\Delta t,\, 0 \le n \le N\}$. Let e^n and $\mathbf{u}^n$ be the approximations of e and $\mathbf{u}$ at time t^n, respectively. We approximate the material time derivative of e at time t^{n+1} by

$$\dot{e}(\mathbf{x},t^{n+1}) \simeq \frac{e^{n+1}(\mathbf{x}) - e^n(\chi^n(\mathbf{x}))}{\Delta t}, \tag{4.5}$$

where $\chi^n(\mathbf{x}) = \mathbf{X}^n(\mathbf{x},t^{n+1};t^n)$ represents the position at time t^n of the material point being at position $\mathbf{x}$ at time t^{n+1}. The previous position $\chi^n(\mathbf{x})$ can be obtained as the solution of the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{X}^n(\mathbf{x},t^{n+1};\tau) = \mathbf{u}^n(\mathbf{X}^n(\mathbf{x},t^{n+1};\tau),\tau)\,, \\ \mathbf{X}^n(\mathbf{x},t^{n+1};t^{n+1}) = \mathbf{x}\,, \end{cases} \tag{4.6}$$

backward in time. We notice that in the solid region, since $\mathbf{u} = \mathbf{0}$, the solution of this Cauchy problem is $\mathbf{X}^n(\mathbf{x},t^{n+1};\tau) = \mathbf{x}$ for any τ, and so equation (4.5) in the solid part is equivalent to the standard Euler discretization without using the method of characteristics. Analogous to (4.5), the material time derivative of the velocity at time t^{n+1} is approximated by

$$\dot{\mathbf{u}}(\mathbf{x},t^{n+1}) \simeq \frac{\mathbf{u}^{n+1}(\mathbf{x}) - \mathbf{u}^n(\chi^n(\mathbf{x}))}{\Delta t}. \tag{4.7}$$

The ordinary differential equation (4.3) must be solved for each node of the mesh with non-null velocity. To compute these solutions we use the same method as the one introduced in

[81], which relies on having a space discretization of the domain, *i.e.*, a mesh. The position $\chi^n(\mathbf{x})$ is approximated by $\chi_h^n(\mathbf{x})$, defined as the extremity of a polygonal curve of vertices $\{\chi_0 = \mathbf{x}, \chi_1, \ldots, \chi_m, \chi_{m+1} = \chi_h^n(\mathbf{x})\}$, computed as follows:

(i) Every χ_i, $i = 1, \ldots, m$, belongs to an edge of the mesh.

(ii) From the known value of χ_i, the position χ_{i+1} is computed as

$$\chi_{i+1} = \chi_i - t_i \mathbf{u}^n(\chi_i), \quad t_i \geq 0\,,$$

with t_i being such that (i) is also accomplished.

(iii) The time of the polygonal curve corresponds to the time step Δt, namely

$$\Delta t = t_0 + t_1 + \ldots + t_m\,.$$

We remark that the problem is solved considering the velocity at the previous time step, namely $\mathbf{u} = \mathbf{u}^n$.

4.1.2 Weak formulation.

Once we have introduced the time discretization, we are going to consider the space discretization. We first recall the weak formulation of the electromagnetic problem introduced as problems **WEPV** and **WEPI** in the previous chapter but now considering the dependence on temperature. Moreover, for the thermal and hydrodynamic problems we introduce the weak formulation of the semi-discretized equations.

Let us denote by T^n and $\mathbf{u}^n$ the temperature and velocity fields at time t^n. We must compute the azimuthal component of the magnetic vector potential A_θ^{n+1}, the temperature T^{n+1} and the velocity $\mathbf{u}^{n+1}$ at time t^{n+1}.

An approximation of A_θ^{n+1} at time t^{n+1} is determined from the solution A'_θ of either Problem **WEPV** or Problem **WEPI**, both defined at the end of Section 3.2.3:

(WEPV) Given $\mathbf{V} = (V_1, \ldots, V_m) \in \mathbb{C}^m$, *find* $A'_\theta : \Omega \longrightarrow \mathbb{C}$ *and* $\lambda' : \Gamma \longrightarrow \mathbb{C}$ *such that*

$$\mathrm{i}\omega \int_\Omega \frac{\sigma(r,z,T^{n+1})}{r} A'_\theta \bar{\psi}'\, drdz + \int_\Omega \frac{1}{\mu(r,z,T^{n+1})r} \mathbf{grad}\, A'_\theta \cdot \mathbf{grad}\, \bar{\psi}'\, drdz$$
$$- \int_\Gamma \frac{1}{\mu(r,z,T^{n+1})r} \lambda' \bar{\psi}'\, d\gamma = -\sum_{k=1}^m \frac{V_k}{2\pi} \int_{\Omega_k} \frac{\sigma(r,z,T^{n+1})}{r} \bar{\psi}'\, drdz,$$
$$\int_\Gamma \frac{1}{\mu(r,z,T^{n+1})r} \bar{\zeta} A'_\theta - \int_\Gamma \frac{1}{\mu(r,z,T^{n+1})} (\mathcal{G}_n A'_\theta) \bar{\zeta}(r,z)\, d\gamma + \int_\Gamma \frac{1}{\mu(r,z,T^{n+1})} (\mathcal{G}\lambda') \bar{\zeta}(r,z)\, d\gamma = 0\,,$$

for all test functions ψ' *and* λ'.

(WEPI) Given $\mathbf{I} = (I_1, \ldots, I_m) \in \mathbb{C}^m$*, find* $A_\theta^{'} : \Omega \longrightarrow \mathbb{C}$*,* $\mathbf{V} \in \mathbb{C}^m$ *and* $\lambda^{'} : \Gamma \longrightarrow \mathbb{C}$ *such that*

$$\mathrm{i}\omega \int_\Omega \frac{\sigma(r,z,T^{n+1})}{r} \, A_\theta^{'} \bar{\psi}' \, drdz + \int_\Omega \frac{1}{\mu(r,z,T^{n+1})r} \mathbf{grad}\, A_\theta^{'} \cdot \mathbf{grad}\, \bar{\psi}' \, drdz$$
$$- \int_\Gamma \frac{1}{\mu(r,z,T^{n+1})r} \, \lambda^{'} \bar{\psi}' \, d\gamma + \sum_{k=1}^m \frac{V_k}{2\pi} \int_{\Omega_k} \frac{\sigma(r,z,T^{n+1})}{r} \bar{\psi}' \, drdz = 0,$$
$$\frac{1}{2\pi} \sum_{k=1}^m \left(\int_{\Omega_k} \frac{\sigma(r,z,T^{n+1})}{r} A_\theta^{'} \, drdz \right) \bar{W}_k + \frac{1}{4\pi^2 \mathrm{i}\omega} \sum_{k=1}^m \left(\int_{\Omega_k} \sigma(r,z,T^{n+1}) \frac{V_k}{r} \, drdz \right) \bar{W}_k = -\frac{1}{2\pi \mathrm{i}\omega} \sum_{k=1}^m I_k \bar{W}_k,$$
$$\int_\Gamma \frac{1}{\mu(r,z,T^{n+1})r} \, \bar{\zeta} A_\theta^{'} \, d\gamma - \int_\Gamma \frac{1}{\mu(r,z,T^{n+1})} (\mathcal{G}_n A_\theta^{'}) \bar{\zeta}(r,z) \, d\gamma + \int_\Gamma \frac{1}{\mu(r,z,T^{n+1})} (\mathcal{G} \lambda^{'}) \bar{\zeta}(r,z) \, d\gamma = 0 \, ,$$

for all test functions ψ'*,* λ' *and for all* $\mathbf{W} \in \mathbb{C}^m$*.*

Then we set $A_\theta^{n+1} = A_\theta'/r$.

Now, let us multiply equation (4.1) discretized in time by a suitable test function and then use a Green's formula. We obtain the following weak formulation of the semi-discretized thermal problem:

(WTP) For each $n = 0, 1, \ldots, N-1$*, find a function* T^{n+1} *such that*

$$\int_{\Omega_T} \frac{1}{\Delta t} e^{n+1} \, Z \, r \, drdz \; + \int_{\Omega_T} k_{eff}(r,z,T^{n+1}) \, \mathbf{grad}\, T^{n+1} \cdot \mathbf{grad}\, Z \, r \, drdz$$
$$= \int_{\Gamma_C} \alpha(T_w - T^{n+1}) Z \, r \, d\Gamma \; + \int_{\Gamma_R} (\alpha(T_c - T^{n+1}) + \gamma(T_r^4 - (T^{n+1})^4)) Z \, r \, d\Gamma$$
$$+ \int_{\Omega_T} \frac{1}{\Delta t} (e^n \circ \chi^n) \, Z \, r \, drdz \; + \int_{\Omega_T} \frac{1}{2\sigma(r,z,T^{n+1})} \, |J^{n+1}|^2 \, Z \, r \, drdz,$$

for all test function Z*.*

Finally, let us consider in equation (4.2) the discretization of the material time derivative $\dot{\mathbf{u}}$ introduced above, multiply this discretized equation and (3.100) by suitable test functions, and integrate in the liquid domain Ω_l. We obtain, after using a Green's formula, the following weak formulation of the semi-discretized hydrodynamic problem:

(WHP) For each $n = 0, 1, \ldots, N-1$*, find functions* $\mathbf{u}^{n+1}$ *and* p^{n+1} *such that* $\mathbf{u}^{n+1} = 0$ *on* Γ_d *and*

$$\frac{1}{\Delta t} \int_{\Omega_l} \rho_0 \mathbf{u}^{n+1} \cdot \mathbf{w} \, r \, drdz + \int_{\Omega_l} \eta_{eff}(r,z,T^{n+1}) \left(\mathbf{grad}\, \mathbf{u}^{n+1} : \mathbf{grad}\, \mathbf{w} \right) \, r \, drdz$$
$$+ \int_{\Omega_l} \eta_{eff}(r,z,T^{n+1}) \left((\mathbf{grad}\, \mathbf{u}^{n+1})^t : \mathbf{grad}\, \mathbf{w} \right) \, r \, drdz - \int_{\Omega_l} p^{n+1} \mathrm{div}\, \mathbf{w} \, r \, drdz =$$
$$-\rho_0 \beta_0 \int_{\Omega_l} (T^{n+1} - T_0) \mathbf{g} \cdot \mathbf{w} \, r \, drdz + \int_{\Omega_l} \frac{\omega\sigma}{2} \mathrm{Im}[A_\theta^{n+1} \mathbf{grad}\, \overline{A}_\theta^{n+1}] \cdot \mathbf{w} \, r \, drdz$$
$$+ \frac{1}{\Delta t} \int_{\Omega_l} \rho_0 (\mathbf{u}^n \circ \chi^n) \cdot \mathbf{w} \, r \, drdz,$$
$$\int_{\Omega_l} \mathrm{div}\, \mathbf{u}^{n+1} \, q \, r \, drdz = 0,$$

for all test functions $\mathbf{w}$ *null on* Γ_d *and* q.

The term $\frac{\omega\sigma}{2}\mathrm{Im}[A_\theta^{n+1}\mathbf{grad}\,\overline{A}_\theta^{n+1}]$ represents Lorentz's force and it has been obtained computing its mean value on a cycle (see Section 3.4).

Since we are assuming that the velocity field and the test function $\mathbf{w}$ are of the form $\mathbf{u}^{n+1} = u_r^{n+1}\mathbf{e}_r + u_z^{n+1}\mathbf{e}_z$ and $\mathbf{w} = w_r\mathbf{e}_r + w_z\mathbf{e}_z$, the equations of problem *(WHP)* can also be written in cylindrical coordinates:

$$
\begin{aligned}
&\frac{1}{\Delta t}\int_{\Omega_l}\rho_0(u_r^{n+1}w_r + u_z^{n+1}w_z)\,r\,drdz \\
&+\int_{\Omega_l}\eta_{eff}(r,z,T^{n+1})\left(\frac{\partial u_r^{n+1}}{\partial r}\frac{\partial w_r}{\partial r} + \frac{\partial u_r^{n+1}}{\partial z}\frac{\partial w_r}{\partial z} + \frac{\partial u_z^{n+1}}{\partial r}\frac{\partial w_z}{\partial r} + \frac{\partial u_z^{n+1}}{\partial z}\frac{\partial w_z}{\partial z} + \frac{1}{r^2}u_r^{n+1}w_r\right) r\,drdz \\
&+\int_{\Omega_l}\eta_{eff}(r,z,T^{n+1})\left(\frac{\partial u_r^{n+1}}{\partial r}\frac{\partial w_r}{\partial r} + \frac{\partial u_r^{n+1}}{\partial z}\frac{\partial w_z}{\partial r} + \frac{\partial u_z^{n+1}}{\partial r}\frac{\partial w_r}{\partial z} + \frac{\partial u_z^{n+1}}{\partial z}\frac{\partial w_z}{\partial z} + \frac{1}{r^2}u_r^{n+1}w_r\right) r\,drdz \\
&-\int_{\Omega_l}p^{n+1}\left(\frac{1}{r}\frac{\partial(rw_r)}{\partial r} + \frac{\partial w_z}{\partial z}\right) r\,drdz = -\rho_0\beta_0\int_{\Omega_l}(T^{n+1}-T_0)(g_r w_r + g_z w_z)\,r\,drdz \\
&+\int_{\Omega_l}\frac{\omega\sigma}{2}\left(\mathrm{Im}\left[A_\theta^{n+1}\frac{\partial\overline{A}_\theta^{n+1}}{\partial r}\right]w_r + \mathrm{Im}\left[A_\theta^{n+1}\frac{\partial\overline{A}_\theta^{n+1}}{\partial z}\right]w_z\right) r\,drdz \\
&+\frac{1}{\Delta t}\int_{\Omega_l}\rho_0[(u_r^n\circ\chi^n)w_r + (u_z^n\circ\chi^n)w_z)]\,r\,drdz, \\
&\int_{\Omega_l}\left(\frac{1}{r}\frac{\partial(ru_r^{n+1})}{\partial r} + \frac{\partial u_z^{n+1}}{\partial z}\right) q\,r\,drdz = 0.
\end{aligned}
$$

We notice that in the previous formulations temperature T^{n+1} appears in the physical properties of the three problems. Moreover, the magnetic potential A_θ^{n+1} is needed to compute the Joule effect in the thermal problem and the Lorentz's force in the hydrodynamic problem. On the contrary the velocity $\mathbf{u}^{n+1}$ does not appear in the electromagnetic problem and hence, in order to compute the term $e^n\circ\chi^n$ with the method of characteristics we use the velocity at time step n, *i.e.*, $\mathbf{u}^n$. Thus, at each time step, the hydrodynamic problem is in fact uncoupled from the two others and it can be solved separately.

4.1.3 Space discretization.

The main difficulty of the spatial discretization relies on the electromagnetic problem, since it is written as an integro-differential equation. For the two other problems we just mention that Problem *(WTP)* has been spatially discretized by a finite element method using triangular Lagrange elements of order one, and Problem *(WHP)* has been discretized by the finite element couple $P1-bubble/P1$, which is also called the *"mini" element.* This couple is known to satisfy the *inf-sup* condition, so that it is stable (see, for instance, [53]).

For the discretization of problems *(WEPV)* and *(WEPI)* we use a hybrid technique, which consists in approximating the field A'_θ by a finite element method, and its normal derivative λ'

by a boundary element method. More precisely, let $\mathcal{T}_h$ be a family of regular triangulations of Ω. We recall that it is formed by the radial sections of the conductor regions. Let us denote by $\mathcal{F}_h$ the induced partitions on the boundary Γ, assumed to be polygonal. Associated with these families of meshes we consider two complex-valued finite element spaces: the space $\mathcal{Y}_h$ consisting of continuous piecewise linear functions defined in Ω and the space $\mathcal{Z}_h$ consisting of piecewise constant functions defined on the boundary Γ. The discrete version of problem *(WEPV)* reads as follows:

Problem PV$_h$.– Find $A'_{\theta h} \in \mathcal{Y}_h$ and $\lambda'_h \in \mathcal{Z}_h$ such that

$$a(A'_{\theta h}, \psi'_h) + b(\lambda'_h, \psi'_h) = -\overline{g(\psi'_h, \mathbf{V}_h)} \quad \forall\, \psi'_h \in \mathcal{Y}_h, \tag{4.8}$$

$$-\overline{b(\zeta_h, A'_{\theta h})} + c(A'_{\theta h}, \zeta_h) + d(\lambda'_h, \zeta_h) = 0 \quad \forall \zeta_h \in \mathcal{Z}_h. \tag{4.9}$$

Moreover, the discrete version of problem *(WEPI)* has the form:

Problem PI$_h$.– Find $A'_{\theta h} \in \mathcal{Y}_h$, $\mathbf{V}_h \in \mathbb{C}^m$ and $\lambda'_h \in \mathcal{Z}_h$ such that

$$a(A'_{\theta h}, \psi'_h) + \overline{g(\psi'_h, \mathbf{V}_h)} + b(\lambda'_h, \psi'_h) = 0 \quad \forall\, \psi'_h \in \mathcal{Y}_h, \tag{4.10}$$

$$g(A'_{\theta h}, \mathbf{W}) + p(\mathbf{V}, \mathbf{W}) = l(\mathbf{W}) \quad \forall\, \mathbf{W} \in \mathbb{C}^m, \tag{4.11}$$

$$-\overline{b(\zeta_h, A'_{\theta h}} + c(A'_{\theta h}, \zeta_h) + d(\lambda'_h, \zeta_h) = 0 \quad \forall \zeta_h \in \mathcal{Z}_h. \tag{4.12}$$

In matrix form, equations (4.8)-(4.9) become

$$\begin{pmatrix} A & B \\ -B^t + C & D \end{pmatrix} \begin{pmatrix} \{A'_{\theta h}\} \\ \{\lambda'_h\} \end{pmatrix} = \begin{pmatrix} \widetilde{G} \\ \{0\} \end{pmatrix}, \tag{4.13}$$

and equations (4.10)-(4.12)

$$\begin{pmatrix} A & G & B \\ G^t & P & 0 \\ -B^t + C & 0 & D \end{pmatrix} \begin{pmatrix} \{A'_{\theta h}\} \\ \mathbf{V}_h \\ \{\lambda'_h\} \end{pmatrix} = \begin{pmatrix} \{0\} \\ L \\ \{0\} \end{pmatrix}, \tag{4.14}$$

where, recalling the notation introduced in Section 3.2.4,

$$
\begin{aligned}
A_{jk} &= a(v_k, v_j) = \int_\Omega \frac{1}{\mu r}\mathbf{grad}\, v_k \cdot \mathbf{grad}\, v_j \, drdz + \mathrm{i}\omega \int_\Omega \frac{\sigma}{r} v_k v_j \, drdz,\\
B_{jk} &= b(w_k, v_j) = -\int_\Gamma \frac{1}{\mu r} w_k v_j \, d\gamma,\\
C_{jk} &= c(v_k, w_j) = -\int_\Gamma \frac{1}{\mu}(\mathcal{G}_{\mathbf{n}}\, v_k) w_j \, d\gamma,\\
D_{jk} &= d(w_k, w_j) = \int_\Gamma \frac{1}{\mu}(\mathcal{G}\, w_k) w_j \, d\gamma,\\
G_{jk} &= g(v_j, \mathbf{e}_k) = -\frac{1}{2\pi}\int_{\Omega_k} \frac{\sigma v_j}{r}\, drdz\,,\\
P_{jk} &= p(\mathbf{e}_k, \mathbf{e}_j) = \frac{\delta_{jk}}{4\pi^2 \mathrm{i}\omega}\int_{\Omega_k} \sigma \frac{1}{r}\, drdz\,,\\
\widetilde{G}_j &= -g(v_j, \mathbf{V}) = -\sum_{k=1}^{m} V_k \frac{1}{2\pi}\int_{\Omega_k} \frac{\sigma v_j}{r} drdz\,,\\
L_j &= l(\mathbf{e}_j) = -\frac{1}{2\pi} I_j,
\end{aligned}
$$

δ_{jk} being the Kronecker's delta, and $\{v_j,\ j = 1, \ldots, N_v\}$, $\{w_j,\ j = 1, \ldots, N_e\}$ and $\{\mathbf{e}_j,\ j = 1, \ldots, m\}$ being the canonical basis in $\mathcal{Y}_h$, $\mathcal{Z}_h$ and $\mathbb{R}^m$, respectively. We have denoted by N_v and N_e the number of vertices and boundary edges of the mesh, respectively. We recall that v_j is the (unique) element of $\mathcal{Y}_h$ which takes the value 1 at the $j-$th vertex and 0 at any other vertex of the mesh. Similarly w_j is the (unique) element of $\mathcal{Z}_h$ which is equal to 1 on the $j-$th boundary edge and 0 on the others. Finally, we notice that vector $\widetilde{G}$ is in fact the matrix-vector product $G\mathbf{V}_h$.

Computation of the matrix.

To solve the electromagnetic problem with a BEM/FEM one has to construct the matrix of the linear system (4.14). The computation of the submatrices A, B, G and P is not difficult, as they have the same form that other matrices appearing in finite element methods. On the contrary, the computation of the matrices C and D requires more complicated techniques, due to the expression of $\mathcal{G}$ and $\mathcal{G}_n$ introduced in equations (3.74) and (3.75), respectively. According to these definitions, one has to compute a double integral on the boundary Γ, along with the computation of $\omega_{p,\alpha}$. Therefore, one is in fact constrained to compute a double integral, the first one defined on Γ, the boundary of the radial section, and the second one defined on Σ, the boundary of the three-dimensional domain.

Arguing as in [86], functions $\omega_{p,\alpha}(r, z, \tilde{r}, \tilde{z})$ with $p \in \{1, 2\}$ and $\alpha \in \{1/2, 3/2\}$ can be written in terms of complete elliptic integrals of the first and second kind, namely

$$\mathrm{K}[\kappa] = \int_0^{\pi/2} (1 - \kappa^2 \sin^2\theta)^{-1/2}\, d\theta, \tag{4.15}$$

$$\mathrm{E}[\kappa] = \int_0^{\pi/2} (1 - \kappa^2 \sin^2\theta)^{1/2}\, d\theta\,, \tag{4.16}$$

and these integrals can be numerically evaluated using the method of the geometric-arithmetic mean (see [1, Sect. 17.6]).

From the definition of $\omega_{p,\alpha}$ given in (3.72) a straightforward computation leads to

$$\omega_{1,1/2}(r,z,\tilde{r},\tilde{z}) = \frac{2\,d}{r\,\tilde{r}}\left\{\left(1-\frac{2\,r\,\tilde{r}}{d^2}\right)\mathrm{K}[\kappa]-\mathrm{E}[\kappa]\right\},$$

with $\kappa^2 = 4\,r\,\tilde{r}/d^2$ and $d^2 = (r+\tilde{r})^2+(z-\tilde{z})^2$. Similarly,

$$\omega_{1,3/2}(r,z,\tilde{r},\tilde{z}) = -\frac{2}{d\,r\,\tilde{r}}\,\mathrm{K}[\kappa]+\left(\frac{2}{\kappa^2}-1\right)\frac{4}{d^3}\left(\frac{1}{1-\kappa^2}\right)\mathrm{E}[\kappa],$$

and

$$\begin{aligned}\omega_{2,3/2}(r,z,\tilde{r},\tilde{z}) = {}& \frac{2}{d\,(d^2-4\,r\,\tilde{r})\,(r\,\tilde{r})^2}\,\{(-d^4+6\,d^2\,r\,\tilde{r}-8\,r^2\,\tilde{r}^2)\,\mathrm{K}[\kappa] \\ & + (d^4-4\,d^2\,r\,\tilde{r}+2\,r^2\,\tilde{r}^2)\mathrm{E}[\kappa]\}.\end{aligned}$$

Remark 4.1. *We notice that some denominators are null when $r=0$, $\tilde{r}=0$, $d=0$ or $d^2=4r\tilde{r}$, which implies $\kappa^2=1$. To avoid dividing by zero it is important to choose a quadrature formula such that the nodes of the formula do not belong to the axisymmetry axis, thus the trapezoidal rule cannot be used to numerically evaluate the integrals. Moreover, in the case $(r,z)=(\tilde{r},\tilde{z})$ one gets $d=4r\tilde{r}=4r^2$ and $\kappa^2=1$. This enforces one to consider different integration formulas for the first and the second integral defined on Γ.*

It is also remarkable the fact that the integrals appearing in the forms $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ and $d(\cdot,\cdot)$ do not depend on the electrical conductivity σ. Hence, when considering a constant magnetic permeability which is the case in most realistic applications, matrices B, C and D can be computed only once. Of course, this is not valid in the case of a moving coil, since a change in the position of the inductor means a change in the position of the boundary Γ, thus affecting the value of the integrals.

4.1.4 Iterative algorithms to solve the couplings and the nonlinearities.

As it was already mentioned when we introduced the weak formulation (see Section 4.1.2), there are several terms coupling the three problems. However, since we are neglecting the velocity in Ohm's law, and the method of characteristics is used with the velocity at the previous time step, the hydrodynamic problem can be solved uncoupled from the two others. Nevertheless, the coupling between the thermal and the electromagnetic models cannot be avoided: the heat source in the thermal equation is the Joule effect and the solution of the electromagnetic problem varies with electrical conductivity, which depends on temperature.

Moreover, the thermal problem *(WTP)* contains several nonlinearities. To treat the nonlinear terms we are going to introduce several iterative algorithms. First of all, we summarize the nonlinearities appearing in the problem:

- The thermal conductivity k depends on temperature.
- The external convection temperature T_w may also depend on temperature. This is in fact what happens in the coil of the furnace, because the temperature of the cooling water inside the coil depends on the temperature of the coil.
- The enthalpy e depends on temperature and it is a multivalued function.
- The radiation boundary condition depends on T^4.

The dependence of the thermal conductivity k can be easily treated, just by taking in the heat equation the thermal conductivity for the temperature at the previous time step (or at the previous iteration of an outer loop). This can be done because the thermal conductivity is a very smooth function. The other nonlinearities, instead, need other iterative algorithms to be solved. In the next sections we will first describe these algorithms separately and then present the complete algorithm for solving the whole coupled problem.

Iterative algorithms for the enthalpy and the radiation boundary condition.

To deal with the multi-valued nonlinear dependence of enthalpy with respect to temperature and with the nonlinear radition boundary condition, which depends on T^4, we make use of an iterative algorithm based on a fixed point procedure. The method was introduced in [16] and it takes into account the following results:

Lemma 4.1. *Let $\mathcal{J}$ be a (possibly multi-valued) maximal monotone operator, and define $\mathcal{J}^\beta := \mathcal{J} - \beta I$, I being the identity operator. Then the following statements are equivalent:*

(i) $q \in \mathcal{J}^\beta(s)\,,$

(ii) $q = \mathcal{J}^\beta_\lambda(s + \lambda q)\,,$

where β and $\lambda > 0$ are real numbers such that $\lambda\beta \leq 1/2$, and $\mathcal{J}^\beta_\lambda$ is the Yosida approximation of the operator $\mathcal{J}^\beta$, which is defined by

$$\mathcal{J}^\beta_\lambda(s) = \frac{[s - (I + \lambda\mathcal{J}^\beta)^{-1}(s)]}{\lambda}, \quad s \in \mathbb{R}. \tag{4.17}$$

Lemma 4.2. *Let $\mathcal{J}$ be a maximal monotone operator and λ and β two real numbers such that $\lambda\beta \leq 1/2$. Then $\mathcal{J}^\beta_\lambda$, the Yosida approximation of $\mathcal{J}^\beta$, is a Lipschitz continuous function with constant equal to $1/\lambda$.*

Proof. The proof easily follows from the results of [84, Lect. 4]. □

We recall that $\mathcal{H}$ denotes the enthalpy multi-valued operator introduced in (3.81), and define the maximal monotone operator

$$\mathcal{G}(T) = |T|T^3\,, \tag{4.18}$$

which coincides with T^4 for any $T \geq 0$. For the time step $n+1$ we introduce the new functions

$$q^{n+1} = e^{n+1} - \beta T^{n+1}\,, \tag{4.19}$$

$$s^{n+1} = |T^{n+1}|(T^{n+1})^3 - \delta T^{n+1}\,. \tag{4.20}$$

According to the definition of the operators $\mathcal{H}$ and $\mathcal{G}$ we have

$$\begin{aligned} q^{n+1}(r,z) &\in \mathcal{H}\left((r,z), T^{n+1}(r,z)\right) - \beta T^{n+1}(r,z) = \mathcal{H}^{\beta}\left((r,z), T^{n+1}(r,z)\right)\,, \\ s^{n+1}(r,z) &= \mathcal{G}(T^{n+1}(r,z)) - \delta T^{n+1}(r,z) = \mathcal{G}^{\delta}(T^{n+1}(r,z))\,, \end{aligned}$$

and due to the result of Lemma 4.1, since both $\mathcal{H}$ and $\mathcal{G}$ are maximal monotone operators, we know that

$$q^{n+1}(r,z) = \mathcal{H}^{\beta}_{\lambda}\left((r,z), T^{n+1}(r,z) + \lambda q^{n+1}(r,z)\right)\,, \quad \text{for } 0 < \lambda \leq \frac{1}{2\beta}\,, \tag{4.21}$$

$$s^{n+1}(r,z) = \mathcal{G}^{\delta}_{\kappa}(T^{n+1}(r,z) + \kappa s^{n+1}(r,z))\,, \quad \text{for } 0 < \kappa \leq \frac{1}{2\delta}\,. \tag{4.22}$$

The idea of the method is to replace, in the discrete version of problem *(WTP)*, the enthalpy e^{n+1} and the term $(T^{n+1})^4$ by the expressions given in (4.19) and (4.20), and then update the multipliers q and s using the expressions (4.21) and (4.22).

At time step t^{n+1} we suppose that q_0^{n+1} and s_0^{n+1} are known. Then at each iteration k of the iterative procedure we successively determine T_k^{n+1}, q_k^{n+1} and s_k^{n+1} by

$$\begin{aligned} &\int_{\Omega_T} \frac{1}{\Delta t}\,\beta T_k^{n+1}\, Z\, r\, drdz \;+\; \int_{\Omega_T} k_{eff}\,\mathbf{grad}\, T_k^{n+1}\cdot \mathbf{grad}\, Z\, r\, drdz + \int_{\Gamma_C} \alpha T_k^{n+1} Z\, r\, d\Gamma \\ &+ \int_{\Gamma_R} (\alpha+\gamma\delta) T_k^{n+1} Z\, r\, d\Gamma \;=\; \int_{\Gamma_C} \alpha T_w\, Z\, r\, d\Gamma \;+\; \int_{\Gamma_R} (\alpha T_c + \gamma T_r^4 - \gamma s_{k-1}^{n+1}) Z\, r\, d\Gamma \\ &+ \int_{\Omega_T} \frac{1}{\Delta t}\,(e^n \circ \chi^n - q_{k-1}^{n+1})\, Z\, r\, drdz \;+\; \int_{\Omega_T} \frac{1}{2\sigma}\,|J^{n+1}|^2\, Z\, r\, drdz \quad \forall Z \in \mathcal{V}_h\,, \\ &q_k^{n+1} = \mathcal{H}^{\beta}_{\lambda}\left(T_k^{n+1} + \lambda q_{k-1}^{n+1}\right)\,, \\ &s_k^{n+1} = \mathcal{G}^{\delta}_{\kappa}(T_k^{n+1} + \kappa s_{k-1}^{n+1})\,. \end{aligned}$$

where $\mathcal{V}_h$ is the space of finite element functions.

Computing the Yosida approximations $\mathcal{H}^{\beta}_{\lambda}$ and $\mathcal{G}^{\delta}_{\kappa}$ at each point requires the solution of the nonlinear equation involved in (4.17). To evaluate $\mathcal{G}^{\delta}_{\kappa}(s)$ first we have to compute y such that

$$y = (I + \kappa\mathcal{G}^{\delta})^{-1}(s)\,.$$

It is easily seen that y satisfies

$$y(1-\delta\kappa) + \kappa y^3|y| - s = 0\,,$$

and this nonlinear equation is solved by the Newton-Raphson method, as it was done in [91]. For the computation of $\mathcal{H}^{\beta}_{\lambda}(s)$ we follow the method proposed in [80]: the multi-valued enthalpy function $\mathcal{H}$ is replaced by a piecewise linear function with a very high slope at T_S. Doing so also the Yosida regularization becomes piecewise linear, and the value of $\mathcal{H}_\lambda^\beta(s)$ can be easily obtained (see [80]).

The performance of the proposed algorithm is known to depend strongly on the choice of the parameters β, λ, δ and κ. In [83] the authors propose to replace the constant parameters by functions depending on $\mathbf{x} = (r, z)$, and introduce an automatic procedure for the computation of the parameters which accelerates the convergence of the method. Following the ideas of that paper, we impose $\beta(r,z)\lambda(r,z) = 1/2$ and $\delta(r,z)\kappa(r,z) = 1/2$, so only the parameters $\beta(r,z)$ and $\delta(r,z)$ have to be chosen. Since $\mathcal{G}$ is a differentiable function for any $T > 0$ we take, as it is proposed in [83],

$$\delta(r,z) = \frac{\partial \mathcal{G}(T^{n+1}(r,z))}{\partial T} = 4|T^{n+1}(r,z)|^3.$$

Since the enthalpy function $\mathcal{H}$ is not differentiable we shall make use of the algorithm proposed in [82] for the choice of a constant parameter β. For the ease of reading, in what follows we drop the time step index $n+1$. Moreover, to simplify the equation let us define the bilinear form

$$a_T(T,Z) := \Delta t \left(\int_{\Omega_T} k_{eff}\, \mathbf{grad}\, T \cdot \mathbf{grad}\, Z\, r drdz + \int_{\Gamma_C} \alpha T Z\, r\, d\Gamma + \int_{\Gamma_R} (\alpha + \gamma\delta) T Z\, r\, d\Gamma \right),$$

and the linear form

$$f_T(Z) := \Delta t \Bigg(\int_{\Gamma_C} \alpha T_w\, Z\, r\, d\Gamma \; + \; \int_{\Gamma_R} (\alpha T_c + \gamma T_r^4 - \gamma s) Z\, r\, d\Gamma \; + \; \int_{\Omega_T} \frac{1}{\Delta t} \left(e^n \circ \chi^n\right) Z\, r\, drdz \\ + \int_{\Omega_T} \frac{1}{2\sigma} |J|^2\, Z\, r\, drdz \Bigg).$$

Using these two definitions, the equations of the problem can be written in the form

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_T} \beta T Z\, r\, drdz + a_T(T,Z) = f_T(Z) - \int_{\Omega_T} q Z\, r\, drdz \quad \forall Z \in \mathcal{V}_h, \\ q = \mathcal{H}_\lambda^\beta(T + \lambda q), \end{array} \right. \tag{4.23}$$

and the equations of the iterative algorithm become

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_T} \beta T_k Z\, r\, drdz + a_T(T_k,Z) = f_T(Z) - \int_{\Omega_T} q_{k-1} Z\, r\, drdz \quad \forall Z \in \mathcal{V}_h, \\ q_k = \mathcal{H}_\lambda^\beta(T_k + \lambda q_{k-1}). \end{array} \right. \tag{4.24}$$

Since $\mathcal{H}$ is a maximal monotone operator and $\lambda\beta = 1/2$, from Lemma 4.2 we know that $\mathcal{H}_\lambda^\beta$ is a Lipschitz function with constant $1/\lambda$, thus subtracting the equations of the systems (4.23) and (4.24), and substituting Z by $T - T_k$ it holds

$$\|q - q_k\|_{L^2}^2 = \left\|\mathcal{H}_\lambda^\beta(T + \lambda q) - \mathcal{H}_\lambda^\beta(T_k + \lambda q_{k-1})\right\|_{L^2}^2 \\ \leq \frac{1}{\lambda^2} \|T - T_k\|_{L^2}^2 + \frac{2}{\lambda}(T - T_k, q - q_{k-1}) + \|q - q_{k-1}\|_{L^2}^2,$$

and

$$(T - T_k, q - q_{k-1}) = -\beta \|T - T_k\|_{L^2}^2 - a_T(T - T_k, T - T_k),$$

where the pairing $(u,v) := \int_{\Omega_T} uv\, r\, drdz$ is the scalar product in $L^2(\Omega)$ expressed in cylindrical coordinates and $\|\cdot\|_{L^2}$ is its corresponding induced norm. From the two previous equations we obtain

$$\|q - q_k\|^2_{L^2} \leq \frac{1}{\lambda^2}\|T - T_k\|^2_{L^2} - \frac{2\beta}{\lambda}\|T - T_k\|^2_{L^2} - \frac{2}{\lambda}a_T(T - T_k, T - T_k) + \|q - q_{k-1}\|^2_{L^2}$$
$$= -\frac{2}{\lambda}a_T(T - T_k, T - T_k) + \|q - q_{k-1}\|^2_{L^2} \ ,$$

where the last equality holds for any choice of parameters such that $\lambda\beta = 1/2$. Since the bilinear form $a_T(\cdot,\cdot)$ is coercive in $H^1(\Omega)$, then

$$\|q - q_k\|^2_{L^2} \leq -\frac{2\alpha_T}{\lambda}\|T - T_k\|^2_1 + \|q - q_{k-1}\|^2_{L^2} \ , \tag{4.25}$$

with α_T the coerciveness constant. Moreover, subtracting again the first equation of the systems and substituting Z by $q - q_{k-1}$ we obtain

$$\|q - q_{k-1}\|^2_{L^2} = -\beta(T - T_k, q - q_{k-1}) - a_T(T - T_k, q - q_{k-1})$$
$$\leq \beta\|T - T_k\|_{L^2}\|q - q_{k-1}\|_{L^2} + \|a_T\|\,\|T - T_k\|_1\|q - q_{k-1}\|_1$$
$$\leq (\beta + C(h)\|a_T\|)\|T - T_k\|_1\|q - q_{k-1}\|_{L^2} \ ,$$

where $C(h)$ is the constant appearing in the inverse inequality $\|u_h\|_1 \leq C(h)\|u_h\|_{L^2}$, $\forall\, u_h \in \mathcal{V}_h$. As a consequence

$$\|q - q_{k-1}\|_{L^2} \leq (\beta + C(h)\|a_T\|)\|T - T_k\|_1 \ , \tag{4.26}$$

and using this last inequality in (4.25) we obtain

$$\|q - q_k\|^2_{L^2} \leq \left(1 - \frac{2\alpha_T}{\lambda(\beta + C(h)\|a_T\|)^2}\right)\|q - q_{k-1}\|^2_{L^2} \ . \tag{4.27}$$

Following the steps in the proof of Theorem 1 of [82] we can prove that the algorithm converges and

$$\|q - q_k\|_{L^2} \leq (C_\beta)^k\|q - q_0\|_{L^2} \ , \qquad \text{with } C_\beta = \sqrt{1 - \frac{4\beta\alpha_T}{(\beta + C(h)\|a_T\|)^2}} \ ,$$

and an easy computation yields that the minimum value of C_β is obtained for $\beta = C(h)\|a_T\|$, which means that this is the optimal value.

To obtain the optimal value of the parameter β from the previous expression one must compute the value of the constant $C(h)$ and the norm $\|a_T\|$. This can be done by solving an eigenvalue problem. Let us suppose that the finite element space $\mathcal{V}_h$ has dimension N_h, equal to the number of nodes of the mesh. Moreover, let us denote by w_i, $i = 1, \ldots, N_h$ the shape functions of the finite element method which form a basis of the space $\mathcal{V}_h$.

Let us introduce a new scalar product in the discrete space $\mathcal{V}_h$, which will be referred as $\tilde{a}(\cdot,\cdot)$, and denote its induced norm by $|||\cdot|||_1$. Reasoning for this norm as we have done before, the optimal value for the parameter is $\beta = \tilde{C}(h)|||a_T|||$, with $\tilde{C}(h)$ the constant of the inequality $|||u_h|||_1 \leq \tilde{C}(h)\|u_h\|_{L^2}$. We consider the following spectral problem:

Find $\Phi \in \mathcal{V}_h$ *and* $\mu \in \mathbb{R}$ *such that*

$$\tilde{a}(\Phi, v) = \mu(\Phi, v), \quad \forall\, v \in \mathcal{V}_h\,. \tag{4.28}$$

Let us denote by $\{\Phi_i\}_{i=1}^{N_h}$ the set of eigenvectors solution of the previous problem which form an orthonormal basis in $\mathcal{V}_h$, with respect to the scalar product $(\cdot,\cdot)$. Then any vector $u_h \in \mathcal{V}_h$ can be expressed as $u_h = \sum_{i=1}^{N_h} u_i \Phi_i$ and it holds

$$|||u_h|||_1^2 = \tilde{a}(u_h, u_h) = \sum_{i=1}^{N_h} \tilde{a}(u_i\Phi_i, u_i\Phi_i) = \sum_{i=1}^{N_h} u_i^2 \mu_i(\Phi_i, \Phi_i) \le \mu_{\max} \|u_h\|_{L^2}^2 \,,$$

where $\mu_{\max}$ is the maximum eigenvalue. Clearly, $\tilde{C}(h) = \sqrt{\mu_{\max}}$.

The scalar product $\tilde{a}(\cdot,\cdot)$ can be chosen such that the shape functions of the finite element method constitute an orthonormal basis of $\mathcal{V}_h$, *i.e.*,

$$\tilde{a}(w_i, w_j) = \delta_{ij}\,.$$

As the shape functions $w_j, j = 1, \ldots, N_h$ form an orthonormal basis for the scalar product $\tilde{a}(\cdot,\cdot)$, the eigenvalue problem (4.28) can be written in terms of these functions, to obtain the equivalent problem

Find $\Phi \in \mathcal{V}_h$ *and* $\mu \in \mathbb{R}$ *such that*

$$\Phi = \mu M \Phi\,, \tag{4.29}$$

where M is the mass matrix of the finite element method. Solving this eigenvalue problem we obtain $\tilde{C}(h) = \sqrt{\mu_{\max}}$, and the norm $|||a_T|||$ is easily computed from the matrix of the finite element discretization, to obtain $\beta = \tilde{C}(h)|||a_T|||$.

We notice that, since constant $\tilde{C}(h)$ tends to infinity as h tends to zero, the constant of convergence C_β tends to one so refining the mesh would slow down the convergence of the method. In the conclusions of [82] the authors suggest to choose the constant parameters computed above in the non-regular regions and to use optimal functions in regular regions, as proposed in [83]. This is in fact what we have done: in the liquid and solid regions, where the enthalpy function is regular, the parameter β is a function and in the mushy region, *i.e.*, in the nodes at temperature T_S, we take the parameter $\beta = \tilde{C}(h)|||a_T|||$.

Iterative algorithm for the temperature of cooling water.

As we said before, the induction coil of the furnace is water-cooled to avoid overheating. Thus in the internal boundary of the coil we consider a convection boundary condition, of the form

$$k(\mathbf{x}, T)\frac{\partial T}{\partial \mathbf{n}} = \alpha(T_w - T) \quad \text{on } \Gamma_{\mathrm{C}}, \tag{4.30}$$

where α is the coefficient of convective heat transfer and T_w is the temperature of the cooling water, which depends on the temperature of the coil. Hence the water temperature T_w depends on the solution of our problem which introduces a nonlinearity in the equations.

To deal with this nonlinearity we will seek the convergence of the heat flux from the coil to the cooling water. From the solution of the thermal problem this heat flux is computed as

$$H = 2\pi \int_{\Gamma_C} k \frac{\partial T}{\partial \mathbf{n}} \, d\Gamma \,, \tag{4.31}$$

where the integral is multiplied by 2π because Γ_C is only a radial section of the boundary. The heat flux is also equal to

$$H = \rho_w c_w Q (T_o - T_i) \,, \tag{4.32}$$

with ρ_w and c_w the density and specific heat of water, respectively, T_i and T_o denote the inlet and outlet temperature of the cooling water, and Q is the water flow rate.

To solve the problem, we assume that the water temperature T_w is constant along the coil. This is a reasonable assumption, since the difference between the inlet and the outlet temperature is seldom higher than 10 °C.

The temperature of the cooling water is computed using an iterative algorithm. Let us suppose that $T_{w,j}$ is known. Then, at iteration $j+1$ we compute T_{j+1} as the solution of the thermal problem *(WTP)* with $T_w = T_{w,j}$. Then, we compute the heat flux H from equation (4.31) and set

$$T_{w,j+1} = T_i + \frac{H}{2\rho_w c_w Q} \,,$$

i.e., $T_{w,j+1}$ is the mean value of the given inlet temperature and the computed outlet temperature.

Remark 4.2. *One could be tempted to avoid the solution of this iterative algorithm by considering an explicit method, just by taking, in the boundary condition (4.30), a temperature T_w computed from the solution at the previous time step. This has been tried but the problem becomes unstable due to a high rise of temperature in the first time steps. However, the iterations of the implicit method can be merged with the iterations of the thermoelectrical coupling so the additional computational cost of the method becomes very low.*

4.1.5 Computation of the hydrodynamic domain.

As said before, the region occupied by the molten metal varies along the time and depends on temperature. We consider a fixed mesh of the region occupied by the material to be heated, *i.e.*, the solid and molten metal in Figure 3.6. At each time step, in order to determine the hydrodynamic domain, we need to compute the position of the boundary of the molten region. To do that, we are obliged to solve the thermal problem previously. More precisely, the enthalpy profile given by the solution of problem *(WTP)* allows us to choose those elements of the mesh belonging to the liquid region. The hydrodynamic computational mesh is then updated. We notice that the mushy region, *i.e.*, the region where the melting temperature has been reached but the material is not completely molten, is not considered as a part of the hydrodynamic domain.

Iterative algorithm for the whole problem.

Now we present the iterative algorithm to solve the three coupled models, along with the nonlinearities. Basically, it consists of three nested loops: the first one for the time discretization, the second one for the thermoelectrical coupling, and the third one for the Bermúdez-Moreno algorithms for the enthalpy and the radiation boundary condition presented above. As we have said before, the hydrodynamic problem is solved at each time step uncoupled from the two others. Moreover, the nonlinearities of the thermal conductivity k and the cooling water temperature T_w are also treated by iterative algorithms and their corresponding loops are in fact merged with that of the thermoelectrical coupling. To make it easier to the reader we present in Figure 4.1 a sketch of the algorithm with the three nested loops.

Algorithm. Let us suppose that the initial temperature, T^0, and velocity, $\mathbf{u}^0$, are known. From T^0 determine the initial enthalpy e^0 and set the temperature of cooling water $T_w^0 = T_i$, with T_i the inlet temperature. Then, at time step n, with $n = 1, \dots, N$ we compute A_θ^n, T^n and $\mathbf{u}^n$ doing the following steps:

1. If $\mathbf{u}^{n-1} \neq \mathbf{0}$, compute $\chi^{n-1}(\mathbf{x})$, the solution of (4.6).
2. Calculate the turbulent viscosity $\eta_t^n = \rho_0 c h^2 |D(\mathbf{u}^{n-1})|$ and the turbulent thermal conductivity $k_t^n = c_0 \eta_t^n / Pr_t$.
3. Set $T_0^n = T^{n-1}$, $e_0^n = e^{n-1}$ and $T_{w,0}^n = T_w^{n-1}$. Then compute A_θ^n, T^n, e^n and T_w^n as the limit of $A_{\theta,j}^n$, T_j^n and $T_{w,j}^n$, following the iterative procedure:
   (a) For $j \geq 1$ let us suppose that T_{j-1}^n and $T_{w,j-1}^n$ are known. Then set $A_{\theta,j}^n = A'_\theta / r$, with A'_θ, λ' and $V \in \mathbb{C}$ the solution of

$$
\begin{aligned}
\mathrm{i}\omega \int_\Omega \frac{\sigma(r,z,T_{j-1}^n)}{r} A'_\theta \bar{\psi}'\, drdz + \int_\Omega \frac{1}{\mu(r,z,T_{j-1}^n) r} \mathbf{grad}\, A'_\theta \cdot \mathbf{grad}\, \bar{\psi}'\, drdz \\
- \int_\Gamma \frac{1}{\mu(r,z,T_{j-1}^n) r} \lambda' \bar{\psi}'\, d\gamma + \sum_{k=1}^m \frac{V_k}{2\pi} \int_{\Omega_k} \frac{\sigma(r,z,T_{j-1}^n)}{r} \bar{\psi}'\, drdz = 0 \quad \forall \psi' \text{ test function}, \\
\frac{1}{2\pi} \sum_{k=1}^m \Big(\int_{\Omega_k} \frac{\sigma(r,z,T_{j-1}^n)}{r} A'_\theta\, drdz \Big) \bar{W}_k + \frac{1}{4\pi^2 \mathrm{i}\omega} \sum_{k=1}^m \Big(\int_{\Omega_k} \sigma(r,z,T_{j-1}^n) \frac{V_k}{r}\, drdz \Big) \bar{W}_k \\
= -\frac{1}{2\pi \mathrm{i}\omega} \sum_{k=1}^m I_k \bar{W}_k \quad \forall \mathbf{W} \in \mathbb{C}^m, \\
\int_\Gamma \frac{1}{\mu(r,z,T_{j-1}^n) r} \bar{\zeta} A'_\theta\, d\gamma - \int_\Gamma \frac{1}{\mu(r,z,T_{j-1}^n)} (\mathcal{G}_n A'_\theta) \bar{\zeta}(r,z)\, d\gamma \\
+ \int_\Gamma \frac{1}{\mu(r,z,T_{j-1}^n)} (\mathcal{G} \lambda') \bar{\zeta}(r,z)\, d\gamma = 0 \quad \forall \lambda' \text{ test function}.
\end{aligned}
$$

   (b) Set $J_j^n = \mathrm{i}\omega\sigma(r,z,T_{j-1}^n) A_{\theta,j}^n - V_k/(2\pi r)$ in Ω_k, with $V_0 = 0$.
   (c) Determine the optimal parameters $\delta_j^n(r,z) = 4|T_{j-1}^n(r,z)|^3$ and $\beta_j^n(r,z)$ with the method described above.

(d) Set $T^n_{j,0} = T^n_{j-1}$ and also

$$\begin{aligned} q^n_{j,0} &= e^n_{j-1} - \beta^n_j T^n_{j-1}\,, \\ s^n_{j,0} &= |T^n_{j-1}|(T^n_{j-1})^3 - \delta^n_j T^n_{j-1}\,. \end{aligned}$$

Then compute T^n_j, q^n_j and s^n_j as the limit of the following iterative procedure:

i. For $k \geq 1$ let us suppose that $q^n_{j,k-1}$ and $s^n_{j,k-1}$ are known. Then compute $T^n_{j,k}$ as the solution of the linear system

$$\begin{aligned} &\int_{\Omega_T} \frac{1}{\Delta t}\,\beta^n_j T^n_{j,k}\, Z\, r\, dr dz \;+\; \int_{\Omega_T} (k(T^n_{j-1}) + k^n_t)\, \mathbf{grad}\, T^n_{j,k} \cdot \mathbf{grad}\, Z\, r\, dr dz + \int_{\Gamma_C} \alpha T^n_{j,k} Z\, r\, d\Gamma \\ &+ \int_{\Gamma_R} (\alpha + \gamma \delta^n_j) T^n_{j,k} Z\, r\, d\Gamma \;=\; \int_{\Gamma_C} \alpha T^n_{w,j-1}\, Z\, r\, d\Gamma \;+\; \int_{\Gamma_R} (\alpha T_c + \gamma T_r^4 - \gamma s^n_{j,k-1}) Z\, r\, d\Gamma \\ &+ \int_{\Omega_T} \frac{1}{\Delta t}\,(e^{n-1} \circ \chi^{n-1} - q^n_{j,k-1})\, Z\, r\, dr dz \;+\; \int_{\Omega_T} \frac{1}{2\sigma(r,z,T^n_{j-1})}\, |J^n_j|^2\, Z\, r\, dr dz \quad \forall Z \text{ test function}\,, \end{aligned}$$

ii. Update multipliers $q^n_{j,k}$ and $s^n_{j,k}$ by the formulas

$$\begin{aligned} q^n_{j,k} &= \mathcal{H}^\beta_\lambda \left(T^n_{j,k} + \lambda q^n_{j,k-1}\right)\,, \\ s^n_{j,k} &= \mathcal{G}^\delta_\kappa (T^n_{j,k} + \kappa s^n_{j,k-1})\,. \end{aligned}$$

(e) Update the value of the enthalpy, and the value of the cooling water temperature by computing

$$e^n_j = q^n_j + \beta^n_j T^n_j\,,$$

$$H = 2\pi \int_{\Gamma_C} k \frac{\partial T^n_j}{\partial \mathbf{n}}\, d\Gamma\,, \quad \text{and} \quad T^n_{w,j} = T_i + \frac{H}{2\rho_w c_w Q}\,.$$

4. Compute the hydrodynamic domain from the value of the enthalpy e^n.

5. Find $\mathbf{u}^n$ and p^n solution of

$$\begin{aligned} &\frac{1}{\Delta t}\int_{\Omega_l} \rho_0 \mathbf{u}^n \cdot \mathbf{w}\, r\, dr dz + \int_{\Omega_l} (\eta_0 + \eta^n_t)\, (\mathbf{grad}\, \mathbf{u}^n : \mathbf{grad}\, \mathbf{w})\; r\, dr dz \\ &+ \int_{\Omega_l} (\eta_0 + \eta^n_t) \left((\mathbf{grad}\, \mathbf{u}^n)^t : \mathbf{grad}\, \mathbf{w}\right)\, r\, dr dz - \int_{\Omega_l} p^n \mathrm{div}\, \mathbf{w}\, r\, dr dz = \\ &\qquad -\rho_0 \beta_0 \int_{\Omega_l} (T^n - T_0)\mathbf{g} \cdot \mathbf{w}\, r\, dr dz \\ &+ \int_{\Omega_l} \frac{\omega\sigma}{2} \left(\mathrm{Im}\left[A^{n+1}_\theta \frac{\partial \overline{A}^{n+1}_\theta}{\partial r} \right] w_r + \mathrm{Im}\left[A^{n+1}_\theta \frac{\partial \overline{A}^{n+1}_\theta}{\partial z} \right] w_z \right)\, r\, dr dz \\ &+ \frac{1}{\Delta t}\int_{\Omega_l} \rho_0 (\mathbf{u}^{n-1} \circ \chi^{n-1}) \cdot \mathbf{w}\, r\, dr dz, \quad \forall \mathbf{w} \text{ test function}, \\ &\int_{\Omega_l} \mathrm{div}\, \mathbf{u}^n\, q = 0, \quad \forall q \text{ test function}. \end{aligned}$$

Remark 4.3. *The initial value of the enthalpy, e^0, can be computed assuming that the temperature at every point is different from T_S, the melting temperature. Otherwise, the enthalpy function for the initial temperature becomes multivalued, and an initial value for e^0 should be provided by the user.*

4.2 Numerical results.

As it was presented in the previous section, the method has been implemented in a computer code using FORTRAN. We now present some numerical results which have been obtained with the use of this code. Two different simulations are considered: the first one is a simulation of an 'academic problem' with known analytical solution, that has been used to validate the computational code. The second one is the numerical simulation of a real industrial melting furnace.

4.2.1 Analytical solution of a thermo-hydrodynamic problem with phase change.

The first simulation we consider consists of a simplified academic problem having an analytical solution, and that has been used to validate the computational code. We notice that the thermoelectrical model had been already implemented and validated in previous works (see [15, 14]). For the present work we have implemented the thermo-electromagneto-hydrodynamic model, but for the validation of the code, due to the difficulty of constructing a problem with analytical solution for the whole system of coupled equations, and since we are neglecting the influence of the velocity on the electromagnetic problem, we are just considering a thermal-hydrodynamic problem with change of phase. This problem with analytical solution is a generalization of another one introduced in [20] for the same purpose of validation.

We consider that our spatial domain is given, in cylindrical coordinates, by $\Delta = (0,1) \times (0,2\pi) \times (-1,1)$. Since we are in an axisymmetrical setting, our domain of computation will be $\Omega = (0,1) \times (-1,1)$. Concerning the time domain, all the computations will be carried out in the time interval $[0,1]$.

For each time t we define the function

$$g(r,z,t) = r^2 + z^2 - h(t), \tag{4.33}$$

where $h(t)$ is a function of time that is solution of a certain ordinary differential equation, as will be shown below. We suppose that the temperature is given by

$$T(r,z,t) = \begin{cases} C_1 g(r,z,t) + 1, & \text{if } g(r,z,t) \geq 0, \\ C_2 g(r,z,t) + 1, & \text{if } g(r,z,t) < 0, \end{cases} \tag{4.34}$$

C_1 and C_2 being two positive constants. With this definition the temperature is a continuous function. The phase change takes place at temperature $T = 1$, and so the free boundary between the solid and the liquid is given by

$$S(t) = \left\{ \mathbf{x} = (r,\theta,z) : r^2 + z^2 = h(t) \right\}, \tag{4.35}$$

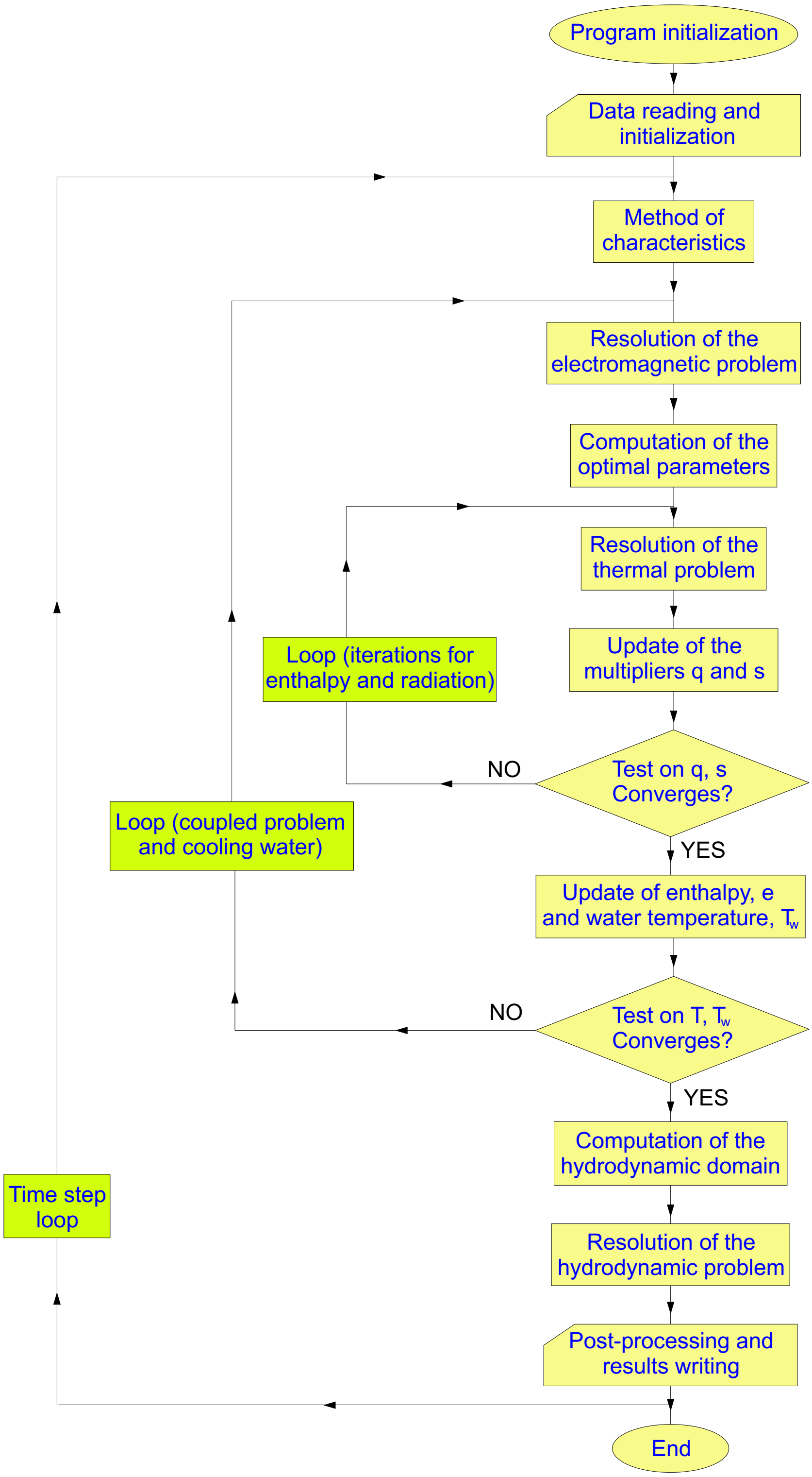

Figure 4.1: Scheme of the algorithm.

i.e., the solid region is the ball centered at the origin and radius $\sqrt{h(t)}$. Moreover, since the temperature is independent of θ, the solution is adequate for an axisymmetrical setting.

Concerning the thermophysical parameters, we suppose that density, thermal conductivity and latent heat are constant, whereas the specific heat is constant in each region, and is given by

$$c(T) \quad = \begin{cases} c_1, & \text{if } T > 1, \\ c_2, & \text{if } T \leq 1, \end{cases} \tag{4.36}$$

where c_1 and c_2 are two positive constants.

In order to obtain a solution of the thermal equation with phase change, the Stefan condition on the interphase must be satisfied. This condition reads

$$\left[k(T) \frac{\partial T}{\partial \mathbf{n}} \right] = \rho L \mathbf{V} \cdot \mathbf{n}, \tag{4.37}$$

where $[\varphi]$ denotes the jump of function φ across the interphase, $\mathbf{n}$ is the unit vector normal to the interphase $S(t)$ outward to the solid region and $\mathbf{V}$ is the velocity of the interphase advance.

For the first term of Stefan condition (4.37), using the definition of g and the fact that on the interphase the normal vector is given by $(r, z)/\sqrt{r^2 + z^2}$ we have

$$\left[k(T) \frac{\partial T}{\partial \mathbf{n}} \right] = k(C_2 \mathbf{grad}\, g \cdot \mathbf{n} - C_1 \mathbf{grad}\, g \cdot \mathbf{n}) = k(C_2 - C_1) \frac{2(r^2 + z^2)}{\sqrt{r^2 + z^2}}\,, \tag{4.38}$$

and since the velocity of the interphase is given by

$$\mathbf{V} = \frac{\partial \left(h(t)^{1/2} \right)}{\partial t} \mathbf{n} = \frac{h'(t)}{\sqrt{h(t)}} \mathbf{n}\,, \tag{4.39}$$

we get that the Stefan condition can be rewritten in the form

$$k(C_2 - C_1) \frac{2(r^2 + z^2)}{\sqrt{r^2 + z^2}} = \rho L \frac{1}{2} \frac{h'(t)}{\sqrt{h(t)}}\,. \tag{4.40}$$

From the values of function $h(t)$ on the interphase $S(t)$, the previous condition is modified to obtain the following ordinary differential equation

$$h'(t) = \frac{4k}{\rho L}(C_2 - C_1) h(t)\,. \tag{4.41}$$

Hence, in order to have a solution satisfying the Stefan condition, function h must be of the form

$$h(t) = C_3\, e^{4k(C_2 - C_1)t/(\rho L)}\,, \tag{4.42}$$

where C_3 is a given constant that allows us to control the velocity of the interphase advance.

Once we know the expression of function h, we can consider the heat equation in terms of the enthalpy. Substituting the Joule effect by a given heat source f, the equation reads

$$\left(\frac{\partial e}{\partial t} + \mathbf{u} \cdot \mathbf{grad}\, e \right) - \operatorname{div} (k\, \mathbf{grad}\, T) = f \ \text{ in } \Omega\,. \tag{4.43}$$

Since the analytical solution of our problem is given by (4.34), the enthalpy can be computed from equation (3.81) and after some straightforward computations one obtains that the heat source f must be

$$f(r,z,t) = \begin{cases} -c_1 h'(t) - 6kC_1 + 2c_1(r,z)\cdot\mathbf{u}\,, & \text{if } g(r,z,t) \geq 0, \\ -c_2 h'(t) - 6kC_2, & \text{if } g(r,z,t) < 0\,, \end{cases}$$

where $\mathbf{u}$ is the velocity field that will be introduced below.

For the boundary condition of the thermal problem we are only considering the radiation part, namely,

$$k\frac{\partial T}{\partial \mathbf{n}} = \gamma(T_r^4 - T^4)\,. \tag{4.44}$$

If we replace the value of the temperature given in (4.34) in this equation, we conclude that T_r must be

$$T_r = \begin{cases} \sqrt[4]{(C_1 g(r,z,t)+1)^4 + \frac{k}{\gamma}\mathbf{n}\cdot(r,z)}\,, & \text{if } g(r,z,t) \geq 0\,, \\ \sqrt[4]{(C_2 g(r,z,t)+1)^4 + \frac{k}{\gamma}\mathbf{n}\cdot(r,z)}\,, & \text{if } g(r,z,t) < 0\,, \end{cases} \tag{4.45}$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary $\partial\Omega$ and k is the thermal conductivity which is constant.

Finally, in order to have a solution of the thermo-hydrodynamical problem, we must define a velocity field in the liquid region. We choose the Stokes' flow around the sphere occupied by the solid region, which is supposed to be centered at the origin and to have radius R. We recall that this flow is the solution of the Stokes stationary equations

$$-\eta\Delta\mathbf{u} + \mathbf{grad}\, p = \mathbf{0} \quad \text{in } \Omega_l\,, \tag{4.46}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_l\,, \tag{4.47}$$

where the liquid domain is given by $\Omega_l = \{(r,z) : r^2 + z^2 \geq R\}$ and equation (4.46) corresponds to the motion equation for low Reynolds number. The velocity field we are interested in is the solution of (4.46)-(4.47) together with conditions

$$\mathbf{u} = \mathbf{0} \qquad \text{on } S(t), \tag{4.48}$$

$$\mathbf{u} = u\mathbf{e}_z,\ p = p_\infty \qquad \text{at } r \to \infty\,. \tag{4.49}$$

To solve the problem, we suppose that the dynamic viscosity is constant and equal to one, ($\eta = 1$). For symmetry reasons we suppose that the velocity field is independent of θ. Then the velocity field solution of equations (4.46)-(4.49) can be expressed in cylindrical coordinates as follows:

$$\begin{aligned} u_r &= \frac{3u}{4}\frac{rz}{r^2+z^2}\left(\frac{R^3}{(r^2+z^2)^{3/2}} - \frac{R}{(r^2+z^2)^{1/2}}\right), \\ u_z &= \frac{3u}{4}\frac{z^2}{r^2+z^2}\left(\frac{R^3}{(r^2+z^2)^{3/2}} - \frac{R}{(r^2+z^2)^{1/2}}\right) \\ &\quad + u\left(1 - \frac{3R}{4(r^2+z^2)^{1/2}} - \frac{R^3}{4(r^2+z^2)^{3/2}}\right). \end{aligned} \tag{4.50}$$

In our case, since the domain occupied by the solid region varies with time, the radius of the ball must be time dependent and we have $R(t) = \sqrt{h(t)}$.

For computational reasons the hydrodynamic domain must be restricted to $\Omega_l \cap \Omega$. On the boundaries of this domain we impose Dirichlet boundary conditions, with the velocity $\mathbf{u}$ given by (4.50) in $\partial\Omega$, and null on the interphase. We notice that the interphase is computed from the numerical solution at each time step.

The problem has been considered with the following physical properties

$$k = 1,\ \rho = 1,\ L = 8,\ \gamma = 1,\ c = \begin{cases} 6, & \text{in } T > 1, \\ 2, & \text{in } T \leq 1, \end{cases}$$

and the constants

$$C_1 = 2,\ C_2 = 1,\ C_3 = 0.25\,.$$

Since the velocity is now the solution of a stationary equation, the computer code has been slightly modified to solve the correct problem. First of all, Lorentz and buoyancy forces have been disabled. Moreover, the time discretization using the method of characteristics is no longer considered, and at each time step the transient Stokes equations are solved until a steady solution is achieved. We notice that even if this method is not the best to solve the steady Stokes equations, it is good for our validation purposes.

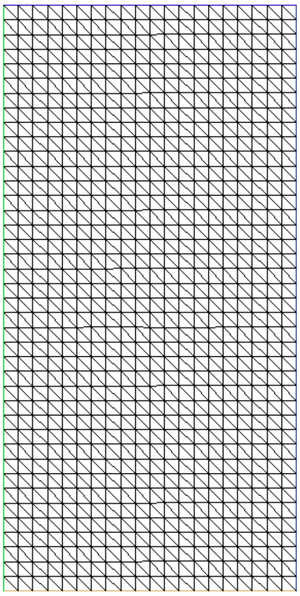

Figure 4.2: Coarsest mesh for the academic problem.

The problem has been solved for four successively refined two-dimensional meshes, with a corresponding reduction of the time step. The numerical results obtained have been compared with the analytical solution. In Figure 4.2 we present the coarsest mesh used for the computation. In Figure 4.3 we compare the temperature for the exact solution and for the finest mesh at the last time step. In Figure 4.4 we present the value of the absolute error for the velocity in the coarsest and the finest meshes. To compare our numerical solution with the analytical one, we have computed the error in $L^\infty - L^2$ norm (in time and space, respectively). In Table 4.1 we present the values of these errors. Figure 4.5 shows the log-log plots of the errors for the computed temperature T and velocity $\mathbf{u}$, respectively, where we can observe a linear convergence of the method.

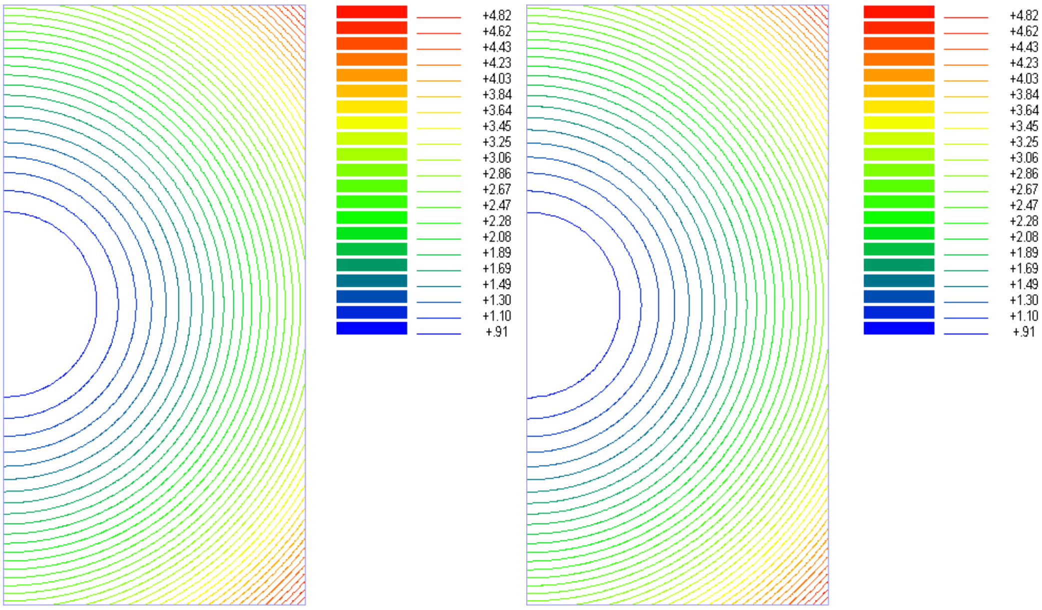

Figure 4.3: Comparison of the temperature for the exact solution (left) and the computed solution (right) in the finest mesh for the last time step.

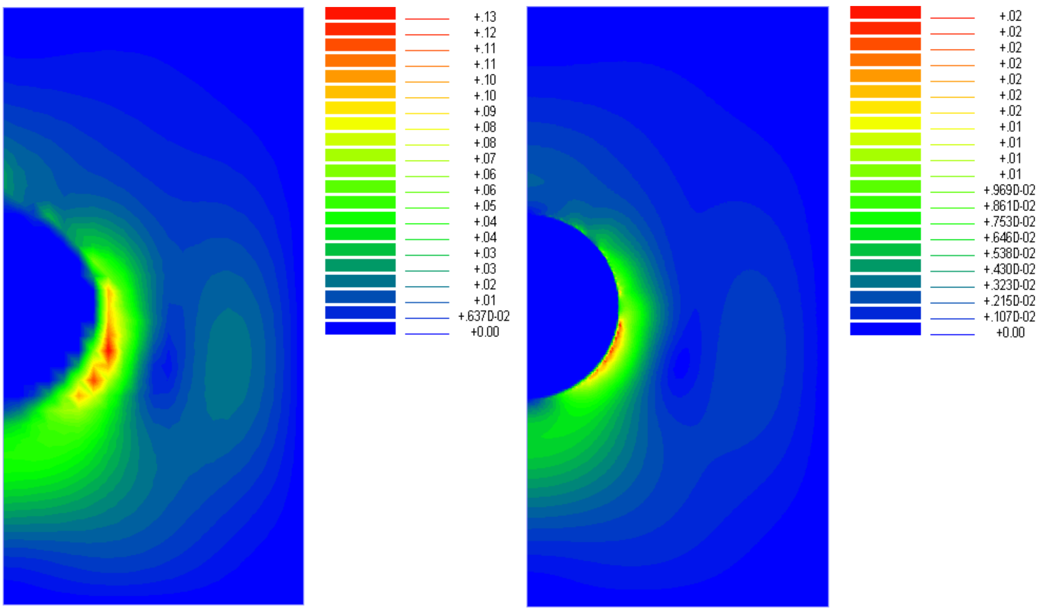

Figure 4.4: Absolute error for the velocity in the coarsest (left) and the finest mesh (right).

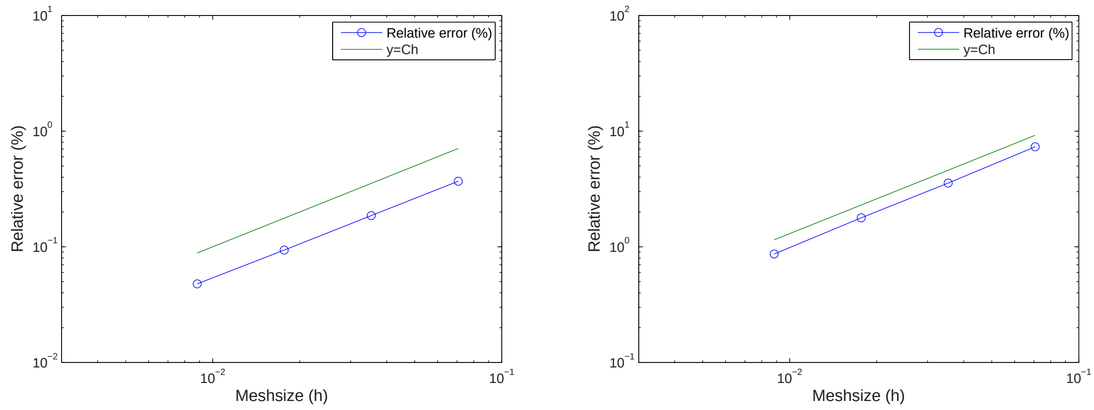

Figure 4.5: Error versus meshsize (log-log scale). Fields T (left) and $\mathbf{u}$ (right).

h	Δt	$L^\infty - L^2(T)$	$L^\infty - L^2(\mathbf{u})$
$h_0 = \sqrt{2}/20$	0.05	3.6851E-03	7.3276E-02
$h_0/2$	0.025	1.8607E-03	3.5593E-02
$h_0/4$	0.0125	9.3735E-04	1.7788E-02
$h_0/8$	0.00625	4.7883E-04	8.6747E-03

Table 4.1: Errors varying the meshsize and the time step.

4.2.2 Simulation of an industrial furnace.

As explained in Chapter 1, the motivation of this work is the simulation of a real furnace used for melting and stirring. A description of the induction furnace and a brief explanation of its design and behaviour have been given in the aforementioned chapter. Now we present a more detailed description of the furnace, considering several details that must be taken into account for the numerical simulation.

Geometry of the furnace.

The inductor of the furnace is a copper helical coil of 12 turns. The coil contains a pipe inside carrying cool water for refrigeration. The radius of the coil section is 16 mm, whereas the radius of the refrigeration tube is 14 mm, which gives a section for the conductor of 1.885e-4 m^2. The radius of curvature of the coil is 260 mm. Finally, the distance between the turns is 12 mm.

Inside the coil a crucible is placed, which contains the metal to be melted. The dimensions of the crucible and the amount of metal contained inside can be changed. For our simulations, we are considering that the height of the crucible is 480 mm and its exterior diameter is 200 mm. The thickness of the crucible is 35 mm in the sides, and 45 mm in the bottom. Moreover, the metal does not completely fills the crucible, but occupies a height of 410 mm.

The crucible is surrounded by an alumina layer to avoid heat losses. The reason to take alumina for this layer is that it is not only a good refractory but also a good electrical insulator. Thus the induction process in the crucible and in the metal is almost unaffected by the presence of alumina. For the same purpose of thermal insulation, there is also placed a lid over the load. For safety reasons, the induction coil is imbedded in the alumina layer. Above this alumina layer there is a layer of another refractory material, called Plibrico.

Finally, the induction furnace rests on a base of concrete, which we shall also consider in our computational domain. Moreover, for safety reasons there is a metal sheet surrounding the furnace, which prevents from high magnetic fields outside the furnace.

The computational domain.

For our numerical simulation we will neglect the presence of the lid above the load, since it would enforce us to solve an internal radiation problem, instead of imposing the convection-radiation boundary condition given in (3.87). The same holds for the external metal sheet, but in this case its influence on the results should be smaller, as it is not subjected to such high temperatures as the lid. However, the internal radiation problem seems that would give more realistic results and could be a possible improvement of the thermal model for future works.

The computational domain for the electromagnetic problem consists of the furnace (load, crucible, refractory materials and coil), the cooling water and the surrounding air, as it is shown in Figure 4.6. As already explained in Section 3.1, the helical coil is replaced by 12 rings to obtain an axisymmetrical geometry. We notice, however, that in Chapter 3 the coil was replaced by rings with toroidal geometry, *i.e.*, by solid tori. Since we are now considering a refrigeration tube along the coil, it would be replaced by hollow tori.

Remark 4.4. *Replacing the coil by hollow tori enforces us to introduce a new assumption in our model. Since the first Betti number of the hollow torus is equal to two, for each hollow torus the space of harmonic functions $\boldsymbol{\mathcal{H}}_\sigma(\Delta_k)$, defined as in (A.19), has dimension two. In this case, the 'cutting' surfaces can be chosen as a vertical surface Σ_1, perpendicular to $\mathbf{e}_\theta$, and a horizontal surface Σ_2, perpendicular to $\mathbf{e}_z$. With this choice the solutions to problem (A.20) are $\eta_1 = \theta/2\pi$ and $\eta_2 = \frac{1}{2\pi}\arcsin\left(\frac{z-z_0}{\left(r_0^2+r^2+2r_0r\sqrt{1-z_0^2/r^2}\right)^{1/2}}\right)$, with (r_0, z_0) the center of the hollow torus in the radial section. Thus, an orthonormal basis for $\boldsymbol{\mathcal{H}}_\sigma(\Delta_k)$ is formed by $\{\boldsymbol{\varrho}_{1,k}, \boldsymbol{\varrho}_{2,k}\}$, with $\boldsymbol{\varrho}_{1,k} = \frac{1}{2\pi r}\mathbf{e}_\theta$ and $\boldsymbol{\varrho}_{2,k} = \mathbf{grad}\,\eta_2$, which is a vector contained in the plane (r, z). Then, reasoning as in Section 3.2, we obtain that*

$$\mathrm{i}\omega\mathbf{A} + \mathbf{E} = -V_1\boldsymbol{\varrho}_{1,k} - V_2\boldsymbol{\varrho}_{2,k} \quad \text{in } \Delta_k, \quad k = 1, \ldots, m.$$

Hence, assuming the current density is of the form $\mathbf{J} = J_\theta(r, z)\mathbf{e}_\theta$ does not imply the same for $\mathbf{A}$, due to the presence of the harmonic function $\boldsymbol{\varrho}_{2,k}$. A further assumption we should do is that V_2 is equal to zero. This is reasonable, because a nonzero value for V_2 would be like imposing a voltage to generate a current around the refrigeration pipe.

The computational domain for the thermal problem is constituted by the metal, the crucible,

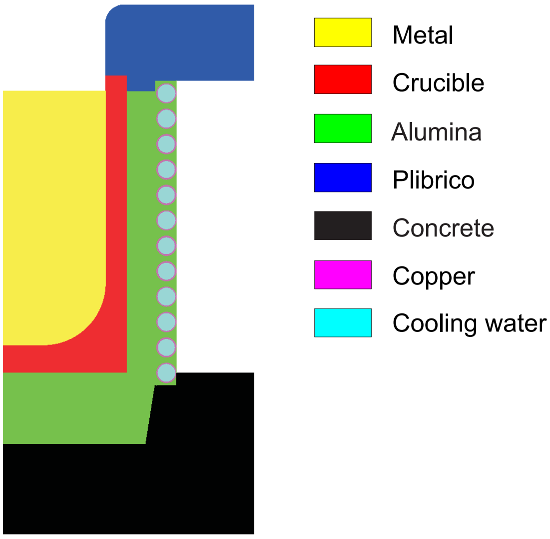

Figure 4.6: Computational domain and materials of the furnace.

the refractory layers and the coil. The computational domain for the hydrodynamic problem is the molten metal and it is computed at each time step, as it was described above.

Thermal boundary conditions.

As explained in Section 3.3.1, the boundary of the thermal domain is formed by the symmetry axis, the inner part of the hollow coil, which is water cooled, and the exterior boundaries of the furnace, which are in contact with air.

For the regions in contact with air we consider a convection-radiation boundary condition of the form

$$k\frac{\partial T}{\partial \mathbf{n}} = \alpha(T_c - T) + \gamma(T_r - T)^4\,.$$

We recall that γ is the product of the Stefan-Boltzmann constant by the emissivity. Thus, for the radiation boundary condition we must know the emissivity of each material and the external radiation temperature T_r. The emissivity is one of the physical properties provided by the company as it will be mentioned in the next section. The external radiation temperature is the temperature of the surfaces placed in front of the boundary. For the exterior boundaries, i.e, the boundaries on concrete, alumina and the external Plibrico, we are taking $T_r = 30°$, the ambient temperature. For the interior boundaries, *i.e.*, those corresponding to the metal, the crucible and the internal Plibrico, we cannot set T_r as the ambient temperature, since the surfaces in front of these boundaries are also in the interior of the furnace and they are heating up during the process. To deal with this problem, we are taking T_r as an averaged value of the temperature of these interior boundaries.

Concerning the convection boundary condition, α is the coefficient of convective heat transfer and T_c is the external convection temperature, *i.e.*, the temperature of the air surrounding the furnace. Again, we are taking this temperature equal to the ambient temperature for the exterior boundaries and as an averaged value of the temperature of the interior boundaries. The coefficient α determines the heat transfer rate between the boundary and the surrounding air. It depends on several parameters such as the properties of the surrounding fluid, its velocity or the geometry of the solid boundary. In general, the values of α are obtained by empirical formulas, which are often expressed in terms of the non-dimensional form of the following convective coefficient, called the Nusselt number:

$$\mathrm{Nu} = \frac{\alpha \mathcal{L}}{k_f}\,, \tag{4.51}$$

where k_f is the thermal conductivity of the fluid, and $\mathcal{L}$ is the characteristic dimension of the surface.

In order to give an expression for the coefficient α we will also need Grashof and Prandtl numbers, which were already introduced in Chapter 2:

$$\mathrm{Gr} = \frac{g\mathcal{L}^3\beta(T-T_c)}{\nu^2}\,, \qquad \mathrm{Pr} = \frac{\nu}{\zeta}\,, \tag{4.52}$$

where g is the magnitude of gravity acceleration, β is the coefficient of thermal expansion, ν is the kinematic viscosity and $\zeta = k_f/\rho c$ is the thermal diffusivity of the fluid. It is also useful to introduce the Rayleigh number:

$$\mathrm{Ra} = \mathrm{GrPr} = \frac{g\beta\mathcal{L}^3(T-T_c)}{\nu\zeta}\,.$$

The empirical values of the Nusselt number recommended in [36, p. 381] for free convection on a *vertical plane surface* are

$$\mathrm{Nu}_{\mathcal{L}} = 0.68 + 0.67\,\mathrm{Ra}^{1/4}\left[1+\left(\frac{0.492}{\mathrm{Pr}}\right)^{9/16}\right]^{-4/9}, \quad 0 < \mathrm{Ra} < 10^9$$

$$\mathrm{Nu}_{\mathcal{L}} = \left\{0.825 + 0.387\,\mathrm{Ra}^{1/6}\left[1+\left(\frac{0.492}{\mathrm{Pr}}\right)^{9/16}\right]^{-8/27}\right\}^2, \quad 10^9 < \mathrm{Ra} < 10^{12}\,.$$

where $\mathrm{Nu}_{\mathcal{L}}$ is an average value of Nu on the involved surface. The same values are recommended for *vertical cylinders*, as long as

$$\frac{\mathcal{D}}{\mathcal{L}} > \frac{35}{\mathrm{Gr}^{1/4}}\,,$$

where $\mathcal{D}$ is the diameter of the cylinder and $\mathcal{L}$ is its length. Similar values but with simpler formulas can also be found in [70].

For free convection past *horizontal plates*, in [36, p. 384] it is recommended to take, for a *heated plate facing up* (or a cooled plate facing down):

$$\begin{aligned} \mathrm{Nu}_{\mathcal{L}} &= 0.54(\mathrm{GrPr})^{1/4}, && \text{if } 10^5 \le \mathrm{GrPr} \le 2\,10^7 \\ \mathrm{Nu}_{\mathcal{L}} &= 0.15(\mathrm{GrPr})^{1/3}, && \text{if } 2\,10^7 \le \mathrm{GrPr} \le 3\,10^{10}\,. \end{aligned}$$

and for a *heated plate facing down* (or a cooled plate facing down):

$$\mathrm{Nu}_{\mathcal{L}} = 0.27(\mathrm{GrPr})^{1/4}, \text{ if } 3\,10^5 \leq \mathrm{GrPr} \leq 3\,10^{10}\,.$$

Then, from the previous expressions of the Nusselt number $\mathrm{Nu}_{\mathcal{L}}$ and using (4.51) and the thermal conductivity of air, one can easily compute the value of the coefficient α.

On the inner part of the coil, we consider a convection boundary condition

$$k\frac{\partial T}{\partial \mathbf{n}} = \alpha(T_w - T)\,,$$

T_w being the temperature of cooling water, which is determined as it was explained before. The coefficient of convective heat transfer, α, is again computed by an empirical formula. In this case we will make use of Reynolds number, which is defined by

$$\mathrm{Re} = \frac{v\mathcal{L}}{\nu}\,,$$

where v is the mean fluid velocity, $\mathcal{L}$ is the characteristic dimension of the pipe, and ν is the kinematic viscosity.

For turbulent heat transfer in a heated coil carrying water, in [56, 7-150] the following value has been recommended:

$$\mathrm{Nu} = 0.021\,\mathrm{Pr}^{0.4}\,\mathrm{Re}^{0.85}\left(\frac{r_w}{R}\right)^{0.1}\,,$$

where r_w is the pipe radius and R is the radius of curvature of the coil. The value of Reynolds number Re is easily computed from the radial section of the pipe r_w, the water mass flow rate Q and the physical properties of water. Then the value of the coefficient α is evaluated from the Nusselt number using expression (4.51) and the thermal conductivity of water.

Physical properties.

The physical properties necessary to perform the numerical simulation are the following:

- Electromagnetic properties: magnetic permeability and electrical conductivity.
- Thermal properties: mass density, specific heat, thermal conductivity, emissivity, latent heat and melting temperature of the metal.
- Hydrodynamic properties: dynamic viscosity.

The thermal and electromagnetic properties must be given for the materials appearing in the induction coil, the crucible, the refractory materials and the metal to be melted. Electromagnetic properties must also be given for air and water. The dynamic viscosity is only necessary for the molten metal. All these properties may depend on temperature.

We consider in all the materials a constant magnetic permeability equal to the vacuum magnetic permeability $\mu = \mu_0 = 4\pi 10^{-7}$ Hm^{-1}. The majority of the other physical properties have been provided by the company.

We will not give the values of the properties used for the simulation, except for the electrical conductivity, because it plays a major role in the results. We notice that the electrical conductivities of the crucible and of the material to be melted depend on temperature. This dependency is really important in the second case, because the load is assumed to be an insulator when solid and a good conductor when molten, as can be seen in Figure 4.7.

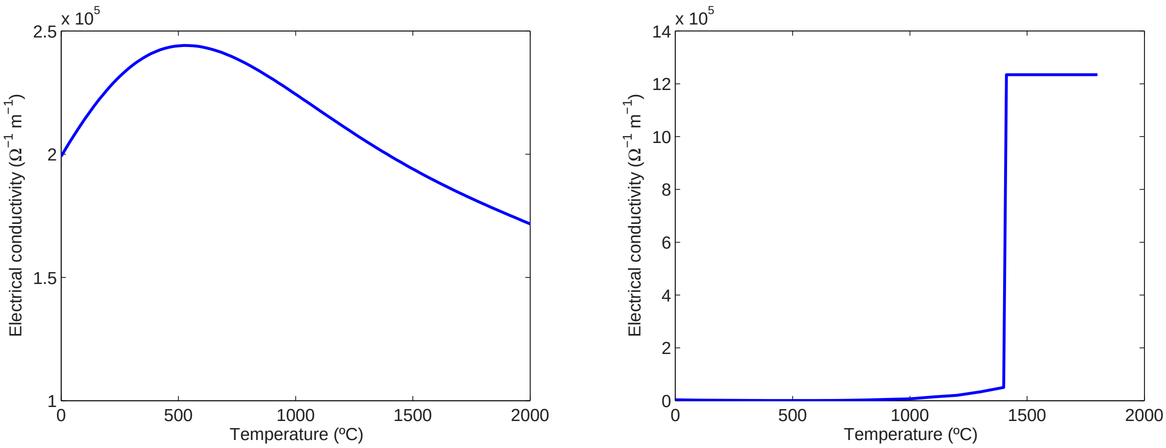

Figure 4.7: Electrical conductivity for the crucible (left) and the load (right).

Working parameters.

In the electromagnetic model that we have presented two working parameters have to be chosen: the first one is the frequency of the alternating current and the second one can be either the current intensity or the voltage. However, when trying to simulate the real furnace, the main data provided are the frequency, the intensity and the power supplied to the furnace. The given frequency can be easily transformed to the angular frequency of our model, but more care has to be taken for the intensity, as we explain below.

When introducing our mathematical model, we have defined the current intensity traversing a surface S as

$$I = \int_S \mathbf{J}(\mathbf{x}) \cdot \boldsymbol{\nu} \,,$$

with $\mathbf{J}$ the complex magnitude of the current density and $\boldsymbol{\nu}$ the normal vector to the surface S, as it was done in equation (3.9). We must notice that I is in fact the complex magnitude of a time harmonic intensity, which is defined from the time harmonic current density $\mathcal{J}$ in the form

$$\mathcal{I}(t) = \int_S \boldsymbol{\mathcal{J}}(\mathbf{x}) \cdot \boldsymbol{\nu} = \int_S \operatorname{Re}\left[e^{\mathrm{i}\omega t}\mathbf{J}(\mathbf{x})\right] \cdot \boldsymbol{\nu} = \operatorname{Re}\left(e^{\mathrm{i}\omega t} \int_S \mathbf{J}(\mathbf{x}) \cdot \boldsymbol{\nu}\right) = \operatorname{Re}\left[e^{\mathrm{i}\omega t} I\right]. \tag{4.53}$$

Actually, in the real furnace the known data is the RMS intensity, namely, the time average of the harmonic intensity during a cycle

$$I_{\mathrm{RMS}} = \left(\frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \mathcal{I}(t)^2\right)^{1/2} ,$$

	Frequency (Hz)	Power (kW)
Simulation 1	500	75
Simulation 2	2650	75

Table 4.2: Working parameters for each simulation.

and considering the expression of $\mathcal{I}$ given in (4.53), and reasoning as in Remark 3.6, we can infer that

$$I_{\mathrm{RMS}} = \frac{|I|}{\sqrt{2}},$$

where $|\cdot|$ denotes the modulus of a complex number. Therefore, from the known RMS intensity and the phase differences at the current entrance areas, we obtain the intensities I_k, $k = 1, \ldots, m$. For our furnace, since the rings represent the connected coil, the phase difference between the current densities will be zero.

In the real application, however, the given data is the power and the intensity is adjusted to obtain that power in the furnace. Moreover, since the electrical conductivity of the materials in the furnace varies with temperature, the intensity is dynamically adjusted during the process. To deal with this difficulty, the algorithm was slightly modified to provide the power as the known data and then to compute the intensity to attain the given power.

Numerical results and discussion.

We have performed two numerical simulations of the furnace with different values of the frequency but the same value for the power, to see how the frequency affects the heating and stirring of the metal. The working parameters we have used are presented in Table 4.2.

In Figure 4.8 we represent the temperature in the furnace for each simulation. As it can be seen the temperatures obtained in the furnace are very similar, but a little higher in the case of the highest frequency. We notice the strong influence of the refrigeration tubes in the temperature: the temperature in the copper coil and the surrounding refractory is about 50 °C, causing a very large gradient in the temperature within the refractory layer. In Figure 4.9 we show a detail of the temperatures in the crucible and in the load. As it can be seen, higher temperatures are reached when working with high frequency. This fact is explained due to the distribution of Ohmic losses, as we will explain below.

In Figure 4.10 we show the Joule effect in the load and the crucible after five minutes, when the metal is still solid, and in Figure 4.11 the same field is represented after three hours, when the metal has been melted. Comparing the results for different frequencies, we can see that for higher frequencies the maximum values of Joule effect are also higher, but due to the skin effect they are concentrated on the external wall of the crucible. Decreasing the frequency allows a better power distribution, at the cost of using higher intensities, thus causing larger power losses in the coil (see Figure 4.13).

Moreover, comparing the results when the load is solid and molten, we see how the high conductivity of the molten metal affects the performance of the furnace. At low frequency the

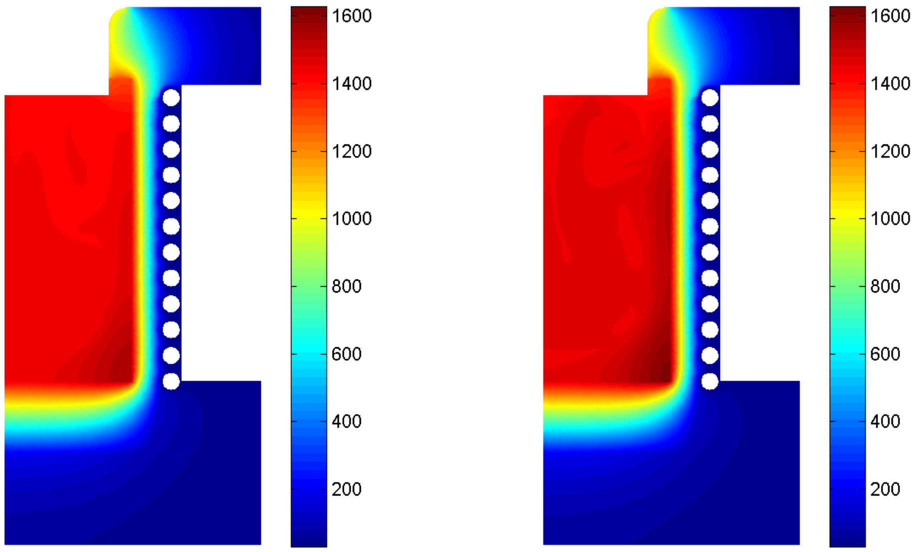

Figure 4.8: Temperature after three hours for Simulation 1 (left) and Simulation 2 (right).

Ohmic losses in the load become higher when the material melts, thus heating the load directly and reducing the crucible temperature (see Figure 4.13 for the Ohmic losses and Figure 4.9 for the crucible temperature). Moreover, the presence of molten material increases the skin effect on the crucible wall. On the contrary, when working with high frequency the power distribution in the furnace remains almost unaffected in the presence of molten material.

We also present in Figure 4.12 the velocity field for both frequencies. When working with low frequency the depth of penetration is higher, and Lorentz's force becomes stronger than buoyancy forces. At high frequency, instead, the low skin depth makes Lorentz's force almost negligible and buoyancy forces become dominant. This can be seen in the figures: at high frequency the molten metal is moving by natural convection, thus it tends to go up near the hot crucible, except in the upper part, probably due to the boundary condition we are imposing. At low frequency magnetic stirring enforces the metal to go down close to the crucible, and a new eddy comes up in the bottom of the furnace.

Finally, in Figure 4.13 we represent the variation in time of the Joule effect in each material, along with the heat losses through the refrigeration tubes. In order to attain the desired power, higher intensities are needed working at low frequency, which causes stronger Ohmic losses in the copper coil. Moreover, at low frequency the load begins to melt after 60 minutes, affecting the performance of the furnace and enforcing to increase the intensity, consequently increasing the power losses in the coil. When working with the high frequency the performance of the furnace is almost unaffected by the presence of molten material. It is also remarkable that at the first time steps the power losses in the coil match the heat losses through the water tubes. When time

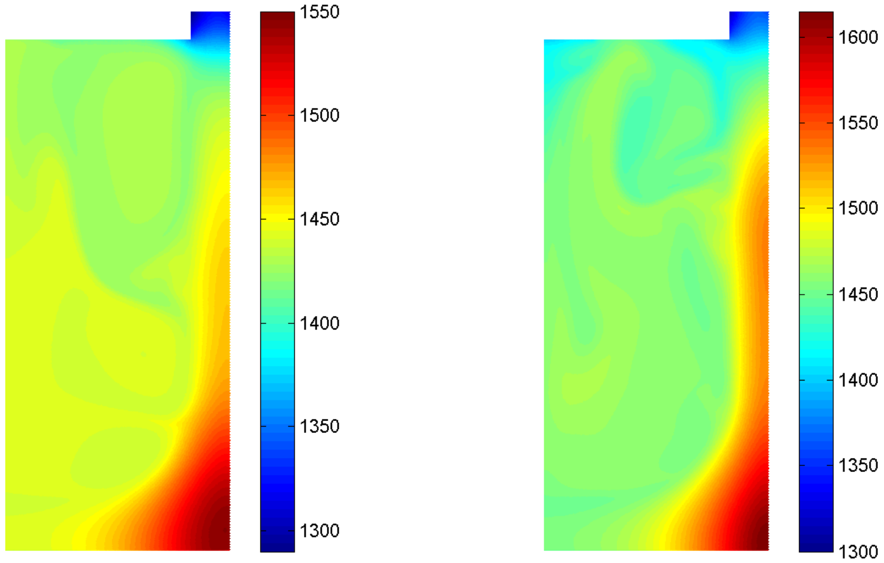

Figure 4.9: Metal temperature after three hours for Simulation 1 (left) and Simulation 2 (right).

increases the heat losses through the tubes become higher due to the heat conduction from the crucible across the refractory layer.

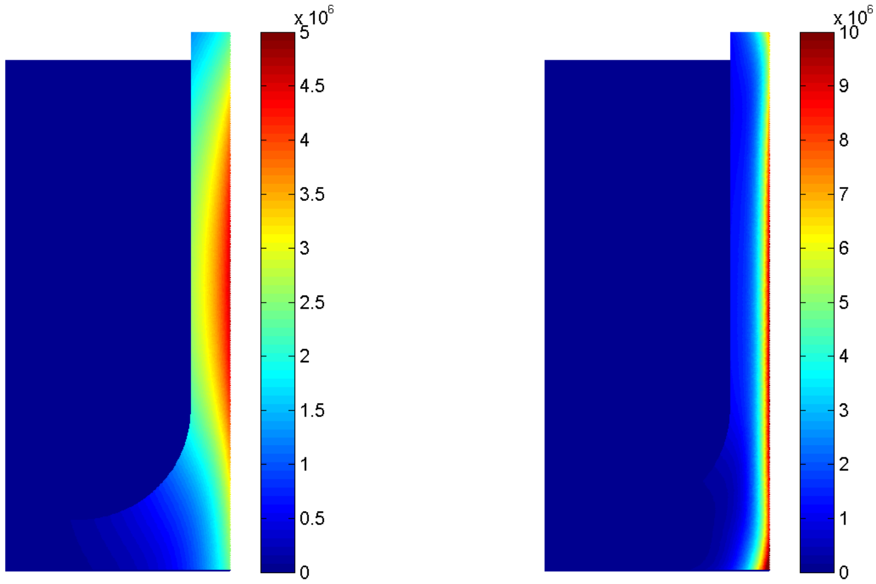

Figure 4.10: Joule effect after five minutes for Simulation 1 (left) and Simulation 2 (right).

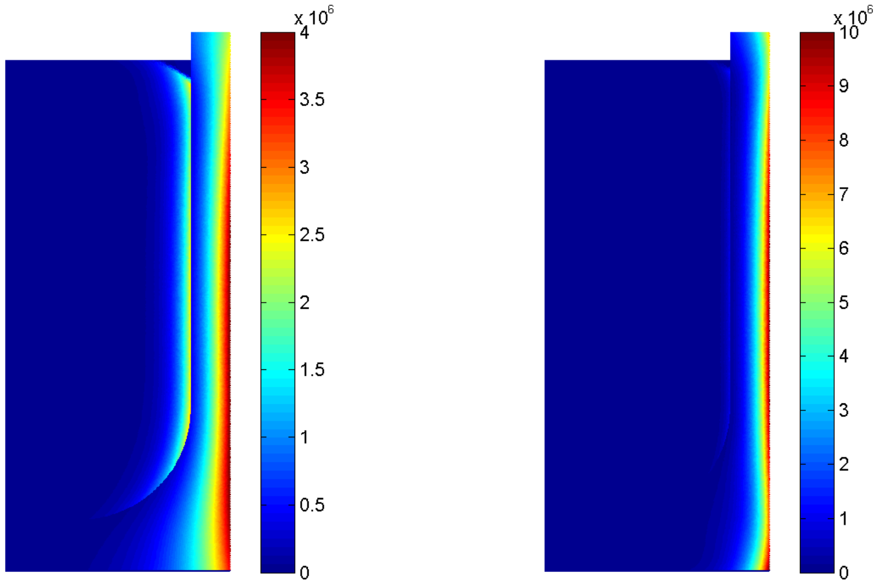

Figure 4.11: Joule effect after three hours for Simulation 1 (left) and Simulation 2 (right).

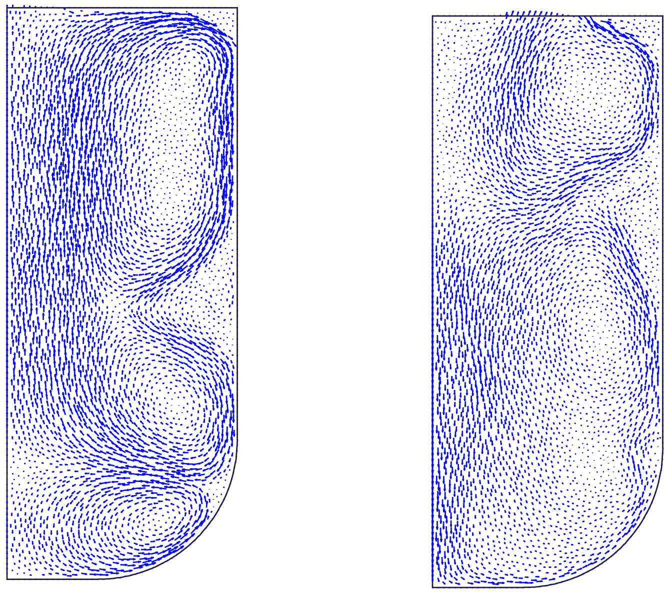

Figure 4.12: Velocity fields after three hours for Simulation 1 (left) and Simulation 2 (right).

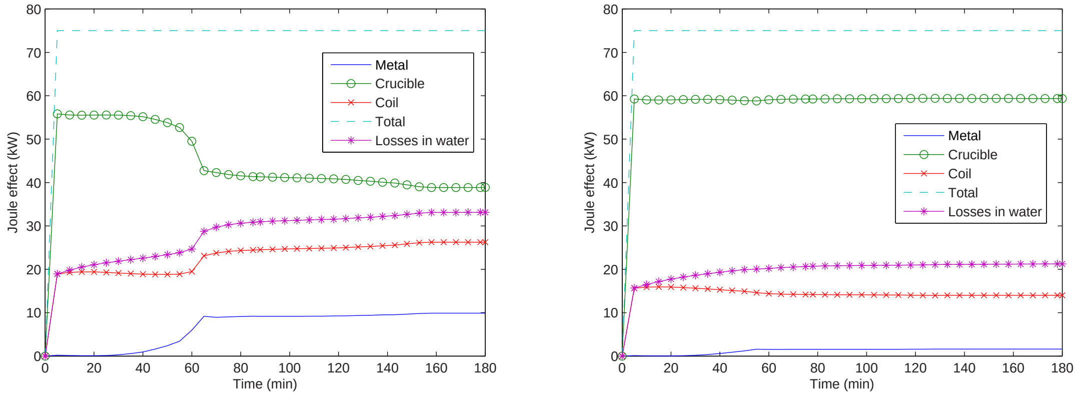

Figure 4.13: Joule effect and heat losses through the tubes, for Simulation 1 (left) and 2 (right).

Chapter 5

A formulation of the eddy current problem in the presence of electric ports.

In Chapter 3 and in the numerical results presented in Chapter 4 the helical coil of the induction furnace was replaced by toroidal rings, in order to achieve a formulation in an axisymmetrical setting. However, imposing the current intensities or the voltage drop in the axisymmetrical geometry, where the coil is replaced by rings, is not compatible with the eddy-currents model, and one of the equations (Ohm's law or Faraday's law) has to be violated, as we already mentioned in Remark 3.1.

In order to impose the current intensity or the voltage drop in the coil without violating any of the equations, one is constrained to maintain the helical geometry of the coil, hence to consider a three-dimensional problem. In this chapter we present a formulation and a finite element approximation of the eddy current model, that has been introduced in [5]. This is a suitable formulation when the full field equations are coupled with circuits. In the domain of the eddy current model, a part of the boundary acts as the interface with the circuit domain and either an intensity current or a voltage drop can be imposed (see, e.g., [27, 43, 59, 65]).

The outline of this chapter is as follows: in Section 1 we recall the eddy currents model and describe our domain. Section 2 is devoted to recall some notation and an orthogonal decomposition presented in Appendix A, that is a key point for the formulation of the problem in the insulator. In Section 3 we obtain the weak formulation of the voltage excitation problem and the current excitation problem. In Section 4 we prove the existence and uniqueness of solution to both problems. In Section 5, we introduce the finite element discretization and obtain the error estimates. Finally, in Section 6 we report some numerical results for two different problems: a test case with a known analytical solution and an application to a metallurgical furnace.

5.1 The eddy currents model.

Throughout this chapter the computational domain will be a simply-connected bounded open set $\Omega \subset \mathbb{R}^3$, with a connected and Lipschitz boundary $\partial\Omega$. It is split into two Lipschitz subdomains, a conducting region Ω_C and a non-conducting region $\Omega_D = \Omega \setminus \overline{\Omega}_C$; the latter is assumed to be non-empty and connected. The conducting region Ω_C is assumed to be simply-connected and not strictly contained in Ω, *i.e.*, $\partial\Omega \cap \partial\Omega_C \neq \emptyset$. (For a more general geometrical situation, see Section 5.3.1.) We shall denote the interface between the two regions by Γ and the different parts of the boundary $\partial\Omega$ by $\Gamma_C = \partial\Omega \cap \partial\Omega_C$ and $\Gamma_D = \partial\Omega \cap \partial\Omega_D$. Moreover, we will suppose that $\Gamma_C = \Gamma_E \cup \Gamma_J$, where Γ_E and Γ_J are two disjoint and connected surfaces on Γ_C ('electric ports'). Therefore, with these notations, we have $\partial\Omega_C = \Gamma_E \cup \Gamma_J \cup \Gamma$, $\partial\Omega_D = \Gamma_D \cup \Gamma$ (see Figure 5.1).

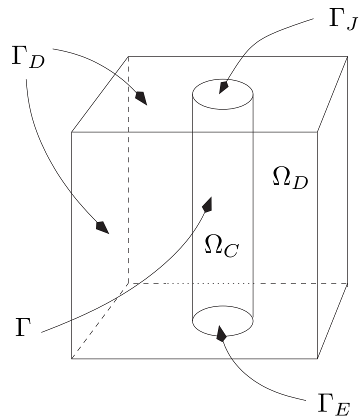

Figure 5.1: The computational domain.

The equations of the eddy-current problem consist of Faraday's law

$$\mathbf{curl}\,\mathbf{E} = -i\omega\boldsymbol{\mu}\mathbf{H} \quad \text{in } \Omega\,, \tag{5.1}$$

and Ampère's law

$$\mathbf{curl}\,\mathbf{H} = \boldsymbol{\sigma}\mathbf{E} \quad \text{in } \Omega\,, \tag{5.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and the magnetic field, respectively, and $\omega \neq 0$ is a given angular frequency. Throughout this chapter the magnetic permeability $\boldsymbol{\mu}$ is assumed to be a symmetric tensor, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$. Concerning the electrical conductivity $\boldsymbol{\sigma}$, the same assumption holds for $\boldsymbol{\sigma}_{|\Omega_C}$, while $\boldsymbol{\sigma}_{|\Omega_D} \equiv \mathbf{0}$ as Ω_D is a non-conducting medium. Equations (5.1)–(5.2) do not completely determine the electric field in Ω_D and it is necessary to demand the condition

$$\operatorname{div}(\boldsymbol{\varepsilon}\mathbf{E}_{|\Omega_D}) = 0, \tag{5.3}$$

where $\boldsymbol{\varepsilon}$ is the electric permittivity, assumed to be a symmetric tensor, uniformly positive definite in Ω_D, with entries belonging to $L^\infty(\Omega)$.

Concerning the boundary conditions, we want to model the electromagnetic fields in the case of an electric current passing along the 'cylinder' Ω_C and impose this electric current as a certain given intensity on Γ_J or as a potential difference between Γ_E and Γ_J. Thus, following [27] we impose the following boundary conditions

$$\begin{aligned} \boldsymbol{\mu}\mathbf{H}\cdot\mathbf{n} = 0 \quad &\text{on} \quad \partial\Omega, &(5.4)\\ \mathbf{E}_C\times\mathbf{n}_C = \mathbf{0} \quad &\text{on} \quad \Gamma_C = \Gamma_E\cup\Gamma_J, &(5.5)\\ \boldsymbol{\varepsilon}\mathbf{E}_D\cdot\mathbf{n}_D = 0 \quad &\text{on} \quad \Gamma_D, &(5.6)\end{aligned}$$

where $\mathbf{E}_S$ and $\mathbf{H}_S$ denote $\mathbf{E}_{|\Omega_S}$ and $\mathbf{H}_{|\Omega_S}$ respectively, $S = C, D$, and $\mathbf{n}_C$ and $\mathbf{n}_D$ denote the unit outward normal vectors to Ω_C and Ω_D, respectively. When considering the boundary of the whole domain Ω the unit outward normal vector is denoted by $\mathbf{n}$.

Moreover we impose either the current intensity traversing Γ_J

$$\int_{\Gamma_J} \mathbf{curl}\,\mathbf{H}_C\cdot\mathbf{n}_C = I, \tag{5.7}$$

or a potential difference. In this respect, the boundary condition (5.4) implies that the tangential component of $\mathbf{E}$ is a gradient. By formal calculations, if we integrate $\mathrm{i}\omega\boldsymbol{\mu}\mathbf{H}$ on any surface S contained in $\partial\Omega$, by using (5.1) and Stokes theorem, we obtain

$$0 = \mathrm{i}\omega\int_S \boldsymbol{\mu}\mathbf{H}\cdot\mathbf{n} = -\int_S \mathbf{curl}\,\mathbf{E}\cdot\mathbf{n} = -\int_{\partial S}\mathbf{E}\cdot\mathbf{t} = -\int_{\partial S}\mathbf{n}\times(\mathbf{E}\times\mathbf{n})\cdot\mathbf{t}\,,$$

with $\mathbf{t}$ a unit vector tangent to ∂S. Hence, since $\partial\Omega$ is simply connected there exists a surface potential v such that $\mathbf{E}\times\mathbf{n} = \mathbf{grad}\,v\times\mathbf{n}$ on $\partial\Omega$ (see [24]). Moreover, boundary condition (5.5) implies that v is constant on Γ_E and on Γ_J. Since v is defined up to a constant, we can take it equal to zero on Γ_E. The voltage $V\in\mathbb{C}$ will be the constant value on Γ_J of the surface electric potential v that is null on Γ_E:

$$\mathbf{E}\times\mathbf{n} = \mathbf{grad}\,v\times\mathbf{n} \quad \text{on } \partial\Omega \text{ with } v_{|\Gamma_J} = V \text{ and } v_{|\Gamma_E} = 0\,. \tag{5.8}$$

Remark 5.1. *The set of boundary conditions (5.4)–(5.6) allows us to assign either the current intensity or the voltage. This is not the case for other boundary conditions such as*

$$\mathbf{E}\times\mathbf{n} = \mathbf{0} \quad \textit{on } \partial\Omega \tag{5.9}$$

or

$$\begin{aligned} \mathbf{E}_C\times\mathbf{n}_C = \mathbf{0} \quad &\textit{on} \quad \Gamma_C = \Gamma_E\cup\Gamma_J\,,\\ \boldsymbol{\varepsilon}\mathbf{E}_D\cdot\mathbf{n}_D = 0 \quad &\textit{on} \quad \Gamma_D\,,\\ \mathbf{H}_D\times\mathbf{n}_D = \mathbf{0} \quad &\textit{on} \quad \Gamma_D\,.\end{aligned} \tag{5.10}$$

In fact, reasoning as in Remark 3.1 it can be proved that the unique solution of the eddy current problem (5.1)–(5.3) with boundary conditions (5.9) or (5.10) is the null solution (see also [9]). □

System (5.1)–(5.7), and its finite element approximation have been studied in [24]. The problem is formulated in terms of the magnetic field and the input current intensity is imposed by means of Lagrange multipliers. In [25] and in [75] the problem is described in terms of a current vector potential and a magnetic scalar potential, using the so-called $\mathbf{T}-\mathbf{T}_0-\phi$ formulation. We want also to mention the paper [23], where both problems of voltage and current excitation have been studied in terms of the electric field, but in a computational domain which reduces to the only conductor Ω_C.

In this work we deal with a new finite element approximation of system (5.1)–(5.6) either with assigned current intensity or assigned voltage. A weak formulation of the problem (5.1)–(5.6) is given considering as main unknowns the electric field in the conductor and the magnetic field in the insulator. The latter is decomposed as the sum of the gradient of a function in $H^1(\Omega_D)$ plus a harmonic field. When the input current intensity is given, this harmonic field is univocally determined, hence the unknowns of the problem reduce to the electric field in the conductor and a scalar magnetic potential in the insulator. On the other hand, when the voltage is given the unknowns of the problem are the electric field in the conductor, a scalar magnetic potential in the insulator and the current intensity. For the finite element approximation, the harmonic field is replaced by the generalized gradient of a piecewise linear function that has a jump of height 1 across a particular surface in Ω_D.

5.2 Notation and preliminaries.

The definition of all the spaces used in this chapter can be found in Appendix A, but in this section we recall some of them that have not been used before. First, we remind that $\mathbf{H}_{0,\Lambda}(\mathbf{curl}\,;\Omega)$ denotes the set of functions in $\mathbf{H}(\mathbf{curl};\Omega)$ with vanishing tangential traces on Λ, and in particular we will make use of $\mathbf{v}_C \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$, the set of functions in $\mathbf{H}(\mathbf{curl}\,;\Omega_C)$ such that their tangential traces are null on the electric ports $\Gamma_C = \Gamma_E \cup \Gamma_J$.

We also mention that the space of tangential traces for $\mathbf{H}(\mathbf{curl};\Omega)$ has been introduced in Section A.1.1, along with a Green's formula which was valid for functions in $\mathbf{H}(\mathbf{curl};\Omega)$. However, for the ease of reading we will express the duality pairings by (surface) integrals. Moreover, for any $\mathbf{v}_C \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$ and $\mathbf{w}_D \in \mathbf{H}(\mathbf{curl}\,;\Omega_D)$ we have

$$\int_\Gamma \mathbf{v}_C \times \mathbf{n}_C \cdot \overline{\mathbf{w}}_D := \int_{\Omega_C} (\mathbf{v}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}} - \mathbf{curl}\,\mathbf{v}_C \cdot \overline{\mathbf{w}})\,, \tag{5.11}$$

where $\mathbf{w}$ is any continuous extension of the trace of $\mathbf{w}_D$, defined on $\partial\Omega_D$, to $\mathbb{R}^3\backslash\overline{\Omega}_D$. We notice that the right hand side of (5.11) does not depend on the extension $\mathbf{w}$ considered, since, given any other extension, $\mathbf{w}_*$, we have $(\mathbf{w}-\mathbf{w}_*)_{|\Omega_C} \in \mathbf{H}_{0,\Gamma}(\mathbf{curl}\,;\Omega_C)$ and thus, using Proposition 3.5 in [45], we know that

$$\int_{\Omega_C} (\mathbf{v}_C \cdot \mathbf{curl}\,(\overline{\mathbf{w}-\mathbf{w}_*}) - \mathbf{curl}\,\mathbf{v}_C \cdot (\overline{\mathbf{w}-\mathbf{w}_*})) = 0\,.$$

In the following it will play a major role the space of Neumann harmonic fields, already introduced in Section A.2, and which is defined as

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{\mu}}(\Omega_D) := \{\mathbf{v}_D \in \mathbf{L}^2(\Omega_D) :\, \mathbf{curl}\,\mathbf{v}_D = \mathbf{0},\, \mathrm{div}\,(\boldsymbol{\mu}\mathbf{v}_D) = 0,\, \boldsymbol{\mu}\mathbf{v}_D \cdot \mathbf{n}_D = 0 \text{ on } \partial\Omega_D\}\,.$$

We recall that this space has finite dimension equal to $\beta_1(\Omega_D)$, the first Betti number of Ω_D. In our particular setting, since the conductor Ω_C is simply connected and 'touches' the boundary of the computational domain in the two contacts, we have $\beta_1(\Omega_D) = 1$. Moreover, the 'cutting' surface Σ, with boundary $\partial\Sigma \subset \partial\Omega_D$, is such that $\Omega_D \setminus \Sigma$ is simply connected (see Figure 5.2), and the basis function can be constructed as $\boldsymbol{\varrho}_D = \mathbf{grad}\,\eta$, where η is the solution (unique up to a constant) of the following problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}\,\mathbf{grad}\,\eta) = 0 & \text{in } \Omega_D \setminus \Sigma\,, \\ \boldsymbol{\mu}\,\mathbf{grad}\,\eta \cdot \mathbf{n}_D = 0 & \text{on } \partial\Omega_D \setminus \partial\Sigma\,, \\ [\eta]_\Sigma = 1\,, & \\ [\boldsymbol{\mu}\,\mathbf{grad}\,\eta \cdot \mathbf{n}_D]_\Sigma = 0\,. & \end{cases} \tag{5.12}$$

Furthermore, we can assume that $\boldsymbol{\varrho}_D$ is chosen such that $\int_{\partial\Gamma_J} \boldsymbol{\varrho}_D \cdot \mathbf{t} = 1$, where $\mathbf{t}$ is the tangential vector counterclockwise oriented with respect to $\mathbf{n}_C$ on Γ_J.

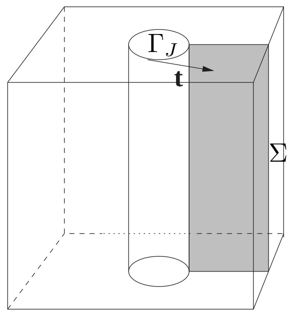

Figure 5.2: The cutting surface.

Finally, we recall that any vector function $\mathbf{v} \in \mathbf{L}^2(\Omega_D)$ such that $\mathbf{curl}\,\mathbf{v}_D = \mathbf{0}$ can be decomposed into the following sum:

$$\mathbf{v}_D = \mathbf{grad}\,\psi_D + \alpha\boldsymbol{\varrho}_D\,, \tag{5.13}$$

with $\psi_D \in H^1(\Omega_D)/\mathbb{C}$ and $\int_{\partial\Gamma_J} \mathbf{v}_D \cdot \mathbf{t} = \alpha$. Moreover the decomposition is $\mathbf{L}^2(\boldsymbol{\mu};\Omega_D)$-orthogonal, in the sense of (A.22).

5.3 Coupled $\mathbf{E}_C/\mathbf{H}_D$ formulation.

Our aim is to introduce and analyze a weak formulation of system (5.1)–(5.6) with assigned current intensity or voltage, where the main unknowns are the electric field in the conductor $\mathbf{E}_C$ and the magnetic field in the insulator $\mathbf{H}_D$. Since $\mathbf{curl}\,\mathbf{H}_D = \mathbf{0}$ we can write $\mathbf{H}_D = \mathbf{grad}\,\psi_D + K\boldsymbol{\varrho}_D$ with $\psi_D \in H^1(\Omega_D)$ and $K \in \mathbb{C}$.

Remark 5.2. *Notice that, by formal calculations, from Stokes Theorem we deduce*

$$I = \int_{\Gamma_J} \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n}_C = \int_{\partial\Gamma_J} \mathbf{H}_C \cdot \mathbf{t} = \int_{\partial\Gamma_J} \mathbf{H}_D \cdot \mathbf{t} = K\,.$$

This means that, when the current intensity is assigned, the main unknowns in our formulation are in fact $\mathbf{E}_C$ *and the magnetic scalar potential* ψ_D. □

Computing the magnetic field from Faraday's equation (5.1) and inserting it in Ampère's law (5.2), we obtain

$$\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C) + i\omega\boldsymbol{\sigma}\mathbf{E}_C = \mathbf{0}.$$

For each $\mathbf{w}_C \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$, by formal integration by parts one finds

$$\int_{\Omega_C} \boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C - \int_\Gamma \boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{w}}_C = 0\,.$$

From Faraday's equation and the matching condition

$$\mathbf{H}_C \times \mathbf{n}_C + \mathbf{H}_D \times \mathbf{n}_D = \mathbf{0} \quad \text{on } \Gamma$$

one has that

$$\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \times \mathbf{n}_C = i\omega\mathbf{H}_D \times \mathbf{n}_D \quad \text{on } \Gamma\,,$$

therefore,

$$\int_{\Omega_C} \boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C - i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{H}_D = 0\,. \tag{5.14}$$

On the other hand, multiplying Faraday's equation by a test function $\mathbf{v}_D = \mathbf{grad}\,\phi_D$ with $\phi_D \in H^1(\Omega_D)$, by integration by parts one has

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \mathbf{grad}\,\overline{\phi}_D = -\int_{\Omega_D} \mathbf{curl}\,\mathbf{E}_D \cdot \mathbf{grad}\,\overline{\phi}_D = \int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \mathbf{grad}\,\overline{\phi}_D\,.$$

From (5.1) and (5.4) we know that $\mathbf{curl}\,\mathbf{E}\cdot\mathbf{n} = 0$ on $\partial\Omega$, then by (A.16) we get $\mathrm{div}_\Gamma\,(\mathbf{E}\times\mathbf{n}) = 0$ on $\partial\Omega$. Denoting by ϕ any extension of ϕ_D in $H^1(\Omega)$, we have

$$\begin{aligned}
&\textstyle\int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \mathbf{grad}\,\overline{\phi}_D \\
&\quad = \textstyle\int_{\partial\Omega} \mathbf{E}\times\mathbf{n}\cdot\mathbf{grad}\,\overline{\phi} + \int_\Gamma \mathbf{E}_D\times\mathbf{n}_D\cdot\mathbf{grad}\,\overline{\phi}_D - \int_{\Gamma_C} \mathbf{E}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\overline{\phi} \\
&\quad = -\textstyle\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\overline{\phi}_D
\end{aligned}$$

because $\mathrm{div}_\Gamma\,(\mathbf{E}\times\mathbf{n}) = 0$ on $\partial\Omega$ and $\mathbf{E}_C\times\mathbf{n}_C = \mathbf{0}$ on Γ_C. Therefore

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \mathbf{grad}\,\overline{\phi}_D = -\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\phi}_D\,. \tag{5.15}$$

In a similar way, taking as test function $\boldsymbol{\varrho}_D$ one obtains

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_D = -\int_{\Omega_D} \mathbf{curl}\,\mathbf{E}_D \cdot \boldsymbol{\varrho}_D = \int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \boldsymbol{\varrho}_D\,.$$

Denoting by $\boldsymbol{\varrho}$ any extension of $\boldsymbol{\varrho}_D$ in $\mathbf{H}(\mathbf{curl}\,;\Omega)$, we have

$$\int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \boldsymbol{\varrho}_D = \int_{\partial\Omega} \mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho} + \int_\Gamma \mathbf{E}_D\times\mathbf{n}_D\cdot\boldsymbol{\varrho}_D\,,$$

and using that $\mathbf{E}\times\mathbf{n} = \mathbf{grad}\, v \times \mathbf{n}$ on $\partial\Omega$ we get

$$\int_{\partial\Omega} \mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho} = \int_{\partial\Omega} \mathbf{grad}\, v\times\mathbf{n}\cdot\boldsymbol{\varrho} = -\int_{\partial\Omega} \boldsymbol{\varrho}\times\mathbf{n}\cdot\mathbf{grad}\, v = \int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{\varrho}\cdot\mathbf{n}\, v\,.$$

Since $\mathbf{curl}\,\boldsymbol{\varrho} = \mathbf{0}$ in Ω_D, $v = V$ on Γ_J and $v = 0$ on Γ_E we obtain, using Stokes Theorem on Γ_J,

$$\int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{\varrho}\cdot\mathbf{n}\, v = V\int_{\Gamma_J} \mathbf{curl}\,\boldsymbol{\varrho}\cdot\mathbf{n}_C = V\int_{\partial\Gamma_J} \boldsymbol{\varrho}_D\cdot\mathbf{t} = V\,.$$

Hence

$$i\omega\int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D\cdot\boldsymbol{\varrho}_D = V - \int_{\Gamma} \mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D\,. \tag{5.16}$$

As we noticed before, $\mathbf{H}_D \in \mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ can be decomposed as $\mathbf{H}_D = \mathbf{grad}\,\psi_D + I\boldsymbol{\varrho}_D$ where $\psi_D \in H^1(\Omega_D)$ and $I \in \mathbb{C}$ is the current intensity. Moreover, as we have already remarked, this decomposition of $\mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ is $\mathbf{L}^2(\boldsymbol{\mu};\Omega_D)$-orthogonal in the sense that

$$\int_{\Omega_D} \boldsymbol{\mu}(\mathbf{grad}\,\varphi_D + K\boldsymbol{\varrho}_D)\cdot(\mathbf{grad}\,\overline{\phi}_D + \overline{Q}\boldsymbol{\varrho}_D) = \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\varphi_D\cdot\mathbf{grad}\,\overline{\phi}_D + K\overline{Q}\int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D$$

for all φ_D, $\phi_D \in H^1(\Omega_D)$ and K, $Q \in \mathbb{C}$. Hence, from (5.14), (5.15) and (5.16), multiplying these two last equations by $-i\omega$, we have that $\mathbf{E}_C$ and $\mathbf{H}_D = \mathbf{grad}\,\psi_D + I\boldsymbol{\varrho}_D$ are such that for each $\mathbf{w}_C \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$ and for each $\phi_D \in H^1(\Omega_D)$ and $Q \in \mathbb{C}$ it holds

$$\begin{aligned}
&\textstyle\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C)\\
&\qquad\qquad -i\omega\textstyle\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\psi_D - i\omega I\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D &&= 0\\
&-i\omega\textstyle\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\overline{\phi}_D + \omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathbf{grad}\,\psi_D\cdot\mathbf{grad}\,\overline{\phi}_D &&= 0\\
&-i\omega\overline{Q}\textstyle\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D + \omega^2 I\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D &&= -i\omega V\overline{Q}\,.
\end{aligned} \tag{5.17}$$

When the voltage V is given and the current intensity I is unknown, these three equations determine $\mathbf{E}_C$, ψ_D and I. On the other hand, when the current intensity I is given, the first two equations are enough to determine the two unknowns of the problem $\mathbf{E}_C$ and ψ_D. The voltage V can be computed using the third equation.

In conclusion we have the following formulations:

Voltage excitation problem

$$\left\{\begin{aligned}
&\text{Find } (\mathbf{E}_C,\psi_D,I) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\times\mathbb{C}\,:\\
&\textstyle\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C)\\
&\qquad\qquad -i\omega\textstyle\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\psi_D - i\omega I\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D &&= 0\\
&-i\omega\textstyle\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\overline{\phi}_D + \omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathbf{grad}\,\psi_D\cdot\mathbf{grad}\,\overline{\phi}_D &&= 0\\
&-i\omega\overline{Q}\textstyle\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D + \omega^2 I\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D &&= -i\omega V\overline{Q}\\
&\text{for all } (\mathbf{w}_C,\phi_D,Q) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\times\mathbb{C}\,.
\end{aligned}\right. \tag{5.18}$$

Current excitation problem

$$
\left\{
\begin{array}{l}
\text{Find } (\mathbf{E}_C, \psi_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,; \Omega_C) \times H^1(\Omega_D)/\mathbb{C}\ : \\[2ex]
\int_{\Omega_C} (\boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}}_C + i\omega \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{w}}_C) - i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \psi_D \\
\qquad\qquad\qquad\qquad\qquad\qquad = i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D \\[2ex]
-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\, \mathbf{grad}\, \psi_D \cdot \mathbf{grad}\, \overline{\phi}_D = 0 \\[2ex]
\text{for all } (\mathbf{w}_C, \phi_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,; \Omega_C) \times H^1(\Omega_D)/\mathbb{C}\,.
\end{array}
\right.
\tag{5.19}
$$

If $(\mathbf{E}_C, \psi_D)$ is the solution of the current excitation problem then the voltage can be computed as

$$
V = \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D + i\omega I \int_{\Omega_D} \boldsymbol{\mu}\, \boldsymbol{\varrho}_D \cdot \boldsymbol{\varrho}_D\,. \tag{5.20}
$$

5.3.1 Adaptation of the formulation to other geometrical settings.

These two formulations can be easily adapted to other cases with more complicated geometries. In what follows we will make the same hypotheses used in [11]: assume that $\beta_1(\Omega_D) = q$ and that there exist q mutually disjoint, orientable two-dimensional manifolds $\Sigma_j \subset \Omega_D$, $j = 1, \ldots, q$ such that $\partial\Sigma_j \subset \partial\Omega_D$ and the open set $\Omega_D \setminus \cup_{k=1}^q \Sigma_k$ is simply connected. These assumptions are valid in most of the geometrical settings one can find in realistic applications and, at the same time, we are preventing complicated topological settings in the conductor, such as knots or links.

First, let us consider a non-connected conductor Ω_C with two connected components, the first one denoted by $\Omega_{C,1}$ is a simply connected set with two electric ports $\partial\Omega \cap \partial\Omega_{C,1} = \Gamma_J \cup \Gamma_E$, as the one used in the previous section; and the second one, denoted by $\Omega_{C,2}$, is a torus shaped conductor strictly contained in Ω, *i.e.*, $\partial\Omega \cap \partial\Omega_{C,2} = \emptyset$, and consequently it has no ports (see Figure 5.3). In this case the space $\boldsymbol{\mathcal{H}}_{\boldsymbol{\mu}}(\Omega_D)$ has dimension $\beta_1(\Omega_D) = 2$ and there are two 'cutting' surfaces, Σ_1 and Σ_2. Denote by η_k, $k = 1, 2$, the functions in $H^1(\Omega_D \setminus \cup_{j=1}^2 \Sigma_j)$ with a jump of magnitude one on Σ_k, solution of the problem analogous to (A.20). Let us set $\boldsymbol{\varrho}_{D,k} = \widetilde{\mathbf{grad}}\, \eta_k$. Then $\{\boldsymbol{\varrho}_{D,1}, \boldsymbol{\varrho}_{D,2}\}$ is a basis of $\boldsymbol{\mathcal{H}}_{\boldsymbol{\mu}}(\Omega_D)$. Moreover, the non-bounding and homologically independent cycles γ_k, $k = 1, 2$, can be chosen such that $\gamma_k = \partial\Xi_{C,k}$, with $\Xi_{C,k}$ an orientable two-dimensional surface contained in $\overline{\Omega}_{C,k}$, and $\int_{\gamma_l} \boldsymbol{\varrho}_{D,k} \cdot \mathbf{t} = \delta_{kl}$, $k, l \in \{1, 2\}$.

In this case the $\mathbf{L}^2(\boldsymbol{\mu}; \Omega_D)-$orthogonal decomposition of $\mathbf{H}^0(\mathbf{curl}\,; \Omega)$ still holds. Thus the magnetic field can be decomposed as $\mathbf{H}_D = \mathbf{grad}\, \psi_D + \sum_{k=1}^2 I_k \boldsymbol{\varrho}_{D,k}$, with $\psi_D \in H^1(\Omega_D)/\mathbb{C}$ and

$$
I_k = \int_{\partial\Xi_{C,k}} \mathbf{curl}\, \mathbf{H}_C \cdot \boldsymbol{\nu}\,.
$$

where $\boldsymbol{\nu}$ is the unit vector normal to $\Xi_{C,k}$ such that $\mathbf{t}$ is counterclockwise oriented with respect to $\boldsymbol{\nu}$ on $\Xi_{C,k}$. Moreover, we have that $\mathbf{E} \times \mathbf{n} = \mathbf{grad}\, v \times \mathbf{n}$ on $\partial\Omega$, and v can be chosen such that

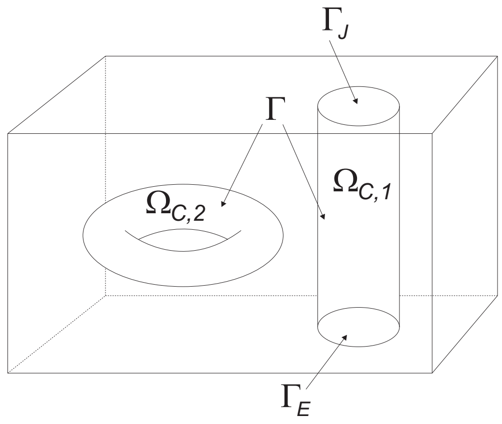

Figure 5.3: A conductor with two ports ($\Omega_{C,1}$) and an internal conductor ($\Omega_{C,2}$), with $\beta_1(\Omega_D) = 2$.

$v_{|\Gamma_J} = V$ and $v_{|\Gamma_E} = 0$. Multiplying Faraday's equation by function $\boldsymbol{\varrho}_{D,l}$ we obtain

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_{D,l} = V \int_{\partial\Gamma_J} \boldsymbol{\varrho}_{D,l} \cdot \mathbf{t} - \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,l}\,,$$

but now $\int_{\partial\Gamma_J} \boldsymbol{\varrho}_{D,2} \cdot \mathbf{t} = 0$. Thus $\mathbf{E}_C$ and $\mathbf{H}_D = \mathbf{grad}\,\psi_D + \sum_{k=1}^2 I_k \boldsymbol{\varrho}_{D,k}$ are such that, for each $\mathbf{w}_C \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$, $\phi_D \in H^1(\Omega_D)$ and $\mathbf{Q} \in \mathbb{C}^2$ it holds

$$\begin{aligned}
&\int_{\Omega_C} (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C) \\
&\qquad -i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\psi_D - i\omega \sum_{k=1}^2 I_k \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,k} = 0 \\
&-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\psi_D \cdot \mathbf{grad}\,\overline{\phi}_D = 0 \\
&-i\omega\overline{Q}_1 \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,1} + \omega^2\overline{Q}_1 \sum_{k=1}^2 I_k \int_{\Omega_D} \boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k} \cdot \boldsymbol{\varrho}_{D,1} \\
&\qquad\qquad\qquad\qquad = -i\omega\overline{Q}_1 V \int_{\partial\Gamma_J} \boldsymbol{\varrho}_{D,1} \cdot \mathbf{t}\,, \\
&-i\omega\overline{Q}_2 \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,2} + \omega^2\overline{Q}_2 \sum_{k=1}^2 I_k \int_{\Omega_D} \boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k} \cdot \boldsymbol{\varrho}_{D,2} = 0\,.
\end{aligned} \tag{5.21}$$

With this variational formulation one can impose the voltage drop in the conductor $\Omega_{C,1}$ and then the current intensities in the internal conductor $\Omega_{C,2}$ are computed as part of the solution. It is also possible to impose the current intensities in $\Omega_{C,1}$ and $\Omega_{C,2}$ by solving the first two equations of (5.21) and then to compute the voltage V from the third equation. However, one cannot affirm that the fourth equation in (5.21) is also satisfied, what would lead to a violation of the eddy-currents model, analogous to the one explained in Remark 3.1.

For the second geometrical setting, let us consider a connected but not simply-connected conductor Ω_C with two electric ports $\partial\Omega_C \cap \partial\Omega = \Gamma_E \cup \Gamma_J$, as the one shown in Figure 5.4. In this case the space $\boldsymbol{\mathcal{H}}_{\boldsymbol{\mu}}(\Omega_D)$ has dimension $q := \beta_1(\Omega_D) > 1$. We can construct a basis of this space as $\{\boldsymbol{\varrho}_{D,1}, \ldots, \boldsymbol{\varrho}_{D,q}\}$ with $\boldsymbol{\varrho}_{D,k} = \widetilde{\mathbf{grad}}\,\eta_k$, and η_k being the corresponding solution to problem (A.20). Once again, the non-bounding cycles $\gamma_1, \ldots, \gamma_q$ can be chosen such that $\gamma_k = \partial\Xi_{C,k}$, where $\Xi_{C,k}$ is an orientable two-dimensional surface contained in $\overline{\Omega}_C$, and $\int_{\gamma_l} \boldsymbol{\varrho}_{D,k} \cdot \mathbf{t} = \delta_{kl}$, $k, l \in \{1, \ldots, q\}$.

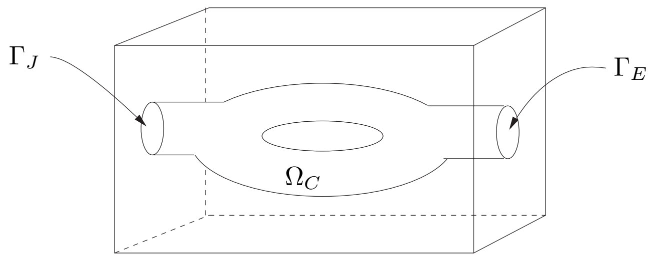

Figure 5.4: A non simply-connected conductor, with $\beta_1(\Omega_D) = 2$.

Since the $\mathbf{L}^2(\boldsymbol{\mu};\Omega_D)$-orthogonal decomposition of the space $\mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ is still valid, the magnetic field in the insulator can be univocally decomposed as $\mathbf{H}_D = \mathbf{grad}\,\psi_D + \sum_{k=1}^q I_k \boldsymbol{\varrho}_{D,k}$ and,

$$I_k = \int_{\Xi_{C,k}} \mathbf{curl}\,\mathbf{H}_C \cdot \boldsymbol{\nu}\,.$$

Multiplying Faraday's equation by the function $\boldsymbol{\varrho}_{D,l}$ and proceeding as in the case of a simply-connected conductor, we obtain

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_{D,l} = V \int_{\partial\Gamma_J} \boldsymbol{\varrho}_{D,l} \cdot \mathbf{t} - \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,l}\,.$$

Hence $\mathbf{E}_C$ and $\mathbf{H}_D = \mathbf{grad}\,\psi_D + \sum_{k=1}^q I_k \boldsymbol{\varrho}_{D,k}$ are such that for each $\mathbf{w}_C \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$, $\phi_D \in H^1(\Omega_D)/\mathbb{C}$ and $\mathbf{Q} \in \mathbb{C}^q$ it holds

$$\begin{aligned}
&\textstyle\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C)\\
&\qquad -i\omega \textstyle\int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\psi_D - i\omega \sum_{k=1}^q I_k \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,k} = 0\\
&-i\omega \textstyle\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\psi_D \cdot \mathbf{grad}\,\overline{\phi}_D = 0\\
&-i\omega\overline{Q}_l \textstyle\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,l} + \omega^2 \overline{Q}_l \sum_{k=1}^q I_k \int_{\Omega_D} \boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k} \cdot \boldsymbol{\varrho}_{D,l}\\
&\qquad\qquad\qquad = -i\omega\overline{Q}_l V \textstyle\int_{\partial\Gamma_J} \boldsymbol{\varrho}_{D,l} \cdot \mathbf{t}\,,\ \forall\, l = 1,\dots,q\,.
\end{aligned} \tag{5.22}$$

The third geometrical setting is a generalization of the previous case for a domain with several conductors with electric ports. Let us suppose that Ω_C is a non-connected set that has p connected components $\Omega_{C,j}$, $j = 1,\dots,p$, each one with two electric ports; then there are p different voltages V_j. In fact, on $\partial\Omega$ we have $\mathbf{E} \times \mathbf{n} = \mathbf{grad}\,v \times \mathbf{n}$, and, setting $\partial\Omega_{C,j} \cap \partial\Omega = \Gamma_{J,j} \cup \Gamma_{E,j}$, with $\Gamma_{J,j}$ and $\Gamma_{E,j}$ disjoint and connected surfaces, we have $v_{|\Gamma_{J,j}} = V_j^1$ and $v_{|\Gamma_{E,j}} = V_j^0$, where V_j^1 and V_j^0 are complex constants. Then the voltages are defined as $V_j = V_j^1 - V_j^0$.

Multiplying Faraday's equation by $\boldsymbol{\varrho}_{D,l}$, a basis function of the space $\boldsymbol{\mathcal{H}}_{\boldsymbol{\mu}}(\Omega_D)$, by a formal integration by parts one has

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_{D,l} = \int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \boldsymbol{\varrho}_{D,l} = \int_{\partial\Omega} \mathbf{E} \times \mathbf{n} \cdot \boldsymbol{\varrho}_l - \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_{D,l}\,,$$

where $\boldsymbol{\varrho}_l$ is any extension of $\boldsymbol{\varrho}_{D,l}$ in $\mathbf{H}(\mathbf{curl}\,;\Omega)$. Moreover

$$\begin{aligned}\int_{\partial\Omega}\mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho}_l=\int_{\partial\Omega}\mathbf{curl}\,\boldsymbol{\varrho}_l\cdot\mathbf{n}\,v &= \sum_{j=1}^p\Big(V_j^1\int_{\Gamma_{J,j}}\mathbf{curl}\,\boldsymbol{\varrho}_l\cdot\mathbf{n}_C+V_j^0\int_{\Gamma_{E,j}}\mathbf{curl}\,\boldsymbol{\varrho}_l\cdot\mathbf{n}_C\Big)\\ &= \sum_{j=1}^p\Big(V_j^1\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}+V_j^0\int_{\partial\Gamma_{E,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}\Big)\\ &= \sum_{j=1}^p V_j\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}\,,\end{aligned}$$

since, denoting by $\Gamma_j=\partial\Omega_{C,j}\setminus(\Gamma_{J,j}\cup\Gamma_{E,j})$, from Stokes Theorem we have

$$\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}+\int_{\partial\Gamma_{E,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}=\int_{\partial\Gamma_j}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}=\int_{\Gamma_j}\mathbf{curl}\,\boldsymbol{\varrho}_{D,l}\cdot\mathbf{n}_C=0\,.$$

Thus, the third equation in (5.22) becomes

$$-i\omega\overline{Q}_l\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,l}+\omega^2\overline{Q}_l\sum_{k=1}^q I_k\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k}\cdot\boldsymbol{\varrho}_{D,l}=-i\omega\overline{Q}_l\sum_{j=1}^p V_j\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}$$

for each $l=1,\dots,q$.

In the voltage excitation problem the p voltages are given, and therefore the unknowns of the problem are the electric field in the conductor, the function ψ_D appearing in the $\mathbf{L}^2(\boldsymbol{\mu};\Omega_D)$-orthogonal decomposition of $\mathbf{H}_D$ and the q intensities, whereas in the current intensity problem the q current intensities are given and the unknowns of the problem are the electric field in the conductor and the function ψ_D. The p voltages can then be computed in the following way: for each $j=1,\dots,p$, let $\boldsymbol{\varrho}_{D,l(j)}$ be a basis function of $\boldsymbol{\mathcal{H}}_{\boldsymbol{\mu}}(\Omega_D)$ corresponding to a non-bounding cycle $\gamma_{l(j)}=\partial\Gamma_{C,l(j)}$ such that $\Gamma_{C,l(j)}\subset\overline{\Omega}_{C,j}$. Then

$$V_j=\left(\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l(j)}\cdot\mathbf{t}\right)^{-1}\left(\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,l(j)}+i\omega\sum_{k=1}^q I_k\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k}\cdot\boldsymbol{\varrho}_{D,l(j)}\right),$$

and this value depends on j but not on the choice of $l(j)$.

Remark 5.3. *In order to carry out a simulation of our induction furnace, we recall its geometrical description, sketched in Figure 1.2. The conductor consists of two connected components: the inductor and the workpiece. The workpiece is a simply connected set formed by the crucible and the metal to be melted, whereas the inductor is the helical coil, which can be thought of as a simply connected set with two ports. Therefore, the geometrical setting for the furnace is similar to the one presented in Figure 5.3, with* $\Omega_{C,1}$ *representing the induction coil and* $\Omega_{C,2}$ *representing the workpiece, but in our case* $\Omega_{C,2}$ *being simply connected. Hence, the first Betti number of* Ω_D *is equal to one, and our formulation would be as (5.21) with* $\boldsymbol{\varrho}_{D,2}=0$. *In fact, this leads to formulation (5.18), but noticing that the conductor* Ω_C *and the interface* Γ *are not connected. For the sake of simplicity in the following we limit ourselves to the case of a connected and simply-connected conductor with two electric ports, i.e., the case depicted in Figure 5.1.*

5.4 Existence and uniqueness of the solution.

Let us define in $\mathbf{H}(\mathbf{curl}\,;\Omega_C)\times\mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ the sesquilinear form

$$\begin{aligned}\mathcal{A}\big((\mathbf{v}_C,\mathbf{u}_D),(\mathbf{w}_C,\mathbf{z}_D)\big):=&\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C+i\omega\boldsymbol{\sigma}\mathbf{v}_C\cdot\overline{\mathbf{w}}_C)\\ &+\omega^2\int_{\Omega_D}\boldsymbol{\mu}\mathbf{u}_D\cdot\overline{\mathbf{z}}_D-i\omega\Big[\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathbf{u}_D+\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\overline{\mathbf{z}}_D\Big]\,.\end{aligned}$$

and the antilinear functionals

$$\begin{array}{rcl} L_V(\mathbf{z}_D) & := & -i\omega c_0 V \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\varrho}_D \cdot \overline{\mathbf{z}}_D \\ L_I(\mathbf{w}_C) & := & i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D \,, \end{array}$$

where V and I are given complex constants and $c_0 = (\int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\varrho}_D \cdot \boldsymbol{\varrho}_D)^{-1}$. Recall that if $\mathbf{z}_D \in \mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ it can be univocally decomposed as $\mathbf{z}_D = \mathbf{grad}\,\phi_D + Q\boldsymbol{\varrho}_D$ with $\phi_D \in H^1(\Omega_D)/\mathbb{C}$ and $Q \in \mathbb{C}$. Then $L_V(\mathbf{z}_D) = -i\omega V\overline{Q}$.

It is easy to see, using the $\mathbf{L}^2(\boldsymbol{\mu};\Omega_D)$-orthogonal decomposition of $\mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ presented in (5.13), that problem (5.18) is equivalent to the following one

$$\begin{cases} \text{Find } (\mathbf{E}_C, \mathbf{H}_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times \mathbf{H}^0(\mathbf{curl}\,;\Omega_D) \,: \\ \mathcal{A}\big((\mathbf{E}_C, \mathbf{H}_D), (\mathbf{w}_C, \mathbf{z}_D)\big) = L_V(\mathbf{z}_D) \\ \text{for all } (\mathbf{w}_C, \mathbf{z}_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times \mathbf{H}^0(\mathbf{curl}\,;\Omega_D)\,, \end{cases} \tag{5.23}$$

whereas problem (5.19) is equivalent to

$$\begin{cases} \text{Find } (\mathbf{E}_C, \psi_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \,: \\ \mathcal{A}\big((\mathbf{E}_C, \mathbf{grad}\,\psi_D), (\mathbf{w}_C, \mathbf{grad}\,\phi_D)\big) = L_I(\mathbf{w}_C) \\ \text{for all } (\mathbf{w}_C, \phi_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C}\,. \end{cases} \tag{5.24}$$

The antilinear functionals $F(\mathbf{w}_C)$ and $L_I(\mathbf{w}_C)$ are clearly continuous in $\mathbf{H}(\mathbf{curl}\,;\Omega_C)$, whereas $L_V(\mathbf{z}_D)$ is continuous in $\mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$ (see (5.11)). Hence the existence and uniqueness of the solution to these two problems follows from Lax-Milgram lemma once we prove that the sesquilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive. This result of coerciveness has been proved in [7]. For the sake of completeness, we present the proof here below.

Proposition 5.1. *The sesquilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive on $\mathbf{H}(\mathbf{curl}\,;\Omega_C) \times \mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$*

Proof. We have

$$\begin{array}{rcl} |\mathcal{A}((\mathbf{w}_C, \mathbf{z}_D), (\mathbf{w}_C, \mathbf{z}_D))|^2 & = & (\int_{\Omega_C} \boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{w}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\mathbf{z}_D \cdot \overline{\mathbf{z}}_D)^2 \\ & & +\omega^2(\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re}\int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D)^2 \,. \end{array}$$

Taking into account that $\mathbf{curl}\,\mathbf{z}_D = \mathbf{0}$ in Ω_D, from the continuity estimate

$$2\left|\int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D\right| \leq k_0 \left(\int_{\Omega_D} |\mathbf{z}_D|^2\right)^{1/2} \left(\int_{\Omega_C} (|\mathbf{w}_C|^2 + |\mathbf{curl}\,\mathbf{w}_C|^2)\right)^{1/2}$$

and the inequality $(A+B)^2 \geq A^2/2 - B^2$ we find

$$
\begin{aligned}
&(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D)^2 \\
&\quad \geq \tfrac{1}{2}(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C)^2 - k_0^2 \left(\int_{\Omega_D} |\mathbf{z}_D|^2\right) \left(\int_{\Omega_C} (|\mathbf{w}_C|^2 + |\mathbf{curl}\, \mathbf{w}_C|^2)\right) \\
&\quad \geq \tfrac{1}{2}(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C)^2 - \delta^{-1} \tfrac{1}{2} k_0^2 (\int_{\Omega_D} |\mathbf{z}_D|^2)^2 \\
&\qquad\qquad - \delta k_0^2 \left(\int_{\Omega_C} |\mathbf{w}_C|^2\right)^2 - \delta k_0^2 \left(\int_{\Omega_C} |\mathbf{curl}\, \mathbf{w}_C|^2\right)^2 ,
\end{aligned}
$$

for each $\delta > 0$. Finally, for each $0 < \gamma \leq 1/2$ we also have

$$
\begin{aligned}
&\omega^2 \left(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D\right)^2 \\
&\quad \geq 2\gamma\omega^2 \left(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D\right)^2 ,
\end{aligned}
$$

so that

$$
\begin{aligned}
&|\mathcal{A}((\mathbf{w}_C, \mathbf{z}_D), (\mathbf{w}_C, \mathbf{z}_D))|^2 \geq (\nu_*^2 - 2\gamma\omega^2\delta k_0^2) \left(\int_{\Omega_C} |\mathbf{curl}\, \mathbf{w}_C|^2\right)^2 \\
&\quad + (\omega^4 \mu_*^2 - \gamma\omega^2\delta^{-1} k_0^2) \left(\int_{\Omega_D} |\mathbf{z}_D|^2\right)^2 + \gamma\omega^2(\sigma_*^2 - 2\delta k_0^2) \left(\int_{\Omega_C} |\mathbf{w}_C|^2\right)^2
\end{aligned}
$$

for some positive constants ν_*, μ_* and σ_*. The proof of the coerciveness of $\mathcal{A}(\cdot,\cdot)$ follows by taking at first δ small enough and then γ small enough. □

Once we have obtained $\mathbf{E}_C$ and $\mathbf{H}_D$, the magnetic field $\mathbf{H}_C$ can be obtained directly from Faraday's law by setting

$$
\mathbf{H}_C = (-i\omega\boldsymbol{\mu})^{-1} \mathbf{curl}\, \mathbf{E}_C ,
$$

while $\mathbf{E}_D$ is the solution to the following problem:

$$
\begin{cases}
\mathbf{curl}\, \mathbf{E}_D = -i\omega\boldsymbol{\mu}\mathbf{H}_D & \text{in } \Omega_D , \\
\mathrm{div}\, (\boldsymbol{\varepsilon}\mathbf{E}_D) = 0 & \text{in } \Omega_D , \\
\mathbf{E}_D \times \mathbf{n}_D = \mathbf{E}_C \times \mathbf{n}_D & \text{on } \Gamma , \\
\boldsymbol{\varepsilon}\mathbf{E}_D \cdot \mathbf{n}_D = 0 & \text{on } \Gamma_D .
\end{cases}
\tag{5.25}
$$

Proposition 5.2. *System (5.25) has a solution, and it is unique.*

Proof. Concerning the uniqueness, we notice that the space

$$
\boldsymbol{\mathcal{H}} := \{\mathbf{v}_D \in \mathbf{L}^2(\Omega_D) |\, \mathbf{curl}\, \mathbf{v}_D = \mathbf{0},\ \mathrm{div}\, (\boldsymbol{\varepsilon}\mathbf{v}_D) = 0,\ \boldsymbol{\varepsilon}\mathbf{v}_D \cdot \mathbf{n}_D = 0 \text{ on } \Gamma_D,\ \mathbf{v}_D \times \mathbf{n}_D = \mathbf{0} \text{ on } \Gamma\}
$$

is trivial in the considered geometrical situation. In fact, given $\mathbf{v}_D \in \boldsymbol{\mathcal{H}}$, one has $\mathbf{curl}\, \mathbf{v}_D = \mathbf{0}$ in $\Omega_D \backslash \Sigma$, that is a simply connected subset. Hence there exists $\psi_* \in H^1(\Omega_D \backslash \Sigma)$ such that $\mathbf{grad}\, \psi_* = \mathbf{v}_D$ and

$$\begin{cases} \operatorname{div}(\boldsymbol{\varepsilon}\,\mathbf{grad}\,\psi_*) = 0 & \text{in } \Omega_D\backslash\Sigma\,, \\ \boldsymbol{\varepsilon}\,\mathbf{grad}\,\psi_* \cdot \mathbf{n}_D = 0 & \text{on } \Gamma_D \setminus \partial\Sigma\,, \\ \psi_* = \kappa^* & \text{on } \Gamma \setminus \partial\Sigma\,, \\ [\psi_*]_\Sigma = c^*\,, \\ [\boldsymbol{\varepsilon}\,\mathbf{grad}\,\psi_* \cdot \mathbf{n}_D]_\Sigma = 0\,, \end{cases} \tag{5.26}$$

κ^* and c^* being constants. Since $\Gamma \cap \Sigma \neq \emptyset$ the constant c^* must be zero; therefore the unique solution of (5.26) is $\psi = \kappa^*$ and consequently $\mathbf{v}_D = \mathbf{0}$. The existence of the solution to (5.26) can be proved as in [3]. □

5.5 Finite element approximation.

The variational formulations (5.18) and (5.19) are not suitable for finite element numerical approximation. In fact, a conforming finite element approximation based directly on them requires that $\boldsymbol{\varrho}_D$ is explicitly known. An alternative approach, that overcomes this difficulty, is based on a different decomposition of $\mathbf{H}_D$.

Let $\boldsymbol{\lambda}_D$ be the generalized gradient of a function $\eta \in H^1(\Omega_D\backslash\Sigma)$ such that $[\eta]_\Sigma = 1$. Then $\mathbf{curl}\,\boldsymbol{\lambda}_D = \mathbf{0}$ and $\int_{\partial\Gamma_J} \boldsymbol{\lambda}_D \cdot \mathbf{t} = 1$, but in general $\boldsymbol{\lambda}_D \notin H(\operatorname{div};\Omega_D)$. From the geometrical assumptions on Ω_D we have that $\boldsymbol{\varrho}_D = \boldsymbol{\lambda}_D + \mathbf{grad}\, g^{\boldsymbol{\lambda}_D}$ for some $g^{\boldsymbol{\lambda}_D} \in H^1(\Omega_D)$. Hence $\mathbf{H}_D = \mathbf{grad}\,\psi_D + I\boldsymbol{\varrho}_D = \mathbf{grad}\,\psi_D + I(\boldsymbol{\lambda}_D + \mathbf{grad}\, g^{\boldsymbol{\lambda}_D}) = \mathbf{grad}\,\widehat{\psi}_D + I\boldsymbol{\lambda}_D$, with $\widehat{\psi}_D \in H^1(\Omega_D)$ that depends on the choice of $\boldsymbol{\lambda}_D$. This alternative decomposition is not $\mathbf{L}^2(\boldsymbol{\mu};\Omega_D)$-orthogonal and as a consequence some additional terms appear in the weak formulation. In fact the voltage excitation problem now reads

$$\begin{cases} \text{Find } (\mathbf{E}_C, \widehat{\psi}_D, I) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}\ : \\[1ex] \int_{\Omega_C} (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C) \\ \qquad\qquad -i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\widehat{\psi}_D - i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\lambda}_D = 0 \\[1ex] -i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\widehat{\psi}_D \cdot \mathbf{grad}\,\overline{\phi}_D + \omega^2 I \int_{\Omega_D} \boldsymbol{\mu}\,\boldsymbol{\lambda}_D \cdot \mathbf{grad}\,\overline{\phi}_D = 0 \\[1ex] -i\omega\overline{Q} \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\lambda}_D + \omega^2\overline{Q} \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\widehat{\psi}_D \cdot \boldsymbol{\lambda}_D + \omega^2 I\overline{Q} \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\lambda}_D \cdot \boldsymbol{\lambda}_D = -i\omega V\overline{Q} \\[1ex] \text{for all } (\mathbf{w}_C, \phi_D, Q) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}\,, \end{cases} \tag{5.27}$$

while the current excitation problem reads

$$\left\{\begin{array}{l}\text{Find } (\mathbf{E}_C, \widehat{\psi}_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \ : \\[2ex] \int_{\Omega_C} (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C) - i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\widehat{\psi}_D \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\lambda}_D \\[2ex] -i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\widehat{\psi}_D \cdot \mathbf{grad}\,\overline{\phi}_D = -\omega^2 I \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\lambda}_D \cdot \mathbf{grad}\,\overline{\phi}_D \\[2ex] \text{for all } (\mathbf{w}_C, \phi_D) \in \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C}\,. \end{array}\right. \tag{5.28}$$

Here below we present two different possible choices of $\boldsymbol{\lambda}_D$ in the framework of finite element approximation.

Let us now propose our finite element approximation schemes. We assume that Ω_C and Ω_D are Lipschitz polyhedral domains, and that $\{\mathcal{T}_h^C\}_h$ and $\{\mathcal{T}_h^D\}_h$ are two families of tetrahedral meshes of Ω_C and Ω_D respectively. We employ the Nédélec curl-conforming edge elements of degree k, $\mathbf{N}_{C,h}^k$, to approximate the functions in $\mathbf{H}(\mathbf{curl}\,;\Omega_C)$ and continuous nodal elements of degree k, $L_{D,h}^k$, to approximate the functions in $H^1(\Omega_D)$. Let us denote $\mathbf{W}_{C,h}^k := \mathbf{N}_{C,h}^k \cap \mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$.

We consider two different approaches. The first one is a conforming method where the function $\boldsymbol{\lambda}_D$ is chosen independently of the mesh, while in the second approach we consider a function $\boldsymbol{\lambda}_D$ which is mesh dependent.

Let us start from the first approach. Let us assume that the family $\{\mathcal{T}_h^D\}_h$ is obtained by refining a coarse mesh $\mathcal{T}_{h^*}^D$. Then we can choose a set of faces of tetrahedra in $\mathcal{T}_{h^*}^D$ such that the union is a 'cutting' surface $\Sigma \subset \Omega_D$. Let us denote η_D^* the piecewise linear function taking value 1 at the nodes on one side of Σ, say Σ^+, and 0 at all the other nodes including those on Σ^-, the other side of Σ. Then we choose $\boldsymbol{\lambda}_D := \widetilde{\mathbf{grad}}\,\eta_D^*$ (see [6]) that is independent of h.

The finite element approximation of the voltage excitation problem reads

$$\left\{\begin{array}{l}\text{Find } (\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}, I_h) \in \mathbf{W}_{C,h}^k \times L_{D,h}^k/\mathbb{C} \times \mathbb{C} \ : \\[2ex] \mathcal{C}((\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}, I_h), (\mathbf{w}_{C,h}, \phi_{D,h}, Q)) = -i\omega V \overline{Q} \\[2ex] \text{for all } (\mathbf{w}_{C,h}, \phi_{D,h}, Q) \in \mathbf{W}_{C,h}^k \times L_{D,h}^k/\mathbb{C} \times \mathbb{C}\,, \end{array}\right. \tag{5.29}$$

where $\mathcal{C}(\cdot,\cdot)$ is the sesquilinear form, defined in $\mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}$, associated to problem (5.27), namely,

$$\begin{array}{l}\mathcal{C}((\mathbf{v}_C, \varphi_D, K), (\mathbf{w}_C, \phi_D, Q) := \\ \quad \int_{\Omega_C} (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{v}_C \cdot \overline{\mathbf{w}}_C) \\ \qquad + \omega^2 \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\varphi_D \cdot \mathbf{grad}\,\overline{\phi}_D + \omega^2 K \overline{Q} \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\lambda}_D \cdot \boldsymbol{\lambda}_D \\ \quad -i\omega \big[\int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\varphi_D + \int_\Gamma \mathbf{v}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\phi}_D \big] \\ \quad -i\omega \big[K \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\lambda}_D + \overline{Q} \int_\Gamma \mathbf{v}_C \times \mathbf{n}_C \cdot \boldsymbol{\lambda}_D \big] \\ \quad + \omega^2 \big[K \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\overline{\phi}_D \cdot \boldsymbol{\lambda}_D + \overline{Q} \int_{\Omega_D} \boldsymbol{\mu}\,\mathbf{grad}\,\varphi_D \cdot \boldsymbol{\lambda}_D \big]\,. \end{array}$$

Analogously, the finite element approximation of the current excitation problem reads

$$\begin{cases} \text{Find } (\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C} : \\ \mathcal{A}((\mathbf{E}_{C,h}, \mathbf{grad}\, \widehat{\psi}_{D,h}), (\mathbf{w}_{C,h}, \mathbf{grad}\, \phi_{D,h})) \\ \qquad = i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\lambda}_D - \omega^2 I \int_{\Omega_D} \boldsymbol{\mu} \boldsymbol{\lambda}_D \cdot \mathbf{grad}\, \overline{\phi}_D \\ \text{for all } (\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C} . \end{cases} \tag{5.30}$$

From the coerciveness of $\mathcal{A}(\cdot,\cdot)$ it is easy to obtain the following result:

Proposition 5.3. *The sesquilinear form $\mathcal{C}(\cdot,\cdot)$ is coercive on $\mathbf{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}$.*

Proof. We notice that

$$\begin{aligned} \left|\mathcal{C}\big((\mathbf{w}_C, \phi_D, Q), (\mathbf{w}_C, \phi_D, Q)\big)\right| &= \left|\mathcal{A}\big((\mathbf{w}_C, \mathbf{grad}\, \phi_D + Q\boldsymbol{\lambda}_D), (\mathbf{w}_C, \mathbf{grad}\, \phi_D + Q\boldsymbol{\lambda}_D)\big)\right| \\ &\geq \alpha(\|\mathbf{w}_C\|^2_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{grad}\, \phi_D + Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3}) , \end{aligned}$$

since $\mathcal{A}(\cdot,\cdot)$ is coercive on $\mathbf{H}(\mathbf{curl}\,;\Omega_C) \times \mathbf{H}^0(\mathbf{curl}\,;\Omega_D)$. Moreover we know that $\boldsymbol{\varrho}_D = \boldsymbol{\lambda}_D + \mathbf{grad}\, g^{\boldsymbol{\lambda}_D}$, and we also have $\int_{\Omega_D} \boldsymbol{\mu}\, \mathbf{grad}\, \varphi_D \cdot \boldsymbol{\varrho}_D = 0$ for each $\varphi_D \in H^1(\Omega_D)$. Since from the assumptions on $\boldsymbol{\mu}$ there exist two positive constants μ_* and μ^* such that $\mu_* \|\mathbf{v}_D\|^2_{(L^2(\Omega_D))^3} \leq \int_{\Omega_D} \boldsymbol{\mu} \mathbf{v}_D \cdot \overline{\mathbf{v}}_D \leq \mu^* \|\mathbf{v}_D\|^2_{(L^2(\Omega_D))^3}$ for all $\mathbf{v}_D \in (L^2(\Omega_D))^3$, it follows that

$$\begin{aligned} &\|\mathbf{grad}\, \phi_D + Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3} = \|\mathbf{grad}\, (\phi_D - Qg^{\boldsymbol{\lambda}_D}) + Q\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3} \\ &\quad \geq \tfrac{1}{\mu^*} \int_{\Omega_D} \boldsymbol{\mu} \left[\mathbf{grad}\, (\phi_D - Qg^{\boldsymbol{\lambda}_D}) + Q\boldsymbol{\varrho}_D\right] \cdot \left[\mathbf{grad}\, (\overline{\phi}_D - \overline{Q}g^{\boldsymbol{\lambda}_D}) + \overline{Q}\boldsymbol{\varrho}_D\right] \\ &\quad = \tfrac{1}{\mu^*} \left(\int_{\Omega_D} \boldsymbol{\mu}\, \mathbf{grad}\, (\phi_D - Qg^{\boldsymbol{\lambda}_D}) \cdot \mathbf{grad}\, (\overline{\phi}_D - \overline{Q}g^{\boldsymbol{\lambda}_D}) + |Q|^2 \int_{\Omega_D} \boldsymbol{\mu} \boldsymbol{\varrho}_D \cdot \boldsymbol{\varrho}_D\right) \\ &\quad \geq \tfrac{\mu_*}{\mu^*} \big(\|\mathbf{grad}\, (\phi_D - Qg^{\boldsymbol{\lambda}_D})\|^2_{(L^2(\Omega_D))^3} + |Q|^2 \|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3}\big) . \end{aligned}$$

Using that $\|f+g\|^2_{(L^2(\Omega_D))^3} \geq (1-\delta)\|f\|^2_{(L^2(\Omega_D))^3} + (1-\frac{1}{\delta})\|g\|^2_{(L^2(\Omega_D))^3}$ for each $\delta > 0$, we obtain

$$\begin{aligned} &\|\mathbf{grad}\, \phi_D + Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3} \\ &\quad \geq \tfrac{\mu_*}{\mu^*}(1-\delta)\|\mathbf{grad}\, \phi_D\|^2_{(L^2(\Omega_D))^3} + \tfrac{\mu_*}{\mu^*}(1-\tfrac{1}{\delta})|Q|^2 \|\mathbf{grad}\, g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3} + \tfrac{\mu_*}{\mu^*}|Q|^2 \|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3} \\ &\quad = \tfrac{\mu_*}{\mu^*}(1-\delta)\|\mathbf{grad}\, \phi_D\|^2_{(L^2(\Omega_D))^3} + \tfrac{\mu_*}{\mu^*}\big[(1-\tfrac{1}{\delta})\|\mathbf{grad}\, g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3} + \|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3}\big]|Q|^2 . \end{aligned}$$

Choosing δ such that

$$\frac{\|\mathbf{grad}\, g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3}}{\|\mathbf{grad}\, g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3} + \|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3}} < \delta < 1 ,$$

we have for some positive constant C

$$\|\mathbf{grad}\, \phi_D + Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3} \geq C(\|\mathbf{grad}\, \phi_D\|^2_{(L^2(\Omega_D))^3} + |Q|^2) ,$$

so the coerciveness of $\mathcal{C}(\cdot,\cdot)$ follows. □

The optimality of the discrete solution of both problems is a consequence of Cea's Lemma: for the voltage excitation problem we have

$$\begin{aligned}&\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{grad}\,\widehat{\psi}_D-\mathbf{grad}\,\widehat{\psi}_{D,h}\|_{(L^2(\Omega_D))^3}+|I-I_h|\\&\quad\le C\inf_{(\mathbf{w}_{C,h},\phi_{D,h})\in\mathbf{W}^k_{C,h}\times L^k_{D,h}}\left(\|\mathbf{E}_C-\mathbf{w}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{grad}\,\widehat{\psi}_D-\mathbf{grad}\,\phi_{D,h}\|_{(L^2(\Omega_D))^3}\right),\end{aligned}$$

and for the current excitation problem we find

$$\begin{aligned}&\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{grad}\,\widehat{\psi}_D-\mathbf{grad}\,\widehat{\psi}_{D,h}\|_{(L^2(\Omega_D))^3}\\&\quad\le C\inf_{(\mathbf{w}_{C,h},\phi_{D,h})\in\mathbf{W}^k_{C,h}\times L^k_{D,h}}\left(\|\mathbf{E}_C-\mathbf{w}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{grad}\,\widehat{\psi}_D-\mathbf{grad}\,\phi_{D,h}\|_{(L^2(\Omega_D))^3}\right).\end{aligned}$$

Therefore, by standard density results, we obtain the convergence of the approximation for both problems. As usual, the precise order of convergence is related to the regularity of the solution $(\mathbf{E}_C,\psi_D)$.

In the second approach the function $\boldsymbol{\lambda}_D$ depends on h because it is the generalized gradient of a piecewise linear function $\eta_{D,h}$ with a jump of magnitude one on a discrete 'cutting' surface Σ_h that depends on the mesh $\{\mathcal{T}_h^D\}_h$. This choice will be denoted by $\boldsymbol{\lambda}_D=\boldsymbol{\lambda}_D^h$. Notice that now we are not assuming that $\{\mathcal{T}_h^D\}_h$ is obtained by refining $\mathcal{T}_{h^*}^D$. This approach is similar to the one analyzed in [24] for the current excitation problem.

The sesquilinear form associated to Problem (5.27) now depends on h

$$\begin{aligned}&\mathcal{C}_h((\mathbf{v}_C,\varphi_D,K),(\mathbf{w}_C,\phi_D,Q)):=\\&\quad\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C+i\omega\boldsymbol{\sigma}\mathbf{v}_C\cdot\overline{\mathbf{w}}_C)\\&\qquad+\omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathbf{grad}\,\varphi_D\cdot\mathbf{grad}\,\overline{\phi}_D+\omega^2K\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\lambda}_D^h\cdot\boldsymbol{\lambda}_D^h\\&\quad-i\omega\left[\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\varphi_D+\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\mathbf{grad}\,\overline{\phi}_D\right]\\&\quad-i\omega\left[K\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D^h+\overline{Q}\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D^h\right]\\&\quad+\omega^2\left[K\int_{\Omega_D}\boldsymbol{\mu}\,\mathbf{grad}\,\overline{\phi}_D\cdot\boldsymbol{\lambda}_D^h+\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\mathbf{grad}\,\varphi_D\cdot\boldsymbol{\lambda}_D^h\right].\end{aligned}$$

However $\mathcal{C}_h((\mathbf{v}_C,\varphi_D,K),(\mathbf{w}_C,\phi_D,Q))=\mathcal{A}((\mathbf{v}_C,\mathbf{grad}\,\varphi_D+K\boldsymbol{\lambda}_D^h),(\mathbf{w}_C,\mathbf{grad}\,\phi_D+Q\boldsymbol{\lambda}_D^h))$. Hence the finite element approximation of the voltage excitation problem with this second approach reads

$$\begin{cases}\text{Find }(\mathbf{E}_{C,h},\widehat{\psi}_{D,h},I_h)\in\mathbf{W}^k_{C,h}\times L^k_{D,h}/\mathbb{C}\times\mathbb{C}\;:\\[2ex]\mathcal{A}((\mathbf{E}_{C,h},\mathbf{grad}\,\widehat{\psi}_{D,h}+I_h\boldsymbol{\lambda}_D^h),(\mathbf{w}_{C,h},\mathbf{grad}\,\phi_{D,h}+Q\boldsymbol{\lambda}_D^h))=-i\omega V\overline{Q}\\[2ex]\text{for all }(\mathbf{w}_{C,h},\phi_{D,h},Q)\in\mathbf{W}^k_{C,h}\times L^k_{D,h}/\mathbb{C}\times\mathbb{C}\,.\end{cases}\tag{5.31}$$

Let us consider now the error estimate. Let us set $\mathbf{H}_{D,h}:=\mathbf{grad}\,\widehat{\psi}_{D,h}+I_h\boldsymbol{\lambda}_D^h\in\mathbf{H}^0(\mathbf{curl};\Omega_D)$. From (5.27) and (5.31), we have the following equation for the error:

$$\mathcal{A}((\mathbf{E}_C-\mathbf{E}_{C,h},\mathbf{H}_D-\mathbf{H}_{D,h}),(\mathbf{w}_{C,h},\mathbf{grad}\,\phi_{D,h}+Q\boldsymbol{\lambda}_D^h))=0$$

for all $\mathbf{w}_{C,h}\in\mathbf{W}^k_{C,h}$, $\phi_{D,h}\in L^k_{D,h}$ and $Q\in\mathbb{C}$. Hence

$$\begin{aligned}&\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{H}_D-\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\\&=\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{H}_D-\mathbf{grad}\,\widehat{\psi}_{D,h}-I_h\boldsymbol{\lambda}_D^h\|_{(L^2(\Omega_D))^3}\\&\quad\le C\inf_{(\mathbf{w}_{C,h},\mathbf{z}_{D,h})\in\mathbf{W}^k_{C,h}\times\mathbf{Z}^k_{D,h}}\left(\|\mathbf{E}_C-\mathbf{w}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}+\|\mathbf{H}_D-\mathbf{z}_{D,h}\|_{(L^2(\Omega_D))^3}\right),\end{aligned}$$

where

$$\mathbf{Z}^k_{D,h} := \mathbf{grad}\, L^k_{D,h} \oplus \text{span}\,\{\boldsymbol{\lambda}^h_D\}\,.$$

An error estimate for the intensity is obtained by noticing that, from (5.13),

$$\int_{\Omega_D} \boldsymbol{\mu}(\mathbf{H}_D - \mathbf{H}_{D,h}) \cdot \boldsymbol{\varrho}_D = (I - I_h) \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\varrho}_D \cdot \boldsymbol{\varrho}_D\,.$$

Hence

$$|I - I_h| \leq C \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\,,$$

where $C = \frac{\mu^*}{\mu_*} \|\boldsymbol{\varrho}_D\|^{-1}_{(L^2(\Omega_D))^3}$.

Remark 5.4. *It is worth noting that a suitable choice of the discrete function* $\mathbf{z}_{D,h}$ *is easily performed. In fact, let us denote by* $\mathbf{N}^k_{D,h}$ *the space of Nédélec curl-conforming edge elements of degree* k *in* $\mathcal{T}^D_h$*, and* $\boldsymbol{\Pi}_{D,h}$ *the interpolation operator. If* $\mathbf{H}_D$ *is so regular that* $\boldsymbol{\Pi}_{D,h}\mathbf{H}_D$ *is well defined, then* $\boldsymbol{\Pi}_{D,h}\mathbf{H}_D \in \mathbf{Z}^k_{D,h}$*. In fact,* $\mathbf{curl}\,(\boldsymbol{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D) = \mathbf{0}$ *and* $\int_{\partial\Gamma_J} (\boldsymbol{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D) \cdot \mathbf{t} = 0$*. Consequently* $\boldsymbol{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D = \mathbf{grad}\,\varphi_D$ *for some* $\varphi_D \in H^1(\Omega_D)$*. Since* $\boldsymbol{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D \in \mathbf{N}^k_{D,h}$*, from Lemma 5.3, Chapter III in [53],* $\varphi_{D|K}$ *is a polynomial of degree* k *for each* $K \in \mathcal{T}_{D,h}$*, therefore* $\varphi_D \in L^k_{D,h}$*.*

As a consequence, from standard interpolation estimates, for a regular solution $(\mathbf{E}_C, \mathbf{H}_D)$ *it is straightforward to specify the order of convergence of the approximation method.*

If one has no information about the regularity of the solution, by a density argument it is possible to prove the convergence of the finite element scheme provided that the permeability coefficient $\boldsymbol{\mu}$ *is regular enough in* Ω_D *(say, a constant as in the usual physical case) or if the family of meshes* $\{\mathcal{T}^D_h\}_h$ *is obtained by refining a coarse mesh* $\mathcal{T}^D_{h^*}$*.*

In fact, when $\boldsymbol{\mu}$ *is constant, we know that the harmonic field* $\boldsymbol{\varrho}_D$ *is regular enough to define the interpolation* $\boldsymbol{\Pi}_{D,h}\boldsymbol{\varrho}_D$ *(see [11]). Since* $\mathbf{H}_D = \mathbf{grad}\,\psi_D + I\boldsymbol{\varrho}_D$*, a density argument applied to* ψ_D *permits to conclude the proof. In the other case, first we note that, as seen in Proposition 5.3, we can write* $\boldsymbol{\varrho}_D = \mathbf{grad}\, g^{\boldsymbol{\lambda}_D} + \boldsymbol{\lambda}_D$*. Then, knowing that* $\{\mathcal{T}^D_h\}_h$ *is a refinement of* $\mathcal{T}^D_{h^*}$*, it follows* $\boldsymbol{\lambda}_D \in \mathbf{N}^k_{D,h}$*. Therefore, as proved above, since* $\mathbf{curl}\,\boldsymbol{\lambda}_D = \mathbf{0}$ *in* Ω_D *we have* $\boldsymbol{\lambda}_D = \boldsymbol{\Pi}_{D,h}\boldsymbol{\lambda}_D \in \mathbf{Z}^k_{D,h}$*, and a density argument for* $\psi_D + g^{\boldsymbol{\lambda}_D}$ *gives the result.* □

For the current excitation problem the finite element approach reads

$$\left\{\begin{array}{l} \text{Find } (\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C}\ : \\[2ex] \mathcal{A}((\mathbf{E}_{C,h}, \mathbf{grad}\,\widehat{\psi}_{D,h}), (\mathbf{w}_{C,h}, \mathbf{grad}\,\phi_{D,h})) \\ \qquad\qquad = i\omega I \int_\Gamma \overline{\mathbf{w}}_{C,h} \times \mathbf{n}_C \cdot \boldsymbol{\lambda}^h_D - \omega^2 I \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\lambda}^h_D \cdot \mathbf{grad}\,\overline{\phi}_{D,h} \\[2ex] \text{for all } (\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C}\,. \end{array}\right. \tag{5.32}$$

Recall that $\mathbf{H}_D = \mathbf{grad}\,\widehat{\psi}_D + I\boldsymbol{\lambda}^h_D$ for some $\widehat{\psi}_D \in H^1(\Omega_D)$ (that in fact depends on h). Setting $\mathbf{H}_{D,h} = \mathbf{grad}\,\widehat{\psi}_{D,h} + I\boldsymbol{\lambda}^h_D$ from (5.28) and (5.32) we have the following equation for the error

$$\begin{aligned} &\mathcal{A}((\mathbf{E}_C - \mathbf{E}_{C,h}, \mathbf{H}_D - \mathbf{H}_{D,h}), (\mathbf{w}_{C,h}, \mathbf{grad}\,\phi_{D,h})) \\ &\quad = \mathcal{A}((\mathbf{E}_C - \mathbf{E}_{C,h}, \mathbf{grad}\,\widehat{\psi}_D - \mathbf{grad}\,\widehat{\psi}_{D,h}), (\mathbf{w}_{C,h}, \mathbf{grad}\,\phi_{D,h})) = 0 \end{aligned}$$

for each $(\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C}$. Therefore, the coerciveness of $\mathcal{A}(\cdot,\cdot)$ gives

$$\begin{aligned}
&\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3} \\
&\quad = \|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{grad}\,\widehat{\psi}_D - \mathbf{grad}\,\widehat{\psi}_{D,h}\|_{(L^2(\Omega_D))^3} \\
&\quad \leq C\big(\|\mathbf{E}_C - \mathbf{w}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{grad}\,\widehat{\psi}_D - \mathbf{grad}\,\phi_{D,h}\|_{(L^2(\Omega_D))^3}\big)
\end{aligned}$$

for each $(\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}$. Therefore

$$\begin{aligned}
&\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3} \\
&\quad \leq C \inf_{(\mathbf{w}_{C,h},\mathbf{z}_{D,h})\in\mathbf{W}^k_{C,h}\times\mathbf{Z}^k_{D,h}(I)} \left(\|\mathbf{E}_C - \mathbf{w}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{z}_{D,h}\|_{(L^2(\Omega_D))^3}\right),
\end{aligned}$$

where

$$\mathbf{Z}^k_{D,h}(I) := \mathbf{grad}\, L^k_{D,h} + I\boldsymbol{\lambda}^h_D\,.$$

The convergence of the approximation scheme can be proved following the arguments presented in Remark 5.4 (the only difference is that now we work with the space $\mathbf{Z}^k_{D,h}(I)$ instead of $\mathbf{Z}^k_{D,h}$, and this fact gives no problem to the procedure).

Once we have obtained $\mathbf{E}_{C,h}$ and $\widehat{\psi}_{D,h}$ we can compute

$$V_h := \int_\Gamma \mathbf{E}_{C,h} \times \mathbf{n}_C \cdot \boldsymbol{\lambda}^h_D + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_{D,h} \cdot \boldsymbol{\lambda}^h_D\,.$$

This quantity is an approximation of the voltage, that, from (5.16), can be written as

$$V := \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_D\,.$$

In fact, let us introduce the auxiliary quantity

$$\widehat{V}_h := \int_\Gamma \mathbf{E}_{C,h} \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_{D,h} \cdot \boldsymbol{\varrho}_D\,.$$

We easily have

$$|V - \widehat{V}_h| \leq C_1(\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3})\,.$$

On the other hand, taking $\mathbf{w}_{C,h} = \mathbf{0}$ in (5.32), it is easy to see that

$$V_h = \int_\Gamma \mathbf{E}_{C,h} \times \mathbf{n}_C\,(\mathbf{grad}\,\phi_{D,h} + \boldsymbol{\lambda}^h_D) + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_{D,h} \cdot (\mathbf{grad}\,\phi_{D,h} + \boldsymbol{\lambda}^h_D)$$

for all $\phi_{D,h} \in L^k_{D,h}$. Thus

$$|\widehat{V}_h - V_h| \leq C_2\left(\|\mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right)\|\boldsymbol{\varrho}_D - (\mathbf{grad}\,\phi_{D,h} + \boldsymbol{\lambda}^h_D)\|_{(L^2(\Omega_D))^3}\,,$$

for all $\phi_{D,h} \in L^k_{D,h}$. Therefore

$$\begin{aligned}
|V - V_h| &\leq C_1\left(\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right) \\
&\quad + C_2\left(\|\mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right)\inf_{z_{D,h}\in\mathbf{Z}^k_{D,h}(1)}\|\boldsymbol{\varrho}_D - z_{D,h}\|_{(L^2(\Omega_D))^3} \\
&\leq \left(C_1 + C_2 \inf_{z_{D,h}\in\mathbf{Z}^k_{D,h}(1)}\|\boldsymbol{\varrho}_D - z_{D,h}\|_{(L^2(\Omega_D))^3}\right) \\
&\qquad\qquad \times\left(\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right) \\
&\quad + C_2\left(\|\mathbf{E}_C\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D\|_{(L^2(\Omega_D))^3}\right)\inf_{z_{D,h}\in\mathbf{Z}^k_{D,h}(1)}\|\boldsymbol{\varrho}_D - z_{D,h}\|_{(L^2(\Omega_D))^3}\,.
\end{aligned}$$

If the permeability coefficient $\boldsymbol{\mu}$ is constant in Ω_D or if the family of meshes $\{\mathcal{T}_h^D\}_h$ is a refinement of a coarse mesh $\mathcal{T}_{h^*}^D$, the convergence can be proved as in Remark 5.4,

5.6 Numerical results.

The finite element method presented above has been implemented in MATLAB, using Nédélec edge elements of first order for the electric field in the conductor and scalar Lagrangian P_1 elements for the magnetic potential in the insulator.

The method has been tested by solving a problem with a known analytical solution. Since this problem has been already presented in [22], we just give a brief description of it and refer the reader to the quoted paper for details.

The conducting domain Ω_C and the whole domain Ω are two coaxial cylinders of radii R_C and R_D, respectively, with height L. An alternating current of intensity $\mathbb{I}(t) = I\cos(\omega t)$ is traversing the conductor in the axial direction. Supposing that the physical parameters $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are constant scalars, the solution of the problem in cylindrical coordinates is given by

$$\begin{aligned}
\mathbf{E}_C(r,\theta,z) &= \frac{I\gamma}{2\pi R_C\boldsymbol{\sigma}}\frac{\mathcal{I}_0(\gamma r)}{\mathcal{I}_1(\gamma R_C)}\mathbf{e}_z \quad \text{in } \Omega_C,\\
\mathbf{H}_C(r,\theta,z) &= \frac{I}{2\pi R_C}\frac{\mathcal{I}_1(\gamma r)}{\mathcal{I}_1(\gamma R_C)}\mathbf{e}_\theta \quad \text{in } \Omega_C,\\
\mathbf{H}_D(r,\theta,z) &= \frac{I}{2\pi r}\mathbf{e}_\theta \quad \text{in } \Omega_D,
\end{aligned}$$

where $\mathcal{I}_0$ and $\mathcal{I}_1$ denote the modified Bessel functions of the first kind and order 0 and 1, respectively, and $\gamma = \sqrt{i\omega\boldsymbol{\mu}\boldsymbol{\sigma}}$. Moreover, for this particular geometry it holds $\boldsymbol{\varrho}_D = \frac{1}{2\pi r}\mathbf{e}_\theta$, so $\mathbf{H}_D = I\boldsymbol{\varrho}_D$.

Once the fields and the function $\boldsymbol{\varrho}_D$ are known, the value of V is computed from the expression (5.20) to obtain

$$V = \frac{\gamma L I}{2\pi\boldsymbol{\sigma}R_C}\frac{\mathcal{I}_0(\gamma R_C)}{\mathcal{I}_1(\gamma R_C)} + i\omega\boldsymbol{\mu}\frac{LI}{2\pi}\ln\left(\frac{R_D}{R_C}\right).$$

For our particular case we have used the following geometry data

$$\begin{aligned}
R_C &= 0.25 \text{ m},\\
R_D &= 0.5 \text{ m},\\
L &= 0.25 \text{ m}.
\end{aligned}$$

For the electromagnetic properties we have considered the values

$$\begin{aligned}
\boldsymbol{\sigma} &= 151565.8\ (\Omega\text{m})^{-1},\\
\boldsymbol{\mu} &= 4\pi 10^{-7}\ \text{Hm}^{-1},\\
\omega &= 50\times 2\pi \text{ rad/s},
\end{aligned}$$

and either assigned current intensity or voltage,

$$I = 10^4 \text{ A}, \quad \text{or} \quad V = 0.08979 + 0.14680i,$$

where the value of V has been analytically computed for an intensity of 10^4 A.

Remark 5.5. *We notice that the value of the voltage V depends on the size of the subdomain Ω_D. This fact is consistent with the expression of the energy conservation equation. To obtain this equation we consider Ampére's law, multiply it by the conjugate of $\mathbf{E}$ and integrate in Ω to get*

$$\int_\Omega \mathbf{curl}\,\mathbf{H} \cdot \overline{\mathbf{E}} = \int_\Omega \boldsymbol{\sigma}\mathbf{E} \cdot \overline{\mathbf{E}}\,, \tag{5.33}$$

then, using a Green's formula, and considering Faraday's law we have

$$-\mathrm{i}\omega \int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_\Omega \boldsymbol{\sigma}\mathbf{E} \cdot \overline{\mathbf{E}} = \int_{\partial\Omega} \overline{\mathbf{E}} \times \mathbf{n} \cdot \mathbf{H}\,. \tag{5.34}$$

Since $\mathbf{E}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ_C and $\mathbf{H}_D = \mathbf{grad}\,\psi_D + I\boldsymbol{\varrho}_D$, reasoning as in Section 5.3 we arrive at the energy conservation equation

$$\mathrm{i}\omega \int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{E}}_C = V\overline{I}\,, \tag{5.35}$$

where the first integral represents the magnetic power and the second one gives the power of Ohmic losses. After an easy computation it can be seen that the equation holds for the analytical solution presented above, and since the value of the integral $\int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}}$ varies with the size of the domain Ω, the voltage V depends also on it.

To test the order of convergence, the problem has been solved in four successively refined meshes, for either assigned intensity or voltage. We notice that the only approximation implemented in our program is that in which the function λ_D^h depends on the mesh, namely, problems (5.31) and (5.32). We present in Tables 5.1 and 5.2 the relative errors of our numerical solutions against the analytical one, that have been set as follows:

$$e_E = \frac{\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)}}{\|\mathbf{E}_C\|_{\mathbf{H}(\mathbf{curl}\,;\Omega_C)}}, \qquad e_V = \frac{|V - V_h|}{|V|}$$
$$e_H = \frac{\|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}}{\|\mathbf{H}_D\|_{(L^2(\Omega_D))^3}}, \qquad e_I = \frac{|I - I_h|}{|I|}.$$

Finally, Figures 5.5 and 5.6 show the plots in a log-log scale of the relative errors versus the degrees of freedom. A linear dependence on the mesh size is obtained for the errors of electric and magnetic fields, either for assigned intensity or voltage. This dependence turns out to be more than linear for the errors of voltage and intensity.

The method has been also applied to a more realistic problem which was presented in [24]. In this case the domain is a cylindrical electric furnace with three electrodes equally distanced. The dimensions of the furnace are the following: furnace height: 2 m.; furnace diameter: 8.88 m.;

Elements	DoF	e_E	e_H	e_V
2304	1684	0.2341	0.1693	0.0312
18432	11240	0.1132	0.0847	0.0089
62208	35580	0.0750	0.0567	0.0048
147456	81616	0.0561	0.0425	0.0018

Table 5.1: Relative errors for assigned intensity.

Elements	DoF	e_E	e_H	e_I
2304	1685	0.2336	0.1685	0.0274
18432	11241	0.1132	0.0847	0.0085
62208	35581	0.0750	0.0566	0.0041
147456	81617	0.0561	0.0425	0.0024

Table 5.2: Relative errors for assigned voltage.

electrodes height: 1.25 m.; electrodes diameter: 1 m.; distance from the center of the electrodes to the wall: 3 m.

The three electrodes inside the furnace are formed by a graphite core of 0.4 m. of diameter, and an outer part of Söderberg paste. The electric current enters the electrodes through horizontal copper bars of rectangular section (0.07 m. $\times$ 0.25 m.), connecting the top of the electrode with the external boundary.

For the simulation we have considered the angular frequency $\omega = 50 \times 2\pi$ rad/s, and the electric conductivities $\sigma = 10^6(\Omega\text{m})^{-1}$ for graphite, $\sigma = 10^4(\Omega\text{m})^{-1}$ for Söderberg paste, and $\sigma = 5 \times 10^6(\Omega\text{m})^{-1}$ for copper. The electrodes are supplied with one-phase current of intensity $7 \times 10^4 A$. Thus, we have imposed the current intensities $I_j = 7 \times 10^4$ using the approach that has been explained in Section 5.3.1 for the case of several conductors with electric ports. With the same notation used there, the boundaries $\Gamma_{E,j}$ correspond to the contacts of the copper bars on the boundary of the furnace and $\Gamma_{J,j}$ to the bottom of the electrodes.

In Figure 5.7 we present the modulus of the magnetic potential, *i.e.*, $|\widehat{\psi}_{D,h} + \sum_{j=1}^{3} I_j \eta_{D,j,h}|$, where $\eta_{D,j,h}$ are the piecewise linear functions with a jump of height 1 on the 'cutting' surfaces $\Sigma_{j,h}$. In Figures 5.8 and 5.9 the modulus of the current density $\mathbf{J}_h = \sigma \mathbf{E}_{C,h}$ on a horizontal and a vertical section of one electrode is shown.

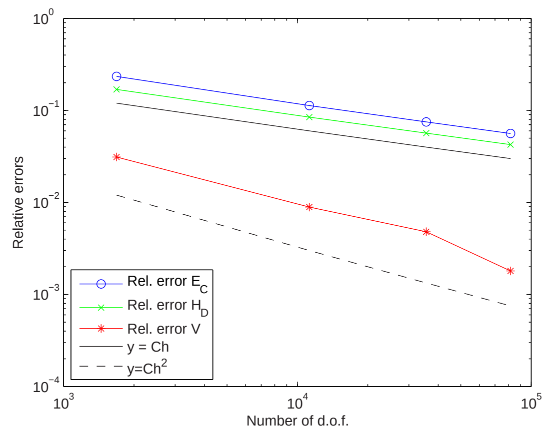

Figure 5.5: Relative error versus number of d.o.f. (assigned intensity).

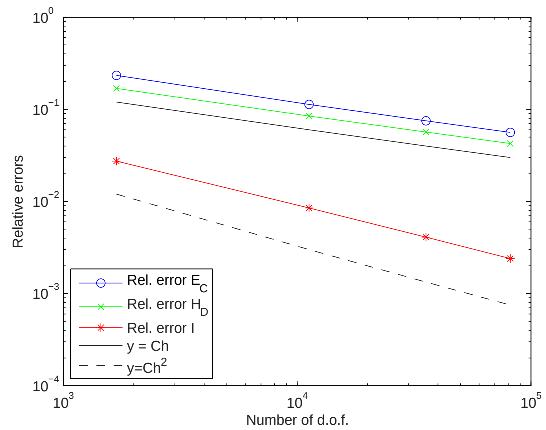

Figure 5.6: Relative error versus number of d.o.f. (assigned voltage).

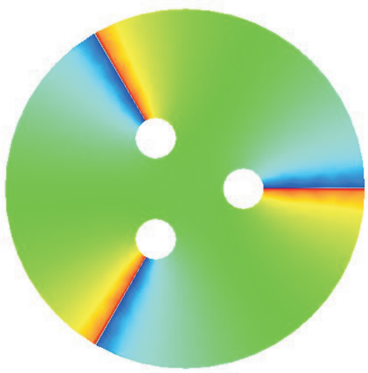

Figure 5.7: Magnetic potential in the dielectric.

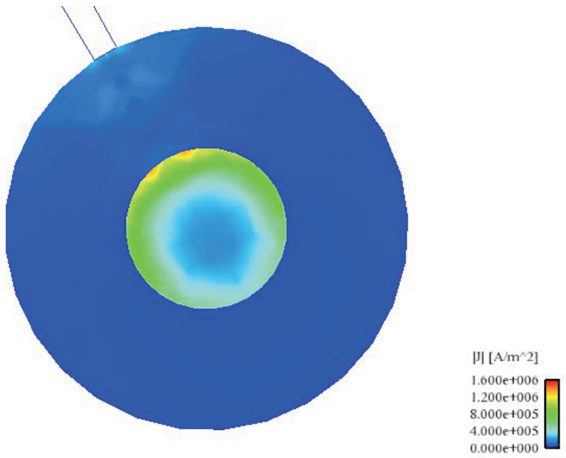

Figure 5.8: $|\mathbf{J}_h|$ on a horizontal section of one electrode.

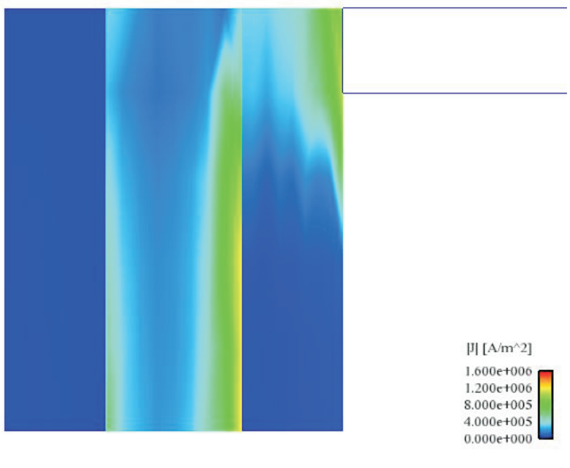

Figure 5.9: $|\mathbf{J}_h|$ on a vertical section of one electrode.

Conclusions

The main objective of this work has been the numerical simulation of an induction heating furnace. To this end, we have introduced a mathematical model, consisting of a nonlinear system of partial differential equations.

In Chapter 2 we have analyzed two different versions of a system of equations concerning a stationary magnetohydrodynamic problem, assuming homogeneous and temperature independent physical properties. The two versions differ on the treatment of the buoyancy term in the hydrodynamic model: in the first one we consider the Boussinesq approximation, whereas in the second we assume certain properties for the density function. When considering the Boussinesq approximation we have proved the existence of solution for the coupled problem under some hypotheses of smallness for the boundary and source data, hence extending the results given in [72] by considering Lipschitz domains and quadratic sources in the heat equation. For the second stationary model, and following the ideas presented in [2], we have proved the result of existence independently of the data size. We have also showed, for both models, the uniqueness of solution under constraints of small source data.

We have introduced in Chapter 3 a mathematical model more suitable for the numerical simulation of the furnace. The main task in this chapter has been the mathematical analysis of the electromagnetic problem, which we have written in terms of the magnetic vector potential with the current voltages acting as Lagrange multipliers. A result of existence and uniqueness of the solution has been proved, with a simple adaptation of a similar result presented in [58].

The mathematical model has been discretized in Chapter 4 using Lagrange-Galerkin methods for the thermal and hydrodynamic subproblems, and a mixed BEM/FEM method for the electromagnetic one. Moreover, we have also presented in this chapter several iterative algorithms to deal with the coupling terms (Joule effect, temperature dependent properties) and the nonlinearities (thermal boundary conditions, change of state). The resulting algorithm has been implemented in FORTRAN and validated by solving a coupled thermal-hydrodynamic problem with phase change. The computational code has been also used to simulate the behaviour of a real induction heating furnace.

Finally, we have introduced and analyzed in Chapter 5 a new formulation of the electromagnetic problem, taking as main unknowns the electric field in the conductor and the magnetic field in the insulator. The main advantage of this formulation is that it permits to impose either the current intensity or the voltage drop in an easy way. Moreover, imposing the electric field as the unknown in the conductor, instead of the magnetic fields (as it is done in [24]) should lead to more accurate calculations of the Joule effect after discretization.

From our point of view there are still many open problems. Concerning the mathematical analysis of the problem, the results of Chapter 2 could be extended to a problem with material dependent properties. Moreover, it seems also possible to consider a formulation of Maxwell equations in terms of the current density **J**, as it is done in [74], which would permit to introduce more realistic boundary conditions. However, the main interest in this field would be the analysis of the coupled mathematical model appearing in Chapter 3, considering the time harmonic electromagnetic model and including temperature dependencies of the material properties and phase change.

About the numerical simulation of the furnace, it would be interesting to compare the numerical results of the axisymmetric electromagnetic formulation presented in Chapter 3 and the one introduced in Chapter 5, in order to know if the geometrical (and topological) modifications necessary to perform the two-dimensional simulation in a cylindrical setting have any influence on the numerical results.

Moreover, there are still some improvements that should be incorporated to the mathematical model to obtain more realistic results. First of all, we should consider an internal radiation boundary condition in the inner walls of the crucible and in the upper boundary of the load. It would be also interesting to introduce a more sophisticated turbulence model, such as the $k-\epsilon$ model, better than the Smagorinsky's one we have used until now. A more challenging problem seems to be the simulation of the displacement of liquid metal during the melting process. However, this is crucial to obtain realistic simulations of the process, and in particular when the physical properties vary at the melting point.

Finally, it should be carried out the numerical analysis of the BEM/FEM introduced in Chapter 4. At this respect, the numerical analysis of a finite element method for the same problem (in a bounded domain) is presented in [21].

Appendix A

Notation and auxiliary results.

This appendix is devoted to introduce some well known definitions and results that are used throughout this work. In Section 1 we introduce the notation for some commonly used function spaces, along with some well known properties and Green's formulas. In Section 2 we present two orthogonal decompositions of the space $\mathbf{L}^2(\Omega)$, which depend on topological properties of the domain Ω.

A.1 Function spaces.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain either of class $\mathcal{C}^{1,1}$ or a Lipschitz polyhedron, in both cases with connected boundary $\partial\Omega$. In this section we introduce several spaces of functions defined on Ω which will be used in different parts of this work.

For a real number $p \geq 1$, $L^p(\Omega)$ denotes the Lebesgue space of (real or complex) scalar functions the p-th power of which are integrable; its vectorial counterpart is denoted by $\mathbf{L}^p(\Omega)$. These spaces are equipped with the norms

$$\|\theta\|_{L^p} := \left(\int_\Omega |\theta(\mathbf{x})|^p \, \mathrm{d}\mathbf{x}\right)^{1/p},$$
$$\|\mathbf{u}\|_{\mathbf{L}^p} := \left(\int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x}\right)^{1/p}.$$

For a non-negative integer m, we denote by $H^m(\Omega)$ the usual m-th order Sobolev space, *i.e.* the space of (real or complex) functions belonging to $L^2(\Omega)$ such that all their distributional derivatives of order less or equal than m also belong to $L^2(\Omega)$. We equip it with the usual norm

$$\|\theta\|_m := \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha\theta\|_{L^2}^2\right)^{1/2}.$$

We denote by $\mathbf{H}^m(\Omega) := (H^m(\Omega))^3$ its vector-valued counterpart, and again by $\|\cdot\|_m$ its norm. We use the convention $H^0(\Omega) = L^2(\Omega)$ and $\mathbf{H}^0(\Omega) = \mathbf{L}^2(\Omega)$. Moreover, for the space $L^2(\Omega)$

(respectively, $\mathbf{L}^2(\Omega)$) we denote its scalar product by $(\theta,\zeta)_\Omega := \int_\Omega \theta\zeta\,\mathrm{d}\mathbf{x}$ (respectively, $(\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}$).

We shall also make use of some subspaces of $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ satisfying certain boundary conditions:

$$\begin{aligned} H_0^1(\Omega) &:= \{\theta \in H^1(\Omega) : \theta_{|\partial\Omega} = 0\}, \\ \mathbf{H}_0^1(\Omega) &:= \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \mathbf{w}_{|\partial\Omega} = \mathbf{0}\}, \\ \mathbf{H}_T^1(\Omega) &:= \{\mathbf{w} \in \mathbf{H}^1(\Omega) : (\mathbf{w}\cdot\mathbf{n})_{|\partial\Omega} = \mathbf{0}\}. \end{aligned}$$

Since $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, and both $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}_T^1(\Omega)$ are closed subspaces of $\mathbf{H}^1(\Omega)$, they are endowed with the corresponding norms $\|\cdot\|_1$ defined above.

To impose the divergence-free condition in the hydrodynamic problem, we will make use of the following subspaces

$$\begin{aligned} \mathbf{Z}(\Omega) &:= \left\{\mathbf{w} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{w} = 0\right\}, \\ \mathbf{Z}_0(\Omega) &:= \left\{\mathbf{w} \in \mathbf{H}_0^1(\Omega) : \operatorname{div}\mathbf{w} = 0\right\}. \end{aligned}$$

In the Navier-Stokes equations, the pressure belongs to a certain closed subspace of $L^2(\Omega)$, namely,

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q\,\mathrm{d}\mathbf{x} = 0 \right\}.$$

For a real number $p \geq 1$, we denote by $W^{1,p}(\Omega)$ the space of functions belonging to $L^p(\Omega)$ such that their first order distributional derivatives also belong to $L^p(\Omega)$. It is endowed with the norm $\|\theta\|_{1,p} := \|\theta\|_{L^p} + \sum_{i=1}^3 \left\|\frac{\partial\theta}{\partial x_i}\right\|_{L^p}$. We define a certain closed subspace of $W^{1,p}(\Omega)$ consisting of functions that satisfy homogeneous Dirichlet boundary conditions

$$W_0^{1,p}(\Omega) := \{\theta \in W^{1,p}(\Omega) : \theta_{|\partial\Omega} = 0\}.$$

We also recall the dual spaces

$$\begin{aligned} H^{-1}(\Omega) &= (H_0^1(\Omega))', \\ \mathbf{H}^{-1}(\Omega) &= (\mathbf{H}_0^1(\Omega))', \\ W^{-1,p'}(\Omega) &= (W_0^{1,p}(\Omega))', \end{aligned}$$

where $1/p' = 1 - 1/p$, for any $p > 1$. Their respective norms are defined as follows:

$$\begin{aligned} \|f\|_{-1} &:= \sup_{\theta \in H_0^1(\Omega),\, \theta \neq 0} \frac{\langle f,\theta\rangle_\Omega}{\|\theta\|_1}, \\ \|\mathbf{f}\|_{-1} &:= \sup_{\mathbf{w} \in H_0^1(\Omega),\, \mathbf{w} \neq \mathbf{0}} \frac{\langle \mathbf{f},\mathbf{w}\rangle_\Omega}{\|\mathbf{w}\|_1}, \\ \|f\|_{-1,p'} &:= \sup_{\theta \in W_0^{1,p}(\Omega),\, \theta \neq 0} \frac{\langle f,\theta\rangle_\Omega}{\|\theta\|_{1,p}}. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality pairing between a function space $V(\Omega)$ defined on the domain Ω and its dual $(V(\Omega))'$. Analogously, we shall denote by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ the duality pairing between a function space $W(\partial\Omega)$ defined on the boundary $\partial\Omega$ and its dual $(W(\partial\Omega))'$.

We also need certain well known trace spaces

$$\begin{aligned} H^{1/2}(\partial\Omega) &:= \{\theta_{|\partial\Omega} : \theta \in H^1(\Omega)\}, \\ \mathbf{H}^{1/2}(\partial\Omega) &:= \{\mathbf{w}_{|\partial\Omega} : \mathbf{w} \in \mathbf{H}^1(\Omega)\}, \end{aligned}$$

endowed with the norms

$$\|q\|_{1/2,\partial\Omega} := \inf_{\theta \in H^1(\Omega),\, \theta_{|\partial\Omega} = q} \|\theta\|_1 \,,$$
$$\|\mathbf{q}\|_{1/2,\partial\Omega} := \inf_{\mathbf{w} \in \mathbf{H}^1(\Omega),\, \mathbf{w}_{|\partial\Omega} = \mathbf{q}} \|\mathbf{w}\|_1 \,,$$

and their respective dual spaces $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))'$, $\mathbf{H}^{-1/2}(\partial\Omega) = (\mathbf{H}^{1/2}(\partial\Omega))'$, equipped with the usual norms:

$$\|g\|_{-1/2,\partial\Omega} := \sup_{w \in H^{1/2}(\partial\Omega), w \neq 0} \frac{\langle g, w \rangle_{\partial\Omega}}{\|w\|_{1/2,\partial\Omega}},$$
$$\|\mathbf{g}\|_{-1/2,\partial\Omega} := \sup_{\mathbf{w} \in \mathbf{H}^{1/2}(\partial\Omega), \mathbf{w} \neq \mathbf{0}} \frac{\langle \mathbf{g}, \mathbf{w} \rangle_{\partial\Omega}}{\|\mathbf{w}\|_{1/2,\partial\Omega}} \,.$$

Let $\Gamma \subset \partial\Omega$ be an open subset of the boundary. We define the spaces (see, for instance, [45])

$$H^{1/2}_{00}(\Gamma) := \left\{ v \in L^2(\Gamma) : \tilde{v} \in H^{1/2}(\partial\Omega) \right\} \,,$$
$$\mathbf{H}^{1/2}_{00}(\Gamma) := \left\{ \mathbf{v} \in \mathbf{L}^2(\Gamma) : \tilde{\mathbf{v}} \in \mathbf{H}^{1/2}(\partial\Omega) \right\} \,,$$

where $\tilde{v}$ and $\tilde{\mathbf{v}}$ are the extensions by zero to $\partial\Omega$. Identifying each field v with its extension by zero $\tilde{v}$, we can write $H^{1/2}_{00}(\Gamma) \subset H^{1/2}(\partial\Omega)$, and also $\mathbf{H}^{1/2}_{00}(\Gamma) \subset \mathbf{H}^{1/2}(\partial\Omega)$. Their dual spaces are respectively denoted by $H^{-1/2}_{00}(\Gamma)$ and $\mathbf{H}^{-1/2}_{00}(\Gamma)$.

Next, we give some well known results about the norms. For any function $\theta \in W^{1,p}_0(\Omega)$, the Poincaré inequality holds:

$$\|\theta\|_{L^p} \leq C(\Omega, p) \sum_{i=1}^{3} \left\| \frac{\partial \theta}{\partial x_i} \right\|_{L^p} , \tag{A.1}$$

where $C(\Omega, p)$ is some constant dependent on the domain Ω and p. This result states that the seminorm

$$|\theta|_{1,p} := \sum_{i=1}^{3} \left\| \frac{\partial \theta}{\partial x_i} \right\|_{L^p} , \tag{A.2}$$

is a norm in $W^{1,p}_0(\Omega)$ equivalent to the usual norm $\|\cdot\|_{1,p}$. Moreover,

$$\|\theta\|_{1,p} \leq (1 + C(\Omega, p))\, |\theta|_{1,p} = C(p)\, |\theta|_{1,p} \,. \tag{A.3}$$

We notice that this result implies the equivalence, in the dual space $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$, of the usual norm $\|\cdot\|_{-1,p'}$ and the dual norm of $|\,\boldsymbol{\cdot}\,|_{1,p}$, which is given by

$$|f|_{-1,p'} := \sup_{\theta \in W_0^{1,p}(\Omega),\, \theta \neq 0} \frac{\langle f, \theta\rangle_\Omega}{|\theta|_{1,p}}. \tag{A.4}$$

In the particular case of spaces $H_0^1(\Omega)$ and $\mathbf{H}_0^1(\Omega)$, Poincaré inequality states the equivalence of the usual norm $\|\cdot\|_1$ and the seminorms $|\cdot|_1$, defined as

$$|\theta|_1 := \|\mathbf{grad}\,\theta\|_0 = \left(\sum_{i=1}^{3} \left\|\frac{\partial\theta}{\partial x_i}\right\|_0^2\right)^{1/2}, \tag{A.5}$$

$$|\mathbf{w}|_1 := \|\mathbf{grad}\,\mathbf{w}\|_0 = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \left\|\frac{\partial w_i}{\partial x_j}\right\|_0^2\right)^{1/2}, \tag{A.6}$$

i.e., there exists a constant C_0 such that

$$\|\theta\|_1 \leq C_0\,|\theta|_1 \quad \forall\,\theta \in H_0^1(\Omega)\,, \tag{A.7}$$

$$\|\mathbf{w}\|_1 \leq C_0\,|\mathbf{w}|_1 \quad \forall\,\mathbf{w} \in \mathbf{H}_0^1(\Omega)\,, \tag{A.8}$$

For the electromagnetic problem we will make use of the Hilbert spaces $\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$, which are defined by

$$\begin{aligned}\mathbf{H}(\mathbf{curl};\Omega) &:= \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\},\\ \mathbf{H}(\mathrm{div};\Omega) &:= \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \mathrm{div}\,\mathbf{w} \in L^2(\Omega)\},\end{aligned}$$

and endowed with the norms

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} &:= \left(\|\mathbf{v}\|_0^2 + \|\mathbf{curl}\,\mathbf{v}\|_0^2\right)^{1/2},\\ \|\mathbf{w}\|_{\mathbf{H}(\mathrm{div};\Omega)} &:= \left(\|\mathbf{w}\|_0^2 + \|\mathrm{div}\,\mathbf{w}\|_0^2\right)^{1/2}.\end{aligned}$$

We will denote by $\mathbf{H}^0(\mathbf{curl}\,;\Omega)$ the closed subspace of functions in $\mathbf{H}(\mathbf{curl};\Omega)$ such that its curl is null, namely,

$$\mathbf{H}^0(\mathbf{curl}\,;\Omega) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl};\Omega) : \mathbf{curl}\,\mathbf{v} = \mathbf{0}\}\,.$$

We denote by $\mathbf{H}_0(\mathbf{curl}\,;\Omega)$ the subspace of $\mathbf{H}(\mathbf{curl};\Omega)$ constituted by functions with vanishing tangential trace on the boundary (see Section A.1.1 for a characterization of the tangential trace), and by $\mathbf{H}_0(\mathrm{div}\,;\Omega)$ the subspace of $\mathbf{H}(\mathrm{div};\Omega)$ constituted by functions with vanishing normal trace on the boundary, namely,

$$\begin{aligned}\mathbf{H}_0(\mathbf{curl}\,;\Omega) &:= \{\mathbf{v} \in \mathbf{H}(\mathbf{curl};\Omega) : \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,\\ \mathbf{H}_0(\mathrm{div}\,;\Omega) &:= \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,.\end{aligned}$$

Moreover, given an open subset $\Lambda \subset \partial\Omega$ we denote by $\mathbf{H}_{0,\Lambda}(\mathbf{curl}\,;\Omega)$ the space of functions in $\mathbf{H}(\mathbf{curl};\Omega)$ with vanishing tangential trace on Λ, *i.e.*,

$$\mathbf{H}_{0,\Lambda}(\mathbf{curl}\,;\Omega) \quad := \quad \{\mathbf{v} \in \mathbf{H}(\mathbf{curl};\Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ in } \mathbf{H}_{00}^{-1/2}(\Lambda)\}\,.$$

To simplify the notation we introduce the spaces

$$\begin{aligned} \mathbf{X}(\Omega) \quad &:= \quad \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)\,, \\ \mathbf{X}_0(\Omega) \quad &:= \quad \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}_0(\mathrm{div};\Omega)\,, \end{aligned}$$

equipped with the norm

$$\|\mathbf{w}\|_{\mathbf{X}} \quad := \quad \left(\|\mathbf{w}\|_0^2 + \|\mathbf{curl}\,\mathbf{w}\|_0^2 + \|\mathrm{div}\,\mathbf{w}\|_0^2\right)^{1/2}\,.$$

It is known that if the domain Ω is bounded and simply connected with Lipschitz boundary, then

$$\|\mathbf{w}\|_0 \leq \tilde{C}_1(\|\mathbf{curl}\,\mathbf{w}\|_0^2 + \|\mathrm{div}\,\mathbf{w}\|_0^2)^{1/2} \quad \forall \mathbf{w} \in \mathbf{X}_0(\Omega)\,. \tag{A.9}$$

The result is a consequence of Lemma 3.6 in chapter 1 of [53]. In fact, the previous result states that the mapping $\mathbf{w} \mapsto |\mathbf{w}|_{\mathbf{X}} := (\|\mathbf{curl}\,\mathbf{w}\|_0^2 + \|\mathrm{div}\,\mathbf{w}\|_0^2)^{1/2}$ defines a norm in $\mathbf{X}_0(\Omega)$ equivalent to the norm $\|\centerdot\|_{\mathbf{X}}$, through the inequality

$$\|\mathbf{w}\|_{\mathbf{X}} \leq C_1\,|\mathbf{w}|_{\mathbf{X}} \quad \forall \mathbf{w} \in \mathbf{X}_0(\Omega)\,, \tag{A.10}$$

with $C_1 = (1 + \tilde{C}_1^2)^{1/2}$.

Remark A.1. *In fact, if the domain Ω is bounded of class $\mathcal{C}^{1,1}$ or it is a bounded convex polyhedron, for any function $\mathbf{w} \in \mathbf{X}_0(\Omega)$ we have the following inequality*

$$\|\mathbf{w}\|_1 \leq C_2(\|\mathbf{curl}\,\mathbf{w}\|_0^2 + \|\mathrm{div}\,\mathbf{w}\|_0^2)^{1/2}, \tag{A.11}$$

with C_2 some constant dependent on Ω. This result states the equality $\mathbf{H}_T^1(\Omega) = \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}_0(\mathrm{div};\Omega) = \mathbf{X}_0(\Omega)$, and the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_{\mathbf{X}}$ in $\mathbf{H}_T^1(\Omega)$. Thus, it would allow us to improve, in the case of regular domains, some of the results appearing in Chapter 2 of this work. The result is presented in chapter 1 of [53]. It is a consequence of Theorem 3.8 and Lemma 3.6 for the case of a bounded domain with $C^{1,1}$ boundary, and of Theorem 3.9 and Lemma 3.6 for the case of a convex polyhedron.

Finally we remind two well known Green's formulas, that will be used throughout this work

$$\int_\Omega \mathbf{u} \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathrm{div}\,\mathbf{u}) v \,\mathrm{d}\mathbf{x} = \langle \mathbf{u} \cdot \mathbf{n}, v\rangle_{\partial\Omega}\,, \quad \forall\, \mathbf{u} \in \mathbf{H}(\mathrm{div};\Omega), \forall\, v \in H^1(\Omega), \tag{A.12}$$

$$\int_\Omega \mathbf{u} \cdot (\mathbf{curl}\,\mathbf{w}) \,\mathrm{d}\mathbf{x} - \int_\Omega (\mathbf{curl}\,\mathbf{u}) \cdot \mathbf{w} \,\mathrm{d}\mathbf{x} = \langle \mathbf{u} \times \mathbf{n}, \mathbf{w}\rangle_{\partial\Omega} \;\; \forall\, \mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega), \forall\, \mathbf{w} \in \mathbf{H}^1(\Omega). \tag{A.13}$$

A.1.1 Characterization of the tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$.

We notice that the second of the previous Green's formulas is not valid when both functions belong to $\mathbf{H}(\mathbf{curl};\Omega)$. In order to extend the use of this formula to a more general situation, it is necessary to characterize the space of tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$. If the domain Ω is smooth this is done in [3]. The results are generalized to the case of Lipschitz polyhedra in [30, 31], and to the case of general Lipschitz domains in [32]. We will first present some of the definitions and results introduced in [3] for smooth domains, but following the notation of [2], and then report the main results of [30, 31] for Lipschitz polyhedra.

Let Ω be a bounded domain either of class $\mathcal{C}^{1,1}$ or a Lipschitz polyhedron. We define the space

$$\mathbf{H}_T^{1/2}(\partial\Omega) := \{\mathbf{w} \in \mathbf{H}^{1/2}(\partial\Omega) : (\mathbf{w}\cdot\mathbf{n})_{|\partial\Omega} = 0\}\,.$$

In the case of Ω being of class $\mathcal{C}^{1,1}$ we can introduce its dual space, denoted by $\mathbf{H}_T^{-1/2}(\partial\Omega)$, which can be identified to $\{\mathbf{w} \in \mathbf{H}^{-1/2}(\partial\Omega) : (\mathbf{w}\cdot\mathbf{n})_{|\partial\Omega} = 0\}$. We also introduce the space $H^{3/2}(\partial\Omega)$ defined as

$$H^{3/2}(\partial\Omega) := \{u_{|\partial\Omega} : u \in H^2(\Omega)\}\,,$$

and its dual space is $H^{-3/2}(\partial\Omega)$. Moreover, given $\varphi \in H^{3/2}(\partial\Omega)$ we denote by $\tilde{\varphi}$ any lifting of φ in $H^2(\Omega)$.

We define the 'tangential gradient operator' $\mathbf{grad}_\Gamma : H^{3/2}(\partial\Omega) \to \mathbf{H}_T^{1/2}(\partial\Omega)$ given by $\mathbf{grad}_\Gamma\,\varphi := \mathbf{n}\times(\mathbf{grad}\,\tilde{\varphi}_{|\partial\Omega}\times\mathbf{n})$ (see, for instance, [2]), and this definition can be seen to be independent of the lifting $\tilde{\varphi}$. In the same paper (see also [3] and [13]) the 'tangential divergence operator' $\mathrm{div}_\Gamma : \mathbf{H}_T^{-1/2}(\partial\Omega) \to H^{-3/2}(\Omega)$ is defined by

$$\langle \mathrm{div}_\Gamma\,\boldsymbol{\lambda}, \varphi\rangle_{\partial\Omega} := -\,\langle \boldsymbol{\lambda}, \mathbf{grad}_\Gamma\,\varphi\rangle_{\partial\Omega} \quad \forall\,\varphi \in H^{3/2}(\partial\Omega), \forall\,\boldsymbol{\lambda} \in \mathbf{H}_T^{-1/2}(\partial\Omega)\,.$$

Following the notation in [2] we introduce the space

$$\mathbf{H}_T^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega) := \{\boldsymbol{\lambda} \in \mathbf{H}_T^{-1/2}(\partial\Omega) : \mathrm{div}_\Gamma\,\boldsymbol{\lambda} \in H^{-1/2}(\partial\Omega)\}\,.$$

It is proved in [3] that the tangential trace operator γ_τ defined as $\gamma_\tau(\mathbf{u}) := \mathbf{u}\times\mathbf{n}$, is linear, continuous and surjective from $\mathbf{H}(\mathbf{curl};\Omega)$ to $\mathbf{H}_T^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$.

As said before, the definitions and results presented above are only valid when the domain Ω is smooth. If Ω is a Lipschitz polyhedron some problems appear. In particular the scalar product $\boldsymbol{\lambda}\cdot\mathbf{n}$ for $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\partial\Omega)$, and the tangential divergence operator are not meaningful anymore. In order to define the space of tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$, in [30, 31] some of the previous definitions and results are generalized, considering the definitions of some spaces and operators face by face, and imposing certain compatibility conditions at the edges of the polyhedron. In the same papers, after giving a precise meaning to the traces of $\mathbf{H}(\mathbf{curl};\Omega)$, the authors also introduce a generalization of the Green's formula which is valid for functions $\mathbf{u}$, $\mathbf{w} \in \mathbf{H}(\mathbf{curl};\Omega)$. In the following we present some of the main results of these papers, and refer the reader to the articles for the full rigorous proofs.

Let Ω be a Lipschitz polyhedron such that its boundary $\partial\Omega$ is split into M open faces $\Gamma_j, j = 1,\ldots,M$, so that $\partial\Omega = \cup_{j=1}^M \overline{\Gamma_j}$. When Γ_i and Γ_j are two adjacent faces, we denote by e_{ij} their 'common' edge. Moreover, for a given face Γ_i, $\mathcal{S}_i$ will denote the set of indices j such that the faces Γ_j have a 'common' edge with Γ_i. Finally, we denote by $\mathbf{u}_i$ the trace $\mathbf{u}_{|\Gamma_i}$, (in particular $\mathbf{n}_i = \mathbf{n}_{|\Gamma_i}$), by $\boldsymbol{\tau}_{ij}$ the unit vector in the direction of the edge e_{ij}, and set $\boldsymbol{\tau}_i = \boldsymbol{\tau}_{ij} \times \mathbf{n}_i$ so that $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij}, \mathbf{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$.

Let us introduce the spaces

$$\mathbf{L}^2_t(\partial\Omega) := \{\boldsymbol{\phi} \in \mathbf{L}^2(\partial\Omega) : \boldsymbol{\phi} \cdot \mathbf{n}_{|\partial\Omega} = 0\}\,,$$
$$\mathbf{H}^{1/2}_-(\partial\Omega) := \{\boldsymbol{\lambda} \in \mathbf{L}^2_t(\partial\Omega) : \boldsymbol{\lambda}_j \in \mathbf{H}^{1/2}(\Gamma_j),\ 1 \le j \le M\}\,.$$

We define the 'tangential components trace' mapping $\pi_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{H}^{1/2}_-(\partial\Omega)$ by $\pi_\tau(\mathbf{u}) := \mathbf{n} \times (\mathbf{u} \times \mathbf{n})_{|\partial\Omega}$, and the 'tangential trace' mapping $\gamma_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{H}^{1/2}_-(\partial\Omega)$ as $\gamma_\tau(\mathbf{u}) := \mathbf{u} \times \mathbf{n}_{|\partial\Omega}$. These mappings can be extended in a unique way to linear continuous mappings from $\mathbf{H}^1(\Omega)$ into $\mathbf{H}^{1/2}_-(\partial\Omega)$ (still denoted by π_τ and γ_τ). We have, for any $\mathbf{u} \in \mathbf{H}^1(\Omega)$

$$\pi_\tau\mathbf{u}(\mathbf{x}) = \mathbf{u}_j(\mathbf{x}) - (\mathbf{u}_j(\mathbf{x}) \cdot \mathbf{n}_j(\mathbf{x}))\mathbf{n}_j(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Gamma_j, 1 \le j \le M\,,$$
$$\gamma_\tau\mathbf{u}(\mathbf{x}) = \mathbf{u}_j(\mathbf{x}) \times \mathbf{n}_j(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Gamma_j, 1 \le j \le M\,.$$

Mappings π_τ and γ_τ are not surjective. In order to characterize their respective ranges we adopt the following notation: for $\psi_i \in H^{1/2}(\Gamma_i)$, $\psi_j \in H^{1/2}(\Gamma_j)$ we say that

$$\psi_i \overset{1/2}{=} \psi_j \text{ on } e_{ij} \Leftrightarrow C(\psi_i, \psi_j) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\psi_i(\mathbf{x}) - \psi_j(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\sigma(\mathbf{x})\, \mathrm{d}\sigma(\mathbf{y}) < \infty\,.$$

Let us introduce the spaces

$$\mathbf{H}^{1/2}_{\|}(\partial\Omega) \;:=\; \{\boldsymbol{\phi} \in \mathbf{H}^{1/2}_-(\partial\Omega) : \boldsymbol{\phi}_j \cdot \boldsymbol{\tau}_{ij} \overset{1/2}{=} \boldsymbol{\phi}_i \cdot \boldsymbol{\tau}_{ij} \text{ on } e_{ij}\ \forall j\,, \forall i \in \mathcal{S}_j\}\,,$$
$$\mathbf{H}^{1/2}_{\perp}(\partial\Omega) \;:=\; \{\boldsymbol{\phi} \in \mathbf{H}^{1/2}_-(\partial\Omega) : \boldsymbol{\phi}_i \cdot \boldsymbol{\tau}_i \overset{1/2}{=} \boldsymbol{\phi}_j \cdot \boldsymbol{\tau}_j \text{ on } e_{ij}\ \forall j\,, \forall i \in \mathcal{S}_j\}\,.$$

These spaces are not closed in $\mathbf{H}^{1/2}_-(\partial\Omega)$. It can be proved that they are Hilbert spaces when endowed with the norms

$$\|\boldsymbol{\phi}\|^2_{\|,1/2,\partial\Omega} := \sum_{j=1}^M \|\boldsymbol{\phi}\|^2_{1/2,\Gamma_j} + \sum_{j=1}^M \sum_{i\in\mathcal{S}_j} C(\boldsymbol{\phi}_i \cdot \boldsymbol{\tau}_{ij}, \boldsymbol{\phi}_j \cdot \boldsymbol{\tau}_{ij})\,,$$
$$\|\boldsymbol{\phi}\|^2_{\perp,1/2,\partial\Omega} := \sum_{j=1}^M \|\boldsymbol{\phi}\|^2_{1/2,\Gamma_j} + \sum_{j=1}^M \sum_{i\in\mathcal{S}_j} C(\boldsymbol{\phi}_i \cdot \boldsymbol{\tau}_i, \boldsymbol{\phi}_j \cdot \boldsymbol{\tau}_j)\,.$$

The mappings $\pi_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_{\|}(\partial\Omega)$ and $\gamma_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_{\perp}(\partial\Omega)$ are linear continuous and surjective. The dual spaces of $\mathbf{H}^{1/2}_{\|}(\partial\Omega)$ and $\mathbf{H}^{1/2}_{\perp}(\partial\Omega)$ will be denoted by $\mathbf{H}^{-1/2}_{\|}(\partial\Omega)$ and $\mathbf{H}^{-1/2}_{\perp}(\partial\Omega)$, respectively.

To make precise the ranges of the mappings π_τ and γ_τ extended to $\mathbf{H}(\mathbf{curl};\Omega)$ we have to introduce some surface operators. Since we are dealing with polyhedra, these operators can be defined face by face. First, let us define the 'tangential gradient operator' as $\mathbf{grad}_{\Gamma_j} u = \mathbf{grad}\, u_{|\Gamma_j} - (\mathbf{grad}\, u_{|\Gamma_j} \cdot \mathbf{n}_j)\mathbf{n}_j$, for $u \in H^2(\Omega)$, and in the same way let us define the 'tangential curl operator' face by face as $\mathbf{curl}_{\Gamma_j} u = \mathbf{grad}\, u_{|\Gamma_j} \times \mathbf{n}_j$. We can define the operators corresponding to the whole boundary $\partial\Omega$ as follows:

$$\begin{aligned}
&\mathbf{grad}_\Gamma : H^2(\Omega) \to \mathbf{H}_-^{1/2}(\partial\Omega),\ \mathbf{grad}_\Gamma\, u(x) = \mathbf{grad}_{\Gamma_j} u(\mathbf{x})\,,\ \text{a.e. } \mathbf{x} \in \Gamma_j\,,\\
&\mathbf{curl}_\Gamma : H^2(\Omega) \to \mathbf{H}_-^{1/2}(\partial\Omega),\ \mathbf{curl}_\Gamma\, u(x) = \mathbf{curl}_{\Gamma_j} u(\mathbf{x})\,,\ \text{a.e. } \mathbf{x} \in \Gamma_j\,,
\end{aligned}$$

and these two operators are linear and continuous. Moreover, it is clear that

$$\begin{aligned}
\mathbf{grad}_\Gamma\, u &= \pi_\tau(\mathbf{grad}\, u)\,,\\
\mathbf{curl}_\Gamma\, u &= \gamma_\tau(\mathbf{grad}\, u)\,.
\end{aligned}$$

For Lipschitz polyhedra we also define the space $H^{3/2}(\partial\Omega)$ as

$$H^{3/2}(\partial\Omega) := \{u_{|\partial\Omega} : u \in H^2(\Omega)\}\,.$$

Given $\varphi \in H^{3/2}(\partial\Omega)$ we define its 'tangential gradient' as $\mathbf{grad}_\Gamma\, \varphi := \mathbf{grad}_\Gamma\, \tilde\varphi$, where $\tilde\varphi \in \mathbf{H}^2(\Omega)$ is such that $\tilde\varphi_{|\partial\Omega} = \varphi$. It can be proved that $\mathbf{grad}_\Gamma\, \varphi$ is independent of the choice of $\tilde\varphi$, and that $\mathbf{grad}_\Gamma \in \mathcal{L}(H^{3/2}(\partial\Omega), \mathbf{H}_\parallel^{1/2}(\partial\Omega))$. In the same way the 'tangential curl operator' $\mathbf{curl}_\Gamma$ can be defined, and $\mathbf{curl}_\Gamma \in \mathcal{L}(H^{3/2}(\Omega), \mathbf{H}_\perp^{1/2}(\partial\Omega))$.

Remark A.2. *In the case of Ω being a domain of class $\mathcal{C}^{1,1}$, the range of both operators γ_τ and π_τ is the space $\mathbf{H}_T^{1/2}(\partial\Omega)$. Furthermore, operators $\mathbf{grad}_\Gamma$ and $\mathbf{curl}_\Gamma$ are defined from $H^{3/2}(\partial\Omega)$ into $\mathbf{H}_T^{1/2}(\partial\Omega)$.*

We also define the 'tangential divergence operator' $\mathrm{div}_\Gamma : \mathbf{H}_\parallel^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega)$, as the adjoint operator of $-\mathbf{grad}_\Gamma$, so that

$$\langle \mathrm{div}_\Gamma\, \boldsymbol\lambda, \varphi\rangle_{\partial\Omega} = -\langle \boldsymbol\lambda, \mathbf{grad}_\Gamma\, \varphi\rangle_{\partial\Omega} \quad \forall\, \varphi \in H^{3/2}(\partial\Omega) \quad \forall\, \boldsymbol\lambda \in \mathbf{H}_\parallel^{-1/2}(\partial\Omega)\,.$$

We can analogously define the operator $\mathrm{curl}_\Gamma : \mathbf{H}_\perp^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega)$ as the adjoint of the operator $\mathbf{curl}_\Gamma \in \mathcal{L}(H^{3/2}(\partial\Omega), \mathbf{H}_\perp^{1/2}(\partial\Omega))$, namely,

$$\langle \mathrm{curl}_\Gamma\, \boldsymbol\lambda, \varphi\rangle_{\partial\Omega} = \langle \boldsymbol\lambda, \mathbf{curl}_\Gamma\, \varphi\rangle_{\partial\Omega} \quad \forall\, \varphi \in H^{3/2}(\partial\Omega) \quad \forall\, \boldsymbol\lambda \in \mathbf{H}_\perp^{-1/2}(\partial\Omega)\,.$$

Now let us set

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega) := \{\boldsymbol\lambda \in \mathbf{H}_\parallel^{-1/2}(\partial\Omega) : \mathrm{div}_\Gamma(\boldsymbol\lambda) \in H^{-1/2}(\partial\Omega)\}\,, \tag{A.14}$$

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega) := \{\boldsymbol\lambda \in \mathbf{H}_\perp^{-1/2}(\partial\Omega) : \mathrm{curl}_\Gamma(\boldsymbol\lambda) \in H^{-1/2}(\partial\Omega)\}\,. \tag{A.15}$$

In Section 3.2 of [30] it is proved that, for any $\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega)$ the following relation holds

$$\mathrm{div}_\Gamma(\gamma_\tau \mathbf{u}) = \mathrm{div}_\Gamma(\mathbf{u} \times \mathbf{n}) = \mathbf{curl}\,\mathbf{u} \cdot \mathbf{n}\,. \tag{A.16}$$

In Theorems 3.9 and 3.10 of the same paper the authors prove that the mappings $\gamma_\tau : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$ and $\pi_\tau : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega)$ are linear and continuous. It is also shown in [31, Th. 5.4] that these mappings are surjective. Moreover, in the same papers it is proved that $(\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega))' = \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$. Thus for any $\mathbf{u}, \mathbf{w} \in \mathbf{H}(\mathbf{curl};\Omega)$ it can be defined the duality pairing $\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{w}) \rangle_{\partial\Omega}$, and the following Green's formula holds:

$$\int_\Omega (\mathbf{curl}\,\mathbf{w}) \cdot \mathbf{u} - \mathbf{w} \cdot (\mathbf{curl}\,\mathbf{u})\,\mathrm{d}\mathbf{x} = \langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{w}) \rangle_{\partial\Omega} \quad \forall\, \mathbf{u}, \mathbf{w} \in \mathbf{H}(\mathbf{curl};\Omega)\,. \tag{A.17}$$

Remark A.3. *As said before, in [31] it is proved that* $(\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega))' = \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$. *However, from the result of that paper we can only affirm that the norms of* $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$ *and* $(\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega))'$ *are equivalent, but not necessarily the same. In order to avoid the use of new constants we will consider for* $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$ *the norm given by* $(\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega))'$, *which will be denoted by* $\|\cdot\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)}$.

Remark A.4. *In the case where* $\mathbf{w} \in \mathbf{H}^1(\Omega)$ *we have*

$$\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{w}) \rangle_{\partial\Omega} = \langle \mathbf{u} \times \mathbf{n}, \mathbf{w} \rangle_{\partial\Omega}$$

where the angles in the left-hand side denote the duality pairing between $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$ *and* $\mathbf{H}_{\perp}^{-1/2}(\mathbf{curl}_\Gamma, \partial\Omega)$, *and the ones in the right-hand side denote the duality pairing between* $\mathbf{H}^{-1/2}(\partial\Omega)$ *and* $\mathbf{H}^{1/2}(\partial\Omega)$. *In fact, this will be the situation when the domain* Ω *is of class* $\mathcal{C}^{1,1}$ *or a convex polyhedron, because our test functions will belong to* $\mathbf{X}_0(\Omega)$ *which is, in this case, contained in* $\mathbf{H}^1(\Omega)$.

Finally, we will also make use of the following lemma:

Lemma A.1. *The space* $\{\pi_\tau(\mathbf{u}) : \mathbf{u} \in \mathbf{X}_0(\Omega)\}$ *is dense in* $\mathbf{H}_{\perp}^{-1/2}(\mathbf{curl}_\Gamma, \partial\Omega)$.

Proof. It is similar to the proof of Lemma 2.4 in [33]. Let us denote by $\Sigma = \cup_{i,j=1}^{M} e_{ij}$ the union of all edges of the polyhedron, and by $\mathbf{H}_S^1(\Omega)$ the functions in $\mathbf{H}^1(\Omega)$ compactly supported in $\overline{\Omega} \setminus \Sigma$. The first step is to prove that the space of tangential traces for $\mathbf{H}_S^1(\Omega)$ is contained in the one for $\mathbf{H}^1(\Omega) \cap \mathbf{X}_0(\Omega)$. Let $\mathbf{v} \in \mathbf{H}_S^1(\Omega)$, its normal component belongs to $H^{1/2}(\Gamma_j)$ on each face, and since $\mathbf{v} \in \mathbf{H}_S^1(\Omega)$ it also belongs to $H^{1/2}(\partial\Omega)$, the space of traces for $\mathbf{H}^1(\Omega)$ (see [30, Th. 2.5]). Hence, there exists $\mathbf{w} \in \mathbf{H}^1(\Omega)$ such that its trace is equal to the normal component of $\mathbf{v}$ and, if we define $\mathbf{z} := \mathbf{v} - \mathbf{w}$, it holds that $\mathbf{z} \in \mathbf{H}^1(\Omega) \cap \mathbf{X}_0(\Omega)$ and $\pi_\tau \mathbf{z} = \pi_\tau \mathbf{v}$.

Now, we recall that $\mathbf{H}_S^1(\Omega)$ is dense in $\mathbf{H}^1(\Omega)$, and also $\mathbf{H}^1(\Omega)$ is dense in $\mathbf{H}(\mathbf{curl};\Omega)$. Since π_τ is continuous and surjective from $\mathbf{H}(\mathbf{curl};\Omega)$ into $\mathbf{H}_{\perp}^{-1/2}(\mathbf{curl}_\Gamma, \partial\Omega)$ the result follows. □

A.2 Two Hodge decompositions of $\mathbf{L}^2(\Omega)$.

It is known that in electromagnetism, and in particular in the eddy currents model, the topological properties of the domain play a crucial role in the mathematical analysis of the model. In this work we present, in Chapters 3 and 5, two different formulations of the eddy currents model, also with different domain configurations. We introduce here some definitions and results used in the aforementioned chapters.

Let $\Omega \subset \mathbb{R}^3$ be an open and connected set, but not necessarily simply connected, with a Lipschitz-continuous boundary Γ. We denote by Γ_k, $k = 1, \ldots, p$, the connected components of Γ, and by $\mathbf{n}$ the outer normal vector to Γ.

We will denote by $\beta_1(\Omega) = q$ the first Betti number of Ω. This number is equal to the dimension of the first cohomology space, a topological invariant of the domain Ω. Roughly speaking, the first Betti number measures the number of independent non-bounding cycles in Ω, and these non-bounding cycles are closed loops that are not a boundary of a two-dimensional manifold contained in Ω. It is known that there exist q orientable two-dimensional manifolds $\Sigma_j \subset \Omega$, $j = 1, \ldots, q$, such that $\partial\Sigma_j \subset \partial\Omega$ and $\Omega \setminus \cup_{j=1}^q \Sigma_j$ has a trivial cohomology space. Moreover, we will denote by $\gamma_j, j = 1, \ldots, q$, the aforementioned independent cycles.

Let $\boldsymbol{\varepsilon}$ be a symmetric tensor field, uniformly positive definite in Ω, and such that $\varepsilon_{ij} \in L^\infty(\Omega)$ for i, $j = 1, 2, 3$. For any functions $\mathbf{u}$, $\mathbf{v} \in \mathbf{L}^2(\Omega)$ let us define the scalar product

$$(\mathbf{u}, \mathbf{v})_{\boldsymbol{\varepsilon}} := \int_\Omega \boldsymbol{\varepsilon}\mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}\,, \tag{A.18}$$

and by $\|\cdot\|_{0,\boldsymbol{\varepsilon}}$ its corresponding induced norm, which is equivalent to the usual norm in $\mathbf{L}^2(\Omega)$, due to the previous assumptions. The space $\mathbf{L}^2(\Omega)$ endowed with this norm will be denoted by $\mathbf{L}^2(\boldsymbol{\varepsilon}; \Omega)$.

We introduce the space of Neumann harmonic fields in Ω:

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{\varepsilon}}(\Omega) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \mathbf{curl}\,\mathbf{w} = \mathbf{0},\ \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{w}) = 0,\ \boldsymbol{\varepsilon}\mathbf{w} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega\}. \tag{A.19}$$

The dimension of this space is known to be equal to $q = \beta_1(\Omega)$ (see, for instance, [11], [40, Ch. IX]). Moreover, it is also possible to construct a basis of $\boldsymbol{\mathcal{H}}_{\boldsymbol{\varepsilon}}(\Omega)$ from the 'cutting' surfaces Σ_j introduced above. Following the construction described in [45], let η_j, $j = 1, \ldots, q$, be the solution to

$$\begin{cases} \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{grad}\,\eta_j) = 0 & \text{in } \Omega \setminus \cup_{k=1}^q \Sigma_k\,, \\ \boldsymbol{\varepsilon}\mathbf{grad}\,\eta_j \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \setminus \cup_{k=1}^q \partial\Sigma_k\,, \\ [\eta_j]_{\Sigma_k} = \delta_{jk}\,, & k = 1, \ldots, q\,, \\ [\boldsymbol{\varepsilon}\mathbf{grad}\,\eta_j \cdot \mathbf{n}]_{\Sigma_k} = 0\,, & k = 1, \ldots, q\,, \end{cases} \tag{A.20}$$

where $[\cdot]_{\Sigma_k}$ denotes the jump across Σ_k of the function between brackets, and δ_{jk} is the Kronecker's delta. Then $\boldsymbol{\varrho}_j := \widetilde{\mathbf{grad}}\,\eta_j$, $j = 1, \ldots, q$ form a basis of $\boldsymbol{\mathcal{H}}_{\boldsymbol{\varepsilon}}(\Omega)$, where $\widetilde{\mathbf{grad}}$ denotes the extension of the gradient computed in $\Omega \setminus \cup_{k=1}^q \Sigma_k$ to Ω. Giving a proper orientation for surfaces Σ_j and cycles γ_j, we can assume that $\int_{\gamma_j} \boldsymbol{\varrho}_j \, ds = 1$.

The following Hodge decomposition holds (see e.g. [6]): given a vector function $\mathbf{v} \in \mathbf{L}^2(\boldsymbol{\varepsilon}; \Omega)$, it can be decomposed into the following sum

$$\mathbf{v} = \boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\mathbf{q} + \mathbf{grad}\,\psi + \sum_{j=1}^q \alpha_j \boldsymbol{\varrho}_j\,, \tag{A.21}$$

with $\mathbf{q} \in \mathbf{H}(\mathbf{curl};\Omega)$ and $\psi \in H^1(\Omega)/\mathbb{C}$. Moreover, this decomposition is $\mathbf{L}^2(\boldsymbol{\varepsilon};\Omega)$-orthogonal, namely,

$$(\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\mathbf{q}, \mathbf{grad}\,\psi)_{\boldsymbol{\varepsilon}} = 0\ ,\ (\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\mathbf{q}, \sum_{j=1}^{q} \alpha_j \boldsymbol{\varrho}_j)_{\boldsymbol{\varepsilon}} = 0\ ,\ (\mathbf{grad}\,\psi, \sum_{j=1}^{q} \alpha_j \boldsymbol{\varrho}_j)_{\boldsymbol{\varepsilon}} = 0\,. \tag{A.22}$$

It is known that if $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ then $\mathbf{q}$ can be taken null, hence

$$\mathbf{v} = \mathbf{grad}\,\psi + \sum_{j=1}^{q} \alpha_j \boldsymbol{\varrho}_j\,, \tag{A.23}$$

and $\int_{\gamma_j} \mathbf{v}\, ds = \alpha_j$.

The second Hodge decomposition we present is based on the space of Dirichlet harmonic functions in Ω, which is defined as

$$\boldsymbol{\mathcal{D}}_{\boldsymbol{\varepsilon}}(\Omega) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \mathbf{curl}\,\mathbf{w} = \mathbf{0},\ \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{w}) = 0,\ \boldsymbol{\varepsilon}\mathbf{w} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}\,, \tag{A.24}$$

and it is known to have finite dimension equal to $\tilde{p} = \beta_2(\Omega)$, with $\beta_2(\Omega)$ the second Betti number of Ω. It is also possible to construct a basis for it: let $\lambda_j \in H^1(\Omega)$ be the solution to

$$\begin{cases} \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{grad}\,\lambda_j) = 0 & \text{in } \Omega\,, \\ \lambda_j = \delta_{jk} & \text{on } \Gamma_k,\ k = 1, \ldots, p\,, \end{cases} \tag{A.25}$$

and denote $\boldsymbol{\pi}_j := \mathbf{grad}\,\lambda_j$, $j = 1, \ldots, p$. It is known that $\boldsymbol{\mathcal{D}}_{\boldsymbol{\varepsilon}}(\Omega) = \mathrm{span}\{\boldsymbol{\pi}_j,\ j = 1, \ldots, p\}$. Moreover, the functions $\boldsymbol{\pi}_j$ can be reordered in such a way that the first $\tilde{p}$ of them, $\{\boldsymbol{\pi}_j\}_{j=1}^{\tilde{p}}$, form a basis of $\boldsymbol{\mathcal{D}}_{\boldsymbol{\varepsilon}}(\Omega)$.

In [10, Lemma 2.1] it is also presented the following Hodge decomposition: given a vector $\mathbf{v} \in \mathbf{L}^2(\boldsymbol{\varepsilon};\Omega)$ it holds that

$$\mathbf{v} = \boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\mathbf{q} + \mathbf{grad}\,\phi + \sum_{j=1}^{\tilde{p}} \alpha_j \boldsymbol{\pi}_j\,, \tag{A.26}$$

with $\mathbf{q} \in \mathbf{H}(\mathbf{curl}\,;\Omega)$ and $\phi \in H_0^1(\Omega)$. Moreover, this decomposition is $\mathbf{L}^2(\boldsymbol{\varepsilon};\Omega)$-orthogonal, namely,

$$(\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\mathbf{q}, \mathbf{grad}\,\phi)_{\boldsymbol{\varepsilon}} = 0\ ,\ (\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\mathbf{q}, \sum_{j=1}^{\tilde{p}} \alpha_j \boldsymbol{\pi}_j)_{\boldsymbol{\varepsilon}} = 0\ ,\ (\mathbf{grad}\,\phi, \sum_{j=1}^{\tilde{p}} \alpha_j \boldsymbol{\pi}_j)_{\boldsymbol{\varepsilon}} = 0\,.$$

It is also known that, for any $\mathbf{w} \in \mathbf{L}^2(\boldsymbol{\varepsilon};\Omega)$, it holds (see [10, eqn. (2.10)])

$$\mathbf{w} \perp \mathbf{grad}\,(H_0^1(\Omega)) \oplus \boldsymbol{\mathcal{D}}_{\boldsymbol{\varepsilon}}(\Omega) \iff \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{w}) = 0 \text{ and } \int_{\Gamma_j} \boldsymbol{\varepsilon}\mathbf{w} \cdot \mathbf{n} = 0 \quad \forall j = 1, \ldots, p\,. \tag{A.27}$$

We notice that the Hodge decompositions presented in this section can be easily generalized to the case of Ω being a non-connected open set, by considering each connected component separately, as it is done in [40].

Appendix B

Solutions by transposition.

For the sake of completeness, we present here some of the results proved by Stampacchia in [94], which are used to show the existence of a solution to the thermal problem with L^1 sources in Chapter 2. For the sake of simplicity we will restrict ourselves to the case $\Omega \subset \mathbb{R}^N$, with $N \geq 3$. The case $N = 2$ can be treated in a similar way, but it would require small changes on the hypotheses (B.3) and (B.4), since $N = 2$ represents a limit case for the Sobolev injections.

Let Ω be a bounded domain contained in $\mathbb{R}^N$, with $N \geq 3$. We consider the elliptic operator

$$\begin{aligned} Lu = &- \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(a_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{N} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(d_j(\mathbf{x}) u \right) + c(\mathbf{x}) u \\ &= -\mathrm{div}\,(A^t(\mathbf{x}) \mathbf{grad}\, u) + \mathbf{b}(\mathbf{x}) \cdot \mathbf{grad}\, u - \mathrm{div}\,(\mathbf{d}(\mathbf{x}) u) + c(\mathbf{x}) u. \end{aligned} \tag{B.1}$$

Let us suppose that

$$a_{ij} \in L^\infty(\Omega), \quad 1 \leq i, j \leq N, \tag{B.2}$$

$$b_i, d_i \in L^N(\Omega), \quad 1 \leq i \leq N, \tag{B.3}$$

$$c \in L^{N/2}(\Omega), \tag{B.4}$$

and that there exists a constant $\nu > 0$ such that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(\mathbf{x}) \xi_i \xi_j \geq \nu |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. in } \Omega. \tag{B.5}$$

Operator L acts from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

Let us consider the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$a(u, v) := \int_\Omega \left(\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{N} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} v + \sum_{j=1}^{N} d_j(\mathbf{x}) \frac{\partial v}{\partial x_j} u + c(\mathbf{x}) uv \right) \mathrm{d}\mathbf{x}, \tag{B.6}$$

which is well defined and continuous due to the hypotheses (B.2)-(B.4). The form $a(u, v)$ is said to be associated to the operator L, and we have

$$\langle Lu, v \rangle_{H^{-1}, H_0^1} = a(u, v) \quad \forall u, v \in H_0^1(\Omega)\,. \tag{B.7}$$

The operator L^* associated with the form $a^*(u,v) := a(v,u)$ is called the formal adjoint operator of L and it is given by

$$\begin{aligned} L^*u &= -\sum_{i=1}^N \sum_{j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(b_i(\mathbf{x})u\right) + \sum_{j=1}^N d_j(\mathbf{x})\frac{\partial u}{\partial x_j} + c(\mathbf{x})u \\ &= -\mathrm{div}\,(A(\mathbf{x})\mathbf{grad}\,u) - \mathrm{div}\,(\mathbf{b}(\mathbf{x})u) + \mathbf{d}(\mathbf{x})\cdot\mathbf{grad}\,u + c(\mathbf{x})u\,. \end{aligned} \tag{B.8}$$

Let us define the Green's operator $G : H^{-1}(\Omega) \longrightarrow H^1_0(\Omega)$ which maps any $f \in H^{-1}(\Omega)$ to the unique element $Gf \in H^1_0(\Omega)$ such that $L(Gf) = f$ in the sense of distributions, *i.e.*, $Gf = u$, where u is the unique solution to the problem

$$\left\{ \begin{array}{l} u \in H^1_0(\Omega), \\ a(u,v) = \langle f, v\rangle_{H^{-1},H^1_0} \quad \forall v \in H^1_0(\Omega). \end{array}\right. \tag{B.9}$$

Under the assumptions

$$c - \sum_{i=1}^N \frac{\partial d_i}{\partial x_i} \geq c_0 > -\infty, \text{ in the sense of distributions,} \tag{B.10}$$

$$a(\cdot,\cdot) \text{ is coercive on } H^1_0(\Omega), \tag{B.11}$$

it has been proved in [94, Th. 4.2], that

$$G \in \mathcal{L}(W^{-1,p}(\Omega), L^\infty(\Omega)), \quad \forall p > N.$$

We can argue in the same way for the adjoint problem: first we define the Green's operator for the adjoint problem $G^* : H^{-1}(\Omega) \longrightarrow H^1_0(\Omega)$, by $L^*(G^*f) = f$ in the sense of distributions, *i.e.*, $G^*f = w$, where w is the unique solution to problem

$$\left\{ \begin{array}{l} w \in H^1_0(\Omega), \\ (a^*(w,v) =)a(v,w) = \langle f, v\rangle_{H^{-1},H^1_0} \quad \forall v \in H^1_0(\Omega). \end{array}\right. \tag{B.12}$$

Next, if we make the analogous assumptions for the adjoint problem, *i.e.*,

$$c - \sum_{i=1}^N \frac{\partial b_i}{\partial x_i} \geq c_0 > -\infty, \text{ in the sense of distributions,} \tag{B.13}$$

$$a(\cdot,\cdot) \text{ is coercive on } H^1_0(\Omega), \tag{B.14}$$

then it holds that

$$G^* \in \mathcal{L}(W^{-1,p}(\Omega), L^\infty(\Omega)), \quad \forall p > N.$$

Remark B.1. *Given an elliptic operator L, and its associated Green operator G, we have defined G^* as the Green operator associated to L^*, the formal adjoint of operator L. It should be noticed that, when $a(\cdot,\cdot)$ is coercive in $H^1_0(\Omega)$, the operator $G^* : H^{-1}(\Omega) \to H^1_0(\Omega)$ is indeed the adjoint operator of $G : H^{-1}(\Omega) \to H^1_0(\Omega)$.*

Our interest is focused on a problem of the form

$$\begin{cases} Lu = f, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{B.15}$$

with $f \in L^1(\Omega)$.

Under assumptions (B.13) and (B.14) we know that $G^* \in \mathcal{L}(W^{-1,p}(\Omega), L^\infty(\Omega))\ \forall p > N$, so we know for its adjoint (see [94, Th. 4.5]),

$$G^{*t} : (L^\infty(\Omega))' \longrightarrow W_0^{1,q}(\Omega), \qquad \forall q = p' < \frac{N}{N-1}. \tag{B.16}$$

Since $L^1(\Omega) \subset (L^\infty(\Omega))'$, we can consider the restriction of G^{*t} to $L^1(\Omega)$, namely

$$G^{*t} : L^1(\Omega) \longrightarrow W_0^{1,q}(\Omega), \qquad 1 < q < \frac{N}{N-1}. \tag{B.17}$$

By definition of the adjoint operator, it holds that

$$\left\langle G^{*t} f, \varphi \right\rangle_{W_0^{1,q}, W^{-1,q'}} = \langle f, G^*\varphi \rangle_{L^1, L^\infty} = \int_\Omega f(G^*\varphi)\, \mathrm{d}\mathbf{x} \qquad \forall f \in L^1(\Omega),\ \forall \varphi \in W^{-1,q'}(\Omega).$$

Thus, given $f \in L^1(\Omega)$, we can think of $G^{*t} f = u$ as the unique solution to the problem

$$\begin{cases} u \in W_0^{1,q}(\Omega), \quad 1 < q < \frac{N}{N-1}, \\ \langle u, \varphi \rangle_{W_0^{1,q}, W^{-1,q'}} = \displaystyle\int_\Omega f(G^*\varphi)\, \mathrm{d}\mathbf{x} \quad \forall \varphi \in W^{-1,q'}(\Omega). \end{cases} \tag{B.18}$$

It is also known that $\mathcal{D}(\Omega)$, the space of indefinitely derivable functions with support contained in Ω, is dense in $W^{-1,q'}(\Omega)$, so the previous problem is equivalent to

$$\begin{cases} u \in W_0^{1,q}(\Omega), \quad 1 < q < \frac{N}{N-1}, \\ \displaystyle\int_\Omega u\varphi\, \mathrm{d}\mathbf{x} = \int_\Omega f(G^*\varphi)\, \mathrm{d}\mathbf{x} \quad \forall \varphi \in \mathcal{D}(\Omega), \end{cases} \tag{B.19}$$

which is also equivalent to

$$\begin{cases} u \in W_0^{1,q}(\Omega), \quad 1 < q < \frac{N}{N-1}, \\ \displaystyle\int_\Omega u(L^*\psi)\, \mathrm{d}\mathbf{x} = \int_\Omega f\psi\, \mathrm{d}\mathbf{x} \quad \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega) \ \text{ such that } L^*\psi \in \mathcal{D}(\Omega). \end{cases} \tag{B.20}$$

A solution to this problem is called solution by transposition to problem (B.15). Moreover, since we have constructed this solution from the linear operator G^{*t}, it follows that the solution by transposition exists and it is unique.

Note that the solution of (B.19) is independent of q, for $q \in (1, N/(N-1))$. Hence the solution by transposition satisfies

$$u \in \bigcap_{1<q<N/(N-1)} W_0^{1,q}(\Omega)\,. \tag{B.21}$$

In order to obtain an estimate for the solution by transposition, we use the following lemma, which is also included in [94].

Lemma B.1. *Let $\varphi(t)$ be a non-negative and non-increasing function, defined for all $t \geq k_0$, and such that for $h > k \geq k_0$ it satisfies*

$$\varphi(h) \leq \frac{C}{(h-k)^\alpha}[\varphi(k)]^\beta,$$

with C, α, β being positive constants. If $\beta > 1$ it holds

$$\varphi(k_0 + d) = 0,$$

with

$$d^\alpha = C[\varphi(k_0)]^{\beta-1} 2^{\alpha\beta/(\beta-1)}.$$

It is worth to note that, when the source data are regular enough, the solution by transposition coincides with the standard weak solution, as it is proved in the following proposition.

Proposition B.2. *Under the assumptions (B.2)-(B.5) and (B.13)-(B.14), for any $f \in L^{2^{*\prime}}(\Omega)$ the solutions to problems (B.9) and (B.20) coincide.*

Proof. Let u be the unique solution to problem (B.9). From the definitions of the operator L^* and the bilinear form $a(\cdot, \cdot)$, we know that

$$a(u, v) = \langle L^* v, u \rangle_{H^{-1}, H_0^1} \quad \forall v \in H_0^1(\Omega)\,,$$

hence

$$\langle L^* v, u \rangle_{H^{-1}, H_0^1} = \int_\Omega f v \, \mathrm{d}\mathbf{x} \quad \forall v \in H_0^1(\Omega)\,,$$

where the right-hand side can be written as a true integral, because $f \in L^{2^{*\prime}}(\Omega)$ and $H^1(\Omega) \subset L^{2^*}(\Omega)$. In particular,

$$\int_\Omega u L^* \psi \, \mathrm{d}\mathbf{x} = \int_\Omega f \psi \, \mathrm{d}\mathbf{x} \quad \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ such that } L^*\psi \in \mathcal{D}(\Omega),$$

and so u is a solution to (B.20). Since the solution by transposition is unique, both solutions must coincide. □

We will now introduce the concept of weak solution when the second member belongs to $L^1(\Omega)$, and prove that the solution by transposition is, in this case, also a weak solution to the problem.

Let us suppose that, for an operator L of the form (B.1), we have

$$a_{ij} \in L^\infty(\Omega), \quad 1 \leq i, j \leq N, \tag{B.22}$$

$$b_i, d_i \in L^{N+\varepsilon}(\Omega), \quad 1 \leq i \leq N, \tag{B.23}$$

$$c \in L^{(N+\varepsilon)/2}(\Omega), \tag{B.24}$$

for a certain $\varepsilon > 0$. We define the bilinear form

$$\begin{aligned} a_q : W_0^{1,q}(\Omega) \times W_0^{1,q'}(\Omega) &\to \mathbb{R} \\ (u, v) &\mapsto a_q(u, v) := \int_\Omega [A^t(\mathbf{x}) \mathbf{grad}\, u \cdot \mathbf{grad}\, v \\ &\quad + (\mathbf{b}(\mathbf{x}) \cdot \mathbf{grad}\, u) v + (\mathbf{d}(\mathbf{x}) \cdot \mathbf{grad}\, v) u + c(\mathbf{x}) u\, v] \, \mathrm{d}\mathbf{x}\,. \end{aligned}$$

Under the assumptions (B.22)-(B.24) the bilinear form $a_q(\cdot,\cdot)$ is well defined for any q such that $(1-\frac{1}{N+\varepsilon})^{-1} \le q < \frac{N}{N-1}$. Moreover, under the same assumptions the bilinear form is continuous.

Let $(1-\frac{1}{N+\varepsilon})^{-1} \le q < \frac{N}{N-1}$. We say that u is a weak solution in $W_0^{1,q}(\Omega)$ to problem (B.15) if

$$\begin{cases} u \in W_0^{1,q}(\Omega), \\ a_q(u,v) = \int_\Omega f v \, \mathrm{d}\mathbf{x}, \quad \forall v \in \mathcal{D}(\Omega). \end{cases} \tag{B.25}$$

Note that the space of test functions in (B.25) can be replaced by $W_0^{1,q'}(\Omega)$.

Proposition B.3. *Under the hypotheses (B.5), (B.13)-(B.14) and (B.22)-(B.24), and for* $(1-\frac{1}{N+\varepsilon})^{-1} \le q < \frac{N}{N-1}$ *the solution to problem (B.20) also solves problem (B.25).*

Proof. Let $f \in L^1(\Omega)$ and $u = G^{*^t} f$ its corresponding solution by transposition, *i.e.*, the solution to problem (B.20). Let $\{f_n\} \subset L^2(\Omega)$ such that $f_n \to f$ strongly in $L^1(\Omega)$. Let u_n be the corresponding weak solution in $H_0^1(\Omega)$ of (B.9) for f_n. By the result of Proposition B.2, u_n is also the solution by transposition, *i.e.*, $u_n = G^{*^t} f_n$. Moreover, since $G^{*^t} \in \mathcal{L}(L^1(\Omega), W_0^{1,q}(\Omega))$ (see eqn. (B.17)), we have $u_n = G^{*^t} f_n \to G^{*^t} f = u$ strongly in $W_0^{1,q}(\Omega)$, for any $1 \le q < N/(N-1)$.

Since $q < N/(N-1)$, we have the inclusions $W_0^{1,q'}(\Omega) \subset H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and $H_0^1(\Omega) \subset W_0^{1,q}(\Omega)$, so

$$a_q(u_n, v) = a(u_n, v) = \int_\Omega f_n v \, \mathrm{d}\mathbf{x} \quad \forall v \in W_0^{1,q'}(\Omega)\,.$$

Hence, due to the continuity of the bilinear form $a_q(\cdot,\cdot)$ we get

$$a_q(u, v) = \int_\Omega f v \, \mathrm{d}\mathbf{x} \quad \forall v \in W_0^{1,q'}(\Omega),$$

and since $u \in W_0^{1,q}(\Omega)$, we have proved that u is also a weak solution in $W_0^{1,q}(\Omega)$, *i.e.*, a solution of (B.25). □

Appendix C

Cylindrical coordinates.

In this appendix we present a brief review of the cylindrical coordinate system, including the expression of the most used operators in cylindrical coordinates.

Definition C.1. *The system of cylindrical coordinates is defined as the mapping*

$$f : \Omega = (0, \infty) \times (0, 2\pi) \times (-\infty, \infty) \to \mathbb{R}^3 \tag{C.1}$$
$$f(r, \theta, z) = (r \cos\theta, r \sin\theta, z)\,. \tag{C.2}$$

Definition C.2. *A local unitary basis for the cylindrical coordinate system is given by*

$$\begin{cases} \mathbf{e}_r(r,\theta,z) = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \\ \mathbf{e}_\theta(r,\theta,z) = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2, \\ \mathbf{e}_z(r,\theta,z) = \mathbf{e}_3. \end{cases} \tag{C.3}$$

where $\{\mathbf{e}_i\}$, $i = 1, \dots, 3$ *is the standard basis in the Cartesian coordinates system.*

Differential operators in cylindrical coordinates.

- *Gradient of scalar field*

$$\mathbf{grad}\,\varphi = \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\varphi}{\partial z}\mathbf{e}_z. \tag{C.4}$$

- *Gradient of a vector field*

$$\begin{aligned} \mathbf{grad}\,\mathbf{w} = {} & \frac{\partial w_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \left[\frac{1}{r}\frac{\partial w_r}{\partial\theta} - \frac{1}{r}w_\theta\right]\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{\partial w_r}{\partial z}\mathbf{e}_r \otimes \mathbf{e}_z \\ & + \frac{\partial w_\theta}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \left[\frac{1}{r}\frac{\partial w_\theta}{\partial\theta} + \frac{1}{r}w_r\right]\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{\partial w_\theta}{\partial z}\mathbf{e}_\theta \otimes \mathbf{e}_z \\ & + \frac{\partial w_z}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial w_z}{\partial\theta}\mathbf{e}_z \otimes \mathbf{e}_\theta + \frac{\partial w_z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z. \end{aligned} \tag{C.5}$$

- *Curl of a vector field*

$$\begin{aligned}\mathbf{curl}\,\mathbf{w} = &\left(\frac{1}{r}\frac{\partial w_z}{\partial \theta} - \frac{\partial w_\theta}{\partial z}\right)\mathbf{e}_r \\ &+ \left(-\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z}\right)\mathbf{e}_\theta \\ &+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r w_\theta\right) - \frac{1}{r}\frac{\partial w_r}{\partial \theta}\right)\mathbf{e}_z.\end{aligned} \tag{C.6}$$

- *Divergence of a vector field*

$$\operatorname{div}\mathbf{w} = \frac{1}{r}\frac{\partial}{\partial r}(r w_r) + \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z}. \tag{C.7}$$

- *Divergence of a tensor field*

$$\begin{aligned}\operatorname{div}\mathbf{S} = &\frac{1}{r}\left[\frac{\partial}{\partial r}(r S_{rr}) + \frac{\partial S_{r\theta}}{\partial \theta} + r\frac{\partial S_{rz}}{\partial z} - S_{\theta\theta}\right]\mathbf{e}_r \\ &+ \left[\frac{\partial S_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{\theta z}}{\partial z} + \frac{1}{r}S_{r\theta} + \frac{1}{r}S_{\theta r}\right]\mathbf{e}_\theta \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r S_{zr}) + \frac{\partial S_{z\theta}}{\partial \theta} + r\frac{\partial S_{zz}}{\partial z}\right]\mathbf{e}_z.\end{aligned} \tag{C.8}$$

- *Laplacian of a scalar field*

$$\begin{aligned}\Delta\varphi &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial \varphi}{\partial z}\right)\right] \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2}.\end{aligned} \tag{C.9}$$

- *Laplacian of a vector field*

$$\Delta\mathbf{w} = \left[\Delta w_r - \frac{1}{r^2}w_r - \frac{2}{r^2}\frac{\partial w_\theta}{\partial \theta}\right]\mathbf{e}_r + \left[\Delta w_\theta - \frac{1}{r^2}w_\theta + \frac{2}{r^2}\frac{\partial w_r}{\partial \theta}\right]\mathbf{e}_\theta + \Delta w_z \mathbf{e}_z. \tag{C.10}$$

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