A fully discrete BEM–FEM method for an exterior elasticity system in the plane

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Abstract

We present a modified version of the usual BEM–FEM coupling for the exterior elasticity problem in the plane, cf. [7]. This new formulation allows us to take advantage of techniques from [13] to compute the boundary integral terms using simple quadrature formulas. We provide error estimates for the Galerkin method and prove that the corresponding fully discrete scheme preserves the optimal rates of convergence.

Key words: exterior boundary value problem; boundary element methods; finite element methods

1 Introduction

The idea of coupling the finite element method (FEM) and the boundary element method (BEM) consists in compensating the deficiencies of each method with the advantages of the other one. Indeed, the FEM can only be used on bounded domains while the BEM requires linear equations with constant coefficients. Often, it is necessary to combine both of them to solve problems in exterior domains.

Much progress has been made in the numerical analysis of these methods since the first BEM–FEM coupling was introduced at the beginning of the eighties, cf. [14]. However, a lot remains to be done before these coupling procedures become popular tools for engineering calculations. For example, little is known about efficient algorithms to solve the

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complicated linear systems that arise from these formulations, cf. [16,10]. It is also difficult to control the effect of numerical integration on the convergence of these methods. The main result of this paper concerns contributions to the analysis of a fully discrete BEM–FEM coupling for an exterior elasticity problem in the plane.

The most popular BEM–FEM formulations are the *Johnson–Nedelec method* (cf. [14]) and the *symmetric method* (cf. [5,12]) which is used for the elasticity problem. It consists in dividing the exterior domain into a bounded inner region and an unbounded outer one by introducing an auxiliary common boundary. Next, the integral representation of the solution in the unbounded domain provides two non–local conditions on the auxiliary boundary for the problem in the inner region.

All authors (cf. [2],[9],[7]) choose a polygonal curve as an auxiliary boundary. At first glance, this election seems to be more suitable to deal with the discrete problem. However, in this case, it is not known how to control the effect of numerical integration on convergence. In this paper, we use a regular curve as an artificial boundary (as in [16,17]) and substitute all terms on this boundary by the corresponding periodic functions. This modified BEM–FEM formulation of the elasticity problem is equivalent to the usual one at the continuous level but it leads to a different Galerkin method that admits a completely discrete version by using elementary quadrature formulas.

The rest of the paper is organized as follows. In section 2, we present a new version of the symmetric BEM–FEM formulation for the elasticity problem and show that the corresponding variational problem is well posed. In section 3, we describe the discretization of the problem and provide an error analysis for the Galerkin scheme. In section 4, we introduce a family of full discretizations of the complete system of equations. Finally, in section 5 we prove that these numerical integration schemes preserve the optimal rates of convergence.

Next we describe some notations used throughout this paper. Let $\mathcal{O}$ be an open set in $\mathbb{R}^2$. We use the Hilbertian Sobolev spaces $H^m(\mathcal{O})$ endowed with their usual norms $\|\cdot\|_{m,\mathcal{O}}$. The inner product of $L^2(\mathcal{O}) = H^0(\mathcal{O})$ is denoted by $(\cdot,\cdot)_{0,\mathcal{O}}$. Finally, the spaces $W^{m,\infty}(\mathcal{O})$ are those Sobolev spaces derived from $L^\infty(\mathcal{O})$ (cf. [1]); we denote their norms and seminorms by $\|\cdot\|_{m,\infty,\mathcal{O}}$ and $|\cdot|_{m,\infty,\mathcal{O}}$, respectively.

We also consider periodic Sobolev spaces. Given a 1–periodic $\mathcal{C}^\infty$ function g, we define its Fourier coefficients

$$\hat{g}(k) := \int_0^1 g(s) e^{-2k\pi\imath s}\, ds, \qquad \forall\, k \in \mathbb{Z}.$$

Then, for each real number r, the 1-periodic Sobolev space H^r is the completion of the space of 1–periodic $\mathcal{C}^\infty$ functions with respect to the norm

$$\|g\|_r := \left(|\hat{g}(0)|^2 + \sum_{k\neq 0} |k|^{2r} |\hat{g}(k)|^2 \right)^{1/2}.$$

It is well known (cf. [19] or [15]) that H^r is a Hilbert space for each r. Moreover, the H^0–inner product

$$(\xi, \eta) := \int_0^1 \xi(s)\eta(s)\, ds,$$

can be extended to represent the duality between H^{-r} and H^r for each r. We will keep the same notation for this duality bracket.

On the other hand, since we will deal with vector unknowns, we need product forms of some spaces. Let H be a normed space. Then, we denote by $\mathbf{H} := H \times H$ the product space endowed with the usual product norm and the corresponding inner product if it exists. We will use the same notation for the inner product and norm of the product space.

We denote vectors and vector–valued functions by small boldface letters. Matrices and matrix–valued functions are denoted by capital boldface letters. The superscript $\top$ will denote transposition of a matrix. Finally, we denote by a dot the Euclidean inner product in $\mathbb{R}^2$ and by a colon the Euclidean inner product in $\mathbb{R}^{2\times 2}$, the space of real 2×2 matrices, i.e.,

$$\mathbf{u}\cdot\mathbf{v} := \sum_{i=1}^{2} u_i v_i, \qquad \mathbf{A}:\mathbf{B} := \sum_{i,j=1}^{2} A_{i,j}B_{i,j}.$$

In all what follows, C denotes a generic constant independent of the discretization parameter h.

2 The model problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ and let us denote by Ω' the complement of its closure $\overline{\Omega}$ in $\mathbb{R}^2$. Let $\mathbf{f}$ be a function with a compact support contained in Ω'. We consider the exterior Dirichlet problem for *the Lamé system*. This consists in finding a displacement vector $\mathbf{u}$ satisfying

$$\begin{aligned} -\sum_{j=1}^{2} \frac{\partial S_{ij}[\mathbf{u}]}{\partial x_j} &= f_i, && \text{in} \quad \Omega', \quad i=1,2, \\ \mathbf{u} &= \mathbf{0}, && \text{on} \quad \Gamma, \\ \mathbf{u}(\mathbf{x}) &= \mathcal{O}(1), && \text{as} \quad |\mathbf{x}| \to +\infty. \end{aligned} \tag{1}$$

We denoted by $\mathbf{S}[\mathbf{u}]$ the stress tensor

$$\mathbf{S}[\mathbf{u}] = \lambda(\nabla\cdot\mathbf{u})\,\mathbf{I} + 2\mu\,\mathbf{E}[\mathbf{u}],$$

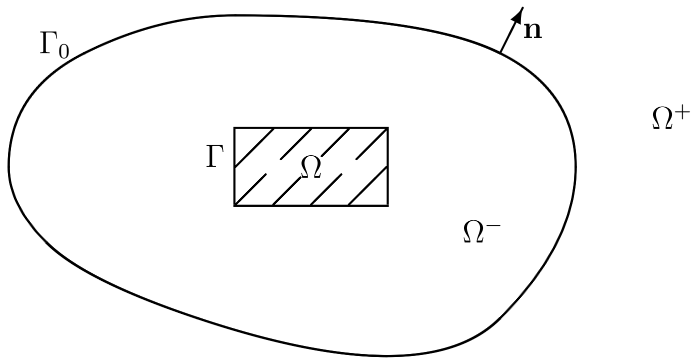

Fig. 1. Geometry of the problem

where $\lambda \geq 0$ and $\mu > 0$ are the Lamé constants, $\mathbf{I}$ is the identity matrix and $\mathbf{E}[\mathbf{u}]$ denotes the strain tensor

$$E_{ij}[\mathbf{u}] := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad 1 \leq i, j \leq 2.$$

Let Ω_0 be a simply connected bounded domain in $\mathbb{R}^2$ with a smooth boundary Γ_0, containing both the support of $\mathbf{f}$ and $\overline{\Omega}$ in its interior. The auxiliar boundary Γ_0 divides Ω' into two subdomains, $\Omega^- := \Omega_0 \cap \Omega'$ and $\Omega^+ := \Omega_0'$. We denote the limit onto Γ_0 of a function defined on Ω^+ or Ω^- by the superscript $+$ or $-$, respectively. Let $\mathbf{n}$ be the unit normal to Γ_0 oriented from Ω^- to Ω^+. We denote by $\mathbf{t}^\pm[\mathbf{u}] := \mathbf{S}[\mathbf{u}]^\pm\mathbf{n}$ the traction operator on Γ_0. Afterwards, problem (1) can be rewritten as an interior problem

$$\begin{aligned} -\sum_{j=1}^{2} \frac{\partial S_{ij}[\mathbf{u}]}{\partial x_j} &= f_i, \quad \text{in} \quad \Omega^-, \quad i = 1, 2, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on} \quad \Gamma, \end{aligned} \tag{2}$$

coupled with the exterior problem

$$\begin{aligned} -\sum_{j=1}^{2} \frac{\partial S_{ij}[\mathbf{u}]}{\partial x_j} &= 0, \qquad \text{in} \quad \Omega^+, \quad i = 1, 2, \\ \mathbf{u}(\mathbf{x}) &= \mathcal{O}(1), \quad \text{as} \quad |\mathbf{x}| \to +\infty, \end{aligned} \tag{3}$$

by means of the transmission conditions

$$\begin{aligned} \mathbf{u}^- &= \mathbf{u}^+, \\ \mathbf{t}^-[\mathbf{u}] &= \mathbf{t}^+[\mathbf{u}]. \end{aligned} \tag{4}$$

The variational formulation of the interior problem follows from completely standard arguments. We multiply the two equations of (2) by a test function v_i such that $v_i|_\Gamma = 0$, integrate over Ω^- and apply a Green's formula to obtain

$$a(\mathbf{u},\mathbf{v}) - \int_{\Gamma_0} \mathbf{t}^-[\mathbf{u}]\cdot\mathbf{v}\, d\sigma = (\mathbf{f},\mathbf{v})_{0,\Omega^-}, \qquad \forall\, \mathbf{v}\in \mathbf{H}^1_\Gamma(\Omega^-), \tag{5}$$

where $\mathbf{H}^1_\Gamma(\Omega^-)$ is the subspace of $\mathbf{H}^1(\Omega^-)$ formed by those functions $\mathbf{v}$ satisfying $\mathbf{v}|_\Gamma = \mathbf{0}$ and

$$a(\mathbf{u},\mathbf{v}) := \int_{\Omega^-} \{\lambda\,(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v}) + 2\mu\,\mathbf{E}[\mathbf{u}] : \mathbf{E}[\mathbf{v}]\}\, d\mathbf{x}, \qquad \forall\, \mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega^-).$$

The bounded bilinear form $a(\cdot,\cdot)$ is elliptic on $\mathbf{H}^1_\Gamma(\Omega^-)$ by virtue of Korn's inequality, i.e., there exists a constant $\alpha > 0$ such that

$$a(\mathbf{v},\mathbf{v}) \ge \alpha\|\mathbf{v}\|^2_{1,\Omega^-}, \qquad \forall\, \mathbf{v}\in\mathbf{H}^1_\Gamma(\Omega^-). \tag{6}$$

Let $\mathbf{U}$ be the fundamental tensor of the Lamé equation,

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = -\frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}\log|\mathbf{x}-\mathbf{y}|\,\mathbf{I} + \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\frac{(\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y})^\top}{|\mathbf{x}-\mathbf{y}|^2}.$$

We denote by $\mathbf{U}^i$ the column vectors of $\mathbf{U}$ and define

$$\mathbf{T}^\pm[\mathbf{U}] := (\mathbf{t}^\pm[\mathbf{U}^1], \mathbf{t}^\pm[\mathbf{U}^2])^\top.$$

Then, we can represent the solution of problem (3) through the Betti–Somigliana formula:

$$\mathbf{u}(\mathbf{x}) = \int_{\Gamma_0} \mathbf{T}^+_\mathbf{y}[\mathbf{U}(\mathbf{x},\mathbf{y})]\mathbf{u}^+(\mathbf{y})\, d\sigma_\mathbf{y} - \int_{\Gamma_0} \mathbf{U}(\mathbf{x},\mathbf{y})\mathbf{t}^+[\mathbf{u}](\mathbf{y})\, d\sigma_\mathbf{y} + \mathbf{c}, \qquad \forall\, \mathbf{x}\in\Omega^+, \tag{7}$$

where $\mathbf{c} = (c_1, c_2)^\top$ is a constant. In relation (7) the subscript $\mathbf{y}$ in operator $\mathbf{T}^+$ denotes differentiation with respect to the $\mathbf{y}$ variables and integration must be understood componentwise.

The symmetric method consists in coupling the variational formulation of the interior problem (5) with two boundary integral equations on Γ_0. These boundary integral equations are derived from (7) and they relate the Cauchy data $\mathbf{u}$ and $\mathbf{t}[\mathbf{u}]$ to each other on the artificial boundary Γ_0.

Letting $\mathbf{x}$ approach Γ_0 in equation (7) and taking into account the jump relations of the layer potentials (cf. [3]), we deduce the first boundary integral equation on Γ_0:

$$\frac{1}{2}\,\mathbf{u}^+(\mathbf{x}) - \int_{\Gamma_0} \mathbf{T}^+_\mathbf{y}[\mathbf{U}(\mathbf{x},\mathbf{y})]\,\mathbf{u}^+(\mathbf{y})\, d\sigma_\mathbf{y} + \int_{\Gamma_0} \mathbf{U}(\mathbf{x},\mathbf{y})\mathbf{t}^+[\mathbf{u}](\mathbf{y})\, d\sigma_\mathbf{y} - \mathbf{c} = \mathbf{0}. \tag{8}$$

We point out that the first integral in (8) exists as a principle value.

The second equation is obtained by applying the traction operator to (7) and using the jump relations of the layer potentials (cf. [3]),

$$\frac{1}{2}\,\mathbf{t}^+[\mathbf{u}](\mathbf{x}) = \int_{\Gamma_0} (\mathbf{T}^+_{\mathbf{x}}\big[\mathbf{T}^+_{\mathbf{y}}[\mathbf{U}(\mathbf{x},\mathbf{y})]\big])^\top \mathbf{u}^+(\mathbf{y}) d\sigma_{\mathbf{y}} - \int_{\Gamma_0} (\mathbf{T}^+_{\mathbf{x}}[\mathbf{U}(\mathbf{x},\mathbf{y})])^\top \mathbf{t}^+[\mathbf{u}](\mathbf{y}) d\sigma_{\mathbf{y}}. \quad (9)$$

Here, the kernel of the first operator on the right hand side is hypersingular. The corresponding operator is obtained by a regularisation of the divergent integral by the usual procedure, cf. [3].

Let $\boldsymbol{x}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ be a smooth regular 1-periodic parametric representation of Γ_0. We can define the parameterized trace onto Γ_0 as the unique extension of the mapping

$$\begin{aligned} \gamma \,:\, \mathcal{C}^\infty(\overline{\Omega}^-) &\longrightarrow H^0 \\ u \quad &\longmapsto \gamma u(\cdot) := u \circ \boldsymbol{x}(\cdot) \end{aligned}$$

to $H^1(\Omega^-)$. By the trace theorem, $\gamma\colon H^1(\Omega^-) \longrightarrow H^{1/2}$ is bounded and onto, cf. theorem 8.15 in [15].

The parameterized versions of the simple and double layer potentials are given by:

$$(\mathcal{V}\boldsymbol{\eta})(s) := \int_0^1 \mathbf{V}(s,t)\, \boldsymbol{\eta}(t)\, dt, \qquad (\mathcal{K}\boldsymbol{\eta})(s) := \int_0^1 \mathbf{K}(s,t)\, \boldsymbol{\eta}(t)\, dt,$$

where $\mathbf{V}(s,t) := \mathbf{U}(\boldsymbol{x}(s), \boldsymbol{x}(t))$ and

$$\begin{aligned} \mathbf{K}(s,t) &= |\boldsymbol{x}'(t)| \mathbf{T}^+_{\boldsymbol{x}(t)}[\mathbf{U}(\boldsymbol{x}(s), \boldsymbol{x}(t))] \\ &= \frac{\mu\, |\boldsymbol{x}'(t)|}{2\pi(\lambda+2\mu)} \left(\frac{(\boldsymbol{x}(s)-\boldsymbol{x}(t))\cdot \mathbf{n}(\boldsymbol{x}(t))}{|\boldsymbol{x}(s)-\boldsymbol{x}(t)|^2}\,\mathbf{I} - \frac{(\boldsymbol{x}(s)-\boldsymbol{x}(t))\cdot \boldsymbol{\tau}(\boldsymbol{x}(t))}{|\boldsymbol{x}(s)-\boldsymbol{x}(t)|^2}\,\widetilde{\mathbf{I}} \right) \\ &+ \frac{\lambda+\mu}{\pi(\lambda+2\mu)} |\boldsymbol{x}'(t)| \frac{(\boldsymbol{x}(s)-\boldsymbol{x}(t))(\boldsymbol{x}(s)-\boldsymbol{x}(t))^\top}{|\boldsymbol{x}(s)-\boldsymbol{x}(t)|^4} (\boldsymbol{x}(s)-\boldsymbol{x}(t))\cdot \mathbf{n}(\boldsymbol{x}(t)). \end{aligned}$$

Here, $\boldsymbol{\tau}$ is the tangent vector to Γ_0 and $\widetilde{\mathbf{I}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

In the sequel, we denote

$$\boldsymbol{\xi}(t) := |\boldsymbol{x}'(t)|\, \mathbf{t}^+[\mathbf{u}](\boldsymbol{x}(t)).$$

Using the representation formula (7), one can easily show that the behaviour of $\mathbf{u}$ at infinity is equivalent to a zero mean–value condition on $\mathbf{t}^+[\mathbf{u}](\mathbf{x})$ on Γ_0. It follows that

$$\int_0^1 \boldsymbol{\xi}(s)\, ds = \int_{\Gamma_0} \mathbf{t}^+[\mathbf{u}](\mathbf{x})\, d\sigma_{\mathbf{x}} = \mathbf{0}.$$

Then, parameterising equation (8), we obtain the following periodic integral equation:

$$(\frac{1}{2}\,\mathcal{I} - \mathcal{K})\,\gamma\mathbf{u}^+ + \mathcal{V}\,\boldsymbol{\xi} - \mathbf{c} = \mathbf{0}, \tag{10}$$

where $\mathcal{I}$ denotes the identity operator and γ is applied componentwise.

On the other hand, we recall the following relation from Gwinner and Stephan (cf. [11])

$$\int_{\Gamma_0} (\mathbf{T}^+_{\mathbf{x}}\big[\mathbf{T}^+_{\mathbf{y}}[\mathbf{U}(\mathbf{x},\mathbf{y})]\big])^\top \mathbf{u}^+(\mathbf{y})\, d\sigma_{\mathbf{y}} = \frac{\partial}{\partial \boldsymbol{\tau}(\mathbf{x})}(\int_{\Gamma_0} \mathbf{U}^*(\mathbf{x},\mathbf{y}) \frac{\partial \mathbf{u}^+(\mathbf{y})}{\partial \boldsymbol{\tau}(\mathbf{y})}\, d\sigma_{\mathbf{y}}),$$

where

$$\mathbf{U}^*(\mathbf{x},\mathbf{y}) = \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\,\{-\log|\mathbf{x}-\mathbf{y}|\,\mathbf{I} + \frac{(\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y})^\top}{|\mathbf{x}-\mathbf{y}|^2}\}.$$

Making use of the parameterization $\boldsymbol{x}(\cdot)$, we obtain

$$-\int_{\Gamma_0}(\int_{\Gamma_0} (\mathbf{T}^+_{\mathbf{x}}\big[\mathbf{T}^+_{\mathbf{y}}[\mathbf{U}(\mathbf{x},\mathbf{y})]\big])^\top \mathbf{u}^+(\mathbf{y})\, d\sigma_{\mathbf{y}})\,\mathbf{v}(\mathbf{x})\, d\sigma_{\mathbf{x}} = (\frac{d}{ds}\gamma\mathbf{v}, \mathcal{V}^*\,\frac{d}{ds}\gamma\mathbf{u}^+), \tag{11}$$

for all $\mathbf{v} \in \mathcal{C}^\infty(\overline{\Omega}^-)^2$, where operator $\mathcal{V}^*$ is formally given by

$$(\mathcal{V}^*\boldsymbol{\xi})(s) := \int_0^1 \mathbf{V}^*(s,t)\,\boldsymbol{\xi}(t)\, dt, \qquad \text{with} \qquad \mathbf{V}^*(s,t) := \mathbf{U}^*(\boldsymbol{x}(s),\boldsymbol{x}(t)).$$

Then, combining equations (9) and (5) and using relation (11), we obtain

$$a(\mathbf{u},\mathbf{v}) + (\frac{d}{ds}\gamma\mathbf{v}, \mathcal{V}^*\,\frac{d}{ds}\gamma\mathbf{u}^+) - ((\frac{1}{2}\mathcal{I} - \mathcal{K}')\,\boldsymbol{\xi}, \gamma\mathbf{v}) = (\mathbf{f},\mathbf{v})_{0,\Omega^-}, \qquad \forall\,\mathbf{v} \in \mathbf{H}^1_\Gamma(\Omega^-), \tag{12}$$

where $\mathcal{K}'$ is the adjoint of $\mathcal{K}$.

Let $H_0^{-1/2}$ be the subspace of $H^{-1/2}$ formed by those functions η satisfying $(\eta, 1) = 0$. Putting together equations (12) and (10) and using the transmission conditions (4) we obtain a weak formulation of problem (1):

$$\begin{aligned}
&\text{find } (\mathbf{u},\boldsymbol{\xi}) \in \mathbf{H}^1_\Gamma(\Omega^-) \times \mathbf{H}_0^{-1/2} \text{ such that}\\
&a(\mathbf{u},\mathbf{v}) + b^*(\frac{d}{ds}\gamma\mathbf{u}, \frac{d}{ds}\gamma\mathbf{v}) - c(\mathbf{v},\boldsymbol{\xi}) = (\mathbf{f},\mathbf{v})_{0,\Omega^-}, \quad \forall\,\mathbf{v} \in \mathbf{H}^1_\Gamma(\Omega^-),\\
&\qquad\qquad c(\mathbf{u},\boldsymbol{\eta}) + b(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \qquad \forall\,\boldsymbol{\eta} \in \mathbf{H}_0^{-1/2},
\end{aligned} \tag{13}$$

where we denoted

$$b(\boldsymbol{\xi},\boldsymbol{\eta}) = (\boldsymbol{\eta}, \mathcal{V}\boldsymbol{\xi}), \qquad b^*(\boldsymbol{\xi},\boldsymbol{\eta}) = (\boldsymbol{\eta}, \mathcal{V}^*\boldsymbol{\xi}) \quad \text{and} \quad c(\mathbf{v},\boldsymbol{\eta}) = (\boldsymbol{\eta}, (\frac{1}{2}\mathcal{I} - \mathcal{K})\,\gamma\mathbf{v}).$$

To prove that problem (13) is well posed we need the following properties of the integral operators $\mathcal{V}$, $\mathcal{K}$ and $\mathcal{V}^*$ defined before.

Lemma 1 *Operators* $\mathcal{V}\colon \mathbf{H}^{-1/2} \to \mathbf{H}^{1/2}$, $\mathcal{K}\colon \mathbf{H}^{1/2} \to \mathbf{H}^{1/2}$ *and* $\mathcal{V}^*\colon \mathbf{H}^{-1/2} \to \mathbf{H}^{1/2}$ *are linear and bounded. Furthermore, there exists a constant* $\beta > 0$ *such that*

$$(\boldsymbol{\eta}, \mathcal{V}\boldsymbol{\eta}) \geq \beta \|\boldsymbol{\eta}\|^2_{-1/2}, \qquad \forall\, \boldsymbol{\eta} \in \mathbf{H}_0^{-1/2} \tag{14}$$

and the operator $-\frac{d}{ds}\mathcal{V}^*\,\frac{d}{ds}\colon \mathbf{H}^{1/2} \longrightarrow \mathbf{H}^{-1/2}$ *is nonnegative, i.e.,*

$$(\frac{d\mathbf{g}}{ds}, \mathcal{V}^*\,\frac{d\mathbf{g}}{ds}) \geq 0, \qquad \forall\, \mathbf{g} \in \mathbf{H}^{1/2}. \tag{15}$$

PROOF. One can easily show that both $\mathcal{V}$ and $\mathcal{K}$ inherit the properties of the classical simple and double layer potentials proved in [7] or [3]. On the other hand, as $\gamma\colon H^1(\Omega^-) \longrightarrow H^{1/2}$ is onto, for any $\mathbf{g} \in \mathbf{H}^{1/2}$, there exists a function $\mathbf{u} \in \mathbf{H}^1(\Omega^-)$ such that $\gamma\mathbf{u} = \mathbf{g}$ and by virtue of relation (11),

$$(\frac{d\mathbf{g}}{ds}, \mathcal{V}^*\,\frac{d\mathbf{g}}{ds}) = -\int_{\Gamma_0}(\int_{\Gamma_0}(\mathbf{T}^+_{\mathbf{x}}[\mathbf{T}^+_{\mathbf{y}}[\mathbf{U}(\mathbf{x},\mathbf{y})]])^\top \mathbf{u}(\mathbf{y})\, d\sigma_{\mathbf{y}})\, \mathbf{u}(\mathbf{x})\, d\sigma_{\mathbf{x}}$$

where the right hand side is nonnegative (cf. [7]). ∎

We denote by $\mathbf{M}$ the product space $\mathbf{H}^1_\Gamma(\Omega^-) \times \mathbf{H}_0^{-1/2}$ endowed with its natural inner product and the induced norm $\|\cdot\|_{\mathbf{M}}$. Consider the bounded bilinear form $A\colon \mathbf{M}\times\mathbf{M} \longrightarrow \mathbb{R}$ obtained by adding the left hand sides of (13), i.e.,

$$A(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = a(\mathbf{u}, \mathbf{v}) + b^*(\frac{d}{ds}\gamma\mathbf{u}, \frac{d}{ds}\gamma\mathbf{v}) - c(\mathbf{v}, \boldsymbol{\xi}) + b(\boldsymbol{\xi}, \boldsymbol{\eta}) + c(\mathbf{u}, \boldsymbol{\eta}),$$

where we denoted the elements of $\mathbf{M}$ by $\hat{\mathbf{u}} := (\mathbf{u}, \boldsymbol{\xi})$ and $\hat{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta})$. It turns out that $A(\cdot,\cdot)$ is $\mathbf{M}$–elliptic since (6), (14) and (15) give

$$A(\hat{\mathbf{v}}, \hat{\mathbf{v}}) \geq \alpha\|\mathbf{v}\|^2_{1,\Omega^-} + \beta\|\boldsymbol{\eta}\|^2_{-1/2} \geq \tilde{\alpha}\|\hat{\mathbf{v}}\|^2_{\mathbf{M}}, \qquad \forall\, \hat{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{M}, \tag{16}$$

with $\tilde{\alpha} := \min\{\alpha, \beta\}$. Let $L\colon \mathbf{M} \longrightarrow \mathbb{R}$ be the bounded linear functional defined by

$$L(\hat{\mathbf{v}}) = (\mathbf{f}, \mathbf{v})_{0,\Omega^-}, \qquad \forall\, \hat{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{M}.$$

With these notations, problem (13) may be written

$$\begin{aligned} &\text{find } \hat{\mathbf{u}} \in \mathbf{M} \text{ such that} \\ &A(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = L(\hat{\mathbf{v}}), \quad \forall\, \hat{\mathbf{v}} \in \mathbf{M}. \end{aligned} \tag{17}$$

Existence and uniqueness of a solution to problem (17) follow immediately from Lax–Milgram lemma.

3 The discrete problem

3.1 Curved triangulation of the bounded domain

For simplicity of exposition we assume that Γ is a polygonal curve. Given a positive integer N and $h := 1/N$, let $\{s_i := i\, h; \quad i = 0, \cdots, N\}$ be the induced uniform partition of $[0,1]$. We denote by Ω_h the polygonal domain whose vertices lying on Γ_0 are $\Delta_h := \{\boldsymbol{x}(s_i)\}_{i=1}^{N}$. Let τ_h be a triangulation of $\overline{\Omega}_h$ by triangles T of diameter h_T not greater than Ch. We assume that any vertex of a triangle lying on the exterior boundary of Ω_h belongs to Δ_h. We also suppose that the family of triangulations $\{\tau_h\}_h$ is regular in the sense of [4].

We obtain from τ_h a triangulation τ_h^- of $\overline{\Omega}^-$ by replacing each triangle of τ_h with one side along the exterior part of $\partial\Omega_h$ by the corresponding curved triangle.

Let T be a curved triangle of τ_h^-. We denote its vertices by $\mathbf{P}_{1,T}$, $\mathbf{P}_{2,T}$ and $\mathbf{P}_{3,T}$, numbered in such a way that there exists an index i such that $\boldsymbol{x}(s_{i-1}) = \mathbf{P}_{2,T}$ and $\boldsymbol{x}(s_i) = \mathbf{P}_{3,T}$. Then, the mapping $\boldsymbol{\varphi} : [0,1] \to \mathbb{R}^2$ defined by

$$\boldsymbol{\varphi}(s) := \boldsymbol{x}\left(s_{i-1} + s\, h\right), \qquad s \in [0,1],$$

is a parameterization of the curved side of T.

Let $\widehat{T}$ be the reference triangle with vertices $\widehat{\mathbf{P}}_1 := (0,0)^\top$, $\widehat{\mathbf{P}}_2 := (1,0)^\top$ and $\widehat{\mathbf{P}}_3 := (0,1)^\top$. Consider the affine mapping $\mathbf{G}_T$ defined by $\mathbf{G}_T(\widehat{\mathbf{P}}_i) = \mathbf{P}_{i,T}$ for $i \in \{1,2,3\}$ and the function $\boldsymbol{\Theta}_T : \widehat{T} \to \mathbb{R}^2$ given by

$$\boldsymbol{\Theta}_T(\hat{\mathbf{x}}) := \frac{\hat{x}_1}{1-\hat{x}_2}\left(\boldsymbol{\varphi}(\hat{x}_2) - (1-\hat{x}_2)\mathbf{P}_{2,T} - \hat{x}_2\mathbf{P}_{3,T}\right), \qquad \forall\, \hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2) \in \widehat{T},$$

where the limiting value has to be taken when $\hat{x}_2$ tends to 1. We then introduce the $\mathcal{C}^\infty$ mapping $\mathbf{F}_T : \widehat{T} \to \mathbb{R}^2$ given by

$$\mathbf{F}_T := \mathbf{G}_T + \boldsymbol{\Theta}_T.$$

It is proved in theorem 22.4 of [20] that $\mathbf{F}_T$ is a $\mathcal{C}^\infty$–diffeomorphism from $\widehat{T}$ onto T. Moreover, $\boldsymbol{\Theta}_T(0,s) = \boldsymbol{\Theta}_T(s,0) = (0,0)^\top$ and $\mathbf{F}_T(s, 1-s) = \boldsymbol{\varphi}(s)$ for all $s \in [0,1]$. Then each side of $\widehat{T}$ is mapped onto the corresponding side of T.

On each curved triangle T, a finite element may be defined by the triplet $(T, P_1(T), \Sigma_T)$, where $P_1(T)$ is the space of functions defined on T with pullback in the space P_1 of polynomials of degree not greater than one:

$$P_1(T) := \{p : T \to \mathbb{R} \;:\; p \circ \mathbf{F}_T \in P_1\}$$

and $\Sigma_T := \{N_{i,T} : i = 1,2,3\}$ is the set of Lagrange functionals: $N_{i,T}(\phi) := \phi(\mathbf{P}_{i,T})$. It is easy to show that Σ_T is $P_1(T)-$unisolvent (cf.[4]). It is also important to note that on each side of T, a function $\phi \in P_1(T)$ is uniquely determined by its nodal values corresponding to that side. On straight triangles we use the classical P_1-finite element.

Under the assumption of regularity of $\{\tau_h\}$, theorem 22.4 in [20] proves that, for curved triangles T, the Jacobian J_T of $\mathbf{F}_T$ does not vanish on a neighborhood of $\widehat{T}$ and the following estimates hold:

$$C_1 h_T^2 \le |J_T(\cdot)| \le C_2 h_T^2, \tag{18}$$

$$|\mathbf{F}_T|_{k,\infty,\widehat{T}} \le C h_T^k, \qquad k = 1, 2, \tag{19}$$

$$|\mathbf{F}_T^{-1}|_{1,\infty,T} \le C h_T^{-1}. \tag{20}$$

These properties of $\mathbf{F}_T$ and the usual technique used in the affine case permit to obtain interpolation error bounds on curved triangles (cf. section 4.3 of [4]). Namely, there exists a constant C independent of T such that

$$|v - \pi_T v|_{1,T} \le \; C h_T \|v\|_{2,T} \quad \forall v \in H^2(T), \tag{21}$$

where $\pi_T v \in P_1(T)$ and is uniquely determined by $\pi_T v(\mathbf{P}_{i,T}) = v(\mathbf{P}_{i,T})$ for $i = 1, 2, 3$. Notice that in the case of straight triangles, we obtain the same estimate with the seminorm of $H^2(T)$ instead of the norm on the right hand side.

3.2 Discrete spaces and Galerkin scheme

We will seek the approximate displacement field in

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathcal{C}^0(\overline{\Omega}^-, \mathbb{R}^2) : \mathbf{v}_h|_T \in \mathbf{P}_1(T), \forall\, T \in \tau_h^-\} \cap \mathbf{H}^1_\Gamma(\Omega^-),$$

where, as usual, $\mathbf{P}_1(T) = P_1(T) \times P_1(T)$. On the other hand, we define

$$\mathbf{H}_h := \{\boldsymbol{\eta}_h \in \mathbf{L}^2(0,1) \;:\; \boldsymbol{\eta}_h|_{(s_{i-1},s_i)} \in \mathbf{P}_0,\; i = 1, \ldots, N\} \cap \mathbf{H}_0^{-1/2},$$

where P_0 is the space of constant functions.

The discrete problem associated to the variational formulation (13) consists in finding $(\mathbf{u}_h, \boldsymbol{\xi}_h) \in \mathbf{V}_h \times \mathbf{H}_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b^*(\frac{d}{ds}\gamma(\mathbf{u}_h), \frac{d}{ds}\gamma(\mathbf{v}_h)) - c(\mathbf{v}_h, \boldsymbol{\xi}_h) &= (\mathbf{f}, \mathbf{v}_h)_{0,\Omega^-}, \qquad && \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ c(\mathbf{u}_h, \boldsymbol{\eta}_h) + b(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) &= 0, && \forall\, \boldsymbol{\eta}_h \in \mathbf{H}_h. \end{aligned} \tag{22}$$

Let us introduce the space $\mathbf{M}_h := \mathbf{V}_h \times \mathbf{H}_h$. Problem (22) can be equivalently written

$$\begin{aligned}&\text{find } \hat{\mathbf{u}}_h \in \mathbf{M}_h \text{ such that}\\ &A(\hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h) = L(\hat{\mathbf{v}}_h), \qquad \forall\, \hat{\mathbf{v}}_h \in \mathbf{M}_h.\end{aligned} \tag{23}$$

The ellipticity of $A(\cdot,\cdot)$ implies that this problem is well posed and we have the following Céa's inequality:

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega^-} + \|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{-1/2} \le C\,\big(\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u}-\mathbf{v}_h\|_{1,\Omega^-} + \inf_{\boldsymbol{\eta}_h\in\mathbf{H}_h} \|\boldsymbol{\xi}-\boldsymbol{\eta}_h\|_{-1/2}\big). \tag{24}$$

Theorem 2 *If* $\mathbf{u}$ *belongs to* $\mathbf{H}^2(\Omega^-)$ *then*

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega^-} + \|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{-1/2} \le Ch\,\|\mathbf{u}\|_{2,\Omega^-}.$$

PROOF. The local interpolation error estimates (21) lead to the following inequality:

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{v}-\mathbf{v}_h\|_{1,\Omega^-} \le Ch\|\mathbf{v}\|_{2,\Omega^-}, \qquad \forall\, \mathbf{v}\in \mathbf{H}^1_\Gamma(\Omega^-)\cap\mathbf{H}^2(\Omega^-) \tag{25}$$

and classical approximation properties in periodic Sobolev spaces (cf. [18]) give

$$\inf_{\boldsymbol{\eta}_h\in\mathbf{H}_h} \|\boldsymbol{\eta}-\boldsymbol{\eta}_h\|_{-1/2} \le Ch\|\boldsymbol{\eta}\|_{1/2}, \qquad \forall\, \boldsymbol{\eta}\in \mathbf{H}_0^{-1/2}\cap\mathbf{H}^{1/2}. \tag{26}$$

We deduce the result from inequalities (25) and (26) together with (24) and the trace theorem. ■

4 Full discretization of the equations

In this section, we describe the different quadratures used to approximate the integrals in (22). We begin by the interior terms. Let $\widehat{Q}$ be a quadrature formula on the reference triangle $\widehat{T}$:

$$\widehat{Q}(\varphi) := \sum_{k=1}^{d_0} \hat{\omega}_k \varphi(\hat{\mathbf{b}}_k) \simeq \int_{\widehat{T}} \varphi(\hat{\mathbf{x}})\, d\hat{\mathbf{x}}.$$

We assume that $\widehat{Q}$ is exact for constant functions; i.e., $\sum_{k=1}^{d_0} \hat{\omega}_k = 1/2$. The corresponding formula Q_T on a given triangle $T \in \tau_h^-$ is obtained by a simple change of variable

$$Q_T(\phi) := \widehat{Q}(|J_T|\,\hat{\phi}) = \sum_{k=1}^{d_0} \hat{\omega}_k |J_T|(\hat{\mathbf{b}}_k)\hat{\phi}(\hat{\mathbf{b}}_k) \simeq \int_T \phi(\mathbf{x})\, d\mathbf{x},$$

where we denoted $\hat{\phi} := \phi \circ \mathbf{F}_T$. We approximate the linear form $L(\cdot)$ by

$$L_h(\hat{\mathbf{v}}_h) := \sum_{T \in \tau_h^-} Q_T(\mathbf{f} \cdot \mathbf{v}_h)$$

on $\mathbf{M}_h$ and the bilinear form $a(\cdot,\cdot)$ by

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T \in \tau_h^-} Q_T(\lambda\,(\nabla \cdot \mathbf{u}_h)(\nabla \cdot \mathbf{v}_h) + 2\,\mu\,\mathbf{E}[\mathbf{u}_h] : \mathbf{E}[\mathbf{v}_h])$$

on $\mathbf{V}_h \times \mathbf{V}_h$.

For the boundary terms, we need a basic quadrature formula on the unit square:

$$\hat{\ell}_2(g) := \sum_{k=1}^{d_1} \eta_k g(\mathbf{x}_k) \simeq \int_0^1 \int_0^1 g(s,t)\, ds dt.$$

We assume that $\hat{\ell}_2$ is exact for polynomial functions of degree not greater than one. In the following, we introduce three different types of approximations:

1. Numerical quadratures must be handled with care when defining an approximation of $b(\cdot,\cdot)$ on $\mathbf{H}_h \times \mathbf{H}_h$ because of the logarithmic singularity of $\mathbf{V}$. Here, we follow [13] and consider the following decomposition of the kernel:

$$\mathbf{V}(s,t) = -C_{\lambda,\mu}\, \log |s-t|^2\, \mathbf{I} + \mathbf{B}(s,t),$$

where $C_{\lambda,\mu} = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}$. Notice that the matrix valued function $\mathbf{B}(\cdot,\cdot)$ is of class $\mathcal{C}^\infty$ in the domain $D_1 = \{(s,t) \in [0,1] \times [0,1] : |s-t| < 1\}$. Now, the strategy consists in using $\hat{\ell}_2$ to approximate the second integral and compute the first one exactly (cf. [13] or [8]); i.e.,

$$\int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_j} \mathbf{V}(s,t)\, ds\, dt \simeq \mathbf{V}_{i,j} := h^2\, \left(\hat{\ell}_2(\mathbf{B}(s_{\underline{i}-1} + h\,\cdot, s_{\underline{j}-1} + h\,\cdot)) - C_{\lambda,\mu}(\log h^2 + B_{\underline{i}-\underline{j}})\mathbf{I} \right),$$

with

$$B_k := \int_0^1 \int_0^1 \log |k + t - s|^2\, dt\, ds, \qquad \forall\, k \in \mathbf{Z}$$

and

$$(\underline{i}, \underline{j}) := \begin{cases} (i,j), & \text{if } \ |i-j| \le N/2, \\ (i, j+N), & \text{if } \ \ i-j \ > N/2, \\ (i, j-N), & \text{if } \ \ j-i \ > N/2. \end{cases}$$

Notice that the periodicity of $\mathbf{V}(\cdot,\cdot)$ allows one to use the indices $(\underline{i}, \underline{j})$ instead of (i,j) and avoid the neighbourhood of the region $\{(s,t); \quad |s-t| = 1\}$.

Let us denote $\boldsymbol{\eta}^i$ the constant value of a given function $\boldsymbol{\eta}_h \in \mathbf{H}_h$ on (s_{i-1}, s_i), $(1 \le i \le N)$. Then, for any $\boldsymbol{\xi}_h$, $\boldsymbol{\eta}_h \in \mathbf{H}_h$, we define

$$b_h(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) := \sum_{i,j=1}^{N} (\boldsymbol{\eta}^i)^\top \mathbf{V}_{i,j}\, \boldsymbol{\xi}^j.$$

2. Notice that if $\mathbf{u}_h$ is a function in $\mathbf{V}_h$, then $\gamma \mathbf{u}_h$ belongs to the space $\mathbf{T}_h$, where

$$T_h = \{\eta_h \in \mathcal{C}^0(\mathbb{R}) \,:\, \eta_h|_{(s_{i-1},s_i)} \in P_1,\ 1 \le i \le N;\ \eta_h(s) = \eta_h(s+1),\ \forall\, s \in \mathbb{R}\}.$$

Hence, for any $\mathbf{v}_h \in \mathbf{V}_h$, $\frac{d}{ds}\gamma \mathbf{v}_h \in \mathbf{H}_h$ and it suffices to approximate $b^*(\cdot,\cdot)$ on $\mathbf{H}_h \times \mathbf{H}_h$ by the same technique given in the previous case. Indeed, the singularity of $\mathbf{V}^*(\cdot,\cdot)$ is removed as above to obtain

$$\int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_j} \mathbf{V}^*(s,t) \simeq \mathbf{V}^*_{i,j} := h^2 \left(\hat{\ell}_2(\mathbf{B}^*(s_{\underline{i}-1} + h\,\cdot, s_{\underline{j}-1} + h\,\cdot)) - C^*_{\lambda,\mu}(\log h^2 + B_{\underline{i}-\underline{j}})\mathbf{I} \right),$$

where $C^*_{\lambda,\mu} = \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)}$ and $\mathbf{B}^*(\cdot,\cdot)$ is a matrix–valued function of class $\mathcal{C}^\infty$ in the domain D_1. Afterwards, we define the perturbed bilinear form

$$b^*_h(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) := \sum_{i,j=1}^{N} (\boldsymbol{\eta}^i)^\top \mathbf{V}^*_{i,j}\, \boldsymbol{\xi}^j.$$

3. It remains to define an approximation $c_h(\mathbf{v}_h, \boldsymbol{\eta}_h)$ of $c(\mathbf{v}_h, \boldsymbol{\eta}_h)$ on $\mathbf{V}_h \times \mathbf{H}_h$. Let $\{\ell_i\}_{i=1}^N$ be the set of nodal basis functions of T_h, i.e., $\ell_i(s_j) = \delta_{i,j}$, for all $1 \le i, j \le N$. Thus, it suffices to provide a quadrature scheme for

$$\int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_{j+1}} \ell_j(t)\, \mathbf{K}(s,t)\, dt\, ds, \tag{27}$$

where $s_{N+1} = 1 + h$.

To treat the singularity of the kernel of operator $\mathcal{K}$ we use the decomposition

$$\mathbf{K}(s,t) = -\frac{\mu}{2\pi(\lambda+2\mu)} \frac{1}{s-t} \widetilde{\mathbf{I}} + \mathbf{C}(s,t).$$

Notice that $\mathbf{C}(\cdot,\cdot)$ is a matrix–valued function of class $\mathcal{C}^\infty$. Then (27) writes

$$-\frac{\mu}{2\pi(\lambda+2\mu)} \int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_{j+1}} \frac{1}{s-t}\, \ell_j(t)\, \widetilde{\mathbf{I}}\, dt\, ds + \int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_{j+1}} \ell_j(t)\, \mathbf{C}(s,t)\, dt\, ds. \tag{28}$$

Computing the first term of (28) exactly we obtain

$$\int_{s_{i-1}}^{s_i}\int_{s_{j-1}}^{s_{j+1}} \frac{1}{s-t}\,\ell_j(t)\,\widetilde{\mathbf{I}}\,dt\,ds = \frac{h}{2}\,A_{i-j}\,\widetilde{\mathbf{I}},$$

where the coefficients A_k are determined by the conditions: $A_{-k} = -A_{k+1}\ \forall k \in \mathbb{N}$, and

$$A_k = (k+1)^2\log(1-\frac{1}{k^2}) - 4(k+\frac{1}{2})\log(1-\frac{1}{k}) - k^2\log(1-\frac{1}{(k-1)^2}) + 4(k-1)\log(1-\frac{1}{k-1}),$$

for $k > 2$ with $A_1 = 4\log 2$ and $A_2 = 9\log 3 - 12\log 2$. The second term of (28) is approximated by $h^2\,\hat{\ell}_2(\mathbf{C}^{ij})$, where we denoted

$$\mathbf{C}^{ij}(s,t) := \ell_j(s_{j-1}+ht)\,\mathbf{C}(s_{i-1}+hs, s_{j-1}+ht) + \ell_j(s_j+ht)\,\mathbf{C}(s_{i-1}+hs, s_j+ht).$$

It follows that

$$c_h(\mathbf{v}_h, \boldsymbol{\eta}_h) = \frac{1}{2}(\boldsymbol{\eta}_h, \gamma\mathbf{v}_h) - \sum_{i,j=1}^{N} (\boldsymbol{\eta}^i)^\top \Big(h^2\,\hat{\ell}_2(\mathbf{C}^{ij}) - \frac{\mu}{2\pi(\lambda+2\mu)}\frac{h}{2}\,A_{i-j}\,\widetilde{\mathbf{I}}\Big)\gamma\mathbf{v}_h(s_j).$$

We are now in a position to write the fully discrete scheme associated to problem (13),

$$\begin{aligned}
&\text{find } \mathbf{u}_h^* \in \mathbf{V}_h \text{ and } \boldsymbol{\xi}_h^* \in \mathbf{H}_h;\\
&a_h(\mathbf{u}_h^*, \mathbf{v}_h) + b_h^*(\frac{d}{ds}\gamma\mathbf{u}_h^*, \frac{d}{ds}\gamma\mathbf{v}_h) - c_h(\mathbf{v}_h, \boldsymbol{\xi}_h^*) = L_h(\mathbf{v}_h), \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \qquad (29)\\
&\qquad b_h(\boldsymbol{\xi}_h^*, \boldsymbol{\eta}_h) + c_h(\mathbf{u}_h^*, \boldsymbol{\eta}_h) = 0, \qquad \forall\, \boldsymbol{\eta}_h \in \mathbf{H}_h.
\end{aligned}$$

5 Analysis of the fully discrete scheme

In this section, we study the stability and convergence of the fully discrete scheme (29). We begin with some bounds related to the five kinds of approximations presented in the last section.

The following results concern estimates on the error committed when approximating the right hand side and the energy form $a(\cdot,\cdot)$. They follow readily from lemma 26.7 and lemma 26.6 in [20].

Lemma 3 *If $\mathbf{f} \in \mathbf{W}^{1,\infty}(\Omega^-)$, then there exists a constant C independent of h such that*

$$|L(\mathbf{v}_h) - L_h(\mathbf{v}_h)| \le Ch\,\|\mathbf{f}\|_{1,\infty,\Omega^-}\|\mathbf{v}_h\|_{1,\Omega^-}, \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h.$$

Lemma 4 *There exists a constant C independent of h such that*

$$|a(\mathbf{u}_h, \mathbf{v}_h) - a_h(\mathbf{u}_h, \mathbf{v}_h)| \leq Ch\, \|\mathbf{u}_h\|_{1,\Omega^-} \|\mathbf{v}_h\|_{1,\Omega^-}, \qquad \forall\, \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

For the bilinear forms associated to the boundary integral operators we have the following results.

Lemma 5 *There exists a constant C independent of h such that*

$$|b(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) - b_h(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h)| \leq Ch\, \|\boldsymbol{\xi}_h\|_{-1/2} \|\boldsymbol{\eta}_h\|_{-1/2}, \qquad \forall\, \boldsymbol{\xi}_h, \boldsymbol{\eta}_h \in \mathbf{H}_h.$$

PROOF. See lemma 11 in [8]. ■

Lemma 6 *There exists a constant C independent of h such that*

$$|b^*(\frac{d}{ds}\gamma\mathbf{u}_h, \frac{d}{ds}\gamma\mathbf{v}_h) - b^*_h(\frac{d}{ds}\gamma\mathbf{u}_h, \frac{d}{ds}\gamma\mathbf{v}_h)| \leq Ch\, \|\mathbf{u}_h\|_{1,\Omega^-} \|\mathbf{v}_h\|_{1,\Omega^-}, \qquad \forall\, \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

PROOF. Lemma 5 shows that there exists a constant $C > 0$ such that

$$|b^*(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) - b^*_h(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h)| \leq Ch\, \|\boldsymbol{\xi}_h\|_{-1/2} \|\boldsymbol{\eta}_h\|_{-1/2}, \qquad \forall\, \boldsymbol{\xi}_h, \boldsymbol{\eta}_h \in \mathbf{H}_h.$$

Therefore,

$$|b^*(\frac{d}{ds}\gamma\mathbf{u}_h, \frac{d}{ds}\gamma\mathbf{v}_h) - b^*_h(\frac{d}{ds}\gamma\mathbf{u}_h, \frac{d}{ds}\gamma\mathbf{v}_h)| \leq Ch\, \|\frac{d}{ds}\gamma\mathbf{u}_h\|_{-1/2} \|\frac{d}{ds}\gamma\mathbf{v}_h\|_{-1/2}$$

and the boundness of operators $\frac{d}{ds} : H^{1/2} \longrightarrow H^{-1/2}$ and γ imply the result. ■

Lemma 7 *There exists a constant C independent of h such that*

$$|c(\mathbf{v}_h, \boldsymbol{\eta}_h) - c_h(\mathbf{v}_h, \boldsymbol{\eta}_h)| \leq Ch^{3/2}\, \|\mathbf{v}_h\|_{1,\Omega^-} \|\boldsymbol{\eta}_h\|_{-1/2}, \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \quad \forall\, \boldsymbol{\eta}_h \in \mathbf{H}_h.$$

PROOF. For any $\mathbf{v}_h \in \mathbf{V}_h$ and $\boldsymbol{\eta}_h \in \mathbf{H}_h$ we have

$$c(\mathbf{v}_h, \boldsymbol{\eta}_h) - c_h(\mathbf{v}_h, \boldsymbol{\eta}_h) = h^2 \sum_{i,j=1}^{N} (\boldsymbol{\eta}^i)^\top\, \hat{e}_2(\mathbf{C}^{ij})\, \gamma\mathbf{v}_h(s_j),$$

where $\hat{e}_2(\cdot) := \int_0^1 \int_0^1 \cdot - \hat{\ell}_2(\cdot)$ is the error functional. Since $\hat{\ell}_2$ is of degree 1 on $D := (0,1) \times (0,1)$, it follows readily from the Bramble-Hilbert lemma that

$$|\hat{e}_2(\mathbf{C}^{ij})| \leq C\, |\mathbf{C}^{ij}|_{2,\infty,D}$$

and the chain rule shows that

$$|\mathbf{C}^{i\,j}|_{2,\infty,D} \leq Ch^2 \|\mathbf{C}\|_{2,\infty,D},$$

Therefore, we have the following estimate

$$|c(\mathbf{v}_h, \boldsymbol{\eta}_h) - c_h(\mathbf{v}_h, \boldsymbol{\eta}_h)| \leq Ch^4 \, \|\mathbf{C}\|_{2,\infty,D} \sum_{i=1}^{N} |\boldsymbol{\eta}^i| \sum_{j=1}^{N} |\gamma \mathbf{v}_h(s_j)|.$$

On the other hand, a well known inverse inequality leads to

$$h \sum_{i=1}^{N} |\boldsymbol{\eta}^i| = \int_0^1 |\boldsymbol{\eta}_h(s)| \, ds \leq \|\boldsymbol{\eta}_h\|_0 \leq Ch^{-1/2} \, \|\boldsymbol{\eta}_h\|_{-1/2} \quad \forall \boldsymbol{\eta}_h \in \mathbf{H}_h$$

and the equivalence of the norms $g \rightarrow \|g\|_0$ and $g \rightarrow (h \sum_{i=1}^{N} g(t_i)^2)^{1/2}$ on T_h together with the trace theorem provide

$$\sum_{j=1}^{N} |\gamma \mathbf{v}_h(s_j)| \leq h^{-1} \left(h \sum_{j=1}^{N} |\gamma \mathbf{v}_h(s_j)|^2 \right)^{1/2} \leq Ch^{-1} \|\gamma \mathbf{v}_h\|_0 \leq Ch^{-1} \|\mathbf{v}_h\|_{1,\Omega^-}.$$

The result is now a direct consequence of the last inequalities. ■

We introduce the bilinear form

$$A_h(\hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h) = a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h^*(\frac{d}{ds}\gamma \mathbf{u}_h, \frac{d}{ds}\gamma \mathbf{v}_h) - c_h(\mathbf{v}_h, \boldsymbol{\xi}_h) + b_h(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) + c_h(\mathbf{u}_h, \boldsymbol{\eta}_h),$$

for all $\hat{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\xi}_h)$ and $\hat{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\eta}_h)$ in $\mathbf{M}_h$. Using the triangular inequality and lemmas 4–7 we deduce the following estimate.

Corollary 8 *There exists a positive constant C such that*

$$|A(\hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h) - A_h(\hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h)| \leq Ch \, \|\hat{\mathbf{u}}_h\|_{\mathbf{M}} \|\hat{\mathbf{v}}_h\|_{\mathbf{M}}, \qquad \forall \, \hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h \in \mathbf{M}_h.$$

This allows us to prove that, if h is sufficiently small, $A_h(\cdot,\cdot)$ is uniformly elliptic on $\mathbf{M}_h$ since

$$A_h(\hat{\mathbf{v}}_h, \hat{\mathbf{v}}_h) \geq (\tilde{\alpha} - Ch) \, \|\hat{\mathbf{v}}_h\|_{\mathbf{M}}^2, \qquad \forall \, \hat{\mathbf{v}}_h \in \mathbf{M}_h.$$

Thus, problem (29), which may be equivalently written

$$\text{find } \hat{\mathbf{u}}_h^* \in \mathbf{M}_h \text{ such that}$$

$$A_h(\hat{\mathbf{u}}_h^*, \hat{\mathbf{v}}_h) = L_h(\hat{\mathbf{v}}_h), \qquad \forall \, \hat{\mathbf{v}}_h \in \mathbf{M}_h,$$

is well posed and we have the following asymptotic error estimate.

Theorem 9 *If* $\mathbf{f} \in \mathbf{W}^{1,\infty}(\Omega^-)$ *and* $\mathbf{u} \in \mathbf{H}^2(\Omega^-)$*, then there exists a constant* $C > 0$ *such that*

$$\|\mathbf{u} - \mathbf{u}_h^*\|_{1,\Omega^-} + \|\boldsymbol{\xi} - \boldsymbol{\xi}_h^*\|_{-1/2} \leq Ch\,(\|\mathbf{u}\|_{2,\Omega^-} + \|\mathbf{f}\|_{1,\infty,\Omega^-}).$$

PROOF. As a consequence of Strang's lemma (cf. [4]) we have the error estimate

$$\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h^*\|_{\mathbf{M}} \leq C\left(\|\hat{\mathbf{u}} - \hat{\mathbf{v}}_h\|_{\mathbf{M}} + \sup_{\hat{\mathbf{z}}_h} \frac{|A(\hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h) - A_h(\hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h)|}{\|\hat{\mathbf{z}}_h\|_{\mathbf{M}}} + \sup_{\hat{\mathbf{z}}_h} \frac{|L(\hat{\mathbf{z}}_h) - L_h(\hat{\mathbf{z}}_h)|}{\|\hat{\mathbf{z}}_h\|_{\mathbf{M}}}\right),$$

for all $\hat{\mathbf{v}}_h \in \mathbf{M}_h$. Hence, the result follows from lemma 3, corollary 8 and the approximation properties (25) and (26). ■

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