# Time-averaged shallow water model: asymptotic derivation and numerical validation*

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**Abstract**

The objective of this paper is to derive, from the Navier-Stokes equations in a shallow domain, a new bidimensional shallow water model able to filter the high frequency oscillations that are produced, when the Reynolds number is increased, in turbulent flows. With this aim, the non-dimensional Navier-Stokes equations are time-averaged, and then asymptotic analysis techniques have been used as in our previous works (Rodríguez and Taboada-Vázquez 2005-2012 [17]-[23]). The small non-dimensional parameter considered, $\varepsilon$, is the quotient between the typical depth of the basin and the typical horizontal length of the domain; and it is studied what happens when $\varepsilon$ becomes small.

Once the new model has been justified, by the method of asymptotic expansions, we perform some numerical experiments. The results of these experiments confirm that this new model is able to approximate analytical solutions of Navier-Stokes equations with more accuracy than classical shallow water models, when high frequency oscillations appear. To reach a given accuracy, the time step for the new model can be much larger (even four hundred times larger) than the time step required for the classical models.

Keywords: Shallow waters, Asymptotic analysis, Reynolds Averaged Navier-Stokes equations (RANS), Large Eddy Simulation (LES), Modelling, Filtering.

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## 1 Introduction

As it is well known, the equations governing the behavior of a fluid are the Navier-Stokes equations. Due to their strong nonlinearity, high frequency oscillations are produced when the Reynolds number is increased, and the flow becomes unstable and turbulent. It is computationally very expensive to solve the equations directly, so at the moment, the most common approach in hydraulic engineering practice is to solve the Reynolds Averaged Navier-Stokes equations, in which the effect of turbulence is modelled rather than solved.

We approximate Navier-Stokes equations using a shallow water model, but if the flow is turbulent, a very small time step must be chosen. This paper is focused on the derivation of a new bidimensional time-averaged shallow water model able to reduce these oscillations, and then able to achieve good results with larger time steps.

Filtering has given good results when working with turbulent Navier-Stokes equations (see [24]). In the literature, we can found that the separation between large and small scales is traditionally assumed to be obtained by applying a spatial filter to the Navier-Stokes equations (see [3, 12]); but time filtering is also suggested by several authors (see [4, 16]). In this work, we shall use a time filter, thus avoiding to model the spacial filtered stress tensor (see [3]).

Asymptotic analysis has been applied successfully to derive and justify shallow water models. The new model, developed from the incompressible Navier-Stokes equations with free surface, has been deduced

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in the spirit of the method proposed in our previous works [17]-[23] and [25]. In order to obtain a shallow water model, we consider a domain with small depth compared with its other dimensions. We use in the sequel the thin-layer assumption and introduce a "small" non-dimensional parameter $\varepsilon = H_C/L_C$ where $H_C$ and $L_C$ are, respectively, the typical scales for the vertical and the horizontal dimensions of the fluid domain of interest.

The outline of the paper is as follows. In the next section we introduce and render non-dimensional the model that serves as our starting point. Then, the time averaging process is described in section 3 and asymptotic analysis is applied following the ideas of [17]-[23] (section 4) to derive our shallow water model that is presented in section 5. We show, in section 6, that this new model is able to obtain a given accuracy using time steps larger than the time steps needed by classical shallow water models. Finally, we make some concluding remarks in section 7.

# 2 The three-dimensional model equations

In this section we present the three-dimensional incompressible Navier-Stokes model that serves as the starting point for our subsequent development. The first subsection gives the basic mass and momentum balance laws for a basin with varying bottom topography and a free top surface, and supplements them with appropriate boundary conditions. In subsection 2.2 we introduce the shallow water scaling, define a non-dimensional parameter and non-dimensionalize the three-dimensional model in terms of that parameter.

## 2.1 Three-dimensional incompressible flow

Let us start with the Navier-Stokes system [13] for incompressible homogeneous fluids, with gravity and Coriolis force, evolving in a sub-domain of $\mathbb{R}^3$. As the domain, the functions and variables involved in this problem depend on $\varepsilon$, we indicate this dependence with superscript $\varepsilon$. Therefore, we have the following general formulation expression:

$$\frac{\partial \vec{\mathbf{U}}^{\varepsilon}}{\partial t^{\varepsilon}} + \left(\vec{\mathbf{U}}^{\varepsilon} \cdot \nabla^{\varepsilon}\right) \vec{\mathbf{U}}^{\varepsilon} = -\frac{1}{\rho_0} \nabla^{\varepsilon} P^{\varepsilon} + \nu \Delta^{\varepsilon} \vec{\mathbf{U}}^{\varepsilon} + \vec{\mathbf{F}}_e^{\varepsilon} \tag{1}$$

$$\operatorname{div} \vec{\mathbf{U}}^{\varepsilon} = 0 \tag{2}$$

and we consider this system for

$$t^{\varepsilon} \in [0, T], \quad (x^{\varepsilon}, y^{\varepsilon}) \in D \subset \mathbb{R}^2, \quad B^{\varepsilon}(x^{\varepsilon}, y^{\varepsilon}) \leq z^{\varepsilon} \leq S^{\varepsilon}(t^{\varepsilon}, x^{\varepsilon}, y^{\varepsilon})$$

where:

- $S^{\varepsilon}(t^{\varepsilon}, x^{\varepsilon}, y^{\varepsilon})$ represents the free surface elevation (unknown) and $B^{\varepsilon}$ the bathymetry (it is not constant and it is supposed to be known). The water height is $H^{\varepsilon} = S^{\varepsilon} - B^{\varepsilon}$ (see Figure 1)
- $\vec{\mathbf{U}}^{\varepsilon} = (U_1^{\varepsilon}(t^{\varepsilon}, x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}), U_2^{\varepsilon}(t^{\varepsilon}, x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}), U_3^{\varepsilon}(t^{\varepsilon}, x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon}))$ is the three-dimensional velocity of the fluid
- $P^{\varepsilon}(t^{\varepsilon}, x^{\varepsilon}, y^{\varepsilon}, z^{\varepsilon})$ is the pressure
- $\rho_0$ denotes the density of the fluid
- $\nu$ is the kinematic viscosity
- $\vec{\mathbf{F}}_e^{\varepsilon} = -g\vec{k} - 2\vec{\Phi} \times \vec{\mathbf{U}}^{\varepsilon}$ is the volume force per unit mass, where $g$ is the gravitational acceleration (assumed constant) and $-2\vec{\Phi} \times \vec{\mathbf{U}}^{\varepsilon}$ is the Coriolis acceleration (where the angular velocity of rotation of the Earth is $\vec{\Phi} = \Phi\left(\sin\varphi\vec{k} + \cos\varphi\vec{j}\right)$ with $\Phi = 7.29 \times 10^{-5}$ rad/s; $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ denote the unit vectors pointing East, North and vertically upward (respectively); $\varphi$, the North latitude, is considered constant).

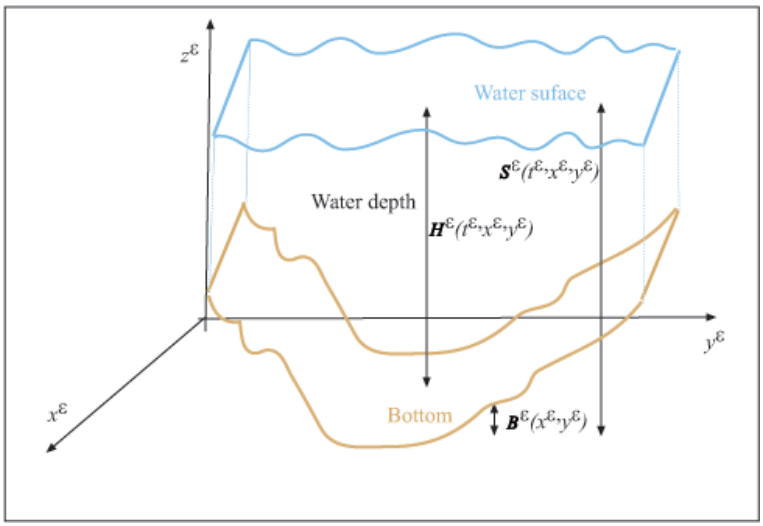


Figure 1: Notations: water height $H^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)$, free surface $S^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)$ and bottom $B^\varepsilon(x^\varepsilon, y^\varepsilon)$

The kinematic continuity condition

$$U_3^\varepsilon = \frac{\partial H^\varepsilon}{\partial t^\varepsilon} + U_1^\varepsilon \frac{\partial S^\varepsilon}{\partial x^\varepsilon} + U_2^\varepsilon \frac{\partial S^\varepsilon}{\partial y^\varepsilon} \quad \text{at } z^\varepsilon = S^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{3}$$

is rewritten in equivalent form (for incompressible fluids):

$$\frac{\partial H^\varepsilon}{\partial t^\varepsilon} + \frac{\partial}{\partial x^\varepsilon} \int_{B^\varepsilon}^{S^\varepsilon} U_1^\varepsilon dz^\varepsilon + \frac{\partial}{\partial y^\varepsilon} \int_{B^\varepsilon}^{S^\varepsilon} U_2^\varepsilon dz^\varepsilon = 0 \tag{4}$$

Equations (1)-(2) must be supplemented by boundary conditions.

- At the bottom,
  - the non-penetration condition is satisfied:

    $$\vec{\mathbf{U}}^\varepsilon \cdot \vec{\mathbf{n}}^\varepsilon = 0 \quad \text{at } z^\varepsilon = B^\varepsilon(x^\varepsilon, y^\varepsilon) \tag{5}$$

    where $\vec{\mathbf{n}}^\varepsilon$ denotes the outward unit normal to the boundary of the domain.
  - tangential forces must be equal to the friction force:

    $$(\mathbf{I} - \vec{\mathbf{n}}^\varepsilon \otimes \vec{\mathbf{n}}^\varepsilon)\mathbf{T}^\varepsilon \vec{\mathbf{n}}^\varepsilon = \vec{\mathbf{F}}_R^\varepsilon \quad \text{at } z^\varepsilon = B^\varepsilon(x^\varepsilon, y^\varepsilon) \tag{6}$$

    Typically, the friction force per unit of surface area is of the form $\vec{\mathbf{F}}_R^\varepsilon = -\rho_0 C_R^\varepsilon |\vec{\mathbf{U}}^\varepsilon| \vec{\mathbf{U}}^\varepsilon$ ($C_R^\varepsilon$ is small), see for example [7] or [26].
- On the free top surface we assume that the only external force acting on the fluid is the wind stress. In particular, we assume that the surface tension and ambient atmospheric pressure variations are negligible. This leads to the boundary condition

  $$\mathbf{T}^\varepsilon \vec{\mathbf{n}}^\varepsilon = -P_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)\vec{\mathbf{n}}^\varepsilon + \vec{\mathbf{F}}_W^\varepsilon \text{ at } z^\varepsilon = S^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{7}$$

  where $\vec{\mathbf{F}}_W^\varepsilon$ is the force of the wind and $P_s^\varepsilon$ is the atmospheric pressure at the surface (supposed known).

The stress tensor ($\mathbf{T}^\varepsilon$) is given by:

$$T_{ij}^\varepsilon = -P^\varepsilon \delta_{ij} + \mu \left( \frac{\partial U_i^\varepsilon}{\partial x_j^\varepsilon} + \frac{\partial U_j^\varepsilon}{\partial x_i^\varepsilon} \right) \quad i,j = 1,2,3 \tag{8}$$

where $x_1^\varepsilon = x^\varepsilon$, $x_2^\varepsilon = y^\varepsilon$, $x_3^\varepsilon = z^\varepsilon$, $\mu = \rho_0 \nu$ is the dynamic viscosity and $\delta_{ij}$ is the Kronecker's delta.

We also suppose that the incoming and outcoming flows are known at each instant (other kind of boundary conditions may be considered). Finally, initial conditions must be imposed too.

## 2.2 Non-dimensionalization

The shallow water approximation is characterized by the smallness of a non-dimensional parameter that we can identify assuming that the typical depth of the basin ($H_C$) is much smaller than the typical horizontal length ($L_C$), i.e., that

$$\frac{H_C}{L_C} = \varepsilon \qquad \text{where} \quad \varepsilon << 1 \tag{9}$$

This small parameter is an aspect ratio.

We introduce the non-dimensional independent variables $t$, $x$, $y$ and $z$ by

$$t = \frac{t^\varepsilon}{T_C}, \quad x = \frac{x^\varepsilon}{L_C}, \quad y = \frac{y^\varepsilon}{L_C}, \quad z = \frac{z^\varepsilon}{\varepsilon L_C} \tag{10}$$

where $T_C$ is a typical time. Recalling (9), the non-dimensional water height and bottom surface are defined by

$$h(t,x,y) = \frac{H^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)}{\varepsilon L_C}, \quad b(x,y) = \frac{B^\varepsilon(x^\varepsilon, y^\varepsilon)}{\varepsilon L_C} \tag{11}$$

so the non-dimensional top surface is given by

$$s(t,x,y) = \frac{S^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)}{\varepsilon L_C} \tag{12}$$

We shall now introduce non-dimensional functions and constants:

$$u_i^\varepsilon(t,x,y,z) = \frac{T_C}{L_C} U_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), \quad i = 1,2,3 \tag{13}$$

$$p^\varepsilon(t,x,y,z) = \frac{T_C^2}{\rho_0 L_C^2} P^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), \quad p_s(t,x,y) = \frac{T_C^2}{\rho_0 L_C^2} P_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{14}$$

$$G = \frac{T_C^2}{L_C} g, \quad \phi = T_C \Phi \tag{15}$$

$$\sigma_{ij}^\varepsilon(t,x,y,z) = \frac{T_C^2}{\rho_0 L_C^2} T_{ij}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \quad i,j = 1,2,3 \tag{16}$$

$$\vec{f}_W^\varepsilon = \frac{T_C^2}{\rho_0 L_C^2} \vec{\mathbf{F}}_W^\varepsilon, \quad \vec{f}_R^\varepsilon = \frac{T_C^2}{\rho_0 L_C^2} \vec{\mathbf{F}}_R^\varepsilon \tag{17}$$

where we have assumed that $p_s$ does not depend on $\varepsilon$.

### 2.2.1 The non-dimensional equations

We now express the three-dimensional incompressible Navier-Stokes system (1), (2) and (4) in terms of the above non-dimensional variables and functions:

$$\begin{aligned} \frac{\partial u_1^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial x} + u_2^\varepsilon \frac{\partial u_1^\varepsilon}{\partial y} + \frac{1}{\varepsilon} u_3^\varepsilon \frac{\partial u_1^\varepsilon}{\partial z} &= -\frac{\partial p^\varepsilon}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u_1^\varepsilon}{\partial x^2} + \frac{\partial^2 u_1^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 u_1^\varepsilon}{\partial z^2}\right) \\ &\quad + 2\phi\left((\sin\varphi)\, u_2^\varepsilon - (\cos\varphi)\, u_3^\varepsilon\right) \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{\partial u_2^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} + u_2^\varepsilon \frac{\partial u_2^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\varepsilon} u_3^\varepsilon \frac{\partial u_2^\varepsilon}{\partial z} &= -\frac{\partial p^\varepsilon}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u_2^\varepsilon}{\partial x^2} + \frac{\partial^2 u_2^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 u_2^\varepsilon}{\partial z^2}\right) \\ &\quad - 2\phi\,(\sin\varphi)\, u_1^\varepsilon \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial u_3^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_3^\varepsilon}{\partial x} + u_2^\varepsilon \frac{\partial u_3^\varepsilon}{\partial y} + \frac{1}{\varepsilon} u_3^\varepsilon \frac{\partial u_3^\varepsilon}{\partial z} &= -\frac{1}{\varepsilon}\frac{\partial p^\varepsilon}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 u_3^\varepsilon}{\partial x^2} + \frac{\partial^2 u_3^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 u_3^\varepsilon}{\partial z^2}\right) \\ &\quad - G + 2\phi\,(\cos\varphi)\, u_1^\varepsilon \end{aligned} \tag{20}$$

$$\frac{\partial u_1^\varepsilon}{\partial x} + \frac{\partial u_2^\varepsilon}{\partial y} + \frac{1}{\varepsilon}\frac{\partial u_3^\varepsilon}{\partial z} = 0 \tag{21}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_b^s u_1^\varepsilon dz + \frac{\partial}{\partial y}\int_b^s u_2^\varepsilon dz = 0 \tag{22}$$

where $\nu\dfrac{T_C}{L_C^2}=\dfrac{1}{Re}$. The non-penetration condition (5) yields:

$$u_3^\varepsilon=\varepsilon\left(u_1^\varepsilon\frac{\partial b}{\partial x}+u_2^\varepsilon\frac{\partial b}{\partial y}\right)\ \text{at}\ z=b \tag{23}$$

The non-dimensional stress tensor can be written:

$$\sigma_{ij}^\varepsilon=-p^\varepsilon\delta_{ij}+\frac{1}{Re}\left(\frac{\partial u_i^\varepsilon}{\partial x_j}+\frac{\partial u_j^\varepsilon}{\partial x_i}\right)\quad i,j=1,2 \tag{24}$$

$$\sigma_{i3}^\varepsilon=\frac{1}{Re}\left(\frac{1}{\varepsilon}\frac{\partial u_i^\varepsilon}{\partial z}+\frac{\partial u_3^\varepsilon}{\partial x_i}\right)\quad i=1,2 \tag{25}$$

$$\sigma_{33}^\varepsilon=-p^\varepsilon+\frac{2}{Re}\frac{1}{\varepsilon}\frac{\partial u_3^\varepsilon}{\partial z} \tag{26}$$

so boundary conditions (6)-(7) result

$$\begin{pmatrix}\varepsilon\dfrac{\partial b}{\partial x}\sigma_{11}^\varepsilon+\varepsilon\dfrac{\partial b}{\partial y}\sigma_{12}^\varepsilon-\sigma_{13}^\varepsilon\\ \varepsilon\dfrac{\partial b}{\partial x}\sigma_{12}^\varepsilon+\varepsilon\dfrac{\partial b}{\partial y}\sigma_{22}^\varepsilon-\sigma_{23}^\varepsilon\\ \varepsilon\dfrac{\partial b}{\partial x}\sigma_{13}^\varepsilon+\varepsilon\dfrac{\partial b}{\partial y}\sigma_{23}^\varepsilon-\sigma_{33}^\varepsilon\end{pmatrix}+\left\{p^\varepsilon-\frac{1}{Re}\frac{2}{1+\left(\varepsilon\dfrac{\partial b}{\partial x}\right)^2+\left(\varepsilon\dfrac{\partial b}{\partial y}\right)^2}\left[\left(\varepsilon\frac{\partial b}{\partial x}\right)^2\frac{\partial u_1^\varepsilon}{\partial x}\right.\right.$$
$$+\varepsilon^2\frac{\partial b}{\partial x}\frac{\partial b}{\partial y}\left(\frac{\partial u_1^\varepsilon}{\partial y}+\frac{\partial u_2^\varepsilon}{\partial x}\right)-\varepsilon\frac{\partial b}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial u_1^\varepsilon}{\partial z}+\frac{\partial u_3^\varepsilon}{\partial x}\right)+\left(\varepsilon\frac{\partial b}{\partial y}\right)^2\frac{\partial u_2^\varepsilon}{\partial y}+\frac{1}{\varepsilon}\frac{\partial u_3^\varepsilon}{\partial z}$$
$$\left.\left.-\varepsilon\frac{\partial b}{\partial y}\left(\frac{1}{\varepsilon}\frac{\partial u_2^\varepsilon}{\partial z}+\frac{\partial u_3^\varepsilon}{\partial y}\right)\right]\right\}\begin{pmatrix}\varepsilon\dfrac{\partial b}{\partial x}\\ \varepsilon\dfrac{\partial b}{\partial y}\\ -1\end{pmatrix}=\sqrt{1+\varepsilon^2\left[\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right]}\begin{pmatrix}f_{R_1}^\varepsilon\\ f_{R_2}^\varepsilon\\ f_{R_3}^\varepsilon\end{pmatrix}$$
$$\text{at}\ z=b \tag{27}$$

$$\begin{pmatrix}-\varepsilon\dfrac{\partial s}{\partial x}\sigma_{11}^\varepsilon-\varepsilon\dfrac{\partial s}{\partial y}\sigma_{12}^\varepsilon+\sigma_{13}^\varepsilon\\ -\varepsilon\dfrac{\partial s}{\partial x}\sigma_{12}^\varepsilon-\varepsilon\dfrac{\partial s}{\partial y}\sigma_{22}^\varepsilon+\sigma_{23}^\varepsilon\\ -\varepsilon\dfrac{\partial s}{\partial x}\sigma_{13}^\varepsilon-\varepsilon\dfrac{\partial s}{\partial y}\sigma_{23}^\varepsilon+\sigma_{33}^\varepsilon\end{pmatrix}=-p_s\begin{pmatrix}-\varepsilon\dfrac{\partial s}{\partial x}\\ -\varepsilon\dfrac{\partial s}{\partial y}\\ 1\end{pmatrix}+\sqrt{1+\left(\varepsilon\frac{\partial s}{\partial x}\right)^2+\left(\varepsilon\frac{\partial s}{\partial y}\right)^2}\begin{pmatrix}f_{W_1}^\varepsilon\\ f_{W_2}^\varepsilon\\ f_{W_3}^\varepsilon\end{pmatrix}$$
$$\text{at}\ z=s \tag{28}$$

# 3 Time averaging process

The formal derivation of our shallow water model has two steps. We first obtain the time-averaged Navier-Stokes equations and then make the shallow water approximation.

We define:

$$\bar{u}_i^\varepsilon(t,x,y,z;\eta)=\frac{1}{2\eta}\int_{t-\eta}^{t+\eta}u_i^\varepsilon(r,x,y,z)dr\quad i=1,2,3 \tag{29}$$

and, in the same way, we also define the other averaged functions of the problem.

If we assume that $0 < \eta << 1$, and we approximate $u_i^\varepsilon$ by its Taylor's expansion for $r \in (t-\eta, t+\eta)$:

$$\begin{aligned} u_i^\varepsilon(r,x,y,z) &= u_i^\varepsilon(t,x,y,z) + \frac{\partial u_i^\varepsilon}{\partial t}(t,x,y,z)(r-t) + \frac{1}{2}\frac{\partial^2 u_i^\varepsilon}{\partial t^2}(t,x,y,z)(r-t)^2 \\ &+ \frac{1}{3!}\frac{\partial^3 u_i^\varepsilon}{\partial t^3}(t,x,y,z)(r-t)^3 + \cdots \end{aligned} \tag{30}$$

then we have:

$$\bar{u}_i^\varepsilon(t,x,y,z;\eta) = u_i^\varepsilon(t,x,y,z) + \frac{\partial^2 u_i^\varepsilon}{\partial t^2}(t,x,y,z)\frac{\eta^2}{3!} + \frac{\partial^4 u_i^\varepsilon}{\partial t^4}(t,x,y,z)\frac{\eta^4}{5!} + \cdots \tag{31}$$

It is easy to demonstrate for regular enough functions $f^\varepsilon$ and $g^\varepsilon$, and small enough $\eta$, that the following equalities are fulfilled:

$$\overline{\frac{\partial f^\varepsilon}{\partial t}}(t,x,y,z;\eta) = \frac{\partial \bar{f}^\varepsilon}{\partial t}(t,x,y,z;\eta) \tag{32}$$

$$f^\varepsilon(t,x,y,z) = \bar{f}^\varepsilon(t,x,y,z;\eta) - \frac{\partial^2 \bar{f}^\varepsilon}{\partial t^2}(t,x,y,z;\eta)\frac{\eta^2}{6} + O(\eta^4) \tag{33}$$

$$\begin{aligned} \overline{f^\varepsilon g^\varepsilon}(t,x,y,z;\eta) &= \bar{f}^\varepsilon(t,x,y,z;\eta)\bar{g}^\varepsilon(t,x,y,z;\eta) \\ &+ \frac{\partial \bar{f}^\varepsilon}{\partial t}(t,x,y,z,\eta)\frac{\partial \bar{g}^\varepsilon}{\partial t}(t,x,y,z,\eta)\frac{\eta^2}{3} + O(\eta^4) \end{aligned} \tag{34}$$

$$\overline{f^\varepsilon g^\varepsilon} = \bar{f}^\varepsilon g^\varepsilon + \frac{\eta^2}{6}\left(2\frac{\partial \bar{f}^\varepsilon}{\partial t}\frac{\partial g^\varepsilon}{\partial t} + \bar{f}^\varepsilon\frac{\partial^2 g^\varepsilon}{\partial t^2}\right) + O(\eta^4) \tag{35}$$

$$\overline{f^\varepsilon(t,x,y,s(t,x,y))} = \bar{f}^\varepsilon\big|_{z=s} \text{ at } z = s \tag{36}$$

Applying (29), (32) and (34) to equations (18)-(21), they yield:

$$\begin{aligned} &\frac{\partial \bar{u}_1^\varepsilon}{\partial t} + \bar{u}_1^\varepsilon\frac{\partial \bar{u}_1^\varepsilon}{\partial x} + \left(\frac{\partial \bar{u}_1^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t \partial x}\right)\frac{\eta^2}{3} + \bar{u}_2^\varepsilon\frac{\partial \bar{u}_1^\varepsilon}{\partial y} + \left(\frac{\partial \bar{u}_2^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t \partial y}\right)\frac{\eta^2}{3} \\ &\quad + \frac{1}{\varepsilon}\left[\bar{u}_3^\varepsilon\frac{\partial \bar{u}_1^\varepsilon}{\partial z} + \left(\frac{\partial \bar{u}_3^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t \partial z}\right)\frac{\eta^2}{3}\right] + O(\eta^4) = -\frac{\partial \bar{p}^\varepsilon}{\partial x} \\ &\quad + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial x^2} + \frac{\partial^2 \bar{u}_1^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial z^2}\right) + 2\phi\left((\sin\varphi)\,\bar{u}_2^\varepsilon - (\cos\varphi)\,\bar{u}_3^\varepsilon\right) \end{aligned} \tag{37}$$

$$\begin{aligned} &\frac{\partial \bar{u}_2^\varepsilon}{\partial t} + \bar{u}_1^\varepsilon\frac{\partial \bar{u}_2^\varepsilon}{\partial x} + \left(\frac{\partial \bar{u}_1^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_2^\varepsilon}{\partial t \partial x}\right)\frac{\eta^2}{3} + \bar{u}_2^\varepsilon\frac{\partial \bar{u}_2^\varepsilon}{\partial y} + \left(\frac{\partial \bar{u}_2^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_2^\varepsilon}{\partial t \partial y}\right)\frac{\eta^2}{3} \\ &\quad + \frac{1}{\varepsilon}\left[\bar{u}_3^\varepsilon\frac{\partial \bar{u}_2^\varepsilon}{\partial z} + \left(\frac{\partial \bar{u}_3^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_2^\varepsilon}{\partial t \partial z}\right)\frac{\eta^2}{3}\right] + O(\eta^4) = -\frac{\partial \bar{p}^\varepsilon}{\partial y} \\ &\quad + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}_2^\varepsilon}{\partial x^2} + \frac{\partial^2 \bar{u}_2^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \bar{u}_2^\varepsilon}{\partial z^2}\right) - 2\phi\,(\sin\varphi)\,\bar{u}_1^\varepsilon \end{aligned} \tag{38}$$

$$\begin{aligned} &\frac{\partial \bar{u}_3^\varepsilon}{\partial t} + \bar{u}_1^\varepsilon\frac{\partial \bar{u}_3^\varepsilon}{\partial x} + \left(\frac{\partial \bar{u}_1^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_3^\varepsilon}{\partial t \partial x}\right)\frac{\eta^2}{3} + \bar{u}_2^\varepsilon\frac{\partial \bar{u}_3^\varepsilon}{\partial y} + \left(\frac{\partial \bar{u}_2^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_3^\varepsilon}{\partial t \partial y}\right)\frac{\eta^2}{3} \\ &\quad + \frac{1}{\varepsilon}\left[\bar{u}_3^\varepsilon\frac{\partial \bar{u}_3^\varepsilon}{\partial z} + \left(\frac{\partial \bar{u}_3^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_3^\varepsilon}{\partial t \partial z}\right)\frac{\eta^2}{3}\right] + O(\eta^4) = -\frac{1}{\varepsilon}\frac{\partial \bar{p}^\varepsilon}{\partial z} - G \\ &\quad + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}_3^\varepsilon}{\partial x^2} + \frac{\partial^2 \bar{u}_3^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \bar{u}_3^\varepsilon}{\partial z^2}\right) + 2\phi\,(\cos\varphi)\,\bar{u}_1^\varepsilon \end{aligned} \tag{39}$$

$$\frac{\partial \bar{u}_1^\varepsilon}{\partial x} + \frac{\partial \bar{u}_2^\varepsilon}{\partial y} + \frac{1}{\varepsilon}\frac{\partial \bar{u}_3^\varepsilon}{\partial z} = 0 \tag{40}$$

To average equation (22), in first place we write $u_1^\varepsilon$ and $u_2^\varepsilon$ using (33) and we get

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_b^s\left(\bar{u}_1^\varepsilon - \frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t^2}\frac{\eta^2}{6}\right)dz + \frac{\partial}{\partial y}\int_b^s\left(\bar{u}_2^\varepsilon - \frac{\partial^2 \bar{u}_2^\varepsilon}{\partial t^2}\frac{\eta^2}{6}\right)dz + O(\eta^4) = 0 \tag{41}$$

This averaging process provides from (23)-(27):

$$\bar{u}_3^\varepsilon = \varepsilon\left(\bar{u}_1^\varepsilon \frac{\partial b}{\partial x} + \bar{u}_2^\varepsilon \frac{\partial b}{\partial y}\right) \text{ at } z = b \tag{42}$$

$$\bar{\sigma}_{ij}^\varepsilon = -\bar{p}^\varepsilon \delta_{ij} + \frac{1}{Re}\left(\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{u}_j^\varepsilon}{\partial x_i}\right) \quad i,j = 1,2 \tag{43}$$

$$\bar{\sigma}_{i3}^\varepsilon = \frac{1}{Re}\left(\frac{1}{\varepsilon}\frac{\partial \bar{u}_i^\varepsilon}{\partial z} + \frac{\partial \bar{u}_3^\varepsilon}{\partial x_i}\right) \quad i = 1,2 \tag{44}$$

$$\bar{\sigma}_{33}^\varepsilon = -\bar{p}^\varepsilon + \frac{2}{Re}\frac{1}{\varepsilon}\frac{\partial \bar{u}_3^\varepsilon}{\partial z} \tag{45}$$

$$\begin{pmatrix} \varepsilon\frac{\partial b}{\partial x}\bar{\sigma}_{11}^\varepsilon + \varepsilon\frac{\partial b}{\partial y}\bar{\sigma}_{12}^\varepsilon - \bar{\sigma}_{13}^\varepsilon \\ \varepsilon\frac{\partial b}{\partial x}\bar{\sigma}_{12}^\varepsilon + \varepsilon\frac{\partial b}{\partial y}\bar{\sigma}_{22}^\varepsilon - \bar{\sigma}_{23}^\varepsilon \\ \varepsilon\frac{\partial b}{\partial x}\bar{\sigma}_{13}^\varepsilon + \varepsilon\frac{\partial b}{\partial y}\bar{\sigma}_{23}^\varepsilon - \bar{\sigma}_{33}^\varepsilon \end{pmatrix} + \left\{ \bar{p}^\varepsilon - \frac{1}{Re}\frac{2}{1+\left(\varepsilon\frac{\partial b}{\partial x}\right)^2 + \left(\varepsilon\frac{\partial b}{\partial y}\right)^2}\left[\left(\varepsilon\frac{\partial b}{\partial x}\right)^2 \frac{\partial \bar{u}_1^\varepsilon}{\partial x}\right.\right.$$
$$+ \varepsilon^2 \frac{\partial b}{\partial x}\frac{\partial b}{\partial y}\left(\frac{\partial \bar{u}_1^\varepsilon}{\partial y} + \frac{\partial \bar{u}_2^\varepsilon}{\partial x}\right) - \varepsilon\frac{\partial b}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial \bar{u}_1^\varepsilon}{\partial z} + \frac{\partial \bar{u}_3^\varepsilon}{\partial x}\right) + \left(\varepsilon\frac{\partial b}{\partial y}\right)^2 \frac{\partial \bar{u}_2^\varepsilon}{\partial y} - \varepsilon\frac{\partial b}{\partial y}\left(\frac{1}{\varepsilon}\frac{\partial \bar{u}_2^\varepsilon}{\partial z}\right.$$
$$\left.\left. + \frac{\partial \bar{u}_3^\varepsilon}{\partial y}\right) + \frac{1}{\varepsilon}\frac{\partial \bar{u}_3^\varepsilon}{\partial z}\right]\right\} \begin{pmatrix} \varepsilon\frac{\partial b}{\partial x} \\ \varepsilon\frac{\partial b}{\partial y} \\ -1 \end{pmatrix} = \sqrt{1+\varepsilon^2\left(\frac{\partial b}{\partial x}\right)^2 + \varepsilon^2\left(\frac{\partial b}{\partial y}\right)^2} \begin{pmatrix} \bar{f}_{R_1}^\varepsilon \\ \bar{f}_{R_2}^\varepsilon \\ \bar{f}_{R_3}^\varepsilon \end{pmatrix} \quad \text{at } z = b \tag{46}$$

Condition (28) is time-averaged too and gives and similar but more complex expression, because $s$ depends on $t$ and we need to use formula (35).

# 4 Asymptotic derivation of the shallow water model

After obtaining the time-averaged Navier-Stokes equations, we make the shallow water approximation by formally expanding solutions of the model in powers of $\varepsilon$.

## 4.1 Asymptotic expansion

Our shallow water model derive from the assumption that the aspect ratio $\varepsilon$ is small. We make the shallow water approximation by formally expanding solutions of the model in powers of $\varepsilon$. We seek a formal solution in the form of an asymptotic series:

$$\begin{cases} \bar{u}_i^\varepsilon = \bar{u}_i^0 + \varepsilon\bar{u}_i^1 + \varepsilon^2\bar{u}_i^2 + \cdots \quad i = 1,2,3 \\ \bar{p}^\varepsilon = \bar{p}^0 + \varepsilon\bar{p}^1 + \varepsilon^2\bar{p}^2 + \cdots \\ \bar{\sigma}_{ij}^\varepsilon = \bar{\sigma}_{ij}^0 + \varepsilon\bar{\sigma}_{ij}^1 + \varepsilon^2\bar{\sigma}_{ij}^2 + \cdots \quad i,j = 1,2,3 \\ \bar{f}_{R_i}^\varepsilon = \varepsilon\bar{f}_{R_i}^1 + \varepsilon^2\bar{f}_{R_i}^2 + \cdots \quad i = 1,2 \\ \bar{f}_{W_i}^\varepsilon = \varepsilon\bar{f}_{W_i}^1 + \varepsilon^2\bar{f}_{W_i}^2 + \cdots \quad i = 1,2 \end{cases} \tag{47}$$

The developments for $\bar{f}_{R_i}^\varepsilon, \bar{f}_{W_i}^\varepsilon$ $(i = 1,2)$ may begin in the term of order 1 because of their small order of magnitude (see [25] for a rigorous justification).

We substitute now this expansion into the system of equations (37)-(46), and grouping the terms multiplied by the same power of $\varepsilon$, we arrive at a series of equations that will allow us to determine $\bar{u}_i^0$, $p^0$, etc. Let us show some particular examples.

The value of $\bar{u}_3^0$ can be found from the incompressibility condition (40) written at the leading order $O(\varepsilon^{-1})$:

$$\frac{\partial \bar{u}_3^0}{\partial z} = 0 \Rightarrow \bar{u}_3^0 = \bar{u}_3^0(t,x,y) \tag{48}$$

We now consider the boundary condition (42); to the leading order it becomes:

$$\bar{u}_3^0 = 0 \text{ for } z = b \tag{49}$$

so

$$\bar{u}_3^0 = 0 \tag{50}$$

In the same way, upon inserting the above expansion into (44), to order $O(\varepsilon^{-1})$, one finds that

$$\frac{\partial \bar{u}_i^0}{\partial z} = 0 \quad i = 1,2 \tag{51}$$

Now, incompressibility equation (40), to the leading order $O(1)$, gives

$$\bar{u}_3^1 = -\int_b^z \left( \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) dz' + \bar{u}_3^1(t,x,y,b) \tag{52}$$

and the boundary condition (42), together with the fact that $\bar{u}_1^0$ and $\bar{u}_2^0$ do not depend on $z$ ((51)), allows us to write:

$$\bar{u}_3^1 = (b-z)\left( \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) + \bar{u}_1^0 \frac{\partial b}{\partial x} + \bar{u}_2^0 \frac{\partial b}{\partial y} \tag{53}$$

Equation (39), to order $O(\varepsilon^{-1})$, together with above expression for $\bar{u}_3^1$, leads us to conclude that $\bar{p}^0$ is constant with respect to the vertical variable:

$$\frac{\partial \bar{p}^0}{\partial z} = \frac{1}{Re} \frac{\partial^2 \bar{u}_3^1}{\partial z^2} = 0 \tag{54}$$

and with the information that we obtain from boundary conditions we have:

$$\bar{p}^0 = \bar{p}_s - \frac{2}{Re}\left( \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) \tag{55}$$

Equation (41), to the leading order $O(1)$, together with (51) becomes

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[ h\left( \bar{u}_1^0 - \frac{\eta^2}{6}\frac{\partial^2 \bar{u}_1^0}{\partial t^2} \right) \right] + \frac{\partial}{\partial y}\left[ h\left( \bar{u}_2^0 - \frac{\eta^2}{6}\frac{\partial^2 \bar{u}_2^0}{\partial t^2} \right) \right] = O(\eta^4) \tag{56}$$

In summary, we are able to derive the following first terms and equations:

$$\bar{u}_3^0 = 0 \tag{57}$$

$$\bar{u}_3^k = (b-z)\left( \frac{\partial \bar{u}_1^{k-1}}{\partial x} + \frac{\partial \bar{u}_2^{k-1}}{\partial y} \right) + \bar{u}_1^{k-1}\frac{\partial b}{\partial x} + \bar{u}_2^{k-1}\frac{\partial b}{\partial y}, \quad k = 1,2 \tag{58}$$

$$\bar{p}^0 = \bar{p}_s - \frac{2}{Re}\left( \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) \tag{59}$$

$$\bar{p}^1 = (s-z)(G - 2\phi(\cos\varphi)\bar{u}_1^0) - \frac{2}{Re}\left( \frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y} \right) \tag{60}$$

$$\bar{\sigma}_{13}^0 = \bar{\sigma}_{23}^0 = 0 \tag{61}$$

$$\frac{\partial \bar{u}_1^k}{\partial z} = \frac{\partial \bar{u}_2^k}{\partial z} = 0 \quad k = 0,1 \tag{62}$$

$$\frac{\partial \bar{u}_1^0}{\partial t}+\bar{u}_1^0\frac{\partial \bar{u}_1^0}{\partial x}+\frac{\eta^2}{3}\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t\partial x}+\bar{u}_2^0\frac{\partial \bar{u}_1^0}{\partial y}+\frac{\eta^2}{3}\frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t\partial y}+O(\eta^4)$$
$$=-\frac{\partial \bar{p}^0}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 \bar{u}_1^0}{\partial x^2}+\frac{\partial^2 \bar{u}_1^0}{\partial y^2}+\frac{\partial^2 \bar{u}_1^2}{\partial z^2}\right)+2\phi\left(\sin\varphi\right)\bar{u}_2^0 \tag{63}$$

$$\frac{\partial \bar{u}_1^1}{\partial t}+\bar{u}_1^0\frac{\partial \bar{u}_1^1}{\partial x}+\bar{u}_1^1\frac{\partial \bar{u}_1^0}{\partial x}+\frac{\eta^2}{3}\left(\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_1^1}{\partial t\partial x}+\frac{\partial \bar{u}_1^1}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t\partial x}\right)+\bar{u}_2^0\frac{\partial \bar{u}_1^1}{\partial y}+\bar{u}_2^1\frac{\partial \bar{u}_1^0}{\partial y}$$
$$+\frac{\eta^2}{3}\left(\frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_1^1}{\partial t\partial y}+\frac{\partial \bar{u}_2^1}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t\partial y}\right)+\bar{u}_3^1\frac{\partial \bar{u}_1^1}{\partial z}+\frac{\eta^2}{3}\frac{\partial \bar{u}_3^1}{\partial t}\frac{\partial^2 \bar{u}_1^1}{\partial t\partial z}+O(\eta^4)$$
$$=-\frac{\partial \bar{p}^1}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 \bar{u}_1^1}{\partial x^2}+\frac{\partial^2 \bar{u}_1^1}{\partial y^2}+\frac{\partial^2 \bar{u}_1^3}{\partial z^2}\right)+2\phi\left[\left(\sin\varphi\right)\bar{u}_2^1-\left(\cos\varphi\right)\bar{u}_3^1\right] \tag{64}$$

$$\frac{\partial \bar{u}_2^0}{\partial t}+\bar{u}_1^0\frac{\partial \bar{u}_2^0}{\partial x}+\frac{\eta^2}{3}\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t\partial x}+\bar{u}_2^0\frac{\partial \bar{u}_2^0}{\partial y}+\frac{\eta^2}{3}\frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t\partial y}+O(\eta^4)$$
$$=-\frac{\partial \bar{p}^0}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 \bar{u}_2^0}{\partial x^2}+\frac{\partial^2 \bar{u}_2^0}{\partial y^2}+\frac{\partial^2 \bar{u}_2^2}{\partial z^2}\right)-2\phi\left(\sin\varphi\right)\bar{u}_1^0 \tag{65}$$

$$\frac{\partial \bar{u}_2^1}{\partial t}+\bar{u}_1^0\frac{\partial \bar{u}_2^1}{\partial x}+\bar{u}_1^1\frac{\partial \bar{u}_2^0}{\partial x}+\frac{\eta^2}{3}\left(\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_2^1}{\partial t\partial x}+\frac{\partial \bar{u}_1^1}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t\partial x}\right)+\bar{u}_2^0\frac{\partial \bar{u}_2^1}{\partial y}+\bar{u}_2^1\frac{\partial \bar{u}_2^0}{\partial y}$$
$$+\frac{\eta^2}{3}\left(\frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_2^1}{\partial t\partial y}+\frac{\partial \bar{u}_2^1}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t\partial y}\right)+\bar{u}_3^1\frac{\partial \bar{u}_2^1}{\partial z}+\frac{\eta^2}{3}\frac{\partial \bar{u}_3^1}{\partial t}\frac{\partial^2 \bar{u}_2^1}{\partial t\partial z}+O(\eta^4)$$
$$=-\frac{\partial \bar{p}^1}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 \bar{u}_2^1}{\partial x^2}+\frac{\partial^2 \bar{u}_2^1}{\partial y^2}+\frac{\partial^2 \bar{u}_2^3}{\partial z^2}\right)-2\phi\left(\sin\varphi\right)\bar{u}_1^1 \tag{66}$$

$$\frac{\partial h}{\partial t}+\frac{\partial(h\bar{u}_1^0)}{\partial x}+\frac{\partial(h\bar{u}_2^0)}{\partial y}-\frac{\eta^2}{6}\left[\frac{\partial}{\partial x}\left(h\frac{\partial^2 \bar{u}_1^0}{\partial t^2}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial^2 \bar{u}_2^0}{\partial t^2}\right)\right]=O(\eta^4) \tag{67}$$

$$\frac{\partial(h\bar{u}_1^1)}{\partial x}+\frac{\partial(h\bar{u}_2^1)}{\partial y}-\frac{\eta^2}{6}\left[\frac{\partial}{\partial x}\left(h\frac{\partial^2 \bar{u}_1^1}{\partial t^2}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial^2 \bar{u}_2^1}{\partial t^2}\right)\right]=O(\eta^4) \tag{68}$$

$$\bar{\sigma}_{ij}^k=-\bar{p}^k\delta_{ij}+\frac{1}{Re}\left(\frac{\partial \bar{u}_i^k}{\partial x_j}+\frac{\partial \bar{u}_j^k}{\partial x_i}\right)\quad i,j=1,2,k=0,1,\cdots \tag{69}$$

$$\bar{\sigma}_{i3}^k=\frac{1}{Re}\left(\frac{\partial \bar{u}_i^{k+1}}{\partial z}+\frac{\partial \bar{u}_3^k}{\partial x_i}\right)\quad i=1,2,k=1,2,\cdots \tag{70}$$

$$\bar{\sigma}_{33}^k=-\bar{p}^k+\frac{2}{Re}\frac{1}{\varepsilon}\frac{\partial \bar{u}_3^{k+1}}{\partial z}\quad k=0,1,\cdots \tag{71}$$

$$\bar{\sigma}_{13}^1=\frac{2}{Re}\left(2\frac{\partial \bar{u}_1^0}{\partial x}+\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial b}{\partial x}+\frac{1}{Re}\left(\frac{\partial \bar{u}_1^0}{\partial y}+\frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial b}{\partial y}-\bar{f}_{R_1}^1\quad \text{at } z=b \tag{72}$$

$$\bar{\sigma}_{23}^1=\frac{1}{Re}\left(\frac{\partial \bar{u}_1^0}{\partial y}+\frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial b}{\partial x}+\frac{2}{Re}\left(\frac{\partial \bar{u}_1^0}{\partial x}+2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial b}{\partial y}-\bar{f}_{R_2}^1\quad \text{at } z=b \tag{73}$$

$$\bar{\sigma}_{13}^1=\frac{2}{Re}\left(2\frac{\partial \bar{u}_1^0}{\partial x}+\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial s}{\partial x}+\frac{1}{Re}\left(\frac{\partial \bar{u}_1^0}{\partial y}+\frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial s}{\partial y}$$
$$+\frac{\eta^2}{6Re}\left[4\frac{\partial}{\partial t}\left(2\frac{\partial \bar{u}_1^0}{\partial x}+\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^2 h}{\partial t\partial x}+2\left(2\frac{\partial \bar{u}_1^0}{\partial x}+\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^3 h}{\partial t^2\partial x}\right.$$
$$\left.+2\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial y}+\frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^2 h}{\partial t\partial y}+\left(\frac{\partial \bar{u}_1^0}{\partial y}+\frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^3 h}{\partial t^2\partial y}\right]$$
$$+O(\eta^4)+\bar{f}_{W_1}^1\ \text{at } z=s \tag{74}$$

$$\bar{\sigma}^1_{23} = \frac{1}{Re}\left(\frac{\partial \bar{u}^0_1}{\partial y} + \frac{\partial \bar{u}^0_2}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{2}{Re}\left(\frac{\partial \bar{u}^0_1}{\partial x} + 2\frac{\partial \bar{u}^0_2}{\partial y}\right)\frac{\partial s}{\partial y}$$
$$+ \frac{\eta^2}{6Re}\left[2\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}^0_1}{\partial y} + \frac{\partial \bar{u}^0_2}{\partial x}\right)\frac{\partial^2 h}{\partial t \partial x} + \left(\frac{\partial \bar{u}^0_1}{\partial y} + \frac{\partial \bar{u}^0_2}{\partial x}\right)\frac{\partial^3 h}{\partial t^2 \partial x}\right.$$
$$\left. + 4\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}^0_1}{\partial x} + 2\frac{\partial \bar{u}^0_2}{\partial y}\right)\frac{\partial^2 h}{\partial t \partial y} + 2\left(\frac{\partial \bar{u}^0_1}{\partial x} + 2\frac{\partial \bar{u}^0_2}{\partial y}\right)\frac{\partial^3 h}{\partial t^2 \partial y}\right]$$
$$+ O(\eta^4) + \bar{f}^1_{W_2} \text{ at } z = s \tag{75}$$
$$\bar{f}^0_{W_1} = \bar{f}^0_{W_2} = \bar{f}^0_{W_3} = \bar{f}^1_{W_3} = 0 \text{ at } z = s \tag{76}$$

## 4.2 First Order Approximation

We can now consider a first order approximation:

$$\tilde{u}^\varepsilon_i = \bar{u}^0_i + \varepsilon \bar{u}^1_i \quad (i = 1, 2), \quad \tilde{u}^\varepsilon_3 = \bar{u}^0_3 + \varepsilon \bar{u}^1_3 + \varepsilon^2 \bar{u}^2_3, \quad \tilde{p}^\varepsilon = \bar{p}^0 + \varepsilon \bar{p}^1 \tag{77}$$

The terms $\bar{u}^k_3$ $(k = 0, 1, 2)$, $\bar{p}^k$ $(k = 0, 1)$ and $\bar{\sigma}^0_{i3}$ $(i = 1, 2)$ are known (see (57)-(61)), so the approximations of the vertical velocity and the pressure are:

$$\begin{aligned}
\tilde{u}^\varepsilon_3 &= \varepsilon \bar{u}^1_3 + \varepsilon^2 \bar{u}^2_3 = \varepsilon\left[(b - z)\left(\frac{\partial \bar{u}^0_1}{\partial x} + \frac{\partial \bar{u}^0_2}{\partial y}\right) + \bar{u}^0_1\frac{\partial b}{\partial x} + \bar{u}^0_2\frac{\partial b}{\partial y}\right] \\
&+ \varepsilon^2\left[(b - z)\left(\frac{\partial \bar{u}^1_1}{\partial x} + \frac{\partial \bar{u}^1_2}{\partial y}\right) + \bar{u}^1_1\frac{\partial b}{\partial x} + \bar{u}^1_2\frac{\partial b}{\partial y}\right] \\
&= \varepsilon\left[(b - z)\left(\frac{\partial \tilde{u}^\varepsilon_1}{\partial x} + \frac{\partial \tilde{u}^\varepsilon_2}{\partial y}\right) + \tilde{u}^\varepsilon_1\frac{\partial b}{\partial x} + \tilde{u}^\varepsilon_2\frac{\partial b}{\partial y}\right] \qquad (78) \\
\tilde{p}^\varepsilon &= \bar{p}_s - \frac{2}{Re}\left(\frac{\partial \bar{u}^0_1}{\partial x} + \frac{\partial \bar{u}^0_2}{\partial y}\right) + \varepsilon\left[(s - z)(G - 2\phi(\cos\varphi)\bar{u}^0_1) - \frac{2}{Re}\left(\frac{\partial \bar{u}^1_1}{\partial x} + \frac{\partial \bar{u}^1_2}{\partial y}\right)\right] \\
&= \bar{p}_s + \varepsilon(s - z)(G - 2\phi(\cos\varphi)\bar{u}^0_1) - \frac{2}{Re}\left(\frac{\partial \tilde{u}^\varepsilon_1}{\partial x} + \frac{\partial \tilde{u}^\varepsilon_2}{\partial y}\right) \qquad (79) \\
&= \bar{p}_s + \varepsilon(s - z)(G - 2\phi(\cos\varphi)\tilde{u}^\varepsilon_1) - \frac{2}{Re}\left(\frac{\partial \tilde{u}^\varepsilon_1}{\partial x} + \frac{\partial \tilde{u}^\varepsilon_2}{\partial y}\right) + O(\varepsilon^2) \qquad (80)
\end{aligned}$$

To obtain closed equations for $\bar{u}^0_1$ and $\bar{u}^0_2$ we have to get rid of the terms $\bar{u}^2_1$ and $\bar{u}^2_2$ in equations (63) and (65). We can accomplish this with the following procedure. First integrate those equations in the vertical variable $z$ from $b$ to $s$:

$$\frac{\partial \bar{u}^0_1}{\partial t} + \bar{u}^0_1\frac{\partial \bar{u}^0_1}{\partial x} + \bar{u}^0_2\frac{\partial \bar{u}^0_1}{\partial y} + \frac{\partial \bar{p}_s}{\partial x} - \frac{1}{Re}\left(3\frac{\partial^2 \bar{u}^0_1}{\partial x^2} + \frac{\partial^2 \bar{u}^0_1}{\partial y^2} + 2\frac{\partial^2 \bar{u}^0_2}{\partial x \partial y}\right) - 2\phi(\sin\varphi)\,\bar{u}^0_2$$
$$- \frac{1}{hRe}\left(\left.\frac{\partial \bar{u}^2_1}{\partial z}\right|_{z=s} - \left.\frac{\partial \bar{u}^2_1}{\partial z}\right|_{z=b}\right) + \frac{\eta^2}{3}\left(\frac{\partial \bar{u}^0_1}{\partial t}\frac{\partial^2 \bar{u}^0_1}{\partial t \partial x} + \frac{\partial \bar{u}^0_2}{\partial t}\frac{\partial^2 \bar{u}^0_1}{\partial t \partial y}\right) = O(\eta^4) \tag{81}$$
$$\frac{\partial \bar{u}^0_2}{\partial t} + \bar{u}^0_1\frac{\partial \bar{u}^0_2}{\partial x} + \bar{u}^0_2\frac{\partial \bar{u}^0_2}{\partial y} + \frac{\partial \bar{p}_s}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2 \bar{u}^0_2}{\partial x^2} + 3\frac{\partial^2 \bar{u}^0_2}{\partial y^2} + 2\frac{\partial^2 \bar{u}^0_1}{\partial y \partial x}\right) + 2\phi(\sin\varphi)\,\bar{u}^0_1$$
$$- \frac{1}{hRe}\left(\left.\frac{\partial^2 \bar{u}^2_2}{\partial z^2}\right|_{z=s} - \left.\frac{\partial^2 \bar{u}^2_2}{\partial z^2}\right|_{z=b}\right) + \frac{\eta^2}{3}\left(\frac{\partial \bar{u}^0_1}{\partial t}\frac{\partial^2 \bar{u}^0_2}{\partial t \partial x} + \frac{\partial \bar{u}^0_2}{\partial t}\frac{\partial^2 \bar{u}^0_2}{\partial t \partial y}\right) = O(\eta^4) \tag{82}$$

Then, we can find the value of $\dfrac{\partial \bar{u}^2_i}{\partial z}$ at the top surface and at the bottom. Using (70) and (58), we write:

$$\frac{\partial \bar{u}^2_i}{\partial z} = Re\,\bar{\sigma}^1_{i3} - \frac{\partial}{\partial x_i}\left[(b - z)\left(\frac{\partial \bar{u}^0_1}{\partial x} + \frac{\partial \bar{u}^0_2}{\partial y}\right) + \bar{u}^0_1\frac{\partial b}{\partial x} + \bar{u}^0_2\frac{\partial b}{\partial y}\right] \quad i = 1, 2 \tag{83}$$

Next, substituting $z$ by $s$ and $b$, we have:

$$\left.\frac{\partial \bar{u}^2_i}{\partial z}\right|_{z=s} - \left.\frac{\partial \bar{u}^2_i}{\partial z}\right|_{z=b} = Re\left(\bar{\sigma}^1_{i3}\big|_{z=s} - \bar{\sigma}^1_{i3}\big|_{z=b}\right) + h\frac{\partial}{\partial x_i}\left(\frac{\partial \bar{u}^0_1}{\partial x} + \frac{\partial \bar{u}^0_2}{\partial y}\right) \quad i = 1, 2 \tag{84}$$

To obtain the value of $\bar{\sigma}_{i3}^1$ at the top surface and at the bottom, boundary conditions (72), (73), (74) and (75) are used

$$\begin{aligned}
\left.\frac{\partial \bar{u}_1^2}{\partial z}\right|_{z=s} - \left.\frac{\partial \bar{u}_1^2}{\partial z}\right|_{z=b} &= 2\left(2\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial h}{\partial x} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial h}{\partial y} \\
&+ \frac{\eta^2}{6}\left[4\frac{\partial}{\partial t}\left(2\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^2 h}{\partial t \partial x} + 2\left(2\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^3 h}{\partial t^2 \partial x}\right. \\
&\left. + 2\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^2 h}{\partial t \partial y} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^3 h}{\partial t^2 \partial y}\right] \\
&+ Re\left(\bar{f}_{W_1}^1 + \bar{f}_{R_1}^1\right) + h\frac{\partial}{\partial x}\left(\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right) + O(\eta^4) \qquad (85)
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial \bar{u}_2^2}{\partial z}\right|_{z=s} - \left.\frac{\partial \bar{u}_2^2}{\partial z}\right|_{z=b} &= 2\left(\frac{\partial \bar{u}_1^0}{\partial x} + 2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial h}{\partial y} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial h}{\partial x} \\
&+ \frac{\eta^2}{6}\left[4\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial x} + 2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^2 h}{\partial t \partial y} + 2\left(\frac{\partial \bar{u}_1^0}{\partial x} + 2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^3 h}{\partial t^2 \partial y}\right. \\
&\left. + 2\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^2 h}{\partial t \partial x} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^3 h}{\partial t^2 \partial x}\right] \\
&+ Re\left(\bar{f}_{W_2}^1 + \bar{f}_{R_2}^1\right) + h\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right) + O(\eta^4) \qquad (86)
\end{aligned}$$

When we substitute expressions (85)-(86) into (81)-(82), we obtain

$$\begin{aligned}
&\frac{\partial \bar{u}_1^0}{\partial t} + \bar{u}_1^0\frac{\partial \bar{u}_1^0}{\partial x} + \bar{u}_2^0\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\eta^2}{3}\left(\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t \partial x} + \frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t \partial y}\right) \\
&\quad = -\frac{\partial \bar{p}_s}{\partial x} + \frac{1}{Re}\left(4\frac{\partial^2 \bar{u}_1^0}{\partial x^2} + \frac{\partial^2 \bar{u}_1^0}{\partial y^2} + 3\frac{\partial^2 \bar{u}_2^0}{\partial x \partial y}\right) + 2\phi\left(\sin\varphi\right)\bar{u}_2^0 \\
&\quad + \frac{1}{hRe}\left\{2\left(2\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial h}{\partial x} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial h}{\partial y}\right. \\
&\quad + \frac{\eta^2}{6}\left[4\frac{\partial}{\partial t}\left(2\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^2 h}{\partial t \partial x} + 2\left(2\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^3 h}{\partial t^2 \partial x}\right. \\
&\quad \left.\left. + 2\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^2 h}{\partial t \partial y} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^3 h}{\partial t^2 \partial y}\right]\right\} \\
&\quad + \frac{1}{h}\left(\bar{f}_{W_1}^1 + \bar{f}_{R_1}^1\right) + O(\eta^4) \qquad (87)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \bar{u}_2^0}{\partial t} + \bar{u}_1^0\frac{\partial \bar{u}_2^0}{\partial x} + \bar{u}_2^0\frac{\partial \bar{u}_2^0}{\partial y} + \frac{\eta^2}{3}\left(\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t \partial x} + \frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t \partial y}\right) \\
&\quad = -\frac{\partial \bar{p}_s}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}_2^0}{\partial x^2} + 4\frac{\partial^2 \bar{u}_2^0}{\partial y^2} + 3\frac{\partial^2 \bar{u}_1^0}{\partial y \partial x}\right) - 2\phi\left(\sin\varphi\right)\bar{u}_1^0 \\
&\quad + \frac{1}{hRe}\left\{\left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial h}{\partial x} + 2\left(\frac{\partial \bar{u}_1^0}{\partial x} + 2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial h}{\partial y}\right. \\
&\quad + \frac{\eta^2}{6}\left[2\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^2 h}{\partial t \partial x} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x}\right)\frac{\partial^3 h}{\partial t^2 \partial x}\right. \\
&\quad \left.\left. + 4\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^0}{\partial x} + 2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^2 h}{\partial t \partial y} + 2\left(\frac{\partial \bar{u}_1^0}{\partial x} + 2\frac{\partial \bar{u}_2^0}{\partial y}\right)\frac{\partial^3 h}{\partial t^2 \partial y}\right]\right\} \\
&\quad + \frac{1}{h}\left(\bar{f}_{W_2}^1 + \bar{f}_{R_2}^1\right) + O(\eta^4) \qquad (88)
\end{aligned}$$

We repeat the process from equations (64) and (66) to obtain closed equations for $\bar{u}_i^1\,(i = 1, 2)$. We

yield:

$$
\begin{aligned}
&\frac{\partial \bar{u}_1^1}{\partial t} + \bar{u}_1^0\frac{\partial \bar{u}_1^1}{\partial x} + \bar{u}_1^1\frac{\partial \bar{u}_1^0}{\partial x} + \bar{u}_2^0\frac{\partial \bar{u}_1^1}{\partial y} + \bar{u}_2^1\frac{\partial \bar{u}_1^0}{\partial y} \\
&\quad + \frac{\eta^2}{3}\left(\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_1^1}{\partial t\partial x} + \frac{\partial \bar{u}_1^1}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t\partial x} + \frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_1^1}{\partial t\partial y} + \frac{\partial \bar{u}_2^1}{\partial t}\frac{\partial^2 \bar{u}_1^0}{\partial t\partial y}\right) \\
&\quad = -\frac{\partial s}{\partial x}G + 2\phi\left(\sin\varphi\right)\bar{u}_2^1 + 2\phi(\cos\varphi)\left(\frac{\partial h}{\partial x}\bar{u}_1^0 + h\frac{\partial \bar{u}_1^0}{\partial x} + \frac{h}{2}\frac{\partial \bar{u}_2^0}{\partial y} - \bar{u}_2^0\frac{\partial b}{\partial y}\right) \\
&\quad + \frac{1}{Re}\left(4\frac{\partial^2 \bar{u}_1^1}{\partial x^2} + \frac{\partial^2 \bar{u}_1^1}{\partial y^2} + 3\frac{\partial^2 \bar{u}_2^1}{\partial x\partial y}\right) + \frac{2}{hRe}\left(2\frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y}\right)\frac{\partial h}{\partial x} \\
&\quad + \frac{1}{hRe}\left(\frac{\partial \bar{u}_1^1}{\partial y} + \frac{\partial \bar{u}_2^1}{\partial x}\right)\frac{\partial h}{\partial y} + \frac{\eta^2}{6h}\left[2\frac{\partial h}{\partial t}(2\phi(\cos\varphi)\bar{u}_1^0 - G)\frac{\partial^2 h}{\partial t\partial x}\right. \\
&\quad + \frac{4}{Re}\frac{\partial}{\partial t}\left(2\frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y}\right)\frac{\partial^2 h}{\partial t\partial x} + \frac{2}{Re}\left(2\frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y}\right)\frac{\partial^3 h}{\partial t^2\partial x} \\
&\quad \left. + \frac{2}{Re}\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^1}{\partial y} + \frac{\partial \bar{u}_2^1}{\partial x}\right)\frac{\partial^2 h}{\partial t\partial y} + \frac{1}{Re}\left(\frac{\partial \bar{u}_1^1}{\partial y} + \frac{\partial \bar{u}_2^1}{\partial x}\right)\frac{\partial^3 h}{\partial t^2\partial y}\right] \\
&\quad + \frac{1}{h}\left(\bar{f}_{W_1}^2 + \bar{f}_{R_1}^2\right) + O(\eta^4)
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
&\frac{\partial \bar{u}_2^1}{\partial t} + \bar{u}_1^0\frac{\partial \bar{u}_2^1}{\partial x} + \bar{u}_1^1\frac{\partial \bar{u}_2^0}{\partial x} + \bar{u}_2^0\frac{\partial \bar{u}_2^1}{\partial y} + \bar{u}_2^1\frac{\partial \bar{u}_2^0}{\partial y} \\
&\quad + \frac{\eta^2}{3}\left(\frac{\partial \bar{u}_1^0}{\partial t}\frac{\partial^2 \bar{u}_2^1}{\partial t\partial x} + \frac{\partial \bar{u}_1^1}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t\partial x} + \frac{\partial \bar{u}_2^0}{\partial t}\frac{\partial^2 \bar{u}_2^1}{\partial t\partial y} + \frac{\partial \bar{u}_2^1}{\partial t}\frac{\partial^2 \bar{u}_2^0}{\partial t\partial y}\right) \\
&\quad = -\frac{\partial s}{\partial y}G - 2\phi\left(\sin\varphi\right)\bar{u}_1^1 + 2\phi(\cos\varphi)\left(\frac{\partial s}{\partial y}\bar{u}_1^0 + \frac{h}{2}\frac{\partial \bar{u}_1^0}{\partial y}\right) \\
&\quad + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}_2^1}{\partial x^2} + 4\frac{\partial^2 \bar{u}_2^1}{\partial y^2} + 3\frac{\partial^2 \bar{u}_1^1}{\partial x\partial y}\right) + \frac{2}{hRe}\left(\frac{\partial \bar{u}_1^1}{\partial x} + 2\frac{\partial \bar{u}_2^1}{\partial y}\right)\frac{\partial h}{\partial y} \\
&\quad + \frac{1}{hRe}\left(\frac{\partial \bar{u}_1^1}{\partial y} + \frac{\partial \bar{u}_2^1}{\partial x}\right)\frac{\partial h}{\partial x} + \frac{\eta^2}{6h}\left[2\frac{\partial h}{\partial t}(2\phi(\cos\varphi)\bar{u}_1^0 - G)\frac{\partial^2 h}{\partial t\partial y}\right. \\
&\quad + \frac{4}{Re}\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^1}{\partial x} + 2\frac{\partial \bar{u}_2^1}{\partial y}\right)\frac{\partial^2 h}{\partial t\partial y} + \frac{2}{Re}\left(\frac{\partial \bar{u}_1^1}{\partial x} + 2\frac{\partial \bar{u}_2^1}{\partial y}\right)\frac{\partial^3 h}{\partial t^2\partial y} \\
&\quad \left. + \frac{2}{Re}\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}_1^1}{\partial y} + \frac{\partial \bar{u}_2^1}{\partial x}\right)\frac{\partial^2 h}{\partial t\partial x} + \frac{1}{Re}\left(\frac{\partial \bar{u}_1^1}{\partial y} + \frac{\partial \bar{u}_2^1}{\partial x}\right)\frac{\partial^3 h}{\partial t^2\partial x}\right] \\
&\quad + \frac{1}{h}\left(\bar{f}_{W_2}^2 + \bar{f}_{R_2}^2\right) + O(\eta^4)
\end{aligned} \tag{90}
$$

Equations (87)-(90) are used to obtain the following equations for $\tilde{u}_i^\varepsilon$ (see (77)):

$$
\begin{aligned}
&\frac{\partial \tilde{u}_1^\varepsilon}{\partial t} + \tilde{u}_1^\varepsilon\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \tilde{u}_2^\varepsilon\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\eta^2}{3}\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial t}\frac{\partial^2 \tilde{u}_1^\varepsilon}{\partial t\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial t}\frac{\partial^2 \tilde{u}_1^\varepsilon}{\partial t\partial y}\right) \\
&\quad = -\frac{\partial \bar{p}_s}{\partial x} - \varepsilon\frac{\partial s}{\partial x}G + 2\phi\left(\sin\varphi\right)\tilde{u}_2^\varepsilon + \frac{1}{Re}\left(4\frac{\partial^2 \tilde{u}_1^\varepsilon}{\partial x^2} + \frac{\partial^2 \tilde{u}_1^\varepsilon}{\partial y^2} + 3\frac{\partial^2 \tilde{u}_2^\varepsilon}{\partial x\partial y}\right) \\
&\quad + \frac{1}{hRe}\left\{2\left(2\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial h}{\partial x} + \left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial h}{\partial y}\right. \\
&\quad + \frac{\eta^2}{6}\left[4\frac{\partial}{\partial t}\left(2\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial^2 h}{\partial t\partial x} + 2\left(2\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial^3 h}{\partial t^2\partial x}\right. \\
&\quad \left.\left. + 2\frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial^2 h}{\partial t\partial y} + \left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial^3 h}{\partial t^2\partial y}\right]\right\} \\
&\quad + (2\phi(\cos\varphi)\tilde{u}_1^\varepsilon - G)\varepsilon\frac{\eta^2}{3h}\frac{\partial h}{\partial t}\frac{\partial^2 h}{\partial t\partial x} + 2\phi(\cos\varphi)\varepsilon\left[\frac{\partial h}{\partial x}\tilde{u}_1^\varepsilon + h\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{h}{2}\frac{\partial \tilde{u}_2^\varepsilon}{\partial y} - \tilde{u}_2^\varepsilon\frac{\partial b}{\partial y}\right] \\
&\quad + \frac{1}{\varepsilon h}\left(\tilde{f}_{W_1}^\varepsilon + \tilde{f}_{R_1}^\varepsilon\right) + O(\eta^4) + O(\varepsilon^2)
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
&\frac{\partial \tilde{u}_2^\varepsilon}{\partial t} + \tilde{u}_1^\varepsilon \frac{\partial \tilde{u}_2^\varepsilon}{\partial x} + \tilde{u}_2^\varepsilon \frac{\partial \tilde{u}_2^\varepsilon}{\partial y} + \frac{\eta^2}{3}\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial t}\frac{\partial^2 \tilde{u}_2^\varepsilon}{\partial t \partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial t}\frac{\partial^2 \tilde{u}_2^\varepsilon}{\partial t \partial y}\right) \\
&= -\frac{\partial \bar{p}_s}{\partial y} - \varepsilon \frac{\partial s}{\partial y} G - 2\phi\,(\sin\varphi)\,\tilde{u}_1^\varepsilon + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}_2^\varepsilon}{\partial x^2} + 4\frac{\partial^2 \tilde{u}_2^\varepsilon}{\partial y^2} + 3\frac{\partial^2 \tilde{u}_1^\varepsilon}{\partial y \partial x}\right) \\
&+ \frac{1}{hRe}\left\{\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial h}{\partial x} + 2\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + 2\frac{\partial \tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial h}{\partial y}\right. \\
&+ \frac{\eta^2}{6}\left[2\frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial^2 h}{\partial t \partial x} + \left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial y} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial^3 h}{\partial t^2 \partial x}\right. \\
&\left.\left. + 4\frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + 2\frac{\partial \tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial^2 h}{\partial t \partial y} + 2\left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + 2\frac{\partial \tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial^3 h}{\partial t^2 \partial y}\right]\right\} \\
&+ (2\phi(\cos\varphi)\tilde{u}_1^\varepsilon - G)\varepsilon\frac{\eta^2}{3h}\frac{\partial h}{\partial t}\frac{\partial^2 h}{\partial t \partial y} + 2\phi(\cos\varphi)\varepsilon\left(\frac{\partial s}{\partial y}\tilde{u}_1^\varepsilon + \frac{\partial \tilde{u}_1^\varepsilon}{\partial y}\frac{h}{2}\right) \\
&+ \frac{1}{\varepsilon h}\left(\tilde{f}_{W_2}^\varepsilon + \tilde{f}_{R_2}^\varepsilon\right) + O(\varepsilon^2) + O(\eta^4)
\end{aligned} \tag{92}
$$

The system of equations for $\tilde{u}_i (i = 1, 2)$ ((91)-(92)) is coupled with the following equation for the water depth deduced from (67)-(68):

$$
\frac{\partial h}{\partial t} + \frac{\partial (h\tilde{u}_1^\varepsilon)}{\partial x} + \frac{\partial (h\tilde{u}_2^\varepsilon)}{\partial y} - \frac{\eta^2}{6}\left[\frac{\partial}{\partial x}\left(h\frac{\partial^2 \tilde{u}_1^\varepsilon}{\partial t^2}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial^2 \tilde{u}_2^\varepsilon}{\partial t^2}\right)\right] = O(\eta^4) \tag{93}
$$

Therefore, if equations (91)-(93) are solved, we obtain the horizontal velocity and the free surface elevation.

# 5 Proposed Shallow Water Model

In this section, we go back the original variables:

$$
H^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon L_C h(t, x, y), \quad B^\varepsilon(x^\varepsilon, y^\varepsilon) = \varepsilon L_C b(x, y), \tag{94}
$$

$$
\tilde{u}_i^\varepsilon(t, x, y) = \frac{T_C}{L_C}\tilde{U}_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon), \quad i = 1, 2, 3, \tag{95}
$$

$$
\tilde{p}^\varepsilon(t, x, y, z) = \frac{T_C^2}{\rho_0 L_C^2}\tilde{P}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), \quad \bar{p}_s(t, x, y) = \frac{T_C^2}{\rho_0 L_C^2}\bar{P}_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{96}
$$

$$
\vec{\tilde{\mathbf{f}}}_W^\varepsilon = \frac{T_C^2}{\rho_0 L_C^2}\vec{\tilde{\mathbf{F}}}_W^\varepsilon, \quad \vec{\tilde{\mathbf{f}}}_R^\varepsilon = \frac{T_C^2}{\rho_0 L_C^2}\vec{\tilde{\mathbf{F}}}_R^\varepsilon \tag{97}
$$

and we present the model that we have derived. We begin summarizing the results that we have achieved in this theorem:

**Theorem 1** *Let us suppose that there exists asymptotic expansion (47). Then approximated solution (94)-(96) verifies*

$$
\frac{\partial H^\varepsilon}{\partial t^\varepsilon} + \nabla^\varepsilon \cdot \left[H^\varepsilon\left(\vec{\tilde{\mathbf{U}}}^\varepsilon - \frac{\eta^2 T_C^2}{6}\frac{\partial^2 \vec{\tilde{\mathbf{U}}}^\varepsilon}{\partial (t^\varepsilon)^2}\right)\right] = \frac{\varepsilon L_C}{T_C} O(\eta^4) \tag{98}
$$

$$\frac{\partial \vec{\tilde{\mathbf{U}}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\tilde{\mathbf{U}}}^\varepsilon \cdot \vec{\tilde{\mathbf{U}}}^\varepsilon + \frac{\eta^2 T_C^2}{3} \nabla \left( \frac{\partial \vec{\tilde{\mathbf{U}}}^\varepsilon}{\partial t^\varepsilon} \right) \cdot \frac{\partial \vec{\tilde{\mathbf{U}}}^\varepsilon}{\partial t^\varepsilon} = -\frac{1}{\rho_0} \nabla \bar{P}_s^\varepsilon + \nu \Delta \vec{\tilde{\mathbf{U}}}^\varepsilon + 3\nu \nabla \left( \nabla \cdot \vec{\tilde{\mathbf{U}}}^\varepsilon \right)$$
$$+ \frac{\nu}{H^\varepsilon} \left\{ \tilde{\mathbf{R}}^\varepsilon \cdot \nabla H^\varepsilon + \frac{\eta^2 T_C^2}{6} \left[ 2 \frac{\partial \tilde{\mathbf{R}}^\varepsilon}{\partial t^\varepsilon} \cdot \nabla \left( \frac{\partial H^\varepsilon}{\partial t^\varepsilon} \right) + \tilde{\mathbf{R}}^\varepsilon \cdot \nabla \left( \frac{\partial^2 H^\varepsilon}{\partial (t^\varepsilon)^2} \right) \right] \right\}$$
$$+ 2\Phi \left( \sin \varphi \right) \begin{pmatrix} \tilde{U}_2^\varepsilon \\ -\tilde{U}_1^\varepsilon \end{pmatrix} + \left( 2\Phi (\cos \varphi) \tilde{U}_1^\varepsilon - g \right) \left[ \nabla S^\varepsilon + \frac{\eta^2 T_C^2}{3H^\varepsilon} \frac{\partial H^\varepsilon}{\partial t^\varepsilon} \nabla \left( \frac{\partial H^\varepsilon}{\partial t^\varepsilon} \right) \right]$$
$$+ 2\Phi (\cos \varphi) \left[ \frac{1}{2} H^\varepsilon \nabla (\tilde{U}_1^\varepsilon) + \begin{pmatrix} -\vec{\tilde{\mathbf{U}}}^\varepsilon \cdot \nabla B^\varepsilon + \frac{1}{2} H^\varepsilon \nabla \cdot \vec{\tilde{\mathbf{U}}}^\varepsilon \\ 0 \end{pmatrix} \right]$$
$$+ \frac{1}{\rho_0 H^\varepsilon} \left( \vec{\tilde{\mathbf{F}}}_W^\varepsilon + \vec{\tilde{\mathbf{F}}}_R^\varepsilon \right) + \frac{L_C}{T_C^2} O(\eta^4) + \frac{L_C}{T_C^2} O(\varepsilon^2) \tag{99}$$

$$\tilde{P}^\varepsilon = \bar{P}_s + \rho_0 \left( S^\varepsilon - z^\varepsilon \right) \left( g - 2\Phi (\cos \varphi) \tilde{U}_1^\varepsilon \right) - 2\mu \nabla^\varepsilon \cdot \vec{\tilde{\mathbf{U}}}^\varepsilon + O(\varepsilon^2) \frac{\rho_0 L_C^2}{T_C^2} \tag{100}$$

$$\tilde{U}_3^\varepsilon = \left( B^\varepsilon - z^\varepsilon \right) \nabla^\varepsilon \cdot \vec{\tilde{\mathbf{U}}}^\varepsilon + \vec{\tilde{\mathbf{U}}}^\varepsilon \cdot \nabla B^\varepsilon \tag{101}$$

*where* $\vec{\tilde{\mathbf{U}}}^\varepsilon = (\tilde{U}_1^\varepsilon, \tilde{U}_2^\varepsilon)$ *is the time-averaged horizontal velocity approximation,* $\bar{P}_s$ *is the time-averaged atmospheric pressure at the surface,* $\vec{\tilde{\mathbf{F}}}_W^\varepsilon$, $\vec{\tilde{\mathbf{F}}}_R^\varepsilon$ *are the wind and friction forces and*

$$\tilde{\mathbf{R}}^\varepsilon = \begin{pmatrix} 4 \frac{\partial \tilde{U}_1^\varepsilon}{\partial x^\varepsilon} + 2 \frac{\partial \tilde{U}_2^\varepsilon}{\partial y^\varepsilon} & \frac{\partial \tilde{U}_1^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{U}_2^\varepsilon}{\partial x^\varepsilon} \\ \frac{\partial \tilde{U}_1^\varepsilon}{\partial y^\varepsilon} + \frac{\partial \tilde{U}_2^\varepsilon}{\partial x^\varepsilon} & 2 \frac{\partial \tilde{U}_1^\varepsilon}{\partial x^\varepsilon} + 4 \frac{\partial \tilde{U}_2^\varepsilon}{\partial y^\varepsilon} \end{pmatrix}$$

We, finally, propose the following model that we have derived neglecting the $O(\varepsilon^2)$ and $O(\eta^4)$ terms in the above equations. For notational convenience, we henceforth drop the $\tilde{}$.

$$\frac{\partial H^\varepsilon}{\partial t^\varepsilon} + \nabla^\varepsilon \cdot \left[ H^\varepsilon \left( \vec{\mathbf{U}}^\varepsilon - \frac{\eta^2 T_C^2}{6} \frac{\partial^2 \vec{\mathbf{U}}^\varepsilon}{\partial (t^\varepsilon)^2} \right) \right] = 0 \tag{102}$$

$$\frac{\partial \vec{\mathbf{U}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{U}}^\varepsilon \cdot \vec{\mathbf{U}}^\varepsilon + \frac{\eta^2 T_C^2}{3} \nabla \left( \frac{\partial \vec{\mathbf{U}}^\varepsilon}{\partial t^\varepsilon} \right) \cdot \frac{\partial \vec{\mathbf{U}}^\varepsilon}{\partial t^\varepsilon} = -\frac{1}{\rho_0} \nabla \bar{P}_s^\varepsilon + \nu \Delta \vec{\mathbf{U}}^\varepsilon + 3\nu \nabla \left( \nabla \cdot \vec{\mathbf{U}}^\varepsilon \right)$$
$$+ \frac{\nu}{H^\varepsilon} \left\{ \bar{\mathbf{R}}^\varepsilon \cdot \nabla H^\varepsilon + \frac{\eta^2 T_C^2}{6} \left[ 2 \frac{\partial \bar{\mathbf{R}}^\varepsilon}{\partial t^\varepsilon} \cdot \nabla \left( \frac{\partial H^\varepsilon}{\partial t^\varepsilon} \right) + \bar{\mathbf{R}}^\varepsilon \cdot \nabla \left( \frac{\partial^2 H^\varepsilon}{\partial (t^\varepsilon)^2} \right) \right] \right\}$$
$$+ 2\Phi \left( \sin \varphi \right) \begin{pmatrix} \bar{U}_2^\varepsilon \\ -\bar{U}_1^\varepsilon \end{pmatrix} + \left( 2\Phi (\cos \varphi) \bar{U}_1^\varepsilon - g \right) \left[ \nabla S^\varepsilon + \frac{\eta^2 T_C^2}{3H^\varepsilon} \frac{\partial H^\varepsilon}{\partial t^\varepsilon} \nabla \left( \frac{\partial H^\varepsilon}{\partial t^\varepsilon} \right) \right]$$
$$+ 2\Phi (\cos \varphi) \left[ \frac{1}{2} H^\varepsilon \nabla (\bar{U}_1^\varepsilon) + \begin{pmatrix} -\vec{\mathbf{U}}^\varepsilon \cdot \nabla B^\varepsilon + \frac{1}{2} H^\varepsilon \nabla \cdot \vec{\mathbf{U}}^\varepsilon \\ 0 \end{pmatrix} \right] + \frac{1}{\rho_0 H^\varepsilon} \left( \vec{\mathbf{F}}_W^\varepsilon + \vec{\mathbf{F}}_R^\varepsilon \right) \tag{103}$$

$$\bar{P}^\varepsilon = \bar{P}_s + \rho_0 \left( S^\varepsilon - z^\varepsilon \right) \left( g - 2\Phi (\cos \varphi) \bar{U}_1^\varepsilon \right) - 2\mu \nabla^\varepsilon \cdot \vec{\mathbf{U}}^\varepsilon \tag{104}$$

$$\bar{U}_3^\varepsilon = \left( B^\varepsilon - z^\varepsilon \right) \nabla^\varepsilon \cdot \vec{\mathbf{U}}^\varepsilon + \vec{\mathbf{U}}^\varepsilon \cdot \nabla B^\varepsilon \tag{105}$$

**Remark 1** In practice, we neglect the term in $\eta^2$ in equation (102), because it does not provide significant improvements in the accuracy of the numerical solutions that we present in section 6, but it causes some problems of stability. Therefore, in what follows, we replace (102) by

$$\frac{\partial H^\varepsilon}{\partial t^\varepsilon} + \nabla^\varepsilon \cdot \left( H^\varepsilon \vec{\mathbf{U}}^\varepsilon \right) = 0 \tag{106}$$

# 6 Numerical experiments

In this section we shall compare model (103)-(106) (we shall refer to it as **NM**) with other shallow water model, not averaged in time, obtained in [17] using the same asymptotic techniques. We shall refer to it

as **SW**, and it can be written as in [18]:

$$\frac{\partial H^\varepsilon}{\partial t^\varepsilon} + \nabla^\varepsilon \cdot \left( H^\varepsilon \vec{\mathbf{U}}^\varepsilon \right) = 0$$

$$\begin{aligned}
\frac{\partial \vec{\mathbf{U}}^\varepsilon}{\partial t^\varepsilon} + \nabla \vec{\mathbf{U}}^\varepsilon \cdot \vec{\mathbf{U}}^\varepsilon = & -\frac{1}{\rho_0} \nabla P_s^\varepsilon + \nu \{ \Delta \vec{\mathbf{U}}^\varepsilon + \frac{1}{H^\varepsilon} [\nabla \vec{\mathbf{U}}^{\varepsilon^T} + \nabla \vec{\mathbf{U}}^\varepsilon] \nabla H^\varepsilon \\
& + \frac{1}{(H^\varepsilon)^2} \nabla [(H^\varepsilon)^2 (\nabla \cdot \vec{\mathbf{U}}^\varepsilon)] \} - g \nabla S^\varepsilon + \frac{1}{\rho_0 H^\varepsilon} \left( \vec{\mathbf{F}}_W^\varepsilon + \vec{\mathbf{F}}_R^\varepsilon \right) + 2\Phi \left\{ (\sin \varphi) \begin{pmatrix} U_2^\varepsilon \\ -U_1^\varepsilon \end{pmatrix} \right. \\
& \left. + (\cos \varphi) \left[ U_1 \nabla S^\varepsilon + \frac{1}{2} H^\varepsilon \nabla (U_1^\varepsilon) + \begin{pmatrix} -\vec{\mathbf{U}}^\varepsilon \cdot \nabla B^\varepsilon + \frac{1}{2} H^\varepsilon \nabla \cdot \vec{\mathbf{U}}^\varepsilon \\ 0 \end{pmatrix} \right] \right\}
\end{aligned} \tag{107}$$

where the horizontal velocity $\vec{\mathbf{U}}^\varepsilon$ is not averaged in time.

In order to compare models (103)-(106) and (107), we shall consider some analytical solutions of Navier-Stokes equations ((1)-(8)) whose velocity rapidly oscillates in time. Then we solve numerically the equations (103)-(106) and (107) for the data provided by the analytical solutions of Navier-Stokes equations, and finally we compute the errors committed by each of the models.

To perform the numerical simulations, we have opted to use MacCormack scheme (see [14]) due to its good stability properties, its easy implementation, and to the fact that has been applied successfully to the resolution of similar problems.

Let us introduce now the first family of exact solutions to Navier-Stokes equations that we shall use to compare models **SW** and **NM**. Horizontal and vertical velocity oscillate in time, but the horizontal velocity depends on variable $x^\varepsilon$ while the vertical velocity depends on variable $z^\varepsilon$:

$$\begin{aligned}
U_1^\varepsilon &= (A_1 + A_2 x^\varepsilon) \sin \left( \frac{2\pi n_1}{T_p} t^\varepsilon \right) + (B_1 + B_2 x^\varepsilon) \cos \left( \frac{2\pi n_1}{T_p} t^\varepsilon \right) \\
&+ (C_1 + C_2 x^\varepsilon) \sin \left( \frac{2\pi n_2}{T_p} t^\varepsilon \right) + (D_1 + D_2 x^\varepsilon) \cos \left( \frac{2\pi n_2}{T_p} t^\varepsilon \right) \\
U_2^\varepsilon &= 0 \\
U_3^\varepsilon &= -z^\varepsilon \left[ A_2 \sin \left( \frac{2\pi n_1}{T_p} t^\varepsilon \right) + B_2 \cos \left( \frac{2\pi n_1}{T_p} t^\varepsilon \right) + C_2 \sin \left( \frac{2\pi n_2}{T_p} t^\varepsilon \right) \right. \\
&+ \left. D_2 \cos \left( \frac{2\pi n_2}{T_p} t^\varepsilon \right) \right] \\
B^\varepsilon &= 0 \\
H^\varepsilon &= E e^{\frac{T_p}{2\pi} \left[ \frac{A_2}{n_1} \cos \left( \frac{2\pi n_1}{T_p} t^\varepsilon \right) - \frac{B_2}{n_1} \sin \left( \frac{2\pi n_1}{T_p} t^\varepsilon \right) + \frac{C_2}{n_2} \cos \left( \frac{2\pi n_2}{T_p} t^\varepsilon \right) - \frac{D_2}{n_2} \sin \left( \frac{2\pi n_2}{T_p} t^\varepsilon \right) \right]} \\
P_s^\varepsilon &= P^\varepsilon (z^\varepsilon = H^\varepsilon) - 2\mu \frac{\partial U_3^\varepsilon}{\partial z^\varepsilon} + F_{W_3}^\varepsilon \\
F_{W_1}^\varepsilon &= F_{W_2}^\varepsilon = 0 \\
\vec{F}_R^\varepsilon &= \vec{0}
\end{aligned} \tag{108}$$

where $A_i, B_i, C_i, D_i, n_i (i = 1, 2)$ and $T_p$ are any real value. We are able to calculate an analytical expression for $P^\varepsilon$ from equation (1), but it is too long and we have decided not to include it here.

We consider that $D$ is a rectangular basin of length 10 meters and width 2 meters with a $100 \times 20$ points grid (that is, the discretization step used is $\Delta x^\varepsilon = \Delta y^\varepsilon = 0.1$). We choose the values of the parameters so the maximum depth is always smaller than 1 meter, and thus the aspect ratio is always smaller than $10^{-1}$. We solve in temporal interval $[0, 10]$ with different time steps.

We introduce these two sets of values for the constants:

$$A_1 = B_1 = 2, C_1 = D_1 = 0.5, A_2 = B_2 = C_2 = D_2 = 0, E = 1 \tag{109}$$

$$A_1 = B_1 = 1, C_1 = D_1 = 0.5, A_2 = B_2 = 0.5, C_2 = D_2 = 0, E = 0.75 \tag{110}$$

With election (109) $U_1^\varepsilon$ does not depend on $x^\varepsilon$, but with election (110) it does. We present in Tables 1 and 3 the infinity-norm errors obtained when we approximate solution (108) using models (103)-(106)

and (107) with these choices of constants. For both examples we take: $\Phi = 0$, $g = 9.8$, $\rho = 998.2$, $\nu = 1.02 \times 10^{-6}$, $T_p = 1$, $n_1 = 0.1$, $n_2 = 100$.

We remark that in model (103)-(106) we must choose $\eta$ and $T_C$, where $T_C$ is the characteristic time and $2\eta$ is the length of the averaging interval for the non dimensional time variable; but the relevant choice here is the value of the product $\eta T_C$ because $2\eta T_C$ is the length of the averaging interval for the dimensional time variable.

| $\Delta t$ | Error bound for $H$ **SW** | Error bound for $H$ **NM** | Error bound for $U_1$ **SW** | Error bound for $\bar{U}_1$ **NM** |
|---|---|---|---|---|
| 0.01 | 0 | 0 | $3.1e3$ | $9.7e-3$ |
| 0.005 | 0 | 0 | $2.6e0$ | $4.8e-3$ |
| 0.001 | 0 | 0 | $4.0e-1$ | $9.7e-4$ |
| 0.0001 | 0 | 0 | $3.8e-2$ | $9.7e-5$ |
| 0.000025 | 0 | 0 | $9.5e-3$ | $2.4e-5$ |

Table 1: Error bounds for example (108) with data (109) and $\eta T_C = 2.5$

In Table 1 we show the error bounds when comparing the computed solution of **SW** with the Navier-Stokes solution (108), and when comparing the computed solution of **NM** with the averaged version of (108). We can see that, to obtain the same order of accuracy than the new model (103)-(106) with $\Delta t = 10^{-2}$, model (107) has to be solved taking $\Delta t = 2.5 \times 10^{-5}$, that is, the time step must be 400 times smaller. In this example, the depth is computed exactly by both models because it just depends on time.

The times (in seconds) required to solve models **SW** and **NM** for the example of Table 1 are presented in Table 2. We have used for these computations a personal computer with an Intel(R) Core(TM) i5 2.80 GHz processor (6 Gb RAM). We can see in this Table 2 that execution time for the new model increased compared with model **SW** for the same time step, but we can observe that it is shorter if we want to obtain the same accuracy. For example in this case, using the new model, in 8.4 seconds we have a more accurate approximation of the solution than the one obtained with model **SW** in 268 seconds. We see that the time needed to achieve the same accuracy with (107) is 1073 seconds.

| | $\Delta t = 10^{-2}$ | $\Delta t = 5 \times 10^{-3}$ | $\Delta t = 10^{-3}$ | $\Delta t = 10^{-4}$ | $\Delta t = 2.5 \times 10^{-5}$ |
|---|---|---|---|---|---|
| SW | 2.9 | 5.4 | 27.0 | 268.5 | **1073.0** |
| NM | **8.4** | 16.4 | 81.5 | 812.6 | 3284.9 |

Table 2: Execution times (in seconds) of the example of table 1

In Table 3 we observe that, when velocity depends on the spatial variable $x^\varepsilon$, the results achieved with the new model are quite better too. When we introduce data (110), we need to introduce a small value for $\eta$ ($\eta T_C = 5 \times 10^{-3}$) to guarantee the convergence of the scheme for $\Delta t = 10^{-4}$. For larger values of $\Delta t$, it is possible to choose larger values of $\eta$.

| $\Delta t$ | Error bound for $H$ **SW** | Error bound for $H$ **NM** | Error bound for $U_1$ **SW** | Error bound for $\bar{U}_1$ **NM** |
|---|---|---|---|---|
| 0.005 | $9.8e-3$ | $9.8e-3$ | $3.1e0$ | $1.4e-2$ |
| 0.001 | $1.9e-3$ | $1.9e-3$ | $4.7e-1$ | $2.7e-3$ |
| 0.0001 | $1.9e-4$ | $1.9e-4$ | $4.4e-2$ | $2.7e-4$ |

Table 3: Error bounds for example (108) with data (110) and $\eta T_C = 5 \times 10^{-3}$

In figure 2 we show the error bounds of table 3. It can be well appreciated that the numerical scheme is first order accurate as it corresponds to MacCormack scheme.

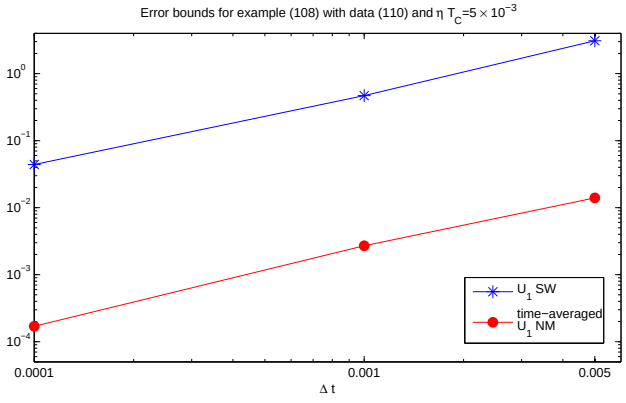


Figure 2: First order accuracy of the numerical scheme

Let us consider now the following solution to Navier-Stokes equations where water depth depends on $x^\varepsilon$:

$$\begin{aligned}
U_1^\varepsilon &= Ax^\varepsilon\left[\sin\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)+\cos\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)+\sin\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)+\cos\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)\right]\\
U_2^\varepsilon &= 0\\
U_3^\varepsilon &= -Az^\varepsilon\left[\sin\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)+\cos\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)+\sin\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)+\cos\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)\right]\\
B^\varepsilon &= 0\\
H^\varepsilon &= B(x^\varepsilon)^m e^{\frac{AT_p(m+1)}{2\pi}\left[\frac{\cos\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)-\sin\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)}{n_1}+\frac{\cos\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)-\sin\left(\frac{2\pi n_2}{T_c}t^\varepsilon\right)}{n_2}\right]}\\
P_s^\varepsilon &= P^\varepsilon(z^\varepsilon=H^\varepsilon)-2\mu\frac{\partial U_3^\varepsilon}{\partial z^\varepsilon}+F_{W_3}^\varepsilon\\
F_{W_1}^\varepsilon &= F_{W_2}^\varepsilon=0\\
\vec{F}_R^\varepsilon &= \vec{0}
\end{aligned}\tag{111}$$

with $m\in\mathbb{N}$, $A$, $B$, $n_i(i=1,2)$ and $T_p$ any real value.

For the following values of the constants

$$A=0.03, B=0.1, n_1=0.1, n_2=100, T_p=1, m=1\tag{112}$$

time step must be very small ($\Delta t=10^{-4}$) to achieve the convergence of model **SW** while the new model gives reasonable results with $\Delta t=10^{-3}$ as we show in Table 4.

| $\Delta t$ | Error bound for $H$ **SW** | Error bound for $H$ **NM** | Error bound for $U_1$ **SW** | Error bound for $\bar{U}_1$ **NM** |
|---|---|---|---|---|
| 0.001 | — | $1.2e-2$ | — | $2.1e-1$ |
| 0.0001 | $7.2e-2$ | $2.9e-3$ | $3.6e-1$ | $5.1e-2$ |

Table 4: Error bounds for example (111) with data (112) and $\eta T_C = 1.3034\times 10^{-2}$

We can observe from Tables 3 and 4 that, when the solution depends on $x^\varepsilon$, we need to take small values of $\eta T_C$. This is due to the fact that approximation (31) (and, as consequence, all formula (32)-(36)) is only valid for small values of $\eta$ (specially when the approximated function is rapidly oscillating), but then, averaged solution is almost equal to exact solution, and we shall also need a very small time step for model (103)-(106). Nevertheless, we see in Table 4 that model (103)-(106) achieves better results than the model (107) even in the worst case.

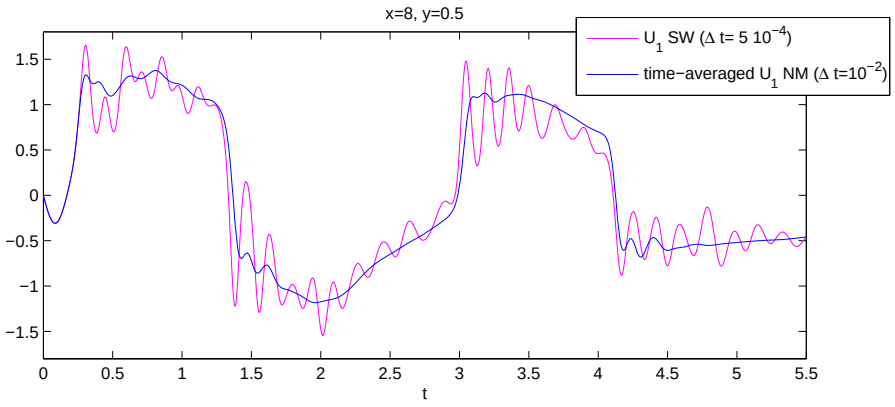


Figure 3: Comparison between $U_1$ **SW** and $\bar{U}_1$ **NM**

Finally, we present a more realistic numerical experiment which does not give a precise solution of the Navier-Stokes equations. Zero-Dirichlet boundary conditions are imposed for horizontal velocity. At the initial time, horizontal velocity is set to be zero and the the water depth is:

$$H_0^\varepsilon(x,y) = \begin{cases} 1 + 0.1\sin(2\pi x) & \text{if } x \in [0,3) \cup (7,10] \\ \\ 1.9 + 0.1\sin(2\pi x) & \text{if } [3,7] \end{cases} \tag{113}$$

In this case, as we have already commented for the previous example, time step must be quite small ($\Delta t = 5 \times 10^{-4}$) to achieve the convergence of model **SW** while the new model gives reasonable results even with $\Delta t = 10^{-2}$. We have plotted together, in figure 3, the approximations of the $U_1$ component of the horizontal velocity that both models provide. We observe that the new model reduces the high frequency oscillations that appear using the shallow water model. If we compare the execution times, we have that the time required to solve model **SW** with this data is 45 seconds while model **NM** runs in just 7 seconds.

# 7 Conclusions

We have used asymptotic analysis to obtain from the time-averaged non-dimensional Navier-Stokes equations a new shallow water model. The new model is able to filter (in some representative cases) the high frequency oscillations and this allows us to choose a much larger time step.

We have made some numerical comparisons between the new model and the shallow water model proposed in [17]. Numerical experiments confirm that this new model is able to obtain a given accuracy using larger time steps than the time step needed by the other shallow water model (see Tables 1, 3-4 and figure 3). In some cases, the time step can be even four hundred times larger. This enhancement leads to much shorter execution times compared with the execution times required by the classical shallow water model to obtain the same precision (see Table 2).

Although the numerical results achieved improve those of the model without time filtering, we have observed not so good results in some cases. We think that it can be caused by the absence of spacial filtering, and it would be convenient to use a combination of spacial and time filters, as suggested in [4], because the use of spacial "projective" filters or modelling of subgride-scale stress tensor is necessary to reduce the number of degrees of freedom of the problem.

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