UNIVERSIDADE DA CORUÑA

PhD. Thesis

Drift-Free Simulation and Libor Market Models

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The undersigned hereby certify that they are supervisors of the Thesis entitled **"Drift-Free Simulation and Libor Market Models"** developed by Marta Pou Bueno inside the Ph.D Program **"Mathematical Modelling and Numerical Methods in Applied Sciences and Engineering"** at the Department of Mathematics (University of A Coruña).

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A mi abuelo Felipe.

"Defiende tu derecho a pensar, porque incluso pensar de manera errónea es mejor que no pensar."

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Abstract

In this work, an efficient procedure to simulate the stochastic dynamics of Libor Market Model that avoids the use of the path dependent drifts in Monte Carlo simulation is proposed. For this purpose, we follow a Drift-Free Simulation methodology, by first simulating certain martingales and then obtaining the involved forward Libor rates in terms of them. More precisely, we propose a particular parameterization of those martingales so that the desired properties of the continuous models can be maintained after the discretization procedure when using either any intermediate forward probability measure or the spot one. Thus, the need of using the terminal probability measure to maintain the desired properties can be overcome. Some numerical results concerning caplets pricing illustrate that the proposed method outperforms other ones existing in the literature. We also explain how this methodology can be adapted to the case of Swap Market Model or any generic market model, and we extend it to the recently appeared multicurve setting. We also present how the proposed technique can be applied in the cross-markets context to price cross-currency, commodity or inflation derivatives, for example. Finally we place the here presented methodology into a graph theoretical framework.

Resumen

En este trabajo presentamos un procedimiento eficaz para simular las dinámicas estocásticas del Modelo de Mercado del Libor, procedimiento que evita el uso de los términos de deriva en la simulación de Monte Carlo. Para este propósito seguimos una metodología de Simulación Sin Derivas simulando en primer lugar ciertas martingalas y obteniendo después los tipos implícitos forward Libor en términos de ellas. En concreto, proponemos una parametrización particular de estas martingalas de modo que las propiedades que posee el modelo continuo se mantengan tras el procedimiento de discretización, tanto bajo cualquier medida de probabilidad forward intermedia como bajo la medida de probabilidad spot. De este modo, se supera la necesidad de usar la medida de probabilidad terminal para mantener esas propiedades deseables. Algunos resultados numéricos relativos a la valoración de caplets ilustran que el método propuesto supera a otros existentes en la literatura. Explicamos también cómo esta metodología puede ser adaptada para el Modelo de Mercado del Swap o cualquier modelo de mercado genérico, y la extendemos para el caso multicurva. Exponemos también cómo la técnica propuesta puede ser aplicada en el contexto de dos economías para valorar derivados tanto de dos monedas, como de ciertas mercancías, como de inflación, entre otros. Finalmente enmarcamos toda la metodología presentada durante el trabajo dentro del ámbito de la teoría de grafos.

Resumo

Neste traballo presentamos un procedemento eficaz para simular as dinámicas estocásticas do Modelo de Mercado do Libor, procedemento que evita o uso dos termos de deriva na simulación de Monte Carlo. Para este propósito seguimos unha metodoloxía de Simulación Sen Derivas simulando nun primer lugar certas martingalas e obtendo despois os tipos implícitos forward Libor en termos delas. En concreto, propoñemos unha parametrización particular destas martingalas de xeito que as propiedades que posee o modelo continuo sexan mantidas tras o procedemento de discretización, tanto baixo calquera medida de probabilidade forward intermedia como baixo a medida de probabilidade spot. De este xeito, se supera a necesidade de facer uso da medida de probabilidade terminal para manter esas propiedades desexables. Alguns resultados numéricos relativos á valoración de caplets ilustran que o método proposto supera a outros xa existentes na literatura. Explicamos tamén cómo esta metodoloxía pode ser adaptada para o Modelo de Mercado do Swap ou para calquera modelo de mercado xenérico, e exténdese para o caso multicurva. Amosamos tamén cómo a técnica proposta pode ser aplicada no contexto de dúas economías para valorar derivados tanto de dúas moedas, como de certas mercancías, como de inflación, entre outros. Finalmente enmarcamos toda a metodoloxía presentada durante o traballo dentro do campo da teoría de grafos.

Abbreviations and Notation

ATM = At The Money;

COTSMM = Coterminal Swap Market Model;

CSA = Credit Support Annex;

CVA = Credit Value Adjustment;

DFS = Drift-Free Simulation;

Eonia = Euro Over Night Index Average;

Euribor = Euro Interbank Offered Rate;

FRA = Forward Rate Agreement;

ITM = In The Money;

Libor = London Interbank Offered Rate;

LMM = Libor Market Model;

OIS = Overnight Indexed Swap;

OTM = Out The Money;

SMM = Swap Market Model;

ZCC = Zero Coupon Curve;

$A_j(t)$ = Annuity at time t for the tenor date T_j;

$\widehat{A}_j(t)$ = Deflated annuities;

$B_j(t)$ = Price at time t of a zero-coupon bond that matures at the tenor date T_j;

$B_j^A(t)$ = Price at time t of a economy A zero-coupon bond that matures at the tenor date T_j;

$B_j^B(t)$ = Price at time t of a economy B zero-coupon bond that matures at the tenor date T_j;

$\widehat{B}_j(t)$ = Deflated bond prices;

$\widehat{B}_j^A(t)$ = Deflated bond prices of the economy A;

$\widehat{B}_j^B(t)$ = Deflated bond prices of the economy B;

$C_{v,w}$ = Path from v to w;

$D_j(t)$ = Differences between consecutive deflated bonds;

E^A = Set of edges of the spanning tree A;

E^G = Set of edges of the graph G;

f_n = Numbers of the Fibonacci sequence;

$F_j(t)$ = Implicit forward rate at time t for the accrual period $[T_j, T_{j+1}]$;

$F_j^A(t)$ = Economy A implicit forward rate at time t for the accrual period $[T_j, T_{j+1}]$;

$F_j^B(t)$ = Economy B implicit forward rate at time t for the accrual period $[T_j, T_{j+1}]$;

$FX_j^{AB}(t)$ = Forward exchange rate at time t for the tenor date T_j ;

G^i = Generations of nodes of the spanning tree A;

$G_i(t)$ = Auxiliary functions in Subsection 4.2.2;

$H_i(t)$ = Auxiliary functions in Subsection 4.2.4 ;

$J_j^{M_l}(t)$ = Differences at time t between implicit rates of two consecutive estimation curves;

$L_j^{M_l}(t)$ = Implicit rates at time t of the estimation curve with accrual period $M_l \Delta T$;

$\text{Num}(t)$ = Numeraire at time t;

Q^n = Forward probability measure (associated to the numeraire B_n);

Q^β = Spot probability measure (associated to the numeraire β);

$S_j(t)$ = Coterminal swap rate at time t for the accrual period $[T_j, T_{N+1}]$;

$S_j^{M_l}(t)$ = Differences at time t between implicit rates of one estimation curve and implicit forward rates;

t_i = Discretization times;

T_j = Tenor dates;

T_{N+1} = Maturity tenor date;

$u_j(t)$ = Terms for the parameterization of DFS in LMM;

$u_j^S(t)$ = Terms for the parameterization of DFS in COTSMM;

V^A = Set of nodes of the spanning tree A;

V^G = Set of nodes of the graph G;

$V_j^{M_l}(t)$ = Estimation curve with accrual period $M_l \Delta T$;

W_j = Brownian motion in the dynamics of $F_j(t)$;

$W_j^A(t)$ = Brownian motion in the dynamics of $F_j^A(t)$;

$W_j^B(t)$ = Brownian motion in the dynamics of $F_j^B(t)$;

$W_j^{J^{M_l}}(t)$ = Brownian motion in the dynamics of $J_j^{M_l}(t)$;

$W_j^{L^{M_l}}(t)$ = Brownian motion in the dynamics of $L_j^{M_l}(t)$;

$W_j^S(t)$ = Brownian motion in the dynamics of $S_j(t)$;

$W_j^{S^{M_l}}(t)$ = Brownian motion in the dynamics of $S_j^{M_l}(t)$;

$X^{AB}(t)$ = Spot exchange rate at time t;

$y_j^{M_l}(t)$ = Terms for the parameterization of DFS in the fourth multicurve setting model;

$z_j^{M_l}(t)$ = Terms for the parameterization of DFS in the second multicurve setting model;

$\beta(t)$ = Bank account at time t;

δt = Time discretization jump;

δ_j = Accrual of the tenor structure;

ΔT = Constant accrual of the tenor structure;

γ_r^A = Correlation between the processes F_r^A and FX_k^{AB};

γ_r^B = Correlation between the processes F_r^B and FX_k^{AB};

$\mu_j(t)$ = Drift term in the dynamics of $F_j(t)$;

$\mu_j^A(t)$ = Drift term in the dynamics of $F_j^A(t)$;

$\mu_j^B(t)$ = Drift term in the dynamics of $F_j^B(t)$;

$\mu_j^{J^{M_l}}(t)$ = Drift term in the dynamics of $J_j^{M_l}(t)$;

$\mu_j^{L^{M_l}}(t)$ = Drift term in the dynamics of $L_j^{M_l}(t)$;

$\mu_j^S(t)$ = Drift term in the dynamics of $S_j(t)$;

$\mu_j^{S^{M_l}}(t)$ = Drift term in the dynamics of $S_j^{M_l}(t)$;

ϕ_N = Number of possible Cross-Market Models;

Φ = Cumulative standard normal distribution function;

$\rho_{r,s}$ = Correlation between the processes F_r^A and F_s^B or correlation between the processes F_r^A and FX_s^{AB};

$\rho_{r,s}^{Z^1,Z^2}$ = Correlation between the processes Z_r^1 and Z_s^2;

$\sigma_j(t)$ = Volatility term in the dynamics of $F_j(t)$;

$\sigma_j^A(t)$ = Volatility term in the dynamics of $F_j^A(t)$;

$\sigma_j^B(t)$ = Volatility term in the dynamics of $F_j^B(t)$;

$\sigma_j^{J^{M_l}}(t)$ = Volatility term in the dynamics of $J_j^{M_l}(t)$;

$\sigma_j^{L^{M_l}}(t)$ = Volatility term in the dynamics of $L_j^{M_l}(t)$;

$\sigma_j^S(t)$ = Volatility term in the dynamics of $S_j(t)$;

$\sigma_j^{S^{M_l}}(t)$ = Volatility term in the dynamics of $S_j^{M_l}(t)$;

$\Sigma_{r,s}$ = Correlation between the processes F_r and F_s;

$\Sigma_{r,s}^A$ = Correlation between the processes F_r^A and F_s^A;

$\Sigma_{r,s}^B$ = Correlation between the processes F_r^B and F_s^B;

$\Sigma_{r,s}^{AB}$ = Correlation between the processes FX_r^{AB} and FX_s^{AB};

$[f(x)]_+ = \max(f(x), 0)$;

$[v]$ = Information of the node v;

$[v, w]$ = Information of the edge $\{v, w\}$;

$[C_{v,w}]$ = Information of the path $C_{v,w}$;

$\{v, w\}$ = Edge from v to w.

Introduction

In this work we study a methodology to simulate several models mainly avoiding the use of the drift terms that appear in the dynamics involved in these models. More precisely, we develop this methodology in the context of Market Models and Cross-Market Models, the purpose of which consists of pricing interest rate derivatives of one economy and of two economies, respectively.

Interest rate derivatives are financial instruments whose payoff depends on some interest rates. The most classical example is an option which pays the positive part of the difference between one interest rate and some strike. The first time that these financial derivatives were traded in organized markets was in the early seventies with the Chicago Board of Options Exchange (CBOE). At that time Fisher Black and Myron Scholes assumed lognormal dynamics for the underlying rate and they proposed the dynamic hedging methodology to obtain the popular Black-Scholes formula for European vanilla options [9]. Since then, the emergence of more complex interest rate derivatives has motivated the need of more involved models. Basically, two types of models have arisen [16]: short rate models and market models. Short rate ones describe the spot interest rate, r_t, evolution via a possibly multi-dimensional driving diffusion process depending on some parameters:

$$dr_t = u(t, r_t)dt + v(t, r_t)dW_t,$$

where W_t denotes a Brownian motion and the expressions of the functions u and v give rise to different short rate models like the equilibrium models, with u and v only

dependent on the spot interest rate, not on the time (Vasicek [61], Cox, Ingersoll and Ross [17]), or the no-arbitrage models, with u and v dependent on the spot interest rate and on the time (Ho and Lee [29], Hull and White [31], Black and Karasinski [11]). One advantage of the equilibrium models is that they allow to obtain analytical formulas for pricing zero coupon bonds or even coupon bearing bonds, but the calibration of their constant parameters to current market data is quite difficult in practice. The no aribtrage models arose in order to overcome this disadvantage but the difficulties for the calibration to the initial curve of discount factors still remain. Alternatively, other models with greater computational complexity arise to solve the previously mentioned problems. An important alternative to short rate models, the approach by Ho and Lee [29] was translated into continuous time by Heath, Jarrow and Morton [27] who developed a general framework for the modeling of interest-rate dynamics. They mainly modeled instantaneous forward rates and they derived an arbitrage-free framework for the stochastic evolution of the entire yield curve, where the forward rates dynamics are fully specified through their instantaneous volatility structure. The main disadvantages of Heath-Jarrow-Morton model are that it is expressed in terms of continuous instantaneous forward rates, that are not directly observed in the market, and that it is not easy to calibrate to prices of actively traded instruments, such as caps. Therefore, taking the Heath, Jarrow and Morton framework as starting point, the most popular family of interest-rate models was introduced: the Market Models, like Libor Market Model (LMM) and Swap Market Model (SMM).

The main reason of the popularity of these market models, introduced in [14, 33, 46], lies in the compatibility with the market formulas for two basic interest rate derivatives: LMM and SMM price caps and swaptions, respectively, according to the standard Black's formula employed in the caps and swaptions markets. This issue constitutes an important fact since caps and swaptions markets are the two main derivative markets of the interest rate options world. Moreover, these models assume

lognormal dynamics for implicit forward Libor or forward swap rates, so they keep these rates positive. It is noteworthy that LMM and SMM are not simultaneously compatible each other. Brigo and Liinev show in [15] that lognormal forward swap rates are distributionally incompatible with lognormal forward Libor rates, and viceversa. Each forward Libor or swap rate is martingale under its own specific probability measure. Nevertheless, when one derivative depends on several interest rates, the dynamics of the rates under a single probability measure are necessary. In this situation, by the change of numeraire technique [48] (to avoid arbitrage), the dynamics under the same probability measure can be obtained, appearing thereby drift terms depending on several rates. From the numerical point of view, it is customary to use Monte Carlo techniques to simulate these dynamics under the common measure [12, 25, 32]. Monte Carlo scheme has a high computational cost and it is complicated to control the error. However, the accuracy of the method does not depend on the dimension, so it is the mainly chosen technique when more than three factors are considered. In this way we avoid the difficulties associated to finite difference or finite element methods for the equivalent formulation in terms of partial derivative equations in case of a large number of spatial dimensions.

Then, in short, the popularity of LMM and SMM mainly comes from their ability to guarantee three desirable features:

1. they avoid arbitrage among bonds,

2. they keep rates positive,

3. they price caps or swaptions according to Black's formula, thus allowing calibration to market data.

However, these desired properties of the continuous formulation can be lost with the discretization, at the simulation level.

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It is complicated to apply an adjustment over the forward rates that guarantees the first property after discretization [3]. So, in order to preclude the lost of the first feature, Glasserman and Zhao propose in [26] to simulate directly the ratios between the bonds and the chosen numeraire (deflated bonds), whose dynamics are drift free. In this way, the lost of the martingale property can be avoided by applying an adjustment over the simulated deflated bonds. We refer to this methodology as the Drift-Free Simulation (DFS) technique. As they only consider one bond or the bank account as possible numeraires, at each time there exists some deflated bond whose dynamics is equal to zero, thus allowing to establish a bijective relation between the deflated bonds and the forward rates. Moreover, in [26] the simulation of some new variables (namely difference between these deflated bonds) is presented in order to guarantee the positivity of the forward rates after the discretization. As pointed out in [4], additional discretization bias has been introduced by using this technique. For this reason, in this thesis a new method based on a parameterization of the martingales introduced in [26] is proposed (see [20]). The main advantage of this method is that it guarantees no arbitrage among bonds and positive rates even when the bank account is used as numeraire. This is an important issue because using the terminal measure for simulation can lead to blowouts in the sample standard deviation when wide time intervals are considered (see [4, 13]).

LMM and SMM can be presented into a general framework like the ones proposed in [23, 52]. Namely, in [23] this general context is presented by studying which market models are admissible through the use of graph theory. In [52] generic expressions for the drift terms appearing in a generic market model are presented. Again, we are interested in the simulation of the stochastic rates avoiding the use of the drift terms. So, taking into account these generalizations, we extend the DFS technique to Generic Market Models.

Some recent versions of LMM pay special attention to capturing the basis between different compounding frequencies by using multiple estimation curves jointly with a reference discount one (see [43, 44], for example). As a first discretization step, these models require a discretization of the discount curve with the guaranteed properties of the proposed method. Moreover, this complete model needs a large amount of data, some of them not easily accessible from the market (EONIA curve volatilities, correlation between curves, etc.). In order to overcome this difficulty, in practice the one curve LMM with an appropriate choice of the input curve and volatility is still used. Anyway we present the DFS technique for this multicurve case.

We also can extend the DFS methodology to the Cross-Market Models. In the context of Cross-Markets we have two economies A and B that are measured in two different units. For example:

- In the cross-currency case, the units are the domestic currency and a foreign one, so we can consider the domestic and the foreign economies plus the forward exchange rates that relate them [2, 16, 45, 59]. There exist numerous financial derivatives that depend on domestic and foreign interest rates; they may be classified into: standard cross-currency products (Cross-currency swap, Cross-crurrency swaption, ...), that are agreements between two parties so that each part pays cash flows referred to one currency in the same currency, and Quanto products (Quanto fra, Quanto caplet/floorlet, Quanto swap, Quanto Swaption, ...), that consist of products that pay cash flows referred to one currency in another currency. One really need a model involving the two currencies to price quanto products.

- In the commodity case, the units are the domestic currency and one commodity, so we can consider the nominal and the commodity economies plus the forward prices of the commodity that relate them [30, 58]. There exist financial products that depend on rates of these economies, like spread options [53]. In this context it is also interesting to obtain the evolution of the convenience yield [24, 47, 53].

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- In the inflation case, the units are the domestic currency and one representative basket of basic products, so we can consider the nominal and the real economies plus the forward inflation rates that relate them [28, 40, 42]. The most popular inflation derivatives are the Inflation Indexed Swaps (IIS). These are agreements between two parties to exchange some flows, in which at least one of the exchanged flows depends on one inflation index. The main IIS traded in the market are the Zero Coupon Inflation Indexed Swaps (ZCIIS) and the Year-On-Year Inflation Indexed Swaps (YYIIS). From these IIS arise the Inflation Indexed Swaptions (IISO). In the market there are also other derivatives like the Caplets/Floorlets Inflation Indexed (C/FII).

The aim of Cross-Market Models is to obtain the evolution of the basic assets of both economies and of the rates that connect them. In this work we present two of the possible Cross-Market Models which are called for us First and Second Cross-Market Model. The DFS for the First Cross-Market Model [21] is completely analogous to the one economy case. On the other hand, the DFS for the Second Cross-Market Model [22] is quite different because it has the advantage that one does not need to parametrize the second economy. We apply the First Cross-Market Model to the cross-currency setting and the Second Cross-Market Model to the commodity case.

Finally, all models that we have presented can be framed into the context of a general connected directed graph, on which we can know the dynamics that govern its evolution. We also can propose a technique of simulation avoiding the drift terms of these dynamics.

The outline of this work is as follows.

Part I deals with DFS techniques for Market Models in the one economy context. Chapter 1 contains an introduction in this subject. In Chapter 2 we develop the LMM case. Moreover, the existing DFS techniques in this setting are described

and an efficient method to guarantee desirable properties after the discretization is proposed. In Chapter 3, DFS methodologies are applied for Coterminal SMM and for Generic Market Models. Finally, in Chapter 4 the technique is extended to the multicurve setting, which has recently appeared after the crisis in 2008.

Part II deals with DFS techniques for Market Models in the two economies context. Thus, in Chapter 5 the Cross-Market Models are introduced, jointly with the criteria to decide the possible models and the general issues of the DFS technique in this setting. Chapter 6 and Chapter 7 contain two particular Cross-Market Models with their corresponding techniques to avoid the presence of the drift terms. More precisely, in Chapter 6 the specific model for the cross-currency example is considered and Chapter 7 is devoted to the commodities case. At the end of the Chapter 7, the inflation derivatives case is addressed.

Part III contains Chapter 8 that deals with an application of graph theory to generic market models.

Appendix A deals with an auxiliary result in matrix algebra to prove certain formula in Generic Market Models of one economy. In Appendix B the number of possible Cross-Carket Models is obtained.

We finish with a short section outlining the main conclusions of this work.

Part I

Drift-Free Simulation Techniques for Market Models

Chapter 1

Market Models and Drift-Free Simulation techniques

An important development in the mathematical modeling for pricing of interest rate derivatives has been the emergence of the most popular family of interest-rate models: the Market Models, the most representative being the Libor Market Model (LMM) and Swap Market Model (SMM). LMM and SMM were introduced in [14, 33, 46], starting from the general Heath-Jarrow-Morton framework [27]. These models price caps and swaptions by the standard Black's formula employed in the caps and swaptions markets, the main markets in the interest-rate derivatives world. Moreover, they assume lognormal dynamics for forward Libor or forward swap rates, so that they keep these rates positive. It is noteworthy that LMM and SMM are not compatible with each other. In [15] Brigo and Liinev show that lognormal forward swap rates are distributionally incompatible with lognormal forward Libor rates, and viceversa. Each forward Libor or swap rate is a martingale under its probability measure. Also the dynamics of each rate under a chosen probability measure can be obtained by the change of numeraire technique [48] (to avoid arbitrage), thus giving rise to the presence of drift terms depending on several rates. From the numerical point of view, the use of Monte Carlo scheme to simulate these dynamics under that chosen measure is quite frequent [12, 25, 32].

Then, in short, the popularity of these models is due mainly to the fact that they keep three desirable features:

1. they avoid arbitrage among bonds,

2. they keep rates positive,

3. they price caps or swaptions according to Black's formula, thus allowing calibration to market data.

However, these properties of the continuous formulation may be lost while discretizing, at the simulation level.

It is complicated to apply an adjustment over the forward rates that guarantees the first property above after the discretization [3]. So, in order to preclude the lost of the first feature, Glasserman and Zhao propose in [26] the direct simulation of the ratios between the bonds and the chosen numeraire (deflated bonds), whose dynamics are free of drift. In this way, the lost of the martingale property can be avoided by applying an adjustment over the simulated deflated bonds. We refer to this methodology as the Drift-Free Simulation (DFS) technique. As they only consider one bond or the bank account as possible numeraires, at each time there exists some deflated bond whose dynamics is equal to zero, thus allowing to establish a bijective relation between the deflated bonds and the forward rates. Moreover, in [26] the simulation of some new variables (namely difference between these deflated bonds) is presented in order to guarantee the positivity of the forward rates after the discretization. As pointed out in [4], additional discretization bias has been introduced by using this technique. For this reason, in [20] a new method based on a parameterization of the martingales introduced in [26] is proposed. The main advantage of this method, that we have presented for LMM, is that it guarantees no arbitrage among bonds and positive rates even when the bank account is used as numeraire. This is an important issue because using the terminal measure for simulation can lead to blowouts in

the sample standard deviation when wide time intervals are considered (see [4, 13]). In Chapter 2 we discuss the existing DFS methodologies and our proposal to simulate the dynamics of LMM, and we examine the impact of these approaches on the errors at pricing caplets. In Chapter 3 we apply these techniques to coterminal SMM.

LMM and SMM can be presented into a general framework like those proposed in [23, 52]. Namely, in [23] this general context is presented by studying which market models are admissible through the use of graph theory. In [52], generic expressions for the drift terms appearing in a generic market model are presented. As before, the cumbersome computation of these drift terms does not allow for an adjustment to avoid arbitrage at the simulation level. Then, again we are interested in the simulation of the stochastic rates avoiding the use of the drift terms involved in their dynamics. So, taking into account these generalizations, also in Chapter 3 we extend the DFS technique to Generic Market Models.

Finally, some recent versions of LMM pay special attention to capturing the basis between different compounding frequencies by using multiple estimation curves jointly with a reference discount one (see [43, 44], for example). As a first discretization step, these models require a discretization of the discount curve with the guaranteed properties of the here proposed method. Moreover, this complete model needs a large amount of data, some of them not easily obtainable from the market (the EONIA curve volatilities, the correlation between curves, etc). We present the DFS technique for the multicurve case in Chapter 4.

Chapter 2

Drift-Free Simulation techniques for Libor Market Model

2.1 Introduction

In this chapter we propose an efficient Drift-Free Simulation (DFS) method to simulate the dynamics of forward Libor rates in the Libor Market Model (LMM) for a single interest rate curve when using as numeraire a bond maturing at any Libor tenor date (forward measure) or the bank account (spot measure). This method, presented in [20], starts from some ideas developed in [26] that replace the simulation of the drifts associated to forward Libor rates by the simulation of appropriate martingales. In this setting, we first describe the methodology introduced in [26] and the analysis carried out in [4]. Next, we present our new method based on a parameterization of the martingales introduced in [26] to ensure the desirable properties (positive martingales and positive forward Libor rates) at the simulation level. Although the positivity of these terms is guaranteed by construction, the martingale property can be lost after the discretization. For this reason we also present a martingale adjustment to ensure the martingale condition. The main advantage of the proposed method is that these conditions are guaranteed even when we use the spot measure.

The numerical results illustrate the comparatively better performance of the proposed method when pricing caplets. Also the adjustment for ensuring the martingale property at discrete level provides slightly better results with the here proposed method.

This chapter is organized as follows. Section 2.2 presents the setting and notation of LMM and describes the DFS methods. We present the new parameterized DFS algorithm and we prove that the desirable properties are ensured with it. We also propose an adjustment to ensure the martingale property after the discretization. Section 2.3 deals with the model calibration procedure. In Section 2.4 we compare our numerical results for pricing caplets with those obtained with alternative DFS techniques.

2.2   Libor Market Model and Drif-Free Simulations

In this section we introduce the standard notation for Libor Market Model (LMM) (see [13, 14, 16, 33, 46, 48, 51], for example). Some existing Drif-Free Simulation (DFS) methodologies are discussed. We also present the new parameterized DFS algorithm with its corresponding adjustment of martingales [20].

We consider a fixed tenor structure $\mathcal{T}:=\{T_0, T_1, ..., T_{N+1}\}$, with $T_0 = 0$, $T_j < T_k, 0 \leq j < k \leq N+1$, and the corresponding accruals $\delta_j = T_{j+1} - T_j$, $0 \leq j \leq N$. We denote by $B_j(t)$, for $j = 0, ..., N+1$, the price at time t of a zero-coupon bond that matures at the tenor date $T_j \geq t$ and by $F_j(t)$, for $j = 0, ..., N$, the value at time $t \leq T_j$ of the forward Libor rate for the accrual period $[T_j, T_{j+1}]$. Bond prices and forward Libor rates are related by the expression

$$F_j(t) = \frac{B_j(t) - B_{j+1}(t)}{\delta_j B_{j+1}(t)}. \tag{2.1}$$

The bank account β is the asset whose value at $t \in (T_{j-1}, T_j]$ is given by

$$\beta(t) = B_j(t) \prod_{l=0}^{j-1} \left(1 + \delta_l F_l(T_l)\right). \tag{2.2}$$

Notice that $\beta(T_j) = \prod_{l=0}^{j-1} \left(1 + \delta_l F_l(T_l)\right)$.

We shall denote by $W(t)$ a (correlated) N-dimensional Wiener process in a certain probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a correlation matrix $\Sigma = (\Sigma_{i,j})$. The natural filtration generated by $W(t)$ is denoted by $\mathcal{F}_t$. Every stochastic process we consider is defined on this probability space.

In the sequel we shall only consider two kinds of numeraires Num: one of the prices of bonds, B_n, with $n = 1, ..., N + 1$ (forward probability measure, Q^n), or the bank account, β (spot probability measure, Q^β).

Once the numeraire Num has been chosen, the notation for the deflated prices of bonds is:

$$\widehat{B}_j(t) \doteq \frac{B_j(t)}{\text{Num}(t)}, \qquad t \leq T_j, \qquad j = 1, \ldots, N + 1. \tag{2.3}$$

Remark 2.2.1. *It is a simple fact, although quite relevant in the sequel, that for any of those possible numeraires and for each interval $(T_{j-1}, T_j]$, $j = 1, ..., N + 1$, there exists an index k such that $d\widehat{B}_k(t) = 0$ along that interval. If the numeraire is one of the prices of bonds B_n, $n \leq N + 1$, then that index k is always n, while for the bank account as numeraire the index k is j along the interval $(T_{j-1}, T_j]$.*

The arbitrage free of the forward Libor rates (so that $\widehat{B}_j$ are martingales, for all j) under each of the numeraires above can be written in a unified way as

$$dF_j(t) = \mu_j(t) F_j(t) dt + \sigma_j(t) F_j(t) dW_j(t), \quad j = 1, ..., N, \tag{2.4}$$

where $\sigma_j(t)$ denotes the volatility term ($\sigma_j(t)$ is a piecewise continuous function) and where $\mu_j(t)$ denotes the drift term, whose value depends on the chosen probability

measure and it is given by

$$\mu_j(t) = -\left(\varepsilon_{k,j+1} \sum_{l \in \theta_{k,j+1}} \frac{\delta_l \sigma_l(t) F_l(t) \Sigma_{lj}}{1 + \delta_l F_l(t)}\right) \sigma_j(t), \tag{2.5}$$

where the value $\varepsilon_{k,j+1}$ and the collection of indexes $\theta_{k,j+1}$ are defined by:

$$\varepsilon_{k,m} = \begin{cases} -1, & \text{if } m > k; \\ 0, & \text{if } m = k; \\ +1, & \text{if } m < k; \end{cases} \quad \theta_{k,m} = \begin{cases} \{k, k+1, ..., m-1\}, & \text{if } m > k; \\ \emptyset, & \text{if } m = k; \\ \{m, m+1, ..., k-1\}, & \text{if } m < k. \end{cases} \tag{2.6}$$

The issue explained in Remark 2.2.1 allows us to know the value of one deflated bond $\widehat{B}_k(t)$ at each time, a fact that we record as the following lemma for future reference.

Lemma 2.2.2. *A biyective relation between the forward Libor rates $\{F_j(t)\}_{j=1}^N$ and the deflated bonds $\{\widehat{B}_j(t)\}_{j=1}^{N+1}$ is established by the following recurrence relation equivalent to (2.1):*

$$\widehat{B}_j(t) = \left(1 + \delta_j F_j(t)\right)\widehat{B}_{j+1}(t), \quad j = 1, ..., N. \tag{2.7}$$

Proof. This lemma is proved directly from the fact of that for each interval $(T_{j-1}, T_j]$, $j = 1, ..., N+1$, there exists an index k such that $d\widehat{B}_k(t) = 0$ along that interval. $\square$

From Lemma 2.2.2, the output of the model can be $\{F_j(t)\}_{j=1}^N$ or $\{\widehat{B}_j(t)\}_{j=1}^{N+1}$ equivalently.

Otherwise, a simulation scheme should maintain the martingale property for the simulated $\widehat{B}_j$, which is more easily achieved by directly simulating the terms $\widehat{B}_j$ and avoiding the cumbersome calculation of the drifts of the forward Libor rates. We refer to this approach as the DFS technique. Besides the simulation scheme must generate positive forward Libor rates F_j and positive and decreasing in j martingales $\widehat{B}_j$.

Next we discuss and compare the existing DFS methodologies for LMM.

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2.2.1 Existing DFS methods for LMM

In this subsection we explain the existing DFS methods.

Implicit Drif-Free Simulation (IDFS)

Taking into account that $\widehat{B}_j(t)$ and $\widehat{B}_{j+1}(t)$ are martingales under the probability measure associated to Num and the recurrence relation (2.7), we obtain the following (equivalent) expressions relating the dynamics of consecutive $\widehat{B}_j$:

$$d\widehat{B}_j(t) \;=\; \delta_j\sigma_j(t)F_j(t)\widehat{B}_{j+1}(t)\,dW_j(t) \;+\; \left(1+\delta_jF_j(t)\right)d\widehat{B}_{j+1}(t), \qquad (2.8)$$

$$d\widehat{B}_{j+1}(t) \;=\; \frac{-\delta_j\sigma_j(t)F_j(t)}{\left(1+\delta_jF_j(t)\right)^2}\widehat{B}_j(t)\,dW_j(t) \;+\; \frac{1}{1+\delta_jF_j(t)}\,d\widehat{B}_j(t). \qquad (2.9)$$

These expressions are relevant for our purposes, because as pointed out in Remark 2.2.1, in each time interval $(t, t+dt]$, $d\widehat{B}_k(t) = 0$ for some k, and then we may compute recursively the dynamics $d\widehat{B}_j(t)$, for $j = k-1, k-2, ...$, using (2.8), and for $j = k+1, k+2, ...$, by (2.9), just in terms of the previously simulated Brownian jumps and the previous levels of forward Libor rates and martingales.

The simulation algorithm goes as follows. For simplicity, we assume the same constant accrual $\delta_j = \Delta T$ for $j = 1, \ldots, N$. For the IDFS procedure we consider a constant simulation time step δt, which is a fraction of the constant accrual[1]. We consider $t_i = i\delta t$ as a generic discretization time.

The steps of the IDFS algorithm are the following:

1. Initialize the values $\{B_j(0)\}_{j=0}^{N+1}$, $\{F_j(0)\}_{j=0}^{N}$ and $\{\widehat{B}_j(0)\}_{j=0}^{N+1}$ from the market data.

2. Simulation of the correlated Brownian motions[2] $\{W_j(t_i)\}_{j=1}^{N}$ with $t_i \leq T_j$.

---

[1]In this way we guarantee that tenor dates are simulation dates, thus avoiding the use of additional complex interpolation procedures.

[2]Box-Muller and Polar-Marsaglia methods for simulation of the correlated Brownian motions can be used (see [19] and [55], for example). We use Polar-Marsaglia method.

3. For each $i = 0, 1, 2, ..., (N+1)(\Delta T / \delta t) - 1$, the computation of the martingale structure and Libor rates at time t_{i+1} follows two steps:

3.1. First, we approximate the whole martingale structure at time t_{i+1}, starting from $d\widehat{B}_k(t_i) = 0$, and $\widehat{B}_k(t_{i+1}) = \widehat{B}_k(t_i) + d\widehat{B}_k(t_i)$. Thus, for $j = k-1, k-2, \ldots$, with $T_j \geq t_{i+1}$, by using the Euler discretization we pose

$$\widehat{B}_j(t_{i+1}) = \widehat{B}_j(t_i) + d\widehat{B}_j(t_i), \tag{2.10}$$

where $d\widehat{B}_j(t_i)$ is given by (2.8). Next, for $j = k, k+1, \ldots, N-1, N$ we pose

$$\widehat{B}_{j+1}(t_{i+1}) = \widehat{B}_{j+1}(t_i) + d\widehat{B}_{j+1}(t_i), \tag{2.11}$$

where $d\widehat{B}_{j+1}(t_i)$ is obtained with (2.9).

3.2. Secondly, we calculate the forward Libor rates at t_{i+1} by using the following equivalent expression to (2.1):

$$F_j(t_{i+1}) = \frac{\widehat{B}_j(t_{i+1}) - \widehat{B}_{j+1}(t_{i+1})}{\delta_j \widehat{B}_{j+1}(t_{i+1})}, \quad \text{with } j \text{ such that } T_j \geq t_{i+1}.$$

Next, we go back to Step 3.1. for the following value of i.

Once we have run the algorithm, we can recover the numeraire and discount bonds values at tenor dates by means of

$$\text{Num}(T_j) = \frac{1}{\widehat{B}_j(T_j)}, \quad j = 1, ..., N+1, \tag{2.12}$$

and

$$B_j(T_i) = \widehat{B}_j(T_i)\text{Num}(T_i), \quad i \leq j. \tag{2.13}$$

If the constraint of recovering only for tenor dates is a problem, we can interpolate the obtained values $\{\text{Num}(T_j)\}_{j=0}^{N+1}$ in order to obtain the whole curve $\{\text{Num}(t_i)\}_{i=0}^{(N+1)(\Delta T / \delta t)}$. In this way we can compute the discount bound prices at all discretized times.

The previously described IDFS procedure combined with the Euler discretization results into a recursive method: to pass from time t_i to t_{i+1} we choose an appropriate order in j (first decreasing from $k-1$ to 0 and next increasing from $k+1$ to $N+1$) to calculate the martingale $\widehat{B}_j$ at time t_{i+1}. Notice also that Euler discretization does not guarantee the required positivity of the martingales $\widehat{B}_j$. In order to avoid these two disadvantages (recursiveness and non positiveness) we explain another simulation procedure proposed in [26].

Explicit Drift-Free Simulation (EDFS)

Now we describe an alternative log-Euler discretization at Step 3 of the IDFS algorithm leading to a non recursive simulation procedure presented in [26] which we call Explicit DFS algorithm.

First, by using (2.8), we have

$$\frac{d\widehat{B}_j(t)}{\widehat{B}_j(t)} = \sigma_j(t)\left(1 - \frac{\widehat{B}_{j+1}(t)}{\widehat{B}_j(t)}\right)dW_j(t) + \frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)}. \tag{2.14}$$

Again, as indicated in Remark 2.2.1, since $d\widehat{B}_k(t) = 0$ for some k, we have that

$$\frac{d\widehat{B}_j(t)}{\widehat{B}_j(t)} = \varepsilon_{k,j} \sum_{l \in \theta_{k,j}} \left(\frac{d\widehat{B}_l(t)}{\widehat{B}_l(t)} - \frac{d\widehat{B}_{l+1}(t)}{\widehat{B}_{l+1}(t)}\right), \tag{2.15}$$

where $\varepsilon_{k,j}$ and $\theta_{k,j}$ are defined in (2.6).

Next, by combining (2.14) and (2.15), we get

$$\frac{d\widehat{B}_j(t)}{\widehat{B}_j(t)} = \varepsilon_{k,j} \sum_{l \in \theta_{k,j}} \sigma_l(t)\left(1 - \frac{\widehat{B}_{l+1}(t)}{\widehat{B}_l(t)}\right)dW_l(t), \tag{2.16}$$

or equivalently,

$$\frac{d\widehat{B}_j(t)}{\widehat{B}_j(t)} = \sigma_j^{\widehat{B}}(t)dW_j^{\widehat{B}}(t), \tag{2.17}$$

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where

$$\sigma_j^{\widehat{B}}(t)^2 = \sum_{l_1,l_2 \in \theta_{k,j}} \left(1 - \frac{\widehat{B}_{l_1+1}(t)}{\widehat{B}_{l_1}(t)}\right)\left(1 - \frac{\widehat{B}_{l_2+1}(t)}{\widehat{B}_{l_2}(t)}\right)\sigma_{l_1}(t)\sigma_{l_2}(t)\Sigma_{l_1,l_2}, \qquad (2.18)$$

and $W_j^{\widehat{B}}(t)$ is an appropriate new Brownian motion.

Therefore, given (2.16) and (2.18), and by applying the log-Euler discretization, we have the following approximation for the martingale structure:

$$\widehat{B}_j(t_{i+1}) = \widehat{B}_j(t_i)\exp\left(\frac{d\widehat{B}_j(t_i)}{\widehat{B}_j(t_i)} - \frac{1}{2}\sigma_j^{\widehat{B}}(t_i)^2\delta t\right), \qquad (2.19)$$

as an alternative to Step 3.1 in the IDFS algorithm.

The rest of the steps of the algorithm are analogous to the previously described IDFS procedure.

In this EDFS method, once we know all the martingales at time t_i we can compute independently the martingales at time t_{i+1}, that is why it is termed *explicit*.

In the EDFS method, the positivity of the deflated bonds $\widehat{B}_j$ is guaranteed by construction, although this is not the case for the forward Libor rates, thus raising the possibility of discount bonds values greater than one. In order to overcome this disadvantage we present another simulation procedure proposed by Glasserman and Zhao in [26].

Glasserman-Zhao Drift-Free Simulation (GZDFS)

As the simulation scheme must generate positive forward Libor rates F_j and positive and decreasing in j martingales $\widehat{B}_j$, in [26] Glasserman and Zhao propose the following DFS approach. They propose to simulate the differences between consecutive

martingales:

$$D_j(t) \doteq \begin{cases} \widehat{B}_j(t) - \widehat{B}_{j+1}(t), & \text{if } j < N+1; \\ \widehat{B}_{N+1}(t), & \text{if } j = N+1; \end{cases} \tag{2.20}$$

whose dynamics are given by

$$\frac{dD_j(t)}{D_j(t)} = \begin{cases} \dfrac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)} + \sigma_j(t)dW_j(t), & \text{if } j < N+1; \\[4mm] \dfrac{d\widehat{B}_{N+1}(t)}{\widehat{B}_{N+1}(t)}, & \text{if } j = N+1; \end{cases} \tag{2.21}$$

or equivalently,

$$\frac{dD_j(t)}{D_j(t)} = \sigma_j^D(t)dW_j^D(t), \tag{2.22}$$

where

$$\sigma_j^D(t)^2 = \begin{cases} \sigma_{j+1}^{\widehat{B}}(t)^2 + \sigma_j(t)^2 + \\[2mm] \quad +2\sigma_j(t)\varepsilon_{k,j+1}\displaystyle\sum_{l\in\theta_{k,j+1}}\left(1 - \dfrac{\widehat{B}_{l+1}(t)}{\widehat{B}_l(t)}\right)\sigma_l(t)\Sigma_{l,j}, & \text{if } j < N+1; \\[6mm] \sigma_{N+1}^{\widehat{B}}(t)^2, & \text{if } j = N+1; \end{cases} \tag{2.23}$$

and $W_j^D(t)$ is an appropriate new Brownian motion.

In this way, with the log-Euler discretization, positive martingales $D_j(t_{i+1})$ are obtained:

$$D_j(t_{i+1}) = D_j(t_i)\exp\left(\frac{dD_j(t_i)}{D_j(t_i)} - \frac{1}{2}\sigma_j^D(t_i)^2\delta t\right). \tag{2.24}$$

With them, the martingales $\widehat{B}_j$ and the forward Libor rates F_j can be obtained from the known martingales $\widehat{B}_k(t_{i+1}) = \widehat{B}_k(t_i)$:

$$\begin{aligned} \widehat{B}_j'(t_{i+1}) &= \varepsilon_{k,j}\sum_{l\in\theta_{k,j}}D_l(t_{i+1}) + \widehat{B}_k(t_{i+1}), \\[2mm] F_j'(t_{i+1}) &= \frac{D_j(t_{i+1})}{\delta_j\left(\varepsilon_{k,j+1}\displaystyle\sum_{l\in\theta_{k,j+1}}D_l(t_{i+1}) + \widehat{B}_k(t_{i+1})\right)}, \end{aligned} \tag{2.25}$$

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or they can be computed from the simulated value $D_{N+1}(t_{i+1})$:

$$\widehat{B}_j''(t_{i+1}) = \sum_{l=j}^{N+1} D_l(t_{i+1}),$$

$$F_j''(t_{i+1}) = \frac{D_j(t_{i+1})}{\delta_j \sum_{l=j+1}^{N+1} D_l(t_{i+1})}. \tag{2.26}$$

So, by using (2.26), the positivity and the decrease in j of $\widehat{B}_j''(t_{i+1})$ and the positivity of $F_j''(t_{i+1})$ are ensured. However, as reported in [26], (2.25) gives better numerical results, although negative martingales and negative forward Libor rates can appear if the measure is not the terminal forward measure (i.e, $k \neq N+1$). Notice that $\varepsilon_{k,j} = -1$ if $j > k$. Moreover, we note that the spot measure is particularly convenient when one is confronted with systematic valuation of a portfolio of instruments of diverse maturities, even further if it contains long dated instruments, because the variance of a Monte Carlo pricing simulation tends to be higher under other measures (see, for instance, [4, 13]).

As pointed out in [4], additional discretization bias has been introduced into the method in (2.26). This is because in (2.25) the known martingale $\widehat{B}_k(t_{i+1}) = \widehat{B}_k(t_i)$ is taken into account, while in (2.26) the simulated martingale $D_{N+1}(t_{i+1})$ is used, so that a system of $N+1$ dynamics is considered for a model with N variables. Therefore it is not ensured that the value of $\widehat{B}_k''(t_{i+1})$ matches the value obtained with $d\widehat{B}_k(t) = 0$. In [4] a re-normalization of the martingales D_j is proposed to ensure the desirable properties and that

$$\sum_{l=k}^{N+1} D_l(t_{i+1}) = \widehat{B}_k(t_{i+1}). \tag{2.27}$$

However, the reason for using the DFS methodology, that is to evolve forward Libor rates by ensuring the martingale property of the discount ratios, has been lost with this technique.

In order to overcome this drawback, in the next subsection we will propose a new parameterized DFS method. Moreover in order to preserve the martingale property of each continuous model in the discrete implementation, we also propose an appropriate martingale adjustment at simulation level.

2.2.2 An alternative Parameterized DFS (PDFS)

In this subsection we propose an alternative parameterized algorithm that guarantees the positivity of the deflated bonds and of the forward Libor rates under any forward or spot probability measure [20].

This procedure is based on the system of stochastic differential equations (2.21) proposed in [26]. It is necessary to impose the following additional constraints:

1. $D_j > 0$, for $j = 1, ..., N+1$. This condition guarantees positive martingales and forward Libor rates.

2. The equality in (2.27) that is necessary to ensure the compatibility of the system.

In order to ensure the previous conditions we propose the simulation of the new terms:

$$u_j(t) \doteq \begin{cases} \ln\left(D_j(t)\right), & \text{if } j = 1, ..., k-1; \\ 0, & \text{if } j = k; \\ \ln\left(D_j(t)\right) - \ln\left(D_k(t)\right), & \text{if } j = k+1, ..., N+1. \end{cases} \tag{2.28}$$

The dynamics of the variables in (2.28) and some useful formulas are summarized in the following lemma.

Lemma 2.2.3. *Given $f_{p,q}$ and $g_{p,q}$ defined by*

$$f_{p,q}(t) \doteq 1 - \frac{\sum\limits_{m=p+1}^{q} \exp\left(u_m(t)\right) + \widehat{B}_k(t)}{\sum\limits_{n=p}^{q} \exp\left(u_n(t)\right) + \widehat{B}_k(t)}, \quad g_{p,q}(t) \doteq 1 - \frac{\sum\limits_{m=p+1}^{q} \exp\left(u_m(t)\right)}{\sum\limits_{n=p}^{q} \exp\left(u_n(t)\right)}, \quad (2.29)$$

each u_j, $j = 1, ..., N+1$, satisfies

$$du_j(t) = \mu^{u_j}(t)dt + \sum_{l=1}^{N} \sigma_l^{u_j}(t)dW_l(t) \qquad (2.30)$$

under the forward or spot probability measure, where

$$\mu^{u_j}(t) = \begin{cases} -\frac{1}{2} \sum\limits_{l_1,l_2=j+1}^{k-1} \sigma_{l_1}(t)\sigma_{l_2}(t)f_{l_1,k-1}(t)f_{l_2,k-1}(t)\Sigma_{l_1,l_2} - \frac{1}{2}\sigma_j(t)^2 - \\ \qquad -\sigma_j(t)\sum\limits_{l=j+1}^{k-1}\sigma_l(t)f_{l,k-1}(t)\Sigma_{l,j}, & \text{if } 1 \leqslant j \leqslant k-1; \\[2em] 0, & \text{if } j = k; \\[2em] -\frac{1}{2}\sum\limits_{l_1,l_2=k}^{j}\sigma_{l_1}(t)\sigma_{l_2}(t)g_{l_1,N+1}(t)g_{l_2,N+1}(t)\Sigma_{l_1,l_2} - \frac{1}{2}\sigma_j(t)^2 + \\ \qquad +\sigma_j(t)\sum\limits_{l=k}^{j}\sigma_l(t)g_{l,N+1}(t)\Sigma_{l,j} + \frac{1}{2}\sigma_k(t)^2(1 - g_{k,N+1}(t))^2, & \text{if } k+1 \leqslant j \leqslant N; \\[2em] -\frac{1}{2}\sum\limits_{l_1,l_2=k}^{N}\sigma_{l_1}(t)\sigma_{l_2}(t)g_{l_1,N+1}(t)g_{l_2,N+1}(t)\Sigma_{l_1,l_2} + \\ \qquad +\frac{1}{2}\sigma_k(t)^2(1 - g_{k,N+1}(t))^2, & \text{if } j = N+1; \\ & \quad (k < N+1), \end{cases}$$

$$(2.31)$$

and

- *if $1 \leqslant j \leqslant k - 1$:*

$$\sigma_l^{u_j}(t) = \begin{cases} 0, & \text{if } l \neq j, ..., k-1; \\ \sigma_j(t), & \text{if } l = j; \\ \sigma_l(t) f_{l,k-1}(t), & \text{if } l = j+1, ..., k-1; \end{cases} \tag{2.32}$$

- *if $j = k$:*

$$\sigma_l^{u_k}(t) = 0, \quad \forall l, \tag{2.33}$$

- *if $k + 1 \leqslant j \leqslant N$:*

$$\sigma_l^{u_j}(t) = \begin{cases} 0, & \text{if } l \neq k, ..., j; \\ -\sigma_k(t), & \text{if } l = k; \\ -\sigma_l(t) g_{l,N+1}(t), & \text{if } l = k+1, ..., j-1; \\ \sigma_j(t)(1 - g_{j,N+1}(t)), & \text{if } l = j; \end{cases} \tag{2.34}$$

- *if $j = N + 1$ $(k < N + 1)$:*

$$\sigma_l^{u_{N+1}}(t) = \begin{cases} 0, & \text{if } l \neq k, ..., N; \\ -\sigma_k(t), & \text{if } l = k; \\ -\sigma_l(t) g_{l,N+1}(t), & \text{if } l = k+1, ..., N. \end{cases} \tag{2.35}$$

Also, we have

$$D_j(t) \;=\; \exp\big(u_j(t)\big), \quad j = 1, ..., k-1,$$

$$D_j(t) \;=\; \frac{\widehat{B}_k(t) \exp\big(u_j(t)\big)}{\displaystyle\sum_{l=k}^{N+1} \exp\big(u_l(t)\big)}, \quad j = k, ..., N+1, \tag{2.36}$$

and

$$\widehat{B}_j(t) = \sum_{l=j}^{N+1} D_l(t), \tag{2.37}$$

$$F_j(t) = \frac{D_j(t)}{\delta_j \displaystyle\sum_{l=j+1}^{N+1} D_l(t)}. \tag{2.38}$$

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With this parameterization the drift and the diffusion terms in (2.30) are globally Lipschitz functions of u_j without requiring any auxiliary function; compared with [26]. The hypotheses of the existence and uniqueness theorem for stochastic differential equations (see [49], for example) can be proved straight forwardly.

Discretization scheme

From Lemma 2.2.3, we propose the Euler discretization:

$$u_j(t_{i+1}) = u_j(t_i) + \mu^{u_j}(t_i)\delta t + \sum_{l=1}^{N} \sigma_l^{u_j}(t_i)(W_l(t_{i+1}) - W_l(t_i)). \qquad (2.39)$$

Then, taking into account that $\widehat{B}_k(t_{i+1}) = \widehat{B}_k(T_{j-1})$, for each $t_{i+1} \in (T_{j-1}, T_j]$, we can approximate the martingales D_j and $\widehat{B}_j$ and the forward Libor rates in the discretized times by using (2.36), (2.37) and (2.38), respectively, that is

$$D_j(t_{i+1}) = \exp\left(u_j(t_{i+1})\right), \quad j = 1, ..., k-1,$$

$$D_j(t_{i+1}) = \frac{\widehat{B}_k(t_{i+1}) \exp\left(u_j(t_{i+1})\right)}{\sum_{l=k}^{N+1} \exp\left(u_l(t_{i+1})\right)}, \quad j = k, ..., N+1, \qquad (2.40)$$

$$(2.41)$$

and

$$\widehat{B}_j(t_{i+1}) = \sum_{l=j}^{N+1} D_l(t_{i+1}), \qquad (2.42)$$

$$F_j(t_{i+1}) = \frac{D_j(t_{i+1})}{\delta_j \sum_{l=j+1}^{N+1} D_l(t_{i+1})}. \qquad (2.43)$$

Theorem 2.2.4. *i)* $\widehat{B}_j$ *are decreasing in* j *and positive terms* $(0 < \widehat{B}_{j+1} < \widehat{B}_j)$. *ii)* F_j *are positive terms* $(F_j > 0)$.

Proof. We know that these conditions are satisfied at time t_0. Since $\widehat{B}_k(t_{i+1}) = \widehat{B}_k(0) > 0$, $\forall t_{i+1} \in (T_0, T_1]$, by (2.40), $D_j(t_{i+1}) > 0$, $\forall t_{i+1} \in (T_0, T_1]$, and by (2.42), $\widehat{B}_j(t_{i+1}) > 0$, $\forall t_{i+1} \in (T_0, T_1]$. Recursively, for $l = 1, ..., N + 1$, since $\widehat{B}_k(t_{i+1}) = \widehat{B}_k(T_{l-1}) > 0$, $\forall t_{i+1} \in (T_{l-1}, T_l]$, by (2.40), $D_j(t_{i+1}) > 0$, $\forall t_{i+1} \in (T_{l-1}, T_l]$, and by (2.42), $\widehat{B}_j(t_{i+1}) > 0$, $\forall t_{i+1} \in (T_{l-1}, T_l]$. Therefore, $D_j(t_{i+1}) > 0$, $\forall t_{i+1} \leqslant T_j$, with $j = 1, ..., N + 1$.

Then, we have:

$$0 < \sum_{l=j+1}^{N+1} D_l(t_{i+1}) < \sum_{l=j}^{N+1} D_l(t_{i+1}),$$

i.e, given (2.42) we prove i).

Finally, by using (2.43), the values $F_j(t_{i+1})$ are quotients of positive terms, so we have ii). $\qquad\square$

Martingale adjustment

The martingale property for $\widehat{B}_j$ or D_j can be lost with the discretization. Since we have simulated directly the martingales, an adjustment for these terms can be applied. The adjustment considered in [26] consists of multiplying each simulated martingale by the ratio of its value at time zero and its simulated mean. In our method we need an adjustment procedure for u_j to ensure that the terms D_j are martingales. Since the dynamics of u_j are coupled for $j = k + 1, ..., N + 1$, in this case the adjustment is not standard. The adjustment for these terms will be an approximation.

- Adjustment for u_j with $j < k$

  By a simple change of variables, we can conclude that the adjustment necessary for the terms u_j, with $j < k$, consists of replacing each simulated value $u_j(t_{i+1})[p]$ by

  $$u_j(t_{i+1})[p] + \alpha_j(t_{i+1}), \tag{2.44}$$

29

where

$$\alpha_j(t_{i+1}) = \ln \left(\frac{D_j(0)}{\frac{1}{N_S} \sum_{p=1}^{N_S} \exp \left(u_j(t_{i+1})[p] \right)} \right). \qquad (2.45)$$

N_S denotes the number of simulations and the index p allows us to distinguish the values obtained with the different simulations.

- Adjustment for u_j with $j > k$

Since the dynamics of u_j are coupled for $j > k$, we have to compute all adjustments $\alpha_{k+1}(t_{i+1}), ..., \alpha_{N+1}(t_{i+1})$ at the same time. So, for each t_{i+1}, we look for the vector $\overrightarrow{\alpha}(t_{i+1}) = \left(\alpha_{k+1}(t_{i+1}), ..., \alpha_{N+1}(t_{i+1}) \right)$ such that

$$G_{i+1}\left(\overrightarrow{\alpha}(t_{i+1}) \right) = \left(D_{k+1}(0), ..., D_{N+1}(0) \right)^t$$

where $G_{i+1} : \mathbb{R}^{N+1-k} \longrightarrow \mathbb{R}^{N+1-k}$ is defined by

$$G_{i+1}\left(\overrightarrow{\alpha}(t_{i+1}) \right) = \left(\begin{array}{c} \frac{1}{N_S} \sum_{p=1}^{N_S} \frac{\widehat{B}_k(t_{i+1})[p] \exp \left(u_{k+1}(t_{i+1})[p] + \alpha_{k+1}(t_{i+1}) \right)}{\sum_{l=k}^{N+1} \exp \left(u_l(t_{i+1})[p] + \alpha_l(t_{i+1}) \right)} \\ \vdots \\ \frac{1}{N_S} \sum_{p=1}^{N_S} \frac{\widehat{B}_k(t_{i+1})[p] \exp \left(u_{N+1}(t_{i+1})[p] + \alpha_{N+1}(t_{i+1}) \right)}{\sum_{l=k}^{N+1} \exp \left(u_l(t_{i+1})[p] + \alpha_l(t_{i+1}) \right)} \end{array} \right). \qquad (2.46)$$

Since we know that $\overrightarrow{\alpha}(t_{i+1})$ will be close to the zero vector of $\mathbb{R}^{N+1-k}$, $\overrightarrow{\Theta} = (0, 0, ..., 0)$, by using just one iteration of Newton method and starting from $\overrightarrow{\Theta} = (0, 0, ..., 0)$, we can calculate the approximated adjustment by

$$\overrightarrow{\alpha}(t_{i+1}) \approx JG_{i+1}(\overrightarrow{\Theta})^{-1} \left(\left(D_{k+1}(0), ..., D_{N+1}(0) \right)^t - G_{i+1}(\overrightarrow{\Theta}) \right), \qquad (2.47)$$

where $JG_{i+1}(\overrightarrow{\Theta})$ is the Jacobian matrix of G_{i+1} in $\overrightarrow{\Theta}$, whose coefficients are given by

$$(JG_{i+1}(\overrightarrow{\Theta}))_{j_1,j_2} = \frac{1}{N_S} \sum_{p=1}^{N_S} \frac{\widehat{B}_k(t_{i+1})[p] \, \exp\left(u_{j_1}(t_{i+1})[p]\right) \sum_{l=k,l\neq j_1}^{N+1} \exp\left(u_l(t_{i+1})[p]\right)}{\left(\sum_{l=k}^{N+1} \exp\left(u_l(t_{i+1})[p]\right)\right)^2},$$

(2.48)

if $j_1 = j_2$, and

$$(JG_{i+1}(\overrightarrow{\Theta}))_{j_1,j_2} = -\frac{1}{N_S} \sum_{p=1}^{N_S} \frac{\widehat{B}_k(t_{i+1})[p] \, \exp\left(u_{j_1}(t_{i+1})[p]\right) \, \exp\left(u_{j_2}(t_{i+1})[p]\right)}{\left(\sum_{l=k}^{N+1} \exp\left(u_l(t_{i+1})[p]\right)\right)^2},$$

(2.49)

if $j_1 \neq j_2$.

See [3] for a similar argument.

2.3   Model Calibration

We are in the context of market models so that we have to introduce into our model as parameters as much market quotes as feasible. So we introduce the Zero Coupon Curve at time zero, $\{B_j(0)\}_{j=0}^{N+1}$, from market data directly. Other free parameters (volatilities and correlations) are considered in next paragraphs so that the model parameters are adjusted to the market information.

- *Adjustment of volatilities:* The structure of volatilities is adjusted to market by imposing that

$$\int_0^{T_j} \sigma_j(t)^2 dt = T_j(\sigma_j^{mar})^2,$$

(2.50)

where σ_j^{mar} is the quoted volatility of the caplet that matures at time T_j.

In practice, for simplicity, we choose constant volatilities, that is, $\sigma_j(t) = \sigma_j^{mar}$, for all t.

- *Adjustment of correlations:* We have to adjust the following correlation matrix:

$$
C = \begin{pmatrix} \Sigma_{11} & \dots & \Sigma_{1N} \\ \vdots & & \vdots \\ \Sigma_{N1} & \dots & \Sigma_{NN} \end{pmatrix}
$$

where Σ_{ij} is the correlation between F_i and F_j.

We adjust these correlations by using swaps volatilities from the market, $\sigma_{j,j+s}^S$, and taking into account the following approximation proposed by Rebonato in [57] that relates the volatilities of forward and swap rates with the correlations between forward rates:

$$
\int_t^{T_j} \sigma_{j,j+s}^S(z)^2 dz \approx
$$

$$
\approx \sum_{r,p=j+1}^{j+s} \frac{\omega_{j+1,j+s}^r(0)\omega_{j+1,j+s}^p(0)F_{r-1}(0)F_{p-1}(0)\Sigma_{r-1,p-1}}{(S_{j,j+s}(0))^2} \int_t^{T_j} \sigma_{r-1}(z)\sigma_{p-1}(z)dz,
$$

$$
(2.51)
$$

for all $t \leqslant T_j$, where

$$
\omega_{j+1,j+s}^l(t) = \frac{B_l(t)}{\displaystyle\sum_{m=j+1}^{j+s} B_m(t)}, \qquad (2.52)
$$

and $S_{j,j+s}$ denotes the value of a forward swap rate for the time interval $[T_j, T_{j+s}]$, that is

$$
S_{j,j+s}(t) = \frac{B_j(t) - B_{j+s}(t)}{\Delta T \displaystyle\sum_{m=j+1}^{j+s} B_m(t)}. \qquad (2.53)
$$

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Note that the approximation (2.51) is known as Rebonato's formula.

Remark 2.3.1. *If matrix C built as above is not positive semidefinite then we have to approximate it by another one that is positive semidefinite maintaining that the absolute value of all their entries is less than or equal to 1, with the diagonal entries equal to 1 (and therefore a correlation matrix). We can do this by parameterizing correlations with some method, as the ones proposed in [56], and then trying to minimize the error between the matrix C and its approximated correlation matrix. Another method that we can apply to approximate the matrix C with one correlation matrix is the spectral decomposition, presented in [32].*

2.4 Numerical Results

In this section we present the results for different examples, mainly in order to illustrate the performance of the proposed DFS method. With these results, we also examine the impact on the error of caplet prices that the martingale adjustment produces. First, we describe the common data for all these examples.

Table 2.1 shows the data for the tenor structure and the Monte Carlo simulation procedure. More precisely, we consider the case $N = 20$ with constant accrual period equal to one year. The time step δt and the number of simulations N_S are also indicated.

N	δ_j	δt	Number of simulations (N_S)
20	1	0.25	400.000

Table 2.1: Parameters of the simulation procedure.

Polar-Marsaglia method is used for simulation of the Brownian motions.

We will consider as numeraire:

$$\text{Num}(t) = \begin{cases} B_{10}(t), & \text{if } t \leq T_{10}, \\ \beta_{10}(t), & \text{if } t > T_{10}, \end{cases}$$

where $\beta_{10}(t)$ corresponds to the bank account starting at time T_{10}, that is

$$\beta_{10}(t) = B_j(t) \prod_{l=10}^{j-1} \left(1 + \delta_l F_l(T_l)\right), \quad t \in (T_{j-1}, T_j], \quad j = 11, \ldots, N+1. \qquad (2.54)$$

Thus, the numeraire is B_{10} until T_{10} and the spot measure starting after T_{10} for $t > T_{10}$. Although this choice neither corresponds to a forward measure nor to a pure spot one, the proposed methodology can also be applied.

Next the data and the obtained results for different examples are presented. In order to illustrate the distribution of the errors across maturity and following the methodology in [34], in the first example a flat structure of interest rates and volatilities is assumed. The second example corresponds to a more realistic situation in which different values both for initial forward rates and for constant volatilities are considered. In both examples the correlation matrix is obtained by using (2.51) from the swap volatilities indicated in Table 2.2 and from the forward volatilities indicated in each example. We approximate the obtained matrix by the spectral decomposition to be a correlation matrix.

j	$\sigma_1^{S_j}$	$\sigma_2^{S_j}$	$\sigma_3^{S_j}$	$\sigma_4^{S_j}$	$\sigma_5^{S_j}$	$\sigma_6^{S_j}$	$\sigma_7^{S_j}$	$\sigma_8^{S_j}$	$\sigma_9^{S_j}$	$\sigma_{10}^{S_j}$	$\sigma_{11}^{S_j}$	$\sigma_{12}^{S_j}$	$\sigma_{13}^{S_j}$	$\sigma_{14}^{S_j}$	$\sigma_{15}^{S_j}$	$\sigma_{16}^{S_j}$	$\sigma_{17}^{S_j}$	$\sigma_{18}^{S_j}$	$\sigma_{19}^{S_j}$	$\sigma_{20}^{S_j}$
1	0,515	0,384	0,340	0,309	0,285	0,269	0,257	0,249	0,243	0,238	0,235	0,232	0,229	0,226	0,223	0,223	0,223	0,223	0,223	0,223
2	0,357	0,290	0,270	0,255	0,243	0,237	0,232	0,229	0,226	0,223	0,221	0,220	0,218	0,217	0,215	0,216	0,216	0,217	0,218	
3	0,284	0,240	0,228	0,219	0,212	0,210	0,208	0,207	0,205	0,205	0,204	0,203	0,202	0,202	0,201	0,202	0,203	0,204		
4	0,237	0,209	0,202	0,197	0,193	0,192	0,190	0,189	0,189	0,188	0,188	0,188	0,188	0,188	0,188	0,189	0,190			
5	0,202	0,185	0,183	0,180	0,178	0,177	0,176	0,175	0,175	0,175	0,175	0,176	0,176	0,176	0,176	0,178				
6	0,185	0,173	0,171	0,169	0,167	0,166	0,166	0,166	0,166	0,166	0,167	0,168	0,168	0,169	0,170					
7	0,169	0,161	0,160	0,158	0,157	0,156	0,156	0,156	0,157	0,158	0,159	0,160	0,161	0,162						
8	0,160	0,154	0,153	0,152	0,151	0,151	0,152	0,152	0,154	0,155	0,156	0,157	0,158							
9	0,152	0,147	0,146	0,146	0,146	0,146	0,147	0,149	0,150	0,152	0,153	0,155								
10	0,143	0,140	0,139	0,140	0,140	0,141	0,143	0,145	0,147	0,149	0,151									
11	0,142	0,138	0,139	0,140	0,141	0,142	0,145	0,147	0,149	0,151										
12	0,140	0,137	0,138	0,140	0,141	0,143	0,146	0,148	0,151											
13	0,139	0,136	0,138	0,140	0,142	0,144	0,148	0,150												
14	0,137	0,135	0,137	0,140	0,142	0,145	0,149													
15	0,136	0,134	0,137	0,140	0,142	0,147														
16	0,140	0,139	0,141	0,144	0,147															
17	0,144	0,143	0,146	0,149																
18	0,148	0,148	0,150																	
19	0,152	0,152																		
20	0,156																			

Table 2.2: Swap volatilities. We simplify the notation: $\sigma_l^{S_j} \equiv \sigma_{j,j+l}^S$.

2.4.1 Example 1: Flat structure of forward rates and volatilities

In this example we assume a flat structure of rates and volatilities. More precisely, in particular we take the initial forward Libor rates $F_j(0) = 0,012464$ (i.e. 1,2464%) and the instantaneous volatility $\sigma_j = 0,3566$ (i.e. 35,66%), for all $j = 1, ..., 20$.

Taking into account the previous data, we compare the Black value at time $t = 0$ of the caplets maturing at time T_j, $1 \leqslant j \leqslant N$ (see, for example [16]), with their value at time $t = 0$ being obtained with different simulation methods:

- the simulation with drifts (DS), by using a log-Euler discretization with predictor correction [39],

- the DFS proposed by Glasserman and Zhao in [26] without martingale adjustment at simulation level (DFS GZ) and the same method with adjustment (DFS GZ WA),

- the new parameterized DFS (DFS P) and the same method with adjustment (DFS P WA).

The simulated value a time $t = 0$, $V(0)$, of the caplet maturing at time T_j is obtained from their simulated value at time T_j, $V(T_j)$, by the following relation:

$$V(0) = \text{Num}(0)\frac{1}{N_S} \sum_{p=1}^{N_S} \frac{V(T_j)[p]}{\text{Num}(T_j)[p]}. \tag{2.55}$$

The index p allows us to distinguish the values obtained with the different simulations.

Figure 2.1 shows the difference in basis points between the Black values for the caplets and the simulated prices with the different methods for different strikes, including the case at-the-money (ATM) and a couple of cases in-the-money (ITM) and out-the-money (OTM). We also present the Black values and the confidence intervals

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for the ATM case with different DFS techniques (see Table 2.3) and the Black values and the confidence intervals with the DFS P with adjustment method for different strikes (see Table 2.4).

j	Black	CI DS	CI DFS GZ	CI DFS GZ WA	CI DFS P
1	1,7208	[1,70665 , 1,72654]	[1,7066 , 1,7265]	[1,70983 , 1,7297]	[1,7066 , 1,7265]
2	2,3910	[2,36613 , 2,39689]	[2,3660 , 2,3967]	[2,37127 , 2,4020]	[2,3660 , 2,3967]
3	2,8772	[2,84317 , 2,88335]	[2,8430 , 2,8832]	[2,85540 , 2,8956]	[2,8430 , 2,8832]
4	3,2642	[3,24227 , 3,29133]	[3,2423 , 3,2913]	[3,24302 , 3,2921]	[3,2423 , 3,2913]
5	3,5860	[3,56603 , 3,62300]	[3,5661 , 3,6230]	[3,56173 , 3,6186]	[3,5661 , 3,6230]
6	3,8598	[3,82407 , 3,88960]	[3,8241 , 3,8896]	[3,83126 , 3,8968]	[3,8241 , 3,8896]
7	4,0965	[4,06990 , 4,14342]	[4,0699 , 4,1434]	[4,06254 , 4,1360]	[4,0699 , 4,1434]
8	4,3033	[4,26067 , 4,34032]	[4,2606 , 4,3402]	[4,26191 , 4,3416]	[4,2606 , 4,3402]
9	4,4852	[4,45683 , 4,54658]	[4,4568 , 4,5466]	[4,44720 , 4,5368]	[4,4568 , 4,5466]
10	4,6459	[4,60964 , 4,69946]	[4,6100 , 4,7000]	[4,60275 , 4,6927]	[4,6036 , 4,6931]
11	4,7884	[4,74243 , 4,83911]	[4,7425 , 4,8394]	[4,74857 , 4,8455]	[4,7357 , 4,8321]
12	4,9148	[4,89157 , 4,99408]	[4,8919 , 4,9947]	[4,86536 , 4,9678]	[4,8833 , 4,9855]
13	5,0272	[4,95825 , 5,06532]	[4,9588 , 5,0662]	[4,97777 , 5,0854]	[4,9497 , 5,0564]
14	5,1271	[5,09500 , 5,20894]	[5,0957 , 5,2100]	[5,07046 , 5,1844]	[5,0858 , 5,1994]
15	5,2157	[5,19552 , 5,31255]	[5,1958 , 5,3131]	[5,17206 , 5,2890]	[5,1860 , 5,3027]
16	5,2941	[5,21426 , 5,33716]	[5,2147 , 5,3380]	[5,22965 , 5,3532]	[5,2054 , 5,3279]
17	5,3633	[5,35458 , 5,48224]	[5,3553 , 5,4834]	[5,31349 , 5,4409]	[5,3455 , 5,4728]
18	5,4241	[5,30307 , 5,43193]	[5,3024 , 5,4315]	[5,34864 , 5,4785]	[5,2929 , 5,4243]
19	5,4773	[5,40717 , 5,53995]	[5,4062 , 5,5392]	[5,40586 , 5,5389]	[5,3980 , 5,5304]
20	5,5234	[5,48730 , 5,62386]	[5,4866 , 5,6235]	[5,46497 , 5,6014]	[5,4775 , 5,6134]

Table 2.3: Example 1: Black values (x10^3) and confidence intervals (CI) (x10^3) obtained with the different DFS techniques. The results for the ATM case are presented. Confidence level = 95%.

As illustrated in Figure 2.1, the proposed method generally exhibits the best performance in all caplet pricing. Furthermore, the martingale adjustment at simulation levels leads to a significant reduction in both DFS techniques.
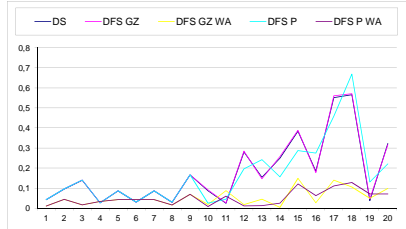
(a) Strike $= F_j(0)$ (ATM).   (b) Strike $= F_j(0) - 0,003$ (ITM).
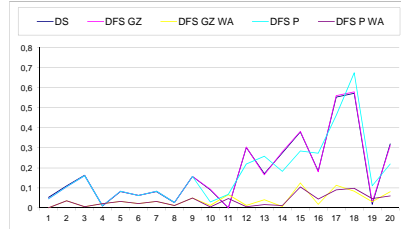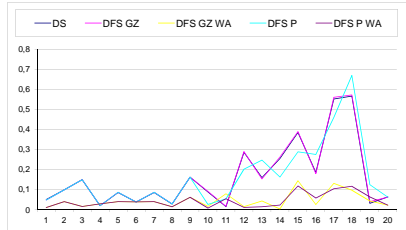
(c) Strike $= F_j(0) - 0,001$ (ITM).   (d) Strike $= F_j(0) + 0,001$ (OTM).
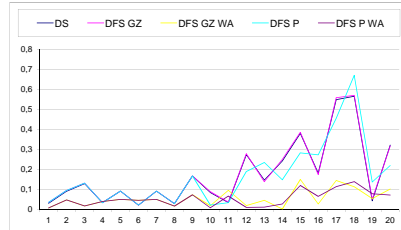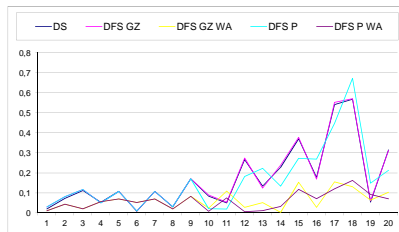
(e) Strike $= F_j(0) + 0,003$ (OTM).

Figure 2.1: Example 1: Differences in basis points between the Black values and simulated caplet prices with different strikes when we consider a flat structure of forward rates and volatilities.

j	ATM (x10^3)	ITM (x10^3)	OTM (x10^3)
1	1,7208 [1,7098, 1,7297]	3,3992 [3,3867 3,4119]	0,8015 [0,7932, 0,8075]
2	2,3910 [2,3713, 2,4020]	3,8456 [3,8240 3,8601]	1,4684 [1,4514, 1,4768]
3	2,8772 [2,8554, 2,8956]	4,1961 [4,1728 4,2181]	1,9834 [1,9637, 1,9989]
4	3,2642 [3,2430, 3,2921]	4,4823 [4,4575 4,5113]	2,4054 [2,3889, 2,4332]
5	3,5860 [3,5617, 3,6186]	4,7221 [4,6947 4,7561]	2,7627 [2,7435, 2,7959]
6	3,8598 [3,8313, 3,8968]	4,9264 [4,8937 4,9635]	3,0713 [3,0458, 3,1072]
7	4,0965 [4,0625, 4,1360]	5,1023 [5,0646 5,1421]	3,3413 [3,3119, 3,3814]
8	4,3033 [4,2619, 4,3416]	5,2549 [5,2119 5,2954]	3,5797 [3,5398, 3,6157]
9	4,4852 [4,4472, 4,5368]	5,3878 [5,3461 5,4393]	3,7917 [3,7570, 3,8431]
10	4,6459 [4,6004, 4,6899]	5,5038 [5,4568 5,5498]	3,9809 [3,9371, 4,0232]
11	4,7884 [4,7461, 4,8426]	5,6051 [5,5599 5,6597]	4,1505 [4,1113, 4,2046]
12	4,9148 [4,8628, 4,9647]	5,6936 [5,6404 5,7454]	4,3026 [4,2525, 4,3514]
13	5,0272 [4,9749, 5,0820]	5,7706 [5,7173 5,8273]	4,4395 [4,3885, 4,4927]
14	5,1271 [5,0679, 5,1813]	5,8374 [5,7780 5,8942]	4,5625 [4,5039, 4,6145]
15	5,2157 [5,1695, 5,2859]	5,8950 [5,8459 5,9651]	4,6732 [4,6281, 4,7419]
16	5,2941 [5,2265, 5,3493]	5,9444 [5,8772 6,0026]	4,7727 [4,7054, 4,8257]
17	5,3633 [5,3109, 5,4377]	5,9862 [5,9306 6,0598]	4,8619 [4,8118, 4,9362]
18	5,4241 [5,3466, 5,4759]	6,0211 [5,9455 6,0771]	4,9419 [4,8621, 4,9891]
19	5,4773 [5,4039, 5,5364]	6,0498 [5,9777 6,1126]	5,0133 [4,9389, 5,0693]
20	5,5234 [5,4625, 5,5984]	6,0728 [6,0098 6,1479]	5,0769 [5,0171, 5,1509]

Table 2.4: Example 1: Black values and confidence intervals obtained with the DFS P WA technique. The results for the ATM case, for one ITM case (Strike $= F_j(0) - 0,003$) and for one OTM case (Strike $= F_j(0) + 0,003$) are presented. Confidence level $= 95\%$.

2.4.2 Example 2: Uneven structure of forward rates and volatilities

For this example the market data is given in Table 2.5, which includes the Zero Coupon Curve and the volatilities of the involved forward Libor rates.

Taking into account these data sets, in Table 2.6 we show the errors in basis points when comparing the martingale values at time zero with the expected values obtained with the parameterized DFS without and with adjustment. We can observe how the proposed martingale adjustment reduces the errors that appear with the discretization, although it is an adjustment based on one approximation. Next, analogously to

j	$B_j(0)$	σ_j
0	1,000000	
1	0,987689	0,5152
2	0,966393	0,3566
3	0,938774	0,2837
4	0,907973	0,2367
5	0,874979	0,2015
6	0,840833	0,1851
7	0,806538	0,1687
8	0,772759	0,1602
9	0,739546	0,1518
10	0,707055	0,1433
11	0,674961	0,1417
12	0,643480	0,1402
13	0,613267	0,1386
14	0,584354	0,1371
15	0,557111	0,1355
16	0,531719	0,1396
17	0,508310	0,1437
18	0,486630	0,1479
19	0,466627	0,1520
20	0,448199	0,1561
21	0,431259	

Table 2.5: Market data for Example 2.

Example 1, the errors for the caplet pricing with different simulation methods and for different strikes are presented in Figure 2.2. We also present the Black values and the confidence intervals for the DFS P with adjustment method (see Table 2.7). Note that the same qualitative behavior as in Example 1 is observed.

Finally, Figure 2.3 shows the differences in basis points with respect to Black formula for the ATM case when the volatility structure of Table 2.5 is shifted up and down by two constants (0,1 and 0,05). We notice that a similar behavior for the ITM and OTM cases has been observed when volatility is shifted in the same way.
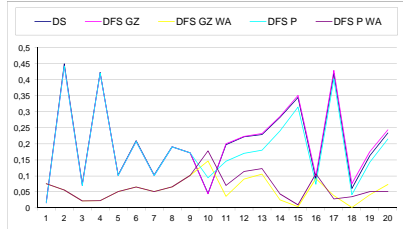
(a) Strike $= F_j(0)$ (ATM).
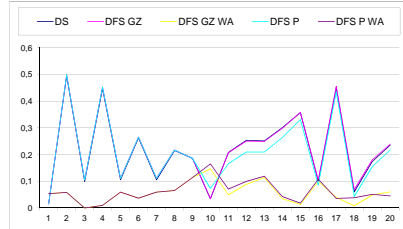
(b) Strike $= F_j(0) - 0,003$ (ITM).
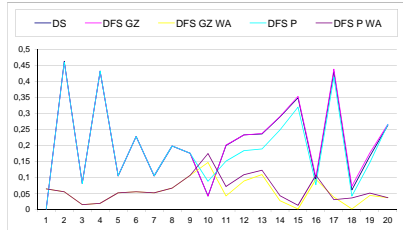
(c) Strike $= F_j(0) - 0,001$ (ITM).
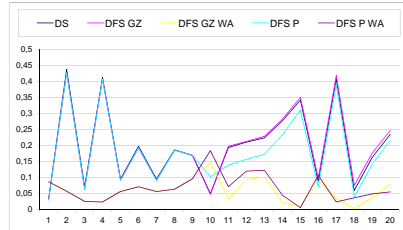
(d) Strike $= F_j(0) + 0,001$ (OTM).
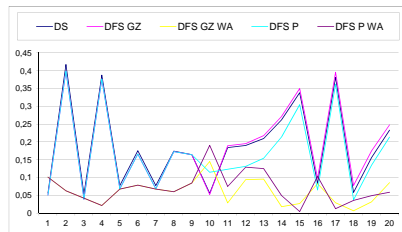
(e) Strike $= F_j(0) + 0,003$ (OTM).

Figure 2.2: Example 2: Differences in basis points between the Black values and simulated caplet prices with different strikes when we consider an uneven structure of forward rates and volatilities.

| j	DFS P	DFS P WA
1	1,34E+00	0,00E+00
2	5,19E+00	0,00E+00
3	2,90E+00	0,00E+00
4	3,21E+00	0,00E+00
5	2,68E+00	0,00E+00
6	2,64E+00	0,00E+00
7	2,17E+00	0,00E+00
8	1,49E+00	0,00E+00
9	6,56E-01	0,00E+00
10	0,00E+00	0,00E+00
11	2,05E-01	2,28E-05
12	7,10E-01	6,17E-03
13	1,37E+00	5,22E-03
14	2,06E+00	4,98E-03
15	2,67E+00	7,69E-03
16	3,35E+00	1,60E-02
17	3,66E+00	1,71E-02
18	4,42E+00	3,19E-02
19	4,44E+00	3,48E-03
20	4,79E+00	2,06E-02
21	5,03E+00	1,26E-02

Table 2.6: Example 2: Errors in basis points when comparing the martingale values at time zero with the expected values obtained with the parameterized DFS without and with adjustment.
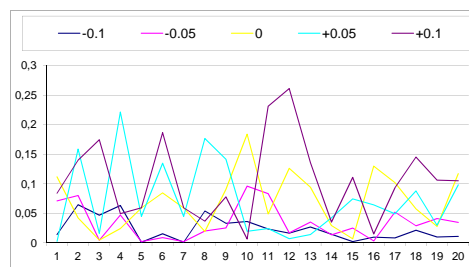
Figure 2.3: Example 2: Differences in basis points between the Black values and simulated caplet prices when moving up and down the volatility structure in the ATM case.

j	ATM ($\times 10^3$)	ITM ($\times 10^3$)	OTM ($\times 10^3$)
1	4,3291 [4,3084, 4,3650]	5,6418 [5,6162, 5,6782]	3,3122 [3,2966, 3,3478]
2	5,4985 [5,4671, 5,5411]	6,7419 [6,7079, 6,7875]	4,4751 [4,4471, 4,5156]
3	5,9780 [5,9356, 6,0160]	7,1746 [7,1317, 7,2175]	4,9710 [4,9293, 5,0043]
4	6,1735 [6,1347, 6,2169]	7,3295 [7,2867, 7,3742]	5,1889 [5,1525, 5,2295]
5	6,0864 [6,0417, 6,1209]	7,2062 [7,1581, 7,2423]	5,1273 [5,0835, 5,1576]
6	6,1508 [6,1046, 6,1840]	7,2193 [7,1734, 7,2577]	5,2285 [5,1833, 5,2579]
7	5,9653 [5,9166, 5,9917]	6,9915 [6,9411, 7,0207]	5,0779 [5,0303, 5,1009]
8	5,9542 [5,9108, 5,9844]	6,9303 [6,8849, 6,9626]	5,1052 [5,0646, 5,1338]
9	5,8510 [5,8055, 5,8763]	6,7818 [6,7330, 6,8077]	5,0387 [4,9969, 5,0637]
10	5,7528 [5,7017, 5,7684]	6,6404 [6,5887, 6,6591]	4,9751 [4,9245, 4,9876]
11	5,8501 [5,8096, 5,8765]	6,6864 [6,6442, 6,7145]	5,1116 [5,0724, 5,1359]
12	5,7960 [5,7523, 5,8168]	6,5855 [6,5417, 6,6094]	5,0960 [5,0524, 5,1138]
13	5,7059 [5,6625, 5,7248]	6,4517 [6,4073, 6,4724]	5,0421 [5,0000, 5,0593]
14	5,5133 [5,4793, 5,5385]	6,2197 [6,1845, 6,2464]	4,8841 [4,8510, 4,9075]
15	5,2558 [5,2266, 5,2831]	5,9263 [5,8949, 5,9539]	4,6590 [4,6324, 4,6864]
16	5,1485 [5,1091, 5,1665]	5,7786 [5,7379, 5,7977]	4,5868 [4,5490, 4,6040]
17	5,0520 [5,0264, 5,0833]	5,6447 [5,6187, 5,6778]	4,5229 [4,4968, 4,5514]
18	4,9251 [4,8940, 4,9491]	5,4835 [5,4511, 5,5082]	4,4261 [4,3961, 4,4490]
19	4,7826 [4,7508, 4,8042]	5,3095 [5,2767, 5,3320]	4,3118 [4,2812, 4,3327]
20	4,6241 [4,6034, 4,6552]	5,1220 [5,0997, 5,1533]	4,1793 [4,1602, 4,2101]

Table 2.7: Example 2: Black values and confidence intervals obtained with the DFS P WA technique. The results for the ATM case, for one ITM case (Strike $= F_j(0) - 0,003$) and for one OTM case (Strike $= F_j(0) + 0,003$) are presented. Confidence level $= 95\%$.

Chapter 3

Drift-Free Simulation techniques for Coterminal Swap Market Model and Generic Market Models

3.1 Introduction

In this chapter we adapt the DFS procedures described in the previous chapter to other Market Models.

First the proposed DFS methodology is adapted to the Coterminal Swap Market Model (COTSMM). As before, in order to avoid the drift-terms appearing in the log-normal dynamics of the implicit coterminal swap rates, the direct simulation of the ratios between annuities and the chosen numeraire (deflated annuities) is considered. This application is mainly formulated by analogy with the technique described in the previous chapter, however we must point out the main difference: while in the LMM case the dynamics of each deflated bond only depends on the dynamics of another deflated bond, in the COTSMM case the dynamics of each deflated annuity depends on the dynamics of two other deflated annuities.

LMM and SMM can be presented into a general framework like those ones proposed in [23, 52]. More precisely, in [23] the general context is presented by studying which market models are admissible through the use of graph theory. In [52], generic expressions for the drift terms appearing in a generic market model are presented. However, like in the previous chapter, the cumbersome computation of the drift terms does not allow for an adjustment to avoid arbitrage at the simulation level. Then, again we are interested in the simulation of the stochastic rates avoiding the use of the drift terms involved in their dynamics. So, taking into account these generalizations, also the DFS technique is extended to the Generic Market Models framework.

This chapter is organized as follows. Section 3.2 presents the standard setting and notation of the COTSMM and the DFS methods. Section 3.3 deals with a generalization of the DFS methods for Generic Market Models. Taking into account that the algorithms are analogous to the previously presented ones, we avoid some obvious explanations.

3.2 Coterminal Swap Market Model and Drift-Free Simulations

First, we consider a fixed tenor structure $\mathcal{T}:=\{T_0, T_1, ..., T_{N+1}\}$, with $T_0 = 0$, $T_j < T_k, 0 \leq j < k \leq N+1$, and the corresponding accruals $\delta_j = T_{j+1} - T_j, 0 \leq j \leq N$. As it is usual in SMM (see [33], for example), the evolution of the Zero Coupon Curve (ZCC), $\{B_j(t)\}_{j=0}^{N+1}$, can be obtained by modeling the implicit coterminal swap rates $\{S_j(t)\}_{j=0}^{N}$ defined by

$$S_j(t) = \frac{B_j(t) - B_{N+1}(t)}{\sum_{m=j+1}^{N+1} \delta_{m-1} B_m(t)}, \quad \forall t \leqslant T_j. \tag{3.1}$$

COTSMM assumes lognormality for these rates under some probability measure, hence:

$$dS_j(t) = \mu_j^S(t)S_j(t)dt + \sigma_j^S(t)S_j(t)dW_j^S(t), \quad j = 1, ..., N, \qquad (3.2)$$

where $\sigma_j^S(t)$ denotes the volatility term ($\sigma_j^S(t)$ is a piecewise continuous function) and where $\mu_j^S(t)$ denotes the drift term, whose specific expression in a non-arbitrage setting depends on the volatilities and the given numeraire.

In this context, the basic assets are the annuities, defined by

$$A_j(t) = \sum_{m=j}^{N+1} \delta_{m-1} B_m(t), \quad \forall t \leqslant T_j, \quad j = 0, 1, ..., N + 1. \qquad (3.3)$$

So, once the numeraire Num has been chosen (a particular bond maturing at any tenor date or the bank account), we introduce the following notation for the deflated annuities:

$$\widehat{A}_j(t) \doteq \frac{A_j(t)}{\text{Num}(t)}, \qquad t \leq T_j, \qquad j = 1, \ldots, N + 1. \qquad (3.4)$$

Remark 3.2.1. *The following relation between the annuities and the prices of bonds (equivalent to (3.3)) holds:*

$$B_j(t) = \begin{cases} \frac{1}{\delta_{j-1}}\big(A_j(t) - A_{j+1}(t)\big), & \text{if } j < N + 1; \\ \frac{1}{\delta_N}A_{N+1}(t), & \text{if } j = N + 1. \end{cases} \qquad (3.5)$$

Therefore, by taking into account Remark 2.2.1, for any of the possible numeraires and for each interval $(T_{j-1}, T_j]$, $j = 1, ..., N + 1$, there exists an index k such that $d\big(\widehat{A}_k(t) - \widehat{A}_{k+1}(t)\big) = 0$ along that interval (we notice that if $k = N + 1$, $\widehat{A}_{k+1}(t)$ is zero).

Thus, in this case it is also possible to establish a biyective relation between the modeled rates $\{S_j(t)\}_{j=0}^N$ and the martingales $\{\widehat{A}_j(t)\}_{j=0}^{N+1}$:

$$\widehat{A}_j(t) = \big(1 + \delta_{j-1}S_j(t)\big)\widehat{A}_{j+1}(t) + \frac{\delta_{j-1}}{\delta_N}\widehat{A}_{N+1}(t), \quad j = 1, ..., N. \qquad (3.6)$$

Next, taking into account identity in (3.6) and that $\widehat{A}_j(t)$, $\widehat{A}_{j+1}(t)$, and $\widehat{A}_{N+1}(t)$ are martingales, we simulate the martingale structure by the DFS procedure, which in the COTSMM is based on

$$d\widehat{A}_j(t) = \left(1 + \delta_{j-1}S_j(t)\right)d\widehat{A}_{j+1}(t) + \delta_{j-1}\widehat{A}_{j+1}(t)\sigma_j^S(t)S_j(t)dW_j^S(t) + \frac{\delta_{j-1}}{\delta_N}d\widehat{A}_{N+1}(t)\,.$$
(3.7)

So, unlike to what happened in the LMM case, in the DFS for COTSMM we need to know previously the dynamics of the martingale $\widehat{A}_{N+1}$.

Taking into account (3.7), we obtain recursively an expression for the dynamics of $\widehat{A}_j(t)$ that only depends on the dynamics of $\widehat{A}_{N+1}(t)$ and on the previous levels of implicit coterminal swap rates and on the previously simulated Brownian jumps. More precisely, we have:

$$d\widehat{A}_j(t) = \left(1 + \delta_{j-1}S_j(t)\right)d\widehat{A}_{j+1}(t) + \delta_{j-1}\widehat{A}_{j+1}(t)\sigma_j^S(t)S_j(t)dW_j^S(t) + \frac{\delta_{j-1}}{\delta_N}d\widehat{A}_{N+1}(t) =$$

$$= \left(1 + \delta_{j-1}S_j(t)\right)\left(1 + \delta_j S_{j+1}(t)\right)d\widehat{A}_{j+2}(t) + \delta_{j-1}\widehat{A}_{j+1}(t)\sigma_j^S(t)S_j(t)dW_j^S(t) +$$

$$+ \left(1 + \delta_{j-1}S_j(t)\right)\delta_j\widehat{A}_{j+2}(t)\sigma_{j+1}^S(t)S_{j+1}(t)dW_{j+1}^S(t) + \left[\frac{\delta_{j-1}}{\delta_N} + \left(1 + \delta_{j-1}S_j(t)\right)\frac{\delta_j}{\delta_N}\right]d\widehat{A}_{N+1}(t) =$$

$$= \left(1 + \delta_{j-1}S_j(t)\right)\left(1 + \delta_j S_{j+1}(t)\right)\left(1 + \delta_{j+1}S_{j+2}(t)\right)d\widehat{A}_{j+3}(t) +$$

$$+ \delta_{j-1}\widehat{A}_{j+1}(t)\sigma_j^S(t)S_j(t)dW_j^S(t) + \left(1 + \delta_{j-1}S_j(t)\right)\delta_j\widehat{A}_{j+2}(t)\sigma_{j+1}^S(t)S_{j+1}(t)dW_{j+1}^S(t) +$$

$$+ \left(1 + \delta_{j-1}S_j(t)\right)\left(1 + \delta_j S_{j+1}(t)\right)\delta_{j+1}\widehat{A}_{j+3}(t)\sigma_{j+2}^S(t)S_{j+2}(t)dW_{j+2}^S(t) +$$

$$+ \left[\frac{\delta_{j-1}}{\delta_N} + \left(1 + \delta_{j-1}S_j(t)\right)\frac{\delta_j}{\delta_N} + \left(1 + \delta_{j-1}S_j(t)\right)\left(1 + \delta_j S_{j+1}(t)\right)\frac{\delta_{j+1}}{\delta_N}\right]d\widehat{A}_{N+1}(t) =$$

$$= \quad \ldots \quad =$$

$$= \sum_{m=j}^{N} X_m^j(t)dW_m^S(t) + V_{N+1}^j(t)d\widehat{A}_{N+1}(t), \quad \forall j = 1, \ldots, N,$$
(3.8)

where

$$X_m^j(t) = \delta_{m-1}\widehat{A}_{m+1}(t)\sigma_m^S(t)S_m(t)\prod_{p=j}^{m-1}\left(1 + \delta_{p-1}S_p(t)\right),$$

$$V_{N+1}^j(t) = \sum_{m=j}^{N+1}\left[\frac{\delta_m - 1}{\delta_N}\prod_{p=j}^{m-1}\left(1 + \delta_{p-1}S_p(t)\right)\right]. \tag{3.9}$$

Then, since $d\widehat{A}_k(t) - d\widehat{A}_{k+1}(t) = 0$ for some index k at each time t, we have:

- If $k = N + 1$:

$$d\widehat{A}_{N+1}(t) = 0. \tag{3.10}$$

- If $k < N + 1$:

$$\sum_{m=k}^{N} X_m^k(t)dW_m^S(t) + V_{N+1}^k(t)d\widehat{A}_{N+1}(t) - \sum_{m=k+1}^{N} X_m^{k+1}(t)dW_m^S(t) -$$

$$- V_{N+1}^{k+1}(t)d\widehat{A}_{N+1}(t) = 0, \tag{3.11}$$

or, equivalently to (3.11):

$$d\widehat{A}_{N+1}(t) = -\frac{\displaystyle\sum_{m=k}^{N} X_m^k(t)dW_m^S(t) - \sum_{m=k+1}^{N} X_m^{k+1}(t)dW_m^S(t)}{V_{N+1}^k(t) - V_{N+1}^{k+1}(t)}, \tag{3.12}$$

where

$$\sum_{m=k}^{N} X_m^k(t)dW_m^S(t) - \sum_{m=k+1}^{N} X_m^{k+1}(t)dW_m^S(t) =$$

$$= X_k^k(t)dW_k^S(t) + \sum_{m=k+1}^{N}\left(X_m^k(t) - X_m^{k+1}(t)\right)dW_m^S(t) =$$

$$= X_k^k(t)dW_k^S(t) +$$

$$+ \sum_{m=k+1}^{N}\left(\delta_{m-1}\sigma_m^S(t)S_m(t)\widehat{A}_{m+1}(t)\prod_{p=k+1}^{m-1}\left(1 + \delta_{p-1}S_p(t)\right)[1 + \delta_{k-1}S_k(t) - 1]\right) =$$

$$= X_k^k(t)dW_k^S(t) + \sum_{m=k+1}^{N}\delta_{k-1}S_k(t)X_m^{k+1}(t), \tag{3.13}$$

and

$$V_{N+1}^k(t) - V_{N+1}^{k+1}(t) =$$

$$= \frac{\delta_{k-1}}{\delta_N} + \sum_{m=k+1}^{N+1} \frac{\delta_{m-1}}{\delta_N} \left(\prod_{p=k}^{m-1} \left(1 + \delta_{p-1}S_p(t)\right) - \prod_{p=k+1}^{m-1} \left(1 + \delta_{p-1}S_p(t)\right) \right) =$$

$$= \frac{\delta_{k-1}}{\delta_N} + \sum_{m=k+1}^{N+1} \frac{\delta_{m-1}}{\delta_N} \prod_{p=k+1}^{m-1} \left(1 + \delta_{k-1}S_k(t)\right)\left[1 + \delta_{k-1}S_k(t) - 1\right] =$$

$$= \frac{\delta_{k-1}}{\delta_N} + \delta_{k-1}S_k(t)V_{N+1}^{k+1}(t). \tag{3.14}$$

Hence, we can compute the dynamics of the martingale $\widehat{A}_{N+1}(t)$ with the following expression:

$$d\widehat{A}_{N+1}(t) = Z_k(t) \sum_{m=k}^{N} Y_m^k(t)dW_m^S(t), \tag{3.15}$$

where

$$Y_m^k(t) = \begin{cases} X_k^k(t), & \text{if } m = k; \\ \delta_{k-1}S_k(t)X_m^{k+1}(t), & \text{if } m = k+1, ..., N; \end{cases}$$

$$Z_k(t) = \begin{cases} \dfrac{-1}{\frac{\delta_{k-1}}{\delta_N} + \delta_{k-1}S_k(t)V_{N+1}^{k+1}(t)}, & \text{if } k < N+1; \\ 0, & \text{if } k = N+1. \end{cases} \tag{3.16}$$

Therefore, once the dynamics of $\widehat{A}_{N+1}(t)$ in (3.15) and the recurrence relation between dynamics of consecutive martingales in (3.7) are obtained, an analogous IDFS algorithm for COTSMM can be considered. Also, in order to avoid the recurrence and to ensure the positivity of the martingales $\widehat{A}_j(t)$, it is possible to apply an EDFS algorithm by means of a log-Euler discretization of the following dynamics:

$$\frac{d\widehat{A}_j(t)}{\widehat{A}_j(t)} = \frac{1}{\widehat{A}_j(t)} \sum_{m=j}^{N} X_m^j(t)dW_m^S(t) + \frac{V_{N+1}^j(t)Z_k(t)}{\widehat{A}_j(t)} \sum_{m=k}^{N} Y_m^k(t)dW_m^S(t), \tag{3.17}$$

if $j = 1, ..., N$, and:

$$\frac{d\widehat{A}_{N+1}(t)}{\widehat{A}_{N+1}(t)} = \frac{Z_k(t)}{\widehat{A}_{N+1}(t)} \sum_{m=k}^{N} Y_m^k(t)dW_m^S(t). \tag{3.18}$$

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Moreover, (3.17) and (3.18) can be written in the form:

$$\frac{d\widehat{A}_j(t)}{\widehat{A}_j(t)} = \sigma_j^{\widehat{A}}(t)dW_j^{\widehat{A}}(t), \tag{3.19}$$

where $dW_j^{\widehat{A}}(t)$ is an appropriate new Brownian motion,

$$\begin{aligned}
\sigma_j^{\widehat{A}}(t)^2 &= \left(\frac{1}{\widehat{A}_j(t)}\right)^2 \sum_{m_1,m_2=j}^{N} X_{m_1}^j(t)X_{m_2}^j(t)\Sigma_{m_1,m_2} + \\
&\quad + \left(\frac{V_{N+1}^j(t)Z_k(t)}{\widehat{A}_j(t)}\right)^2 \sum_{m_1,m_2=k}^{N} Y_{m_1}^k(t)Y_{m_2}^k(t)\Sigma_{m_1,m_2} + \\
&\quad + 2\frac{V_{N+1}^j(t)Z_k(t)}{\widehat{A}_j(t)^2} \sum_{m_1=j}^{N}\sum_{m_2=k}^{N} X_{m_1}^j(t)Y_{m_2}^k(t)\Sigma_{m_1,m_2}, \tag{3.20}
\end{aligned}$$

if $j = 1, ..., N$, and

$$\sigma_{N+1}^{\widehat{A}}(t)^2 = \left(\frac{Z_k(t)}{\widehat{A}_{N+1}(t)}\right)^2 \sum_{m_1,m_2=k}^{N} Y_{m_1}^k(t)Y_{m_2}^k(t)\Sigma_{m_1,m_2}. \tag{3.21}$$

As in the LMM case, the positivity of the martingales is guaranteed with the EDFS algorithm, although it is not the same for the COTSMM case. Then, analogously to the approach of Glasserman and Zhao in [26], we can simulate the following martingales:

$$D_j^S(t) \doteq \begin{cases} \dfrac{\widehat{A}_j(t)}{\delta_{j-1}} - \dfrac{\widehat{A}_{j+1}(t)}{\delta_{j-1}} - \dfrac{\widehat{A}_{N+1}(t)}{\delta_N}, & \text{if } j = 1, ..., N; \\ \widehat{A}_{N+1}(t), & \text{if } j = N+1; \end{cases} \tag{3.22}$$

the dynamics of which are given by

$$\frac{dD_j^S(t)}{D_j^S(t)} = \begin{cases} \dfrac{d\widehat{A}_{j+1}(t)}{\widehat{A}_{j+1}(t)} + \sigma_j^S(t)dW_j^S(t), & \text{if } j = 1, ..., N; \\ \dfrac{d\widehat{A}_{N+1}(t)}{\widehat{A}_{N+1}(t)}, & \text{if } j = N+1. \end{cases} \tag{3.23}$$

Notice that, under any forward probability or under the spot probability measures, in the COTSMM a log-Euler discretization of the GZDFS algorithm results

to be enough to ensure the positivity of martingales and coterminal swap rates. So, a priori one may think that a parameterization is not necessary in this case. However, this is not the case because if $k \neq N + 1$ an additional discretization bias has been introduced into the IDFS, EDFS and GZDFS methods for COTSMM. This is because, at the simulation level, from the computed value $\widehat{A}_{N+1}(t_{i+1})$ we can obtain $\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1}) \neq \widehat{A}_k(t_i) - \widehat{A}_{k+1}(t_i)$, although the dynamics of the martingale $\widehat{A}_{N+1}(t)$ has been constructed to ensure $d\widehat{A}_k(t) - d\widehat{A}_{k+1}(t) = 0$.

So, for $k \neq N + 1$ we need to impose the following additional constraints:

1. $D_j^S > 0$, for $j = 1, ..., N + 1$. This condition guarantees positive martingales and coterminal swap rates.

2. $\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1}) = \widehat{A}_k(t_i) - \widehat{A}_{k+1}(t_i)$, i.e.:

$$\delta_{k-1} D_k^S(t_{i+1}) + \frac{\delta_{k-1}}{\delta_N} D_{N+1}^S(t_{i+1}) = \widehat{A}_k(t_i) - \widehat{A}_{k+1}(t_i). \qquad (3.24)$$

In order to ensure conditions 1 and 2 we propose the simulation of the following new processes:

$$u_j^S(t) \doteq \begin{cases} \ln\left(D_j^S(t)\right), & \text{if } j \neq k, \\ \ln\left(\frac{D_{N+1}^S(t)}{\delta_N}\right) - \ln\left(D_k^S(t)\right), & \text{if } j = k, \end{cases} \qquad (3.25)$$

with $j = 1, ..., N$.

The dynamics of $u_j^S(t)$ and some useful formulas are summarized in the following lemma. Notice that the dynamics are given in terms of the dynamics of $D_j^S(t)$ because they become intractable in terms of $u_j^S(t)$.

Lemma 3.2.2. *Each u_j^S, $j = 1, ..., N$, satisfies*

$$du_j^S(t) = \frac{dD_j^S(t)}{D_j^S(t)} - \frac{1}{2} \frac{dD_j^S(t)}{D_j^S(t)} \frac{dD_j^S(t)}{D_j^S(t)}, \qquad (3.26)$$

if $j \neq k$, and

$$du_j^S(t) = \left(\frac{dD_{N+1}^S(t)}{D_{N+1}^S(t)} - \frac{dD_k^S(t)}{D_k^S(t)} \right) - \frac{1}{2} \left(\frac{dD_{N+1}^S(t)}{D_{N+1}^S(t)} \frac{dD_{N+1}^S(t)}{D_{N+1}^S(t)} - \frac{dD_k^S(t)}{D_k^S(t)} \frac{dD_k^S(t)}{D_k^S(t)} \right),$$
$$(3.27)$$

if $j = k$.

Also, we have

$$D_j^S(t) = \exp\left(u_j^S(t) \right), \quad j = 1, ..., N, \quad j \neq k,$$

$$D_k^S(t) = \frac{\left[\widehat{A}_k(t) - \widehat{A}_{k+1}(t) \right]}{\delta_{k-1} \left[1 + \exp\left(u_k^S(t) \right) \right]},$$

$$D_{N+1}^S(t) = \frac{\delta_N \left[\widehat{A}_k(t) - \widehat{A}_{k+1}(t) \right] \exp\left(u_k^S(t) \right)}{\delta_{k-1} \left[1 + \exp\left(u_k^S(t) \right) \right]}, \qquad (3.28)$$

and

$$\widehat{A}_j(t) = \sum_{l=j}^{N} \delta_{l-1} D_l^S(t) + \left(\sum_{l=j}^{N+1} \frac{\delta_{l-1}}{\delta_N} \right) D_{N+1}^S(t), \qquad (3.29)$$

$$S_j(t) = \frac{D_j^S(t)}{\sum_{l=j}^{N} \delta_{l-1} D_l^S(t) + \left(\sum_{l=j}^{N+1} \frac{\delta_{l-1}}{\delta_N} \right) D_{N+1}^S(t)}. \qquad (3.30)$$

From Lemma 3.2.2, by analogy with the discretization scheme in Chapter 2, we propose the Euler discretization:

$$u_j^S(t_{i+1}) = u_j^S(t_i) + du_j^S(t_i). \qquad (3.31)$$

Then, taking into account that $\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1}) = \widehat{A}_k(T_{j-1}) - \widehat{A}_{k+1}(T_{j-1})$, for each $t_{i+1} \in (T_{j-1}, T_j]$, we can approximate the martingales D_j^S and $\widehat{A}_j$ and the forward swap rates S_j in the discretized times by using (3.28), (3.29) and (3.30), respectively.

Theorem 3.2.3. *i) $\widehat{A}_j$ are decreasing in j and positive terms ($0 < \widehat{A}_{j+1} < \widehat{A}_j$). ii) S_j are positive terms ($S_j > 0$).*

Proof. We know that these conditions are satisfied at time t_0. Since $\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1}) = \widehat{A}_k(0) - \widehat{A}_{k+1}(0) > 0$, $\forall t_{i+1} \in (T_0, T_1]$, by (3.28), we obtain that $D_j^S(t_{i+1}) > 0$, and by using (3.29), we guarantee that $0 < \widehat{A}_{j+1}(t_{i+1}) < \widehat{A}_j(t_{i+1})$, $\forall t_{i+1} \in (T_0, T_1]$. Next, by arguing recursively for $l = 1, ..., N$, since $\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1}) = \widehat{A}_k(T_{l-1}) - \widehat{A}_{k+1}(T_{l-1}) > 0$, $\forall t_{i+1} \in (T_{l-1}, T_l]$, then by using (3.28), we obtain $D_j^S(t_{i+1}) > 0$, and by applying (3.29), we get $0 < \widehat{A}_{j+1}(t_{i+1}) < \widehat{A}_j(t_{i+1})$, $\forall t_{i+1} \in (T_{l-1}, T_l]$. Therefore, we deduce that $0 < \widehat{A}_{j+1}(t_{i+1}) < \widehat{A}_j(t_{i+1})$, $\forall t_{i+1} \leqslant T_j$, with $j = 1, ..., N$. Then, we have proved i).

Taking into account (3.30), the values of $S_j(t_{i+1})$ are quotients of positive terms, so we have ii). $\qquad\square$

Finally, once the algorithm has been applied, we can recover the numeraire and the annuities at tenor dates by means of the following expressions:

$$\text{Num}(T_j) = \begin{cases} \dfrac{\delta_{j-1}}{\widehat{A}_j(T_j) - \widehat{A}_{j+1}(T_j)}, & \text{if } j \leqslant N, \\ \dfrac{\delta_N}{\widehat{A}_{N+1}(T_j)}, & \text{if } j = N+1, \end{cases} \tag{3.32}$$

and

$$A_j(T_i) = \widehat{A}_j(T_i)\text{Num}(T_i), \quad i \leq j. \tag{3.33}$$

Again, if required, we can recover these values for times different from the tenor dates by means of an appropriate interpolation procedure.

The forward Libor rates can be also computed by the following expression (equivalent to (2.1)):

$$F_j(t) = \begin{cases} \dfrac{\delta_j \widehat{A}_j(t) - (\delta_j + \delta_{j-1})\widehat{A}_{j+1}(t) + \delta_{j-1}\widehat{A}_{j+2}(t)}{\delta_{j-1}\delta_j(\widehat{A}_{j+1}(t) - \widehat{A}_{j+2}(t))}, & \text{if } j < N; \\ \dfrac{\delta_N \widehat{A}_N(t) - (\delta_N + \delta_{N-1})\widehat{A}_{N+1}(t)}{\delta_{N-1}\delta_N \widehat{A}_{N+1}(t)}, & \text{if } j = N. \end{cases} \tag{3.34}$$

As in the LMM case, we can apply an adjustment to ensure the martingale property at the simulation level. The adjustment is analogous to the adjustments of

the Chapter 2 in the IDFS, EDFS and GZDS methods and in the parameterization method when $j \neq k$, although it has to be explained with more detail for the parameterization method when $j = k$. In the case $j = k$, by following an analogous approach to the one of the previous chapter, we can guarantee the martingale property of $D_k^S(t_{i+1})$ at the simulation level by replacing the obtained value $u_k^S(t_{i+1})[p]$ with $u_k^S(t_{i+1})[p] + \alpha_k^S(t_{i+1})$, where

$$
\alpha_k^S(t_{i+1}) \approx \frac{D_k^S(0) - \dfrac{1}{N_S} \displaystyle\sum_{p=1}^{N_S} \dfrac{\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1})}{\delta_{k-1}\big(1 + \exp(u_k^S(t_{i+1})[p])\big)}}{-\dfrac{1}{N_S} \displaystyle\sum_{p=1}^{N_S} \dfrac{\big(\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1})\big)\exp(u_k^S(t_{i+1})[p])}{\delta_{k-1}\big(1 + \exp(u_k^S(t_{i+1})[p])\big)^2}}, \tag{3.35}
$$

with N_S denoting the number of simulations. The index p allows us to distinguish the values obtained with the different simulations. Notice that the value of $\widehat{A}_k(t_{i+1}) - \widehat{A}_{k+1}(t_{i+1})$ is known at each time (see Remark 3.2.1).

3.3   Generic Market Models and Drift-Free Simulations

In this section we explain how the DFS approach can be applied to Generic Market Models like those ones presented in [23, 52].

We consider a tenor structure $\mathcal{T}:=\{T_0, T_1, ..., T_{N+1}\}$, with $T_0 = 0$, $T_j < T_k, 0 \leq j < k \leq N + 1$, and the corresponding accruals $\Delta T = T_{j+1} - T_j$, $0 \leq j \leq N$[1].

Let Num be the chosen numeraire. As before we shall only consider two kinds of numeraires: one of the prices of bonds, B_n, with $n = 1, ..., N+1$ or the bank account β.

---

[1]For simplicity we consider constant accrual, which is not a drawback since in practice it often takes constant.

Our aim is to obtain the evolution of the Zero Coupon Curve (ZCC) $\{B_j(t)\}_{j=1}^{N+1}$, under the probability measure associated to the numeraire Num.

We suppose that the ZCC can be expressed in terms of the implicit rates $\{Z_j(t)\}_{j=1}^{N}$, that is

$$\left(Z_1(t), ..., Z_1(t)\right) = f\left(\widehat{B}_1(t), ..., \widehat{B}_{N+1}(t)\right), \tag{3.36}$$

where

$$\widehat{B}_j(t) = \frac{B_j(t)}{\text{Num}(t)}, \tag{3.37}$$

and f is an invertible function.

We consider a complete and connected graph G with $N+1$ nodes that represent implicit rates. In [23], following [18], it is showed that one set of implicit rates is an admissible market model if and only if the edges that represent that implicit rates form a spanning tree of G. Thus, the set of rates $\{Z_j(t)\}_{j=1}^{N}$ has been chosen so as to form a spanning tree of the graph G [7]. For example, if we present the rates modeled with LMM and the ones modeled with Coterminal or Constant Maturity SMM in a graph G as indicated in Figure 3.1, these models are consistent because they are spanning trees of the graph G, so that we get a model without redundancies and the other remaining rates can be calculated from the modeled ones.

Let $\{BA_j(t)\}_{j=1}^{N+1}$ denotes the set of basic assets. We assume that

$$BA_j(t) = \sum_{m=1}^{N+1} \lambda_m^j B_m(t), \tag{3.38}$$

with

$$\lambda_m^j = \begin{cases} 0, & \text{if } m < j; \\ 1, & \text{if } j \leqslant m \leqslant l_j; \\ 0, & \text{if } m > l_j; \end{cases} \tag{3.39}$$

for some index $l_j \geqslant j$.
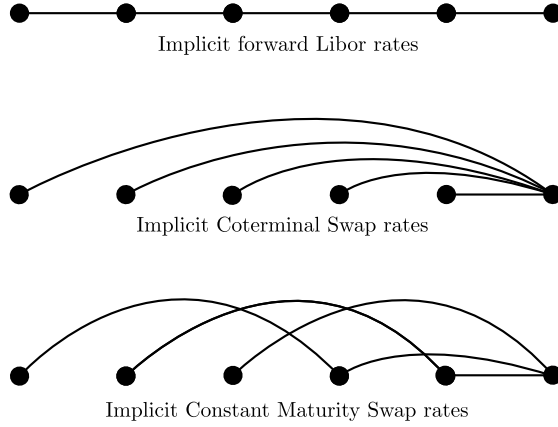
Figure 3.1: Spanning trees of the graph G associated to the implicit rates.

We introduce the notation for the deflated basic assets in the form:

$$\widehat{BA}_j(t) \doteq \frac{BA_j(t)}{\text{Num}(t)}, \qquad t \leq T_j, \qquad j = 1, \ldots, N+1. \tag{3.40}$$

We consider Generic Market Models that assume lognormal dynamics for the processes selected to express the evolution of the ZCC, $\{Z_j(t)\}_{j=1}^N$. Hence, we have:

$$dZ_j(t) = \mu_j(t)Z_j(t)dt + \sigma_j(t)Z_j(t)dW_j(t), \quad j = 1, \ldots, N, \tag{3.41}$$

where $\sigma_j(t)$ denotes the volatility term ($\sigma_j(t)$ is a piecewise continuous function) and where $\mu_j(t)$ denotes the drift term, whose value is the one which makes $\{\widehat{BA}_j(t)\}_{j=1}^{N+1}$ martingales, so it depends on the chosen probability measure.

Now, we are interested in the simulation of $\{Z_j(t)\}_{j=1}^N$ avoiding the use of the drift terms.

As we can see in Appendix A, from (3.38) we obtain that for all $j = 1, \ldots, N+1$ there exist some indexes $\nu_m^j \in \{-1, 0, 1\}$ such that

$$B_j(t) = \sum_{m=1}^{N+1} \nu_m^j BA_m(t), \tag{3.42}$$

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or, equivalently,

$$\widehat{B}_j(t) = \sum_{m=1}^{N+1} \nu_m^j \widehat{BA}_m(t). \tag{3.43}$$

Therefore, we have:

- $\left(Z_1(t), ..., Z_1(t)\right) = f\left(\sum_{m=1}^{N+1} \nu_m^1 \widehat{BA}_m(t), ..., \sum_{m=1}^{N+1} \nu_m^{N+1} \widehat{BA}_m(t)\right)$, with f an invertible function.

- As previously indicated, since we shall only consider one bond or the bank account as numeraire, for each interval $(T_{j-1}, T_j]$, $j = 1, ..., N+1$, there exists an index k such that $d\widehat{B}_k(t) = 0$ along that interval, or equivalently, $\sum_{m=1}^{N+1} \nu_m^k d\widehat{BA}_m(t) = 0$. This issue allows us to calculate the value of one martingale at each time (as a function of the others).

Then, a biyective relation between the implicit rates $\{Z_j(t)\}_{j=1}^N$ and the martingales $\{\widehat{BA}_j(t)\}_{j=1}^{N+1}$ is established. So we can apply a DFS technique to a Generic Market Model analogously as we proceed in the LMM and the COTSMM.

Chapter 4

Drift-Free Simulation techniques in a multicurve context

4.1 Introduction

Among other consequences, the financial crisis that exploded in August 2007 affected the until then standard foundations of interest rates modeling. Prior to the crisis, the use of a single yield curve, both for discounting and projecting (or forwarding), was enough to describe the whole market of interbanking interest rate products.

Since August 2007, some changes in the market rates showed situations that seemed to be arbitrage violations: basis swap spreads, between two different Euribor rates, or between Euribor and Eonia (Euro Over Night Index Average) rates, increased; FRA and two deposit rates became no longer equivalent (that is, $T_1 \times T_2$ FRA and deposits to T_1 and T_2); interest rate swaps linked to different tenors became different (and ordered from the lowest frequency to the highest frequency), etc. These new rate situations can no longer be reproduced with one single interest rate curve.

Moreover, the market is now considering credit and liquidity risks that were negligible before the crisis (see [6, 50], for example).

For instance, lending cash once at a six-month rate, and lending cash twice at rolling three-month rates are no longer equivalent. The lender prefers three-month rolling because in three months he/she has the option of changing to a higher rated counterparty (in case of downgrade); and he/she has also the option of keeping the money in case of liquidity problems after three months.

Moreover, the use of collateral agreements in order to avoid counterparty risk has also increased. Most of these contracts include a Credit Support Annex (CSA) and then yield the overnight (Eonia) rate.

Therefore, a new framework, referred as Overnight Indexed Swap (OIS) discounting, dual curve, or CSA discounting has been developed. It is mainly based on the assumption of the coexistence of multiple curves:

- One curve for discounting;

- Jointly with different curves for projecting (associated to each tenor).

Clearly, the presence of multiple curves gives rise to the requirement of the corresponding volatility structures. However, the standard volatility quotes in the market are not enough in this multicurve setting, so that new techniques to obtain volatilities for non standard tenors are being developed (see [37]).

For collateralized contracts the standard discounting curve is now the Eonia; for non-collateralized ones the debate is still in progress and involves funding costs, impact of liquidity, Credit Value Adjustment (CVA) measurement, etc. (see [54], for example).

This new framework requires a reformulation of the most basic assumptions and pricing formulas (see [5, 43]), as well as a whole rethinking of the bootstrapping of

the yield curves (see [1]). Also, the dynamic models for multiple curves have to be developed afresh (see [35, 36, 44]).

In the next section, we extend the LMM to this multicurve setting. For this purpose, different alternative formulations are described and the pros and cons of each one are discussed. Also the corresponding DFS techniques are introduced and described.

4.2   LMM extension to multicurve setting and DFS techniques

As in previous chapters, we consider a tenor structure $\mathcal{T}:=\{T_0, T_1, ..., T_{N+1}\}$, with $T_0 = 0$, $T_j < T_k, 0 \le j < k \le N + 1$, and the corresponding accruals $\delta_j = T_{j+1} - T_j$, $0 \le j \le N$. For simplicity, we consider constant accruals $\delta_j = \Delta T$, for all j. Again, for $j = 0, ..., N + 1$, $B_j(t)$ denotes the price at time t of the instrument that pays 1 at time $T_j \geqslant t$, and $B_j(t)$ can be expressed in terms of the implicit forward rates $F_j(t)$, $j = 0, ..., N$, $t \leqslant T_j$, as indicated in (2.1). Let $V_j^{M_l}(t)$ denote the price at time $t \leqslant T_j$ of the instrument that pays $R(T_j, T_{j+M_l})M_l\Delta T$ at time T_{j+M_l}, with $R(T_j, T_{j+M_l})$ denoting the Euribor that has been fixed at time T_j.

Now, the aim is to obtain the joint evolution of the discount curve $\{B_j(t)\}_{j=1}^{N+1}$ and of the estimation curves $\{V_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$. The new terms, $\{V_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, can be expressed in terms of the implicit rates $\{L_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, where

$$L_j^{M_l}(t) = \frac{V_j^{M_l}(t)}{M_l\Delta T B_{j+M_l}(t)}, \quad j = 0, ..., N + 1 - M_l, \quad t \leqslant T_j. \tag{4.1}$$

Remark 4.2.1. *Notice that each $L_j^{M_l}(t)$ is a martingale under the corresponding probability measure Q^{j+M_l}. This is a relevant point when obtaining Black-like formulas for caplets (see [43]).*

The extension of LMM to multicurve context is mainly based on modeling the joint evolution of:

- $\{F_j(t)\}_{j=1}^N$,

- $\{L_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$,

either directly or through differences between them.

In this chapter we develop four ways to extend the LMM to the multicurve setting:

1. Modeling the joint evolution of the rates $\{F_j(t)\}_{j=1}^N$ and $\{L_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$.

2. Modeling the joint evolution of the rates $\{F_j(t)\}_{j=1}^N$ and the spreads[1] $\{S_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, where

$$S_j^{M_l}(t) = L_j^{M_l}(t) - F_j(t). \tag{4.2}$$

3. Modeling the joint evolution of the rates $\{L_j^{M_p}(t)\}_{j=1}^{N+1-M_p}$, for some index p, and the spreads $\{S_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $i = l, ..., x$.

4. Modeling the joint evolution of the rates $\{F_j(t)\}_{j=1}^N$ and the differences $\{J_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, where

$$J_j^{M_l}(t) = L_j^{M_l}(t) - L_j^{M_{l-1}}(t), \tag{4.3}$$

with $L_j^{M_0}(t) \equiv F_j(t)$.

The first three formulations have been presented by Mercurio in [43, 44], who also explained the relative advantages and drawbacks of them. We add the fourth formulation because it can cover the more realistic situations of positive differences between the implicit rates of consecutive curves.

---

[1]Note the difference between this notation for the spreads and the notation for the implicit swap rates in Section 3.2.

Remark 4.2.2. *The fourth formulation guarantees positive differences between the implicit rates of consecutive curves. Although this situation is observed in practice, as a further study we propose to analyze a formulation which shows the realistic preference of lending consecutively twice for three months instead of lending once during six months.*

In order to compare the different formulations, we take into account that the only market data are the volatilities of the implicit rates $\{L_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$. However, if more information were available the preferences would change because, even though there is not an explicit arbitrage between the level of rates, the market is showing an order relation in different tenors that we prefer to preserve in the formulation. In particular, when we use the Eonia curve as discounting curve, the market is showing that $L_j^{M_l}(t) > L_j^{M_{l-1}}(t) > F_j(t)$. In this sense, although the first formulation takes into consideration more market parameters, it does not guarantee the order shown in the market.

The second formulation considers the order relation $L_j^{M_l}(t) > F_j(t)$, although not the order between different implicit rates of estimation curves. Moreover the volatilities of the spreads are not explicitly quoted in the market, so that further techniques to approximate them have been proposed [44].

The third formulation takes into account the market volatilities of one of the estimation curves, although only accounts for the order between one estimation curve and the others. For the special (and important) case of just two modeled curves, i.e., $x = 1$, this formulation preserves the order and it is market model.

Finally, the fourth formulation we add preserves a correct order and it requires the development of the way to incorporate information about the volatilities not explicitly quoted in the market until now.

In any case, with the market data available so far, in all presented formulations this model seems overparameterized. If more market information is not available, then one can resort to simplifications of the volatility and correlation structures: flat structures, principal components, functional forms with reduced number of parameters, and so on.

Next, we formulate the models and we explain the DFS technique for each alternative. For all of them, the DFS technique is based on the no arbitrage condition that is equivalent to imposing that

$$\widehat{B}_j(t) \doteq \frac{B_j(t)}{\text{Num}(t)} \quad \text{and} \quad \widehat{V}_j^{M_l}(t) \doteq \frac{V_j^{M_l}(t)}{\text{Num}(t)}, \quad l = 1, ..., x, \tag{4.4}$$

are martingales under the probability measure associated to a previously chosen numeraire Num. The martingales $\widehat{B}_j(t)$ are analogous to the ones introduced in the previous chapters.

4.2.1 Multicurve LMM 1: modeling $\{F_j\}_{j=1}^{N}$ and $\{L_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$

Let $Z(t) = \left(W_1(t), ..., W_N(t), W_1^{L^{M_1}}(t), ..., W_{N+1-M_1}^{L^{M_1}}(t), ..., W_1^{L^{M_x}}(t), ..., W_{N+1-M_x}^{L^{M_x}}(t) \right)$ denote a correlated Dim-dimensional Wiener process in a certain filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with

$$Dim = N + \sum_{l=1}^{x}(N + 1 - M_l) \tag{4.5}$$

and let the associated correlations be denoted by

- $dW_r dW_s = \rho_{r,s}^{F,F}$,

- $dW_r dW_s^{L^{M_l}} = \rho_{r,s}^{F,L^{M_l}}$,

- $dW_r^{L^{M_l}} dW_s^{L^{M_n}} = \rho_{r,s}^{L^{M_l},L^{M_n}}$.

Moreover, let us assume lognormal dynamics for the implicit rates $\{F_j(t)\}_{j=1}^N$ and $\{L_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, under the probability measure $\mathbf{P}$. Hence, we assume the following dynamics:

$$dF_j(t) = \mu_j(t)F_j(t)dt + \sigma_j(t)F_j(t)dW_j(t), \quad j = 1, ..., N,$$
$$dL_j^{M_1}(t) = \mu_j^{L^{M_1}}(t)L_j^{M_1}(t)dt + \sigma_j^{L^{M_1}}(t)L_j^{M_1}(t)dW_j^{L^{M_1}}(t), \quad j = 1, ..., N+1-M_1,$$
$$\vdots$$
$$dL_j^{M_x}(t) = \mu_j^{L^{M_x}}(t)L_j^{M_x}(t)dt + \sigma_j^{L^{M_x}}(t)L_j^{M_x}(t)dW_j^{L^{M_x}}(t), \quad j = 1, ..., N+1-M_x,$$
$$(4.6)$$

where $\mu_j(t)$ and $\mu_j^{L^{M_l}}(t)$ denote the drift terms and $\sigma_j(t)$ and $\sigma_j^{L^{M_l}}(t)$ denote the volatility terms.

We are interested in the DFS simulation of $\{F_j(t)\}_{j=1}^N$ and $\{L_j^{M_l}(t)\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$. For the implicit forward rates $F_j(t)$, $j = 1, ..., N$, the DFS technique is the one described in the second chapter. Next, assuming that $\{\widehat{B}_j(t)\}_{j=1}^{N+1}$ and $\{F_j(t)\}_{j=1}^N$ have been computed in the best conditions (i.e., as positive terms, with the PDFS technique proposed in this thesis), the procedure to obtain the rest of terms for each $l = 1, ..., x$ is described in next paragraphs.

Taking into account that $\widehat{V}_j^{M_l}(t)$ and $\widehat{B}_{j+M_l}(t)$ are martingales under $\mathbf{P}$ and the following (no recurrence) relation (equivalent to (4.1)):

$$\widehat{V}_j^{M_l}(t) = M_l \Delta T L_j^{M_l}(t)\widehat{B}_{j+M_l}(t), \quad j = 1, ..., N+1-M_l, \qquad (4.7)$$

we obtain the following dynamics:

$$d\widehat{V}_j^{M_l}(t) = M_l \Delta T L_j^{M_l}(t)d\widehat{B}_{j+M_l}(t) + M_l \Delta T \widehat{B}_{j+M_l}(t)\sigma_j^{L^{M_l}}(t)L_j^{M_l}(t)dW_j^{L^{M_l}}(t) \quad (4.8)$$

that can be discretized by the Euler method to obtain the martingale $\widehat{V}_j^{M_l}(t)$. Next, by using $\widehat{V}_j^{M_l}(t)$, the corresponding implicit rate $L_j^{M_l}(t)$ by (4.1) can be obtained.

In order to guarantee a positive martingale $\widehat{V}_j^{M_l}(t)$, we propose the following log-Euler discretization:

$$\widehat{V}_j^{M_l}(t_{i+1}) = \widehat{V}_j^{M_l}(t_i) \exp\left(\frac{d\widehat{V}_j^{M_l}(t_i)}{\widehat{V}_j^{M_l}(t_i)} - \frac{1}{2} \frac{d\widehat{V}_j^{M_l}(t_i)}{\widehat{V}_j^{M_l}(t_i)} \frac{d\widehat{V}_j^{M_l}(t_i)}{\widehat{V}_j^{M_l}(t_i)} \right), \qquad (4.9)$$

where

$$\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} = \frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} + \sigma_j^{L^{M_l}}(t)dW_j^{L^{M_l}}(t) =$$

$$= \varepsilon_{k,j+M_l} \sum_{p \in \theta_{k,j+M_l}} \sigma_p(t)\left(1 - \frac{\widehat{B}_{p+1}(t)}{\widehat{B}_p(t)} \right) dW_p(t) + \sigma_j^{L^{M_l}}(t)dW_j^{L^{M_l}}(t), \qquad (4.10)$$

and

$$\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} \frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} = \left(\sum_{p_1,p_2 \in \theta_{k,j+M_l}} \sigma_{p_1}(t)\sigma_{p_2}(t)\left(1 - \frac{\widehat{B}_{p_1+1}(t)}{\widehat{B}_{p_1}(t)} \right)\left(1 - \frac{\widehat{B}_{p_2+1}(t)}{\widehat{B}_{p_2}(t)} \right)\rho_{p_1,p_2}^{F,F} + \right.$$

$$\left. + \sigma_j^{L^{M_l}}(t)^2 + 2\sigma_j^{L^{M_l}}(t)\varepsilon_{k,j+M_l} \sum_{p \in \theta_{k,j+M_l}} \sigma_p(t)\left(1 - \frac{\widehat{B}_{p+1}(t)}{\widehat{B}_p(t)} \right)\rho_{p,j}^{F,L^{M_l}} \right) dt. \qquad (4.11)$$

The values of $\varepsilon_{k,j+M_l}$ and $\theta_{k,j+M_l}$ are given by (2.6). The index k is such that $d\widehat{B}_k(t) = 0$.

As we have a no recursive algorithm that guarantees the positivity of the martingales $\widehat{V}_j^{M_l}(t)$, then the positivity of the implicit rates $L_j^{M_l}(t)$ is also ensured.

As in the previous chapters, it is possible to apply an adjustment at simulation level to ensure the martingale property of $\widehat{V}_j^{M_l}(t)$ after the discretization. This adjustment consists of multiplying each simulated martingale by the ratio of its value at time zero and its mean.

4.2.2 Multicurve LMM 2: modeling $\{F_j\}_{j=1}^N$ and $\{S_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$

Let $Z(t) = \left(W_1(t), ..., W_N(t), W_1^{S^{M_1}}(t), ..., W_{N+1-M_1}^{S^{M_1}}(t), ..., W_1^{S^{M_x}}(t), ..., W_{N+1-M_x}^{S^{M_x}}(t) \right)$ denote correlated Dim-dimensional Wiener process in a certain filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with Dim given by (4.5) and

- $dW_r dW_s = \rho_{r,s}^{F,F}$,

- $dW_r dW_s^{S^{M_l}} = \rho_{r,s}^{F,S^{M_l}}$,

- $dW_r^{S^{M_l}} dW_s^{S^{M_n}} = \rho_{r,s}^{S^{M_l},S^{M_n}}$.

Moreover, let us assume lognormal dynamics for the implicit rates $\{F_j\}_{j=1}^N$ and for the spreads $\{S_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, under the probability measure $\mathbf{P}$. Hence, we assume the following dynamics:

$$dF_j(t) = \mu_j(t) F_j(t) dt + \sigma_j(t) F_j(t) dW_j(t), \quad j = 1, ..., N,$$
$$dS_j^{M_1}(t) = \mu_j^{S^{M_1}}(t) S_j^{M_1}(t) dt + \sigma_j^{S^{M_1}}(t) S_j^{M_1}(t) dW_j^{S^{M_1}}(t), \quad j = 1, ..., N+1-M_1,$$
$$\vdots$$
$$dS_j^{M_x}(t) = \mu_j^{S^{M_x}}(t) S_j^{M_x}(t) dt + \sigma_j^{S^{M_x}}(t) S_j^{M_x}(t) dW_j^{S^{M_x}}(t), \quad j = 1, ..., N+1-M_x,$$

$$(4.12)$$

where $\mu_j(t)$ and $\mu_j^{S^{M_l}}(t)$ denote the drift terms and $\sigma_j(t)$ and $\sigma_j^{S^{M_l}}(t)$ denote the volatility terms.

Again, we are interested in the DFS simulation of $\{F_j\}_{j=1}^N$ and $\{S_j^{M_l}\}_{j=1}^{N+1-M_l}$ and we suppose that we have calculated $\{\widehat{B}_j(t)\}_{j=1}^{N+1}$ and $\{F_j(t)\}_{j=1}^N$ as positive terms with the PDFS technique. In what follows we explain how to obtain the rest of terms for each $l = 1, ..., x$.

Taking into account that $\widehat{V}_j^{M_l}(t)$ and $\widehat{B}_{j+M_l}(t)$ are martingales under $\mathbf{P}$ and the following (no recurrence) relation (equivalent to (4.1)):

$$\widehat{V}_j^{M_l}(t) = M_l \Delta T (S_j^{M_l}(t) + F_j(t)) \widehat{B}_{j+M_l}(t), \quad j = 1, ..., N + 1 - M_l, \tag{4.13}$$

we obtain the dynamics:

$$
\begin{aligned}
d\widehat{V}_j^{M_l}(t) &= M_l \Delta T \widehat{B}_{j+M_l}(t) \left(\sigma_j^{S^{M_l}}(t) S_j^{M_l}(t) dW_j^{S^{M_l}}(t) + \sigma_j(t) F_j(t) dW_j(t) \right) \quad + \\
&+ M_l \Delta T (S_j^{M_l}(t) + F_j(t)) d\widehat{B}_{j+M_l}(t),
\end{aligned}
\tag{4.14}
$$

that can be discretized by the Euler method to obtain the martingale $\widehat{V}_j^{M_l}(t)$. With $\widehat{V}_j^{M_l}(t)$ we can compute the corresponding spread $S_j^{M_l}(t)$ with the following expression (equivalent to (4.13)):

$$S_j^{M_l}(t) = \frac{\widehat{V}_j^{M_l}(t)}{M_l \Delta T \widehat{B}_{j+M_l}(t)} - F_j(t). \tag{4.15}$$

Next, the implicit rates $L_j^{M_l}(t) = S_j^{M_l}(t) + F_j(t)$ can be computed.

In order to obtain the positive martingales $\widehat{V}_j^{M_l}(t)$, we propose a log-Euler discretization like the one proposed in (4.9), where

$$
\begin{aligned}
\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} &= \frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} + \frac{1}{S_j^{M_l}(t) + F_j(t)} \left(\sigma_j^{S^{M_l}}(t) S_j^{M_l}(t) dW_j^{S^{M_l}}(t) + \sigma_j(t) F_j(t) dW_j(t) \right) = \\
&= \varepsilon_{k,j+M_l} \sum_{p \in \theta_{k,j+M_l}} \sigma_p(t) \left(1 - \frac{\widehat{B}_{p+1}(t)}{\widehat{B}_p(t)} \right) dW_p(t) + \\
&+ \frac{1}{S_j^{M_l}(t) + F_j(t)} \left(\sigma_j^{S^{M_l}}(t) S_j^{M_l}(t) dW_j^{S^{M_l}}(t) + \sigma_j(t) F_j(t) dW_j(t) \right),
\end{aligned}
\tag{4.16}
$$

and

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$$\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} \frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} = \left(\sum_{p_1,p_2\in\theta_{k,j+M_l}} \sigma_{p_1}(t)\sigma_{p_2}(t)\left(1-\frac{\widehat{B}_{p_1+1}(t)}{\widehat{B}_{p_1}(t)}\right)\left(1-\frac{\widehat{B}_{p_2+1}(t)}{\widehat{B}_{p_2}(t)}\right)\rho_{p_1,p_2}^{F,F}+ \right.$$

$$+\frac{1}{(S_j^{M_l}(t)+F_j(t))^2}\left(\sigma_j^{S^{M_l}}(t)^2 S_j^{M_l}(t)^2 + \sigma_j(t)^2 F_j(t)^2 + 2\sigma_j^{S^{M_l}}(t)\sigma_j(t)S_j^{M_l}(t)F_j(t)\rho_{j,j}^{F,S^{M_l}}\right)+$$

$$\left.+\frac{2\varepsilon_{k,j+M_l}}{S_j^{M_l}(t)+F_j(t)}\sum_{p\in\theta_{k,j+M_l}}\left(\sigma_j^{S^{M_l}}(t)S_j^{M_l}(t)\rho_{p,j}^{F,S^{M_l}}+\sigma_j(t)F_j(t)\rho_{p,j}^{F,F}\right)\sigma_p(t)\left(1-\frac{\widehat{B}_{p+1}(t)}{\widehat{B}_p(t)}\right)\right) dt.$$

$$(4.17)$$

Notice that we have a no recursive algorithm that guarantees the positivity of the martingales $\widehat{V}_j^{M_l}(t)$, although the positivity of the spreads $S_j^{M_l}(t)$ can not be ensured.

By combining (2.1) and (4.15), we have the following expression to compute the spreads:

$$S_j^{M_l}(t) = \frac{\widehat{V}_j^{M_l}(t)\widehat{B}_{j+1}(t) - M_l\widehat{B}_{j+M_l}(t)(\widehat{B}_j(t)-\widehat{B}_{j+1}(t))}{M_l\Delta T\widehat{B}_{j+M_l}(t)\widehat{B}_{j+1}(t)}. \qquad (4.18)$$

Therefore, since $M_l\Delta T\widehat{B}_{j+M_l}(t)\widehat{B}_{j+M_l}(t)$ is a positive term, if we ensure that

$$\widehat{V}_j^{M_l}(t) > \frac{M_l\widehat{B}_{j+M_l}(t)D_j(t)}{\widehat{B}_{j+1}(t)}, \qquad (4.19)$$

where $D_j(t)$ is given by (2.20), then the positivity of the spreads is guaranteed.

In order to obtain (4.19), we propose the following parameterization of $\widehat{V}_j^{M_l}(t)$:

$$\widehat{V}_j^{M_l}(t) = \frac{M_l \widehat{B}_{j+M_l}(t) D_j(t)}{\widehat{B}_{j+1}(t)} + e^{z_j^{M_l}(t)}. \tag{4.20}$$

Then, we simulate the new terms $z_j^{M_l}(t)$, the dynamics of which are given by

$$
\begin{aligned}
dz_j^{M_l}(t) &= g_1(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} + g_2(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} + g_3(t)\frac{dD_j(t)}{D_j(t)} \quad + \\
&+ g_4(t)\frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)} + g_5(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} \quad + \\
&+ g_6(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)}\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} + g_7(t)\frac{dD_j(t)}{D_j(t)}\frac{dD_j(t)}{D_j(t)} \quad + \\
&+ g_8(t)\frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)}\frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)} + g_9(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} \quad + \\
&+ g_{10}(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{dD_j(t)}{D_j(t)} + g_{11}(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)} \quad + \\
&+ g_{12}(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)}\frac{dD_j(t)}{D_j(t)} + g_{13}(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)}\frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)} \quad + \\
&+ g_{14}(t)\frac{dD_j(t)}{D_j(t)}\frac{d\widehat{B}_{j+1}(t)}{\widehat{B}_{j+1}(t)},
\end{aligned}
\tag{4.21}
$$

where $g_i(t) \equiv G_i(M_l, \widehat{V}_j^{M_l}(t), \widehat{B}_{j+M_l}(t), D_j(t), \widehat{B}_{j+1}(t))$ with

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$$G_1(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{Y_1(t)Y_4(t)}{Y_1(t)Y_4(t) - cY_2(t)Y_3(t)},$$

$$G_i(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{-cY_2(t)Y_3(t)}{Y_1(t)Y_4(t) - cY_2(t)Y_3(t)}, \quad i = 2, 3,$$

$$G_4(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{cY_2(t)Y_3(t)}{Y_1(t)Y_4(t) - cY_2(t)Y_3(t)},$$

$$G_5(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{-Y_1(t)^2 Y_4(t)^2}{2\big(Y_1(t)Y_4(t) - cY_2(t)Y_3(t)\big)^2},$$

$$G_i(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{-c^2 Y_2(t)^2 Y_3(t)^2}{2\big(Y_1(t)Y_4(t) - cY_2(t)Y_3(t)\big)^2}, \quad i = 6, 7,$$

$$G_8(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{-2cY_1(t)Y_2(t)Y_3(t)Y_4(t) + c^2 Y_2(t)^2 Y_3(t)^2}{2\big(Y_1(t)Y_4(t) - cY_2(t)Y_3(t)\big)^2},$$

$$G_i(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = G(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)), \quad i = 9, 10, 13, 14$$

$$G_i(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = -G(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)), \quad i = 11, 12, \quad (4.22)$$

with

$$G(c, Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = \frac{cY_1(t)Y_2(t)Y_3(t)Y_4(t)}{\big(Y_1(t)Y_4(t) - cY_2(t)Y_3(t)\big)^2}. \quad (4.23)$$

Again, it is also possible to apply an adjustment on $z_j^{M_l}(t)$ such that $V_j^{M_l}(t)$ is martingale after the discretization. By denoting by N_S the number of the simulations, the obtained value for each $z_j^{M_l}(t)[p]$ is replaced by

$$z_j^{M_l}(t)[p] + \alpha_j^{M_l}(t), \quad (4.24)$$

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where

$$\alpha_j^{M_l}(t) = \ln \left(\frac{\widehat{V}_j^{M_l}(0) - \frac{1}{N_S} \sum_{p=1}^{N_S} \frac{M_l \widehat{B}_{j+M_l}(t)[p] D_j(t)[p]}{\widehat{B}_{j+1}(t)[p]}}{\frac{1}{N_S} \sum_{p=1}^{N_S} e^{z_j^{M_l}(t)[p]}} \right). \tag{4.25}$$

4.2.3 Multicurve LMM 3: modeling $\{L_j^{M_p}\}_{j=1}^{N+1-M_p}$ and $\{F_j\}_{j=N+2-M_p}^{N}$, for some index p, and $\{S_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$

Let $Z(t) = \left(W_1^{L^{M_p}}(t), ..., W_{N+1-M_p}^{L^{M_p}}(t), W_{N+2-M_p}(t), ..., W_N(t), W_1^{S^{M_1}}(t), ..., W_{N+1-M_1}^{S^{M_1}}(t), ..., W_1^{S^{M_x}}(t), ..., W_{N+1-M_x}^{S^{M_x}}(t) \right)$ denote a correlated Dim-dimensional Wiener process in a certain filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with Dim given by (4.5) and

- $dW_r dW_s = \rho_{r,s}^{F,F}$,

- $dW_r^{L^{M_p}} dW_s^{L^{M_p}} = \rho_{r,s}^{L^{M_p},L^{M_p}}$,

- $dW_r^{S^{M_l}} dW_s^{S^{M_n}} = \rho_{r,s}^{S^{M_l},S^{M_n}}$,

- $dW_r dW_s^{L^{M_p}} = \rho_{r,s}^{F,L^{M_p}}$,

- $dW_r dW_s^{S^{M_n}} = \rho_{r,s}^{F,S^{M_n}}$,

- $dW_r^{L^{M_p}} dW_s^{S^{M_n}} = \rho_{r,s}^{L^{M_p},S^{M_n}}$.

Moreover, let us assume lognormal dynamics for the implicit rates $\{L_j^{M_p}\}_{j=1}^{N+1-M_p}$ and $\{F_j\}_{j=N+2-M_p}^{N}$, for some index p, and $\{S_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, under the probability measure $\mathbf{P}$. Hence, we assume the following dynamics:

$$dL_j^{M_p}(t) = \mu_j^{L^{M_p}}(t)L_j^{M_p}(t)dt + \sigma_j^{L^{M_p}}(t)L_j^{M_p}(t)dW_j^{L^{M_p}}(t), \quad j = 1, ..., N+1-M_p,$$

$$dF_j(t) = \mu_j(t)F_j(t)dt + \sigma_j(t)F_j(t)dW_j(t), \quad j = N+2-M_p, ..., N,$$

$$dS_j^{M_1}(t) = \mu_j^{S^{M_1}}(t)S_j^{M_1}(t)dt + \sigma_j^{S^{M_1}}(t)S_j^{M_1}(t)dW_j^{S^{M_1}}(t), \quad j = 1, ..., N+1-M_1,$$

$$\vdots$$

$$dS_j^{M_x}(t) = \mu_j^{S^{M_x}}(t)S_j^{M_x}(t)dt + \sigma_j^{S^{M_x}}(t)S_j^{M_x}(t)dW_j^{S^{M_x}}(t), \quad j = 1, ..., N+1-M_x,$$

$$(4.26)$$

where $\mu_j^{L^{M_p}}(t)$, $\mu_j(t)$ and $\mu_j^{S^{M_l}}(t)$ denote the drift terms and $\sigma_j^{L^{M_p}}(t)$, $\sigma_j(t)$ and $\sigma_j^{S^{M_l}}(t)$ denote the volatility terms.

In this case, the equations that appear by assuming the no-arbitrage conditions do not allow isolating particular conditions for each of the variables. In particular, we are interested in the DFS simulation of $\{L_j^{M_p}\}_{j=1}^{N+1-M_p}$ and $\{F_j\}_{j=N+2-M_p}^{N}$, for some index p, and $\{S_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$. Unlike the other cases, in this one it is not possible to use the DFS methodology for the implicit forward rates $F_j(t)$ from the Chapter 2, so we do not deal with the third model.

4.2.4 Multicurve LMM 4: modeling $\{F_j\}_{j=1}^{N}$ and $\{J_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$

Let $Z(t) = \left(W_1(t), ..., W_N(t), W_1^{J^{M_1}}(t), ..., W_{N+1-M_1}^{J^{M_1}}(t), ..., W_1^{J^{M_x}}(t), ..., W_{N+1-M_x}^{J^{M_x}}(t)\right)$ denote a correlated Dim-dimensional Wiener process in a certain filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with Dim given by (4.5) and the following correlation structures:

- $dW_r dW_s = \rho_{r,s}^{F,F}$,

- $dW_r dW_s^{J^{M_l}} = \rho_{r,s}^{F,J^{M_l}}$,

- $dW_r^{J^{M_l}} dW_s^{J^{M_n}} = \rho_{r,s}^{J^{M_l}, J^{M_n}}$.

Moreover, let us assume lognormal dynamics for the implicit rates $\{F_j\}_{j=1}^N$ and for the spreads $\{J_j^{M_l}\}_{j=1}^{N+1-M_l}$, $l = 1, ..., x$, under the probability measure $\mathbf{P}$. Hence, we assume the following dynamics:

$$dF_j(t) = \mu_j(t) F_j(t) dt + \sigma_j(t) F_j(t) dW_j(t), \quad j = 1, ..., N,$$

$$dJ_j^{M_1}(t) = \mu_j^{J^{M_1}}(t) J_j^{M_1}(t) dt + \sigma_j^{J^{M_1}}(t) J_j^{M_1}(t) dW_j^{J^{M_1}}(t), \quad j = 1, ..., N+1-M_1,$$

$$\vdots$$

$$dJ_j^{M_x}(t) = \mu_j^{J^{M_x}}(t) J_j^{M_x}(t) dt + \sigma_j^{J^{M_x}}(t) J_j^{M_x}(t) dW_j^{J^{M_x}}(t), \quad j = 1, ..., N+1-M_x,$$

$$(4.27)$$

where $\mu_j(t)$ and $\mu_j^{J^{M_l}}(t)$ denote the drift terms and $\sigma_j(t)$ and $\sigma_j^{J^{M_l}}(t)$ denote the volatility terms.

Again, we are interested in the DFS technique of $\{F_j\}_{j=1}^N$ and $\{J_j^{M_l}\}_{j=1}^{N+1-M_l}$ and we assume that we have calculated $\{\widehat{B}_j(t)\}_{j=1}^{N+1}$ and $\{F_j(t)\}_{j=1}^N$ as positive terms with the PDFS technique. Next, we describe how to obtain the rest of terms for each $l = 1, ..., x$.

Taking into account that $\widehat{V}_j^{M_l}(t)$ and $\widehat{B}_{j+M_l}(t)$ are martingales under $\mathbf{P}$ and the following (no recurrence) relation (equivalent to (4.1)):

$$\widehat{V}_j^{M_l}(t) = M_l \Delta T \left(\sum_{s=1}^l J_j^{M_s}(t) + F_j(t) \right) \widehat{B}_{j+M_l}(t), \quad j = 1, ..., N+1-M_l, \quad (4.28)$$

we obtain the following dynamics:

$$d\widehat{V}_j^{M_l}(t) = M_l \Delta T \left(\sum_{s=1}^l J_j^{M_s}(t) + F_j(t) \right) d\widehat{B}_{j+M_l}(t) + \qquad (4.29)$$

$$+ \quad M_i \Delta T \widehat{B}_{j+M_l}(t) \left(\sum_{s=1}^l \sigma_j^{J^{M_s}}(t) J_j^{M_s}(t) dW_j^{J^{M_s}}(t) + \sigma_j(t) F_j(t) dW_j(t) \right),$$

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that can be discretized by the Euler method to obtain the martingale $\widehat{V}_j^{M_l}(t)$. By using this martingale, we compute the corresponding spread $J_j^{M_l}(t)$ defined in (4.3) by the following expression (equivalent to (4.28)):

$$J_j^{M_l}(t) = \frac{\widehat{V}_j^{M_l}(t)}{M_l \Delta T \widehat{B}_{j+M_l}(t)} - \Big(\sum_{s=1}^{l-1} J_j^{M_s}(t) + F_j(t) \Big). \qquad (4.30)$$

Next, the implicit rates $L_j^{M_l}(t) = \sum_{s=1}^{l} J_j^{M_s}(t) + F_j(t)$ can be computed.

However, in order to obtain a positive martingale $\widehat{V}_j^{M_l}(t)$, we propose a log-Euler discretization like the one proposed in (4.9), where

$$\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} = \frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} +$$
$$+ \frac{1}{\displaystyle\sum_{s=1}^{l} J_j^{M_s}(t) + F_j(t)} \left(\sum_{s=1}^{l} \sigma_j^{J^{M_s}}(t) J_j^{M_s}(t) dW_j^{J^{M_s}}(t) + \sigma_j(t) F_j(t) dW_j(t) \right) =$$
$$= \varepsilon_{k,j+M_l} \sum_{p \in \theta_{k,j+M_l}} \sigma_p(t) \left(1 - \frac{\widehat{B}_{p+1}(t)}{\widehat{B}_p(t)} \right) dW_p(t) +$$
$$+ \frac{1}{\displaystyle\sum_{s=1}^{l} J_j^{M_s}(t) + F_j(t)} \left(\sum_{s=1}^{l} \sigma_j^{J^{M_s}}(t) J_j^{M_s}(t) dW_j^{J^{M_s}}(t) + \sigma_j(t) F_j(t) dW_j(t) \right),$$

$$(4.31)$$

and

$$\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} = \left(\sum_{p_1,p_2\in\theta_{k,j+M_l}} \sigma_{p_1}(t)\sigma_{p_2}(t)\left(1-\frac{\widehat{B}_{p_1+1}(t)}{\widehat{B}_{p_1}(t)}\right)\left(1-\frac{\widehat{B}_{p_2+1}(t)}{\widehat{B}_{p_2}(t)}\right)\rho_{p_1,p_2}^{F,F}+\right.$$

$$+\frac{1}{\left(\sum_{s=1}^{l} J_j^{M_s}(t)+F_j(t)\right)^2}\left(\sum_{s_1,s_2=1}^{i} \sigma_j^{JM_{s_1}}(t)\sigma_j^{JM_{s_2}}(t)J_j^{M_{s_1}}(t)J_j^{M_{s_2}}(t)\rho_{j,j}^{JM_{s_1},JM_{s_2}}+\right.$$

$$+\quad \sigma_j(t)^2 F_j(t)^2 + 2\sigma_j(t)F_j(t)\sum_{s=1}^{l}\sigma_j^{JM_s}(t)J_j^{M_s}(t)\rho_{j,j}^{F,JM_s})+$$

$$+\quad \frac{2\varepsilon_{k,j+M_l}}{\sum_{s=1}^{l} J_j^{M_s}(t)+F_j(t)}\sum_{p\in\theta_{k,j+M_l}}(\sum_{s=1}^{l}\sigma_j^{JM_s}(t)J_j^{M_s}(t)\rho_{p,j}^{F,JM_s}+$$

$$+\sigma_j(t)F_j(t)\rho_{p,j}^{F,F})\sigma_p(t)\left(1-\frac{\widehat{B}_{p+1}(t)}{\widehat{B}_p(t)}\right)\right)\right) dt. \tag{4.32}$$

Notice that we have a no recursive algorithm that guarantees the positivity of the martingales $\widehat{V}_j^{M_l}(t)$, although the positivity of $J_j^{M_l}(t)$ defined by (4.30) is not ensured.

By combining (2.1) and (4.3), we have the following expression to compute the consecutive differences:

$$J_j^{M_l}(t) = \begin{cases} \frac{\widehat{V}_j^{M_l}(t)\widehat{B}_{j+1}(t)-M_l\widehat{B}_{j+M_l}(t)(\widehat{B}_j(t)-\widehat{B}_{j+1}(t))}{M_l\Delta T\widehat{B}_{j+M_l}(t)\widehat{B}_{j+1}(t)}, & \text{if } l=1; \\[3mm] \frac{\widehat{V}_j^{M_l}(t)M_{l-1}\widehat{B}_{j+M_{l-1}}(t)-\widehat{V}_j^{M_{l-1}}(t)M_l\widehat{B}_{j+M_l}(t)}{\Delta TM_{l-1}M_l\widehat{B}_{j+M_{l-1}}(t)\widehat{B}_{j+M_l}(t)}, & \text{if } l>1. \end{cases} \tag{4.33}$$

Therefore, the $l=1$ case is like the $l=1$ case in Section 4.2.2, and for the $l>1$ case, since $\Delta TM_{l-1}M_l\widehat{B}_{j+M_{l-1}}(t)\widehat{B}_{j+M_l}(t)$ is a positive term, if we ensure that

$$\widehat{V}_j^{M_l}(t) > \frac{M_l\widehat{V}_j^{M_{l-1}}(t)\widehat{B}_{j+M_l}(t)}{M_{l-1}\widehat{B}_{j+M_{l-1}}(t)}, \tag{4.34}$$

then the positivity of the differences $J_j^{M_l}(t)$ is guaranteed.

In order to obtain (4.34), we propose the following parameterization of $\widehat{V}_j^{M_l}(t)$:

$$\widehat{V}_j^{M_l}(t) = \frac{M_l \widehat{V}_j^{M_{l-1}}(t) \widehat{B}_{j+M_l}(t)}{M_{l-1} \widehat{B}_{j+M_{l-1}}(t)} + e^{y_j^{M_l}(t)}. \tag{4.35}$$

Then, we simulate the new terms $y_j^{M_l}(t)$, the dynamics of which are given by

$$
\begin{aligned}
dy_j^{M_l}(t) \;=\; & h_1(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} + h_2(t)\frac{d\widehat{V}_j^{M_{l-1}}(t)}{\widehat{V}_j^{M_{l-1}}(t)} + h_3(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} \quad + \\
+\;\; & h_4(t)\frac{d\widehat{B}_{j+M_{l-1}}(t)}{\widehat{B}_{j+M_{l-1}}(t)} + h_5(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)} \quad + \\
+\;\; & h_6(t)\frac{d\widehat{V}_j^{M_{l-1}}(t)}{\widehat{V}_j^{M_{l-1}}(t)}\frac{d\widehat{V}_j^{M_{l-1}}(t)}{\widehat{V}_j^{M_{l-1}}(t)} + h_7(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)}\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} \quad + \\
+\;\; & h_8(t)\frac{d\widehat{B}_{j+M_{l-1}}(t)}{\widehat{B}_{j+M_{l-1}}(t)}\frac{d\widehat{B}_{j+M_{l-1}}(t)}{\widehat{B}_{j+M_{l-1}}(t)} + h_9(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{V}_j^{M_{l-1}}(t)}{\widehat{V}_j^{M_{l-1}}(t)} \quad + \\
+\;\; & h_{10}(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} + h_{11}(t)\frac{d\widehat{V}_j^{M_l}(t)}{\widehat{V}_j^{M_l}(t)}\frac{d\widehat{B}_{j+M_{l-1}}(t)}{\widehat{B}_{j+M_{l-1}}(t)} \quad + \\
+\;\; & h_{12}(t)\frac{d\widehat{V}_j^{M_{l-1}}(t)}{\widehat{V}_j^{M_{l-1}}(t)}\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)} + h_{13}(t)\frac{d\widehat{V}_j^{M_{l-1}}(t)}{\widehat{V}_j^{M_{l-1}}(t)}\frac{d\widehat{B}_{j+M_{l-1}}(t)}{\widehat{B}_{j+M_{l-1}}(t)} \quad + \\
+\;\; & h_{14}(t)\frac{d\widehat{B}_{j+M_l}(t)}{\widehat{B}_{j+M_l}(t)}\frac{d\widehat{B}_{j+M_{l-1}}(t)}{\widehat{B}_{j+M_{l-1}}(t)}, \tag{4.36}
\end{aligned}
$$

where $h_i(t) \equiv G_i(\frac{M_l}{M_{l-1}}, \widehat{V}_j^{M_l}(t), \widehat{V}_j^{M_{l-1}}(t), \widehat{B}_{j+M_l}(t), \widehat{B}_{j+M_{l-1}}(t))$ with the functions G_i given by (4.22).

In this case, it is also possible to apply an adjustment on $y_j^{M_l}(t)$ at simulation level, such that $V_j^{M_l}(t)$ is martingale. Thus, the value obtained for each $y_j^{M_l}(t)[p]$ is replaced by

$$y_j^{M_l}(t)[p] + \beta_j^{M_l}(t), \tag{4.37}$$

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where

$$\beta_j^{M_l}(t) = \ln \left(\frac{\widehat{V}_j^{M_l}(0) - \frac{1}{N_S} \sum_{p=1}^{N_S} \dfrac{M_l \widehat{V}_j^{M_{l-1}}(t)[p] \widehat{B}_{j+M_l}(t)[p]}{M_{l-1} \widehat{B}_{j+M_{l-1}}(t)[p]}}{\frac{1}{N_S} \sum_{p=1}^{N_S} e^{y_j^{M_l}(t)[p]}} \right). \qquad (4.38)$$

Part II

Drift-Free Simulation Techniques for Cross-Market Models

Chapter 5

Cross-Market Models and Drift-Free Simulation techniques

In the context of Cross-Markets we have two economies A and B that are measured in two different units. For example:

- in the cross-currency case the units are the domestic currency and a foreign one, so we can consider the domestic and the foreign economies plus the forward exchange rates that relate them [45],

- in the commodity case the units are the domestic currency and one commodity, so we can consider the nominal and the commodity economies plus the forward prices of the commodity that relate them [58],

- in the inflation case the units are the domestic currency and one basket of basic products, so we can consider the nominal and the real economies plus the forward inflation rates that relate them [16].

The aim of Cross-Market Models is to obtain the evolution of the basic assets of both economies. With the same tenor structure of dates considered in previous chapters, the basic assets are:

- $B_j^A(t)$ (for $j = 1, ..., N+1$ and $t \leqslant T_j$), that is the price at time t in units of the economy A of the product that pays one unity of the economy A at time T_j,

- $B_j^B(t)$ (for $j = 1, ..., N+1$ and $t \leqslant T_j$), that is the price at time t in units of the economy B of the product that pays one unity of the economy B at time T_j.

These basic assets can be expressed by means of implicit forward rates F_j^A and F_j^B in the form:

$$\frac{B_j^A(t)}{B_{j+1}^A(t)} = 1 + \delta_j F_j^A(t), \quad \frac{B_j^B(t)}{B_{j+1}^B(t)} = 1 + \delta_j F_j^B(t), \tag{5.1}$$

with $j = 1, ..., N$.

Moreover, we also have the rates that connect both economies allowing us to change from units of the economy A to units of the economy B, and viceversa. These are the spot exchange rate $X^{AB}(t)$ (the price in units of the economy A that you have to pay at time t to receive one unity of the economy B at the same time t), and the implicit forward exchange rates $FX_j^{AB}(t)$, for $j = 1, ..., N+1$ (the agreed price at time $t \leqslant T_j$ in units of the economy A that you have to pay at a future time T_j to receive one unity of the economy B at the same time T_j). The implicit forward exchange rates can be calculated as follows:

$$FX_j^{AB}(t) = \frac{B_j^B(t)X^{AB}(t)}{B_j^A(t)}. \tag{5.2}$$

Notice that $FX_j^{AB}(T_j) = X^{AB}(T_j)$.

In this setting we have to model N forward rates of the economy A, F_j^A, N forward rates of the economy B, F_j^B, and the $N+1$ forward exchange rates that relate both economies, FX_j^{AB}. Therefore, in general we have to choose between $3N+1$ rates to formulate our Cross-Market Model.

If we present all these rates in a graph G as indicated in Figure 5.1, where

$$\frac{1}{1 + \delta_j F_j^A(t)} FX_{j+1}^{AB}(t) = FX_j^{AB}(t) \frac{1}{1 + \delta_j F_j^B(t)}, \tag{5.3}$$

by using language of graph theory we can demonstrate that we have to choose for our model the rates corresponding to the $2N + 1$ edges of any of the possible spanning trees of G, see [8, 23]. In this way, we get a model without redundancies (because we do not have cycles), so that the other remaining N rates can be calculated (because all nodes are connected).
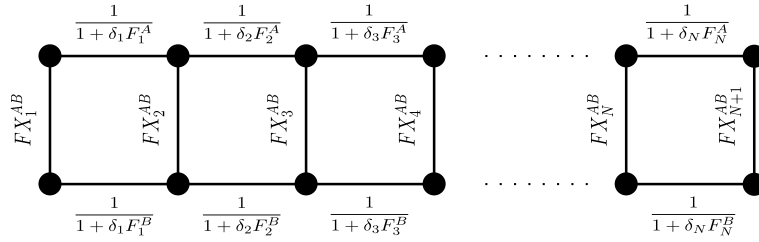
Figure 5.1: Graph associated to the rates appearing in Cross-Market Models.

In our three examples, the economy A is measured in the domestic currency. As we are in the LMM context we choose to model all the implicit forward rates of this economy. The rest of the edges depend on the context where we are, the market data and so on. Once we choose to model all the domestic forward Libor rates, we have ϕ_N choices, where

$$\phi_N = 2\frac{\varphi^{2N-1} - (1 - \varphi)^{2N-1}}{\sqrt{5}} + 3\frac{\varphi^{2N-2} - (1 - \varphi)^{2N-2}}{\sqrt{5}} + \frac{\varphi^{2N-3} - (1 - \varphi)^{2N-3}}{\sqrt{5}} \quad (5.4)$$

with $\varphi = \frac{1+\sqrt{5}}{2}$ (see Appendix B, for details).

In this work we present two of these alternative formulations, see Figures 5.2 and 5.3, and we use them for the three cross-markets we consider: cross-currency, commodity and inflation.

As explained later, the model in Figure 5.2 will be the one chosen for the two currencies context, while in the commodities setting the chosen model is sketched in Figure 5.3. These choices are motivated by the quoted assets in the market in each
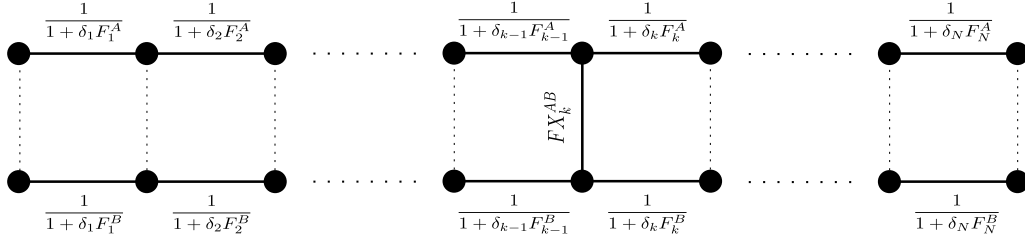
Figure 5.2: The First Cross-Market Model. We model all the implicit forward rates of both economies, F_j^A and F_j^B, $j = 1, ..., N$, and the forward exchange rate FX_k^{AB} that is martingale.
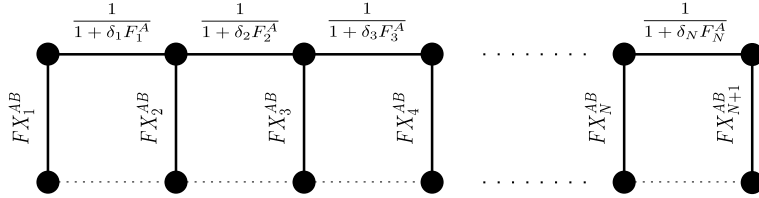
Figure 5.3: The Second Cross-Market Model. We model all the implicit forward rates of the economy A, F_j^A, $j = 1, ..., N$, and all the forward exchange rates, FX_j^{AB}, $j = 1, ..., N + 1$.

case. The liquidity of the inflation market is quite low, so the two models are usually presented as possible alternatives, see [42]. In Figure 5.4 the quotes are indicated.

In the next chapters we explain the formulation of these two models and we extend the DFS procedures explained in Chapter 2 to these Cross-Market Models. In both cases, once we have chosen the numeraire Num (into the economy A), the DFS methodology is based on the simulation of the following deflated bonds:

$$\widehat{B}_j^A(t) \doteq \frac{B_j^A(t)}{\text{Num}(t)}, \quad \widehat{B}_j^B(t) \doteq \frac{B_j^B(t)X^{AB}(t)}{\text{Num}(t)}, \qquad t \le T_j, \qquad j = 1, \dots, N+1, \quad (5.5)$$

which are martingales under the probability measure associated to the chosen numeraire Num. We only consider one economy A bond or the bank account of the economy A as possible numeraires, so that in both cases the DFS procedure for the economy A part is completely analogous to the technique indicated in Chapter 2.

CROSS-CURRENCY

t T_j

$B_j^d(t)$

EURO

$X^{df}(t)$ $FX_j^{df}(t)$

DOLLAR

$B_j^f(t)$

COMMODITIES

t T_j

$B_j^d(t)$

EURO

$X^{dc}(t)$ $FX_j^{dc}(t)$

COMMODITY

$B_j^c(t)$

INFLATION

t T_j

$B_j^d(t)$

EURO

$X^{db}(t)$ $FX_j^{db}(t)$

BASKET

$B_j^b(t)$
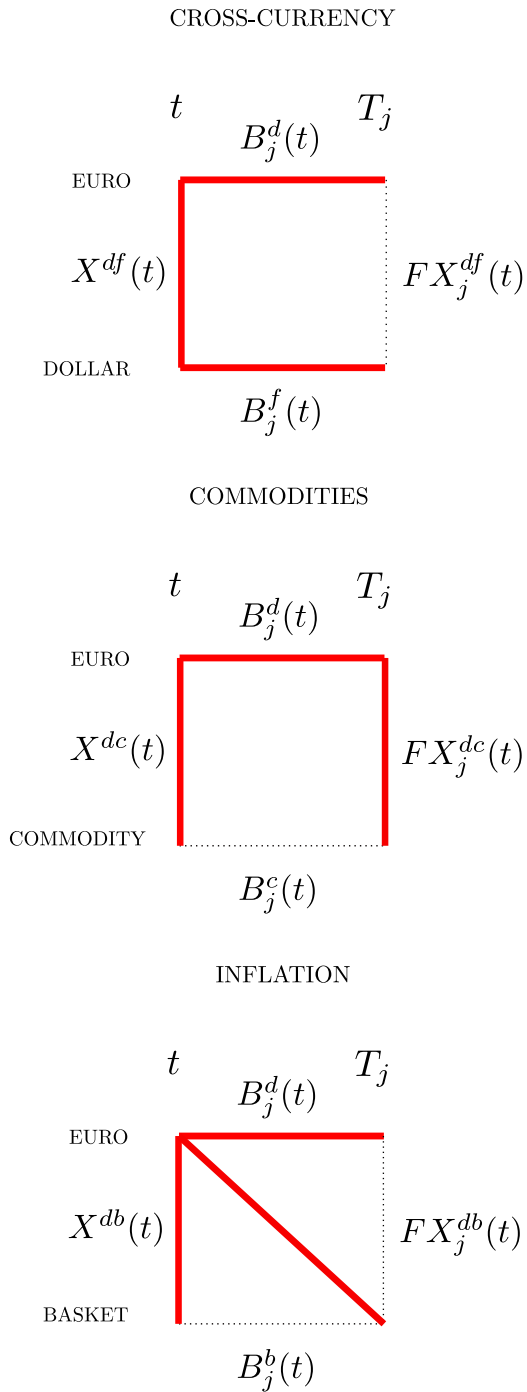
Figure 5.4: The assets that quote in the market are marked by a thicker red line.

Chapter 6

Drift-Free Simulation techniques for a First Cross-Market Model

6.1 Introduction

In this chapter we present the formulation of the First Cross-Market Model as well as the associated DFS technique. This technique was proposed in [21] and it is based on the DFS technique for one economy and in the fact that the forward exchange rate whose dynamic is given by the First Cross-Market Model matches some of the economy B martingales. This fact allows to establish a bijective relation between the economy B forward rates and the economy B martingales, analogously to the one economy DFS methodology.

In [21] we also apply this technique to price derivatives that depend on forward rates of two currency markets [2, 16, 45, 59]. Quanto derivatives justify the need of two currencies models: as an example, we price a quanto caplet.

This chapter is organized as follows. Section 6.2 presents the formulation of the First Cross-Market Model while Section 6.3 describes the DFS methods for this model. Section 6.4 deals with the model calibration procedure. In Section 6.5 we expose the

dynamics of the rates that are not modeled. Finally, this methodology is applied to the two currencies context in Section 6.6.

6.2   Formulation

In this section we formulate the First Cross-Market Model. As previously indicated, in this case we model all the forward rates F_j^A and F_j^B, $j = 1, ..., N$, and the forward exchange rate FX_k^{AB} that is martingale. Thus, we choose FX_k^{AB} where k is the index such that $d\widehat{B}_k^A(t) = 0$, i.e., k is n if the numeraire is the bond B_n^A and it is j along the interval $(T_{j-1}, T_j]$ if the numeraire is the bank account of the economy A. We select the forward exchange rate that is martingale because we are interested in the simulation of these rates avoiding the use of the drift terms.

In order to formulate the model we denote by $Z(t) = (W_1^A(t), ..., W_N^A(t), W_1^B(t), ...,$ $W_N^B(t), W_k^{AB}(t))$ a correlated $(2N+1)$-dimensional Wiener process in a certain probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a correlation matrix

$$C = \begin{pmatrix} \Sigma^A & \rho & \gamma^A \\ \rho & \Sigma^B & \gamma^B \\ \gamma^A & \gamma^B & 1 \end{pmatrix} \quad \in \mathcal{M}^{(2N+1)\times(2N+1)},$$

so that Σ^A $(\Sigma^B) \in \mathcal{M}^{N \times N}$ denotes the matrix of correlations of the forward rates of the economy A (B), $\rho \in \mathcal{M}^{N \times N}$ denotes the matrix of correlations among the forward rates of the economy A and the forward rates of the economy B, and γ^A (γ^B) $\in \mathcal{M}^{N \times 1}$ denotes the matrix of correlations among the forward rates of the economy A (B) and the forward exchange rate. The natural filtration spanned by $Z(t)$ is denoted by $\mathcal{F}_t$. Every stochastic process we consider is defined on this probability space.

LMM assumes lognormal forward rates under any probability measure. Hence, we

assume the following system of dynamics:

$$dF_j^A(t) = \mu_j^A(t)F_j^A(t)dt + \sigma_j^A(t)F_j^A(t)dW_j^A(t), \quad j = 1, ..., N,$$

$$dF_j^B(t) = \mu_j^B(t)F_j^B(t)dt + \sigma_j^B(t)F_j^B(t)dW_j^B(t), \quad j = 1, ..., N,$$

$$dFX_k^{AB}(t) = \sigma_k^{AB}(t)FX_k^{AB}(t)dW_k^{AB}(t), \tag{6.1}$$

where $\mu_j^A(t)$ and $\mu_j^B(t)$ denote the drift terms, $\sigma_j^A(t)$, $\sigma_j^B(t)$ and $\sigma_k^{AB}(t)$ denote the volatility terms and $W_j^A(t)$, $W_j^B(t)$ and $W_k^{AB}(t)$ denote Brownian motions under $\mathbf{P}$, for the forward rates of the economy A, for the forward rates of the economy B and for the foward exhange rate, respectively. Notice that the drift term of the forward exchange rate is zero because we choose the forward exchange rate that is martingale.

The drift terms depend on the chosen probability measure. In fact, if we consider Q^n then $F_{n-1}^A(t)$ and $F_{n-1}^B(t)$ are martingales, and if we consider Q^β the terms that are martingales depend on the time interval, that is, $F_{j-1}^A(t)$ and $F_{j-1}^B(t)$ are martingales if $t \in (T_{j-1}, T_j]$. For these two particular measures we have:

$$\mu_j^A(t) = -\left(\varepsilon_{k,j+1} \sum_{l \in \theta_{k,j+1}} \frac{\delta_l \sigma_l^A(t)F_l^A(t)\Sigma_{lj}^A}{1 + \delta_l F_l^A(t)} \right) \sigma_j^A(t), \tag{6.2}$$

$$\mu_j^B(t) = -\left(\varepsilon_{k,j+1} \sum_{l \in \theta_{k,j+1}} \frac{\delta_l \sigma_l^B(t)F_l^B(t)\Sigma_{lj}^B}{1 + \delta_l F_l^B(t)} + \sigma_k^{AB}(t)\gamma_j^B \right) \sigma_j^B(t), \tag{6.3}$$

where k is the index such that $d\widehat{B}_k^A(t) = 0$ and $d\widehat{B}_k^B(t) = dFX_k^{AB}(t)$, see Remark 2.2.1, and $\varepsilon_{k,j+1}$ and $\theta_{k,j+1}$ are defined in (2.6). These values can be obtained by taking into account (5.1) and the martingales defined in (5.5).

We are interested in the simulation of these rates avoiding the use of the drift terms. For this purpose, in the next section we extend the DFS methodology for one economy to the First Cross-Market Model case.

6.3 DFS technique

Once we have chosen the numeraire Num, DFS procedures are based on the simulation of the terms $\widehat{B}_j^A, \widehat{B}_j^B, j = 1, ..., N+1$ (see (5.5)), that are martingales under the probability measure associated to the numeraire Num.

The DFS procedure for the economy A and the economy B is based on the following (recurrence) relations:

$$\widehat{B}_j^A(t) = \left(1 + \delta_j F_j^A(t)\right)\widehat{B}_{j+1}^A(t), \quad \widehat{B}_j^B(t) = \left(1 + \delta_j F_j^B(t)\right)\widehat{B}_{j+1}^B(t), \qquad (6.4)$$

connecting consecutive deflated bonds and a forward rate in both economies.

Then, DFS procedure (with or without adjustment) for the economy A is carried out as indicated in Chapter 2, by taking into account that now $F_j \equiv F_j^A$, $B_j \equiv B_j^A$ and $\widehat{B}_j \equiv \widehat{B}_j^A$. The construction of the economy B structure is totally analogous with one main difference: while the calculus of the martingale structure in the economy A starts from the known martingale dynamics $d\widehat{B}_k^A(t) = 0$ (remember Remark 2.2.1), the calculus in the economy B starts from the martingale dynamics $d\widehat{B}_k^B(t) = dFX_k^{AB}(t)$. Notice that we just need to assume lognormality for $FX_k^{AB}(t)$, which is consistent with assuming lognormality for all forward Libor rates (see [45], for example).

Therefore, for the IDFS a previous step to the calculation of the economy B martingales is the computation of $FX_k^{AB}(t)$ from its drift-free dynamics, which is given in (6.1). Once we know the value of $\widehat{B}_k^B(t) = FX_k^{AB}(t)$ we can compute the rest of the economy B martingales by the recurrence relation (6.4). As in the case of martingale $\widehat{B}_j^A$, we can apply the analogous adjustment to guarantee the martingale property of $\widehat{B}_j^B(t)$ at discrete level.

For the rest of the DFS procedures (EDFS, GZDFS and PDFS), we apply the methodology described in Chapter 2, although taking into account that now $d\widehat{B}_k^B(t) =$

$dFX_k^{AB}(t)$ and replacing (2.15) and (2.16) by

$$
\begin{aligned}
\frac{d\widehat{B}_j^B(t)}{\widehat{B}_j^B(t)} &= \frac{dFX_k^{AB}(t)}{FX_k^{AB}(t)} + \varepsilon_{k,j} \sum_{l \in \Theta_{k,j}} \left(\frac{d\widehat{B}_l^B(t)}{\widehat{B}_l^B(t)} - \frac{d\widehat{B}_{l+1}^B(t)}{\widehat{B}_{l+1}^B(t)} \right) \\
&= \frac{dFX_k^{AB}(t)}{FX_k^{AB}(t)} + \varepsilon_{k,j} \sum_{l \in \Theta_{k,j}} \sigma_l^B \left(1 - \frac{\widehat{B}_{l+1}^B(t)}{\widehat{B}_l^B(t)} \right) dW_l^B(t).
\end{aligned} \qquad (6.5)
$$

Moreover, while in the economy A case we calculate the numeraire and the discount bonds at the tenor dates with (2.12) and (2.13), in the economy B case we compute the spot exchange at time T_j with the expression

$$
X^{AB}(T_j) = \widehat{B}_j^B(T_j)\mathrm{Num}(T_j), \quad \forall j = 1, ..., N + 1. \qquad (6.6)
$$

Finally, we compute the economy B Zero Coupon curve at tenor dates with

$$
B_j^B(T_i) = \frac{\widehat{B}_j^B(T_i)\mathrm{Num}(T_i)}{X^{AB}(T_i)}, \quad i \leq j. \qquad (6.7)
$$

6.4   Model calibration

Since we are in the context of market models, we have to introduce into our model as parameters as much market information as feasible. We first introduce the following market data:

$$
\{B_j^A(0)\}_{j=0}^{N+1}, \quad \{B_j^B(0)\}_{j=0}^{N+1}, \quad X^{AB}(0).
$$

For this reason, the First Cross-Market Model is chosen for settings in which the quoted assets in the market are these terms, for example the cross-currency setting (see Figure 6.1).

Other free parameters (volatilities and correlations) are considered in next paragraphs so that the model parameters are adjusted to the market.
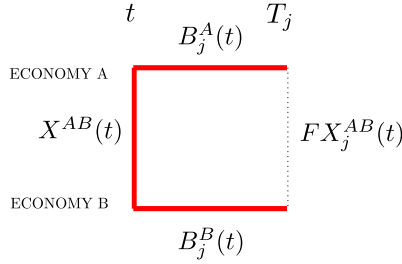
Figure 6.1: The assets that quote in the cross-currency markets are marked by a thicker red line.

- *Adjustment of volatilities:* The structure of volatilities is adjusted to market by imposing that

$$
\begin{aligned}
\int_0^{T_j} \sigma_j^A(t)^2 dt &= T_j(\sigma_j^{A^{mar}})^2, \\
\int_0^{T_j} \sigma_j^B(t)^2 dt &= T_j(\sigma_j^{B^{mar}})^2, \\
\int_0^{T_k} \sigma_k^{AB}(t)^2 dt &= T_k(\sigma_k^{AB^{mar}})^2,
\end{aligned}
\tag{6.8}
$$

where $\sigma_j^{A^{mar}}$ and $\sigma_j^{B^{mar}}$ are quoted volatilities of caplets of each economy and $\sigma_k^{AB^{mar}}$ is a quoted volatility of a call option on the forward exchange rate.

In practice, for simplicity, we choose constant volatilities, that is,

$$
\sigma_j^A(t) = \sigma_j^{A^{mar}}, \quad \sigma_j^B(t) = \sigma_j^{B^{mar}}, \quad \sigma_k^{AB}(t) = \sigma_k^{AB^{mar}}, \quad \forall t.
$$

- *Adjustment of correlations:* We have to adjust the following correlation matrix:

$$
C = \begin{pmatrix}
\Sigma_{11}^A & \cdots & \Sigma_{1N}^A & \rho_{11} & \cdots & \rho_{1N} & \gamma_1^A \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
\Sigma_{N1}^A & \cdots & \Sigma_{NN}^A & \rho_{N1} & \cdots & \rho_{NN} & \gamma_N^A \\
\rho_{11} & \cdots & \rho_{N1} & \Sigma_{11}^B & \cdots & \Sigma_{1N}^B & \gamma_1^B \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
\rho_{1N} & \cdots & \rho_{NN} & \Sigma_{N1}^B & \cdots & \Sigma_{NN}^B & \gamma_N^B \\
\gamma_1^A & \cdots & \gamma_N^A & \gamma_1^B & \cdots & \gamma_N^B & 1
\end{pmatrix}.
$$

First, we adjust the correlations within each economy, $(\Sigma_{ij}^A)_{i,j=1}^N$ and $(\Sigma_{ij}^B)_{i,j=1}^N$, by using the swaps volatilities from the market in each economy and taking into account the approximation (2.51) that relates the volatilities of forward and swap rates with the correlations between forward rates within an economy.

As the other correlations cannot be calculated analogously, we obtain them from historical data and assume they are constant, that is,

$$\rho_{ij} = \rho, \quad \gamma_i^A = \gamma^A, \quad \gamma_i^B = \gamma^B, \quad \forall i, j = 1, ..., N.$$

See Remark 2.3.1 to fix the matrix C if it is not actually a correlation matrix.

6.5 Dynamics of the remaining of forward exchange rates

In this section we present the dynamics of those forward exchange rates which are not modeled by the First Cross-Market Model.

In Figure 6.2 we can see that for all $j = 1, ..., N + 1$, with $j \neq k$, the forward exchange rate FX_j^{AB} depends on the forward exchange rate FX_k^{AB} and the forward rates F_p^A and F_p^B with $p \in \theta_{k,j}$, where $\theta_{k,j}$ is given by (2.6).

In fact, we have the following relations:

$$FX_j^{AB}(t) = \left(\prod_{p \in \theta_{k,j}} \frac{1 + \delta_p F_p^B(t)}{1 + \delta_p F_p^A(t)} \right) FX_k^{AB}(t), \quad \text{if } j < k,$$

$$FX_j^{AB}(t) = \left(\prod_{p \in \theta_{k,j}} \frac{1 + \delta_p F_p^A(t)}{1 + \delta_p F_p^B(t)} \right) FX_k^{AB}(t), \quad \text{if } j > k. \tag{6.9}$$
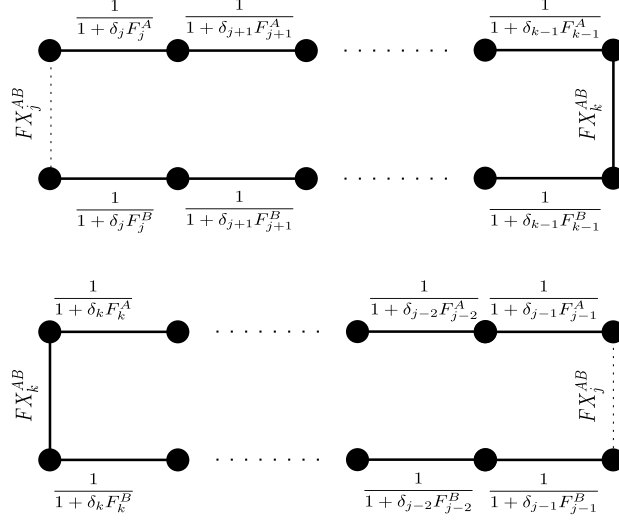
Figure 6.2: Relation between the different processes.

Therefore, by using Ito's Lemma, we have the following no lognormal dynamics:

$$
\begin{aligned}
\frac{dFX_j^{AB}(t)}{FX_j^{AB}(t)} = & \left[-\varepsilon_{k,j} \sum_{p \in \theta_{k,j}} \frac{\delta_p \mu_p^A(t) F_p^A(t)}{1 + \delta_p F_p^A(t)} + \varepsilon_{k,j} \sum_{p \in \theta_{k,j}} \frac{\delta_p \mu_p^B(t) F_p^B(t)}{1 + \delta_p F_p^B(t)} + \Gamma_{kj}(t) - \right. \\
& - \frac{1}{2} \varepsilon_{k,j} \sum_{p \in \theta_{k,j}} \frac{\delta_p \sigma_k^{AB}(t) \sigma_p^A(t) F_p^A(t) \gamma_p^A}{1 + \delta_p F_p^A(t)} + \frac{1}{2} \varepsilon_{k,j} \sum_{p \in \theta_{k,j}} \frac{\delta_p \sigma_k^{AB}(t) \sigma_p^B(t) F_p^B(t) \gamma_p^B}{1 + \delta_p F_p^B(t)} + \\
& + \frac{1}{2} \sum_{p,q \in \theta_{k,j}, p \neq q} \frac{\delta_p \delta_q \sigma_p^A(t) F_p^A(t) \sigma_q^A(t) F_q^A(t) \Sigma_{pq}^A}{(1 + \delta_p F_p^A(t))(1 + \delta_q F_q^A(t))} + \\
& + \frac{1}{2} \sum_{p,q \in \theta_{k,j}, p \neq q} \frac{\delta_p \delta_q \sigma_p^B(t) F_p^B(t) \sigma_q^B(t) F_q^B(t) \Sigma_{pq}^B}{(1 + \delta_p F_p^B(t))(1 + \delta_q F_q^B(t))} - \\
& \left. - \frac{1}{2} \sum_{p,q \in \theta_{k,j}} \frac{\delta_p \delta_q \sigma_p^A(t) F_p^A(t) \sigma_q^B(t) F_q^B(t) \rho_{pq}}{(1 + \delta_p F_p^A(t))(1 + \delta_q F_q^B(t))} \right] dt - \varepsilon_{k,j} \sum_{p \in \theta_{k,j}} \frac{\delta_p \sigma_p^A(t) F_p^A(t)}{1 + \delta_p F_p^A(t)} dW_p^A(t) + \\
& + \varepsilon_{k,j} \sum_{p \in \theta_{k,j}} \frac{\delta_p \sigma_p^B(t) F_p^B(t)}{1 + \delta_p F_p^B(t)} dW_p^B(t) + \sigma_k^{AB}(t) dW_k^{AB}(t), \quad (6.10)
\end{aligned}
$$

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where $\varepsilon_{k,j}$ is given by (2.6) and

$$\Gamma_{kj}(t) = \begin{cases} \displaystyle\sum_{p\in\theta_{k,j}} \frac{-\delta_p^2 \sigma_p^A(t)^2 F_p^A(t)^2}{(1+\delta_p F_p^A(t))^2}, & \text{if } j < k; \\[3ex] \displaystyle\sum_{p\in\theta_{k,j}} \frac{\delta_p^2 \sigma_p^B(t)^2 F_p^B(t)^2}{(1+\delta_p F_p^B(t))^2}, & \text{if } j > k. \end{cases} \tag{6.11}$$

6.6 Example of the First Cross-Market Model: Two currencies

Cross-Currency markets can be framed into Cross-Markets. As previously indicated, the First Cross-Market Model is chosen for the two currencies setting, in which the units are the domestic currency and a foreign one (see Figure 5.2). Therefore, we have the domestic and the foreign economies plus the forward exchange rates that relate them. So if $A \equiv d$ and $B \equiv f$, by analogy with Chapter 5, we can redefine B_j^d, B_j^f, F_j^d, F_j^f, X^{df} and FX_j^{df}. This is the usual notation in the bibliography.

In the next subsection, we present the cross-currency derivatives and explain why the cross-currency model is needed.

6.6.1 Cross-currency derivatives: Quanto Caplets

Nowadays, there are numerous financial derivatives on rates of different currency markets (see [16], for example). They can be mainly classified into:

- Standard cross-currency products (Cross-currency swap, Cross-crurrency swaption...): These contracts are agreements between two parties so that each part pays cash flows referred to one currency in the same currency.

- Quanto products (Quanto fra, Quanto caplet/floorlet, Quanto swap, Quanto

Swaption...): They consist of products that pay cash flows referred to one currency in another currency. As it is later explained, in the pricing of these derivatives there is one term that does not affect the pricing of the corresponding derivatives on one currency (fra, caplet/floorlet, swap, swaption,...). This term is known as quanto adjustment.

In view of the previous description, Quanto products justify the need of two currencies models, because for the other kind of products the pricing of the two payments can be made in each currency and then we can obtain the calculated values in the same currency by using the current value of the spot exchange rate. Here we present a Quanto Caplet/Floorlet pricing methodology.

Let $Q_j(t)$ denote the price at time t of a Quanto Caplet/Floorlet that matures at time T_j, so that it pays at T_{j+1}.

We distinguish two kinds of Quanto Caplets/Floorlets:

1. The Quanto Caplet/Floorlet with invested notional Not that pays at T_{j+1} the amount $\delta_j\big[\omega(F_j^f(T_j) - K^f)\big]_+$Not in the domestic currency, where the invested notional is paid back at the payment date and ω is $+1$ in the Quanto Caplet case and -1 in the Quanto Floorlet case. The payoff of this product is

$$Q_j(T_j) = \delta_j\big[\omega(F_j^f(T_j) - K^f)\big]_+ \text{Not} B_{j+1}^d(T_j). \tag{6.12}$$

Hence, for all $t < T_j$, the Quanto Caplet/Floorlet price is given by

$$Q_j(t) = B_{j+1}^d(t)\delta_j E_{Q^{j+1}}\left[\big[\omega(F_j^f(T_j) - K^f)\big]_+\big|\mathcal{F}_t\right]\text{Not}, \tag{6.13}$$

where Q^{j+1} is the probability measure associated to the numeraire $B_{j+1}^d(t)$. Moreover, we know that under Q^{j+1} we have

$$\frac{dF_j^f(t)}{F_j^f(t)} = -\gamma_{j,j+1}^f \sigma_j^f(t)\sigma_{j+1}^{fd}(t)dt + \sigma_j^f(t)dW_j^f(t), \tag{6.14}$$

where $\gamma^f_{j,j+1}$ denotes the correlation between F^f_j and FX^{df}_{j+1}. So, by a proposition in [16], p. 919, we obtain the Black formula

$$Q_j(t) = B^d_{j+1}(t)\delta_j\omega\left(F^f_j(t)e^{\Omega^f_j}\Phi(\omega d_+) - K^f\Phi(\omega d_-)\right)\text{Not}, \qquad (6.15)$$

where

$$\Omega^f_j = \int_t^{T_j} -\gamma^f_{j,j+1}\sigma^f_j(z)\sigma^{df}_{j+1}(z)dz,$$

$$d_\pm = \frac{\ln\left(\dfrac{F^f_j(t)}{K^f}\right) + \Omega^f_j \pm \frac{1}{2}\int_t^{T_j}\sigma^f_j(z)^2 dz}{\left(\int_t^{T_j}\sigma^f_j(z)^2 dz\right)^{\frac{1}{2}}},$$

and Φ denotes the cumulative standard normal distribution function.

2. The Quanto Caplet/Floorlet with invested notional Not that pays at T_{j+1} the amount $\delta_j\left[\omega(F^d_j(T_j) - K^d)\right]_+\text{Not}$ in the foreign currency, where ω is $+1$ in the Quanto Caplet case and -1 in the Quanto Floorlet case. The payoff of this derivative is

$$Q_j(T_j) = \delta_j\left[\omega(F^d_j(T_j) - K^d)\right]_+\text{Not}B^f_{j+1}(T_j)X^{df}(T_j). \qquad (6.16)$$

Hence, for all $t < T_j$, the Quanto Caplet/Floorlet price is given by

$$Q_j(t) = B^d_{j+1}(t)\delta_j E_{Q^{j+1}}\left[\left[\omega(F^d_j(T_j) - K^d)\right]_+ FX^{df}_{j+1}(T_j)|\mathcal{F}_t\right]\text{Not}. \qquad (6.17)$$

Moreover, we know that under Q^{j+1} we have

$$\frac{d\left(F^d_j(t)FX^{df}_{j+1}(t)\right)}{F^d_j(t)FX^{df}_{j+1}(t)} = \gamma^d_{j,j+1}\sigma^d_j(t)\sigma^{df}_{j+1}(t)dt + \sigma^d_j dW^d_j(t) + \sigma^{df}_{j+1}dW^{df}_{j+1}(t), \quad (6.18)$$

where $\gamma^d_{j,j+1}$ is the correlation between F^d_j and FX^{df}_{j+1}. So, using the same proposition of [16] that in the previous case, we can deduce the Black formula

$$Q_j(t) = B^d_{j+1}(t)\delta_j FX^{df}_{j+1}(t)\omega\left(F^d_j(t)e^{\Omega^d_j}\Phi(\omega d_+) - K^d\Phi(\omega d_-)\right)\text{Not}, \qquad (6.19)$$

with

$$\Omega_j^d = \int_t^{T_j} \gamma_{j,j+1}^d \sigma_j^d(z) \sigma_{j+1}^{df}(z) dz,$$

$$d_\pm = \frac{\ln\left(\dfrac{F_j^d(t)}{K^d}\right) + \Omega_j^d \pm \frac{1}{2} \int_t^{T_j} \sigma_j^d(z)^2 dz}{\left(\int_t^{T_j} \sigma_j^d(z)^2 dz\right)^{\frac{1}{2}}}.$$

Furthermore, we can obtain the value of Quanto Caplets/Floorlets at time t by using simulation under Q (probability measure associated to the numeraire Num) and taking into account that

$$Q_j(t) = \text{Num}(t) E_Q\left[\frac{Q_j(T_j)}{\text{Num}(T_j)}|\mathcal{F}_t\right]. \tag{6.20}$$

In next section we present the results obtained by applying the DFS procedures for pricing Quanto Caplets/Floorlets.

6.6.2 Numerical results

In this subsection, we present the same data used for the different DFS methods described in previous sections and the results for the martingales and caplet pricing in each currency (dollar and euro). We also present the results for the pricing of one Quanto Caplet. Thus, in Tables 6.1 to 6.10 we consider the dollar as the domestic currency and the euro as the foreign one.

Table 6.1 shows the data for the Monte Carlo simulation procedure. More precisely, we consider the case $N = 10$. The accrual period is constant and equal to one year. The time step and the number of simulations are also indicated. Polar-Marsaglia method is used for simulation of the Brownian motions. The domestic and foreign market data are given in Tables 6.2 and 6.3, respectively. They include the Zero Coupon Curves, the volatilities of the involved forward Libor and swap rates in

both economies. Table 6.4 shows the spot exchange rate and the volatilities of the forward exchange rates. As the correlation matrix is completed with constant input data, these data are shown in Table 6.5. We approximate the obtained matrix by the spectral decomposition to be a correlation matrix.

N	δ_j	δt	Number of simulations (N_S)
10	1	0.25	400.000

Table 6.1: Parameters of the simulation procedure.

j	$B_j^d(0)$	σ_j^d	$\sigma_{j,j+1}^{Sd}$	$\sigma_{j,j+2}^{Sd}$	$\sigma_{j,j+3}^{Sd}$	$\sigma_{j,j+4}^{Sd}$	$\sigma_{j,j+5}^{Sd}$	$\sigma_{j,j+6}^{Sd}$	$\sigma_{j,j+7}^{Sd}$	$\sigma_{j,j+8}^{Sd}$	$\sigma_{j,j+9}^{Sd}$	$\sigma_{j,j+10}^{Sd}$
0	1,000000											
1	0,990761	0,7666	0,7666	0,5925	0,5015	0,4501	0,4240	0,3938	0,3764	0,3684	0,3565	0,3404
2	0,980002	0,5095	0,5095	0,4445	0,4063	0,3842	0,3632	0,3534	0,3386	0,3308	0,3221	
3	0,954435	0,3950	0,3950	0,3650	0,3509	0,3319	0,3232	0,3165	0,3031	0,2998		
4	0,922703	0,3378	0,3378	0,3272	0,3102	0,3032	0,2945	0,2876	0,2783			
5	0,887878	0,3132	0,3132	0,2966	0,2893	0,2820	0,2736	0,2692				
6	0,851701	0,2893	0,2893	0,2781	0,2695	0,2622	0,2546					
7	0,815186	0,2655	0,2655	0,2596	0,2497	0,2425						
8	0,779710	0,2485	0,2485	0,2413	0,2336							
9	0,744830	0,2315	0,2315	0,2230								
10	0,710892	0,2145	0,2145									
11	0,677054											

Table 6.2: Domestic market data.

j	$B_j^f(0)$	σ_j^f	$\sigma_{j,j+1}^{Sf}$	$\sigma_{j,j+2}^{Sf}$	$\sigma_{j,j+3}^{Sf}$	$\sigma_{j,j+4}^{Sf}$	$\sigma_{j,j+5}^{Sf}$	$\sigma_{j,j+6}^{Sf}$	$\sigma_{j,j+7}^{Sf}$	$\sigma_{j,j+8}^{Sf}$	$\sigma_{j,j+9}^{Sf}$	$\sigma_{j,j+10}^{Sf}$
0	1,000000											
1	0,987689	0,5152	0,5152	0,3844	0,3402	0,3088	0,2847	0,2689	0,2568	0,2495	0,2434	0,2381
2	0,966393	0,3566	0,3566	0,2896	0,2702	0,2549	0,2428	0,2370	0,2325	0,2289	0,2256	
3	0,938774	0,2837	0,2837	0,2395	0,2281	0,2190	0,2121	0,2098	0,2081	0,2065		
4	0,907973	0,2367	0,2367	0,2086	0,2019	0,1971	0,1931	0,1917	0,1903			
5	0,874979	0,2015	0,2015	0,1854	0,1826	0,1802	0,1780	0,1767				
6	0,840833	0,1851	0,1851	0,1732	0,1711	0,1690	0,1673					
7	0,806538	0,1687	0,1687	0,1610	0,1596	0,1579						
8	0,772759	0,1602	0,1602	0,1538	0,1528							
9	0,739546	0,1518	0,1518	0,1467								
10	0,707055	0,1433	0,1433									
11	0,674961											

Table 6.3: Foreign market data.

$X^{df}(0)$	σ_1^{df}	σ_2^{df}	σ_3^{df}	σ_4^{df}	σ_5^{df}	σ_6^{df}	σ_7^{df}	σ_8^{df}	σ_9^{df}	σ_{10}^{df}	σ_{11}^{df}
1.511	0,1365	0,1311	0,1256	0,1202	0,1147	0,1106	0,1064	0,1023	0,0981	0,0940	0,0899

Table 6.4: Spot and forward exchange rates market data.

ρ	γ^d	γ^f
0.7	0.2	0.3

Table 6.5: Correlation data.

We consider as numeraire:

$$\text{Num}(t) = \begin{cases} B_5^d(t), & \text{if } t \le T_5, \\ \beta_5^d(t), & \text{if } t > T_5, \end{cases}$$

where $\beta_5^d(t)$ corresponds to the domestic bank account starting at time T_5, that is,

$$\beta_5^d(t) = B_j^d(t) \prod_{l=5}^{j-1} \left(1 + \delta_l F_l^d(T_l)\right), \quad t \in (T_{j-1}, T_j], \quad j = 6, \dots, N+1. \qquad (6.21)$$

In this case the numeraire is B_5^d until T_5 and the spot measure starting after T_5 for $t > T_5$. Although this choice neither corresponds to a forward measure nor to a pure spot one, the proposed methodology can be applied.

Taking into account this information, in each currency we compare:

- the value of the martingales at time $t = 0$ jointly with the expected values obtained with different simulation methods (see Tables 6.6 and 6.7 and Figure 6.3),

- the Black value of the caplets ATM of each economy with the one obtained with the simulations at time $t = 0$ (see Tables 6.8 and 6.9 and Figure 6.4).

Finally, Table 6.10 shows the market price, (6.15) and (6.19), and the simulated prices in Case 1 and Case 2 for both the Quanto Caplet and the Quanto Floorlet that mature

at time T_5 (so that they pay at time T_6), without and with adjustment when using the previously described DFS methods. We also present the obtained confidence intervals from the PDFS with adjustment method (see Table 6.11).

For all DFS methods, Tables 6.6 and 6.7 illustrate how the proposed adjustment guarantees the martingale property for $\widehat{B}_j^d$ and $\widehat{B}_j^f$ at discrete level. On the other hand, Figures 6.3a and 6.3b show that the numerical results are better in the PDFS method for the domestic currency (even when compared with the recently introduced GZDFS method) and in the IDFS one for the foreign currency. In any case, the results for the foreign martingale with the IDFS method are very close to those ones with the PDFS method. As illustrated by Figures 6.4a and 6.4b, the PDFS method exhibits the best behavior for caplet pricing in both currencies. Finally, the PDFS method provides the best results in the quanto pricing for the Caplet 2 and Floorlet 1, and it results to be the second best method in the Caplet 1 case. So, taking into account its results and its positiveness preserving property, in general PDFS method should be chosen for the cross-currency simulation.

j	$\widehat{B}_j^d(0)$	IDFS	IDFS WA	EDFS	EDFS WA	GZDFS	GZDFS WA	PDFS	PDFS WA
1	1,115875	1,115886	1,115875	1,115888	1,115875	1,115910	1,115875	1,115910	1,115875
2	1,103757	1,103853	1,103757	1,103855	1,103757	1,103869	1,103757	1,103869	1,103757
3	1,074962	1,074947	1,074962	1,074949	1,074962	1,074963	1,074962	1,074963	1,074962
4	1,039223	1,039260	1,039223	1,039260	1,039223	1,039269	1,039223	1,039269	1,039223
5	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000
6	0,959254	0,959235	0,959254	0,959235	0,959254	0,959227	0,959254	0,959253	0,959254
7	0,918129	0,918048	0,918129	0,918050	0,918129	0,918036	0,918129	0,918083	0,918129
8	0,878172	0,878017	0,878172	0,878020	0,878172	0,878005	0,878172	0,878066	0,878171
9	0,838888	0,838771	0,838888	0,838775	0,838888	0,838752	0,838888	0,838820	0,838888
10	0,800663	0,800441	0,800663	0,800447	0,800663	0,800419	0,800663	0,800492	0,800664
11	0,762553	0,762317	0,762553	0,762323	0,762553	0,762293	0,762553	0,762367	0,762556

Table 6.6: Expected value of the domestic martingales, $E(\widehat{B}_j^d(T_j))$, obtained from the different Drift-Free Simulation methods without and with adjustment (WA), compared with $\widehat{B}_j^d(0)$.
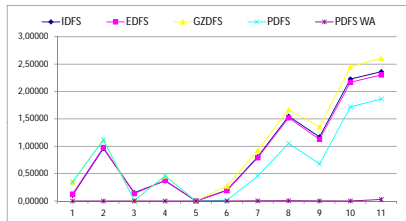
(a) Domestic martingales.
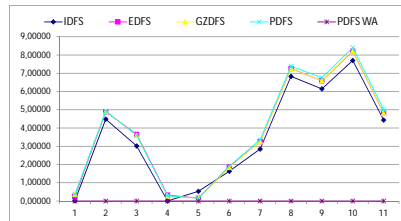
(b) Foreign martingales.

Figure 6.3: Differences in basis points between the martingale values at time 0 and their expected values at their expiration date with different DFS techniques.

(a) Domestic caplets.

(b) Foreign caplets.

Figure 6.4: Differences in basis points between the Black values and simulated caplet prices with different DFS techniques.

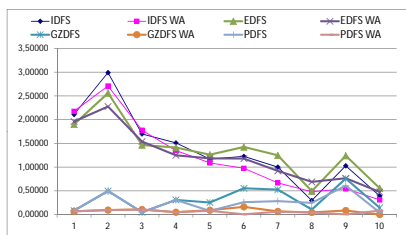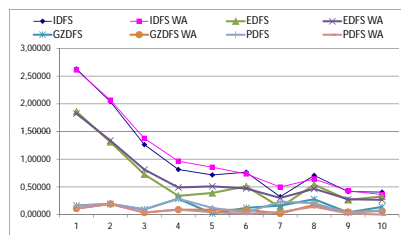| j	$\widehat{B}_j^f(0)$	IDFS	IDFS WA	EDFS	EDFS WA	GZDFS	GZDFS WA	PDFS	PDFS WA
1	1,680858	1,680860	1,680858	1,680885	1,680858	1,680895	1,680858	1,680895	1,680858
2	1,644618	1,645066	1,644618	1,645107	1,644618	1,645110	1,644618	1,645110	1,644618
3	1,597615	1,597917	1,597615	1,597981	1,597615	1,597975	1,597615	1,597975	1,597615
4	1,545198	1,545197	1,545198	1,545230	1,545198	1,545226	1,545198	1,545226	1,545198
5	1,489049	1,488996	1,489049	1,489033	1,489049	1,489033	1,489049	1,489033	1,489049
6	1,430939	1,431101	1,430939	1,431127	1,430939	1,431123	1,430939	1,431129	1,430939
7	1,372574	1,372859	1,372574	1,372900	1,372574	1,372900	1,372574	1,372910	1,372574
8	1,315089	1,315772	1,315089	1,315815	1,315089	1,315814	1,315089	1,315828	1,315089
9	1,258567	1,259181	1,258567	1,259225	1,258567	1,259226	1,258567	1,259243	1,258567
10	1,203273	1,204043	1,203273	1,204094	1,203273	1,204095	1,203273	1,204114	1,203273
11	1,148656	1,149099	1,148656	1,149138	1,148656	1,149137	1,148656	1,149158	1,148656

Table 6.7: Expected value of the foreign martingales, $E(\widehat{B}_j^f(T_j))$, obtained from the different Drift-Free Simulation methods without and with adjustment (WA), compared with $\widehat{B}_j^f(0)$.

j	BLACK	IDFS	IDFS WA	EDFS	EDFS WA	GZDFS	GZDFS WA	PDFS	PDFS WA
1	0,003212	0,003423	0,003429	0,003402	0,003408	0,003220	0,003219	0,003220	0,003219
2	0,007193	0,007492	0,007464	0,007449	0,007420	0,007242	0,007202	0,007242	0,007202
3	0,008495	0,008665	0,008673	0,008642	0,008649	0,008500	0,008506	0,008500	0,008506
4	0,009209	0,009361	0,009344	0,009351	0,009334	0,009240	0,009215	0,009240	0,009215
5	0,009905	0,010022	0,010014	0,010031	0,010023	0,009930	0,009914	0,009897	0,009897
6	0,010112	0,010234	0,010209	0,010254	0,010230	0,010167	0,010128	0,010138	0,010112
7	0,009739	0,009840	0,009806	0,009864	0,009832	0,009792	0,009746	0,009768	0,009733
8	0,009581	0,009611	0,009630	0,009631	0,009650	0,009571	0,009586	0,009557	0,009577
9	0,009217	0,009320	0,009272	0,009341	0,009294	0,009293	0,009226	0,009278	0,009217
10	0,008984	0,009024	0,009015	0,009040	0,009032	0,008999	0,008984	0,008989	0,008976

Table 6.8: Value at the zero time of domestic caplets ATM that mature at T_j. The Black values and the ones obtained from the different Drift-Free Simulation methods without and with adjustment are presented.

j	BLACK	IDFS	IDFS WA	EDFS	EDFS WA	GZDFS	GZDFS WA	PDFS	PDFS WA
1	0,006541	0,006804	0,006802	0,006727	0,006724	0,006558	0,006552	0,006558	0,006552
2	0,008308	0,008512	0,008515	0,008439	0,008442	0,008327	0,008327	0,008327	0,008327
3	0,009033	0,009159	0,009171	0,009105	0,009114	0,009023	0,009037	0,009023	0,009037
4	0,009328	0,009410	0,009425	0,009362	0,009377	0,009299	0,009319	0,009299	0,009319
5	0,009197	0,009268	0,009282	0,009236	0,009248	0,009195	0,009205	0,009184	0,009201
6	0,009294	0,009371	0,009368	0,009345	0,009341	0,009306	0,009303	0,009298	0,009293
7	0,009013	0,009046	0,009063	0,009027	0,009043	0,008997	0,009015	0,008990	0,009009
8	0,008997	0,009067	0,009061	0,009051	0,009043	0,009025	0,009014	0,009017	0,009011
9	0,008841	0,008883	0,008884	0,008868	0,008869	0,008845	0,008846	0,008838	0,008839
10	0,008692	0,008733	0,008729	0,008725	0,008719	0,008706	0,008699	0,008700	0,008692

Table 6.9: Value at the zero time of foreign caplets ATM that mature at T_j. The Black values and the ones obtained from the different Drift-Free Simulation methods without and with adjustment are presented.

CASE	MARKET	IDFS	IDFS WA	EDFS	EDFS WA	GDFS	GDFS WA	PDFS	PDFS WA
Caplet 1	0,005559	0,005611	0,005620	0,005573	0,005580	0,005542	0,005549	0,005537	0,005548
Caplet 2	0,015901	0,016884	0,016870	0,016897	0,016884	0,016793	0,016765	0,016713	0,016712
Floorlet 1	0,006611	0,006724	0,006715	0,006675	0,006668	0,006641	0,006636	0,006638	0,006630
Floorlet 2	0,014157	0,014069	0,014084	0,014083	0,014097	0,013938	0,013951	0,013924	0,013925

Table 6.10: Value at time $t = 0$ of Cases 1 and 2 of Quanto Caplets/Floorlets that mature at T_5 with Not $= 1$ and strikes $K^f = F_5^f(0)$ and $K^d = F_5^d(0)$. The market values and the ones obtained from Drift-Free Simulation methods are presented.

CASE	MARKET	CONFIDENCE INTERVAL
Caplet 1	0,005559	[0,005179 , 0,005917]
Caplet 2	0,015901	[0,015804 , 0,017621]
Floorlet 1	0,006611	[0,006603 , 0,006656]
Floorlet 2	0,014157	[0,001388 , 0,026463]

Table 6.11: Value at time $t = 0$ of Cases 1 and 2 of Quanto Caplets/Floorlets that mature at T_5 with Not $= 1$ and strikes $K^f = F_5^f(0)$ and $K^d = F_5^d(0)$. The market values and the confidence intervals obtained from PDFS WA are presented. Confidence level $= 95\%$.

Chapter 7

Drift-Free Simulation techniques for a Second Cross-Market Model

7.1 Introduction

In this chapter we present the formulation of a Second Cross-Market Model as well as the associated DFS technique. This technique has been proposed in [22]. More precisely, in [22] we also apply the DFS technique to price commodity derivatives [30, 58]. As an example, we price a spread option [53]. In the context of commodity derivatives it is also interesting to obtain the evolution of the convenience yield [24, 47, 53].

This chapter is organized as follows. Section 7.2 presents the formulation of the Second Cross-Market Model. Section 7.3 describes the DFS methods for this model. Section 7.4 deals with the model calibration procedure. In Section 7.5 we explain the dynamics of the rates which are not directly modeled. The DFS methodology is applied to the commodities context in Section 7.6. Finally, a third example of a cross-market concerning inflation derivatives is presented in 7.7.

7.2 Formulation

In this section we formulate the Second Cross-Market Model. As previously indicated, in this case we model the forward rates of the economy A, $F_j^A(t)$, $j = 1, ..., N$, and the forward exchange rates, $FX_j^{AB}(t)$, $j = 1, ..., N + 1$.

In order to formulate the model we denote by $Z(t) = (W_1^A(t), ..., W_N^A(t), W_1^{AB}(t), ..., W_{N+1}^{AB}(t))$ a correlated $(2N + 1)$-dimensional Wiener process in a certain probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a correlation matrix

$$C = \begin{pmatrix} \Sigma^A & \rho \\ \rho & \Sigma^{AB} \end{pmatrix} \quad \in \mathcal{M}^{(2N+1)\times(2N+1)},$$

so that $\Sigma^A \in \mathcal{M}^{N \times N}$ denotes the matrix of correlations among the forward rates of the economy A, $\Sigma^{AB} \in \mathcal{M}^{(N+1)\times(N+1)}$ denotes the matrix of correlations among forward exchange rates, and $\rho \in \mathcal{M}^{N \times (N+1)}$ denotes the matrix of correlations among the forward rates of the economy A and forward exchange rates. The natural filtration spanned by $Z(t)$ is denoted by $\mathcal{F}_t$. Every stochastic process we consider is defined on this probability space.

LMM assumes lognormal forward rates under any probability measure. Hence, we assume the following system of dynamics:

$$dF_j^A(t) = \mu_j^A(t)F_j^A(t)dt + \sigma_j^A(t)F_j^A(t)dW_j^A(t), \quad j = 1, ..., N,$$
$$dFX_j^{AB}(t) = \mu_j^{AB}(t)FX_j^{AB}(t)dt + \sigma_j^{AB}(t)FX_j^{AB}(t)dW_j^{AB}(t), \quad j = 1, ..., N + 1,$$

$$(7.1)$$

where $\mu_j^A(t)$ and $\mu_j^{AB}(t)$ denote the drift terms, $\sigma_j^A(t)$ and $\sigma_j^{AB}(t)$ denote the volatility terms and $W_j^A(t)$ and $W_j^{AB}(t)$ denote Brownian motions under $\mathbf{P}$, for the forward rates of the economy A and for the forward exchange rates, respectively.

The drift terms depend on the chosen probability measure. In fact, if we consider Q^n then $F_{n-1}^A(t)$ and $FX_n^{AB}(t)$ are martingales, and if we consider Q^β the terms that

are martingales depend on the time interval, that is, $F_{j-1}^A(t)$ and $FX_j^{AB}(t)$ are martingales if $t \in (T_{j-1}, T_j]$. For these two particular measures:

$$\mu_j^A(t) = -\left(\varepsilon_{k,j+1} \sum_{l \in \theta_{k,j+1}} \frac{\delta_l \sigma_l^A(t) F_l^A(t) \Sigma_{lj}^A}{1 + \delta_l F_l^A(t)}\right) \sigma_j^A(t), \qquad (7.2)$$

$$\mu_j^{AB}(t) = -\left(\varepsilon_{k,j} \sum_{l \in \theta_{k,j}} \frac{\delta_l \sigma_l^A(t) F_l^A(t) \rho_{lj}}{1 + \delta_l F_l^A(t)}\right) \sigma_j^{AB}(t), \qquad (7.3)$$

where k is the index such that $d\widehat{B}_k^A(t) = 0$ and $d\widehat{B}_k^B(t) = dFX_k^{AB}(t)$, see Remark 2.2.1, and $\varepsilon_{k,j}$ and $\theta_{k,j}$ are defined in (2.6). These values can be obtained by taking into account (5.1) and (5.2) and the martingales defined in (5.5).

We are again interested in the simulation of the rates in (7.1) avoiding the use of the drift terms. For this purpose, in the next section we extend the DFS methodology developed in Chapter 2 for one economy to the Second Cross-Market Model case.

7.3   DFS technique

As in the First Cross-Market Model case, once we have chosen the numeraire Num, the DFS procedures are based on the simulation of the terms $\widehat{B}_j^A, \widehat{B}_j^B, j = 1, ..., N+1$ (see (5.5)), that are martingales under the probability measure associated to the numeraire Num.

While the DFS procedure for the economy A was based on the following (recurrence) relation:

$$\widehat{B}_j^A(t) = \left(1 + \delta_j F_j^A(t)\right) \widehat{B}_{j+1}^A(t), \qquad (7.4)$$

and on that $d\widehat{B}_k^A(t) = 0$ for some k at each time t, the DFS technique to simulate forward exchange rates is based on the following (no recurrence) relation:

$$\widehat{B}_j^B(t) = \widehat{B}_j^A(t) FX_j^{AB}(t). \qquad (7.5)$$

Then, DFS procedure (with or without adjustment) for the economy A part is completely analogous to the technique indicated in Chapter 2, just taking into account that now $F_j \equiv F_j^A$, $B_j \equiv B_j^A$ and $\widehat{B}_j \equiv \widehat{B}_j^A$. The construction for the forward exchange rates is explained in next paragraphs.

Taking into account that $\widehat{B}_j^B(t)$ is a martingale and the relation (7.5), we obtain the following expression relating the dynamics of $\widehat{B}_j^B$ and $\widehat{B}_j^A$:

$$d\widehat{B}_j^B(t) = \widehat{B}_j^A(t)\sigma_j^{AB}(t)FX_j^{AB}(t)dW_j^{AB}(t) + FX_j^{AB}(t)d\widehat{B}_j^A(t). \qquad (7.6)$$

Therefore, we can compute $d\widehat{B}_j^B(t)$ in terms of the previously simulated Brownian jumps, the previously calculated dynamics of the economy A martingales and the previous levels of the economy B martingales and of the forward exchange rates. Notice that now the recursiveness of the economy A case has been lost. Then, by using the Euler discretization we compute the corresponding martingales. Moreover, by using (7.5) and the following formula (equivalent to (5.1)):

$$F_j^B(t) = \frac{\widehat{B}_j^B(t) - \widehat{B}_{j+1}^B(t)}{\delta_j \widehat{B}_{j+1}^B(t)}, \qquad (7.7)$$

we can obtain the forward exchange rates and the forward rates of the economy B, respectively. Finally, we can obtain the spot exchange rates, X^{AB}, and the economy B discount bonds values, B_j^B, at tenor dates by means of

$$X^{AB}(T_j) = FX_j^{AB}(T_j), \quad j = 1, \ldots, N+1, \qquad (7.8)$$

and

$$B_j^B(T_i) = \frac{\text{Num}(T_i)\widehat{B}_j^B(T_i)}{X^{AB}(T_i)}, \quad i \leqslant j. \qquad (7.9)$$

As in the economy A case, a log-Euler discretization ensures the positiveness of the martingales $\widehat{B}_j^B$:

$$\widehat{B}_j^B(t_{i+1}) = \widehat{B}_j^B(t_i) \exp\left(\frac{d\widehat{B}_j^B(t_i)}{\widehat{B}_j^B(t_i)} - \frac{1}{2}\sigma_j^{\widehat{B}^B}(t_i)^2 \delta t \right), \quad j = 1, \ldots, N+1, \qquad (7.10)$$

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where

$$\frac{d\widehat{B}_j^B(t)}{\widehat{B}_j^B(t)} = \sigma_j^{AB}(t)dW_j^{AB}(t) + \frac{d\widehat{B}_j^A(t)}{\widehat{B}_j^A(t)} =$$

$$= \sigma_j^{AB}(t)dW_j^{AB}(t) + \varepsilon_{k,j} \sum_{l \in \theta_{k,j}} \sigma_l^A(t) \left(1 - \frac{\widehat{B}_{l+1}^A(t)}{\widehat{B}_l^A(t)}\right) dW_l^A(t), \qquad (7.11)$$

and

$$\sigma_j^{\widehat{B}^B}(t)^2 = \sigma_j^{AB}(t)^2 + 2\sigma_j^{AB}(t)\varepsilon_{k,j} \sum_{l \in \theta_{k,j}} \sigma_l^A(t) \left(1 - \frac{\widehat{B}_{l+1}^A(t)}{\widehat{B}_l^A(t)}\right) \rho_{l,j} +$$

$$+ \sum_{l_1,l_2 \in \theta_{k,j}} \sigma_{l_1}^A(t)\sigma_{l_2}^A(t) \left(1 - \frac{\widehat{B}_{l_1+1}^A(t)}{\widehat{B}_{l_1}^A(t)}\right) \left(1 - \frac{\widehat{B}_{l_2+1}^A(t)}{\widehat{B}_{l_2}^A(t)}\right) \Sigma_{l_1,l_2}^A, \qquad (7.12)$$

where, as in Chapter 2, k is the index such that $d\widehat{B}_k^A(t) = 0$.

Notice that in this procedure we obtain positivity of the forward exchange rates FX_j^{AB} as a consequence of the positivity of the martingales $\widehat{B}_j^A$ and $\widehat{B}_j^B$, .

Aditionally, it is also possible to perform an adjustment for ensuring the martingale property after the discretization. This adjustment is completely analogous to the one presented in Chapter 2.

7.4   Model calibration

As indicated in the First Cross-Market Model case, since we are in the context of market models, we have to introduce into our model as parameters as much market information as feasible. For this purpose, we first introduce the following market data:

$$\{B_j^A(0)\}_{j=0}^{N+1}, \quad \{FX_j^{AB}(0)\}_{j=0}^{N+1}, \quad X^{AB}(0).$$

For this reason, the Second Cross-Market Model is chosen for settings in which the quoted assets in the market are the previous data, as it is the case of the commodities

setting (see Figure 7.1).
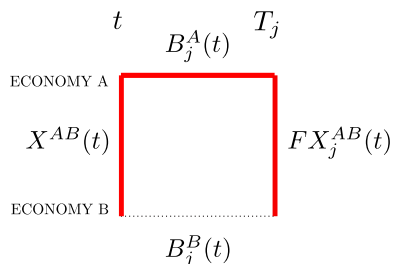
Figure 7.1: The assets that quote in the commodity markets are marked by a thicker red line.

Other free parameters (volatilities and correlations) are considered in next paragraphs, so that these model parameters are adjusted to the market.

- *Adjustment of volatilities:* The structure of volatilities is adjusted to market by imposing that

$$
\begin{aligned}
\int_0^{T_j} \sigma_j^A(t)^2 dt &= T_j(\sigma_j^{A^{mar}})^2, \\
\int_0^{T_j} \sigma_j^{AB}(t)^2 dt &= T_j(\sigma_j^{AB^{mar}})^2,
\end{aligned} \tag{7.13}
$$

  where $\sigma_j^{A^{mar}}$ are quoted volatilities of caplets of economy A and $\sigma_j^{AB^{mar}}$ are quoted volatilities of call options on forward exchange rates.

In practice, for simplicity, we choose constant volatilities, that is,

$$
\sigma_j^A(t) = \sigma_j^{A^{mar}}, \quad \sigma_j^{AB}(t) = \sigma_j^{AB^{mar}}, \quad \forall t.
$$

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- *Adjustment of correlations:* We have to adjust the following correlation matrix:

$$
C = \begin{pmatrix}
\Sigma^A_{11} & \cdots & \Sigma^A_{1N} & \rho_{11} & \cdots & \rho_{1,N+1} \\
\vdots & & \vdots & \vdots & & \vdots \\
\Sigma^A_{N1} & \cdots & \Sigma^A_{NN} & \rho_{N1} & \cdots & \rho_{N,N+1} \\
\rho_{11} & \cdots & \rho_{N,1} & \Sigma^{AB}_{11} & \cdots & \Sigma^{AB}_{1,N+1} \\
\vdots & & \vdots & \vdots & & \vdots \\
\rho_{1,N+1} & \cdots & \rho_{N,N+1} & \Sigma^{AB}_{N+1,1} & \cdots & \Sigma^{AB}_{N+1,N+1}
\end{pmatrix}.
$$

First, we adjust the correlations within the economy A, $(\Sigma^A_{ij})^N_{i,j=1}$, by using swaps volatilities from the market in economy A and taking into account the approximation indicated in (2.51), fully analogous to the one economy case.

As the other correlations cannot be calculated analogously, we obtain them from historical data and assume they are constant, that is,

$$
\Sigma^{AB}_{ij} = \Sigma^{AB}, \forall i, j = 1, ..., N+1, \quad \rho_{ij} = \rho, \forall i = 1, ..., N, \forall j = 1, ..., N+1.
$$

Again, if necessary we can approximate the resulting matrix to be a correlation matrix, see Remark 2.3.1.

7.5 Dynamics of the economy B forward rates

In this case, it is easier to obtain the rest of the process of the graph.

In Figure 7.2 we can see that for all $j = 1, ..., N$ the forward rate F^B_j only depends on the values F^A_j, FX^{AB}_j and FX^{AB}_{j+1}. In this way, we can obtain the dynamics of the forward rate F^B_j by using Ito's Lemma and taking into account that

$$
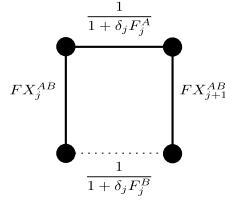\frac{1}{1 + \delta_j F^A_j(t)} FX^{AB}_{j+1}(t) = FX^{AB}_j(t) \frac{1}{1 + \delta_j F^B_j(t)}. \tag{7.14}
$$

Figure 7.2: Relation between the different processes.

Then, we have the following no lognormal dynamics:

$$
dF_j^B(t) = \frac{FX_j^{AB}(t)}{FX_{j+1}^{AB}(t)}[F_j^A(t)\big(\mu_j^A(t) + \sigma_j^A(t)\sigma_j^{AB}(t)\rho_{jj} - \sigma_j^A(t)\sigma_{j+1}^{AB}(t)\rho_{j,j+1}\big)+
$$

$$
+ \frac{1 + \delta_j F_j^A(t)}{\delta_j}\big(\mu_j^{AB}(t) - \mu_{j+1}^{AB}(t) + \sigma_{j+1}^{AB}(t)^2 - \sigma_j^{AB}(t)\sigma_{j+1}^{AB}(t)\Sigma_{j,j+1}^{AB}\big)]dt+
$$

$$
+ \frac{FX_j^{AB}(t)}{FX_{j+1}^{AB}(t)}[\sigma_j^A(t)F_j^A(t)dW_j^A(t) + \frac{1 + \delta_j F_j^A(t)}{\delta_j}\big(\sigma_j^{AB}(t)dW_j^{AB}(t) - \sigma_{j+1}^{AB}(t)dW_{j+1}^{AB}(t)\big)].
$$

$$(7.15)$$

7.6 Example of the Second Cross-Market Model: Commodities

A commodity is any marketable item produced to satisfy wants or needs. In general, its price is practically the same regardless of who provides it. For example, oil, paper, milk or electricity are commodities.

Commodity markets can be understood as Cross-Markets. As mentioned before, for the commodity case the Second Cross-Market Model is preferable, see Figure 5.3. In this setting, the units are the domestic currency and one standard quantity of one commodity, so that we have the domestic (nominal) and the commodity economies plus the forward prices of the commodity that relate them. So, if we consider the notation $A \equiv d$ and $B \equiv c$, by analogy with Chapter 5, we can redefine B_j^d, B_j^c, F_j^d, F_j^c, X^{dc} and FX_j^{dc}. In the commodity setting there exists a specific notation (see [58]): the spot exchange and the forward exchange rates are the spot price and the

forward prices of the commodity, respectively. B_j^c and F_j^c are introduced so that we can proceed analogously as in the general frame of cross-market models, although they have no practical sense. However, in next paragraphs we will argue how the forward rate F_j^c matches the convenience yield Y_j, that is a concept that appears when consumption commodities are treated (see [30], for example).

In the next subsections, we present the convenience yield and one commodity derivative, that is, the Spread Options.

7.6.1 The convenience yield

A singular aspect of commodity markets is the presence of the convenience yield, that can be defined as the benefit obtained from holding the spot commodity that is not obtained from holding the forward contract. It is assumed to be constant in the seminal paper [10]. After noting the evidence of the stochastic behavior of the convenience yield, recently in [24, 47] an original model with stochastic interest rates and convenience yield has been proposed. More precisely, a Heath-Jarrow-Morton methodology for interest rates is used while a mean reversion stochastic model for convenience yield is chosen. A first attempt to use LMM for commodities market appears in [53]. It is mainly based on its ability to fit observed market data and on the analogy provided by the multi-currency model introduced in [59],

As there exist financial derivatives where the underlying is the convenience yield, our aim is to model the evolution of the implicit forward rates and the forward prices of the commodity in order to obtain the evolution of the convenience yield.

Attending to their financial purposes, in [30] the commodities are classified into two kinds:

- Investment commodities: These are assets held for investment purposes. For

example, securities, bonds, gold, silver...

- Consumption commodities: These are assets held for consumption purposes. For example, copper, oil...

In the investment commodity case, the futures prices can be exactly calculated by no arbitrage arguments. However, in the consumption commodity case, these prices are not so precise, because there can be exist benefits from the ownership of the commodity that are no obtained by the contract holder.

If we consider that we are trading with investment commodities and r is the risk-free rate, then we have:

$$FX_j^{dc}(t) = X^{dc}(t)e^{r(T_j-t)}. \tag{7.16}$$

This relation is obtained by the following no arbitrage arguments, see [30]:

a) $FX_j^{dc}(t) \leqslant X^{dc}(t)e^{r(T_j-t)}$, because otherwise if $FX_j^{dc}(t) > X^{dc}(t)e^{r(T_j-t)}$ then one investor can obtain profit without risk by means of the following strategy:

1. At time t, the investor borrows $X^{dc}(t)$ units of the domestic currency at an interest rate r for $T_j - t$ years and, with this money, he/she buys one unit of the commodity. Moreover the investor enters in a short position such that he/she agrees to sell this unit of the commodity at time T_j with the price $FX_j^{dc}(t)$.

2. At time T_j, the investor sells the commodity for the agreed price, $FX_j^{dc}(t)$, and he/she returns the borrowed money, $X^{dc}(t)e^{r(T_j-t)}$. Therefore, if we assume no storage cost, the investor obtains $FX_j^{dc}(t) - X^{dc}(t)e^{r(T_j-t)} > 0$.

b) $FX_j^{dc}(t) \geqslant X^{dc}(t)e^{r(T_j-t)}$, because otherwise if $FX_j^{dc}(t) < X^{dc}(t)e^{r(T_j-t)}$ then one investor can obtain profit without risk by using the following strategy:

1. At time t, the investor sells one unit of the commodity at the price $X^{dc}(t)$ and he/she invests this quantity at interest rate r for $T_j - t$ years. Moreover

the investor enters in a long position such that he/she agrees to buy one unit of the commodity at time T_j with the price $FX_j^{dc}(t)$.

2. At time T_j, the investor buys the commodity for the agreed price, $FX_j^{dc}(t)$, and he/she recovers the invested money, $X^{dc}(t)e^{r(T_j-t)}$. Therefore, the investor obtains $X^{dc}(t)e^{r(T_j-t)} - FX_j^{dc}(t) > 0$.

Then, $FX_j^{dc}(t) \leqslant X^{dc}(t)e^{r(T_j-t)}$ and $FX_j^{dc}(t) \geqslant X^{dc}(t)e^{r(T_j-t)}$, so we conclude the relation (7.16).

However, when the commodities are obtained for their value for consumption purposes, and not like an investment, the argument in b) fails. This is because commodity users must feel that there exist benefits from ownership of the physical commodity that are not obtained by the contract holder. These benefits can include the ability to profit from temporary local shortages or the ability to keep a production process running. These benefits are collected by the term known as convenience yield and denoted by y. This term is defined such that

$$FX_j^{dc}(t) = X^{dc}(t)e^{(r-y)(T_j-t)} \tag{7.17}$$

Notice that in the investment commodities case the value $y = 0$ is chosen to avoid arbitrage opportunities.

We are interested in obtaining the evolution of the convenience yield by simulating the rates F_j^d and FX_j^{dc}. Thus, in order to get it in terms of our basic assets (bonds instead risk-free rates) we present an analogous argument to the one of a) in [30].

The following relation is satisfied:

$$FX_{j+1}^{dc}(t)B_{j+1}^d(t) \leqslant FX_j^{dc}(t)B_j^d(t), \tag{7.18}$$

because if $FX_{j+1}^{dc}(t)B_{j+1}^d(t) > FX_j^{dc}(t)B_j^d(t)$ one person can obtain profit without risk by the following strategy: at time t, the person pays $FX_j^{dc}(t)B_j^d(t)$, so he/she

will receive at time T_j one unit of the commodity, and receives $FX_{j+1}^{dc}(t)B_{j+1}^d(t)$, so he/she will have to give at time $T_j + 1$ one unit of the commodity. Thus, he/she obtains $FX_{j+1}^{dc}(t)B_{j+1}^d(t) - FX_j^{dc}(t)B_j^d(t) > 0$, without risk and costs (if we assume that there is not storage cost).

Next, taking into account (7.18), we obtain

$$FX_{j+1}^{dc}(t) \leqslant FX_j^{dc}(t)\frac{B_j^d(t)}{B_{j+1}^d(t)}. \tag{7.19}$$

Then the convenience yield is defined as the rate Y_j such that

$$FX_{j+1}^{dc}(t)(1 + \delta_j Y_j(t)) = FX_j^{dc}(t)\frac{B_j^d(t)}{B_{j+1}^d(t)}, \tag{7.20}$$

that in turn, by using (5.2), can be expressed in the form

$$Y_j(t) = \frac{B_j^c(t) - B_{j+1}^c(t)}{\delta_j B_{j+1}^c(t)}. \tag{7.21}$$

Therefore, in this context the forward rate $F_j^c(t)$ matches the convenience yield $Y_j(t)$.

Finally, depending on the evolution of the commodity price, two possible situations arise:

- The situation where the price of the commodity for future delivery is higher than the spot price, i.e., than a nearer future delivery. This situation is known as *contango* and it involves an upward sloping forward curve. In this case the convenience yield is negative.

- The situation where the price of the commodity for future delivery is lower than the spot price, i.e., than a nearer future delivery. This situation is known as *backwardation* and it involves an downward sloping forward curve. This is not a normal market condition and it suggest shortcomings in the supply of the commodity. However, many commodity markets are often in backwardation, especially when the seasonal aspect is taken into account (for example, with perishable commodities). In this situation the convenience yield is positive.

7.6.2 Commodity derivatives: Spread Options

In this subsection we present the pricing of Spread Options on forward prices of the commodity taking into account the approximation approach proposed in [53].

Spread options can be used to hedge the risk of the difference between two forward prices with different maturities.

Let $CSpr_{i,j}(t)$ denote the price at time t of the call option on the spread

$$Spr_{i,j}(T_i) = FX_j^{dc}(T_i) - FX_i^{dc}(T_i), \tag{7.22}$$

with $T_i < T_j$. Then, we have:

$$CSpr_{i,j}(T_i) = \left(FX_j^{dc}(T_i) - FX_i^{dc}(T_i) - K\right)_+, \tag{7.23}$$

where K is the considered strike. Hence, for all $t < T_i$, the Spread Option price is given by

$$CSpr_{i,j}(t) = B_i^d(t) E_{Q^i}\left[\left(FX_j^{dc}(T_i) - FX_i^{dc}(T_i) - K\right)_+|\mathcal{F}_t\right], \tag{7.24}$$

where Q^i is the probability measure associated to the numeraire $B_i^d(t)$. Moreover, we know that under Q^i we have

$$
\begin{aligned}
dFX_i^{dc}(t) &= \sigma_i^{dc}(t)FX_i^{dc}(t)dW_i^{dc}(t), \\
dFX_j^{dc}(t) &= \mu_j^{dc}(t)FX_j^{dc}(t)dt + \sigma_j^{dc}(t)FX_j^{dc}(t)dW_j^{dc}(t),
\end{aligned} \tag{7.25}
$$

with

$$\mu_j^{dc}(t) = \left(\sum_{l=i}^{j-1} \frac{\delta_l \sigma_l^d(t) F_l^d(t) \rho_{lj}}{1 + \delta_l F_l^d(t)}\right) \sigma_j^{dc}(t). \tag{7.26}$$

Following [38], in order to can use Margrabe's formula, we rewrite (7.24) as

$$CSpr_{i,j}(t) = B_i^d(t) E_{Q^i}\left[\left(FX_j^{dc}(T_i) - (FX_i^{dc}(T_i) + K)\right)_+|\mathcal{F}_t\right], \tag{7.27}$$

where

$$d\big(FX_i^{dc}(t) + K\big) = \frac{FX_i^{dc}(t)}{FX_i^{dc}(t) + K}\sigma_i^{dc}(t)\big(FX_i^{dc}(t) + K\big)dW_i^{dc}(t). \tag{7.28}$$

Next, by freezing some terms appearing in $\mu_j^{dc}(t)$, as proposed in [53],

$$\mu_j^{dc}(t) \approx \Upsilon_j^{dc}(t) = \left(\sum_{l=i}^{j-1} \frac{\delta_l \sigma_l^d(t) F_l^d(0)\rho_{lj}}{1 + \delta_l F_l^d(0)}\right)\sigma_j^{dc}(t), \tag{7.29}$$

and by applying Margrabe's formula [41], we have

$$CSpr_{i,j}(t) = B_i^d(t)\left[\exp\left(\int_t^{T_i}\Upsilon_j^{dc}(z)dz\right)FX_j^{dc}(t)\Phi(d_+) - \big(FX_i^{dc}(t) + K\big)\Phi(d_-)\right], \tag{7.30}$$

where

$$d_{\pm} = \frac{\ln\left(\dfrac{FX_j^{dc}(t)}{FX_i^{dc}(t) + K}\right) + \int_t^{T_i}\left(\Upsilon_j^{dc}(z) \pm \frac{1}{2}\Lambda_{i,j}^2(z)\right)dz}{\left(\int_t^{T_i}\Lambda_{i,j}^2(z)dz\right)^{\frac{1}{2}}},$$

$$\Lambda_{i,j}^2(t) = \sigma_j^{dc}(t)^2 + \left(\frac{FX_i^{dc}(t)}{FX_i^{dc}(t) + K}\sigma_i^{dc}(t)\right)^2 - 2\sigma_j^{dc}(t)\left(\frac{FX_i^{dc}(t)}{FX_i^{dc}(t) + K}\sigma_i^{dc}(t)\right)\Sigma_{ij}^{dc},$$

and Φ denotes the cumulative standard normal distribution function.

Furthermore, we can obtain the value of the Spread Option at time t simulating under Q (probability measure with Num as numeraire) and taking into account that

$$CSpr_{i,j}(t) = \text{Num}(t)E_Q\left[\frac{CSpr_{i,j}(T_i)}{\text{Num}(T_i)}|\mathcal{F}_t\right]. \tag{7.31}$$

In next section we present some numerical results obtained by applying the DFS procedures for pricing Spread Options.

7.6.3   Numerical results

In this section, we present the data used for the most convenient DFS methods described in previous sections, that are PDFS for the domestic economy and the log-Euler DFS (LE DFS) for the forward prices of the commodities, and the obtained

results for martingales in both cases and caplet pricing in the domestic economy. We also present the obtained results for pricing one Spread Option.

Table 7.1 shows the data for the Monte Carlo simulation procedure. More precisely, we consider the case $N = 10$. The accrual period is constant and equal to one year. The time step and the number of simulations are also indicated. Polar-Marsaglia method is used for simulation of the Brownian motions. The domestic data and the forward exchange rate data are given in Tables 7.2 and 7.3, respectively. These data include the zero coupon curves at time zero and the volatilities of the involved rates. Table 7.2 also shows the swap volatilities to obtain the correlations between the implicit forward rates of the domestic economy. The correlation matrix is completed with constant input data. These data are shown in Table 7.4. We approximate the obtained matrix by the spectral decomposition to be a correlation matrix.

N	δ_j	δt	Number of simulations (N_S)
10	1	0,25	400.000

Table 7.1: Parameters of the simulation procedure.

j	$B_j^d(0)$	σ_j^d	$\sigma_{j,j+1}^{S^d}$	$\sigma_{j,j+2}^{S^d}$	$\sigma_{j,j+3}^{S^d}$	$\sigma_{j,j+4}^{S^d}$	$\sigma_{j,j+5}^{S^d}$	$\sigma_{j,j+6}^{S^d}$	$\sigma_{j,j+7}^{S^d}$	$\sigma_{j,j+8}^{S^d}$	$\sigma_{j,j+9}^{S^d}$	$\sigma_{j,j+10}^{S^d}$
0	1,000000											
1	0,987689	0,5152	0,5152	0,3844	0,3402	0,3088	0,2847	0,2689	0,2568	0,2495	0,2434	0,2381
2	0,966393	0,3566	0,3566	0,2896	0,2702	0,2549	0,2428	0,2370	0,2325	0,2289	0,2256	
3	0,938774	0,2837	0,2837	0,2395	0,2281	0,2190	0,2121	0,2098	0,2081	0,2065		
4	0,907973	0,2367	0,2367	0,2086	0,2019	0,1971	0,1931	0,1917	0,1903			
5	0,874979	0,2015	0,2015	0,1854	0,1826	0,1802	0,1780	0,1767				
6	0,840833	0,1851	0,1851	0,1732	0,1711	0,1690	0,1673					
7	0,806538	0,1687	0,1687	0,1610	0,1596	0,1579						
8	0,772759	0,1602	0,1602	0,1538	0,1528							
9	0,739546	0,1518	0,1518	0,1467								
10	0,707055	0,1433	0,1433									
11	0,674961											

Table 7.2: Domestic market data.

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j	$FX_j^{dc}(0)$	σ_j^{dc}
0	0,5	
1	0,6	0,1365
2	0,7	0,1311
3	0,8	0,1256
4	0,9	0,1202
5	1,0	0,1147
6	1,1	0,1106
7	1,2	0,1064
8	1,3	0,1023
9	1,4	0,0981
10	1,5	0,0940
11	1,6	0,0899

Table 7.3: Forward exchange rates market data.

Σ^{dc}	ρ
0,4	0,7

Table 7.4: Correlation data.

We consider as numeraire:

$$\text{Num}(t) = \begin{cases} B_5^d(t), & \text{if } t \leq T_5, \\ \beta_5^d(t), & \text{if } t > T_5, \end{cases}$$

where

$$\beta_5^d(t) = B_j^d(t) \prod_{l=5}^{j-1} \left(1 + \delta_l F_l^d(T_l)\right), \quad t \in (T_{j-1}, T_j], \quad j = 6, \ldots, N+1. \tag{7.32}$$

In this case the numeraire is B_5^d until T_5 and the spot measure starting after T_5 for $t > T_5$. Although this choice neither corresponds to a forward measure nor to a pure spot one, the proposed methodology can be applied.

Taking into account this information, we compare:

- the value of the martingales at time $t = 0$ jointly with the expected values obtained with the simulation method, without and with adjustment at simulation level (see Table 7.5),

- the Black value of the caplets ATM of the domestic economy (see, for example [16]) with the one obtained with the simulation at time $t = 0$ (see Table 7.6).

Finally, Table 7.7 shows the market price (7.30) and the simulated price with adjustment for one Spread Option that matures at time T_5 with $T_j = T_7$. It also shows the difference in basic points (bbpp) between the simulated price and the market one and the confidence interval.

j	$\widehat{B}_j^d(0)$	PDFS	PDFS WA	$\widehat{B}_j^c(0)$	LE DFS	LE DFS WA
1	1,128814	1,128859	1,128814	0,677288	0,677269	0,677288
2	1,104475	1,104562	1,104475	0,773133	0,773180	0,773133
3	1,072910	1,072966	1,072910	0,858328	0,858592	0,858328
4	1,037708	1,037731	1,037708	0,933937	0,934602	0,933937
5	1,000000	1,000000	1,000000	1,000000	1,001306	1,000000
6	0,960975	0,960928	0,960975	1,057073	1,056973	1,057073
7	0,921779	0,921713	0,921779	1,106135	1,106138	1,106135
8	0,883174	0,883096	0,883173	1,148126	1,148940	1,148126
9	0,845215	0,845084	0,845216	1,183302	1,183610	1,183302
10	0,808082	0,807925	0,808082	1,212123	1,212015	1,212123
11	0,771402	0,771203	0,771402	1,234244	1,233795	1,234244

Table 7.5: Expected values obtained from the DFS methods, without and with adjustment (WA), compared with the martingales values at time 0.

Table 7.5 illustrates how the proposed adjustment guarantees the martingale property for $\widehat{B}_j^d$ and $\widehat{B}_j^c$ at discrete level, although in the case of $\widehat{B}_j^d$ with $j > 5$ it is an adjustment based on an approximation. Moreover, Table 7.6 shows that the PDFS WA technique provides better results than PDFS in the domestic caplets pricing. Finally, we obtain the price of a Spread Option by simulation which is similar to the

j	BLACK	PDFS	PDFS WA
1	0,004329	0,004333	0,004328
2	0,005498	0,005510	0,005510
3	0,005978	0,005984	0,005983
4	0,006173	0,006185	0,006185
5	0,006086	0,006109	0,006103
6	0,006151	0,006164	0,006161
7	0,005965	0,005996	0,005984
8	0,005954	0,005956	0,005955
9	0,005851	0,005865	0,005859
10	0,005753	0,005762	0,005760

Table 7.6: Value at the zero time of domestic caplets ATM that mature at T_j. The Black values and the ones obtained from the PDFS method without and with adjustment are presented.

Market Value	Simulated Price	Difference in bbpp	Confidence Interval
0,0010781	0,0010924	0,3849550	[0,0010342 , 0,0011506]

Table 7.7: Results for the Spread Option that matures at time T_5 with $T_j = T_7$ and $K = FX_5^{dc}(0)$, being the DFS with adjustment the chosen technique. Confidence level = 95%.

one computed with the market formula.

7.7 Inflation

Another example where we can apply the Second Cross-Market Model setting is the inflation case, where the units are the domestic currency and one basket of basic products. Therefore, we have the domestic (nominal) and the basket (real) economies plus the forward inflation index that relates them. So if $A \equiv d$ and $B \equiv b$, by analogy with Section 5, we can redefine B_j^d, B_j^b, F_j^d, F_j^b, X^{db} and FX_j^{db}. As in the commodity case, in the inflation setting there exists a specific notation (see [42]): the spot exchange

and the forward exchange rates are the inflation index, I, and the forward inflation indexes, $\mathcal{I}_j$, respectively. Again, as in the commodity case, B_j^b and F_j^b are introduced so that we can proceed analogously as in the general frame of cross-market models, although they have no practical sense.

The most popular inflation derivatives are the Inflation Indexed Swaps (IIS). The IIS are agreements between two parties to exchange some flows, in which at least one of the exchanged flows depends on one inflation index. The main IIS traded in the market are the Zero Coupon Inflation Indexed Swaps (ZCIIS) and the Year-On-Year Inflation Indexed Swaps (YYIIS). Based on IIS the Inflation Indexed Swaptions (IISO) arise. The IISO are options that give the right to enter into an IIS contract at one future date and with one previously specified fixed rate. In the market there exist also other derivatives like the Caplets/Floorlets Inflation Indexed (C/FII) [42].

As we can see in [42], the ZCIIS are quoted in the market in terms of the rate of the fixed payment that makes the value of the contract at time T_0 equal to zero and we can obtain the values of the basket bonds $B_1^b, ..., B_N^b$ from the quoted rates $K_1, ..., K_N$ of the ZCIIS that mature at times $T_1, ..., T_N$, respectively. This follows from the pricing formula of these products, that is independent on the chosen model. Moreover, [42] develops the pricing of the YYIIS for three different models. Two of them are models analogous to Black, and Mercurio in [42] refers to them as the First Market Model and the Second Market Model. These First and Second Market Models are consistent with the ones presented in the Figures 5.2 and 5.3, respectively.

Domestic volatilities can be calibrated automatically with (domestic) caps, but in the inflation setting volatilities of the basket forward rates F_j^b (for the First Cross-Market Model case) and volatilities of the inflation forward rates $\mathcal{I}_j$ (for the Second Cross-Market Model case) are very difficult to estimate. In [42] Mercurio proposes a

calibration procedure by using some parameterization of the volatilities and correlations that have to be estimated.

Part III

Some graphs tools for Market Models

Chapter 8

Connecting graphs with Generic Market Models

In this chapter we frame the methodology presented throughout this thesis into a graph theoretical context.

Our approach is the following:

1. At time zero, no arbitrage means that Kirchoff's laws are satisfied.

2. Further, the evolution that we will impose on the nodes of the chosen spanning tree prevents intertemporal arbitrage opportunities.

3. Since, by construction, at every time t Kirchoff's laws are satisfied, there is no arbitrage opportunities buying and selling (instantaneously) at time t.

Next we detail our approach, that is sketched in Figure 8.1.

All models that we have described so far can be framed into the context of a general connected directed graph $G = (V^G, E^G)$, where V^G, the set of nodes of G, represents the prices of basic assets and E^G, the set of edges of G, contains information connecting the values of those nodes.

For all $v \in V^G$, $[v]$ denotes the value of the node v. For all $v, w \in V^G$, $\{v, w\} \in E^G$ denotes the edge that goes from v to w and $[v, w]$ denotes the value of the edge $\{v, w\}$ that is given by:

$$[v, w] = \frac{[w]}{[v]}. \tag{8.1}$$

Observed that $[v, w] = 1/[w, v]$.

We assume that the values of nodes are prices.

The value in each node and each edge at time zero is defined by the market so that there is no external arbitrage. This, as we shall see shortly, translates into the equivalent fact that Kirchoff's laws are satisfied [60].

Given the prices P_1 and P_2 of the basic assets represented in the nodes v_1 and v_2, respectively, i.e., with $[v_1] = P_1, [v_2] = P_2$, for any two paths from v_1 to v_2,

$$
\begin{aligned}
C_{v_1, v_2} &= \big\{ \{v_1, a_1\}, \{a_1, a_2\}, ..., \{a_x, v_2\} \big\} \subseteq E^G, \\
\widetilde{C}_{v_1, v_2} &= \big\{ \{v_1, b_1\}, \{b_1, b_2\}, ..., \{b_y, v_2\} \big\} \subseteq E^G,
\end{aligned}
$$

we have

$$[C_{v_1, v_2}] = \prod_{\{v, w\} \in C_{v_1, v_2}} [v, w] = \frac{[a_1]}{[v_1]} \frac{[a_2]}{[a_1]} ... \frac{[v_2]}{[a_x]} = \frac{P_2}{P_1},$$

$$[\widetilde{C}_{v_1, v_2}] = \prod_{\{\widetilde{v}, \widetilde{w}\} \in \widetilde{C}_{v_1, v_2}} [\widetilde{v}, \widetilde{w}] = \frac{[b_1]}{[v_1]} \frac{[b_2]}{[b_1]} ... \frac{[v_2]}{[b_y]} = \frac{P_2}{P_1}.$$

Therefore, no external arbitrage is equivalent to $[C_{v_1, v_2}] = [\widetilde{C}_{v_1, v_2}]$, or equivalently, to

$$\sum_{\{v, w\} \in C_{v_1, v_2}} \ln [v, w] - \sum_{\{\widetilde{v}, \widetilde{w}\} \in \widetilde{C}_{v_1, v_2}} \ln [\widetilde{v}, \widetilde{w}] = 0, \tag{8.2}$$

i.e., the graph G is a conservative field that verifies Kirchoff's laws.

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The graph G is given at time zero, and it satisfies Kirchoff's laws. Next, we describe how to evolve the graph by analogy with the methodology previously applied for the market models, that is, so that there is no internal arbitrage.

Step 1. In market models, we first choose the model.

Since G is a connected graph, there exists at least one spanning tree of G. We choose one of the possible spanning trees of G, $A = (V^A, E^A)$, which defines the way to go from one node to another one in a unique way. Moreover, we assume that the value in each edge of A, $\{v, w\} \in E^A$, is specified by some function $f_{\{v,w\}}$ depending on one stochastic process $Y_{\{v,w\}}$ whose dynamic is given by

$$\frac{dY_{\{v,w\}}(t)}{Y_{\{v,w\}}(t)} = \mu_{\{v,w\}}(t)dt + \sigma_{\{v,w\}}(t)dW_{\{v,w\}}(t). \tag{8.3}$$

If $|V^G|$ denotes the number of nodes of the graph G, then we have $|V^G| - 1$ stochastic process, one for each edge of A.

Step 2. Once we have chosen the numeraire, we compute the drift terms to avoid internal arbitrage.

The analogous step in graph context is given by the following theorem:

Theorem 8.0.1. *Once we have chosen a root node $R \in V^A$, the dynamics of $Y_{\{v,w\}}$ are defined by imposing that $[C_{R,v}]$ is a martingale for all $v \in V^A$, where $C_{R,v}$ denotes the unique path in A from R to v.*

Proof. Once we have chosen a root node $R \in V^A$, the subsequent generations

of nodes of A are defined by:

$$
\begin{aligned}
G^1 &= \{R\} \\
G^2 &= \{\text{Neighboring nodes to } R\} \\
G^i &= \{\text{Neighboring nodes to nodes of } G^{i-1}\} \setminus G^{i-2}, \quad i = 3, ..., a, \quad (8.4)
\end{aligned}
$$

where a is the height of A.

Since A is a spanning tree, for each node $v^l \in G^l$ ($l = 2, ..., a$) there exists a unique path $C_{R,v^l} = \left\{ \{a_1^l, b_1^l\}, \{a_2^l, b_2^l\}, ..., \{a_{l-1}^l, b_{l-1}^l\} \right\}^1$ and

$$
[C_{R,v^l}] = \prod_{m=1}^{l-1} f_{\{a_m^l, b_m^l\}}\left(Y_{\{a_m^l, b_m^l\}}\right). \tag{8.5}
$$

We redefine $f_m^l \equiv f_{\{a_m^l, b_m^l\}}$, $Y_m^l \equiv f_{\{a_m^l, b_m^l\}}$, $\mu_m^l \equiv \mu_{\{a_m^l, b_m^l\}}$, $\sigma_m^l \equiv \sigma_{\{a_m^l, b_m^l\}}$ to simplify the notation.

Therefore, by (8.3) and Ito's Lemma, we have:

$$
\begin{aligned}
\frac{d[C_{R,v^l}](t)}{[C_{R,v^l}](t)} = &\left[\sum_{m=1}^{l-1} \left(\frac{\frac{\partial f_m^l}{\partial Y_m^l}(Y_m^l(t))Y_m^l(t)}{f_m^l(Y_m^l(t))} \right) \mu_m^l(t) + \frac{1}{2} \sum_{m=1}^{l-1} \left(\frac{\frac{\partial^2 f_m^l}{\partial Y_m^{l^2}}(Y_m^l(t))\sigma_m^l(t)^2 Y_m^l(t)^2}{f_m^l(Y_m^l(t))} \right) + \right. \\
&\left. + \frac{1}{2} \sum_{\substack{m_1, m_2=1 \\ m_1 \neq m_2}}^{l-1} \left(\frac{\frac{\partial f_{m_1}^l}{\partial Y_{m_1}^l}(Y_{m_1}^l(t))\frac{\partial f_{m_2}^l}{\partial Y_{m_2}^l}(Y_{m_2}^l(t))\sigma_{m_1}^l(t)\sigma_{m_2}^l(t)Y_{m_1}^l(t)Y_{m_2}^l(t)\Sigma_{m_1,m_2}^l}{f_{m_1}^l(Y_{m_1}^l(t))f_{m_2}^l(Y_{m_2}^l(t))} \right) \right] dt + ...,
\end{aligned}
$$
$$\tag{8.6}$$

where Σ_{m_1,m_2}^l is the correlation between $Y_{m_1}^l$ and $Y_{m_2}^l$.

---

[1]Notice that $a_1^l = R$, $b_{l-1}^l = v^l$ and $b_j^l = a_{j+1}^l$, $\forall j = 1, ..., l-2$.

Next, we impose that $[C_{R,v^l}]$ is a martingale, so that:

$$\sum_{m=1}^{l-1}\left(\frac{\frac{\partial f_m^l}{\partial Y_m^l}(Y_m^l(t))Y_m^l(t)}{f_m^l(Y_m^l(t))}\right)\mu_m^l(t)+\frac{1}{2}\sum_{m=1}^{l-1}\left(\frac{\frac{\partial^2 f_m^l}{\partial Y_m^{l2}}(Y_m^l(t))\sigma_m^l(t)^2Y_m^l(t)^2}{f_m^l(Y_m^l(t))}\right)+$$

$$+\frac{1}{2}\sum_{\substack{m_1,m_2=1\\m_1\neq m_2}}^{l-1}\left(\frac{\frac{\partial f_{m_1}^l}{\partial Y_{m_1}^l}(Y_{m_1}^l(t))\frac{\partial f_{m_2}^l}{\partial Y_{m_2}^l}(Y_{m_2}^l(t))\sigma_{m_1}^l(t)\sigma_{m_2}^l(t)Y_{m_1}^l(t)Y_{m_2}^l(t)\Sigma_{m_1,m_2}^l}{f_{m_1}^l(Y_{m_1}^l(t))f_{m_2}^l(Y_{m_2}^l(t))}\right)=0,$$

$$(8.7)$$

or, equivalently,

$$\mu_{l-1}^l=\left[-\sum_{m=1}^{l-2}\left(\frac{\frac{\partial f_m^l}{\partial Y_m^l}(Y_m^l(t))Y_m^l(t)}{f_m^l(Y_m^l(t))}\right)\mu_m^l(t)-\frac{1}{2}\sum_{m=1}^{l-1}\left(\frac{\frac{\partial^2 f_m^l}{\partial Y_m^{l2}}(Y_m^l(t))\sigma_m^l(t)^2Y_m^l(t)^2}{f_m^l(Y_m^l(t))}\right)-\right.$$

$$\left.-\frac{1}{2}\sum_{\substack{m_1,m_2=1\\m_1\neq m_2}}^{l-1}\left(\frac{\frac{\partial f_{m_1}^l}{\partial Y_{m_1}^l}(Y_{m_1}^l(t))\frac{\partial f_{m_2}^l}{\partial Y_{m_2}^l}(Y_{m_2}^l(t))\sigma_{m_1}^l(t)\sigma_{m_2}^l(t)Y_{m_1}^l(t)Y_{m_2}^l(t)\Sigma_{m_1,m_2}^l}{f_{m_1}^l(Y_{m_1}^l(t))f_{m_2}^l(Y_{m_2}^l(t))}\right)\right]/$$

$$/\left(\frac{\frac{\partial f_{l-1}^l}{\partial Y_{l-1}^l}(Y_{l-1}^l(t))Y_{l-1}^l(t)}{f_{l-1}^l(Y_{l-1}^l(t))}\right).$$

$$(8.8)$$

Therefore, from each node of the generation G^2 we can recursively compute the drift terms of its descendants.

$\square$

Step 3. We obtain the values of the rates that were not modeled.

In graph context, once we have obtained the values on the nodes in the spanning tree A at each time, we can compute the values in the edges that are not included in A by using Kischoff's laws, thus avoiding the arbitrage at each time.

DFS in the graph context:

Finally, if $f_{\{v,w\}}$ is an invertible function for all edge $\{v,w\} \in E^A$, we can also present the DFS methodology into this graph framework.

Once we have chosen a root node $R \in V^A$, the generations of nodes of A in (8.4) are defined. For all node $v^l \in G^l$, we denote by $d(v^l)$ one descendant of v^l, i.e., one node of G^{l+1} that is neighbor of v^l. Then, we have:

$$\left[C_{R,d(v)}\right](t) = \left[C_{R,v}\right](t)\left[v, d(v)\right](t) = \left[C_{R,v}\right](t) f_{\{v,d(v)\}}\left(Y_{\{v,d(v)\}}(t)\right). \qquad (8.9)$$

Therefore, by Ito's Lemma and taking into account (8.3) and that $\left[C_{R,v}\right](t)$ and $\left[C_{R,d(v)}\right](t)$ are martingales, we have

$$d\left[C_{R,d(v)}\right](t) = f_{\{v,d(v)\}}\left(Y_{\{v,d(v)\}}(t)\right) d\left[C_{R,v}\right](t) +$$

$$+ \left[C_{R,d(v)}\right] \frac{\partial f_{\{v,d(v)\}}}{\partial Y_{\{v,d(v)\}}}\left(Y_{\{v,d(v)\}}(t)\right) \sigma_{\{v,d(v)\}}(t) Y_{\{v,d(v)\}}(t) dW_{\{v,d(v)\}}(t).$$
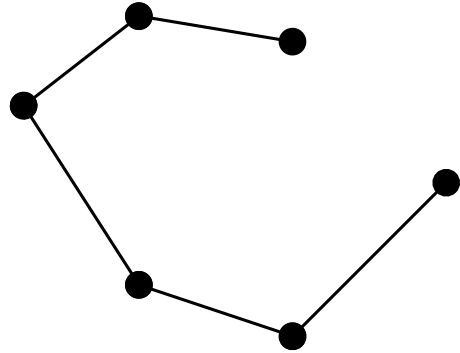
$$(8.10)$$

At each time t, we have $d[C_{R,R}](t) = 0$. From this condition, for all node $v_j^2 \in G^2$ we can compute recursively the dynamics $d\left[C_{R,v_j^2}\right](t)$, $d\left[C_{R,v_j^3}\right](t)$, $d\left[C_{R,v_j^4}\right](t)$, ..., where $v_j^l = d(v_j^{l-1})$, $\forall l = 3, 4, ...$. Then, from the values of the paths obtained by discretization of these dynamics, we can recover the values of the processes by

$$Y_{v,d(v)}(t) = f_{v,d(v)}^{-1}\left(\frac{\left[C_{R,d(v)}\right](t)}{\left[C_{R,v}\right](t)}\right). \qquad (8.11)$$
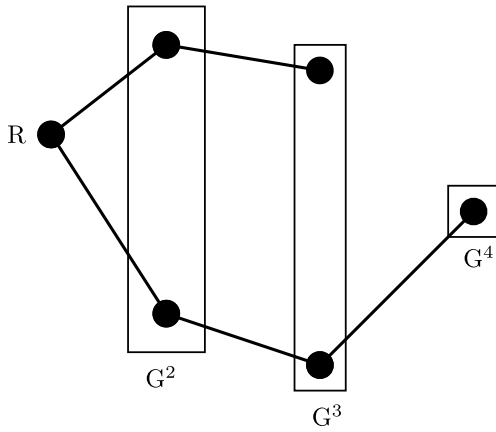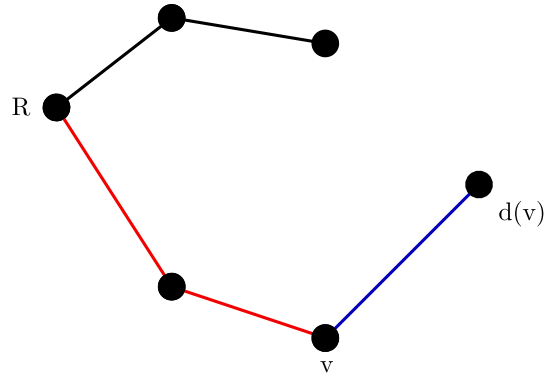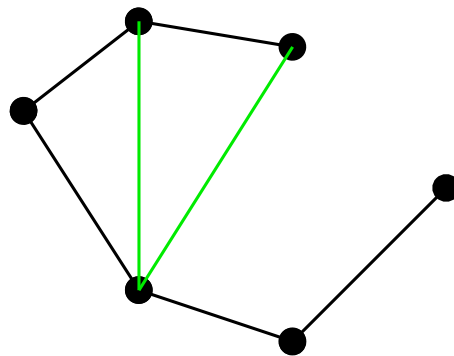
132
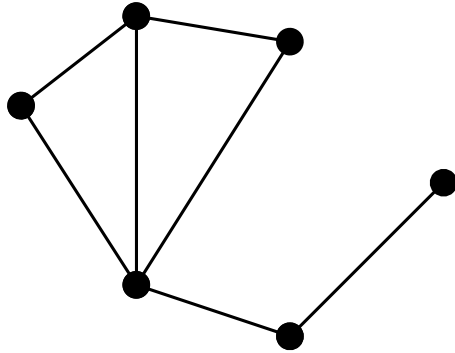
1. Connected graph G.

2. Spanning tree A.

3. Root node R $\Rightarrow$ The generations are defined.

4. Drift-Free Simulation.
$[C_{R,d(v)}] = [C_{R,v}]\ [v, d(v)]$

5. The graph is completed by Kirchoff's laws.

Figure 8.1: Outline of our approach.

Appendix A

An auxiliary result in matrix algebra to proof (3.42)

In this appendix we present the main algebraic results to be used in the proof of (3.42).

Lemma A.0.2. *Let M be a matrix in $\mathcal{M}^{d \times d}$ and let us assume that for each $i = 1, ..., d$ there exists an index l_i, with $l_i \geqslant i$, such that*

$$M_{ij} = \begin{cases} 0, & \text{if } j < i; \\ 1, & \text{if } i \leqslant j \leqslant l_i; \\ 0, & \text{if } j > l_i. \end{cases} \tag{A.1}$$

If $v = (v_1, ... v_d)$ is an orthogonal vector to the first $d - 1$ rows of M with $v_d = 1$, then for all $i = 1, ..., d$ we have:

(a_i) $\displaystyle\sum_{k=i}^{m} v_k \in \{-1, 0, 1\}$, $\forall m = i, i+1, ..., d$,

(b_i) $\displaystyle\sum_{k=m}^{d} v_k \in \{-1, 0, 1\}$, $\forall m = i, i+1, ..., d$,

(c_i) $\displaystyle\sum_{k=m}^{\widetilde{m}} v_k \in \{-1, 0, 1\}$, $\forall m = i+1, i+2, ..., d$, $\forall \widetilde{m} = m, m+1, ..., d$.

Proof. Since $v_d = 1$, then (a_d), (b_d) and (c_d) are trivial.

For each $i = 1, ..., d - 1$, v is orthogonal to the row $(M_{i1}, ..., M_{id})$, so we have:

$$\sum_{k=1}^{d} M_{ik} v_k = 0 \quad \Rightarrow \quad \sum_{k=i}^{l_i} v_k = 0 \quad \Rightarrow \quad v_i = - \sum_{k=i+1}^{l_i} v_k. \tag{A.2}$$

Taking into account (A.2), we can prove by induction in $i = d - 1, ..., 1$ the properties (a_i), (b_i) and (c_i).

- $i = d - 1$

  From (A.2), we have

  $$v_{d-1} = - \sum_{k=d}^{l_{d-1}} v_k = \begin{cases} 0, & \text{if } l_{d-1} = d - 1; \\ -v_d = -1, & \text{if } l_{d-1} = d. \end{cases}$$

  Therefore, as $v_d = 1$ and $v_{d-1} \in \{-1, 0\}$, then (a_{d-1}), (b_{d-1}) and (c_{d-1}) are satisfied.

  $\vdots$

- $i = n$

  We assume that (a_n), (b_n) and (c_n) are satisfied.

- $i = n - 1$

  From (A.2), we have

  $$v_{n-1} = - \sum_{k=n}^{l_{n-1}} v_k = \begin{cases} 0 \in \{-1, 0, 1\}, & \text{if } l_{n-1} = n - 1; \\ - \sum_{k=n}^{l_{n-1}} v_k \in \{-1, 0, 1\}, \text{ through } (a_n), & \text{if } l_{n-1} > n - 1. \end{cases}$$

  Therefore, we have

  $(\mathbf{a}_{n-1})$ $\displaystyle\sum_{k=n-1}^{m} v_k \in \{-1, 0, 1\}$, $\forall m = n - 1, n, ..., d$, since:

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– if $m = n - 1$, then:

$$\sum_{k=n-1}^{n-1} v_k = v_{n-1} \in \{-1, 0, 1\};$$

– if $m = n, ..., d$, then:

$$\sum_{k=n-1}^{m} v_k = v_{n-1} + \sum_{k=n}^{m} v_k = -\sum_{k=n}^{l_{n-1}} v_k + \sum_{k=n}^{m} v_k =$$

$$= \begin{cases} \displaystyle\sum_{k=n+1}^{m} v_k \in \{-1, 0, 1\} \text{ through } (\text{c}_n), & \text{if } l_{n-1} < m; \\ 0 \in \{-1, 0, 1\}, & \text{if } l_{n-1} = m; \\ -\displaystyle\sum_{k=m+1}^{l_{n-1}} v_k \in \{-1, 0, 1\} \text{ through } (\text{c}_n), & \text{if } l_{n-1} > m. \end{cases}$$

$(\mathbf{b}_{n-1})$ $\displaystyle\sum_{k=m}^{d} v_k \in \{-1, 0, 1\}$, $\forall m = n - 1, n, ..., d$, since:

– if $m = n - 1$, then:

$$\sum_{k=n-1}^{d} v_k \in \{-1, 0, 1\} \text{ through } (\text{a}_{n-1});$$

– if $m = n$, then:

$$\sum_{k=n}^{d} v_k \in \{-1, 0, 1\} \text{ through } (\text{a}_n);$$

– if $m = n + 1, ..., d$, then:

$$\sum_{k=m}^{d} v_k \in \{-1, 0, 1\} \text{ through } (\text{c}_n).$$

$(\mathbf{c}_{n-1})$ $\displaystyle\sum_{k=m}^{\widetilde{m}} v_k \in \{-1, 0, 1\}$, $\forall m = n, n + 1, ..., d$, $\forall \widetilde{m} = m, m + 1, ..., d$, since:

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– if $m = n$ and $\widetilde{m} = n, n + 1, ..., d$, then:

$$\sum_{k=n}^{\widetilde{m}} v_k \in \{-1, 0, 1\} \text{ through } (a_n);$$

– if $m = n + 1, ..., d$ and $\widetilde{m} = m, m + 1, ..., d$, then:

$$\sum_{k=m}^{\widetilde{m}} v_k \in \{-1, 0, 1\} \text{ through } (c_n);$$

□

Theorem A.0.3. *Let M be a matrix in $\mathcal{M}^{d \times d}$ and assume that for each $i = 1, ..., d$ there exists an index l_i, with $l_i \geqslant i$, such that*

$$M_{ij} = \begin{cases} 0, & \text{if } j < i; \\ 1, & \text{if } i \leqslant j \leqslant l_i; \\ 0, & \text{if } j > l_i. \end{cases} \tag{A.3}$$

Then, the coefficients of the inverse matrix M^{-1} belong to the set $\{-1, 0, 1\}$.

Proof. Since M is an upper triangular matrix with diagonal elements equal to 1, there exists the inverse matrix of M, M^{-1}, which is an upper triangular matrix too. Therefore, for all $p = 1, ..., d$, $M_{ip}^{-1} = 0 \in \{-1, 0, 1\}$ if $i > p$.

It remains to prove that $M_{ip}^{-1} \in \{-1, 0, 1\}$ for all $p = 1, ..., d$ and $i = 1, ..., p$.

For this purpose we consider the following submatrix of M:

$$A_p = \begin{pmatrix} M_{11} & \cdots & M_{1p} \\ \vdots & & \vdots \\ M_{p1} & \cdots & M_{pp} \end{pmatrix},$$

for all $p = 1, 2, ..., d$. Then, $\left(M_{1p}^{-1}, ..., M_{pp}^{-1} \right)$ is orthogonal to the first $p - 1$ rows of A_p and $M_{pp}^{-1} = 1$. Hence, from comdition (a_p) in the previous Lemma A.0.2, we obtain that $M_{1p}^{-1}, ..., M_{pp}^{-1} \in \{-1, 0, 1\}$.

□

In order to obtain (3.42), we can formulate (3.38) in matrix form and apply Theorem A.0.3 with $d = N + 1$.

Appendix B

Number of possible Cross-Market Models

In this appendix we obtain the number of possible Cross-Market Models (5.4). Notice that this number coincides with the number of the possible cross-market models which include the implicit forward rates of the economy A among the modeled rates.

Let T_N denote the maturity date. We consider the graph G_N in Figure B.1 with $2(N+1)$ nodes and $3N+1$ edges. Let Φ_N denote the set of possible spanning trees of G_N containing all the edges of the first row (see Figure B.2). Our aim is to obtain the number ϕ_N of the trees that are in Φ_N.

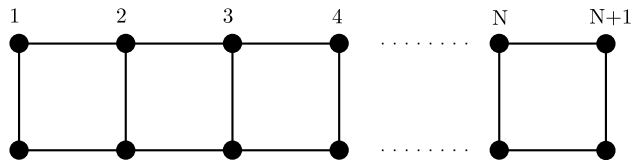

Figure B.1: Graph G with $2(N+1)$ nodes and $3N+1$ edges.

If we consider the edge between the last nodes of each row as the last edge, we can classify the trees of Φ_N into two kinds according to their last edge:
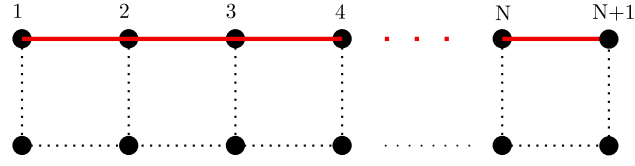
Figure B.2: We consider the spanning trees of G that contain the edges of the first row (marked by a thicker red line).

- Type 1: Trees which include the last edge.

- Type 2: Trees which do not include the last edge.

Then, $\phi_N = \phi_N^1 + \phi_N^2$, where ϕ_N^1 and ϕ_N^2 are the numbers of trees of Type 1 and Type 2, respectively.

For the case $N = 1$ (see Figure B.3), we have $\phi_1^1 = 2$ and $\phi_1^2 = 1$, so that $\phi_1 = 3$.
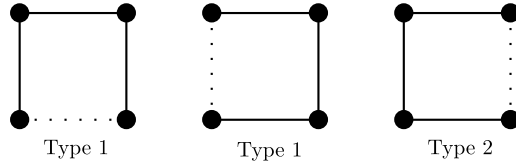
Figure B.3: Trees of Φ_1.

Moreover, by expanding each Type 1 tree of Φ_N we can obtain two Type 1 trees and one Type 2 tree of Φ_{N+1} (see Figure B.4) and by expanding each Type 2 tree of Φ_N we can obtain one Type 1 tree and one Type 2 tree of Φ_{N+1} (see Figure B.5).

Then, we have

$$\begin{pmatrix} \phi_{N+1}^1 \\ \phi_{N+1}^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi_N^1 \\ \phi_N^2 \end{pmatrix}, \tag{B.1}$$
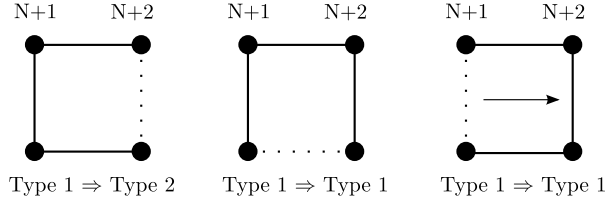
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Figure B.4: Possibilities to expand type 1 trees.
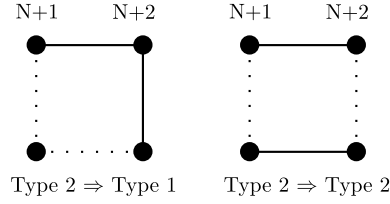
Figure B.5: Possibilities to expand type 2 trees.

and, therefore,

$$\begin{pmatrix} \phi^1_{N+1} \\ \phi^2_{N+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^N \begin{pmatrix} \phi^1_1 \\ \phi^2_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^N \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{B.2}$$

Moreover, we can prove by induction that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^N = \begin{pmatrix} f_{2N+1} & f_{2N} \\ f_{2N} & f_{2N-1} \end{pmatrix}, \tag{B.3}$$

where the numbers f_n are the numbers of the Fibonacci sequence, i.e.,

$$\begin{cases} f_0 = 0, \\ f_1 = 1, \\ f_n = f_{n-1} + f_{n-2}, & \text{if } n > 1, \end{cases} \tag{B.4}$$

or, equivalently,

$$f_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}, \tag{B.5}$$

with $\varphi = \dfrac{1 + \sqrt{5}}{2}$.

Then, taking into account (B.2) and (B.3), we have

$$\begin{pmatrix} \phi_N^1 \\ \phi_N^2 \end{pmatrix} = \begin{pmatrix} f_{2(N-1)+1} & f_{2(N-1)} \\ f_{2(N-1)} & f_{2(N-1)-1} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{B.6}$$

Therefore, we obtain

$$\phi_N = \phi_N^1 + \phi_N^2 = 2f_{2N-1} + 3f_{2N-2} + f_{2N-3}, \tag{B.7}$$

or, equivalently, (5.4).

Conclusions

The objective of this work has been to propose an efficient procedure to simulate the stochastic dynamics of Market Models for interest rates that avoids the use of the drift dependent paths in Monte Carlo simulation. For this purpose, we have followed a Drift-Free Simulation methodology, by first simulating certain martingales and then obtaining the involved forward rates in terms of them. More precisely, we propose an original parameterization that ensures two properties of the continuous model after the discretization: positivity of the rates and martingale property for the deflated bonds. These properties are guaranteed even when we use the spot probability measure. We have presented this methodology into the one economy and two economies settings.

In the one economy context first we have considered the Libor Market Model case. Then, the developed Drift-Free Simulation methodology has been adapted to the Coterminal Swap Market Model. This application has been mainly formulated by analogy with the technique described for Libor Market Model, however we must point out the main difference: while in the Libor Market Model case the dynamics of each deflated bond only depends on the dynamics of another deflated bond, in the Coterminal Swap Market Model case the dynamics of each deflated annuity depends on the dynamics of two other deflated annuities. Moreover, we have also studied the Generic Market Model context and the recently appeared multicurve LMM that tries to explain the new behavior of interest rate curves after the financial crisis in 2008.

Clearly, the presence of multiple curves gives rise to the requirement of the corresponding volatility structures. However, the standard volatility quotes in the market are not enough in this multicurve setting, so that new techniques to obtain volatilities for non standard tenors are still being developed. In order to overcome this difficulty, in practice the one curve Libor Market Model with an appropriate choice of the input curve and volatility is still used.

In the two economy framework we have considered two Cross-Market Model cases, thus applying the first one to the two currencies setting and the second one to the commodities setting. Finally, we have framed the proposed methodology into the graph theory.

The algorithms concerning to the Drift-Free Simulation techniques have been implemented in C.

We have obtained results for the Libor Market Model case in one economy context, and also for the two currencies and the commodities cases in the two economies setting.

When we have considered the one economy Libor Market Model, the obtained results for different examples have been presented. In order to illustrate the distribution of the errors across maturity, in the first example a flat structure of interest rates and volatilities has been assumed. The second example has corresponded to a more realistic situation in which different values both for initial forward rates and for constant volatilities have been considered. In the flat structure example, the proposed method generally exhibits the best performance in all caplet pricing. Furthermore, the martingale adjustment at simulation levels leads to a significant reduction. In the uneven structure example, we have shown the errors in basis points when comparing the martingale values at time zero with the expected values obtained with

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the parameterized Drift-Free Simulation without and with adjustment. We have observed how the proposed martingale adjustment reduces the errors that appear with the discretization, although it is an adjustment based on one approximation. Next, analogously to first example, the errors for the caplet pricing with different simulation methods and for different strikes have been presented. Note that the same qualitative behavior as in first example is observed.

Having in view the previous comments, the proposed method not only guarantees after the discretization the desired properties of the continuous model, but also it provides the best results when compared with other techniques in the literature.

When we have applied the First Cross-Market Model to the two currencies setting, we have compared the value of the martingales at time zero jointly with the expected values obtained with different simulation methods. Also, we have compared the Black value of the ATM caplets of each economy with the ones obtained by simulations at time $t = 0$. For all Drift-Free Simulation methods, the proposed adjustment guarantees the martingale property at discrete level in each economy and the numerical results in the caplets pricing are better in the here proposed Parameterized Drift-Free Simulation method for the domestic currency and in the Implicit Drift-Free Simulation one for the foreign currency. In any case, the results for the foreign martingale with the Implicit Drift-Free Simulation method are very close to those ones with the Parameterized Drift-Free Simulation method. Moreover, we have priced Quanto Caplets and Quanto Floorlets and we have observed that the here proposed Parameterized Drift-Free Simulation method provides the best results in the quanto pricing for the previously refered as Caplet 2 and Floorlet 1, and it results to be the second best method for the Caplet 1. So, taking into account its results and its positiveness preserving property, in general the proposed Parameterized Drift-Free Simulation method exhibits the best performance for the cross-currency simulation.

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When we have applied a second Cross-Market Model to the commodity setting, we have made a comparison in each economy analogous to the one economy case and we have observed similar behaviors. Moreover, we have obtained a Spread Option price by simulation which is similar to the one computed with the market formula.

Resumen extenso

En este trabajo proponemos una metodología para simular diferentes modelos de tipos de interés evitando el uso de los términos de derivas que aparecen en las dinámicas involucradas en dichos modelos. Más concretamente, analizamos esta metodología en el contexto de los modelos de mercado (*Market Models*) y de los modelos de mercados cruzados (*Cross-Market Models*), cuyo propósito fundamental consiste en plantear las dinámicas de los procesos subyacentes para valorar derivados de tipos de interés de una economía y derivados de tipos de interés de dos economías, respectivamente.

Los derivados de tipos de interés son instrumentos financieros cuya función de pago depende de determinados tipos de interés. El ejemplo más clásico es una opción que paga la parte positiva de la diferencia entre un tipo de interés variable y un tipo fijo (*strike*). La primera vez que estos derivados financieros se comercializaron en mercados organizados fue a principios de los setenta del siglo XX en el *Chicago Board of Options Exchange (CBOE)*. En esa época, Fisher Black y Myron Scholes asumieron dinámica lognormal para el tipo de interés subyacente y propusieron la metodología de cobertura dinámica a fin de obtener la popular fórmula de Black-Scholes para opciones *vanilla* europeas [9]. Desde entonces, la aparición de derivados de tipos de interés cada vez más complejos ha motivado la necesidad de conocer la evolución de los tipos involucrados. Básicamente, dos clases de modelos surgen para obtener esta evolución [16]: modelos de tipo a corto (*short rate models*) y modelos de mercado (*market models*). Los modelos de tipo a corto describen la evolución del tipo de interés instantáneo (*spot*), r_t, mediante un proceso de difusión multidimensional

dependiente de varios parámetros o funciones. Estos modelos se pueden escribir de modo general en la forma:

$$dr_t = u(t, r_t)dt + v(t, r_t)dW_t,$$

donde W_t denota un movimiento Browniano y las expresiones de las funciones u y v dan lugar a diferentes modelos de tipo a corto, como los modelos de equilibrio, con u y v dependientes sólo del tipo de interés instantáneo, independientes del tiempo (Vasicek [61], Cox, Ingersoll y Ross [17]), o los modelos de no arbitraje, con u y v dependientes del tipo de interés instantáneo y del tiempo (Ho y Lee [29], Hull y White [31], Black y Karasinski [11]). Una ventaja de los modelos de equilibrio es que permiten obtener fórmulas analíticas para valorar los bonos cupón cero (*zero coupon bonds*) o incluso bonos que pagan cupones (*coupon bearing bonds*), pero el calibrado de sus parámetros constantes a los datos de mercado resulta casi imposible en la práctica. Los modelos de no arbitraje surgieron a fin de superar esta desventaja, no obstante las dificultades en el calibrado de la curva inicial de factores de descuento aún se mantiene. Alternativamente, para solventar los problemas mencionados surgen otros modelos con mayor complejidad computacional. Como primera alternativa importante a los modelos de tipo a corto, el enfoque de Ho y Lee [29] fue llevado a tiempo continuo por Heath, Jarrow y Morton [27], quienes desarrollaron un marco general para la evolución estocástica de la curva de rendimiento completa, en el que las dinámicas de los tipos *forward* están completamente especificadas a través de su estructura de volatilidades instantáneas. Las principales desventajas del modelo Heath-Jarrow-Morton son: que está expresado en términos de tipos *forward* instantáneos continuos, que no son observables directamente en mercado, y que no es fácil su calibrado a precios de instrumentos muy activamente comercializados, como los *caps*. Con el objetivo de superar estas desventajas, tomando el marco de Heath, Jarrow y Morton, se introdujo la familia más popular en el momento actual de modelos de tipos de interés: los modelos de mercado (*Market Models*), siendo el *Libor Market Model (LMM)* y el *Swap Market Model (SMM)* los más extendidos.

La principal razón a la que deben su popularidad estos modelos de mercado, introducidos en [14, 33, 46], es su compatibilidad con las fórmulas de mercado de dos derivados de tipos de interés básicos: *LMM* y *SMM* valoran *caps* y *swaptions*, respectivamente, de acuerdo con la fórmula de Black estándar empleada en los mercados de *caps* y *swaptions*. Esta cuestión constituye un hecho importante puesto que dichos mercados son los principales mercados de derivados en el mundo de los derivados sobre tipos de interés. Los modelos LMM y SMM asumen dinámicas lognormales para los tipos implícitos *forward Libor* o *forward Swap*, es decir, mantienen dichos tipos positivos. Hay que destacar que *LMM* y *SMM* no son compatibles el uno con el otro. Brigo y Liinev prueban en [15] que los tipos *forward swap* lognormales resultan distribucionalmente incompatibles con los tipos *forward Libor* lognormales, y viceversa. Cada tipo *forward Libor* o *swap* es martingala bajo su correspondiente medida de probabilidad. No obstante, cuando un derivado depende de varios tipos *forward Libor* o *swap* es necesario ultilizar la misma medida de probabilidad. En esta situación, haciendo uso de la técnica de cambio de numerario [48], pueden obtenerse las dinámicas de cada tipo bajo una medida de probabilidad común (de modo que estén libres de arbitraje). Bajo esta medida de probabilidad común, aparecen términos de deriva que dependen de varios tipos. Desde un punto de vista numérico, es frecuente el uso de técnicas de Monte Carlo para simular estas dinámicas bajo la medida común [12, 25, 32]. Aunque el esquema de Monte Carlo tiene un elevado coste computacional y es complicado controlar el error cometido, es la técnica elegida principalmente cuando se condieran más de tres factores (debido a que la precisión del método no depende de la dimensión). De este modo se evitan las dificultades que surgen, en caso de un elevado número de dimensiones espaciales, cuando se utizan métodos de diferencias o elementos finitos para resolver numéricamente las formulaciones equivalentes en términos de problemas de ecuaciones en derivadas parciales.

Resumiendo lo anterior, la popularidad de *LMM* y *SMM* viene, principalmente, de su habilidad para garantizar tres características deseables:

1. evitan arbitraje entre los diferentes bonos,

2. mantienen los tipos positivos,

3. valoran los *caps* o *swaptions* de acuerdo con la fórmula de Black, permitiendo el calibrado a mercado.

Sin embargo, estas tres propiedades de la formulación continua pueden perderse con la discretización, al realizar la simulación.

Es complicado realizar un ajuste sobre los tipos *forward* que garantice la primera propiedad tras la discretización [3]. Por lo cual, a fin de impedir la pérdida de la primera característica, Glasserman y Zhao proponen en [26] simular directamente los cocientes entre bonos y el numerario elegido (*deflated bonds*), cuyas dinámicas están libres de deriva. De este modo, la pérdida de la propiedad de martingalidad puede evitarse aplicando un ajuste sobre los *deflated bonds* simulados. Nos referimos a esta metodología con el nombre de Simulación Sin Derivas (SSD). Debido a que sólo consideran como posibles numerarios un bono o la cuenta bancaria, en cada tiempo existe un *deflated bond* cuya dinámica es cero. Este hecho permite establecer una relación biyectiva entre los *deflated bonds* y los tipos *forward*. A mayores, en [26] se presenta la simulación de unas nuevas variables (definidas como diferencias entre éstos *deflated bonds*) con el fin de garantizar la positividad de los tipos *forward* tras la discretización. Como se indica en [4], un sesgo de discretización adicional fue introducido con el uso de esta técnica. Por esta razón, en esta tesis se propone un nuevo método basado en una parametrización de las martingalas introducidas en [26] (see [20]). La principal ventaja de este método, que hemos presentado para el *LMM* y para el *SMM*, es que garantiza la ausencia de arbitraje entre bonos y los tipos positivos, incluso cuando se elige la cuenta bancaria como numerario. Ésta es una cuestión importante puesto que el uso de la medida terminal para simular puede conducir a la explosión de la desviación estándar muestral cuando se consideran amplios intervalos de tiempo.

LMM y *SMM* se pueden presentar dentro de un marco general como el propuesto en [23, 52]. En [23] este contexto general se introduce mediante el estudio de qué modelos de mercado son admisibles a través del uso de teoría de grafos. En [52] se presentan expresiones genéricas para los términos de deriva que aparecen en un modelo de mercado genérico. De nuevo, estamos interesados en la simulación de los tipos estocásticos evitando el uso de términos de deriva. Por lo tanto, teniendo en cuenta estas generalizaciones, extendemos la técnica de la SSD a modelos de mercado genéricos.

Algunas versiones recientes de *LMM* prestan especial atención a la captura de *basis* entre diferentes frecuencias de capitalización, mediante el uso de curvas de estimación junto con una curva de descuento de referencia (ver [43, 44], por ejemplo). Como un primer paso en la discretización, estos modelos requieren una discretización de la curva de descuento con las propiedades garantizadas por el método propuesto. A mayores, este modelo completo necesita una amplia cantidad de datos, algunos de los cuales no se obtienen fácilmente del mercado (las volatilidades de la curva *EONIA*, la correlación entre curvas, etc.). A fin de superar esta dificultad, en la práctica, aún se utiliza el *LMM* de una curva con una apropiada elección de la curva y las volatilidades. En cualquier caso, presentamos la técnica de la SSD para el caso de diferentes alternativas en el modelo multicurva.

También extendemos la metodología de la SSD a los modelos de mercados cruzados (*cross-markets*). En el contexto de los modelos de mercados cruzados tenemos dos economías, A y B, que se miden en dos unidades diferentes. Por ejemplo:

- En el caso dos monedas (*cross-currency*) las unidades son la moneda nacional y una extranjera, por lo tanto podemos considerar las economías nacional y extranjera, junto con el tipo de cambio *forward* que las relaciona [2, 16, 45, 59]. Existen muchos derivados financieros que dependen de tipos de interés nacionales y extranjeros, que pueden clasificarse en: productos *cross-currency*

estándar (*Cross-currency swap, Cross-crurrency swaption, ...*), que son acuerdos entre dos partes de modo que cada parte realiza pagos referenciados a una moneda en esa misma moneda, y productos *Quanto* (*Quanto fra, Quanto caplet/floorlet, Quanto swap, Quanto Swaption, ...*), que son productos que realizan pagos referenciados a una moneda en otra moneda. El segundo tipo de productos es el que realmente justifica la necesidad de los modelos de dos monedas, puesto que en el primer caso la valoración de los dos pagos puede realizarse en cada moneda, obteniendo a posteriori los valores en la misma moneda mediante el uso del valor del tipo de cambio actual de esa moneda.

- En el caso de productos básicos (*commodities*) las unidades son la moneda nacional y un producto básico, con lo cual tenemos la economía nominal y la del producto básico, junto con el precio *forward* de dicho producto básico que relaciona ambas economías [30, 58]. Existen productos financieros que dependen de los tipos de estas economías, como los *spread options* [53]. En este contexto también resulta interesante obtener la evolución del rendimiento de conveniencia (*convenience yield*).

- En el caso de inflación las unidades son la moneda nacional y una cesta representativa de productos básicos, de modo que podemos considerar las economías nominal y real, junto con el índice de inflación *forward* que las relaciona [28, 40, 42]. Los derivados de inflación más populares son los *swaps* indiciados de inflación (*inflation indexed swaps*), que son acuerdos entre dos partes para intercambiar ciertos flujos, en los cuales al menos uno de los flujos intercambiados depende de un índice de inflación. Los principales *swaps* indiciados de inflación comercializados en el mercado son los *swaps* indiciados de inflación cupón cero (*zero coupon inflation indexed swaps*) y los *swaps* indiciados de inflación año a año (*year-on-year inflation indexed swaps*). A partir de éstos *swaps* surgen las opciones sobre ellos (*inflation indexed swaptions*). En el mercado hay también otros derivados como los *caplets* o *floorlets* indiciados de inflación.

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El objetivo de los modelos de mercados cruzados es obtener la evolución de los activos básicos de ambas economías y de los tipos que las conectan. En este trabajo presentamos dos de los posibles modelos de mercados cruzados a los que llamamos primer y segundo modelos de mercados cruzados. La SSD para el primer modelo de mercados cruzados [21] es completamente análoga a la técnica para una economía, mientras que la SSD para el segundo modelo de mercados cruzados [22] es diferente, teniendo en cuenta que en la SSD para la economía B no es necesaria la parametrización propuesta. Aplicamos el primer modelo de mercados cruzados al contexto de dos monedas y el segundo modelo de mercados cruzados al caso de productos básicos. Para el caso de inflación se pueden utilizar los dos modelos cruzados.

Todos los modelos que hemos presentado se pueden enmarcar dentro del contexto de un grafo dirigido conexo general, en el que podemos conocer las dinámicas que gobiernan su evolución. Proponemos también una técnica de simulación evitando el uso de los términos de deriva de dichas dinámicas.

Finalmente, resumimos las conclusiones de este trabajo. Los algoritmos relativos a las técnicas de la Simulación Sin Derivas se han implementado en C. Hemos obtenido resultados para el *Libor Market Model* en el contexto de una economía, así como para los casos de dos monedas y de productos básicos dentro del marco de dos economías. Cuando hemos considerado el *Libor Market Model* de una economía, se han presentado los resultados obtenidos para diferentes ejemplos. A fin de ilustrar la distribución de los errores a través del tiempo, en un primer ejemplo hemos asumido una estructura de tipos de interés y volatilidades plana. El segundo ejemplo considerado se corresponde a una situación más realista en la que se asumen diferentes valores para los tipos *forward* y para las volatilidades de éstos. En el ejemplo de la estrucutra plana, el método propuesto exhibe un mejor comportamiento en la valoración de *caplets*. Además, el ajuste de martingala a nivel de simulación conduce a una significante reducción de error. En el ejemplo de la estructura no plana, hemos

mostrado los errores en puntos básicos cuando se comparan los valores de martingalas en tiempo cero con los valores esperados obtenidos con el método propuesto sin y con ajuste. Se puede observar cómo dicho ajuste reduce los errores que aparecen debido a la discretización, a pesar de que es un ajuste basado en una aproximación. A mayores, de modo análogo al primer ejemplo, se han presentado los errores en la valoración de *caplets*, para diferentes *strikes*, obtenidos con diferentes métodos de simulación, notando que se ha observado el mismo comportamiento cualitativo que en el primer ejemplo. Teniendo en cuenta todo ello, podemos concluir que el método propuesto no sólo garantiza ciertas propiedades del modelo continuo tras la discretización, sino que también proporciona mejores resultados respecto de otras técnicas de la literatura. En el campo de dos economías hemos realizado comparaciones en cada economía análogas a las expuestas hasta ahora, obteniendo comportamientos similares. A mayores, cuando hemos aplicado el primer modelo de mercados cruzados al contexto de dos monedas, hemos valorado *Quanto Caplets* y *Quanto Floorlets* observando que el método propuesto proporciona los mejores resultados en la valoración de los llamados *Quanto Caplet 2* y *Quanto Floorlet 1* y resulta ser el segundo mejor método para el *Quanto Caplet 1*. Teniendo en cuenta sus resultados y su propiedad de garantizar la positividad de los tipos, en general, la SSD basada en la parametrización exhibe el mejor funcionamiento para la simulación en el contexto de dos monedas. Por último, cuando hemos aplicado el segundo modelo de mercados cruzados al contexto de productos básicos, hemos obtenido el precio de un *spread option* mediante simulación, observando su similitud con el precio calculado mediante la fórmula de mercado.

El esquema de la memoria de esta tesis es el siguiente.

La Parte I aborda las técnicas de la SSD para modelos de mercado en el contexto de una economía. En particular, el Capítulo 1 contiene una introducción en esta parte. En el Capítulo 2 exponemos el *LMM*. Además, se describen las técnicas de la

SSD existentes y se propone un método eficiente para garantizar las propiedades deseables tras la discretización. En el Capítulo 3, las metodologías de la SSD se aplican al *SMM* coterminal y a los modelos de mercado genéricos. Finalmente, en el Capítulo 4 la técnica se extiende para el ámbito de multicurva, recientemente aparecido tras la crisis de 2008.

La Parte II aborda las técnicas de la SSD para los modelos de mercado en el contexto de dos economías. En particular, en el Capítulo 5 se introducen los modelos de mercados cruzados, junto con el criterio para decidir los modelos posibles y cuestiones generales de la técnica de SSD en este ámbito. Los Capítulos 6 y 7 contienen dos modelos de mercados cruzados particulares con sus correspondiente técnicass SSD para evitar la presencia de los términos de deriva. En concreto, en el Capítulo 6 se considera el modelo específico para el ejemplo de dos monedas y el Capítulo 7 se dedica al caso de productos básicos. Recogemos al final del Capítulo 7 el caso de derivados de inflación, susceptible de ser tratado con ambos modelos de mercados cruzados.

La Parte III contiene el Capítulo 8, en el que se presenta una aplicación de la teoría de grafos a los modelos de mercado genéricos.

El Apéndice A recoge un resultado auxiliar dentro del álgebra de matrices para probar cierta fórmula dentro de los modelos de mercado genéricos de una economía. En el Apéndice B se obtiene el número de modelos de mercados cruzados posibles.

Finalizamos con una breve sección que resume las principales conclusiones de este trabajo.

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