UNIVERSIDADE DA CORUÑA

PhD. Thesis

Numerical methods to price interest rate derivatives based on LIBOR Market Model for forward rates

AUTORA:

María Suárez Taboada

Departamento de Matemáticas
Universidade da Coruña

UNIVERSIDADE DA CORUÑA

PhD. Thesis

Numerical methods to price interest rate derivatives based on LIBOR Market Model for forward rates

AUTORA:

María Suárez Taboada

DIRECTOR:

Carlos Vázquez Cendón

Departamento de Matemáticas

Universidade da Coruña

UNIVERSIDADE DA CORUÑA

The undersigned hereby certifies that he is supervisor of the Thesis entitled **"Numerical methods to price interest rate derivatives based on LIBOR Market Model for forward rates"** developed by María Suárez Taboada inside the Ph.D Program **"Mathematical Modelling and Numerical Methods in Applied Sciences and Engineering"** at the Department of Mathematics (University of A Coruña).

Dated: May 28, 2012

Ph.D. Advisor: Carlos Vázquez Cendón

Table of Contents

List of Tables

List of Figures

Agradecimientos

Durante el desarrollo de esta tesis doctoral he tenido la suerte de trabajar con gente magnífica tanto profesional como personalmente y quiero dedicarles unas líneas porque, de distintas maneras, hicieron posible este trabajo.

Me gustaría agradecer al director de esta tesis doctoral, Prof. Carlos Vázquez Cendón, su voto de confianza en septiembre del 2006, sin el cual esto no hubiese sido posible y mucho menos en la recta final de este trabajo.

Querría agradecer también al Prof. Alfredo Bermúdez el haberme brindado la oportunidad de empezar el doctorado trabajando con él, siendo una fuente inagotable de energía y entusiasmo por la investigación.

Me gustaría mencionar a Josechu con el que tuve la suerte de trabajar y asistir como alumna a múltiples conferencias y cursos. Aparte de ser brillante y motivar ante cualquier proyecto, quisiera destacarlo por su calidad humana. En esa etapa de trabajo con él, después de haber abierto el camino en las Finanzas, quiero agradecer a María R. Nogueiras su ayuda.

El apoyo del Prof. Andrea Pascucci durante mis estancias en Bologna ha sido fundamental. Su profesionalidad y capacidad de trabajo fueron muy importantes en el desarrollo de gran parte de esta tesis doctoral.

Agradecer también al Prof. Christoph Reisinger la oportunidad de realizar una estancia en el Mathematical Institute, University of Oxford y la cálida acogida de las *chicas* Karolina y Sigrid y del *gentleman* Jan (o Pepe para los amigos).

Por mis primeros años en el doctorado, quiero dar las gracias al Departamento de Matemática Aplicada de la Universidade de Santiago de Compostela y en particular a Dolores, Elena, Ferrín, Fran, Óscar, Pere y Rafa M. por su ayuda en innumerables ocasiones. No puedo olvidarme tampoco de los becarios Andrés, Cris, Laura, Luz, Marta, Rafa y Tere S. que me facilitaron las cosas en esos primeros años.

Me gustaría dedicar unas líneas al Departamento de Matemáticas de la Universidade da Coruña. A Íñigo, por su ayuda desde el principio así como sus palabras de ánimo en el último año. A Suso, gracias por echarme una mano para pasarme al Kile y tus "vaciles" de primera hora. A Teresa, por aconsejarme en las primeras horas de docencia. A Ana y José Antonio por su gran ayuda con C++ y "el código". Además, Ana, fue un placer compartir docencia contigo y muchas gracias por estar ahí estos últimos meses.

No puedo olvidarme de mis compañeros del Laboratorio 2.1 y "allegados". En una primera etapa, Adri, Bruño, Eli, Gonsalito, Moli, Raquel y Xocas. En una segunda etapa y hasta ahora: Al, Carmiña, Marta, Paula, Álvaro, Jose, Brais, Miguelón y Jezú. Gracias por compartir tantos cafés, comidas a la canasta y demás con nuestro laboratorio "amigo", con Porta al frente. Gracias Martiña, Álvaro y Jose por resolverme las dudas de última hora. Carmen se merece una mención especial por las horas compartidas durante el proyecto de máster, su paciencia, ayuda y por estar siempre en "esa" silla, ;).

En los últimos años de la carrera y en los primeros de doctorado estuvo Rebeca, porque no hay nada que una más que unas prácticas de Elementos Finitos.

Desde el minuto uno ha estado mi amiga Diana. Literalmente, desde el primer día de facultad y hasta ahora, con ese sentido del humor tan exarcebado que nos permitió aprobar duras asignaturas a la primera, que era como sortear socavones en Montero Ríos, no?.

Tampoco puedo olvidarme de mis amigos de las Xuntanzas: Bolu, Bics, Di, María, Óscar , Sancho y Tono. No tengo palabras para agradeceros todos esos momentos, esas inyecciones de energía, risas y por estar seguros de que "el fin del mundo nos pillará bailando". Pablito, a pesar de ser el amigo raro y en la distancia, siempre has sabido estar cerca.

Por cierto, piscuisa, ¡estás en todas! Te agradezco infinitamente tus palabras de ánimo pero sobre todo, tantísimos momentos que hemos pasado (¡y los que nos quedan!). Unos muy buenos y otros no tan buenos pero siempre has estado ahí. Ahora sé que escuchar cuarteto hace que las EDP's sean más llevaderas.

A mi compañero, que no se escapó ante la pregunta "¿me lo dices porque se verifica en general o para un caso particular?", estando a mi lado siempre. Siempre que las matemáticas y el espíritu libre lo permitían, apoyándome y animándome a seguir pasara lo que pasara.

Por último y sabiendo de antemano que no me llegarán las palabras, quiero agradecer a mi familia todo porque sin ellos absolutamente nada de esto sería posible. A mi padre, por saber "tener" una hija a la que le gustan las Matemáticas, a mi abuela, por ser mi otra madre, a mis hermanos Raquel y Manu, por existir, sois lo mejor que me ha pasado y a mi madre, por ser como es, por su apoyo incondicional y confianza en mí. Doctorarse es un paso más pero que no tendría ningún sentido sin vosotros. Muchísimas gracias por estar conmigo. Os quiero.

A Coruña, 2012.

Este trabajo de investigación ha sido cofinanciado por los siguientes proyectos:

- Beca FPI con referencia BES-2008-004492 asociada al proyecto MTM2007-67596-C02-01.
- Proyecto de la Xunta de Galicia con referencia 2006-2009 PGIDIT06PXIB105230PR.
- Proyecto del MCINN con referencia MTM2010-21135-C02-01.
- Proyecto de la Xunta de Galicia con referencia INCITE09105339PR .
- Ayuda de referencia competitiva CN2011/004.
- Fondos FEDER.

Para as dúas C's e as súas vogais.

Introduction

In this work we study some mathematical models for pricing specific financial derivatives. More precisely, the study covers the modeling, mathematical analysis and numerical solution of the ratchet cap, spread option and stock loan pricing problems.

The ratchet cap consists of an interest rate derivative product that can be decomposed in ratchet caplet contracts, for which the associated strike is recursively defined in terms of a set of forward LIBOR rates. In the spread option contract the payoff depends on the relation between the difference of two LIBOR rates and a fixed strike, so that call and put versions can be considered. In the stock loan contract mainly the borrower of a loan owns shares on a stock that are used as collateral of the loan, so that it can be framed as a derivative on the stock. In both cases, the usual methodology in mathematical finance for option pricing also allows to obtain different models that can be formulated in terms of partial differential equations problems. After stating these models, their mathematical analysis allows to obtain existence and uniqueness of solution, as well as some qualitative and/or regularity properties. Furthermore, having in view the practical pricing requirements, we develop different numerical methods to solve the models and provide the fair prices of the financial products here treated.

Financial derivatives are a type of financial instruments, the price of which depends on other underlying products. The most classical example appears in option markets on assets. The starting point of trading these financial derivatives in organized markets dates back to early 1970s with the Chicago Board of Options Exchange (CBOE) in Chicago (USA). Almost at the same time, the seminal papers by Black and

Scholes [11] and Merton [42] provided the dynamic hedging methodology to obtain the popular Black-Scholes partial differential equation model and Black-Scholes formula for European vanilla options. Since then, the complexity of option-like products and other derivatives has greatly increased and different partial differential equation models have been proposed accordingly to price them [66, 65]. Among the different mathematical tools involved in the statement of the model, the consideration of a geometric Brownian motion that governs the dynamics of the underlying (asset price, in this case) results to be a key point. Also notice that the short maturity of option contracts allows the use of either constant or deterministic time dependent interest rates, however this is not the case for long term products, such as bonds.

Among the large variety of derivatives, when the underlying is a particular interest rate or a set of them, the class of interest rate derivatives arises. Thus, the benefits or payoffs associated to an interest rate derivative depend on the level of certain interest rates. One of the most simple examples of this kind of derivatives consists of a bond periodically paying coupons that depend on certain floating rate. As in the case of options, when trying to solve the associated pricing problem, the question about the suitable models for the dynamics of the involved interest rates arises. Unlike in the case of options, the long term of contracts and the behavior of the involved rates motivates the consideration of stochastic interest rate models. In the literature a lot of effort has been devoted to develop such models, that nowadays can be mainly classified into two classes: short rate models and market models. In the book by Brigo and Mercurio [13] a comprehensive presentation of the different families of interest rates and their modeling can be obtained.

Short rate models are mainly based in one-factor dynamics for the spot rate process, r_t, that is

$$dr_t \;=\; u(t, r_t)\, dt \;+\; w(t, r_t)\, dW_t\,,$$

where the different particular expressions for the functions u and w give rise to a variety of models and W_t denotes a Brownian motion (see [66, 13], for example). The popular models of Vasicek (1977) [63], Dothan (1978) [20] and Cox-Ingersoll-Ross

(1985) [16] can be framed in this setting. One advantage of these models comes from the possibility of obtaining analytical formulas for pricing zero coupon bonds or even coupon bearing bonds. Furthermore, they are an easy first step to explain more general and convenient models. Nevertheless, these models result to be endogenous in the sense that they provide the term structure as an output and the calibration of their constant parameters to the current market term structure results to be almost impossible in practice. A first attempt to overcome this drawback is provided by the inclusion of time dependency in the involved functions u and w, as it is proposed in Hull-White model (1990) [33], or the consideration of two factor models [13]. Nevertheless, as it is also based on the unobservable values in the market of the instantaneous short rates and its variance, the drawbacks related to the difficulty of calibration to the initial curve of discount factors still remain.

The general Heath-Jarrow-Morton model [32], that appears in 1992, constitutes the first alternative to short rate models in continuous time. In this general model, the instantaneous forward rates are modeled and an arbitrage-free methodology for the stochastic evolution of the entire yield curve is proposed, so that the forward-rate dynamics is appropriately defined in terms of their instantaneous volatility structure. Heath-Jarrow-Morton model is also considered as the starting point of market models.

Among a large variety of interest rates, the LIBOR (London Interbank Offer Rate) represents the most important interbank rate usually considered as reference for contracts and also the rate at which large international banks lend money to each other. Moreover, forward rates are a class of interest rates which are valid for future periods. They can be locked in today for an investment in a future period (for example, from one to two years from now). In the LIBOR Market Model, the forward LIBOR rates and their dynamics are chosen as the underlying interest rates in the ratchet cap and spread option contracts.

Since the seminal papers by Brace, Gatarek and Musiela [12], Jashmidian [35] and Miltersen, Sandmann and Sondermann [44], the LIBOR Market Model has been widely used to model the evolution of forward LIBOR rates. It is based in the general Heath-Jarrow-Morton framework and has become one of the most popular interest rate market models, mainly due to its agreement with analytical Black pricing formulas used in the market for caps and floors, which are the most traded interest derivatives. Moreover, its parameters can be calibrated with market data and liquid products.

From the numerical point of view, in the LIBOR Market Model framework most of the pricing of interest rate derivatives is carried out by means of Monte Carlo simulation, taking advantage of its general applicability to almost all interest rate derivatives. More precisely, the most traded interest rate derivatives, such as vanilla caps and floors, discrete barrier caps and floors, discrete barrier digital caps and floors, spread options and ratchets can be priced (see Brigo and Mercurio [13] or Pelsser [53], for example). As soon as the derivative depends on a set of LIBOR rates, a common measure has to be used in the formulation of the stochastic differential equations for the dynamics of the different involved rates, thus giving rise to the appearance of drift terms. In this case, by introducing appropriate auxiliary martingales, some drift-free simulation techniques have been recently introduced (see [31, 10, 22], for example). However, the main limitation of Monte Carlo simulation comes from the excessively long computational times, specially when the prices of a lot of derivatives in a portfolio are required.

Sometimes, as in the pricing of many other financial derivatives, an alternative and more efficient numerical technique in LIBOR Market Model setting turns out from the formulation of the pricing problem in terms of partial differential equations. This approach for pricing derivatives is more classically addressed in option pricing (see, for instance, Pascucci [50] and Wilmott [65]). Feynman-Kàc theorem allows to obtain a representation formula of the price of many financial derivatives as the solution of Cauchy problem associated to parabolic (sometimes degenerated) partial differential equations (see [50], for example). In case of the ratchet cap contract, as the payoff depends on a set of forward LIBOR rates, the price of each ratchet caplet is obtained from the solution of a sequence of Cauchy problems, with increasing spatial dimension for decreasing time intervals. The rigorous statement of this complex PDE model and its mathematical analysis represents an original part of this work, as in the literature we have only found the reference [55] where a simpler case is posed and numerically solved. In this thesis also this particular case is framed in the general one and mathematically analyzed. In the case of the call (put) spread option on LIBOR rates also the PDE methodology can be carried out.

The first attempt to pose a rigorous pricing model for stock loans appears in [67], in which the stock dividends are collected by the lender until redemption so that a PDE model analogous to the American vanilla call option with time dependent strike is posed. The pricing of American vanilla call options can be formulated in terms of complementarity problems associated to classical Black-Scholes equations (see [66], for example), so that their mathematical analysis can be framed in the theory of degenerated parabolic variational inequalities (see, [18] or [34], for example). It is well-known the interpretation of the American pricing model as a free boundary problem in which not only the option price but also the exercise region has to be determined. Although usually free boundary problems are associated to linear parabolic equations, the consideration of transaction costs in vanilla European options, for example, gives rise to (double obstacle) free boundary problems associated to nonlinear equations [2]. In [67] the stock loan with infinite maturity is related to the American perpetual option to obtain an analytical formula. More recently, in [17] different possible regimes of dividend yield distribution lead to the corresponding free boundary problems in the finite maturity setting for the stock loan contract. Those ones corresponding to dividend yield gained by the lender before redemption, reinvested dividend returned to borrower on redemption and dividend always delivered to borrower lead to parabolic variational inequalities in one spatial dimension very similar to the one governing American vanilla options. Nevertheless, the most interesting case arises when the accumulative dividend yield is returned to the borrower on redemption. Under this specification of the stock loan contract, the introduction of an auxiliary stochastic process and the use of dynamic hedging methodology allows to represent the stock loan price as the solution of an obstacle problem associated to a Kolmogorov equation. This model is posed in [17] and the existence of a free boundary (redemption boundary in the case of stock loans) is analyzed, assuming that existence of solution has been obtained. An original part of the present work is the proof of existence of solution and its uniqueness in the set of functions with polynomial growth by using the techniques recently applied in [45] for Asian options with arithmetic averaging and early exercise opportunity. Furthermore, the anisotropic regularity of solution is analyzed by means of the techniques developed in [26] for hypoelliptic parabolic equations.

Having in view the application in practice of the here considered models, their mathematical analysis needs to be completed with their numerical solution. These numerical methods can be implemented as software toolboxes in appropriate programming languages to be handled by practitioners. Generally, the numerical methods to price financial derivatives can be classified into Monte Carlo simulation, binomial trees and the numerical or analytical solution of the PDE models.

As it has been indicated before, in LIBOR Market Model the most used in the literature is Monte Carlo simulation. Binomial trees can also be used and some examples of pricing interest rate derivatives for the very close Swap Market Model are included in [15].

In quantitative finance, initially the most extended methods for the numerical solution of PDE models were the classical ones of finite differences for parabolic equations governing European derivatives prices, with some additional projection techniques for the products with early exercise opportunity, such as American options or callable bonds [66]. However, other numerical techniques already used in computational fluid dynamics have been also applied to computational finance, such as finite volumes [68], finite elements [41, 57] or characteristics methods (semilagrangian schemes) for time discretization [64, 5, 21, 29]. A rigorous presentation of finite differences and finite elements methods in option pricing problems can be found in the text [1].

In the particular setting of the numerical solution of PDE problems for the ratchet cap pricing based on LIBOR Market Model, we have only found the work of Pietersz

[55], in which a parabolic PDE on two spatial dimensions is posed and a comparison between Monte Carlo simulation and an explicit finite differences scheme is presented. Concerning the stock loan problem, to our knowledge only the reference [17] addresses the numerical solution by means of a forward shooting method proposed in [4] for Asian options of American style.

In the present work, we propose the use of an unified methodology for the time-space discretization in all PDE problems, which is based on the higher order Crank-Nicolson Lagrange-Galerkin method initially proposed in [60] for a convection-diffusion equation with constant coefficients, and extended in [6, 7] to a wider framework of convection-diffusion-reaction (possibly degenerated) PDE problems. Moreover, these methods have been also successfully applied to the pricing of Asian options without early exercise opportunity in [8]. The advantage of the characteristics methods for time discretization arises in the case of convection dominated problems, in which spurious oscillations can appear when unsuitable numerical methods are applied. In the case of ratchet cap and spread option contracts, an original semianalytical technique is proposed and compared with the previous one and a crude Monte Carlo simulation. In the case of stock loans, the Lagrange-Galerkin technique is combined with the augmented Lagrangian active set technique proposed in [37] to treat the unilateral constraint in a mixed formulation setting. This method has been successfully used in [9] to price Asian options with arithmetic averaging and early exercise opportunity. In the case of stock loans, the application of the numerical methods allows not only to obtain their price but also the redemption and no redemption regions, as well as the optimal redee-ming boundary separating both regions. Also the theoretical properties proved in [17] about these regions contribute to validate the performance of the numerical methods.

The outline of the thesis memoir is as follows.

Chapter 1 is devoted to the presentation of the functional framework about parabolic partial differential equations with variable coefficients to be used in the mathematical analysis of the forthcoming models of ratchet cap and spread options on forward LIBOR rates. It mainly contains the definitions and main results concerning existence and uniqueness of solutions.

In Chapter 2 the rigorous formulation of an original PDE model for pricing the ratchet cap contract is posed. Next, the mathematical analysis of the general model to obtain the existence of solution is developed. A particular simpler case is also mathematically analyzed. For this case, different numerical techniques to obtain the ratchet cap price are based on semianalytical solutions, Lagrange-Galerkin methods and Monte Carlo simulation are described. Next, different numerical examples are presented to illustrate the performance of the numerical methods.

In Chapter 3 a PDE model for pricing call and put spread options on forward LIBOR rates is rigorously posed, mathematically analyzed and numerically solved with the same techniques of Chapter 2. Furthermore, some numerical examples are presented.

In Chapter 4 the mathematical analysis of the model for pricing stock loan contracts, when the accumulative dividend yield associated to the stock is returned by the lender to the borrower on redemption, is carried out. More precisely, the model can be framed as an obstacle problem associated to a Kolmogorov equation, so that existence and uniqueness in the set of solutions with polynomial growth can be obtained. Next, for the numerical solution of the problem the combination of Crank-Nicolson Lagrange-Galerkin with the augmented Lagrangian active set method is described. Some numerical examples illustrate the theoretical properties of the optimal redeeming boundary.

In Chapter 5 the main conclusions of this work are summarized.

In Annexe A some intermediate calculus related to the analytical approximation proposed in chapter 2 for the pricing of ratchet caplets are included. The same kind of computations are required for the spread option in chapter 3.

Chapter 1

Functional framework

1.1 Introduction

It is well-known that the price of financial contracts, such as options or interest rate derivatives, could be defined in terms of the solutions of Cauchy problems associated to partial differential equations. Throughout this work several interest rate derivatives will be described in order to get their price via the numerical solution of partial differential equations, the use of some fundamental solutions and the development of suitable Monte Carlo simulation techniques. Therefore, mainly due to the first two approaches, it is necessary to introduce a functional framework and some results of classical theory of partial differential equations. In particular, the results for the mathematical analysis of Black-Scholes models are based on the classical theory for parabolic equations with constant and variable coefficients. Furthermore, as long as the new financial products get more complex, the theory and techniques required to deal with them become more sophisticated.

In this chapter we introduce the main notations and results related to the classical theory of parabolic partial differential equations that constitute the basis of the mathematical analysis of the models appearing in next chapters. However, taking into account that this is not a guide neither a text book about this topic, the reader is addressed to the more specific literature, such as [28] or [50], for example.

1.2 Parabolic PDE problems with variable coefficients

In the literature, some pricing problems in finance can be formulated as problems associated to general parabolic operators with variable coefficients taking the form

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} + \sum_{j=1}^{n} b_j \partial_{x_j} - a_0 - \partial_t, \quad \text{in } (0,T)\times\mathbb{R}^n, \tag{1.1}$$

where a_{ij}, b_j and a_0 are given real functions defined in $(0,T)\times\mathbb{R}^n$.

This is also the case of the PDE models associated to the interest rate derivatives pricing problems appearing in this thesis. We point out that sometimes in the present work it results convenient to rewrite the operator $\mathcal{L}$ in divergence form, so that for a given function ϕ, defined in $(0,T)\times\mathbb{R}^n$, we can write

$$\mathcal{L}\phi = \partial_t\phi - Div(A\nabla\phi) + \vec{v}\cdot\nabla\phi + l\phi. \tag{1.2}$$

Clearly, the functional coefficients appearing in expressions (1.1) and (1.2) can be related by the following identities:

$$A_{ii} = a_{ii},\ A_{ji} = a_{ji},\ v_j = \partial_{x_j} a_{jj} + \sum_{i=1}^{n} \partial_{x_i} a_{ij} - b_j,\ l = a_0. \tag{1.3}$$

Note that (1.2) represents a linear convection-diffusion-reaction operator with diffusion tensor A, velocity vector $\vec{v}$ and reaction coefficient l.

Thus, in many financial situations appearing in the present work, the pricing model can be formulated in terms of a Cauchy problem associated to the operator $\mathcal{L}$ in the form

$$\begin{cases} \mathcal{L}u = f, & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0,\cdot) = \varphi, & \text{on } \mathbb{R}^n, \end{cases} \tag{1.4}$$

where f and φ are given functions defined in $(0,T)\times\mathbb{R}^n$ and $\mathbb{R}^n$, respectively.

In the financial setting, it is classical to prove that the solution of problem (1.4) represents the value of a self-financing strategy and the non-negativity of the solution avoids arbitrage opportunities (see [50], for example).

In the present chapter, mainly following the text [50], the main notations, definitions and some conditions that guarantee the existence and uniqueness of solution for the Cauchy problem (1.4) are presented. Also the results for the corresponding Cauchy-Dirichlet problem in bounded domains are summarized.

For this purpose, let us introduce some hypotheses about the regularity and growth on the coefficients involved in the operator, which will be necessary later on:

(H_1) The coefficients a_{ij}, b_j and a_0 are real-valued functions. The matrix $A = (a_{ij})$ is symmetric and positive semi-definite for any point of the considered domain. Moreover, the coefficient a_0 is bounded from below.

(H_2) There exists a constant M such that

$$| a_{ij}(t,x) | \leq M, \; | b_j(t,x) | \leq M(1+ | x |), \; | a_0(t,x) | \leq M(1+ | x |^2)$$

for all $(t,x) \in (0,T) \times \mathbb{R}^n$ and $i, j, = 1, \cdots, n$.

More precisely, in the case of a bounded domain $\Omega \subset \mathbb{R}^n$, under hypothesis ($H_1$) the weak maximum principle can be obtained, that in turn implies the uniqueness of solution. For the unbounded domain case, hypothesis (H_2) is additionally required. We note that the assumption of both hypotheses is not enough to obtain the existence of solution.

Furthermore, under the much stronger assumption of existence of fundamental solution, another uniqueness result can be obtained.

1.2.1 Uniqueness of solution

First, we present some results concerning the uniqueness of solution for the case of the following Cauchy-Dirichlet problem posed on a bounded domain:

$$\begin{cases} \mathcal{L}u = f, & \text{in } (0,T) \times \Omega, \\ u(0,\cdot) = \varphi, & \text{in } \Sigma, \end{cases} \tag{1.5}$$

where Ω is a bounded set in $\mathbb{R}^n$ and $\Sigma = \partial((0,T) \times \Omega) - (\{T\} \times \Omega)$.

At this point it is important to recall the concept of *classical solution* of the Cauchy-Dirichlet problem (1.5).

Definition 1.2.1. *Let* $f \in C((0,T) \times \Omega)$ *and* $\varphi \in C(\Sigma)$. *A classical solution of the Cauchy-Dirichlet problem (1.5) is a function* $u \in C^{1,2}((0,T)\times\Omega)\cap C(((0,T)\times\Omega)\cup\Sigma)$ *satisfying (1.5).*

Remark 1.2.1. *In the previous definition we use the notation* $C^{1,2}((0,T) \times \Omega)$ *for the space of functions with continuous first order derivatives in the first variable and continuous second order derivatives in the second variable.*

In this setting we have the following weak maximum principle and comparison results, the proof of which can be found in [50].

Theorem 1.2.2. *Let* $u \in C^{1,2}((0,T) \times \Omega) \cap C(((0,T) \times \Omega) \cup \Sigma)$ *such that* $\mathcal{L}u \geq 0$ *on* $(0,T) \times \Omega$ *and assume that hypothesis (*H_1*) holds. If* $u \leq 0$ *on* Σ, *then* $u \leq 0$ *in* $(0,T) \times \Omega$.

Corollary 1.2.1. *Let* $u, v \in C^{1,2}((0,T)\times\Omega)\cap C(((0,T)\times\Omega)\cup\Sigma)$ *such that* $\mathcal{L}u \leq \mathcal{L}v$ *on* $(0,T) \times \Omega$ *and* $u \geq v$ *on* Σ, *then* $u \geq v$ *in* $(0,T) \times \Omega$. *Particularly, in this case there exists at most one classical solution of the Cauchy-Dirichlet problem (1.5).*

Once the Cauchy-Dirichlet problem has been previously treated, we present the results for the Cauchy problem (1.4).

Theorem 1.2.3. *Let* $u \in C^{1,2}((0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *such that* $\mathcal{L}u \leq 0$ *on* $(0,T) \times \mathbb{R}^n$, $u \geq 0$ *on* $\mathbb{R}^n$ *and assume that hypotheses (*H_1*) and (*H_2*) hold. If* u *satisfies that*

$$u(t,x) \geq -C\exp(C \mid x \mid^2), \quad (t,x) \in (0,T) \times \mathbb{R}^n \tag{1.6}$$

for some positive constant C, *then* $u \geq 0$ *in* $(0,T) \times \mathbb{R}^n$.

From the previous theorem, the following uniqueness result follows.

Theorem 1.2.4. *Under the hypotheses of the previous theorem, there exists at most one classical solution* $u \in C^{1,2}((0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *of the Cauchy problem (1.4) such that condition*

$$| u(t,x) | \leq C \exp(C \mid x \mid^2), \quad (t,x) \in (0,T) \times \mathbb{R}^n \tag{1.7}$$

holds for some positive constant C.

In order to obtain uniqueness of solution under stronger assumptions, it is convenient to introduce the definition of fundamental solution.

Definition 1.2.2. *A fundamental solution of the operator* $\mathcal{L}$*, with a pole in* $(s,y) \in \mathbb{R}^n$*, is a function* $\Gamma(\cdot,\cdot;s,y)$ *defined on* $(s,\infty) \times \mathbb{R}^n$ *such that for every* $\varphi \in C(\mathbb{R}^n)$ *and bounded, the function*

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;s,y)\varphi(y)dy \tag{1.8}$$

is a classical solution of the Cauchy problem

$$\begin{cases} \mathcal{L}u = 0, & \text{in } (s,+\infty) \times \mathbb{R}^n, \\ u(s,\cdot) = \varphi, & \text{on } \mathbb{R}^n. \end{cases} \tag{1.9}$$

Next, for a fixed $\lambda > 0$, we introduce the function

$$\Gamma_\lambda(t,x) = \frac{1}{(2\pi\lambda t)^{n/2}} \exp\left(-\frac{\mid x \mid^2}{2t\lambda}\right), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^n. \tag{1.10}$$

Remark 1.2.5. *Notice that* Γ_λ *is the fundamental solution, with pole at* $s = 0$ *and* $y = 0$*, of the operator* $\frac{\lambda}{2}\Delta - \partial_t$.

Now, let us introduce the following hypothesis:

(H_3) The operator $\mathcal{L}$ has a fundamental solution Γ. Furthermore, there exists $\lambda > 0$ such that for every $T > 0$, $t \in (s, s+T)$, $x, y \in \mathbb{R}^n$, the following estimates hold:

$$\begin{aligned} \frac{1}{M}\Gamma_{\frac{1}{\lambda}}(t-s, x-y) &\leq \Gamma(t,x;s,y) \leq M\Gamma_\lambda(t-s,x-y), \\ | \partial_{y_k}\Gamma(t,x;s,y) | &\leq \frac{M}{\sqrt{t-s}}\Gamma_\lambda(t-s,x-y), \end{aligned}$$

with M a positive constant depending on T.

Let us recall that uniqueness results have already been obtained by assuming hypotheses (H_1) and (H_2). However, as stated in [50], if we additionally assume that (H_3) is satisfied, then $\mathcal{L}$ has a fundamental solution Γ, such that the function

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(y)dy,\ (t,x) \in (0,T) \times \mathbb{R}^n \tag{1.11}$$

is a classical solution of the Cauchy problem (1.4) for a given $\varphi \in C(\mathbb{R}^n)$ such that

$$| \varphi(y) | \leq c \exp(c \mid y \mid^{\gamma}),\ y \in \mathbb{R}^n$$

with c, γ positive constants and $\gamma < 2$.

Finally, we state a result that guarantees the existence of a unique solution which is bounded from below by a constant. In particular, the result allows to state the uniqueness of a positive solution. For this purpose, we introduce the following hypothesis:

(H_4) The operator $\mathcal{L}$ admits an adjoint operator given by

$$\mathcal{L}^* = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\partial_{x_i x_j} + \sum_{j=1}^{n} b_j^* - a_0^* + \partial_t,$$

with the coefficients

$$\begin{aligned} b_i^* &= -b_i + \sum_{j=1}^{n} \partial_{x_i} a_{ij} \\ a_0^* &= a_0 - \frac{1}{2}\sum_{i,j=1}^{n} \partial_{x_i x_j} a_{ij} + \sum_{j=1}^{n} \partial_{x_j} b_j \end{aligned}$$

verifying analogous growth conditions to the ones in hypothesis (H_2).

Theorem 1.2.6. *If we assume hypotheses* (H_1)*,* (H_2)*,* (H_3) *and* (H_4)*, then there exists at most one function* $u \in C^{1,2}((0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *that is bounded from below and is the classical solution of problem (1.4).*

The proof of the previous theorem is based on the following one, which in turn is a generalization of Theorem 1.2.4.

Theorem 1.2.7. *If we assume hypotheses (H_1), (H_2), (H_3) and (H_4), then there exists at most one classical solution $u \in C^{1,2}((0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ of problem (1.4). Furthermore, for this solution, there exists a constant C such that*

$$\int_{\mathbb{R}^n} | u(t,x) | \exp(-C | x |^2) \, dx < \infty, \tag{1.12}$$

for every $0 \leq t \leq T$

Finally, next theorem states that non-negative solutions of problem (1.4) satisfy the estimate (1.12), so that uniqueness of non-negative solutions is obtained.

Theorem 1.2.8. *If we assume hypotheses (H_1), (H_2) and (H_4) and $u \in C^{1,2}((0,T) \times \mathbb{R}^n)$ is a non-negative function such that $\mathcal{L}u \leq 0$, then*

$$\int_{\mathbb{R}^n} \Gamma(t,x;s,y) \, u(s,y) \, dy \leq u(t,x), \tag{1.13}$$

for every $x \in \mathbb{R}^n$ and $0 < s < t < T$.

1.2.2 Existence of solution

As in the case of uniqueness, also analogous results to the ones related to existence of solution for parabolic operators with constant and variable coefficients are required in order to ensure the existence of solution for the models which are considered throughout this work.

In order to present these results, some definitions concerning to specific functional spaces are first introduced (see [50], for example).

Definition 1.2.3. *Let M be a compact subset of $\mathbb{R}^n$. A function $f : M \subset \mathbb{R}^n \to \mathbb{R}$ is Hölder continuous of exponent α ($0 < \alpha < 1$) in M if there exists a constant C such that:*

$$| f(x) - f(y) | \leq C | x - y |^{\alpha}, \quad x, y \in M. \tag{1.14}$$

Definition 1.2.4. *A function $f : (0,T) \times \mathbb{R}^n \to \mathbb{R}$ is locally Hölder continuous in variable x, uniformly in variable t, with exponent α ($0 < \alpha < 1$) if for every compact set M in $\mathbb{R}^n$ there exists a constant C, such that:*

$$| f(t,x) - f(t,y) | \leq C | x - y |^{\alpha}, \quad x, y \in M, \ t \in (0,T). \tag{1.15}$$

Both previous definitions motivate the corresponding norms to introduce the associated normed spaces. However, as we are dealing with parabolic equations, we introduce the more useful definition of the parabolic *Hölder* spaces, which result to be more natural for these equations as they give to the time variable a different exponent from the spatial variable.

Definition 1.2.5. *For the exponent α $(0 < \alpha < 1)$ and the domain $O = (0,T) \times \mathbb{R}^n$, with $T > 0$, we denote by $C_P^\alpha(O)$ the space of bounded functions u, such that there exists a constant C satisfying:*

$$| u(t,x) - u(s,y) | \leq C \left(| t - s |^{\alpha/2} + | x - y |^{\alpha} \right), \; t, s \in (,T), \; x, y \in \mathbb{R}^n. \tag{1.16}$$

Therefore, the space $C_P^\alpha(O)$ can be equipped with the norm

$$|| u ||_{C_P^\alpha(O)} = \sup_{(t,x)\in O} | u(t,x) | + \sup_{(t,x)\in O, (t,x)\neq(s,y)} \frac{| u(t,x) - u(s,y) |}{| t - s |^{\alpha/2} + | x - y |^{\alpha}} \tag{1.17}$$

Moreover, the function u is said to be locally parabolic Hölder continuous of exponent α, which is denoted by $u \in C_{P,loc}^\alpha(O)$, if $u \in C_P^\alpha(M)$ for any compact M such that $\overline{M} \subset O$.

In order to state a result for the existence of solution to the Cauchy problem (1.4), we introduce a first hypothesis that assumes that the operator $\mathcal{L}$ is uniformly parabolic.

(H_5) $\mathcal{L}$ is a uniformly parabolic operator. More precisely, there exists a positive constant λ such that

$$\lambda^{-2} | \psi |^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\psi_i\psi_j \leq \lambda^2 | \psi |^2, \; t \in \mathbb{R}, \; x, \; \psi \in \mathbb{R}^n.$$

Next, we introduce hypothesis (H_6), that mainly assumes that the coefficients involved in the expression of the operator (1.1) are bounded and parabolic *Hölder* continuous functions.

(H_6) The coefficients are bounded and parabolic *Hölder* continuous functions with exponent $\alpha \in (0,1)$, that is:

$$a_{ij},\ b_j,\ a_0 \in \mathcal{C}_P^{\alpha}(\mathbb{R}^{n+1}),\ 1 \leq i,j \leq n.$$

The forthcoming hypothesis (H_7) concerns to the growth and regularity conditions on the functions defining the second member of the equation and the initial datum. They are similar to the ones imposed in the case of the Cauchy problem associated to the heat operator.

(H_7) The functions φ and f are continuous and there exist some positive constants c and $\gamma < 2$, such that

$$| \varphi(x) | \leq c \exp(c \mid x \mid^{\gamma}), \qquad x \in \mathbb{R}^n, \tag{1.18}$$

$$| f(t,x) | \leq c \exp(c \mid x \mid^{\gamma}), \qquad (t,x) \in (0,T) \times \mathbb{R}^n. \tag{1.19}$$

Furthermore, f is locally *Hölder* continuous in x and uniformly in t.

Then, under hypotheses (H_5), (H_6) and (H_7), we can state the following theorem that guarantees the existence of solution to problem (1.4).

Theorem 1.2.9. *If hypotheses (H_5), (H_6) and (H_7) are satisfied, then the operator $\mathcal{L}$ has a positive fundamental solution $\Gamma = \Gamma(t,x;s,y)$, which is defined for x, $y \in \mathbb{R}^n$ and $t > s$. Furthermore, for every functions φ and f satisfying (H_7), the function u defined by*

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(y)dy - \int_0^t \int_{\mathbb{R}^n} \Gamma(t,x;s,y)f(s,y)dyds, \tag{1.20}$$

with $(t,x) \in (0,T) \times \mathbb{R}^n$ and by $u(0,x) = \varphi(x)$, is a classical solution of the Cauchy problem (1.4).

Note that if the conditions of the previous theorem are satisfied, then Theorem 1.2.4 implies that the function defined by expression (1.20) is the unique solution of (1.4) such that condition (1.7) is satisfied.

In reference [50] an alternative hypothesis weaker than (H_7) jointly with a locally integrable initial datum can be assumed to obtain the corresponding results of existence of solution.

Chapter 2

Ratchet cap

2.1 Introduction

Since the seminal papers by Brace, Gatarek and Musiela [12], Jashmidian [35] and Miltersen, Sandmann and Sondermann [44], the LIBOR Market Model (LMM) has become one of the more popular interest rate market models due to its agreement with market pricing formulas. Also it is referred as Lognormal Forward LIBOR Model (LFM) in the book of Brigo and Mercurio [13].

As indicated in [12], LMM provides a class of term structure model with lognormal volatility so that market forward rates do not explode, are positive and mean reverting. More precisely, it models the dynamics of LIBOR forward rates so that pricing caps and floors is consistent with Black formulas used in the market. Moreover, their parameters can be calibrated with market data and liquid products. Nevertheless, an exact calibration of the initial curve of discount factors and a clear covariances structure of forwards rates is difficult to achieve, specially for models which are not analytically tractable.

Also notice that LMM results to be not compatible with the Swap Market Model (also referred in [13] as lognormal forward-swap model (LSM)) as forward swap rates cannot be lognormal under their own measure in LMM.

In the LMM framework most of the pricing is carried out by means of Monte

Carlo simulation taking advantage of its general applicability to almost all interest rate financial derivatives. However, the main limitation comes from the long computational times, specially when a lot of prices are required. Alternative numerical techniques in LMM setting arise from the pricing problem formulation in terms of partial differential equations (PDE). This approach is more classically addressed in option pricing (see Pascucci [50] and Wilmott [65], for example).

In the present chapter we first pose the appropriate PDE model for pricing the interest rate derivative known as ratchet cap (compounded of ratchet caplets), which is described, for example, in [13]. The ratchet caplet payoff depends on a variable strike in terms of the reset value of all previous forward LIBOR rates. The number of involved rates increases as the time interval approaches to ratchet cap maturity. In the work of Pietersz [54], the particular choice of some parameter reduces this dependence to only the last two previous LIBOR rates. In this particular setting a parabolic PDE on two spatial dimensions (the two LIBOR rates) is obtained and a comparison between Monte Carlo simulation and explicit finite differences for the PDE model is presented in [54].

In this chapter we address the general case where the strike depends on all previous LIBOR rates so that the dimension of the PDE domain increases. In this setting we obtain the existence and uniqueness of classical solution for the PDE problem. Moreover, this solution is expressed in terms of the fundamental solution of each associated operator and provides a numerical algorithm to compute the solution. Then, for the particular case of Pietersz [54], the results obtained with a new numerical method based on an analytical approximation by using the fundamental solution associated to a constant coefficient operator, the proposed Crank-Nicolson-characteristics method combined with finite elements and the more classical Monte Carlo simulation are compared.

Most of the original results in this chapter are included in references [51] and [62].

2.2 Some basics on LIBOR Market Model for forward rates

As it has been indicated in the preface, we can find in the market loads of interest rate derivatives. In the present chapter, we first describe the characteristics and pricing formulas for the simplest ones, the caps, and mainly focus on the ratchet cap that depends on a set of forward LIBOR rates.

First of all, when dealing with a set of LIBOR forward rates, we need a time structure (tenor) to introduce some important definitions and notations to be handled in the LIBOR Market Model. As different forward rates associated to LMM are involved, for $N \in \mathbb{Z}^+$ let us consider the following tenor $\mathcal{T} = \{T_0, T_1, ..., T_N\}$, with

$$0 < T_0 < T_i < T_k, \quad 1 \leq i < k \leq N.$$

Thus, we can consider N LIBOR forward rates, L^1, $L^2, \ldots, L^N$ associated to the previous tenor. More precisely, the i-th forward rate L^i is fixed at time T_{i-1} and accounts for the period $[T_{i-1}, T_i]$. We denote by $(L^i_t)_{t \leq T_{i-1}}$ the value process of the i-th forward LIBOR rate. In general, forward LIBOR rates are characterized by the time instant at which the rate is fixed (expiry) and the maturity.

In an arbitrage free-market, the price of any attainable claim is uniquely given, either by the value of the associated replicating strategy, or by the risk neutral expectation of the discounted claim payoff under any of the equivalent (risk-neutral) martingale measures. The computation of such expectation under stochastic interest rates is very complex so that a change of measure could be useful. For that, a reference asset (which is known as numeraire) is chosen to normalize the asset prices with respect to it. In general, a numeraire is the reference financial instrument with positive price in terms of which relative prices are expressed and it is identifiable with a self-financing strategy (see [13], for example).

It is well-known that a zero coupon bond is a contract that guarantees its holder the payment of a unit of currency at a future and fixed time (maturity date), without intermediate payments. Let us denote by B^i the zero coupon bond that matures at time T_i and let B_t^i denote its value at time $t < T_i$.

If we consider B^i as numeraire, then the usual no arbitrage hypothesis in financial pricing guarantees the existence of a martingale measure Q^i associated with the numeraire B^i, such that the process (L_t^i) is a martingale under Q^i. This measure Q^i is often called the forward measure for the maturity T_i. Note that all probability measures are considered in the space $(\Omega, \mathcal{F})$, where Ω denotes the sample space and $\mathcal{F}$ represents the filtration spanned by the Wiener process $\mathcal{W}$.

Note that LIBOR forward rates can be related to zero coupon bonds by the following relation [13]:

$$1 + \delta_i L_t^i = \frac{B_t^{i-1}}{B_t^i}, \; i = 1, \dots N.$$

In the standard LIBOR market model, the dynamics of the forward rate L^i under the probability measure Q^i are given by the stochastic differential equation

$$dL_t^i \;=\; L_t^i \sigma^i(t)\, d\mathcal{W}_t^i, \tag{2.1}$$

where

- $\mathcal{W} = \left(\mathcal{W}^1, \dots, \mathcal{W}^N\right)$ is a N-dimensional Brownian motion with instantaneous covariance matrix $\rho = (\rho_{i,j})$, that is

$$d\mathcal{W}_t^i d\mathcal{W}_t^j = \rho_{i,j} dt.$$

- σ^i is the deterministic volatility of the i-th forward LIBOR rate. We assume that it is bounded to guarantee the existence and uniqueness of solution for the stochastic differential equation (2.1).

- $\delta_i = T_i - T_{i-1}$ denotes the i-th accrual factor.

Thus, LIBOR market model can be set up by specifying the dynamics of each forward LIBOR rate with respect to the corresponding terminal probability measure. This results to be enough when pricing interest rate derivatives that only depend on one of the forward LIBOR rates, as it is the case of a caplet, for example.

Nevertheless, when we try to price interest rate derivatives depending on a set of forward LIBOR rates we need to express the dynamics of all the underlying forward LIBOR rates under the same probability measure. For this purpose, we can apply a change of numeraire technique by using the Radon-Nykodim Theorem and *Itô′s* Lemma (see [50] more details). These techniques can be applied to our setting, so that from (2.1) we get

$$dL_t^j = -L_t^j \sigma^j(t) \sum_{h=j+1}^{i} \frac{\rho_{j,h}\delta_h \sigma^h(t) L_t^h}{1+\delta_h L_t^h}\, dt + L_t^j \sigma^j(t)\, d\mathcal{W}_t^j, \tag{2.2}$$

for $j < i$.

Remark 2.2.1. *We highlight that when pricing interest derivatives depending on the set of forwards $\{L^1, \ldots, L^i\}$ in this work, we will use the measure Q^i. This is the reason why we do not include the expression of the dynamics of L^j for $j > i$ that can be found in [13], for example.*

The above dynamics, either in the different particular measures (2.1) or in the common one (2.2), constitute the LIBOR market model, which is also known as lognormal forward-LIBOR model in [13]. The second name comes from the fact that the distribution of the process (L_t^i) is lognormal under the measure Q^i. Notice that this martingale property is lost for $j < i$ when using the common probability measure Q^i.

The role of terms $\rho_{i,j}$ becomes important when pricing derivatives depending on a set of forward LIBOR rates. It denotes the correlation between the forward LIBOR rates L^i and L^j at time t, with both $t < T_{i-1}$ and $t < T_{j-1}$.

Taking into account financial products depending on several rates, prices will depend on the terminal correlation between different forward rates. The instantaneous correlation summarizes the degree of dependence between changes of different forward rates and instead, the terminal correlation summarizes the dependence between changes of different forward rates at a given terminal time-instant.

2.3 Financial derivatives: cap and ratchet cap contracts

2.3.1 Cap and caplets. Black formula

Interest rate derivatives are financial instruments whose payoffs depend on the level of certain interest rates. Due to this dependence, its pricing in Black-Scholes framework results to be more complex than for equities. In fact, the behavior of the underlying, forward LIBOR interest rates, is more complex than the dynamics of shares prices, usually assumed to be lognormal.

Cap and swaption markets are the two main markets in the interest rate derivatives world, so that the compatibility with market formulas is a very desirable property. That is, in order to price some financial contracts quoting in the market, it is important to match the prices obtained throughout different techniques and the ones quoted in the market.

In this section we introduce one of the simplest interest rates derivatives: the cap contracts. A cap contract is a financial product that can be additively decomposed into several caplet contracts. We can say that a caplet is an example of an European claim which only depends on one forward rate. Due to that, it is clear that we must use the martingale measure associated to the underlying rate in order to price the product.

First of all, we describe more precisely the cap contract. Assuming the previous notation for the tenor, the cap contract is signed at time T_0 and the last caplet payoff is fixed at time T_{N-1} and paid at time T_N. Thus, for $i = 1, \ldots, N$, let us consider the caplet i as the contract that fixes the forward rate L_t^i at time $t = T_{i-1}$ to the value $\bar{L}^i = L_{T_{i-1}}^i$ and pays at time T_i the amount

$$M\, \delta_i\, (\bar{L}^i - K_i)^+ \,,$$

where M denotes the notional and K_i is the constant fixed strike associated to caplet i. Figure 2.1 sketches the i-th caplet time structure.

If we denote by L_i the spatial variable corresponding to the i-th forward LIBOR rate, the caplet payoff at time T_i is given by:

$$\varphi_i(L_i) = M\,\delta_i\,(L_i - K_i)^+ \tag{2.3}$$

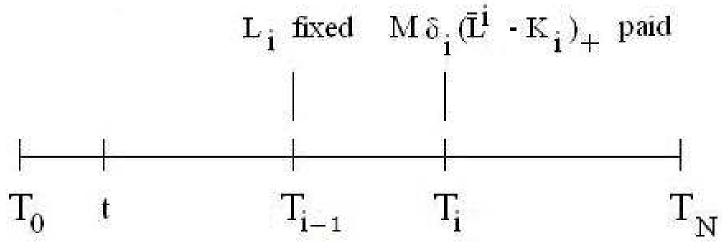

Figure 2.1: Time structure for the caplet i

For the time $t < T_{i-1}$ let us introduce the (absolute) price of caplet i, C_t^i. Taking into account the payoff expression (2.3), clearly the price of caplet i only depends on the forward LIBOR L^i. If we use the bond B^i as numeraire, we can define the relative (or discounted) price of caplet i as $\Pi_t^i = \frac{C_t^i}{B_t^i}$.

The fundamental results of pricing theory state that the price of any attainable financial product divided by the numeraire is a martingale. So, for each time $t < T_{i-1}$, we have:

$$\begin{aligned}
\Pi_t^i &= \frac{C_t^i}{B_t^i} = \\
&= \mathbb{E}^{Q^i}\left[\frac{\varphi_i(L_{T_{i-1}}^i)}{B^i(T_i)} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{Q^i}\left[\varphi_i(L_{T_{i-1}}^i) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{Q^i}\left[(\bar{L}^i - K_i)^+ \mid \mathcal{F}_t\right].
\end{aligned} \tag{2.4}$$

Once we can express the relative caplet price Π_t^i as an expectation, it is straightforward to obtain the partial differential equation (PDE) model. For this purpose, we assume that there exists a function u^i such that $\Pi_t^i = u^i(t, L_t^i)$ and we use the Feynman-Kâc theorem (see [13], for example). This theorem is the link between the solutions of PDEs and their expressions in terms of expectations. Thus, applying Feynman-Kâc theorem we get the following PDE model for the function u^i:

Find $u^i : [0, T_{i-1}] \times [0, \infty) \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u^i + \frac{1}{2}(\sigma^i)^2(t) L_i^2 \partial_{L_i L_i} u^i = 0, &\qquad \text{in } (0, T_{i-1}) \times \mathbb{R}^+ \\ u^i(T_{i-1}, L_i) = M\, \delta_i\, (L_i - K_i)^+, &\qquad \text{in } \mathbb{R}^+. \end{aligned}$$

The well-known solution of the previous problem provides the Black formula for caplets, just by taking into account that

$$C_t^i \;=\; B_t^i\, \Pi_t^i \;=\; B_t^i\, u^i(t, L_t^i) \tag{2.5}$$

and that

$$u^i(t, L_i) = M\delta_i(L_i \mathcal{N}(d^+(t, L_i)) - K_i \mathcal{N}(d^-(t, L_i))), \quad t \in (0, T_{i-1}) \tag{2.6}$$

$$d^{\pm}(t, L_i) = \frac{\log(\frac{L_i}{K_i}) \pm \frac{1}{2}(\bar{\sigma}^i)^2(t, T_{i-1})}{\bar{\sigma}^i(t, T_{i-1})}, \tag{2.7}$$

$$\bar{\sigma}^i(t, T_{i-1}) = \sqrt{\int_t^{T_{i-1}} (\sigma^i(s))^2 ds}, \tag{2.8}$$

where $\mathcal{N}$ denotes the standard normal distribution and $\bar{\sigma}^i(t, T_{i-1})$ is the integrated instantaneous variance multiplied by the time interval length. Notice that it is different from the quantity $\frac{1}{T_{i-1}-t}\bar{\sigma}^i(t, T_{i-1})$ which is termed as T_{i-1}-caplet volatility and it is always standardized with respect to time, that is, it is understood as an average instantaneous variance over time.

We notice that correlations between different rates have no impact on the caplet price as the caplet payoff only involves one of the forward LIBOR rates, so that the expectations do not affect two or more forward rates.

Once the prices of all caplets have been obtained, the price of the cap can be computed by the formula:

$$C(t, L_t^1, \ldots, L_t^N) = \sum_{i=1}^{N} B_t^i \, \Pi_t^i, \quad t \leq T_0.$$

Remark 2.3.1. *Analogously to caps and caplets, floors and floorlets could be defined and priced, the only difference being that the payoff function for the floorlet is given by the expression:*

$$\varphi_i(L_i) = M \, \delta_i \, (K_i - L_i)^+.$$

Moreover, a Black formula for floorlets is available.

In financial markets, it is a extended practice to price caps by using the Black formula and to quote, instead of the price, the volatility parameter that enters such formula. The market cap volatility is then simply defined as a parameter that must be plugged into the Black formula to recover the right market cap price.

Analogously, we can do the same for caplets but there is an important difference. Cap volatilities assume that caplets concurring to same cap share the same volatility, while caplet volatilities are allowed to be different for caplets concurring to the same cap.

2.3.2 Ratchet cap and ratchet caplet contracts

In the same way as the previously described cap contract can be additively decomposed into caplets, the ratchet cap is a contract that can be additively decomposed into simpler contracts, called ratchet caplets, and the payments associated to each ratchet caplet are similar to caplet payments. However, while in the case of caplets the strike is a fixed rate, in the case of ratchet caplets the strike is variable and depends on earlier LIBOR resets (see [53], for example).

More precisely, the ratchet caplet payoff, which is paid at time T_i, is given by

$$M\delta_i(\bar{L}^i - K_i)^+, \tag{2.9}$$

where $\bar{L}^i = L^i_{T_{i-1}}$, δ_i denotes the accrual, M represents the notional and the strike K_i is recursively defined as follows:

$$\begin{cases} K_1 & \text{is given,} \\ K_{j+1} = \left(a\bar{L}^j + bK_j + c\right)^+ & \text{for } 1 < j < i, \end{cases} \tag{2.10}$$

with a, b and c real parameters. So, we can see a ratchet cap as a cap where the strike is updated at each caplet, based on the previous "realization" of the relevant interest rate.

In particular, we remark that K_{j+1} is a function of $\bar{L}^1, \ldots, \bar{L}^j$. This feature of the product gives rise to a numerical difficulty because when $b \neq 0$ the dimension of the PDE model increases, as well as the number of underlying forward LIBOR rates, when time goes by.

Figure 2.2 sketches the i-th caplet time structure.

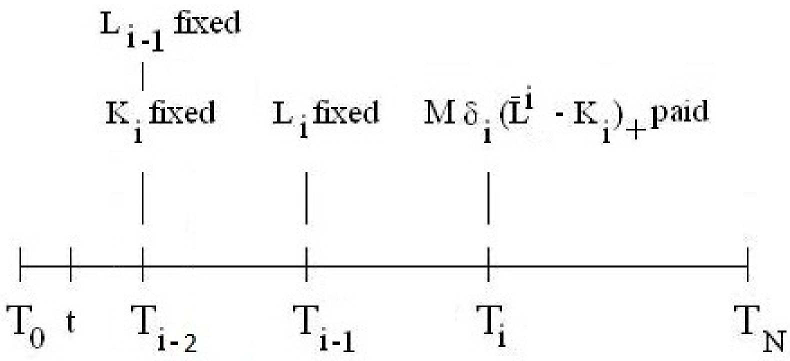

Figure 2.2: Time structure for the ratchet caplet i

As in the case of caplets, by usual no-arbitrage arguments, the relative or discounted price of the i-th ratchet caplet is given by

$$\Pi^i_t = \mathbb{E}^{Q^i}\left[M\delta_i(\bar{L}^i - K_i)^+ \mid \mathcal{F}_t\right], \qquad t \leq T_{i-1},$$

and the corresponding absolute price is equal to $R^i_t = B^i_t \Pi^i_t$.

2.4 PDE models for the ratchet caplets pricing

In this section the PDE model and the main results concerning the existence and uniqueness of solution for the PDE model governing a ratchet caplet price are presented.

Since we are in a Markovian framework, by using Feynman-Kâc theorem, we can obtain a representation of the relative price of the i-th ratchet caplet in terms of solutions of a sequence of Cauchy problems. This specific representation leads us to a scheme which will be suitable and useful to solve numerically the PDE model.

Analogously as the PDE model to price a caplet, we denote by L_i the real variable corresponding to the i-th forward LIBOR rate for $i = 1, \dots, N$ and we set $T_{-1} = 0$ by convention.

For simplicity, in this section we consider the notional $M = 1$ and the accrual $\delta_i = 1$. In the general case, due to linearity, we just need to multiply the here obtained ratchet caplet price by the amount $M\delta_i$.

Theorem 2.4.1. *For a fixed index $i \in \{1, \dots, N\}$, assume that for $j = 1, \dots, i$ the matrix $(\rho_{h,k}\sigma^h(t)\sigma^k(t))_{h,k=j,\dots,i}$ is bounded and uniformly positive definite. Then we have*

$$\Pi_t^i = u^{i,j}(t, L_t^j, L_t^{j+1}, \dots, L_t^i; K_j), \qquad t \in [T_{j-2}, T_{j-1}],\ j = 1, \dots, i, \tag{2.11}$$

where $K_j = K_j\left(\bar{L}^1, \dots, \bar{L}^{j-1}\right)$ is defined in (2.10) and the function

$$u^{i,j} = u^{i,j}(t, L_j, L_{j+1}, \dots, L_i; K), \qquad t \in [T_{j-2}, T_{j-1}],\ L_j, L_{j+1}, \dots, L_i > 0,\ K \geq 0,$$

is uniquely defined by the following backward recursion starting from $j = i$:

- *$u^{i,i}$ is the unique non-negative solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{i,i}u^{i,i} = 0, & \text{in } (T_{i-2}, T_{i-1}) \times \mathbb{R}_+, \\ u^{i,i}(T_{i-1}, L_i; K) = (L_i - K)^+, & \text{in } \mathbb{R}_+ \end{cases} \tag{2.12}$$

where $\mathcal{L}^{i,i}$ *is the two-dimensional operator*

$$\mathcal{L}^{i,i} = \frac{\left(\sigma^i(t)L_i\right)^2}{2}\partial_{L_iL_i} + \partial_t \tag{2.13}$$

- $u^{i,j}$, *with* $j < i$, *is the unique non-negative solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{i,j}u^{i,j} = 0, & \text{in } (T_{j-2}, T_{j-1}) \times \mathbb{R}_+^{i-j+1}, \\ u^{i,j}(T_{j-1}, L_j, L_{j+1}, \dots, L_i; K) & \\ \qquad = u^{i,j+1}(T_{j-1}, L_{j+1}, L_{j+2}, \dots, L_i; (aL_j + bK + c)^+), & \text{in } \mathbb{R}_+^{i-j+1}, \end{cases} \tag{2.14}$$

where $\mathcal{L}^{i,j}$ *is the following* $(i-j+2)$*-dimensional operator acting in the variables* $t, L_j, L_{j+1}, \dots, L_i$*:*

$$\begin{aligned} \mathcal{L}^{i,j} &= \frac{1}{2}\sum_{h,k=j}^{i} \rho_{h,k}\sigma^h(t)\sigma^k(t)L_hL_k\partial_{L_hL_k} \\ &\quad - \sum_{h=j}^{i-1}\sum_{k=h+1}^{i} \rho_{h,k}\sigma^h(t)\sigma^k(t)\frac{\delta_kL_k}{1+\delta_kL_k}L_h\partial_{L_h} + \partial_t. \end{aligned} \tag{2.15}$$

Proof. Throughout the proof we will use results from Chapter 1 in the thesis taken from [50].

First step. We show that the functions $\{u^{i,j}\}_{j=1,\dots,i}$ are well defined recursively as the solutions of problems (2.12)-(2.15).

First, in the case $i = j$, by the change of variable $x_i = \log L_i$, i.e. setting

$$\bar{u}^{i,i}(t, x_i; K) = u^{i,i}(t, e^{x_i}; K), \qquad x_i \in \mathbb{R}, \tag{2.16}$$

problem (2.12) becomes

$$\begin{cases} \bar{\mathcal{L}}^{i,i}\bar{u}^{i,i} = 0, & \text{in } (T_{i-2}, T_{i-1}) \times \mathbb{R}, \\ \bar{u}^{i,i}(T_{i-1}, x_i; K) = (e^{x_i} - K)^+, & \text{in } \mathbb{R}, \end{cases} \tag{2.17}$$

where $\bar{\mathcal{L}}^{i,i}$ is the backward heat operator

$$\bar{\mathcal{L}}^{i,i} = \frac{(\sigma^i(t))^2}{2}(\partial_{x_ix_i} - \partial_{x_i}) + \partial_t. \tag{2.18}$$

Since by assumption $(\sigma^i)^2$ is bounded from above and below by positive constants, standard results of the theory of parabolic PDEs, previously indicated in chapter 1, ensure that problem (2.17) has a unique non-negative classical solution given by

$$\bar{u}^{i,i}(t, x_i; K) = \int_{\mathbb{R}} \bar{\Gamma}^{i,i}(t, x_i; T_{i-1}, y_i)\,(e^{y_i} - K)^+ \, dy_i, \tag{2.19}$$

where $\bar{\Gamma}^{i,i}$ denotes the Gaussian fundamental solution of $\bar{\mathcal{L}}^{i,i}$:

$$\bar{\Gamma}^{i,i}(t, x; T, y) = \frac{1}{\bar{\sigma}^i(t,T)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - x + \frac{1}{2}\left(\bar{\sigma}^i(t,T)\right)^2}{\bar{\sigma}^i(t,T)}\right)^2\right), \tag{2.20}$$

for $x, y \in \mathbb{R}$ and $t < T$, with

$$\bar{\sigma}^i(t,T) = \sqrt{\int_t^T \left(\sigma^i(s)\right)^2 ds}\,. \tag{2.21}$$

From the representation formula (2.19) we also get the following estimate:

$$\left|\bar{u}^{i,i}(t, x_i; K)\right| \le C e^{C|x_i|^2}, \qquad (t, x_i) \in [T_{i-2}, T_{i-1}] \times \mathbb{R},\ K \ge 0, \tag{2.22}$$

for some positive constant C.

Next, by backward induction we show existence and uniqueness of $u^{i,j}$ for $j < i$. First, by the change of variables $x_j = \log L_j$, that is, by setting

$$\bar{u}^{i,j}\left(t, x_j, x_{j+1}, \dots, x_i; K\right) = u^{i,j}(t, e^{x_j}, e^{x_{j+1}}, \dots, e^{x_i}; K), \qquad x_j, x_{j+1}, \dots, x_i \in \mathbb{R},$$

problem (2.14) can be rewritten as follows:

$$\begin{cases} \bar{\mathcal{L}}^{i,j}\bar{u}^{i,j} = 0, & \text{in } (T_{j-2}, T_{j-1}) \times \mathbb{R}^{i-j+1}, \\ \bar{u}^{i,j}(T_{j-1}, x_j, x_{j+1}, \dots, x_i; K) & \\ \qquad = \bar{u}^{i,j+1}\left(T_{j-1}, x_{j+1}, x_{j+2}, \dots, x_i; (ae^{x_j} + bK + c)^+\right), & \text{in } \mathbb{R}^{i-j+1} \end{cases} \tag{2.23}$$

where

$$\begin{aligned} \bar{\mathcal{L}}^{i,j} &= \frac{1}{2}\sum_{h,k=j}^{i} \rho_{h,k}\sigma^h(t)\sigma^k(t)\partial_{x_h x_k} - \frac{1}{2}\sum_{k=j}^{i}(\sigma^k(t))^2\partial_{x_k} \\ &\quad - \sum_{h=j}^{i-1}\sum_{k=h+1}^{i} \rho_{h,k}\sigma^h(t)\sigma^k(t)\frac{\delta_k e^{x_k}}{1+\delta_k e^{x_k}}\partial_{x_h} + \partial_t \end{aligned}$$

is a second order differential operator that, by assumption, is uniformly parabolic and has bounded coefficients. Note in particular, that $\frac{\delta_k e^{x_k}}{1+\delta_k e^{x_k}} \in (0,1)$ for $x_k \in \mathbb{R}$. Then we recall that $\bar{\mathcal{L}}^{i,j}$ has a fundamental solution $\bar{\Gamma}^{i,j}$ satisfying the following Gaussian upper bound:

$$\begin{aligned}\bar{\Gamma}^{i,j}&(t, x_j, x_{j+1}, \dots, x_i; T, y_j, y_{j+1}, \dots, y_i)\\ &\leq C\Gamma^{i,j}_{\text{heat}}(t, x_j, x_{j+1}, \dots, x_i; T, y_j, y_{j+1}, \dots, y_i)\end{aligned} \tag{2.24}$$

for $x_j, y_j, \dots, x_i, y_i \in \mathbb{R}$, where C is a positive constant only dependent on $T-t$ and $\Gamma^{i,j}_{\text{heat}}$ is the Gaussian fundamental solution of a suitable parabolic operator with constant coefficients.

Now let us assume that $\bar{u}^{i,j+1}$ is a continuous and non-negative function satisfying the growth condition

$$\left|\bar{u}^{i,j+1}\left(t, x_{j+1}, \dots, x_i; K\right)\right| \leq Ce^{C|(x_{j+1},\dots,x_i)|^2} \tag{2.25}$$

for $(t, x_{j+1}, \dots, x_i) \in [T_{j-1}, T_j] \times \mathbb{R}^{i-j}$ and $K \geq 0$, with C some positive constant. Then problem (2.23) has unique non-negative classical solution given by

$$\begin{aligned}\bar{u}^{i,j}(t, x_j, \dots, x_i; K) := \int_{\mathbb{R}^{i-j+1}} &\bar{\Gamma}^{i,j}(t, x_j, \dots, x_i; T, y_j, \dots, y_i)\\ \times\ \bar{u}^{i,j+1}&\left(T_{j-1}, y_{j+1}, \dots, y_i; (ae^{y_j} + bK + c)^+\right) dy_j \cdots dy_i\end{aligned} \tag{2.26}$$

for $K \geq 0$ and $(t, x_j, \dots, x_i) \in [T_{j-2}, T_{j-1}] \times \mathbb{R}^{i-j+1}$. Moreover, combining estimates (2.24) and (2.25) with formula (2.26), we also deduce

$$\left|\bar{u}^{i,j}(t, x_j, \dots, x_i; K)\right| \leq Ce^{C|(x_j,\dots,x_i)|^2},$$

for $(t, x_j, \dots, x_i) \in [T_{j-2}, T_{j-1}] \times \mathbb{R}^{i-j+1}$ and $K \geq 0$, with some positive constant C. From the above results, a simple backward inductive argument shows that the functions $\{u^{i,j}\}_{j=1,\dots,i}$ are well defined as in the statement, in a unique way.

Second step.

We prove formula (2.11) by backward induction on j. Since $u^{i,i}(t, L_i; K)$ is a classical solution to problem (2.12), by Feynman-Kâc theorem we have

$$u^{i,i}(t, L_i; K_i) = \mathbb{E}^{Q^i}\left[(L^i_{T_{i-1}} - K_i)^+ \mid \mathcal{F}_t\right] = \Pi^i_t, \qquad t \in [T_{i-2}, T_{i-1}]$$

and this proves the thesis for $j = i$.

Now we assume that (2.11) is valid for a generic $j+1$ and we prove it for j. Hence we assume that

$$\Pi_t^i = u^{i,j+1}(t, L_t^{j+1}, L_t^{j+2}, \dots, L_t^i; K_{j+1}), \qquad t \in [T_{j-1}, T_j].$$

Since the process Π_t^i is a Q^i-martingale, we have for $t \in [T_{j-2}, T_{j-1}]$

$$\begin{aligned}
\Pi_t^i &= \mathbb{E}^{Q^i}\left[\Pi_{T_{j-1}}^i \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{Q^i}\left[u^{i,j+1}(T_{j-1}, L_{T_{j-1}}^{j+1}, L_{T_{j-1}}^{j+2}, \dots, L_{T_{j-1}}^i; K_{j+1}) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{Q^i}\left[u^{i,j+1}\left(T_{j-1}, L_{T_{j-1}}^{j+1}, L_{T_{j-1}}^{j+2}, \dots, L_{T_{j-1}}^i; \left(aL_{T_{j-1}}^j + bK_j + c\right)^+\right) \mid \mathcal{F}_t\right] \\
&= u^{i,j}(t, L_t^j, L_t^{j+1}, \dots, L_t^i; K_j),
\end{aligned}$$

where we have sequentially used the induction hypothesis, the expression of K_{j+1} in (2.10) and the Feynman-Kâc theorem, since $u^{i,j}$ is the unique non-negative classical solution to problem (2.14). □

As we mentioned at the beginning of this section, Theorem 2.4.1 provides an algorithm for the computation of the ratchet price via PDEs techniques. Indeed, according to formula (2.11), Π_t^i can be computed by solving recursively the Cauchy problems (2.12)-(2.14) starting from the last period $[T_{i-2}, T_{i-1}]$ back to the first period $[0, T_0]$.

By formula (2.11) the discounted price Π_t^i of the i-th ratchet caplet in the last period $[T_{i-2}, T_{i-1}]$ is given in terms of the solution $u^{i,i}$ of problem (2.17) and this corresponds to the price of a standard caplet with strike K_i. More precisely, by the Black formula, we have

$$\Pi_t^i = u^{i,i}(t, L_t^i; K_i) = L_t^i \mathcal{N}(d^+(t, L_t^i)) - K_i \mathcal{N}(d^-(t, L_t^i)) \qquad t \in (T_{i-2}, T_{i-1}), \quad (2.27)$$

where $\mathcal{N}$ denotes the normal cumulative distribution function and

$$d^{\pm}(t, L_i) = \frac{\log(\frac{L_i}{K_i}) \pm \frac{1}{2}\left(\bar{\sigma}^i(t, T_{i-1})\right)^2}{\bar{\sigma}^i(t, T_{i-1})}$$

with $\bar{\sigma}^i(t, T_{i-1})$ as in (2.21).

Note that at each step the dimension of the Cauchy problems increases by one due to the dependence of an additional forward rate.

We emphasize that the problems (2.12) and (2.14) depend on the parameter K. Moreover, at each step it must be solved for *any* value of K since the solution $u^{i,j+1}(T_j, L_{j+1}, L_{j+2}, \ldots, L_i; K)$ enters as final condition of the subsequent Cauchy problem posed on $(T_{j-2}, T_{j-1}) \times \mathbb{R}_+^{i-j+1}$ as a function of $L_{j+1}, L_{j+2}, \ldots, L_i$ and K. This fact puts severe restrictions on the applicability of standard numerical techniques for PDEs and the use of sparse grids seems an appropriate numerical technique when the constraint $b \neq 0$ in order to cope with the increasing of spatial dimensions of the PDE associated to the subsequent ratchet caplets.

2.5 Ratchet caplet pricing with $b = 0$

The ratchet caplet pricing problem turns out to be definitely easier when the parameter b is null since in this case the strike K_i depends only on the forward rate L^{i-1} and not on the previous ones. More precisely, in this case the payoff of the i-th ratchet caplet given by (2.10) is equal to

$$(\bar{L}^i - K_i)^+, \quad \text{with} \quad K_i = (a\bar{L}^{i-1} + c)^+,$$

so that it only depends on two LIBOR rates.

Remark 2.5.1. *Also as in previous section, for simplicity we consider the notional $M = 1$ and the accrual $\delta_i = 1$. Notice that in the formulas and numerical approximations appearing in this section, the derivative prices for the different values taken by M and δ_i can be obtained just by multiplying the ratchet caplet price by these quantities.*

Then, the discounted price Π_t^i is given by the Black formula (2.27) for $t \in [T_{i-2}, T_{i-1}]$ and by

$$\Pi_t^i = \mathbb{E}^{Q^i}\left[\left(L^i_{T_{i-1}} - (aL^{i-1}_{T_{i-2}} + c)^+\right)^+ \mid \mathcal{F}_t\right] = u^{i,i-1}(t, L_t^{i-1}, L_t^i),\ t \in [0, T_{i-2}], \quad (2.28)$$

where $u^{i,i-1}$ is the non-negative solution of the Cauchy problem

$$\begin{cases} \mathcal{L}^{i,i-1}u^{i,i-1} = 0, & \text{in } (0, T_{i-2}) \times \mathbb{R}^2_+, \\ u^{i,i-1}(T_{i-2}, L_{i-1}, L_i) = u^{i,i}(T_{i-2}, L_i; (aL_{i-1} + c)^+), & \text{in } \mathbb{R}^2_+, \end{cases} \tag{2.29}$$

with $u^{i,i}$ is given by (2.27) and

$$\begin{aligned} \mathcal{L}^{i,i-1} &= \frac{1}{2}\left(\sigma^{i-1}(t)L_{i-1}\right)^2 \partial_{L_{i-1}L_{i-1}} + \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)L_{i-1}L_i\partial_{L_{i-1}L_i} \\ &+ \frac{1}{2}\left(\sigma^i(t)L_i\right)^2 \partial_{L_iL_i} - \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)\frac{\delta_i L_i}{1+\delta_i L_i}L_{i-1}\partial_{L_{i-1}} + \partial_t. \end{aligned} \tag{2.30}$$

Note that in this particular case, as the strike only depends on L_{i-1}, the definition of Π_t^i in (2.11) written in terms of $u^{i,j}$ in $[T_{j-2}, T_{j-1}]$, actually does not depend on j. This is taken in account in (2.28) where notation $u^{i,i-1}$ is used for the large interval $[0, T_{i-2}]$.

Therefore, the final condition for ratchet caplet i is posed on the (L_{i-1}, L_i)-plane at time T_{i-2} and the price is computed for $t < T_{i-2}$ at each point of the plane (see Figure 2.3).

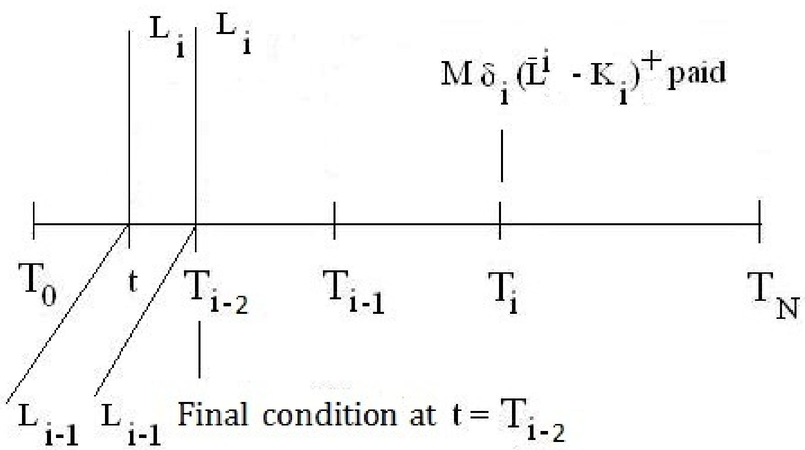

Figure 2.3: Sketch of the final condition for the ratchet caplet i.

Once the relative ratchet caplet prices are obtained, the absolute price of the ratchet cap is given by:

$$R(t, L_t^1, \ldots, L_t^N) = \sum_{i=1}^{N} B_t^i \Pi_i(t, L_t^{i-1}, L_t^i), \quad t \le T_0 \tag{2.31}$$

for a set of LIBOR rates $(L_t^1, \ldots, L_t^N)$ at the pricing date $t \le T_0$.

In order to price ratchet caplet i, by using the change of variables

$$\bar{u}^{i,i-1}(t, x_{i-1}, x_i; K) = u^{i,i-1}(t, e^{x_{i-1}}, e^{x_i}; K), \qquad x_{i-1}, x_i \in \mathbb{R}, t < T_{i-2},$$

problem (2.14) can be rewritten as follows:

$$\begin{cases} \bar{\mathcal{L}}^{i,i-1}\bar{u}^{i,i-1} = 0, & \text{in } (0, T_{i-2}) \times \mathbb{R}^2, \\ \bar{u}^{i,i-1}(T_{i-2}, x_{i-1}, x_i; K) = u^{i,i}\left(T_{i-2}, e^{x_i}; (ae^{x_{i-1}} + c)^+\right), & \text{in } \mathbb{R}^2, \end{cases} \tag{2.32}$$

where

$$\begin{aligned} \bar{\mathcal{L}}^{i,i-1} =& \frac{\left(\sigma^{i-1}(t)\right)^2}{2}\left(\partial_{x_{i-1}x_{i-1}} - \partial_{x_{i-1}}\right) + \frac{\left(\sigma^{i}(t)\right)^2}{2}\left(\partial_{x_i x_i} - \partial_{x_i}\right) \\ & + \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)\partial_{x_{i-1}x_i} - \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)\frac{\delta_i e^{x_i}}{1 + \delta_i e^{x_i}}\partial_{x_{i-1}} + \partial_t. \end{aligned} \tag{2.33}$$

In terms of the representation formulas (2.19)-(2.26), we have

$$\begin{aligned} &\bar{u}^{i,i-1}(t, x_{i-1}, x_i; K) \\ &= \int_{\mathbb{R}^2} \bar{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \cdot u^{i,i}\left(T_{i-2}, e^{y_i}; (ae^{y_{i-1}} + c)^+\right) dy_i dy_{i-1} \\ &= \int_{\mathbb{R}^2} \bar{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \\ &\qquad \times \int_{\mathbb{R}} \bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i)\left(e^{\eta_i} - (ae^{y_{i-1}} + c)^+\right)^+ d\eta_i dy_i dy_{i-1}, \end{aligned} \tag{2.34}$$

where $\bar{\Gamma}^{i,i}$ is the Gaussian fundamental solution of $\bar{\mathcal{L}}^{i,i}$, whose explicit expression is given in (2.20) and $\bar{\Gamma}^{i-1,i}$ is the (unknown) fundamental solution of $\bar{\mathcal{L}}^{i,i-1}$.

In the forthcoming sections, we consider three different methods for pricing the ratchet caplet in the case that the parameter b is equal to zero in the expression (2.10): a semi-analytical formula, a characteristics-finite elements based numerical method and a Monte Carlo simulation technique.

For the sake of simplicity, throughout this section we consider constant volatilities in time, the extension to time dependent volatilities resulting very easy.

2.5.1 Analytical approximation

In the previous section, the ratchet caplet price, $\bar{u}^{i,i-1}$, was given in terms of the solution of the Cauchy problem (2.32). In order to obtain an analytical approximation of the ratchet caplet price, we will use classical theory of fundamental solutions because we have a representation in terms of solutions of a sequence of Cauchy problems.

We get an analytical approximation of the ratchet caplet price by starting from the integral representation for $\bar{u}^{i,i-1}$, that is:

$$\begin{aligned} &\bar{u}^{i,i-1}(t, x_{i-1}, x_i; K) \\ &= \int_{\mathbb{R}^2} \bar{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \cdot u^{i,i}\left(T_{i-2}, e^{y_i}; (ae^{y_{i-1}} + c)^+\right) dy_i dy_{i-1} \\ &= \int_{\mathbb{R}^2} \bar{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \\ &\qquad \times \int_{\mathbb{R}} \bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i)\left(e^{\eta_i} - (ae^{y_{i-1}} + c)^+\right)^+ d\eta_i dy_i dy_{i-1}\,, \end{aligned} \tag{2.35}$$

for $K \geq 0$ and $t \in (0, T_{i-2})$.

We first recall the expression of the Gaussian fundamental solution $\bar{\Gamma}^{i,i}$ of the one dimensional heat operator

$$\bar{\mathcal{L}}^{i,i} = \frac{(\sigma^i)^2}{2}(\partial_{y_i y_i} - \partial_{y_i}) + \partial_t,$$

given by (2.20) with $\bar{\sigma}^i(t, T_{i-1}) = \sigma^i\sqrt{T_{i-1} - t}$, so that

$$\bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i) = \frac{1}{\sigma^i\sqrt{2\pi\delta_{i-1}}} \exp\left[-\frac{1}{2}\left(\frac{2(\eta_i - y_i) + (\sigma^i)^2\delta_{i-1}}{2\sigma^i\sqrt{\delta_{i-1}}}\right)^2\right], \tag{2.36}$$

for $y_i, \eta_i \in \mathbb{R}$ and $T_{i-2} < T_{i-1}$, $\delta_{i-1} = T_{i-1} - T_{i-2}$.

An expression of $\bar{\Gamma}^{i,i-1}$ is not possible to compute explicitly because it has got non constant coefficients. Thus, we approximate the fundamental solution $\bar{\Gamma}^{i,i-1}$ by means of the fundamental solution $\widetilde{\Gamma}^{i,i-1}$ of the constant coefficients operator

$$\begin{aligned}\widetilde{\mathcal{L}}^{i,i-1} :=& \frac{(\sigma^{i-1})^2}{2}\left(\partial_{x_{i-1}x_{i-1}} - \partial_{x_{i-1}}\right) + \frac{(\sigma^{i})^2}{2}\left(\partial_{x_i x_i} - \partial_{x_i}\right)\\ &+ \rho_{i-1,i}\,\sigma^{i-1}\sigma^i\ \partial_{x_{i-1}x_i} - \bar{c}_i\,\rho_{i-1,i}\ \sigma^{i-1}\sigma^i\partial_{x_{i-1}} + \partial_t,\end{aligned}$$

which is obtained by freezing the variable coefficient $\delta_i e^{x_i}(1+\delta_i e^{x_i})^{-1}$ appearing in (2.33) to the value defined by the spot, i.e:

$$\bar{c}_i = \frac{\delta_i L_i^0}{1+\delta_i L_i^0}. \tag{2.37}$$

Now, its fundamental solution $\widetilde{\Gamma}^{i,i-1}$ is given by

$$\widetilde{\Gamma}^{i,i-1}(t,x_{i-1},x_i;T_{i-2},y_{i-1},y_i) \ = \ \frac{\exp(F(t,x_{i-1},x_i;T_{i-2},y_{i-1},y_i))}{2\pi\sigma^i\sigma^{i-1}(T_{i-2}-t)\sqrt{1-\rho_{i-1,i}^2}}, \tag{2.38}$$

for $x_{i-1}, x_i, y_{i-1}, y_i \in \mathbb{R}$, $t < T_{i-2}$, where:

$$\begin{aligned} & F(t,x_{i-1},x_i;T_{i-2},y_{i-1},y_i) = \\ = \ & \frac{1}{8\left(1-\rho_{i-1,i}^2\right)}\Big[(\sigma^{i-1})^2(t-T_{i-2}) + 4(x_{i-1}+x_i-y_i-y_{i-1}) \\ & + 8\bar{c}_i\ \rho_{i-1,i}^2\ (y_{i-1}-x_{i-1}) + (\sigma^i)^2(t-T_{i-2})\left(1+4(-1+\bar{c}_i)\ \bar{c}_i\ \rho_{i-1,i}^2\right) \\ & + \frac{4(x_i-y_i)^2}{(\sigma^{i-1})^2(t-T_{i-2})} + \frac{4(x_{i-1}-y_{i-1})^2}{(\sigma^i)^2(t-T_{i-2})} \\ & + \frac{2\sigma^i(-1+2\bar{c}_i)\left((\sigma^{i-1})^2(t-T_{i-2})+2(x_i-y_i)\right)\rho_{i-1,i}}{\sigma^{i-1}} \\ & - \frac{4\left((\sigma^{i-1})^2(t-T_{i-2})+2(x_i-y_i)\right)(x_{i-1}-y_{i-1})\rho_{i-1,i}}{\sigma^i\sigma^{i-1}(t-T_{i-2})}\Big]. \end{aligned} \tag{2.39}$$

Thus we get the following analytical approximation formula:

$$\begin{aligned} & \bar{u}^{i,i-1}(t,x_{i-1},x_i;K) \\ & = \int_{\mathbb{R}^2}\bar{\Gamma}^{i,i-1}(t,x_{i-1},x_i;T_{i-2},y_{i-1},y_i) \\ \times & \int_{\mathbb{R}}\bar{\Gamma}^{i,i}(T_{i-2},y_i;T_{i-1},\eta_i)\left(e^{\eta_i}-(ae^{y_{i-1}}+c)^+\right)^+ d\eta_i dy_i dy_{i-1} \\ & \approx \int_{\mathbb{R}^2}\widetilde{\Gamma}^{i,i-1}(t,x_{i-1},x_i;T_{i-2},y_{i-1},y_i) \\ \times & \int_{\mathbb{R}}\bar{\Gamma}^{i,i}(T_{i-2},y_i;T_{i-1},\eta_i)\left(e^{\eta_i}-(ae^{y_{i-1}}+c)^+\right)^+ d\eta_i dy_i dy_{i-1} \end{aligned} \tag{2.40}$$

Notice that formula (2.40) involves a triple integral but two of them can be computed analytically. We defer all the explicit formulas to the Appendix A.

Finally, the approximation of the relative price is given by

$$\Pi_t^i \equiv u^{i,i-1}(t, L_t^{i-1}, L_t^i; K) = \bar{u}^{i,i-1}(t, \log\, L_t^{i-1}, \log\, L_t^i; K).$$

2.5.2 Crank-Nicolson Lagrange Galerkin method

In the literature we can find different applications of the classical method of characteristics of first order (introduced in [56]) for the numerical solution of financial problems, as for example in Vázquez [64] for European and American vanilla options, in D'Halluin, Forsyth and Labahn [19] for Asian options under jump-diffusion models or in Farto and Vázquez [21] for callable bonds with notice pricing under Vasicek and CIR interest rate models.

Other alternative finite differences numerical schemes for Kolmogorov equations appearing in the Hobson-Rogers stochastic volatility model are proposed in Di Francesco and Pascucci [24], Di Francesco, Foschi and Pascucci [23]. Indeed, the classical characteristics method has been adapted and combined with finite differences by González-Gaspar and Vázquez in [29] to solve this stochastic volatility model.

Recently, the here used Crank-Nicholson characteristic methods of order two for general convection-diffusion-reaction equations (eventually degenerated) have been proposed and analyzed numerically in Bermúdez, Nogueiras and Vázquez [6, 7]. More precisely, in [6] the time discretization scheme is analyzed for a variable coefficient convection-(possibly degenerate) diffusion-reaction equation with mixed Dirichlet-Robin boundary conditions. Firstly, the proposed second order time discretization scheme is rigorously introduced for exact and approximate characteristics. Next, under not much restrictive hypotheses on the data, the $l^\infty(L^2)$ stability is proved and $l^\infty(L^2)$ error estimates of order $O(\Delta t^2)$ are obtained. In [7], Lagrange-Galerkin schemes using different finite elements spaces are analyzed and different quadrature formulas are proposed for practical implementation. Furthermore, the previous methods have been applied to price Asian options in Bermúdez, Nogueiras and Vázquez

[8].

In this section we study the numerical solution to the ratchet caplet pricing problem (2.29) by a Crank-Nicholson-characteristics method combined with finite elements. Some difficulties in the numerical solution are due to the fact that the spatial domain is unbounded in both forwards directions. Therefore, domain truncation and boundary conditions are proposed as a previous step to perform a finite element numerical approximation of the solution.

On the other hand, the diffusion matrix degenerates at the axis (i.e. convection dominates near the axes) so we propose a higher order Lagrange-Galerkin methods. Thus, we use a combination of the Crank-Nicholson characteristics method for the time discretization and piecewise quadratic finite elements method for the spatial discretization.

Unbounded domain. Divergence form

In order to rewrite the problem (2.32) as an initial value problem in divergence form, we introduce the new time variable $\tau = T_{i-2} - t$ and pose the equivalent problem:

$$\partial_\tau u^{i,i-1} + \vec{v} \cdot \nabla u^{i,i-1} - Div(A \nabla u^{i,i-1}) = 0 \qquad \text{in} \quad (0, T_{i-2}) \times \mathbb{R}^2_+ \,, \tag{2.41}$$

$$u^{i,i-1}(0, L_{i-1}, L_i) = u^{i,i}\left(t = T_{i-2}, L_i; (aL_{i-1} + c)^+\right) \qquad \text{in} \quad \mathbb{R}^2_+ \,, \tag{2.42}$$

where:

$$A(L_{i-1}, L_i) = \begin{pmatrix} \frac{1}{2}(\sigma^{i-1})^2 L_{i-1}^2 & \frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i L_{i-1} L_i \\ \frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i L_{i-1} L_i & \frac{1}{2}(\sigma^i)^2 L_i^2 \end{pmatrix}, \tag{2.43}$$

$$\vec{v}(L_{i-1}, L_i) = \begin{pmatrix} \frac{\rho_{i-1,i}\delta_i L_{i-1} L_i}{1+\delta_i L_i}\sigma^{i-1}\sigma^i + (\sigma^{i-1})^2 L_{i-1} + \frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i L_{i-1} \\ \frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i L_i + (\sigma^i)^2 L_i \end{pmatrix} \tag{2.44}$$

Truncated domain

As in most problems arising in finance, the numerical solution with finite differences, finite volumes or finite elements requires the approximation of the original problem in an unbounded domain by another one posed in a bounded computational domain. This technique is known as localization procedure, that has to be performed so that the truncation by the bounded domain and the associated boundary conditions do not affect the solution in the region of financial interest. For the classical problem of European vanilla options and Dirichlet boundary conditions, a rigorous analysis has been carried out in [36]. In general, the required boundary conditions at the new boundaries of the bounded domain are obtained with financial and/or mathematical arguments.

For the localization purpose, let us consider both L_{i-1}^{∞} and L_i^{∞} large enough real numbers suitably chosen and let the bounded domain $\Omega = (0, L_{i-1}^{\infty}) \times (0, L_i^{\infty})$ with Lipschitz boundary Γ, such that $\Gamma = \Gamma_1^+ \bigcup \Gamma_2^+ \bigcup \Gamma_1^- \bigcup \Gamma_2^-$, where $\Gamma_1^- = \Gamma \cap \{L_{i-1} = 0\}$, $\Gamma_2^- = \Gamma \cap \{L_i = 0\}$, $\Gamma_1^+ = \Gamma \cap \{L_{i-1} = L_{i-1}^{\infty}\}$, $\Gamma_2^+ = \Gamma \cap \{L_i = L_i^{\infty}\}$.

Then, problem (2.41)-(2.42) is replaced by

Find $u^{i,i-1} : [0, T_{i-2}] \times \Omega \to \mathbb{R}$ such that

$$\partial_\tau u^{i,i-1} + \vec{v} \cdot \nabla u^{i,i-1} - Div(A \nabla u^{i,i-1}) = 0 \qquad \text{in } (0, T_{i-2}) \times \Omega\,, \tag{2.45}$$

$$u^{i,i-1}(0, L_{i-1}, L_i) = u^{i,i}\left(t = T_{i-2}, L_i; (aL_{i-1} + c)^+\right) \qquad \text{in } \Omega\,, \tag{2.46}$$

where A and $\vec{v}$ are defined in (2.43)-(2.44).

Next, by applying the theory of second order partial differential equations with nonnegative characteristics that can be found in [47] and taking into account the expression of the matrix A and the vector $\vec{v}$, only boundary conditions at Γ_1^+ and Γ_2^+ are required.

More precisely, following the ideas in [47], for simplicity let us introduce the notation

$$x_1 = L_{i-1}, \quad x_2 = L_i. \tag{2.47}$$

Then, operator (2.30) associated to the Cauchy problem (2.29) can be written in the form:

$$\mathcal{L}^* = \sum_{i,j=1}^{2} a_{ij}^* \frac{\partial^2}{\partial x_i x_j} + \sum_{j=1}^{2} b_j^* \frac{\partial}{\partial x_j} + \frac{\partial}{\partial t}, \tag{2.48}$$

where the involved data are defined as follows

$$A^*(x_1,x_2) = (a^*_{ij}) \;=\; \begin{pmatrix} \frac{1}{2}(\sigma^1 x_1)^2 & \frac{1}{2}\rho_{1,2}\sigma^1\sigma^2 x_1 x_2 \\ \frac{1}{2}\rho_{1,2}\sigma^1\sigma^2 x_1 x_2 & \frac{1}{2}(\sigma^2 x_2)^2 \end{pmatrix}, \tag{2.49}$$

$$v^*(x_1,x_2) = (b^*_j) \;=\; \begin{pmatrix} -\rho_{1,2}\sigma^1\sigma^2 \dfrac{\delta x_1 x_2}{1+\delta x_2} \\ 0 \end{pmatrix}. \tag{2.50}$$

Thus, in terms of the inwards normal vector to the boundary of Ω, $\vec{m} = (m_1, m_2)$, we introduce the following subsets of Γ:

$$\Sigma_1 \;=\; \{(x_1,x_2) \in \Gamma \quad , \sum_{i,j=1}^{2} a^*_{ij} m_i m_j > 0\}, \tag{2.51}$$

$$\Sigma_2 \;=\; \left\{(x_1,x_2) \in \Gamma - \Sigma_1 \quad , \sum_{i=1}^{2}\left(b^*_i - \sum_{j=1}^{2} \frac{\partial a^*_{ij}}{\partial x_j}\right) m_i < 0\right\}. \tag{2.52}$$

As indicated in [47], the boundary conditions at $\Sigma_1 \cup \Sigma_2$ for the initial boundary value problem associated to (2.48) are required. So, when considering each boundary of Ω, we obtain:

- On boundary Γ_1^+ : $\quad x_1 = x_1^\infty, \quad 0 \le x_2 \le x_2^\infty, \quad \vec{m} = (-1, 0)$

$$\sum_{i,j=1}^{2} a^*_{ij} m_i m_j = \frac{1}{2}(\sigma^1 x_1^\infty)^2 > 0$$

- On boundary Γ_2^+ : $\quad 0 \le x_1 \le x_1^\infty, \quad x_2 = x_2^\infty, \quad \vec{m} = (0, -1)$

$$\sum_{i,j=1}^{2} a^*_{ij} m_i m_j = \frac{1}{2}(\sigma^2 x_2^\infty)^2 > 0$$

- On boundary Γ_1^- : $\quad x_1 = 0, \quad 0 \le x_2 \le x_2^\infty, \quad \vec{m} = (1, 0)$

$$\sum_{i,j=1}^{2} a^*_{ij} m_i m_j = 0$$

$$\sum_{i=1}^{2}\left(b^*_i - \sum_{j=1}^{2} \frac{\partial a^*_{ij}}{\partial x_j}\right) m_i =$$

$$-\rho_{1,2}\sigma^1\sigma^2 \frac{\delta x_1 x_2}{1+\delta x_2} - \sigma^1 x_1 - \frac{1}{2}\rho_{1,2} x_1 \sigma^1 \sigma^2 = 0$$

- On boundary Γ_2^- : $\quad 0 \leq x_1 \leq x_1^\infty, \quad x_2 = 0, \quad \vec{m} = (0,1)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = 0$$

$$\sum_{i=1}^{2} \left(b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = -\frac{1}{2}\rho_{1,2} x_2 \sigma^1 \sigma^2 - \sigma^2 x_2 = 0$$

Therefore, we obtain that $\Sigma_1 = \Gamma_1^+ \cup \Gamma_2^+$ and $\Sigma_2 = \emptyset$.

Next, we propose the following Dirichlet boundary conditions

$$u^{i,i-1}(\tau, L_{i-1}, L_i) = u^{i,i}(t = T_{i-2}, L_{i-1}, L_i) \qquad \text{on } [0, T_{i-2}] \times \Gamma_1^+ , \tag{2.53}$$

$$u^{i,i-1}(\tau, L_{i-1}, L_i) = u^{i,i}(t = T_{i-2}, L_{i-1}, L_i) \qquad \text{on } [0, T_{i-2}] \times \Gamma_2^+ . \tag{2.54}$$

Remark 2.5.2. *These conditions are slightly different from those ones proposed in [62], which are based on financial arguments. More precisely, in [62] we select the bounded domain such that $L_i^\infty < aL_{i-1}^\infty$ and then we propose*

$$u^{i,i-1}(T_{i-1}, L_{i-1}, L_i) = 0 \qquad \text{on } \Gamma_1^+ , \tag{2.55}$$

$$u^{i,i-1}(T_{i-1}, L_{i-1}, L_i) = M\delta_i (L_i - aL_{i-1} - c)^+ \qquad \text{on } \Gamma_2^+ . \tag{2.56}$$

The results are very close each other with both conditions.

Crank-Nicolson characteristics discretization

For the time discretization of the PDE problem defined by equations (2.45)-(2.46) and (2.53)- (2.54), a method of characteristics is proposed. This method can be framed into the more general setting of upwinding methods, which take in account the local direction of the flux. Specifically, the method of characteristics is based on a finite differences scheme for the discretization of the material derivative, i.e., the time derivative along the characteristic lines of the convective part of the equation. In this section we will also introduce the variational formulation for the time discretized problem, for which the notation introduced in [46], for example, is followed.

First, we define the characteristics curve through $L = (L_{i-1}, L_i)$ at time $\bar{\tau}$, $X_e(L, \bar{\tau}; \tau)$, which verifies:

$$\partial_\tau X_e(L, \bar{\tau}; \tau) = \vec{v}(X_e(L, \bar{\tau}; \tau)), \quad X_e(L, \bar{\tau}; \bar{\tau}) = L\,. \tag{2.57}$$

The final value problem (2.57) can be exactly solved and we obtain:

$$\begin{aligned} X_e^1(L, \bar{\tau}; \tau) &= L_{i-1} \exp\left(\frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i + (\sigma^{i-1})^2 + \frac{\delta_i L_i \rho_{i-1,i}\sigma^{i-1}\sigma^i}{1+\delta_i L_i}(\bar{\tau} - \tau)\right) \\ X_e^2(L, \bar{\tau}; \tau) &= L_i \exp\left((\frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i + (\sigma^i)^2)(\bar{\tau} - \tau)\right) \end{aligned}$$

Now, for $i = 1, ..., N$, let us consider the time step $\Delta\tau = \frac{T_{i-1}}{N}$ and the time meshpoints $\tau_n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by characteristics method is given by:

$$\frac{Du^{i,i-1}}{D\tau} \approx \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta\tau}$$

where $X_e^n(L) := X_e(L, \tau^{n+1}; \tau^n)$, the components of which are given by

$$\begin{aligned} X_e^{n,1}(L) &= L_{i-1} \exp\left(\frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i + (\sigma^{i-1})^2 + \frac{\delta_i L_i \rho_{i-1,i}\sigma^{i-1}\sigma^i}{1+\delta_i L_i}\Delta\tau\right) \\ X_e^{n,2}(L) &= L_i \exp\left((\frac{1}{2}\rho_{i-1,i}\sigma^{i-1}\sigma^i + (\sigma^i)^2)\Delta\tau\right). \end{aligned}$$

Remark 2.5.3. *Figure 2.4 shows that the velocity field at the boundaries does not point to the interior of the domain. So, for small enough time steps the points $X_e^n(L)$ belong to the domain.*

Next, we consider a Crank-Nicolson scheme around $(X_e(L, \tau_{n+1}; \tau), \tau)$ for $\tau = \tau_{n+\frac{1}{2}}$. So, for n=0,...,N-1, the time discretized equation can be written as:

Find $(u^{i,i-1})^{n+1}$ such that:

$$\begin{aligned} \frac{(u^{i,i-1})^{n+1}(L) - (u^{i,i-1})^n(X_e^n(L))}{\Delta\tau} &- \frac{1}{2}Div(A\nabla(u^{i,i-1})^{n+1})(L) \\ -\frac{1}{2}Div(A\nabla(u^{i,i-1})^n)(X_e^n(L)) &= 0\,. \end{aligned} \tag{2.58}$$

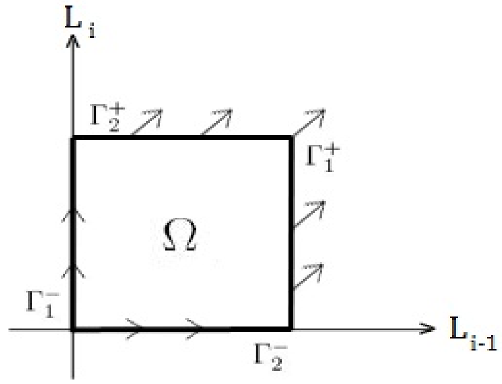

Figure 2.4: Velocity field in the domain Ω.

In order to state the weak formulation for the semidiscretizated problem, we use the following lemma that appears in [8] for $w = A\nabla V^n$ and $X(x) = X_e^n(x)$:

Lemma 2.5.4. *Let* $X : \bar{\Omega} \to X(\bar{\Omega})$, $X \in \mathrm{C}^2(\bar{\Omega})$, *be an invertible vector valued function. Let* $\boldsymbol{F}=\nabla X$ *and assume that* $\boldsymbol{F}^{-1} \in \mathrm{C}^1(\bar{\Omega})$. *Then,*

$$\begin{aligned} \int_\Omega Div\boldsymbol{w}(X(\boldsymbol{x}))\psi(\boldsymbol{x})\,dx &= \int_\Gamma \boldsymbol{F}^{-T}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{w}(X(\boldsymbol{x}))\psi(\boldsymbol{x})\,dA_x - \\ \int_\Omega \boldsymbol{F}^{-1}(\boldsymbol{x})\boldsymbol{w}(X(\boldsymbol{x}))\cdot\nabla\psi(\boldsymbol{x})\,dx &- \int_\Omega Div\boldsymbol{F}^{-T}\cdot\boldsymbol{w}(X(\boldsymbol{x}))\psi(\boldsymbol{x})\,dx, \end{aligned}$$

being $\boldsymbol{w} \in \mathrm{H}^1(X(\Omega))$ *a vector valued function and* $\psi \in \mathrm{H}^1(\Omega)$ *a scalar function.*

Now, multiplying equation (2.58) by a suitable test function ψ and integrating in Ω, we have:

$$\begin{aligned} &\int_\Omega \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta\tau}\psi\,dL \\ -\frac{1}{2}\int_\Omega Div(A\nabla(u^{i,i-1})^{n+1})\psi\,dL - \frac{1}{2}\int_\Omega \big(Div(A\nabla(u^{i,i-1})^n)\big)\circ X_e^n\psi\,dL &= 0\,. \quad (2.59) \end{aligned}$$

Applying Lemma 3.4 that appears in [8] and the usual Green's formula, equation (2.59) is equivalent to:

$$\begin{aligned}
\int_\Omega \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta\tau} \psi dL + \frac{1}{2} \int_\Omega A\nabla (u^{i,i-1})^{n+1} \nabla\psi \, dL \\
+ \frac{1}{2} \int_\Omega (F_e^n)^{-1} (A\nabla (u^{i,i-1})^n) \circ X_e^n \nabla\psi \, dL \\
+ \frac{1}{2} \int_\Omega Div (F_e^n)^{-t} (A\nabla (u^{i,i-1})^n) \circ X_e^n \psi \, dL \\
= \frac{1}{2} \int_\Gamma \vec{n} \cdot A\nabla (u^{i,i-1})^{n+1} \psi \, dA_L \\
+ \frac{1}{2} \int_\Gamma (F_e^n)^{-t} \vec{n} \cdot (A\nabla (u^{i,i-1})^n) \circ X_e^n \psi \, dA_L \quad . \qquad (2.60)
\end{aligned}$$

Notice that the tensor $(F_e^n)^{-t} = (\nabla X_e(x, \tau_{n+1}; \tau_n))^{-t}$ can be expressed in the form

$$(\mathbf{F}_e^n)^{-t} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix},$$

with

$$\begin{aligned}
b_{11}(L) &= \exp\left(\frac{1}{2} \rho_{i-1,i} \sigma^{i-1} \sigma^i + (\sigma^{i-1})^2 + \frac{\delta_i L_i \rho_{i-1,i} \sigma^{i-1} \sigma^i}{1 + \delta_i L_i} \Delta\tau \right) \\
b_{21}(L) &= L_{i-1} \Delta t \left(\frac{\delta_i \rho_{i-1,i} \sigma^{i-1} \sigma^i (1 + \delta_i L_i) - \delta_i^2 \rho_{i-1,i} \sigma^{i-1} \sigma^i L_i}{(1 + \delta_i L_i)^2} \right) \\
&\quad \times \exp\left((\frac{1}{2} \rho_{i-1,i} \sigma^{i-1} \sigma^i + (\sigma^i)^2) \Delta\tau \right) \\
b_{22}(L) &= \exp\left((\frac{1}{2} \rho_{i-1,i} \sigma^{i-1} \sigma^i + (\sigma^i)^2) \Delta\tau \right).
\end{aligned}$$

Next, let us precise the boundary integrals appearing in formulation (2.60).

First, notice that we have $\vec{n} \cdot A\nabla (u^{i,i-1})^{n+1} = 0$ on $\Gamma_1^- \cup \Gamma_2^-$ and $\psi = 0$ on $\Gamma_1^+ \cup \Gamma_2^+$. Therefore, the first boundary integral on the right hand side of equation (2.60) vanishes. Moreover, for the second integral, we have

$$\int_\Gamma (F_e^n)^{-t} \vec{n} \cdot (A\nabla (u^{i,i-1})^n) \circ X_e^n \psi \, dA_L = \int_{\Gamma_1^- \cup \Gamma_2^-} g^n \psi dA_L, \qquad (2.61)$$

where $g^n : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is given by,

$$
\mathrm{g}^n(L)=\begin{cases} -\frac{1}{2}((F_e^n)^{-t})_{21}(\sigma^i)^2 L_i^2 \dfrac{\partial (u^{i,i-1})^n}{\partial L_i}(X_e^n(L)) & on \quad \Gamma_1^- \\ \\ -\frac{1}{2}((F_e^n)^{-t})_{12}(\sigma^{i-1})^2 L_{i-1}^2 \dfrac{\partial (u^{i,i-1})^n}{\partial L_{i-1}}(X_e^n(L)) & on \quad \Gamma_2^- \end{cases}
$$

Therefore, equation (2.60) becomes

$$
\begin{aligned}
\int_\Omega \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta t}\psi\, dL + \frac{1}{2}\int_\Omega A\nabla(u^{i,i-1})^{n+1}\nabla\psi\, dL \\
+\frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla(u^{i,i-1})^n)\circ X_e^n \nabla\psi\, dL \\
+\frac{1}{2}\int_\Omega Div(F_e^n)^{-t}(A\nabla(u^{i,i-1})^n)\circ X_e^n \psi\, dL \\
= \frac{1}{2}\int_{\Gamma_1^-\cup\Gamma_2^-} g^n\psi\, dA_L, \qquad (2.62)
\end{aligned}
$$

for all $\psi \in H^1_{0,\Gamma_D}(\Omega)$, where the involved functional sets are,

$$
\begin{aligned}
H^1_{\Gamma_D}(\Omega) &= \{\psi \in H^1(\Omega)/\psi_{|_{\Gamma_1^+}} = 0, \psi_{|_{\Gamma_2^+}} = M\delta_i(L_i - aL_{i-1} - c)_+\}, \\
H^1_{0,\Gamma_D}(\Omega) &= \{\psi \in H^1(\Omega)/\psi_{|_{\Gamma_D}} = 0\}.
\end{aligned}
$$

Finite elements discretization

As we mention at the beginning of the section, we use the previously described characteristics-Crank-Nicholson method for the time discretization jointly with finite elements for spatial discretization. For this purpose, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω.

Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. Now, let us define the subset of finite elements V_h and the space of test functions V_{h,Γ_D}:

$$
\begin{aligned}
V_h &= \{\varphi_h \in \mathcal{C}^0(\bar{\Omega}) : \varphi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \\
V_{h,\Gamma_D} &= \{\varphi_h \in V_h : \quad \varphi_h = 0, \quad on \quad \Gamma_D\},
\end{aligned}
$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$. The numerical analysis of the method for a more general equation can be found in [6, 7].

Therefore, if $u_h^{i,i-1}$ denotes the finite element approximation of $u^{i,i-1}$, the discretized equation (2.62) is

$$\begin{aligned}\int_\Omega \frac{(u_h^{i,i-1})^{n+1} - (u_h^{i,i-1})^n \circ X_e^n}{\Delta t}\psi_h \, dL + \frac{1}{2}\int_\Omega A\nabla(u_h^{i,i-1})^{n+1}\nabla\psi_h \, dL \\ +\frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla(u_h^{i,i-1})^n) \circ X_e^n \nabla\psi_h \, dL \\ +\frac{1}{2}\int_\Omega Div(F_e^n)^{-t}(A\nabla(u_h^{i,i-1})^n) \circ X_e^n \psi_h \, dL \\ = \frac{1}{2}\int_{\Gamma_1^- \cup \Gamma_2^-} g^n \psi_h \, dA_L, \ \forall \psi_h \in V_{h,\Gamma_D}\end{aligned}$$

Then, the fully discretized problem is posed as follows

For $(u_h^{i,i-1})^0 \in V_h$, find $u_h^{i,i-1} = \{(u_h^{i,i-1})^n\}_{n=1}^N \in [V_h]^N$ such that for $n = 0, ..., N-1$ we have:

$$\frac{1}{\Delta t} < \mathcal{D}_E^{n+1}[u_h^{i,i-1}], \psi_h > + < \mathcal{M}^n[u_h^{i,i-1}], \psi_h > = < \mathcal{N}^n, \psi_h >, \ \forall \psi \in V_h. \tag{2.63}$$

where

$$\begin{aligned} < \mathcal{D}_E^{n+1}[u_h^{i,i-1}], \psi_h > &= \int_\Omega \frac{(u_h^{i,i-1})^{n+1} - (u_h^{i,i-1})^n \circ X_e^n}{\Delta t}\psi_h \, dL \\ < \mathcal{M}^n[u_h^{i,i-1}], \psi_h > &= \frac{1}{2}\int_\Omega A\nabla(u_h^{i,i-1})^{n+1}\nabla\psi_h \, dL \\ &+\frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla(u_h^{i,i-1})^n) \circ X_e^n \nabla\psi_h \, dL \\ &+\frac{1}{2}\int_\Omega Div(F_e^n)^{-t}(A\nabla(u_h^{i,i-1})^n) \circ X_e^n \psi_h \, dL \\ < \mathcal{N}^n, \psi_h > &= \frac{1}{2}\int_{\Gamma_1^- \cup \Gamma_2^-} g^n \psi_h \, dA_L \end{aligned}$$

with $\mathcal{M}^n[\psi_h] \in (H^1(\Omega))'$, for $\Pi_h \in \mathcal{C}^0(H^1(\Omega))$ and $\mathcal{N}^n \in (H^1(\Omega))'$.

In [6] the convergence of the previous scheme is analyzed for a more general equation and the method is unconditionally stable in case of exact integration of the integral terms. Also, for academic cases of constant coefficients convection-diffusion and pure convection equations the study of the different quadrature formulas to compute the involved integral terms is carried out.

First, in this setting some stability results in one spatial dimension are obtained. Thus, in case of trapezoidal or Simpson formulas for the case of pure convection stability can be proved, in the presence of an additional diffusive term the stability region is smaller for lower Peclet numbers, thus this formulas are convenient for convection dominated problems. These results can be extended to higher spatial dimensions when product of one dimensional finite element spaces and quadrature formulas are considered. Notice that the piecewise quadratic finite elements over quadrangular meshes are a particular case of product finite element spaces.

In all presented examples which are out of the scope of academic ones, we use a product three point Gauss-Legendre quadrature formula to approximate all the integral terms appearing in the fully discretized problem, without observing any numerical instabilities and providing better results than Simpson rule.

2.5.3 Monte Carlo simulation

In this section the particular use of a Monte Carlo simulation technique to approximate the price of the ratchet caplets in the case $b = 0$ is briefly described. In a forthcoming section for the case $b \neq 0$ is considered.

First, for the ratchet caplet i ($i = 1, \ldots, N$) we consider the terminal probability measure, Q^i, associated to the numeraire B^i, so that in this probability measure the stochastic differential equations governing the dynamics of forward LIBOR L^{i-1} and L^i (entering in the i-th ratchet caplet contract for $b = 0$) are given by:

$$dL_t^{i-1} = -L_t^{i-1}\sigma^{i-1}(t)\frac{\rho_{i-1,i}\delta_i\sigma^i(t)L_t^i}{1+\delta_i L_t^i}\,dt + L_t^{i-1}\sigma^{i-1}(t)\,d\mathcal{W}_t^{i-1}, \qquad (2.64)$$

$$dL_t^i = L_t^i\sigma^i(t)\,d\mathcal{W}_t^i. \qquad (2.65)$$

Note that due to the choice of the probability measure and the fact that the price of the ratchet caplet i only depends on L^{i-1} and L^i, then only the correlation $\rho_{i-1,i}$ is involved.

We assume that we are interested in the price at time $t = 0$ of the ratchet caplet i which is signed up at time T_0 and pays at time T_i.

For Monte Carlo simulation, we consider the uniform time grid t_n, $n = 0,\ 1,\ 2, \dots, N_t$, such that $t_{N_t} = T_i$ and includes the tenor dates between T_0 and T_i. So, we first draw a path for the correlated Brownian motions W^{i-1} and W^i at the sampling dates t_n, $n = 0,\ 1,\ 2, \dots, N_t - 1$. For this purpose, we first consider two independent Brownian motions, Y^1 and Y^2, whose simulation can be obtained by

$$
\begin{aligned}
Y^1(t_{n+1}) &= Y^1(t_n) + \sqrt{\Delta_n}\, \epsilon_n^1, \quad Y^1(t_0) = 0, \\
Y^2(t_{n+1}) &= Y^2(t_n) + \sqrt{\Delta_n}\, \epsilon_n^2, \quad Y^2(t_0) = 0,
\end{aligned}
$$

where $\Delta_n = t_{n+1} - t_n$, although in practice we consider $\Delta_n = \Delta$ as a constant, and where ϵ_n^1 and ϵ_n^2 are drawings from standard normal distributions. Next, the correlated Brownian motions are defined by

$$
\begin{aligned}
W^i(t_n) &= Y^1(t_n), \\
W^{i-1}(t_n) &= \rho_{i-1,i} Y^1(t_n) + \sqrt{1 - \rho_{i-1,i}^2}\, Y^2(t_n).
\end{aligned}
$$

As the simulation of LIBOR forward rates is a particular case of simulating the solution of stochastic differential equations, we have several possibilities for the numerical solution. In this work we apply an Euler-Maruyama scheme [38], so that we compute the forward LIBOR rates at each sampling date by using the following discretization scheme for $n = 0,\ 1,\ 2, \dots, N_t - 1$:

$$
\begin{aligned}
L_{t_{n+1}}^{i-1} &= L_{t_n}^{i-1} - \frac{\delta_i \rho_{i-1,i} \sigma^i(t_n) L_{t_n}^i}{1 + \delta_i L_{t_n}^i} \sigma^{i-1}(t_n) L_{t_n}^{i-1} \Delta_n + \\
&\quad + \sigma^{i-1}(t_n) L_{t_n}^{i-1} (W^{i-1}(t_{n+1}) - W^{i-1}(t_n)) \qquad (2.66) \\
L_{t_{n+1}}^i &= L_{t_n}^i + \sigma^i(t_n) L_{t_n}^i (W^i(t_{n+1}) - W^i(t_n)), \qquad (2.67)
\end{aligned}
$$

starting from the given values of $L^{i-1}(t_0)$ and $L^i(t_0)$, because at the pricing date $t = 0$ the forward LIBOR rates are known for sure.

Once we have the samples of the forward rates at the tenor dates we can compute the discount bond (numeraire), B^i, at the tenor dates by using the formula

$$B^i_{T_j} = \prod_{k=n}^{i-1} (1 + \delta_k L_k(T_j))^{-1}, \quad j \leq i. \tag{2.68}$$

Finally, using that the discounted price of a derivative must be a martingale under the terminal measure Q^i, the price of the ratchet caplet at time $t = 0$ is given by

$$R(0, L_0^{i-1}, L_0^i) = B_0^i \, \mathbb{E}^i \left[\frac{R(T_i, L_{T_{i-1}}^{i-1}, L_{T_i}^i)}{B^i_{T_i}} \right] = B_0^i \, \mathbb{E}^i \left[M\delta_i(\bar{L}^i - K_i)^+ \right] \tag{2.69}$$

where $\mathbb{E}^i$ denotes the expectation under the terminal measure Q^i.

The above described Monte Carlo technique has been used to obtain the results that appear in the forthcoming section of numerical results for the case $b = 0$.

As it has been pointed out in previous paragraphs, Monte Carlo simulation requires the generation of large quantities of random numbers with specific statistical properties to build the dynamics of LIBOR forward rates.

Actually, we can compute quickly and easily sequences of random numbers, however sometimes maybe they are not truly random, so they are referred as pseudo-random numbers. This means that we can compute a huge quantity of random numbers with identic statistical properties, but they are not simulated in a really independent way, thus leading to specific errors in the approximation. In order to overcome this kind of problem, Box-Muller and Polar-Marsaglia methods can be used [38]. Polar-Marsaglia method avoids the computation of trigonometric functions and in consequence, the computational time is lower with respect to Box-Muller. Moreover, it is more efficient computationally when a large quantity of random numbers are required.

2.5.4 Numerical results for the ratchet cap with b=0

In this section we show some numerical results for an academic test and an example of a real ratchet cap pricing problem to illustrate the performance of the proposed numerical methods.

First, we consider an academic test to check the second order Lagrange-Galerkin method for finite elements $\mathcal{Q}_h^2$ introduced in previous section and applied to price interest rate derivatives such as ratchet caps. The results provided by the numerical method are compared with the exact solution of the academic problem.

Next, in the real case example obtained from [54], we compare the computed prices obtained with the different proposed numerical techniques.

FORTRAN scientific computing language has been chosen for the implementation of the Lagrange-Galerkin numerical methods, MATLAB for Monte Carlo and MATHEMATICA for the computations related to the approximation by means of fundamental solutions. As indicated in the previous section, we only consider the case b equal to zero in the expression (2.10).

Academic test

The objective of this academic test is mainly checking the good performance of the second order Lagrange-Galerkin method applied to solve the PDE models stated in the previous section to price ratchet caplets when parameter b is equal to zero in expression (2.10).

For this purpose, we consider the following problem posed in an unbounded domain:

$$\begin{cases} \mathcal{L}u = f, & \text{in } (0,T) \times \mathbb{R}_+^2, \\ u(T,x,y) = 1, & \text{in } \mathbb{R}_+^2, \end{cases} \tag{2.70}$$

with

$$\mathcal{L} = \frac{1}{2}(\sigma^x x)^2 \partial_{xx} + \rho_{x,y}\sigma^x\sigma^x xy\partial_{xy} + \frac{1}{2}(\sigma^y y)^2 \partial_{yy} - \rho_{x,y}\sigma^x\sigma^y \frac{\delta y}{1+\delta y} x\partial_x + \partial_t,$$

where the following volatilities, correlation and accrual parameters

$$\sigma^x = 0.242,\ \sigma^y = 0.242,\ \rho_{x,y} = 1,\ \delta = 0.49833$$

have been taken. Moreover, the maturity $T = 1$ and the second member function

$$f = \mathcal{L}u_e \tag{2.71}$$

have been considered where

$$u^e(t,x,y) = \exp(\alpha(T-t)(x+y)), \qquad t \in [0,T], \quad (x,y) \in \mathbb{R}^+. \tag{2.72}$$

The idea is to introduce a nonzero second member in the ratchet caplet equation and an appropriate terminal condition such that the exact solution can be analytically provided by expression (2.72).

In the localization procedure we consider a bounded domain $\Omega = [0,3K] \times [0,3K]$, the given parameter K being the strike of the first ratchet caplet in the forthcoming real ratchet cap example, so that $3K = 0.144$. Moreover, as we also take $T = 1$ and $\alpha = 22$, then the maximum of the analytical solution is given by $u^e(0,3K,3K) = 564.5337$.

So, we consider the following Cauchy problem in divergence form already posed in the bounded domain:

$$\begin{aligned} \partial_\tau u + \vec{v} \cdot \nabla u - Div(A \nabla u) = f \quad & \text{in} \quad (0,T] \times \Omega \\ u(0,x,y) = 1 \quad & \text{in} \quad \Omega \\ u(\tau,x,y) = u^e(T-\tau,x,y) \quad & \text{on} \quad (0,1) \times \Gamma_D \end{aligned}$$

where $\Gamma_D = \Gamma_1^+ \cup \Gamma_2^+$ and

$$\begin{aligned} A(x,y) &= \begin{pmatrix} \frac{1}{2}(\sigma^x)^2 x^2 & \frac{1}{2}\rho_{x,y}\sigma^x\sigma^y xy \\ \frac{1}{2}\rho_{x,y}\sigma^x\sigma^y xy & \frac{1}{2}(\sigma^y)^2 y^2 \end{pmatrix}, \\ \vec{v}(x,y) &= \begin{pmatrix} \frac{\delta xy}{1+\delta y}\rho_{x,y}\sigma^x\sigma^y + (\sigma^x)^2 x + \frac{1}{2}\rho_{x,y}\sigma^x\sigma^y x \\ \frac{1}{2}\rho_{x,y}\sigma^x\sigma^y y + (\sigma^y)^2 y \end{pmatrix}, \\ f(\tau,x,y) &= \left(\partial_\tau u^e + \vec{v} \cdot \nabla u^e - Div(A \nabla u^e)\right)(\tau,x,y). \end{aligned}$$

Throughout this section the spatial quadrangular meshes are structured, uniform and with edges parallel to the axis. This is motivated by the fact that in [46] analogous results for quadrangular and triangular meshes were found, although quadrangular elements seem to be more efficient.

In Table 2.1 some data concerning the used meshes are presented. Moreover, in Table 2.2 and Table 2.3 the error in the $l^\infty((0,T);l^2(\Omega))$ norm for each mesh and different time steps are shown. More precisely, Table 2.2 contains the results using a Gauss-Legendre quadrature formula in all the integrals appearing in the variational formulation while Table 2.3 presents those ones corresponding to Simpson quadrature formula. As expected, the error decreases as the number of time mesh points increases or the spatial meshes becomes finer. Notice that for each finer fixed mesh in space, a second order convergence is clearly observed, until reaching the rounding error for finer time meshes. These results are in agreement with the theoretically stated order in time in [7] under certain assumptions on a general convection-diffusion-reaction (possibly degenerated) equation.

	N. Elem	**N. Nodes**
Mesh 2	4	9
Mesh 4	16	81
Mesh 8	64	289
Mesh 16	256	1089
Mesh 32	1024	4225
Mesh 64	4096	16641
Mesh 96	9216	37249

Table 2.1: FEM meshes data.

NT	**Mesh 2**	**Mesh 4**	**Mesh 8**	**Mesh 16**	**Mesh 32**	**Mesh 64**
10	2.0159×10^{-3}	2.7522×10^{-3}	2.5478×10^{-3}	1.9773×10^{-3}	1.4033×10^{-3}	1.0909×10^{-3}
$\mathbf{10^2}$	1.1440×10^{-3}	7.8230×10^{-5}	2.6811×10^{-5}	1.9992×10^{-5}	1.4139×10^{-5}	1.0937×10^{-5}
$\mathbf{10^3}$	1.1525×10^{-3}	7.6045×10^{-5}	7.0675×10^{-6}	6.0155×10^{-7}	1.4688×10^{-7}	1.0927×10^{-7}
$\mathbf{10^4}$	1.1528×10^{-3}	7.6193×10^{-5}	7.0897×10^{-6}	5.7718×10^{-7}	3.1795×10^{-7}	3.4992×10^{-8}
$\mathbf{10^5}$	1.1529×10^{-3}	7.6206×10^{-5}	7.0922×10^{-6}	2.4162×10^{-6}	3.9664×10^{-8}	2.9971×10^{-9}

Table 2.2: Error for different meshes and number of time steps (NT) in the academic test, when using Gauss-Legendre quadrature formulas.

NT	**Mesh 2**	**Mesh 4**	**Mesh 8**	**Mesh 16**	**Mesh 32**	**Mesh 64**
10	1.8552×10^{-3}	2.5966×10^{-3}	2.5040×10^{-3}	1.9466×10^{-3}	1.3818×10^{-3}	1.0807×10^{-3}
$\mathbf{10^2}$	5.9552×10^{-4}	1.8053×10^{-4}	4.2185×10^{-5}	2.1019×10^{-5}	1.4050×10^{-5}	1.0863×10^{-5}
$\mathbf{10^3}$	5.8425×10^{-4}	1.6594×10^{-4}	2.3973×10^{-5}	2.5240×10^{-6}	3.8031×10^{-7}	1.1890×10^{-7}
$\mathbf{10^4}$	5.8385×10^{-4}	1.6570×10^{-4}	2.3843×10^{-5}	2.4174×10^{-6}	3.1795×10^{-7}	3.4992×10^{-8}
$\mathbf{10^5}$	5.8382×10^{-4}	1.6569×10^{-4}	2.3840×10^{-5}	2.4162×10^{-6}	3.1756×10^{-7}	3.4815×10^{-8}

Table 2.3: Error for different meshes and number of time steps (NT) in the academic test, when using Simpson quadrature formulas.

Comparison of the different pricing techniques

Next we show other numerical tests to illustrate the performance of the analytical approximation of Section 2.5.1, the finite elements solution of Section 2.5.2 and a standard Monte Carlo simulation technique described in Section 2.5.3. They are mainly intended to illustrate the comparison in terms of computational cost.

In Table 2.4 and Table 2.5 we show the data for the different tests. More precisely, Table 2.4 shows the constant volatilities, correlations and accrual jointly with the constants appearing in the strike definition (2.10) while Table 2.5 shows the forward LIBOR spot values for the different tests.

Concerning to the used standard Monte Carlo simulation technique, the 99% confidence intervals related to 500.000 simulations are determined. The computational time for each price is approximately 26 minutes on a Intel(R) Core(TM)2 Duo CPU T8100 2.10 GHz.

In the finite elements method, the used spatial quadrangular meshes are structured, uniform and with edges parallel to the axis with 4096 elements and 16641 nodes. Pricing a ratchet caplet using finite elements takes about 26 minutes. Note that this method provides simultaneously the prices for the 16641 mesh nodes, prices for any LIBOR values can be easily obtained from them by a standard bilinear interpolation from the prices at mesh nodes.

The analytical approximation method results to be very fast as it gives one ratchet caplet price in about 0.031 seconds, while keeping a good level of precision, especially for not too long maturities.

Finally, comparison of the computed numerical results are shown in Table 2.6, Table 2.7 and Table 2.8 for different spot values of the forward LIBOR rates.

Index frequency	Semi Annual
δ_i	0.5
$\rho_{i-1,i}$	0.8
σ^i	0.2
σ^{i-1}	0.2
a, b, c	0.9, 0.0, 0.01
t	0

Table 2.4: Numerical data (I).

$T0FR$	**At-the-money**	**In-the-money**	**Out-of-the-money**
L_0^{i-1}	0.05	0.03	0.06
L_0^i	0.05	0.05	0.05

Table 2.5: Numerical data (II).

T_{i-2}	T_{i-1}	**Monte Carlo**	**Anal. Appr.**	**Fin. Elem.**
0.5	1.0	[0.0014551, 0.0015599]	0.00150373	0.00151720
1.5	2.0	[0.0021582, 0.0023049]	0.00220159	0.00224088
2.5	3.0	[0.0027734, 0.0029599]	0.00280634	0.00287529
3.5	4.0	[0.0033225, 0.0035454]	0.00334588	0.00344715
4.5	5.0	[0.0038577, 0.0041172]	0.00383673	0.00397233

Table 2.6: Tests results for ratchet caplet i At-the-money ($L_0^{i-1} = 0.05$, $L_0^i = 0.05$).

T_{i-2}	T_{i-1}	**Monte Carlo**	**Anal. Appr.**	**Fin. Elem.**
0.5	1.0	[0.012981, 0.013218]	0.0130957	0.0131012
1.5	2.0	[0.013164, 0.013462]	0.0132759	0.0132989
2.5	3.0	[0.013371, 0.013722]	0.0135010	0.0135448
3.5	4.0	[0.013606, 0.014004]	0.0137462	0.0138108
4.5	5.0	[0.013902, 0.014346]	0.0139994	0.0140839

Table 2.7: Tests results for ratchet caplet i In-the-money ($L_0^{i-1} = 0.03$, $L_0^i = 0.05$).

T_{i-2}	T_{i-1}	**Monte Carlo**	**Anal. Appr.**	**Fin. Elem.**
0.5	1.0	[0.0002542, 0.0002981]	0.0002741	0.0002807
1.5	2.0	[0.0005890, 0.0006765]	0.0006273	0.0006430
2.5	3.0	[0.0009783, 0.0010905]	0.0010120	0.0010458
3.5	4.0	[0.0013795, 0.0015256]	0.0014000	0.0014557
4.5	5.0	[0.0017733, 0.0019517]	0.0017812	0.0018621

Table 2.8: Tests results for ratchet caplet i Out-of-the-money ($L_0^{i-1} = 0.06$, $L_0^i = 0.05$).

Real ratchet caplets and ratchet cap examples

Along this section, first the numerical results obtained when pricing different real ratchet caplets are shown. These results were obtained with the second-order characteristic method combined with $\mathcal{Q}_h^2$ finite elements, that is, $(\mathcal{LG})_2/\mathcal{Q}_h^2$. Also, we have fixed the upper bounds of the computational bounded domain to $3K$.

Notice that we take the notional $M = 10,000$ and the pricing date $t = T_0 = 0$, which corresponds to 10 Oct 2001 and coincides with the strike fixing date in Table 2.9, so that for computing the first ratchet caplet price ($i = 1$), $R_1(0, L_0, L_1)$, the initial condition corresponds to time $T_0 = 0$. Thus, to obtain the price at $t = 0$ it is not necessary to solve the corresponding PDE, but just to compute the interpolated initial condition at the forwards data pairs (L_0^0, L_0^1).

Market data and characteristics of the first ratchet caplet are shown in Table 2.9.

In Table 2.10 the ratchet caplet computed prices for the forwards $L_0^0 = L_0^1 = 0.0323903$ with different meshes are shown.

By using Monte Carlo simulation the price is equal to 0.120237. In Figure 2.5, the first ratchet caplet price at $t = 0$ is shown in terms of the two involved forward LIBOR rates.

Currency	EURO
Index Name	EURIBOR
Strike fixing date	10 Oct 2001
Forward fixing date	10 Apr 2002
Payment date	10 Oct 2002
Index frequency	Semi Annual
Day count	ACT/360
Fixed strike rate	4.8% (0.048)
Volatility	0.242
Correlation	1
Expiry	0.49683
Accrual	0.508333
Payment discount factor (B_0^2)	0.966618

Table 2.9: First ratchet caplet data.

Mesh 2	**Mesh 8**	**Mesh 32**	**Mesh 64**	**Mesh 96**
-4.3269	0.3147	0.1088	0.1190	0.1197

Table 2.10: First ratchet caplet price, $R_1(0, 0.0323903, 0.0323903)$, for different meshes. Monte Carlo pricing is 0.120237.

Next, in Table 2.11 the data for the second ratchet caplet are shown. Notice that we need to solve the correspondig PDE, the final condition of which is posed at time T_1 (corresponding to 10 Apr 2002). In Table 2.12 the computed prices for $(L_0^1, L_0^2) = (0.0323903, 0.0354491)$ with different meshes and numbers of time steps are shown. Monte Carlo simulation provides the second ratchet caplet price equal to

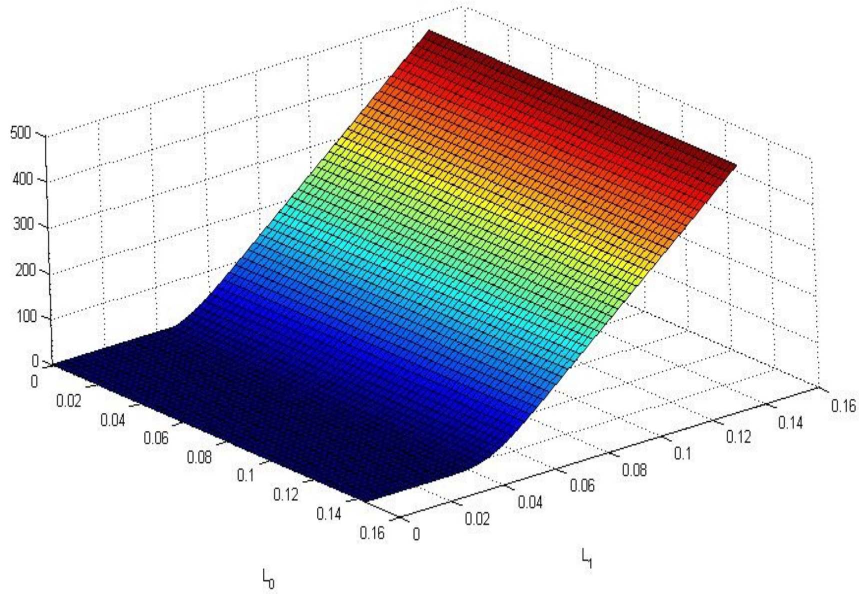

Figure 2.5: First ratchet caplet (Mesh 64, Time steps 1000).

4.98708. Moreover in Figure 2.6 the computed ratchet caplet price function at $t = 0$ is displayed in terms of the corresponding two forward rates.

Finally, the third ratchet caplet data and computed prices (Monte Carlo price is equal to 5.72940) are shown in Tables 2.13 and 2.14, respectively. Figure 2.7 shows the price function at $t = 0$ in terms of L_2 and L_3.

In order to price a real ratchet cap (composed of a set of ratchet caplets) the required market data and the product characteristics are shown in Tables 2.15 and 2.16. Thus, once the different involved ratchet caplets have been computed, the price of the ratchet cap is provided by adding the prices of each ratchet caplet at time $t = 0$ and considering the forwards appearing in column T0FR of Table 2.16 provides the results presented in Table 2.18.

The ratchet cap price computed by Monte Carlo simulation ([55]) is equal to 52.5583 which is very close to the one obtained with the finest mesh and smallest time step. Monte Carlo simulation has been carried out with 200,000 paths and 250 time steps of the Euler-Maruyama scheme.

Currency	EURO
Index name	EURIBOR
Strike fixing date	10 Apr 2002
Forward fixing date	10 Oct 2002
Payment date	10 Apr 2003
Index frequency	Semi Annual
Day count	ACT/360
Volatility	0.228481
Correlation	1
Expiry	1
Accrual	0.505556
Payment discount factor (B_0^3)	0.949599
β,**a**, **b**, **c**	1.0, 0.9, 0.0, 1.0%

Table 2.11: Second ratchet caplet data.

	Mesh 2	**Mesh 8**	**Mesh 32**	**Mesh 64**	**Mesh 96**
10	2.927363	5.057679	4.976746	4.9498720	4.946677
$\mathbf{10^2}$	2.912453	5.047475	4.976309	4.949729	4.946172
$\mathbf{10^3}$	2.910926	5.046458	4.976268	4.949725	4.946161
$\mathbf{10^4}$	2.910773	5.046357	4.976264	4.949724	4.946160
$\mathbf{10^5}$	2.910758	5.046347	4.976263	4.94973	4.946159

Table 2.12: Second ratchet caplet price, $R_2(0, 0.0323903, 0.0354491)$, for different meshes and time steps. Monte Carlo pricing is 4.95703.

Note that the analytical formula cannot be applied for the case of correlations equal to one, thus we present another example taken from [55], in which the market correlations $\rho_{i-1,i}$ are shown in Table 2.19. For this new set of market correlations and the same other data as in previous real case, the computed results with the three different methods are shown in Table 2.20.

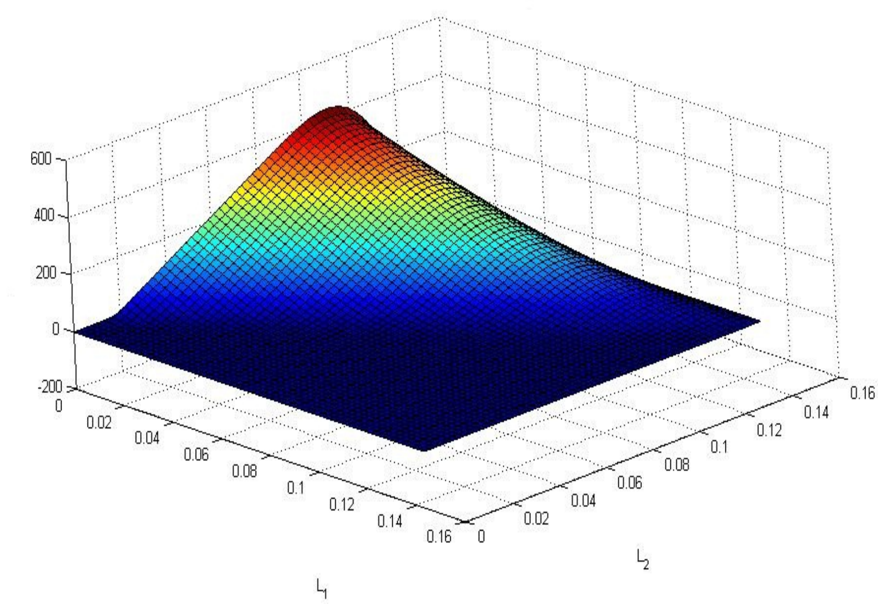

Figure 2.6: Second ratchet caplet (Mesh 64, Time steps 1000).

2.6 Numerical methods for the ratchet caplet with $b \neq 0$

As it has been mentioned in a previous section, in the general case with $b \neq 0$ the spatial dimension of the Cauchy problem associated to each ratchet caplet increases as soon as we move backwards in the intervals between tenor dates. Therefore, the analytical approximation seems a very difficult task for higher dimensions and a possible solution for the use of Crank Nicolson Lagrange Galerkin techniques or other numerical methods for the PDE problems can be provided by the recently developed sparse grid methods, already applied for option pricing in [40, 39, 59, 58], for example. Thus, we just present a classical Monte Carlo simulation strategy for LIBOR Market Model in next section in order to compare the numerical results to those ones obtained for a case with $b = 0$.

Currency	EURO
Index name	EURIBOR
Strike fixing date	10 Oct 2002
Forward fixing date	10 Apr 2003
Payment date	10 Oct 2003
Index frequency	Semi Annual
Day count	ACT/360
Correlation	1
Volatility	0.202308
Expiry	1.49863
Accrual	0.508333
Payment discount factor (B_0^4)	0.931105
β, **a**, **b**, **c**	1.0, 0.9, 0.0, 1.0%

Table 2.13: Third ratchet caplet data.

	Mesh 2	**Mesh 8**	**Mesh 32**	**Mesh 64**	**Mesh 96**
10	9.168823	6.682162	5.636706	5.609913	5.619255
$\mathbf{10^2}$	9.110199	6.683876	5.633042	5.619419	5.620347
$\mathbf{10^3}$	9.104151	6.684039	5.632686	5.619411	5.620376
$\mathbf{10^4}$	9.103545	6.684056	5.632651	5.619411	5.620380
$\mathbf{10^5}$	9.103484	6.684057	5.632647	5.619401	5.620382

Table 2.14: Third ratchet caplet price, $R_3(0, 0.0354491, 0.0390755)$, for different meshes and time steps. Monte Carlo pricing is 5.759119.

2.6.1 Monte Carlo simulation

In this section the particular use of a Monte Carlo simulation technique to approximate the price of the ratchet caplets in the case $b \neq 0$.

For the ratchet caplet j ($j = 1, \ldots, N$) we consider the terminal probability measure, Q^j, associated to the numeraire B^j, so that in this probability measure the stochastic differential equations governing the dynamics of forward LIBOR entering in the j-th ratchet caplet contract are given by equations (2.2).

We assume that we are interested in the price at time $t = 0$ of the ratchet caplet

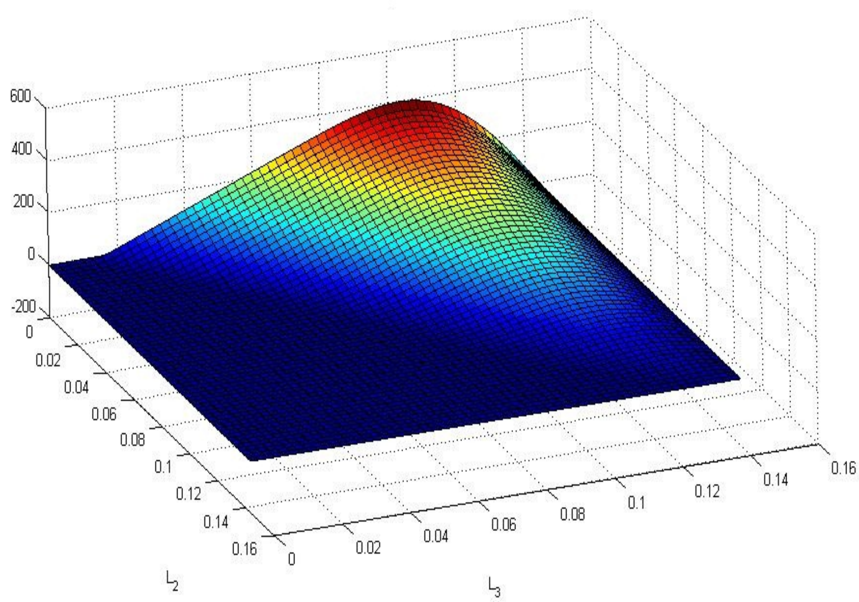

Figure 2.7: Third ratchet caplet (Mesh 64, Time steps 1000).

j which is signed up at time T_0 and pays at time T_j.

As it has been described in the case $b = 0$, we consider the uniform time grid t_n, $n = 0,\ 1,\ 2, \ldots,\ N_t$, such that $t_{N_t} = T_j$ and includes the tenor dates between T_0 and T_j.

As the simulation of LIBOR forward rates is a particular case of simulating the solution of stochastic differential equations, we have several possibilities for the numerical solution. In this work we apply an Euler-Maruyama scheme [38], so that we compute the forward LIBOR rates at each sampling date by using the following discretization scheme for $n = 0,\ 1,\ 2, \ldots, N_t - 1$:

$$\begin{aligned} L^k_{t_{n+1}} &= L^k_{t_n} - \sum_{h=k+1}^{j} \frac{\delta_h \rho_{h,k} \sigma^h(t_n) L^h_{t_n}}{1 + \delta_h L^h_{t_n}} \sigma^k(t_n) L^k_{t_n} \Delta_n + \\ & \quad + \sigma^k(t_n) L^k_{t_n} (W^k(t_{n+1}) - W^k(t_n)), \quad k = 1, \ldots, j-1, \qquad (2.73) \\ L^j_{t_{n+1}} &= L^j_{t_n} + \sigma^j(t_n) L^j_{t_n} (W^j(t_{n+1}) - W^j(t_n)), \qquad (2.74) \end{aligned}$$

starting from the values of $L^k(t_0)$, $k = 1, \ldots, j$, because at the pricing date $t = 0$ the forward LIBOR rates are known for sure.

Currency	EURO
Index name	EURIBOR
Rates-Volatilities	Data from 10 Oct 2001
First strike fixing date	10 Oct 2001
First strike rate	4.8% (0.048)
First forward fixing date	10 Apr 2002
Index frequency	Semi Annual
Tenor	5 Y
Day count	ACT/360
β, **a**, **b**, **c**	1.0, 0.9, 0.0, 1.0%

Table 2.15: Ratchet cap general data

Once we have the samples of the forward rates at the tenor dates we can compute the discount bond (numeraire), B^j, at the tenor dates by using the formula

$$B^j_{T_n} = \prod_{k=n}^{j-1} (1 + \delta_k L_k(T_n))^{-1}. \tag{2.75}$$

Finally, using that the discounted price of a derivative must be a martingale under the terminal measure Q^j, the price of the ratchet caplet j, R_j, at time $t = 0$ is given by

$$\begin{aligned} R_j(0, L^1_0, \ldots, L^{j-1}_0, L^j_0) &= B^j_0 \mathbb{E}^j \left[\frac{R_j(T_j, L^1_{T_0}, \ldots, L^{j-1}_{T_{j-2}}, L^j_{T_{j-1}})}{B^j_{T_j}} \right] \\ &= B^j_0 \mathbb{E}^j \left[M\delta_j(\overline{L}^j - K_j)^+ \right], \end{aligned} \tag{2.76}$$

where $\mathbb{E}^j$ denotes the expectation under the terminal measure Q^j.

The above described Monte Carlo technique has been used to obtain the results that appear in the forthcoming section of numerical results, in which the case with $b \neq 0$ is included.

Fixing date	Payment date	Accrual	Volatility	Expiry	PDF	T0FR
10 Apr 2002	10 Oct 2002	0.508333	0.242	0.49863	0.966618	0.0323903
10 Oct 2002	10 Apr 2003	0.505556	0.228481	1	0.949599	0.0354491
10 Apr 2003	10 Oct 2003	0.508333	0.202308	1.49863	0.931105	0.0390755
10 Oct 2003	10 Apr 2004	0.513889	0.196635	2	0.911389	0.0420952
10 Apr 2004	10 Oct 2004	0.505556	0.191161	2.50685	0.89127	0.0446503
10 Oct 2004	10 Apr 2005	0.505556	0.180336	3.00548	0.87087	0.0463364
10 Apr 2005	10 Oct 2005	0.505556	0.169901	3.50411	0.84998	0.0486138
10 Oct 2005	10 Apr 2006	0.505556	0.169616	4.00274	0.828783	0.0505905
10 Apr 2006	10 Oct 2006	0.508333	0.169385	4.50137	0.807112	0.0528180
10 Oct 2006	10 Apr 2007	0.505556	0.160095	5.00274	0.785758	0.0537565

Table 2.16: Ratchet cap general data (PDF: Payment discount factor, T0FR: Time zero forward rate)

2.6.2 Numerical results for $b \neq 0$

In this section we show an example of pricing for a general ratchet cap allowing b different from zero in the expression (2.10). The numerical technique used here is Monte Carlo simulation described in previous section, by taking $100,000$ paths and 100 time steps for the Euler-Maruyama scheme.

As it has been described in the previous section, when b is different from zero, we need a correlation matrix in order to suitably compute the dynamics of all the involved LIBOR forward rates in each ratchet caplet. So, we cannot use the correlation data appearing in [55] where only the correlations required for the case $b = 0$ are indicated. Therefore, we consider the same data as in Table (2.15) and Table (2.16) jointly with the correlation matrix appearing in page 78.

Expiry date	**Monte Carlo**	**Fin. Elem.**
0.49863	[0.08035, 0.16103]	0.119052
1	[4.46309, 4.99531]	4.949725
1.49863	[5.30064, 5.87787]	5.619411
2	[5.76613, 6.41552]	6.048858
2.50685	[5.63189, 6.28277]	6.234265
3.00548	[4.56902, 5.14389]	4.882059
3.50411	[5.24919, 5.85409]	5.598571
4.00274	[5.79944, 6.47193]	6.214945
4.50137	[6.75106, 7.49735]	7.180383
5.00274	[4.63946, 5.21721]	4.942808

Table 2.17: Tests results for ratchet caplets (discounted prices) with all correlations equal to 1.

	Mesh 2	**Mesh 8**	**Mesh 32**	**Mesh 64**	**Mesh 96**
10	181.328153	62.991668	51.761137	51.792480	51.789511
10^2	179.863369	62.944707	51.927481	51.790246	51.777014
10^3	179.710736	62.938902	51.925654	51.788980	51.779073
10^4	179.695409	62.938329	51.925487	51.788865	51.778974
10^5	179.693877	62.938272	51.925469	51.788855	51.778971

Table 2.18: Ratchet cap price (R) for different meshes and time steps. Monte Carlo pricing is 52.5583.

Correlation	**Value**
$\rho_{1,2}$	0.819825
$\rho_{2,3}$	0.752715
$\rho_{3,4}$	0.939065
$\rho_{4,5}$	0.907290
$\rho_{5,6}$	0.893844
$\rho_{6,7}$	0.975612
$\rho_{7,8}$	0.921198
$\rho_{8,9}$	0.919544
$\rho_{9,10}$	0.956855

Table 2.19: Market correlations.

Expiry date	**Monte Carlo**	**Fin. Elem.**	**Analytical**
0.49863	[0.08035, 0.16103]	0.119056	0.101565
1	[5.70828, 6.64748]	6.17175	6.45604
1.49863	[8.71595, 9.94146]	8.97657	9.42576
2	[7.03822, 8.14347]	7.32275	7.82406
2.50685	[7.99605, 9.23802]	8.76002	8.92893
3.00548	[7.80544, 9.04676]	8.23551	8.92812
3.50411	[6.23903, 7.24893]	6.62653	6.94819
4.00274	[8.95355, 10.32293]	9.51259	9.39194
4.50137	[10.62527, 12.17423]	11.07908	10.9521
5.00274	[6.89349, 8.00426]	7.39245	7.46256

Table 2.20: Results for ratchet caplets with market correlation in Table 2.19.

$$C = \begin{pmatrix}
1.0 & 0.819825 & 0.792337 & 0.885672 & 0.86543 & 0.88654 & 0.73452 & 0.80119 & 0.81562 & 0.79732 \\
0.819825 & 1.0 & 0.752715 & 0.723451 & 0.76567 & 0.89654 & 0.83211 & 0.87654 & 0.82333 & 0.84156 \\
0.792337 & 0.752715 & 1.0 & 0.939065 & 0.91234 & 0.88111 & 0.79676 & 0.86712 & 0.75432 & 0.80155 \\
0.885672 & 0.723451 & 0.939065 & 1.0 & 0.76543 & 0.77722 & 0.71343 & 0.81675 & 0.82334 & 0.91123 \\
0.86543 & 0.76567 & 0.91234 & 0.76543 & 1.0 & 0.80123 & 0.88564 & 0.79876 & 0.80111 & 0.82345 \\
0.88654 & 0.89654 & 0.88111 & 0.77722 & 0.80123 & 1.0 & 0.91234 & 0.89992 & 0.84532 & 0.86667 \\
0.73452 & 0.83211 & 0.79676 & 0.71343 & 0.88564 & 0.91234 & 1.0 & 0.89766 & 0.88723 & 0.80001 \\
0.80119 & 0.87654 & 0.86712 & 0.81675 & 0.79876 & 0.89992 & 0.89766 & 1.0 & 0.83321 & 0.82034 \\
0.81562 & 0.82333 & 0.75432 & 0.82334 & 0.80111 & 0.84532 & 0.88723 & 0.83321 & 1.0 & 0.95555 \\
0.79732 & 0.84156 & 0.80155 & 0.91123 & 0.82345 & 0.86667 & 0.80001 & 0.82034 & 0.95555 & 1.0
\end{pmatrix}$$

Correlation matrix for the real ratchet cap example with $b \neq 0$.

In Table 2.21 the results obtained for different values of b are shown. As expected, a monotone behavior with respect to the parameter b is observed.

In practice, as it is the case of other model parameters, this correlation matrix has to be calibrated from market data and sometimes an additional adjustment to obtain a positive defined matrix is required.

Expiry date	**b**= 0.0	**b**= 0.01	**b**= 0.025	**b**= 0.05	**b**= 0.075	**b**= 0.1
0.49863	0.118320	0.107638	0.116599	0.114724	0.121690	0.126388
1	6.637526	6.018332	5.178980	3.973709	3.041199	2.371222
1.49863	11.018768	10.219395	9.286535	7.931483	6.487005	5.449291
2	8.227067	7.61439346	6.679627	5.394552	4.173885	3.188410
2.50685	13.308241	12.682557	11.502850	9.658426	8.073740	6.678742
3.00548	12.477173	11.659358	10.570663	8.765664	7.2050844	5.865793
3.50411	11.460219	10.448779	9.292035	7.591007	5.890608	4.469814
4.00274	12.919596	12.167242	10.831986	8.727991	7.054128	5.619187
4.50137	18.276307	17.197203	15.734764	13.391551	11.254968	9.231583
5.00274	10.037807	9.285889	8.185133	6.399956	4.803908	3.607201

Table 2.21: Results for ratchet caplets for different values of b and correlation matrix C with Monte Carlo simulation.

Chapter 3

Spread Option

3.1 Introduction

In the previous chapter, mainly the description, mathematical modeling and numerical solution of the ratchet cap pricing problem have been addressed. One of the main features of the enclosed ratchet caplet type contracts is the presence of a payoff that involves a set of forward rates. So, these products are an example of financial derivatives whose payoff depends on multiple underlying LIBOR rates.

In the present chapter, with the same previously developed methodologies, other interest rate derivatives are treated. More precisely, we will focus on the widely extended spread options on forward LIBOR rates, which can be framed into the slightly more general rate based spread options. However, taking into account that they are more popular, we maintain the name of spread option for the chapter title and its sections. For both contracts, a PDE pricing model is posed, the existence and uniqueness of solution is obtained and three alternative numerical methods are proposed. These methods are the analytical approximation, the Crank Nicolson characteristics method for time discretization combined with piecewise quadratic finite elements and the Monte Carlo simulation.

A part of the original numerical results concerning the numerical methods are included in reference [61].

3.2 Financial product

In a spread option contract, the evolution of any two different underlying financial products is observed and at contract maturity date a payoff involving the difference between the prices of these two financial products is paid out. Spread options can be designed with different underlying products: two equities, two bonds, two interest rates, etc.

In the present work, we consider a spread option contract on two different forward LIBOR rates. More precisely, this contract is signed with maturity at time T_2. So, previously at time T_0 the first forward LIBOR rate is fixed to $L^1_{T_0} = \bar{L}^1$ and next at time T_1 the second forward LIBOR rate is fixed to $L^2_{T_1} = \bar{L}^2$. Taking into account this notation, the payoff of the spread option at maturity is

$$M\delta\left(\eta(\bar{L}^2 - \bar{L}^1 - K)\right)^+, \tag{3.1}$$

where M denotes the notional, δ represents the duration of the period when the contract is alive, K is the strike that is fixed at the beginning of the contract, and $\eta = 1$ ($\eta = -1$) corresponds to the call (put) spread option.

In the terminology of [53], a rate based spread option is a more general product that pays a weighed sum of two interest rates, whenever this sum exceeds the fixed strike K in the case of a call, or if sum is below the strike for the case of a put, otherwise the payment is zero. More precisely, for a fixed positive real number a and a negative one c, at time T_2 the buyer of the spread option receives the amount

$$M\delta\left(\eta(a\bar{L}^2 + c\bar{L}^1 - K)\right)^+ \tag{3.2}$$

where a and c denote the weighing factors, usually a is positive and c is negative. Notice that for the choice $a = 1$ and $c = -1$ we recover the spread option as a particular case.

For the sake of simplicity, hereafter we just consider the case of a call option, so that the parameter $\eta = 1$, and we take the notional $M = 1$ and $\delta = 1$.

3.3 PDE mathematical model

In this section, we pose the PDE mathematical model governing a (rate based) call spread option and the main results concerning the existence and uniqueness of solution for the PDE model are presented. For this purpose, let us first introduce some details about LIBOR Market Model specially focused on the pricing a spread option.

As we consider a spread option which depends on two different forward LIBOR rates, let us first consider an appropriate time structure $\mathcal{T} := \{T_0, T_1, T_2\}$ with $0 < T_0 < T_1 < T_2$. Thus, for $t < T_{i-1}$ we consider $(L_t^i)_{t \leq T_{i-1}}$, $i = 1, 2$ as the value process of the $i - th$ forward LIBOR rate.

As we mentioned in the previous chapter, for $i = 1, 2$, the LIBOR Market model assumes that there exists a probability measure Q^i associated to the numeraire B^i, such that the stochastic process (L_t^i) is a martingale and verifies the following stochastic differential equation:

$$dL_t^i \;=\; L_t^i \sigma^i(t)\, d\mathcal{W}_t^i, \tag{3.3}$$

where

- $\mathcal{W} = (\mathcal{W}^1, \mathcal{W}^2)$ denotes a 2-dimensional Brownian motion with covariance matrix ρ (i.e., $d\mathcal{W}_t^i d\mathcal{W}_t^j = \rho_{i,j} dt$). Actually, this matrix is characterized by the correlation coefficient $\rho_{1,2}$ between both LIBOR rates.

- σ^i is the deterministic volatility of the i-th LIBOR forward rate.

- $\delta_i = T_i - T_{i-1}$ denotes the i-th accrual factor.

As argued in the case of ratchet caplets, taking into account that the payoffs (3.1) and (3.2) involve two rates, we need to express the dynamics of the two forward LIBOR rates under the same probability measure. So, by using a change of numeraire technique, we can use Q^2 (i.e, B^2 as numeraire) as common measure to have

$$dL_t^1 \;=\; -L_t^1 \sigma^1(t) \frac{\rho_{1,2}\delta_2 \sigma^2(t) L_t^2}{1 + \delta_2 L_t^2} dt + L_t^1 \sigma^1(t)\, d\mathcal{W}_t^2 \tag{3.4}$$

jointly with the stochastic equation (3.3) with $i = 2$ for L_t^2 dynamics.

Next, by the usual no arbitrage arguments, in the case of the rate based call spread option the relative or discounted price is given by

$$\Pi_t = \mathbb{E}^{Q^2}\left[(a\bar{L}_2 + c\bar{L}_1 - K)^+ \mid \mathcal{F}_t\right], \quad t \leq T_0 \tag{3.5}$$

and the absolute price is equal to $S_t = B_t^2 \Pi_t$.

Remark 3.3.1. *Clearly, for the choice $a = 1$ and $c = -1$ in expression (3.5) the relative price of the call spread option is obtained.*

It is important to notice that since a (rate based) call spread option can be understood as very close to a ratchet caplet with the parameter b equal to zero, we can deduce a very close PDE problem by using the Feynman-Kâc theorem, therefore obtaining a similar result to the one in Theorem 1.4.1. Thus, the proof of the following theorem is also analogous to the one of the Theorem 1.4.1.

For $i = 1, 2$ we denote by L_i the real variable corresponding to the i-th forward LIBOR rate and we set $T_{-1} = 0$ by convention.

Theorem 3.3.2. *Let us assume that the matrix $(\rho_{h,k}\sigma^h(t)\sigma^k(t))_{h,k=1,2}$ is bounded and uniformly positive definite. Then, we have*

$$\Pi_t = u^{2,1}(t, L_t^1, L_t^2; K_1), \quad t \in [0, T_0] \tag{3.6}$$

where $K_1 = K$ is given and the function

$$u^{2,1} = u^{2,1}(t, L_1, L_2; K_1), t \in [0, T_0], L_1, L_2 > 0, K_1 \geq 0.$$

Moreover, we have

$$\Pi_t = u^{2,2}(t, L_t^1, L_t^2; K_2), \quad t \in [T_0, T_1], \tag{3.7}$$

where $K_2 = K_2(\bar{L}^1) = -c\bar{L}^1 + K_1$ and the function

$$u^{2,2} = u^{2,2}(t, L_2; K_2), t \in [T_0, T_1], L_2 > 0, K_2 \geq 0.$$

The functions $u^{2,2}$ and $u^{2,1}$ are uniquely recursively defined as follows

- $u^{2,2}$ *is the unique non-negative solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{2,2}u^{2,2} = 0 \quad in \quad (T_0, T_1) \times \mathbb{R}_+ \\ u^{2,2}(T_1, L_2; K_2) = (aL_2 - K_2(L_1))^+, \quad in \quad \mathbb{R}_+ \end{cases}$$

where $\mathcal{L}^{2,2}$ *is the two-dimensional operator*

$$\mathcal{L}^{2,2} = \frac{(\sigma^2(t)L_2)^2}{2}\partial_{L_2L_2} + \partial_t.$$

- $u^{2,1}$ *is the unique non-negative solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{2,1}u^{2,1} = 0 \quad in \quad (0, T_0) \times \mathbb{R}^2_+ \\ u^{2,1}(T_0, L_1, L_2; K) = u^{2,2}(T_0, L_2; (-cL_1 + K)^+), \quad in \quad \mathbb{R}^2_+ \end{cases}$$

where $\mathcal{L}^{2,1}$ *is the following*

$$\mathcal{L}^{2,1} = \frac{1}{2}\sum_{h,k=1}^{2} \rho_{h,k}\sigma^h(t)\sigma^k(t)L_hL_t\partial_{L_hL_k} - \rho_{1,2}\sigma^1(t)\sigma^2(t)\frac{\delta_2L_2}{1+\delta_2L_2}L_1\partial_{L_1} + \partial_t.$$

3.4 Numerical solution

3.4.1 Finite Elements

In order to state the problem as a IVP in divergencial form, we consider the change of time variable $\tau = T_0 - t$. Then, with a certain abuse of notation by maintaining the same notation for the unknown in the new time variable, we pose the problem for $u^{2,1}$ in the form:

$$\partial_\tau u^{2,1} + \vec{v} \cdot \nabla u^{2,1} - Div(A\nabla u^{2,1}) = 0 \qquad \text{in} \quad (0, T_0) \times \mathbb{R}^2_+ \tag{3.8}$$

$$u^{2,1}(0, L_1, L_2) = u^{2,2}(t = T_0, L_2; (-cL_1 + K)^+) \qquad \text{in} \quad \mathbb{R}^2_+ \tag{3.9}$$

where:

$$A(L_1, L_2) = \begin{pmatrix} \frac{1}{2}(\sigma^1)^2 L_1^2 & \frac{1}{2}\rho_{1,2}\sigma^1\sigma^2 L_1 L_2 \\ \frac{1}{2}\rho_{1,2}\sigma^1\sigma_2 L_1 L_2 & \frac{1}{2}(\sigma^2)^2 L_2^2 \end{pmatrix} \tag{3.10}$$

$$\vec{v}(L_1, L_2) = \begin{pmatrix} \dfrac{\delta_2 L_1 L_2}{1+\delta_2 L_2}\rho_{1,2}\sigma^1\sigma^2 + (\sigma^1)^2 L_1 + \dfrac{1}{2}\rho_{1,2}\sigma^1\sigma^2 L_1 \\ \frac{1}{2}\rho_{1,2}\sigma^1\sigma^2 L_2 + (\sigma^2)^2 L_2 \end{pmatrix} \tag{3.11}$$

Next step in the numerical solution is the localization process. For this purpose, let us consider both L_1^∞ and L_2^∞ large enough real numbers suitably chosen and let the bounded domain $\Omega = (0, L_1^\infty) \times (0, L_2^\infty)$ with Lipschitz boundary Γ, such that $\Gamma = \Gamma_1^+ \bigcup \Gamma_2^+ \bigcup \Gamma_1^- \bigcup \Gamma_2^-$, where $\Gamma_1^- = \Gamma \cap \{L_1 = 0\}$, $\Gamma_2^- = \Gamma \cap \{L_2 = 0\}$, $\Gamma_1^+ = \Gamma \cap \{L_1 = L_1^\infty\}$, $\Gamma_2^+ = \Gamma \cap \{L_2 = L_2^\infty\}$ as shown Figure 2.4. Then, problem (2.41)-(2.42) is replaced by

Find $u^{2,1} : [0, T_0] \times \Omega \to \mathbb{R}$ such that

$$\partial_\tau u^{2,1} + \vec{v} \cdot \nabla u^{2,1} - Div(A \nabla u^{2,1}) = 0 \qquad \text{in } (0, T_0] \times \Omega\,, \tag{3.12}$$

$$u^{2,1}(0, L_1, L_2) = u^{2,2}(t = T_0, L_2; (L_1 + K)^+) \qquad \text{in } \Omega\,, \tag{3.13}$$

where the matrix A and the vector $\vec{v}$ are defined in (3.10) and (3.11), respectively. Moreover, using [47] and taking into account the expression of the matrix A and the vector $\vec{v}$, the boundary condition are only required at Γ_1^+ and Γ_2^+, so that we consider the additional boundary conditions for a call (put) spread option

$$u^{2,1}(\tau, L_1, L_2) = u^{2,2}(t = T_0, L_2; (L_1 + K)^+) \qquad \text{on } [0, T_0] \times \Gamma_1^+, \tag{3.14}$$

$$u^{2,1}(\tau, L_1, L_2) = u^{2,2}(t = T_0, L_2; (L_1 + K)^+) \qquad \text{on } [0, T_0] \times \Gamma_2^+, \tag{3.15}$$

where the appropriate expression is considered in $u^{2,2}$ for the case of a call or a put.

Remark 3.4.1. *Here we note the small improvement of the here proposed choice of the boundary conditions (3.14)-(3.15) with respect to the ones proposed in the recent paper [61], where the boundary conditions for the call are*

$$u^{2,1}(\tau, L_1, L_2) = 0 \qquad \textit{on } [0, T_0] \times \Gamma_1^+ \tag{3.16}$$

$$u^{2,1}(\tau, L_1, L_2) = L_2 \qquad \textit{on } [0, T_0] \times \Gamma_2^+ \tag{3.17}$$

while for the put would be

$$u^{2,1}(\tau, L_1, L_2) = L_1 \qquad on\ [0, T_0] \times \Gamma_1^+ \tag{3.18}$$

$$u^{2,1}(\tau, L_1, L_2) = 0 \qquad on\ [0, T_0] \times \Gamma_2^+ \tag{3.19}$$

which are based on considerations taking into account the corresponding payoff expression. Additionally to the more coherent choice of (3.14)-(3.15), in practice we observed that this choice allows the use of smaller computational domains to get more accurate solutions. More precisely, the domain truncation with values of $4K$ *in the new choice provides the same kind of results that the truncation with values of* $15K$ *in the case of using conditions (3.16)-(3.17).*

As we mentioned at the beginning of this section, we use the characteristics-Crank-Nicolson method for the time discretization jointly with finite elements for spatial discretization. For this purpose, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T.

Now, let us define the subset of finite elements V_h and the space of test functions V_{h,Γ_D}:

$$V_h = \{\varphi_h \in \mathcal{C}^0(\bar{\Omega}) : \varphi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \;\; V_{h,\Gamma_D} = \{\varphi_h \in V_h : \quad \varphi_h = 0, \quad on \quad \Gamma_D\},$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of piecewise continuous functions on $\bar{\Omega}$.

Therefore, if $u_h^{2,1}$ denotes the finite element approximation of $u^{2,1}$, the discretized equation associated to (2.60) is

$$\begin{aligned}\int_\Omega \frac{(u_h^{2,1})^{n+1} - (u_h^{2,1})^n \circ X_e^n}{\Delta t} \psi_h \, dL + \frac{1}{2} \int_\Omega A \nabla (u_h^{2,1})^{n+1} \nabla \psi_h \, dL \\ + \frac{1}{2} \int_\Omega (F_e^n)^{-1} (A \nabla (u_h^{2,1})^n) \circ X_e^n \nabla \psi_h \, dL \\ + \frac{1}{2} \int_\Omega Div (F_e^n)^{-t} (A \nabla (u_h^{2,1})^n) \circ X_e^n \psi_h \, dL &= \\ = \frac{1}{2} \int_{\Gamma_1^- \cup \Gamma_2^-} g^n \psi_h \, dA_L, \qquad \forall \psi_h \in V_{h,\Gamma_D}.\end{aligned} \tag{3.20}$$

3.4.2 Analytical Approximation

The pricing problem of a spread option contract is analogous to the ratchet caplet one with the parameter b equal to zero. Thus, in the interval $[T_0, T_1]$ the discounted spread price Π_t is given by

$$\Pi_t = u^{2,2}(t, L_t^2; K_2(L_t^1)) = aL_t^2\mathcal{N}(d^+(t, L_t^2)) - K_2(L_t^1)\mathcal{N}(d^-(t, L_t^2)), \qquad t \in [T_0, T_1], \tag{3.21}$$

where $\mathcal{N}$ denotes the normal cumulative distribution function and

$$d^{\pm}(t, L_2) = \frac{\log\left(\frac{L_2}{K_2}\right) \pm \frac{1}{2}\left(\bar{\sigma}^2(t, T_1)\right)^2}{\bar{\sigma}^2(t, T_1)},$$

with $\bar{\sigma}^2(t, T_1)$ as in (2.21). Moreover, in the interval $[0, T_0]$ the discounted price satisfies

$$\Pi_t = E^{Q^2}\left[\left(L_{T_1}^2 - (-cL_{T_0}^1 + K)\right)^+ \mid \mathcal{F}_t\right] = u^{2,1}(t, L_t^1, L_t^2),\ t \in [0, T_0], \tag{3.22}$$

where $u^{2,1}$ is the non-negative solution of the Cauchy problem

$$\begin{cases} \mathcal{L}^{2,1}u^{2,1} = 0, & \text{in } (0, T_0) \times \mathbb{R}_+^2, \\ u^{2,1}(T_0, L_1, L_2) = u^{2,2}(T_0, L_2; (-cL_1 + K)^+), & \text{in } \mathbb{R}_+^2, \end{cases} \tag{3.23}$$

with $u^{2,2}$ given by formula (3.21) and

$$\begin{aligned} \mathcal{L}^{2,1} &= \frac{1}{2}\left(\sigma^1(t)L_1\right)^2 \partial_{L_1L_1} + \rho_{1,2}\sigma^1(t)\sigma^2(t)L_1L_2\partial_{L_1L_2} \\ &\quad + \frac{1}{2}\left(\sigma^2(t)L_2\right)^2 \partial_{L_2L_2} - \rho_{1,2}\sigma^1(t)\sigma^2(t)\frac{\delta_2 L_2}{1 + \delta_2 L_2}L_1\partial_{L_1} + \partial_t. \end{aligned} \tag{3.24}$$

Notice that in the rate based spread option the strike only depends on L^1 and the definition of Π_t^2 by means of $u^{2,1}$ is used for the large interval $[0, T_0]$.

Next, by the change of variables

$$\bar{u}^{2,1}(t, x_1, x_2; K) = u^{2,1}(t, e^{x_1}, e^{x_2}; K), \qquad x_1, x_2 \in \mathbb{R}, t < T_0\,,$$

problem 3.23 can be rewritten as follows:

$$\begin{cases} \bar{\mathcal{L}}^{2,1}\bar{u}^{2,1} = 0, & \text{in } (0,T_0)\times\mathbb{R}^2, \\ \bar{u}^{2,1}(T_0,x_1,x_2;K) = u^{2,2}\left(T_0, e^{x_2}; (-ce^{x_1}+K)^+\right), & \text{in } \mathbb{R}^2, \end{cases} \tag{3.25}$$

where

$$\begin{aligned} \bar{\mathcal{L}}^{2,1} =& \frac{(\sigma^1(t))^2}{2}(\partial_{x_1x_1}-\partial_{x_1}) + \frac{(\sigma^2(t))^2}{2}(\partial_{x_2x_2}-\partial_{x_2}) \\ &+ \rho_{1,2}\sigma^1(t)\sigma^2(t)\partial_{x_1x_2} - \rho_{1,2}\sigma^1(t)\sigma^2(t)\frac{\delta_2 e^{x_2}}{1+\delta_2 e^{x_2}}\partial_{x_1} + \partial_t. \end{aligned} \tag{3.26}$$

By the analogous use of the representation formulas (2.19)-(2.26) in the ratchet caplet case with $b=0$, we have

$$\begin{aligned} &\bar{u}^{2,1}(t,x_1,x_2;K) \\ =& \int_{\mathbb{R}^2} \bar{\Gamma}^{2,1}(t,x_1,x_2;T_0,y_1,y_2)\cdot u^{2,2}\left(0, ae^{y_2}; -ce^{y_1}+K\right) dy_2dy_1 \\ =& \int_{\mathbb{R}^2} \bar{\Gamma}^{2,1}(t,x_1,x_2;T_0,y_1,y_2) \\ &\qquad \times \int_{\mathbb{R}} \bar{\Gamma}^{2,2}(T_0,y_2;T_1,\eta_2)\left(ae^{\eta_2}-(-ce^{y_1}+K)\right)^+ d\eta_2dy_2dy_1\,, \end{aligned} \tag{3.27}$$

where $K\geq 0$, $t\in(0,T_0)$ and $\bar{\Gamma}^{2,2}$ is the Gaussian fundamental solution of $\bar{\mathcal{L}}^{2,2}$, whose explicit expression is given in (3.21) and $\bar{\Gamma}^{2,1}$ is the (unknown) fundamental solution of $\bar{\mathcal{L}}^{2,1}$.

In the previous section, the ratchet caplet price was given in terms of the solution $\bar{u}^{2,1}$ to the Cauchy problem (3.25). In order to obtain an analytical approximation of the spread option price, we will used classical theory of fundamental solutions because we have a representation in terms of solutions of a sequence of Cauchy problems.

We first recall the expression of the Gaussian fundamental solution $\bar{\Gamma}^{2,2}$ of the one dimensional heat operator

$$\bar{\mathcal{L}}^{2,2} = \frac{(\sigma^2)^2}{2}(\partial_{y_2y_2}-\partial_{y_2}) + \partial_t,$$

given by (2.20) with $\bar{\sigma}^2(t,T_1) = \sigma^2\sqrt{T_1-t}$ (cf. (2.21)), so that

$$\bar{\Gamma}^{2,2}(T_0,y_2;T_1,\eta_2) = \frac{1}{\sigma^2\sqrt{2\pi\delta_1}}\exp\left[-\frac{1}{2}\left(\frac{2(\eta_2-y_2)+(\sigma^2)^2\delta_1}{2\sigma^2\sqrt{\delta_1}}\right)^2\right], \tag{3.28}$$

for $y_2, \eta_2 \in \mathbb{R}$ and $T_0 < T_1$ ($\delta_1 = T_1 - T_0$).

It is not possible to compute explicitly an expression of $\bar{\Gamma}^{2,1}$ because the operator $\mathcal{L}^{2,1}$ in (3.26) has got non constant coefficients. Thus, we approximate the fundamental solution $\bar{\Gamma}^{2,1}$ by means of the fundamental solution $\widetilde{\Gamma}^{2,1}$ of the constant coefficients operator

$$\begin{aligned}\widetilde{\mathcal{L}}^{2,1} :=& \frac{(\sigma^1)^2}{2}\left(\partial_{x_1x_1} - \partial_{x_1}\right) + \frac{(\sigma^2)^2}{2}\left(\partial_{x_2x_2} - \partial_{x_2}\right)\\ &+ \rho_{1,2}\,\sigma^1\sigma^2\,\partial_{x_1x_2} - \bar{c}_2\,\rho_{1,2}\,\sigma^1\sigma^2\partial_{x_1} + \partial_t,\end{aligned}$$

which is obtained by freezing the variable coefficient $\delta_2 e^{x_2}(1+\delta_2 e^{x_2})^{-1}$ appearing in (3.26) to the value defined by the spot, i.e:

$$\bar{c}_2 = \frac{\delta_2 L_2^0}{1+\delta_2 L_2^0}. \tag{3.29}$$

Now, its fundamental solution $\widetilde{\Gamma}^{2,1}$ is given by

$$\widetilde{\Gamma}^{2,1}(t, x_1, x_2; 0, y_1, y_2) = \frac{\exp(F(t, x_1, x_2; 0, y_1, y_2))}{2\pi\sigma^2\sigma^1(T_0 - t)\sqrt{1-\rho_{1,2}^2}}, \tag{3.30}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $t < 0$, where:

$$\begin{aligned}&F(t, x_1, x_2; 0, y_1, y_2) =\\ =\;& \frac{1}{8\left(1-\rho_{1,2}^2\right)}\Big[(\sigma^1)^2(t-0) + 4(x_1 + x_2 - y_2 - y_1)\\ &+ 8\bar{c}_2\,\rho_{1,2}^2\,(y_1 - x_1) + (\sigma^2)^2(t - T_0)\left(1 + 4(-1+\bar{c}_2)\,\bar{c}_2\,\rho_{1,2}^2\right)\\ &+ \frac{4(x_2-y_2)^2}{(\sigma^1)^2(t-T_0)} + \frac{4(x_1-y_1)^2}{(\sigma^2)^2(t-T_0)}\\ &+ \frac{2\sigma^2(-1+2\bar{c}_2)\left((\sigma^1)^2(t-T_0) + 2(x_2-y_2)\right)\rho_{1,2}}{\sigma^1}\\ &- \frac{4\left((\sigma^1)^2(t-T_0) + 2(x_2-y_2)\right)(x_1-y_1)\rho_{1,2}}{\sigma^2\sigma^1(t-T_0)}\Big].\end{aligned} \tag{3.31}$$

Thus, we get the following analytical approximation formula:

$$\begin{aligned}
&\bar{u}^{2,1}(t, x_1, x_2; K) \\
&= \int_{\mathbb{R}^2} \bar{\Gamma}^{2,1}(t, x_1, x_2; T_0, y_1, y_2) \\
&\times \int_{\mathbb{R}} \bar{\Gamma}^{2,2}(T_0, y_2; T_1, \eta_2) \left(ae^{\eta_2} - (-ce^{y_1} + K)\right)^+ d\eta_2 dy_2 dy_1 \\
&\approx \int_{\mathbb{R}^2} \widetilde{\Gamma}^{2,1}(t, x_1, x_2; T_0, y_1, y_2) \\
&\times \int_{\mathbb{R}} \bar{\Gamma}^{2,2}(T_0, y_2; T_1, \eta_2) \left(ae^{\eta_2} - (-ce^{y_1} + K)\right)^+ d\eta_2 dy_2 dy_1.
\end{aligned}$$

Notice that formula (3.32) involves a triple integral, however two of them can be computed analytically. We defer all the explicit formulas to the Appendix A.

Finally, the approximation of the (rate based) call spread price in the interval $(0, T_0)$ is given by

$$\Pi_t \equiv u^{2,1}(t, L_t^1, L_t^2; K) = \bar{u}^{2,1}(t, \log\, L_t^1, \log\, L_t^2; K).$$

3.5 Numerical results

In this section we show some numerical results of a call spread option pricing problem to illustrate the performance of the proposed numerical methods and PDE model. We compare the computed prices obtained with the PDE model with those ones from the Monte Carlo simulation.

FORTRAN scientific computing language and Mathematica have been chosen for the implementation of the PDE numerical methods and MATLAB for the second one.

3.5.1 Example of spread option pricing

In this section we show the computed prices for a call spread option by means of the finite element methods. The financial data and characteristics of this call spread option are shown in Table 3.1. Particularly, notice that constant volatilities and correlation have been considered.

Concerning the numerical methods, the computational domain $\Omega = (0, 4K) \times (0, 4K)$ has been chosen and the data for the different finite element meshes over Ω are shown in Table 3.2.

First, call spread option computed prices with the different meshes and time steps are presented in Table 3.3. Notice the convergence of the computed numerical results as the mesh becomes finer in time and space.

Monte Carlo simulation provides the price 6.9552×10^{-4} for the call spread option after $100,000$ simulations. The computational times for the PDE are smaller than those ones required by Monte Carlo simulation.

Index frequency	Annual
Fixed strike rate (K)	2.8% (0.028)
Payoff parameters (a, c)	1.0, -1.0
Volatilities (σ_1, σ_2)	(0.2, 0.2)
Correlation (ρ)	1
Accrual (δ_1, δ_2)	(1.0, 1.0)
Fixing date 1st rate (T_1)	1.0
Fixing date 2nd rate (T_2)	2.0
Payment discount factor (B_0^3)	0.84116

Table 3.1: Call spread option data.

	N. Elem	**N. Nodes**
Mesh 2	4	9
Mesh 8	64	289
Mesh 32	1024	4225
Mesh 64	4096	16641

Table 3.2: FEM meshes data.

Figure 3.1 shows the call spread option price function at $t = 0$ on the computational domain in terms of L_0^1 and L_0^2 with the data of Table 3.1 .

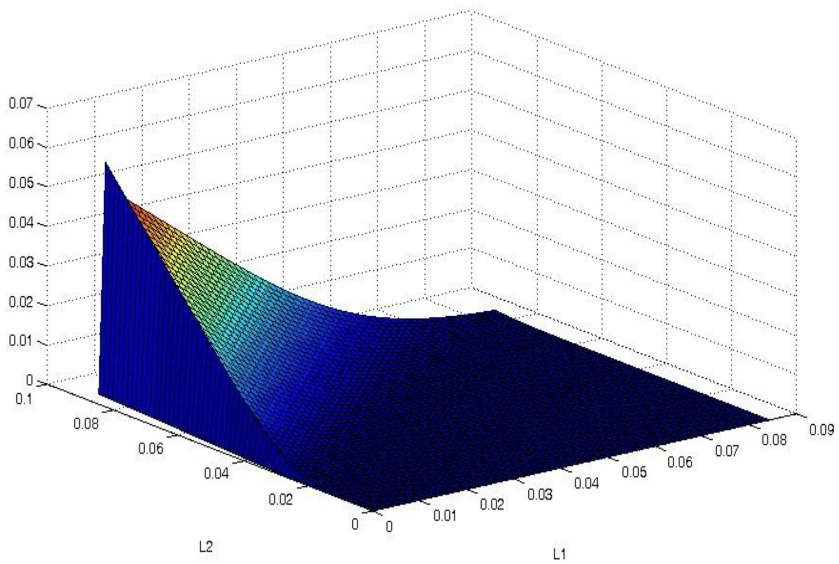

Figure 3.1: Spread option computed prices by the numerical methods applied to the PDE model with Mesh 64 and 100000 time steps.

	Mesh 2	**Mesh 8**	**Mesh 32**	**Mesh 64**
10	6.7847×10^{-4}	6.91406×10^{-4}	6.9391×10^{-4}	6.9402×10^{-4}
10^2	6.8888×10^{-4}	6.9104×10^{-4}	6.9379×10^{-4}	6.9397×10^{-4}
10^3	$6.8996\ \times10^{-4}$	6.9099×10^{-4}	6.9378×10^{-4}	6.9398×10^{-4}
10^4	6.9007×10^{-4}	6.9099×10^{-4}	6.9378×10^{-4}	6.9398×10^{-4}

Table 3.3: Spread option price, $S(0, 0.0822, 0.0822)$, for different meshes and time steps.

3.5.2 Comparison between Monte Carlo, finite elements method and analytical approximations

In this section we present a comparison among the three previously described numerical methods. The financial data of the rate based spread options are given in Table 3.4.

Note that in the Tables appearing in this section different parameters a and K have been considered and that $a = 1$ corresponds to the classical call spread option.

Index frequency	Annual
Payoff parameter (c)	-1.0
Volatilities (σ_1, σ_2)	(0.2, 0.2)
Correlation (ρ)	0.8
Accrual (δ)	1.0
Fixing date 1st rate (T_1)	1.0
Fixing date 2nd rate (T_2)	2.0
Payment discount factor (B_0^3)	0.84116

Table 3.4: Rate based call spread option data.

Concerning the numerical methods, $200,000$ simulations with 250 steps each have been used in Monte Carlo simulation while 1000 time steps and mesh 64 over the domain $\Omega = (4K, 4K)$ have been considered for the finite element method.

K/a	0	0.2449	0.4898	0.7347	1
0.010	7.2002×10^{-2}	5.1955×10^{-2}	3.2037×10^{-2}	1.4783×10^{-2}	4.4174×10^{-3}
0.0144	6.7539×10^{-2}	4.7522×10^{-2}	2.7879×10^{-2}	1.2211×10^{-2}	3.6577×10^{-3}
0.0188	6.3139×10^{-2}	4.3128×10^{-2}	2.3877×10^{-2}	9.7446×10^{-3}	2.7491×10^{-3}
0.0232	5.8739×10^{-2}	3.8750×10^{-2}	2.0134×10^{-2}	7.6698×10^{-3}	2.0474×10^{-3}
0.0280	5.3939×10^{-2}	3.4029×10^{-2}	1.6425×10^{-2}	5.8249×10^{-3}	1.4724×10^{-3}

Table 3.5: Rate based spread option price with finite elements with mesh 64 and 1000 time steps.

K/a	0	0.2449	0.4898	0.7347	1
0.010	7.2000×10^{-2}	5.1869×10^{-2}	3.1836×10^{-2}	1.4339×10^{-2}	4.0031×10^{-3}
0.0144	6.7600×10^{-2}	4.7470×10^{-2}	2.7599×10^{-2}	1.1439×10^{-2}	2.9488×10^{-3}
0.0188	6.3200×10^{-2}	4.3073×10^{-2}	2.3536×10^{-2}	8.9749×10^{-3}	2.1497×10^{-3}
0.0232	5.8801×10^{-2}	3.8689×10^{-2}	1.9731×10^{-2}	6.9356×10^{-3}	1.5537×10^{-3}
0.0280	5.4001×10^{-2}	3.3954×10^{-2}	1.5966×10^{-2}	5.1568×10^{-3}	1.0818×10^{-3}

Table 3.6: Rate based spread option price with the analytical approximation.

K/a	0	0.2449	0.4898	0.7347	1
0.0100	7.2226×10^{-2}	5.2110×10^{-2}	3.2105×10^{-2}	1.4613×10^{-2}	4.1458×10^{-3}
0.0144	6.7754×10^{-2}	4.7717×10^{-2}	2.7891×10^{-2}	1.1637×10^{-2}	3.0406×10^{-3}
0.0188	6.3449×10^{-2}	4.3267×10^{-2}	2.3864×10^{-2}	9.2062×10^{-3}	2.2211×10^{-3}
0.0232	5.9056×10^{-2}	3.8953×10^{-2}	1.9948×10^{-2}	7.0543×10^{-3}	1.6208×10^{-3}
0.0280	5.4171×10^{-2}	3.4262×10^{-2}	1.6168×10^{-2}	5.3146×10^{-3}	1.1153×10^{-3}

Table 3.7: Rate based spread option price by Monte Carlo with $200,000$ simulations and 250 time steps in Euler scheme.

Chapter 4

Stock Loan

4.1 Introduction

A stock loan is a contract between a lender (for example, a bank) and a borrower (for example, a client of the bank). The borrower owns a share of a stock which acts as the collateral of the loan obtained from the lender. At any time before (or at) loan maturity, the borrower may recover the stock by repaying the lender the principal and the fixed interest rate associated to the loan. Otherwise, the borrower can surrender the stock instead of paying the loan. The product is a way of financing in which the stocks are employed as the only guarantee for the loan, being the secured feature one advantage with respect to traditional loans. The stock loan price must be here understood as the fair price the lender should charge to the borrower and this is the target of the pricing problem here addressed. On the other hand, as the borrower has the option to redeem, the question about the optimal redeeming strategy arises.

An important feature from the financial and mathematical point of view are the contract specifications concerning the destination of the accumulative dividends associated to the stock: they can be either gained by the lender or by the borrower and, in both cases, also either before or on redemption. The first attempt of a quantitative analysis to price a stock loan contract appears in [67], where the case in which the dividends of the stock are collected by the lender until redemption and not credited to the borrower. Thus, under this dividend treatment by the contract, the pricing problem is formulated as an American call option with a time dependent strike. Moreover, in [67] the authors deduce a pricing formula when the maturity of the loan is infinite by analogy with the American perpetual option with time varying exercise value. Finally, the paper indicates different interesting open problems, among them there are some of those ones treated in [17]. Thus, in [17] the different PDE based pricing models for the finite maturity case subjected to different possibilities of dividend yield distribution are presented and the mathematical analysis mainly focus on the properties of the redeeming boundary, which is the unknown free boundary that separates the redemption region from the no redemption one, thus characterizing the optimal redemption policy to be followed by the borrower. More precisely, the first three situations analyzed in [17] correspond to the cases of dividend gained by the lender before redemption, reinvested dividend returned to the borrower on redemption and dividend always delivered to the borrower, and all lead to one-dimensional Black-Scholes variational inequalities. From the mathematical point of view, the most complex case arises when the accumulative dividend yield is returned to the borrower on redemption. In this fourth case, the introduction of a path dependent variable allows to pose an obstacle problem associated to an ultraparabolic PDE of Kolmogorov type, as in the case of Asian options with continuous arithmetic averaging. For this case, in [17] the existence of a redeeming boundary and their properties are analyzed.

In this chapter, when the accumulative dividend yield is returned to the borrower on redemption, the mathematical analysis of the PDE model for the stock loan pricing problem is carried out. For this purpose, the techniques developed in [48] and [45] to study obstacle problems associated to hypoelliptic equations of Kolmogorov type are applied. Secondly, as the analytical solution cannot be obtained, we propose a numerical method to approximate the solution. More precisely, first the unbounded domain is truncated to a large enough computational bounded domain with suitable boundary conditions. Next, taking into account that the Kolmogorov equation is strongly convection-dominated, we propose the characteristics method to discretize the material derivative associated to the time derivative and first order spatial derivatives terms. This technique has been previously used in other related financial problems such as in [8] for arithmetic Asian options with American feature, or the ones treated in previous chapters of this thesis. Furthermore, in order to deal with the nonlinearity associated to the obstacle condition (free boundary), the augmented Lagrangian active set method proposed in [37] is used. For Asian options, this method has been compared with an alternative duality method in [9]. For the discretization in the asset and accumulative dividend variables, a piecewise quadratic finite elements method is considered, so that the joint time and spatial discretization falls in the frame of the so called Lagrange-Galerkin methods. In order to validate the performance of the proposed numerical techniques, we verify all qualitative properties theoretically proved in [17] about the redemption region and the optimal redeeming boundary .

The main original results of the present chapter are included in reference [52].

4.2 Financial product and formulation of the pricing model

We assume that the risk neutral price of the stock evolves according to the classical geometric Brownian motion dynamics

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \tag{4.1}$$

where $\{W_t\}_{t\geq 0}$ denotes a standard real Brownian motion on a suitable filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. The parameters r, δ and σ denote the risk-free interest rate, the dividend yield and the volatility respectively. Hereafter we assume that the dividend yield is a positive constant, that is

$$\delta > 0. \tag{4.2}$$

As indicated in the introduction of this chapter, we consider a stock loan financial product in which the borrower receives a loan from the lender who in turn receives the stocks as collateral. The load contract states that the accumulative dividend yield associated to the stock will be returned to the borrower on redemption. Redemption can take place at any time before or on the loan maturity and in case of no redemption, the lender maintains the stocks. The parameters of the stock loan contract are the principal value, K, the continuously compounded interest rate, γ, of the loan and the maturity, T.

Let us assume that the initial date for the loan is $t = 0$. Then, the intrinsic value of the stock loan is given by

$$S_t - K\exp(\gamma t) + \int_0^t \delta\exp(r(t-u))\, S_u\, du, \qquad t \in [0,T]. \tag{4.3}$$

Taking into account certain analogy with options based on continuous averages (Asian option) allowing early exercise (American feature), we introduce the auxiliary path dependent stochastic process

$$I_t = \int_0^t \delta\exp(r(t-u))\, S_u\, du, \qquad t \in [0,T], \tag{4.4}$$

so that expression (4.3) can be easily written as

$$S_t - K\exp(\gamma t) + I_t, \quad t \in [0,T]. \tag{4.5}$$

In order to obtain the stochastic differential equation for the process I_t, we can consider the more general case

$$I_t = g(t)\int_0^t f(u, S_u)du,$$

where dI_t is given by

$$dI_t = \left(g(t)f(t,S_t) + \frac{g'(t)I_t}{g(t)} \right) dt.$$

Thus, for the particular case $g(t) = \delta\exp(rt)$ and $f(t,S) = \exp(-rt)S$, that corresponds to (4.4), we get

$$dI_t = (\delta S_t + rI_t)dt. \tag{4.6}$$

By using classical arguments, such us $It\hat{o}'s$ lemma and dynamic hedging methodology (see [50], for example), the unique price of the stock loan contract which avoids the introduction of arbitrage opportunities is given by the process $v_t = V(t, S_t, I_t)$, where V is the solution of the following free boundary problem:

$$\begin{cases} \min\{-\mathcal{L}[V], V - \Psi\} = 0 \\ V(T, S, I) = \Psi(T, S, I), \quad (t, S, I) \in [0, T) \times \mathbb{R}^2_+. \end{cases} \tag{4.7}$$

In (4.7)

$$\mathcal{L} = \frac{\sigma^2 S^2}{2}\partial_{SS} + (r - \delta)S\partial_S + (\delta S + rI)\partial_I + \partial_t - r \tag{4.8}$$

is the Kolmogorov operator related to processes (S_t, I_t) defined in (4.1) and (4.6), and

$$\Psi(t, S, I) = (S - K\exp(\gamma t) + I)^+, \quad t \in [0, T], \quad S > 0. \tag{4.9}$$

is the payoff/obstacle function.

The proof of the existence of solutions to (4.7) is a delicate matter. In fact, on the one hand, it is well-known that obstacle problems do not generally admit classical (smooth) solutions. On the other hand, $\mathcal{L}$ is not a uniformly parabolic operator and the classical PDE theory of generalized solutions does not apply to problem (4.7). We emphasize that, differently from the standard Black-Scholes case, (4.7) is a two-dimensional time-dependent problem and does not admit dimension reduction. Indeed, the solution V is a function of the time variable t and the spatial variables S and I. Operator $\mathcal{L}$ *is not uniformly parabolic* because only the first order derivative with respect to I appears; in other terms, we have two spatial variables but only one source of diffusion, i.e. one Brownian motion.

We mention that operators similar to $\mathcal{L}$ were recently studied in [26], [48] and [45] in connection with the analysis of American Asian options. Taking into account certain analogies between stock loans and Asian options, in Section 4.3. we give some result on the existence and optimal regularity of solutions. Moreover, we prove the following stochastic representation formula for the solution to (4.7):

$$v_t = \sup_{u \in \mathcal{T}_{[t,T]}} \mathbb{E}\left[\exp(-r(u-t))\max\left(S_u - K\exp(\gamma u) + I_u, 0\right) \mid S_t = S,\ I_t = I\right], \tag{4.10}$$

where $\mathcal{T}_{[t,T]}$ denotes the set of optimal $(\mathcal{F}_t)$-stopping times and $\mathbb{E}$ represents the expected value under the risk neutral measure.

The first equation in (4.7) can be equivalently decomposed in the form

$$\begin{cases} \mathcal{L}[V] \leq 0 & \text{in} \quad (0,T) \times \mathbb{R}_+^2, \\ V \geq \Psi & \text{in} \quad (0,T) \times \mathbb{R}_+^2, \\ \mathcal{L}[V] \cdot (V - \Psi) = 0 & \text{in} \quad (0,T) \times \mathbb{R}_+^2, \end{cases} \tag{4.11}$$

where the third equation is known as a *complementarity condition* of the also called *complementarity problem* (4.7).

As in most financial contracts with early exercise opportunity, associated to problem (4.7) we can distinguish the redemption region

$$R_0 = \{(t,S,I) \in [0,T) \times \mathbb{R}_+^2 / V(t,S,I) = \Psi(t,S,I)\}, \tag{4.12}$$

the no-redemption region

$$R_+ = \{(t,S,I) \in [0,T) \times \mathbb{R}_+^2 / V(t,S,I) > \Psi(t,S,I)\} \tag{4.13}$$

and the optimal redeeming boundary that separates both regions

$$\Sigma = \overline{R}_0 \cap \overline{R}_+. \tag{4.14}$$

For each time $t \in [0,T)$, the notation $R_0(t)$, $R_+(t)$ and $\Sigma(t)$ identifies in the SI-plane the set of points located at time t in the redemption region, no redemption region and optimal redeeming boundary, respectively. Also, the optimal redeeming boundary identifies the critical price of the stock at which it is worth redeeming the loan.

By assuming the existence of solution and exploiting the regularity of the solution V, in [17] some qualitative properties of the optimal redeeming boundary have been proved. Specifically, early redemption never happens for $r > \gamma$, while when $r = \gamma$ it is optimal to hold the loan before maturity, even if in some occasion early redemption may be optimal as well. Furthermore, for $r < \gamma$ the redemption region is non-empty. These results are summarized in the following proposition proved in [17]:

Proposition 4.2.1. *It we assume that* $\delta > 0$ *then*

1. *If* $r > \gamma$ *then* $R_0 = \emptyset$.

2. *If* $r = \gamma$ *then it is optimal to hold the loan before maturity.*

3. *If* $r < \gamma$ *then*

$$\{(t, S, I)/I \geq K \exp(\gamma t)\} \subset R_0 \tag{4.15}$$

and the optimal redeeming boundary can be parameterized by the curve S^* *in the* tI*-plane in the form:*

$$S^* : [0, T) \times (0, K \exp(\gamma t)) \to (0, +\infty)$$

such that

$$R_0 = \{(t, S, I)/S \geq S^*(t, I)\}$$

and S^* *is monotonically decreasing in* t *and* I*, with*

$$\lim_{t \to T} S^*(t, I) = \exp(\gamma T) K - I. \tag{4.16}$$

The above result is based on the existence and other properties of the solution V, which we examine in detail in the forthcoming sections. In particular, we put some emphasis on the regularity properties of generalized solutions because those properties also give some hint for the efficient numerical solution of (4.7): specifically, we will show that the numerical schemes can take advantage of the degenerate structure of $\mathcal{L}$ as a strongly convection-dominated operator.

4.3 Mathematical analysis

In this section we prove the existence of solutions to problem (4.7) in the anisotropic Sobolev spaces $\mathcal{S}^p$ defined in (4.18) as well as some regularity results. It is interesting to notice that the regularity in $\mathcal{S}^p$ is optimal for this kind of problems and gives a clear picture of the peculiar properties of the solution. Other notions of generalized solutions (for instance, in the viscosity or variational sense) can be considered as well, albeit the stochastic representation (4.10) entails uniqueness among different solutions.

4.3.1 Existence and uniqueness of solution

In order to study the existence and uniqueness of solution we first introduce a suitable functional setting. So, we denote by

$$Y = (r-\delta)S\partial_S + (rI + \delta S)\partial_I + \partial_t \tag{4.17}$$

the first order part of $\mathcal{L}$.

For any domain $\Omega \subset \mathbb{R}^3$ and $p \geq 1$, we define the anisotropic Sobolev spaces

$$\mathcal{S}^p(\Omega) = \{U \in L^p(\Omega) \mid \partial_S U,\, \partial_{SS}U,\, YU \in L^p(\Omega)\} \tag{4.18}$$

endowed with the semi-norm

$$\|U\|_{\mathcal{S}^p} = \|U\|_{L^p} + \|\partial_S U\|_{L^p} + \|\partial_{SS}U\|_{L^p} + \|YU\|_{L^p}.$$

If $U \in \mathcal{S}^p(H)$ for any compact subset $H \subseteq \Omega$, then we write $U \in \mathcal{S}^p_{\text{loc}}(\Omega)$. Next, we introduce the notion of *strong solution* of the free boundary problem (4.7).

Definition 4.3.1 (Strong solution)**.** *A strong solution to problem (4.7) is a function* $V \in \mathcal{S}^1_{loc} \cap C((0,T] \times \mathbb{R}^2_+)$ *which satisfies the differential inequality a.e. in* $(0,T) \times \mathbb{R}^2_+$ *and the final condition in the pointwise sense.*

Although the goal of this section is the proof of the existence of a strong solution to problem (4.7), in the sense of Definition 4.3.1, as an intermediate result we first construct a supersolution.

Definition 4.3.2. *A function* $\bar{V} \in C^2([0,T) \times \mathbb{R}^2_+) \cap C([0,T] \times \mathbb{R}^2_+)$ *such that*

$$\mathcal{L}\bar{V} \leq 0 \quad \textit{and} \quad \bar{V} \geq \Psi \quad \textit{in } (0,T) \times \mathbb{R}^2_+, \tag{4.19}$$

is called a supersolution to problem (4.7).

As the following lemma shows, it is not difficult to give the explicit expression of a supersolution to (4.7) with Ψ as in (4.9).

Lemma 4.3.1. *For any β and q suitably large constants, the function*

$$\bar{V}(t,S,I) = \beta e^{-qt}\sqrt{S^2+I^2} \tag{4.20}$$

is a super-solution to problem (4.7).

Proof. We have

$$\mathcal{L}\bar{V}(t,S,I) = \frac{\beta e^{-qt}}{2\left(I^2+S^2\right)^{3/2}} W(S,I)$$

where

$$W(S,I) = -2\left(S^2+I^2\right)\left(q\left(S^2+I^2\right)+\delta S\left(S-I\right)\right)+\sigma^2 S^2 I^2.$$

Therefore $\mathcal{L}\bar{V} \leq 0$ if and only if $W(S,I) \leq 0$. By using repeatedly the elementary inequality

$$SI \leq \frac{S^2+I^2}{2},$$

we have

$$\begin{aligned} W(S,I) &\leq \frac{S^2+I^2}{2}\left(\left(-4q+\sigma^2\right)\left(S^2+I^2\right)-4\delta S\left(S-I\right)\right) \\ &\leq \frac{S^2+I^2}{2}\left(S^2\left(-4q-2\delta+\sigma^2\right)+I^2\left(-4q+2\delta+\sigma^2\right)\right). \end{aligned}$$

Thus $W(S,I) \leq 0$ if q is positive and suitably large. Once q is fixed, it is clear that there exists $\beta > 0$ such that

$$\bar{V}(t,S,I) \geq \Psi(t,S,I), \qquad (t,S,I) \in (0,T)\times\mathbb{R}^2_+,$$

and therefore $\bar{V}$ is a supersolution. □

Now we prove the main result of this section. Generally speaking, we study problem (4.7) in the framework of hypoelliptic equations of Kolmogorov type. The obstacle problem for a general class of degenerate parabolic operators including (4.8) was first studied for the free boundary problem for arithmetic Asian options in [45].

As it appears in next theorem, we previously introduce the concept of function with polynomial growth.

Definition 4.3.3. *A function $f : [0,T] \times \mathbb{R}_+^2 \to \mathbb{R}$ has polynomial growth if*

$$|f(t,S,I)| \le C\,(1 + S^p + I^p)\,, \qquad (t,S,I) \in [0,T] \times \mathbb{R}_+^2,$$

for some positive constants C, p.

Theorem 4.3.2. *There exists a strong solution V of problem* (4.7) *with Ψ as in* (4.9)*: we have that $V \in \mathcal{S}^p_{loc}\left((0,T] \times \mathbb{R}_+^2\right)$ for any $p \ge 1$ and*

$$V \le \bar{V} \tag{4.21}$$

where $\bar{V}$ is the supersolution in (4.20)*. Moreover, V is the unique solution with polynomial growth of problem* (4.7)*, which solves the optimal stopping problem* (4.10)*.*

Proof. Following [45], let $D_\rho(x_1,x_2)$ denote the Euclidean ball centered at $(x_1,x_2) \in \mathbb{R}^2$, with radius ρ. We consider the sequence of domains $O_n = D_n\left(n + \frac{1}{n}, 0\right) \cap D_n\left(0, n + \frac{1}{n}\right)$ covering $\mathbb{R}_+^2$. For any $n \in \mathbb{N}$, the cylinder $H_n = (0,T) \times O_n$ is a $\mathcal{L}$-*regular domain* in the sense that the Cauchy-Dirichlet problem for $\mathcal{L}$ is well-posed because it is possible to find a barrier function (cf. Remark 3.1 in [26]) at any point of the "parabolic" boundary

$$\partial_P H_n := \partial H_n \setminus (\{0\} \times O_n).$$

In particular, since $\mathcal{L}$ satisfies condition (4.25) on any H_n, then by Theorem 3.1 in [26], for any $n \in \mathbb{N}$, problem

$$\begin{cases} \max\{\mathcal{L}U - f, \Psi - U\} = 0 & \text{in } H_n, \\ U|_{\partial_P H_n} = \Psi \end{cases} \tag{4.22}$$

has a strong solution $U \in \mathcal{S}^p_{\text{loc}}(H_n) \cap C(H_n \cup \partial_P H_n)$. Moreover, for every $p \ge 1$ and $H \subset\subset H_n$ there exists a positive constant C, only depending on $H, H_n, p, \|\Psi\|_{L^\infty(H_n)}$ such that

$$\|U_n\|_{\mathcal{S}^p(H)} \le C. \tag{4.23}$$

Next we consider a sequence of cut-off functions $\chi_n \in C_0^\infty(\mathbb{R}_+^2)$, such that $\chi_n = 1$ on O_{n-1}, $\chi_n = 0$ on $\mathbb{R}_+^2 \setminus O_n$ and $0 \leq \chi_n \leq 1$. We set

$$\Psi_n(t,S,I) = \chi_n(S,I)\Psi(t,S,I) + (1-\chi_n(S,I))\bar{V}(t,S,I)\,,$$

where $\bar{V}$ is the supersolution in (4.20), and we denote by V_n the strong solution to (4.22) with $\Psi = \Psi_n$. By the comparison principle we have $\Psi \leq V_{n+1} \leq V_n \leq \bar{V}$. Therefore, by (4.23), for every compact set H and $n \in \mathbb{N}$ such that $H_n \supset H$ we have

$$\|V_n\|_{\mathcal{S}^p(H)} \leq C, \qquad p \geq 1,$$

for some constant C depending on H and p but not on n. Then we can pass to the limit as $n \to \infty$, on compact subsets of $(0,T) \times \mathbb{R}_+^2$, to get a strong solution of $\max\{\mathcal{L}V - f, \Psi - V\} = 0$ in the space $\mathcal{S}^p_{\text{loc}}$. A standard argument based on barrier functions shows that $V(t,\cdot)$ is continuous up to $t = T$ and attains the final datum. Finally, the uniqueness and the Feynman-Kac representation of strong solutions is a consequence of the local summability properties of the transition density of the process (cf. [45], Theorem 1-ii)) and it can be proved as in [48], Theorem 4.3. □

4.3.2 Anisotropic regularity of solutions

In this section we analyze the regularity properties of V and, in particular, the anisotropic Hölder continuity of V is compared with the classical Euclidean regularity.

For greater convenience, we put $x = (S,I)$ and, using the matrix notation, we rewrite the vector field Y in (4.17) as

$$Y = \langle Bx, \nabla_x \rangle + \partial_t$$

where B is the convection matrix

$$B = \begin{pmatrix} r-\delta & 0 \\ \delta & r \end{pmatrix}$$

and ∇_x is the gradient in the variables x. It is possible to introduce a functional setting, induced by the convection field Y, which is natural for the study of the interior regularity of strong solutions. Let us first consider an operator in $\mathbb{R}^3$ of the form

$$\bar{\mathcal{L}} = \bar{a}(t,x)\partial_{x_1x_1} + Y, \tag{4.24}$$

with Y as in (4.17). It is known (cf. [25], Theorem 1.4) that under the assumption (4.2) (i.e. $\delta > 0$) and if the coefficient $\bar{a}$ is a smooth function such that

$$\frac{1}{\mu} \leq \bar{a} \leq \mu \qquad \text{on } \mathbb{R}^3, \tag{4.25}$$

where μ is a positive constant, then $\bar{\mathcal{L}}$ has a fundamental solution which can be globally estimated by Gaussian functions from above and below. Moreover, if the coefficient $\bar{a}$ in (4.24) is constant, then operator $\bar{\mathcal{L}}$ is invariant[1] w.r.t. the left translations in the group law

$$(\tau,\xi) * (t,x) = (\tau + t, x + e^{tB}\xi), \tag{4.26}$$

where the exponential matrix of B is equal to

$$\exp(tB) = \exp(t(r-\delta)) \begin{pmatrix} 1 & 0 \\ \exp(t\delta) - 1 & \exp(t\delta) \end{pmatrix}.$$

Since the function $\bar{a}(t,S,I) = \frac{\sigma^2 S^2}{2}$ verifies the non-degeneracy condition (4.25) on any compact subset of $\mathbb{R} \times \mathbb{R}^2_+$, then the pricing operator $\mathcal{L}$ is *locally* of the form (4.24). Consequently, it is natural to characterize the interior regularity of solutions to $\mathcal{L}$ in terms of the group law (4.26). Indeed, the following embedding theorem holds (cf. [26]).

[1] $\bar{\mathcal{L}}$ is left-$*$-invariant if

$$\bar{\mathcal{L}}U\left((\tau,\xi) * (t,x)\right) = \left(\bar{\mathcal{L}}U\right)\left((\tau,\xi) * (t,x)\right).$$

Theorem 4.3.3 (Embedding theorem)**.** *Let O, Ω be bounded domains of $\mathbb{R}^3$ such that $O \subset\subset \Omega$ and $p > 6$. There exists a positive constant c, only dependent on B, Ω, O and p, such that*

$$\|U\|_{C_B^{1,\alpha}(O)} \le c\|U\|_{\mathcal{S}^p(\Omega)}, \qquad \alpha = 1 - \frac{6}{p}, \tag{4.27}$$

for any $u \in \mathcal{S}^p(\Omega)$. In (4.27), $C_B^{1,\alpha}$ is the anisotropic Hölder space defined by the following norms[2]*:*

$$\|U\|_{C_B^{0,\alpha}(\Omega)} = \sup_{\Omega} |U| + \sup_{\substack{(t,x),(\tau,\xi)\in\Omega \\ (t,x)\neq(\tau,\xi)}} \frac{|U(t,x) - U(\tau,\xi)|}{\|(\tau,\xi)^{-1} * (t,x)\|_B^{\alpha}},$$

$$\begin{aligned}\|U\|_{C_B^{1,\alpha}(\Omega)} = {} & \|U\|_{C_B^{0,\alpha}(\Omega)} + \|\partial_{x_1} U\|_{C_B^{0,\alpha}(\Omega)} \\ & + \sup_{\substack{(t,x),(\tau,\xi)\in\Omega \\ (t,x)\neq(\tau,\xi)}} \frac{|U(t,x) - U(\tau,\xi) - (x_1 - \xi_1)\partial_{x_1} U(\tau,\xi)|}{\|(\tau,\xi)^{-1} * (t,x)\|_B^{1+\alpha}},\end{aligned}$$

where $\|\cdot\|_B$ is the anisotropic norm in $\mathbb{R}^3$ defined by

$$\|(t, x_1, x_2)\|_B = |t|^{\frac{1}{2}} + |x_1| + |x_2|^{\frac{1}{3}}.$$

As a consequence of Theorem 4.3.3, the strong solutions to problem (4.7) belong locally to the space $C_B^{1,\alpha}$ for any $\alpha < 1$. Actually, according to the recent results in [27], the solutions to (4.7) belong to the class $\mathcal{S}^{\infty}_{\text{loc}}$ and this regularity is optimal.

Now we briefly compare the intrinsic notion of $C_B^{1,\alpha}$-regularity with the more familiar regularity in the standard Euclidean sense. First notice that, for any *bounded* domain Ω, there exists a positive constant c_Ω such that

$$\begin{aligned}\|(\tau,\xi)^{-1} * (t,x)\|_B &= \|(t-\tau, x-\xi) + (0, (\mathrm{Id}_2 - e^{(t-\tau)B})\xi)\|_B \\ &\le c_\Omega |(t-\tau, x-\xi)|^{\frac{1}{3}}, \qquad (\tau,\xi),(t,x) \in \Omega,\end{aligned}$$

where Id_2 is the identity matrix in $\mathbb{R}^2$. It immediately follows that

$$C_B^{0,\alpha}(\Omega) \subseteq C^{0,\frac{\alpha}{3}}(\Omega)$$

where $C^{0,\alpha}$ denotes the standard Euclidean Hölder space.

[2] *We adopt the notation $x = (S, I)$ and $\xi = (S', I')$.*

Remark 4.3.4 (Euclidean regularity). *If* $U \in C_B^{1,\alpha}(\Omega)$ *then* $U, \partial_S U \in C^{0,\frac{\alpha}{3}}(\Omega)$ *and also*

$$|U((t,x) * (\tau,0)) - U(t,x)| = |U((t+\tau, e^{\tau B}x)) - U(t,x)| \leq c_\Omega |\tau|^{\frac{1+\alpha}{2}}. \qquad (4.28)$$

Estimate (4.28) *is equivalent to the Hölder regularity of order* $\frac{1+\alpha}{2}$ *along the integral curves of* Y*. As a matter of fact, if we identify* Y *with the vector field* $Y(t,x) = (1, Bx)$*, then* $\gamma(\tau) := (t+\tau, e^{\tau B}x)$ *is the integral curve of* Y *starting from* (t,x)*, that is the solution of the problem*

$$\begin{cases} \dot{\gamma}(\tau) = Y(\gamma(\tau)), \\ \gamma(0) = (t,x). \end{cases}$$

Notice that the $C_B^{1,\alpha}$*-regularity of* U *does not imply the existence of the Euclidean derivative* $\partial_I U$*: roughly speaking, since* ∂_I *is obtained by commuting* ∂_S *and* Y

$$[\partial_S, Y] = \partial_S Y - Y \partial_S = (r-\delta)\partial_S + \delta \partial_I,$$

then intrinsically it has to be considered a third order derivative.

Keeping in mind the above remarks, in the numerical solution of problem (4.7) we adopt the natural approach of using a semi-Lagrangian method for time discretization, that mainly consists of a finite differences scheme along the integral curves of the convective part Y of the equation.

4.4 Numerical methods

In order to enumerate the numerical techniques, the main difficulties and the way to overcome them numerically are briefly outlined. First, a localization technique is used to cope with the initial formulation in an unbounded domain. Also, as the diffusive term is strongly degenerated, the PDE can be understood as an example of extreme convective dominated case, so that we propose a Crank-Nicolson characteristics time discretization scheme combined with a piecewise quadratic Lagrange finite element method. For the inequality constraints associated to the early redemption opportunity, we propose a mixed formulation and the use of an augmented Lagrangian active set technique.

4.4.1 Divergence form and localization in a bounded domain

Taking into account that we apply finite elements methods based on variational formulation, we first rewrite the PDE in (4.7) in divergence form. For simplicity, we introduce the new time variable $\tau = T - t$ and pose the equivalent problem:

$$\overline{\mathcal{L}}[V] \geq 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^2_+\,, \tag{4.29}$$

$$V \geq \overline{\Lambda} \quad \text{in} \quad (0,T) \times \mathbb{R}^2_+\,, \tag{4.30}$$

$$\overline{\mathcal{L}}[V] \cdot (V - \overline{\Lambda}) = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^2_+\,, \tag{4.31}$$

$$V(0,S,I) = \overline{\Lambda}(0,S,I) \quad \text{in} \quad \mathbb{R}^2_+\,, \tag{4.32}$$

where the new operator and obstacle are respectively given by

$$\overline{\mathcal{L}}[V] = \partial_\tau V + \vec{v} \cdot \nabla V - Div(A \nabla V) + rV, \tag{4.33}$$

$$\overline{\Lambda}(\tau, S, I) = \Psi(T - \tau, S, I), \tag{4.34}$$

with

$$A(S,I) = \begin{pmatrix} \frac{1}{2}\sigma^2 S^2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.35}$$

$$\vec{v}(S,I) = \begin{pmatrix} (\sigma^2 - r + \delta)S \\ -(\delta S + rI) \end{pmatrix}. \tag{4.36}$$

As in most problems arising in finance, the numerical solution with finite differences, finite volumes or finite elements requires the approximation of the original problem in an unbounded domain by another one posed in a bounded computational domain. This technique is known as localization procedure, that has to be performed so that the truncation by the bounded domain and the associated boundary conditions do not affect the solution in the region of financial interest. For the classical problem of European vanilla options and Dirichlet boundary conditions, a rigorous analysis has been carried out in [36]. In general, the required boundary conditions at the new boundaries of the bounded domain are obtained with financial and/or mathematical arguments.

For the localization purpose, let us consider both S^∞ and I^∞ large enough real numbers suitably chosen and let the bounded domain be $\Omega = (0, S^\infty) \times (0, I^\infty)$, with Lipschitz boundary Γ, such that $\Gamma = \Gamma_1^+ \bigcup \Gamma_2^+ \bigcup \Gamma_1^- \bigcup \Gamma_2^-$, where $\Gamma_1^- = \Gamma \cap \{S = 0\}$, $\Gamma_2^- = \Gamma \cap \{I = 0\}$, $\Gamma_1^+ = \Gamma \cap \{S = S^\infty\}$, $\Gamma_2^+ = \Gamma \cap \{I = I^\infty\}$.

Then, problem (4.29)-(4.32) is replaced by the following one:

Find $V : [0, T] \times \Omega \to \mathbb{R}$, such that

$$\overline{\mathcal{L}}[V] \geq 0 \qquad \text{in } (0, T] \times \Omega\,, \tag{4.37}$$

$$V \geq \overline{\Lambda} \qquad \text{in } \ (0, T) \times \Omega\,, \tag{4.38}$$

$$\overline{\mathcal{L}}[V] \cdot (V - \overline{\Lambda}) = 0 \qquad \text{in } \ (0, T) \times \Omega\,, \tag{4.39}$$

$$V(0, S, I) = \overline{\Lambda}(0, S, I) \qquad \text{in } \ \Omega\,, \tag{4.40}$$

We note that in a certain abuse of notation, we maintain the use of V also for the solution in the new time variable.

Next, by applying the theory of second order partial differential equations with nonnegative characteristics that can be found in [47] and taking into account the expression of the matrix A and the vector $\vec{v}$, only boundary conditions at Γ_1^+ and Γ_2^+ are required.

More precisely, following the ideas in [47], for simplicity let us introduce the notation

$$x_1 = S, \quad x_2 = I. \tag{4.41}$$

Then, the operator associated to the Cauchy problem (4.7) can be written in the form:

$$\mathcal{L}^* = \sum_{i,j=1}^{2} a_{ij}^* \frac{\partial^2}{\partial x_i x_j} + \sum_{j=1}^{2} b_j^* \frac{\partial}{\partial x_j} + l^* + \frac{\partial}{\partial t}, \tag{4.42}$$

where the involved data are defined as follows

$$A^*(x_1, x_2) = (a^*_{ij}) = \begin{pmatrix} \dfrac{\sigma^2 x_1^2}{2} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.43}$$

$$v^*(x_1, x_2) = (b^*_j) = \begin{pmatrix} (r-\delta)x_1 \\ \delta x_1 + r x_2 \end{pmatrix}, \tag{4.44}$$

$$l^*(x_1, x_2) = -r. \tag{4.45}$$

Thus, in terms of the inwards normal vector to the boundary of Ω, $\vec{m} = (m_1, m_2)$, we introduce the following subsets of Γ:

$$\Sigma^1 = \{(x_1, x_2) \in \Gamma \quad , \sum_{i,j=1}^{2} a^*_{ij} m_i m_j > 0\}, \tag{4.46}$$

$$\Sigma^2 = \left\{(x_1, x_2) \in \Gamma - \Sigma_1 \quad , \sum_{i=1}^{2} \left(b^*_i - \sum_{j=1}^{2} \frac{\partial a^*_{ij}}{\partial x_j} \right) m_i < 0 \right\}. \tag{4.47}$$

As indicated in [47], the boundary conditions at $\Sigma_1 \cup \Sigma_2$ for the initial boundary value problem associated to (4.42) are required. So, considering each boundary of Ω, we get:

- On boundary Γ_1^+ : $\quad x_1 = x_1^\infty, \quad 0 \le x_2 \le x_2^\infty, \quad \vec{m} = (-1, 0)$

$$\sum_{i,j=1}^{2} a^*_{ij} m_i m_j = a^*_{11} m_1^2 = \frac{\sigma^2 x_1^2}{2} > 0$$

- On boundary Γ_2^+ : $\quad 0 \le x_1 \le x_1^\infty, \quad x_2 = x_2^\infty, \quad \vec{m} = (0, -1)$

$$\sum_{i,j=1}^{2} a^*_{ij} m_i m_j = a^*_{11} m_1^2 = 0$$

$$\sum_{i=1}^{2} \left(b^*_i - \sum_{j=1}^{2} \frac{\partial a^*_{ij}}{\partial x_j} \right) m_i = -(\delta x_1 + r x_2^\infty) < 0$$

- On boundary Γ_1^- : $x_1 = 0, \quad 0 \le x_2 \le x_2^\infty, \quad \vec{m} = (1,0)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0$$

$$\sum_{i=1}^{2} \left(b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = (-\sigma^2 + r - \delta) x_1 = 0$$

- On boundary Γ_2^- : $0 \le x_1 \le x_1^\infty, \quad x_2 = 0, \quad \vec{m} = (0,1)$

$$\sum_{i,j=1}^{2} a_{ij}^* m_i m_j = a_{11}^* m_1^2 = 0$$

$$\sum_{i=1}^{2} \left(b_i^* - \sum_{j=1}^{2} \frac{\partial a_{ij}^*}{\partial x_j} \right) m_i = \delta x_1 > 0$$

Therefore, we obtain that $\Sigma_1 = \Gamma_1^+$ and $\Sigma_2 = \Gamma_2^+$, so that $\Sigma_1 \cup \Sigma_2 = \Gamma_1^+ \cup \Gamma_2^+$. Next, we propose the following nonhomogeneous Neumann conditions:

$$\frac{\partial V}{\partial S}(t,S,I) = g_1(t,S,I) \qquad \text{on } [0,T] \times \Gamma_1^+ , \tag{4.48}$$

$$\frac{\partial V}{\partial I}(t,S,I) = g_2(t,S,I) \qquad \text{on } [0,T] \times \Gamma_2^+ , \tag{4.49}$$

the functions g_1 and g_2 being defined by

$$g_1(t,S,I) = \frac{\partial \overline{\Lambda}}{\partial S}(0,S,I) = 1, \quad (t,S,I) \in [0,T] \times \Gamma_1^+, \tag{4.50}$$

$$g_2(t,S,I) = \frac{\partial \overline{\Lambda}}{\partial I}(0,S,I) = 1, \quad (t,S,I) \in [0,T] \times \Gamma_2^+, \tag{4.51}$$

which are derived from the exercise value function $\overline{\Lambda}$, provided that we choose the bounded domain satisfying the condition

$$\min(S_\infty, I_\infty) > K \exp(\gamma T), \tag{4.52}$$

that guarantees the inequality

$$S + I - K\exp(\gamma T) > 0, \quad \forall (S,I) \in \Gamma_1^+ \cup \Gamma_2^+. \tag{4.53}$$

Notice that condition (4.52) is satisfied by the data in the forthcoming test examples.

Moreover, we propose a mixed formulation to deal with obstacle problem by introducing the multiplier $P : [0,T] \times \Omega \longrightarrow \mathbb{R}$, so that we can replace equations (4.37)-(4.39) by the equation

$$V_\tau - Div\,(A\nabla V) + \vec{v} \cdot \nabla V + rV + P = 0 \quad \text{ in } (0,T) \times \Omega, \tag{4.54}$$

and the complementarity conditions

$$V \geq \overline{\Lambda}, \quad P \leq 0, \quad (V - \overline{\Lambda}) \cdot P = 0 \quad \text{ in } (0,T) \times \Omega. \tag{4.55}$$

This kind of mixed formulations have been previously used in early exercise Asian options with arithmetic averaging in [8] or in pension plans with early retirement opportunity pricing problems in [14], for example. In practice, we will apply the mixed formulation (4.54)-(4.55) to the fully discretized problem.

4.4.2 Discretization in time

Very often, in differential equations for pricing financial products, the diffusive term is quite small relative to the convective one for some regions of the domain or due to the presence of particular values of the involved parameters. This is specially reinforced in the case of the equations here considered for the stock loans pricing, due to the fact that there is no diffusion in one of the spatial dimensions. In such circumstances numerical schemes present difficulties.

A relatively large variety of ideas and approaches have been proposed in widely different contexts to solve these difficulties and the characteristics method for time discretization constitutes a possible upwinding scheme that leads to symmetric and stable approximations, reducing temporal errors and allowing for large timesteps without loss of accuracy.

The classical method of characteristics of first order has been introduced in [56] and first applied for the resolution of financial problems in Vázquez [64] for vanilla options and in D'Halluin, Forsyth and Labahn [19] for pricing Asian options. Also in the framework of the Hobson-Rogers stochastic volatility model, it has been applied to price European and American vanilla options in [29].

More recently, the higher order Crank-Nicolson Lagrange-Galerkin method has been analyzed in [6] and [7] for a general possibly degenerated convection-diffusion-reaction equation and applied to the pricing problem of Asian options with continuous arithmetic averaging in [8, 9].

In order to cope with the extremely convection dominated feature that appears in the Kolmogorov equation associated to the stock loan model, we use the Crank-Nicolson Lagrange-Galerkin method to approximate the material derivative

$$\frac{D}{D\tau} = \partial_\tau + \vec{v} \cdot \nabla \tag{4.56}$$

For this purpose, we define the characteristics curve through the point (S, I) at time $\bar{\tau}$, $X_e(x, \bar{\tau}; \tau)$, which verifies the following final value problem:

$$\partial_\tau X_e((S,I),\bar{\tau};\tau) = \vec{v}(X_e((S,I),\bar{\tau};\tau)), \quad X_e((S,I),\bar{\tau};\bar{\tau}) = (S,I)\,. \tag{4.57}$$

The final value problem (4.57) can be exactly solved, so that depending on the parameter values, we obtain:

- If $\sigma^2 - r + \delta = 0$:

$$\begin{aligned} X_e^1((S,I),\bar{\tau};\tau) &= S \\ X_e^2((S,I),\bar{\tau};\tau) &= -\frac{\delta}{r}S + \exp(r(\bar{\tau}-\tau))\left(I + \frac{S\delta}{r}\right) \end{aligned}$$

- If $\sigma^2 - r + \delta \neq 0$:

$$\begin{aligned} X_e^1((S,I),\bar{\tau};\tau) &= S\exp(-(\sigma^2 - r + \delta)(\bar{\tau} - \tau)) \\ X_e^2((S,I),\bar{\tau};\tau) &= \frac{-\delta S\exp(-(\sigma^2 - r + \delta)(\bar{\tau} - \tau))}{\sigma^2 + \delta} + \\ &\quad \exp(r(\bar{\tau} - \tau))\left(I + \frac{S\delta}{\sigma^2 + \delta}\right) \end{aligned}$$

Next, in order to describe the time discretization taking into account previous computations, for $N > 0$ let us consider the time step $\Delta\tau = \frac{T}{N}$ and the time mesh-points $\tau^n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. Then, at time $\tau^{n+\frac{1}{2}}$ the material derivative approximation by characteristics method is given by:

$$\frac{DV}{D\tau} \approx \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau},$$

where $X_e^n(S, I) := X_e(S, I, \tau^{n+1}; \tau^n)$, the components of which are given by

- If $\sigma^2 - r + \delta = 0$:

$$X_e^{n,1}(S,I) = S, \quad X_e^{n,2}(S,I) = -\frac{\delta}{r}S + \exp(r\Delta\tau)\left(I + \frac{S\delta}{r}\right)$$

- If $\sigma^2 - r + \delta \neq 0$:

$$\begin{aligned} X_e^{n,1}(S,I) &= S\exp(-(\sigma^2 - r + \delta)\Delta\tau), \\ X_e^{n,2}(S,I) &= \frac{-\delta S\exp(-(\sigma^2 - r + \delta)\Delta\tau)}{\sigma^2 + \delta} \\ &\quad + \exp(r\Delta\tau)\left(I + \frac{S\delta}{\sigma^2 + \delta}\right) \end{aligned}$$

The velocity field $\vec{v}$ is shown in Figures 4.1 and 4.2 for the conditions $\sigma^2 - \delta + r > 0$ and $\sigma^2 - \delta + r > 0$, respectively.

Remark 4.4.1. *Note that the velocity field at the boundary Γ_2^+ points towards the interior of the domain if $\sigma^2 - r + \delta \geq 0$ (see Figure 4.1). Also, if the quantity $\sigma^2 - r + \delta < 0$ then the velocity field at the boundaries Γ_2^+ and Γ_1^+ points towards the*

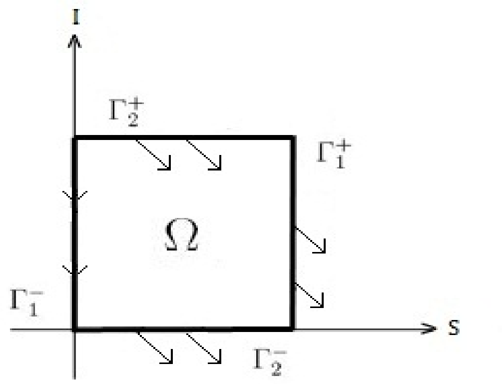

Figure 4.1: Velocity field in the domain Ω for $\sigma^2 - \delta + r > 0$.

interior of the domain (see Figure 4.2). So, even for small enough time steps, the point $X_e^n(S,I)$ may not belong to the domain and some approximations will be used. More precisely, if the point $X_e^n(S,I)$ is located outside the domain, we use a suitable Taylor approximation at the corresponding boundary, taking in account the functions appearing in the Neumann boundary conditions (4.48) and (4.49).

Next, if we consider a Crank-Nicolson scheme around the point $(X_e((S,I),\tau^{n+1};\tau),\tau)$ with $\tau = \tau^{n+\frac{1}{2}}$ for $n = 0,...,N-1$, then the time discretized PDE operator can be written as follows:

$$\begin{aligned} \overline{\mathcal{L}}[V]\left(X_e((S,I),t^{n+1};t^{n+\frac{1}{2}}),t^{n+\frac{1}{2}}\right) &\approx \\ \frac{V^{n+1}(S,I) - V^n(X_e^n(S,I))}{\Delta\tau} - \frac{1}{2}Div(A\nabla V^{n+1})(S,I) \\ -\frac{1}{2}Div(A\nabla V^n)(X_e^n(S,I)) + \frac{1}{2}(rV^{n+1}(S,I)) + \frac{1}{2}(rV^n(X_e^n(S,I))) &. \end{aligned} \quad (4.58)$$

For simplicity, let us introduce the notation $\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}$:

$$\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}}(S,I) = \overline{\mathcal{L}}[V]\left(X_e((S,I),\tau^{n+1};\tau^{n+\frac{1}{2}}),\tau^{n+\frac{1}{2}}\right). \quad (4.59)$$

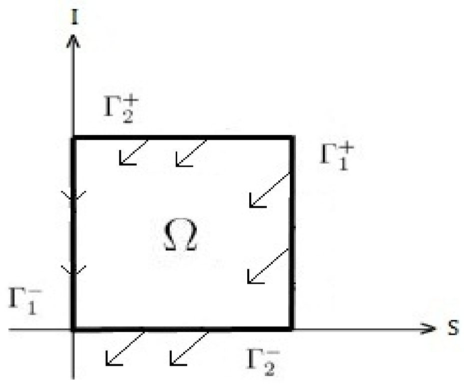

Figure 4.2: Velocity field in the domain Ω for $\sigma^2 - \delta + r < 0$.

In order to state the weak formulation for the semidiscretized problem, we use a Lemma that appears in [8, 46], multiplying the terms in (4.58) by a suitable test function ψ and integrating in Ω, we have:

$$\begin{aligned} \left((\overline{\mathcal{L}}[V])^{n+\frac{1}{2}}, \psi \right) \approx \int_\Omega \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau} \psi \, dSdI \\ -\frac{1}{2} \int_\Omega Div(A \nabla V^{n+1}) \psi dSdI - \frac{1}{2} \int_\Omega (Div(A \nabla V^n)) \circ X_e^n \psi \, dSdI \\ +\frac{1}{2} \int_\Omega r V^{n+1} \psi \, dSdI + \frac{1}{2} \int_\Omega (rV^n) \circ X_e^n \psi \, dSdI \end{aligned} \tag{4.60}$$

where notation dA is used for the integration measure in Γ.

Next, applying Lemma 3.4 that appears in [8, 46] and the usual Green's formula, expression (4.60) is equivalent to:

$$
\begin{aligned}
\left(\left(\overline{\mathcal{L}}[V]\right)^{n+\frac{1}{2}},\psi\right) \approx& \\
\int_\Omega \frac{V^{n+1}-V^n\circ X_e^n}{\Delta\tau}\psi\,dSdI + \frac{1}{2}\int_\Omega A\nabla V^{n+1}\nabla\psi\,dSdI& \\
+\frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla V^n)\circ X_e^n\nabla\psi\,dSdI& \\
+\frac{1}{2}\int_\Omega (Div(F_e^n)^{-t}(A\nabla V^n))\circ X_e^n\psi\,dSdI& \\
+\frac{1}{2}\int_\Omega rV^{n+1}\psi\,dSdI& \\
-\frac{1}{2}\int_\Gamma \vec{n}\cdot A\nabla V^{n+1}\psi\,dA& \\
+\frac{1}{2}\int_\Omega (rV^n)\circ X_e^n\psi\,dSdI& \\
-\frac{1}{2}\int_\Gamma \left((F_e^n)^{-t}\vec{n}\cdot(A\nabla V^n)\right)\circ X_e^n\psi\,dA& \quad . \qquad (4.61)
\end{aligned}
$$

Notice that the tensor $(F_e^n)^{-t}(S,I) = (\nabla X_e(S,I,\tau_{n+1};\tau_n))^{-t}$ can be easily computed and takes the form

$$
(\mathbf{F}_e^n)^{-t} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix},
$$

where the tensor components are actually independent of S and I. More precisely, by taking into account the different cases depending on the value of $\sigma^2 - r + \delta$, we have:

- If $\sigma^2 - r + \delta = 0$:

$$
b_{11} = \exp(r\Delta\tau), \quad b_{22} = 1, \quad b_{12} = \frac{\delta}{r}(1-\exp(r\Delta\tau)).
$$

- If $\sigma^2 - r + \delta \neq 0$:

$$
\begin{aligned}
b_{11} &= \exp(r\Delta\tau), \\
b_{22} &= \exp(-(\sigma^2 - r + \delta)\Delta\tau), \\
b_{12} &= \frac{\delta\exp(-(\sigma^2 - r + \delta)\Delta\tau)}{\sigma^2+\delta} - \frac{\delta\exp(r\Delta\tau)}{\sigma^2+\delta}.
\end{aligned}
$$

Next, let us precise the boundary integrals appearing in formulation (4.61). First, notice that we have $\vec{n} \cdot A\nabla V^{n+1} = 0$ on $\Gamma_1^- \cup \Gamma_2^-$ and we can use the Neumann boundary conditions on $\Gamma_1^+ \cup \Gamma_2^+$. Therefore, in the first boundary integral on the right hand side of equation (4.61) we can introduce the function

$$\bar{g}^{n+1} = \begin{cases} a_{11}\, g_1^{n+1} & \text{on} \quad \Gamma_1^+ \\ a_{11}\, g_2^{n+1} & \text{on} \quad \Gamma_2^+ \end{cases} \tag{4.62}$$

Moreover, for the second integral, we have

$$\int_\Gamma \big((F_e^n)^{-t}\vec{n} \cdot (A\nabla V^n)\big) \circ X_e^n \psi \, dA = \int_\Gamma g^n \psi \, dA, \tag{4.63}$$

where $g^n : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is given by

$$g^n(S,I) = \begin{cases} 0 & \text{on} \quad \Gamma_1^- \\ -((F_e^n)^{-t})_{11} a_{11}(X_e^n(S,I)) \dfrac{\partial V}{\partial I}(X_e^n(S,I)) & \text{on} \quad \Gamma_1^+ \\ -\frac{1}{2}((F_e^n)^{-t})_{12} a_{11}(X_e^n(S,I)) \dfrac{\partial V^n}{\partial I}(X_e^n(S,I)) & \text{on} \quad \Gamma_2^- \\ -((F_e^n)^{-t})_{12} a_{11}(X_e^n(S,I)) \dfrac{\partial V}{\partial I}(X_e^n(S,I)) & \text{on} \quad \Gamma_2^+ \end{cases}$$

Therefore, expression (4.61) becomes

$$\begin{aligned} \Big(\big(\overline{\mathcal{L}}[V]\big)^{n+\frac{1}{2}}, \psi\Big) \approx \; & \int_\Omega \frac{V^{n+1} - V^n \circ X_e^n}{\Delta\tau} \psi \, dSdI + \frac{1}{2}\int_\Omega A\nabla V^{n+1} \nabla\psi \, dSdI \\ & + \frac{1}{2}\int_\Omega \big((F_e^n)^{-1}(A\nabla V^n)\big) \circ X_e^n \nabla\psi \, dSdI \\ & + \frac{1}{2}\int_\Omega r V^{n+1} \psi dSdI \\ & + \frac{1}{2}\int_\Omega \big(Div(F_e^n)^{-t}(A\nabla V^n)\big) \circ X_e^n \psi \, dSdI \\ & - \frac{1}{2}\int_\Gamma g^n(S,I)\psi \, dA \\ & + \frac{1}{2}\int_\Omega (rV^n) \circ X_e^n \psi \, dSdI - \frac{1}{2}\int_{\Gamma_1^+ \cup \Gamma_2^+} \bar{g}^{n+1} \psi \, dA \end{aligned} \tag{4.64}$$

for all $\psi \in H^1(\Omega)$.

4.4.3 Finite elements discretization

In order to obtain the fully discretized problem, we combine the previously describe time discretization with a finite elements based spatial discretization. For this purpose, we consider a family of quadrangular meshes $\{\tau_h\}$ of the domain Ω. Associated to the mesh $\{\tau_h\}$, let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ denotes the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. Now, let us define the finite elements space V_h:

$$\mathcal{V}_h = \left\{\varphi_h \in \mathcal{C}^0(\bar{\Omega}) : \varphi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\right\},$$

where $\mathcal{C}^0(\bar{\Omega})$ denotes the space of continuous functions on $\bar{\Omega}$.

Therefore, if $V_h \in \mathcal{V}_h$ denotes the finite element approximation of $V \in H^1(\Omega)$, then the spatial discretization of (4.64) can be written in the form

$$\begin{aligned}
\left(\left(\overline{\mathcal{L}}[V_h]\right)^{n+\frac{1}{2}}, \psi_h\right) \approx \\
\int_\Omega \frac{V_h^{n+1} - V_h^n \circ X_e^n}{\Delta\tau} \psi_h \, dSdI + \frac{1}{2}\int_\Omega A\nabla V_h^{n+1} \nabla\psi_h \, dSdI \\
+\frac{1}{2}\int_\Omega \left((F_e^n)^{-1}(A\nabla V_h^n)\right) \circ X_e^n \nabla\psi_h \, dSdI \\
+\frac{1}{2}\int_\Omega rV_h^{n+1}\psi_h dSdI \\
+\frac{1}{2}\int_\Omega \left(Div(F_e^n)^{-t}(A\nabla V_h^n)\right) \circ X_e^n \psi_h \, dSdI \\
-\frac{1}{2}\int_\Gamma g_h^n(S,I)\psi_h \, dA \\
+\frac{1}{2}\int_\Omega (rV_h^n) \circ X_e^n \psi_h dSdI - \frac{1}{2}\int_{\Gamma_1^+ \cup \Gamma_2^+} \bar{g}^{n+1}\psi_h \, dA
\end{aligned} \tag{4.65}$$

for all $\psi_h \in \mathcal{V}_h$.

4.4.4 Mixed formulation and augmented Lagrangian active set method

Once the previous discretizations have been applied, we are led to the following fully discretized complementarity problem at each time step n:

$$M_h V_h^n \geq b_h^{n-1}, \quad V_h^n \geq \overline{\Lambda}_h, \quad \left(M_h V_h^n - b_h^{n-1}\right) \cdot \left(V_h^n - \overline{\Lambda}_h\right) = 0, \tag{4.66}$$

where M_h denotes the finite elements matrix, V_h^n denotes de vector containing the values of the solution at the nodes of the finite element mesh and $\overline{\Lambda}_h$ is the vector of the node values of the function $\overline{\Lambda}$. So, the corresponding mixed formulation of the complementarity problem (4.66) can be written in the form

$$M_h V_h^n + P_h^n = b_h^{n-1}, \tag{4.67}$$

jointly with the complementarity conditions

$$V_h^n \geq \overline{\Lambda}_h, \quad P_h^n \leq 0, \quad P_h^n \cdot \left(V_h^n - \overline{\Lambda}_h\right) = 0, \tag{4.68}$$

where P_h^n denotes the vector containing the nodal values of the multiplier.

The basic iteration of the augmented Lagrangian active set algorithm has been introduced in [37] and mainly consists of two steps. In the first one the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not), and in the second step, a reduced linear system associated to the inactive part is solved. Although the algorithm can be used in bilateral problems (in case of upper and lower constraints), we use the algorithm for unilateral problems, which are based on the augmented Lagrangian formulation. The method has been already successfully used when pricing early exercise Asian option with continuous arithmetic mean [9] and pension plans [14]. We address the reader to both papers for further details on the algorithm.

4.5 Numerical results

After verifying the performance of the numerical methods with some academic test problems with analytical solution, the real test proposed in [17], in which the authors apply the shooting grid method introduced in [4] is carried out. More precisely, the financial data appearing in the stock loan are the following:

$$\sigma = 0.4,\ r = 0.05,\ \delta = 0.03,\ \gamma = 0.09,\ K = 0.7,\ T = 3. \tag{4.69}$$

We note that for the previous data set, the relation $r < \gamma$ holds, so that Proposition 4.2.1 states the existence of a redeeming boundary and that the redemption region always contains a specific known region. We notice that the numerical methods confirm these results.

After using different meshes, time discretization steps and parameters of the numerical method, we show the results obtained for the localization parameters $S^\infty = 3K$ and $L^\infty = 3K$, a quadrangular finite elements mesh with 4096 elements and 16641 nodes, and the time step $\Delta\tau = 0.001$.

Note that the particular choice of the bounded domain guarantees that condition (4.52) is satisfied.

Figure 4.3 shows the computed optimal redeeming boundary at the times to maturity $\tau = T - t = 0$, 1, and 3, which coincide with those ones presented in [17] by using different scales in the axes. It has been computed by taking into account that the multiplier passes from zero value in the redemption region to a negative value in the no redemption region. The redemption region is located above the redeeming boundary curve and condition (4.15) is numerically satisfied. On the other hand, clearly the limit property (4.16) is also illustrated by Figure 4.3.

Figure 4.4 shows the computed stock loan value for $r < \gamma$ with the data in (4.69) at $t = 0$ which qualitatively resembles the kind of results obtained for Asian options with early exercise opportunity (see [8, 9], for example).

Additionally to the example appearing in [17], also the tests corresponding to cases with $r = \gamma$ and $r \geq \gamma$ have been performed and the results stated in Proposition 4.2.1

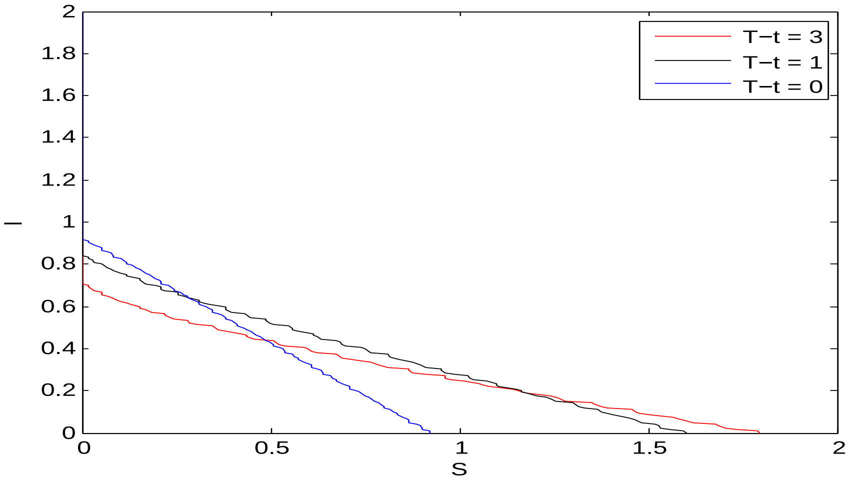

Figure 4.3: Optimal redeeming boundary for the data $\sigma = 0.4$, $r = 0.05$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

have been satisfied.

More precisely, in a second example for the case $r = \gamma$ the following financial data set has been chosen:

$$\sigma = 0.4, \ r = 0.09, \ \delta = 0.03, \ \gamma = 0.09, \ K = 0.7, \ T = 3. \tag{4.70}$$

For these data, Figure 4.5 shows the computed stock loan prices for $t = 0$. The computations have been performed with the same parameters of the numerical methods than the previous case.

Taking into account that the no redemption region corresponds to the points where the multiplier is identically zero, we note that in this case the multiplier vanishes in the whole domain, thus confirming the theoretical property proved for this case in Proposition 4.2.1.

Finally, in a third example for the case $r > \gamma$ the following financial data set has

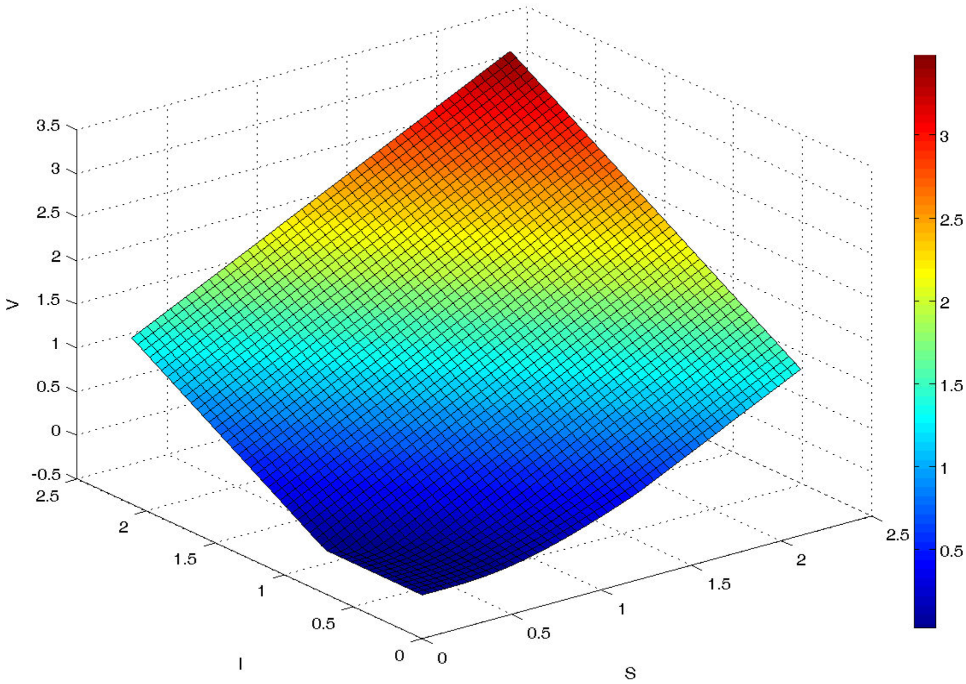

Figure 4.4: Stock loan price at $t = 0$ for the data $\sigma = 0.4$, $r = 0.05$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

been taken:

$$\sigma = 0.4,\ r = 0.13,\ \delta = 0.03,\ \gamma = 0.09,\ K = 0.7,\ T = 3. \tag{4.71}$$

For these data, Figure 4.6 shows the computed stock loan prices for $t = 0$. The computations have been performed with the same parameters of the numerical methods than in previous examples.

We also mention that in this case the multiplier vanishes in the whole domain, thus confirming the theoretical property proved for this case in Proposition 4.2.1.

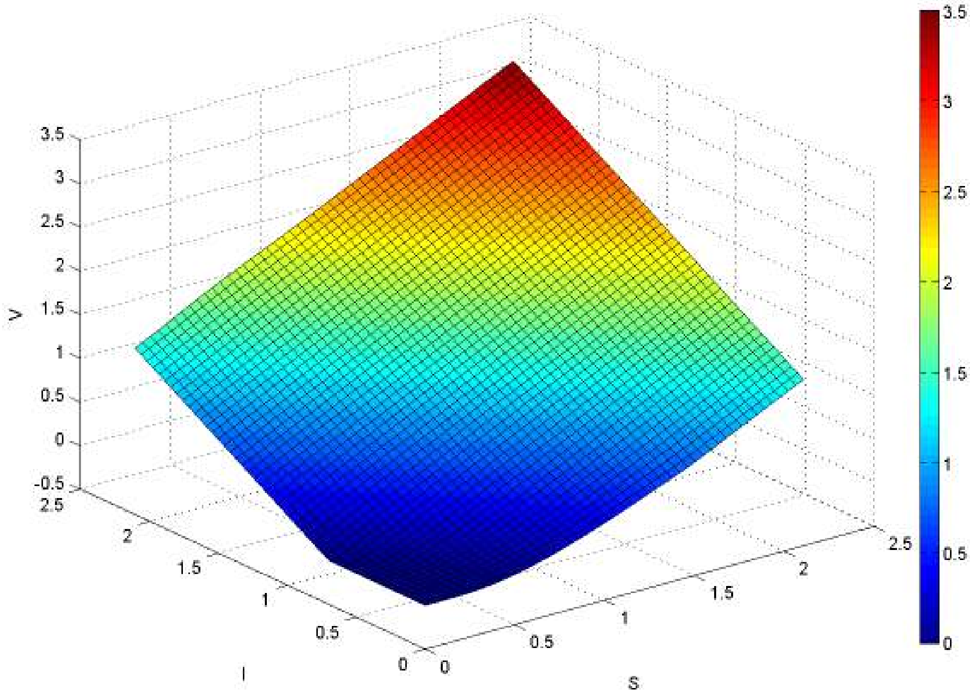

Figure 4.5: Stock loan price at $t = 0$ for the data $\sigma = 0.4$, $r = 0.09$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

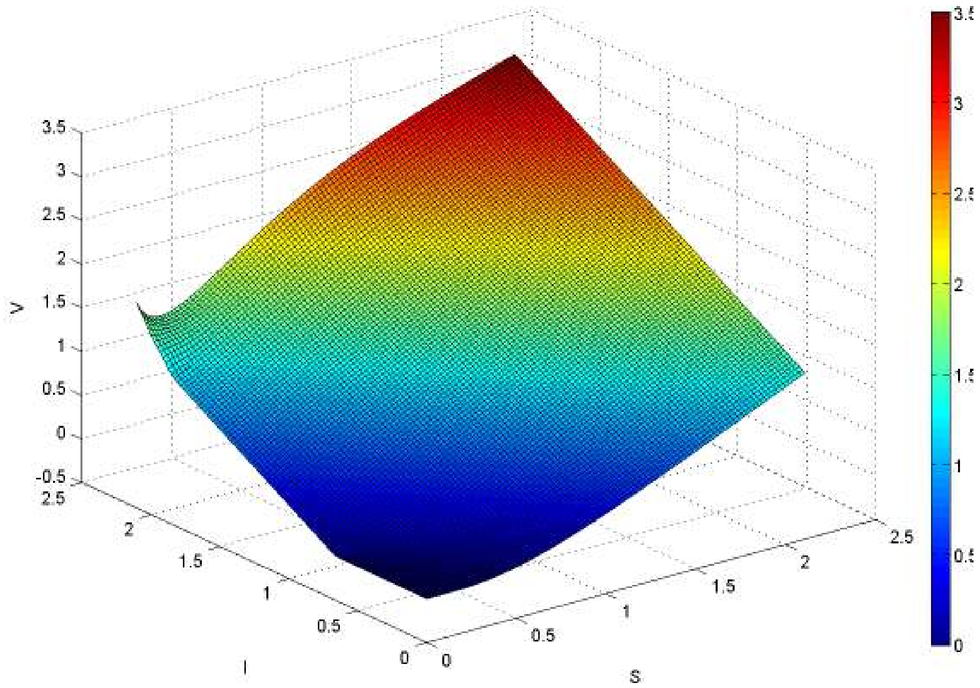

Figure 4.6: Stock loan price at $t = 0$ for the data $\sigma = 0.4$, $r = 0.11$, $\delta = 0.03$, $\gamma = 0.09$, $K = 0.7$ and $T = 3$.

Chapter 5

Conclusions

The objective of this work has been the mathematical analysis and numerical solution of the models that arise in pricing problems for several financial derivatives: ratchet cap, spread option and stock loan. These models are mainly posed in terms of partial differential equation of parabolic and Kolmogorov type, either in the form of Cauchy problems or in the form of obstacle ones. Once the models have been posed, their mathematical analysis allows to obtain the existence and uniqueness of solution under certain assumptions, as well as some qualitative and regularity properties of the solution. Moreover, the numerical methods provide rigorous tools to obtain in practice the fair prices of the financial products. The algorithms developing the numerical methods have been implemented in FORTRAN, MATLAB and Mathematica.

For the ratchet cap contract, in the LIBOR Market Model setting a rigorous and original PDE modeling of the general case has been posed, thus leading to a sequence of nested Cauchy problems for which the main result of existence and uniqueness has been obtained. The particular case with $b = 0$ leads to a simpler case where only two LIBOR rates are involved in the price of each ratchet caplet. Thus, by freezing a variable coefficient and using the concept of fundamental solutions, an analytical approximation can be obtained to price the ratchet cap. Alternatively, after a localization procedure and the statement of appropriate boundary condition for a Cauchy-boundary value problem, a Crank-Nicolson Lagrange-Galerkin method can be used to price the ratchet cap. Both previous methods can be compared with the third classical alternative based on Monte Carlo simulation. The numerical results confirm that the computed prices for different data sets are very close each other for the three methods. The main advantages of the finite elements approach are the precision and the fact that a large quantity of prices can be computed in a single execution at a relatively moderate computational cost. The advantages of the analytical approximation are also the precision and the very fast computing of a single price. The treatment of the case $b \neq 0$ is limited in this work to the use of Monte Carlo simulation, as the use of the analytical approximation faces the computation of very complex nested integrals in increasing dimension and the application of finite elements techniques would require the use of specific techniques in higher dimensions, such as possibly sparse grids [40, 59, 58].

The spread option contract on two LIBOR rates results to be very close to the ratchet caplet case with $b = 0$, so that most of the previous comments can be considered. Also the mathematical analysis provides the existence and uniqueness of solution and the three numerical alternatives can be handled to price this product. Notice that the methodology could be applied to other options on two LIBOR rates.

The rigorous quantitative treatment of the stock loan pricing problem in the literature is very recent. For pricing this product, when the cumulative dividend yield associated to the stock is delivered by the lender to the borrower on redemption, an obstacle problem associated to a Kolmogorov equation can be posed. In this work, the existence and uniqueness of solution is analyzed. Furthermore, appropriate numerical methods for solving the problem are proposed. This methods mainly consist of a Crank-Nicholson characteristics technique for the time discretization of the convection dominated Kolmogorov equation combined with piecewise quadratic Lagrange finite elements and an augmented Lagrangian active set technique to treat the nonlinearity associated to the (unilateral) obstacle condition. The validation of the performance of the proposed numerical methods is partly validated by verifying some qualitative properties about the redemption region and redeeming boundary theoretically proved in the literature.

As a whole, by using some tools of applied mathematics to quantitative finance problems, the present work tries to contribute to the understanding and rigorous statement of mathematical models for pricing complex real financial derivative, to their rigorous mathematical analysis and to the development of software tools that can be used in practice and that are based on deeply studied numerical techniques.

Appendix A

Computations related to the analytical approximation

In this Appendix we detail the intermediate computations related to the analytical approximation of the ratchet caplet price when $b = 0$ treated in chapter 2, i.e. the payoff only depends on two forward LIBOR rates. The same kind of computations can be addressed for the base rated call spread option treated in chapter 3.

So, starting from expression (3.32) for the ratchet caplet case, we can first compute explicitly

$$\begin{aligned} & G_3(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}, \eta_i) \\ & \qquad = \int_{\mathbb{R}} \widetilde{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i) dy_i, \end{aligned}$$

obtaining the following expression:

$$G_3(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}, \eta_i) = \frac{\exp(M_i)}{(2\pi)^{3/2}(\sigma^i)^2 \sigma^{i-1} \tau \sqrt{\delta_{i-1}(1-\rho_{i-1,i}^2)}},$$

where we recall that $\tau = T_{i-2} - t$ and M_i is given by

$$
\begin{aligned}
M_i = &- \frac{\left(\frac{1}{2}(\sigma^i)^2\delta_{i-1} + \eta_i - y_i\right)^2}{2(\sigma^i)^2\delta_{i-1}} \\
&- \frac{(\sigma^{i-1})^2\tau - 4(x_{i-1} + x_i - y_i - y_{i-1}) + \frac{4(x_i-y_i)^2}{(\sigma^{i-1})^2\tau} + \frac{4(x_{i-1}-y_{i-1})^2}{(\sigma^i)^2\tau}}{8\left(1-\rho_{i-1,i}^2\right)} \\
&+ \frac{\sigma^i(2\bar{c}_i - 1)\left((\sigma^{i-1})^2\tau - (x_i - y_i)\right)\rho_{i-1,i}}{4\sigma^{i-1}\left(1-\rho_{i-1,i}^2\right)} \\
&- \frac{\left((\sigma^{i-1})^2\tau + 2(x_i - y_i)\right)(x_{i-1} - y_{i-1})\rho_{i-1,i}}{2\tau\sigma^i\sigma^{i-1}\left(1-\rho_{i-1,i}^2\right)} \\
&+ \frac{8\bar{c}_i\rho_{i-1,i}^2(-x_{i-1} + y_{i-1}) - (\sigma^i)^2\tau\left(1 + 4(-1+\bar{c}_i)\bar{c}_i(\rho_{i-1,i})^2\right)}{8\left(1-\rho_{i-1,i}^2\right)}.
\end{aligned}
$$

Once G_3 has been obtained, we compute

$$
\begin{aligned}
&G_4(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}) \\
&= \int_{\log(ae^{y_{i-1}}+c)}^{\infty} G_3(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}, \eta_i)(e^{\eta_i} - (ae^{y_{i-1}} + c))d\eta_i\,,
\end{aligned}
$$

the value of which is given by

$$
\begin{aligned}
G_4(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}) = &\frac{\exp(H_i)}{4\sigma^i\sqrt{-\pi B_i\tau((\sigma^i)^2\delta_{i-1} + (\sigma^{i-1})^2\tau(1-\rho_{i-1,i}^2))}} \\
&\times\left(-2 + K\exp\left(\frac{1+S_i^1}{4B_i}\right)\cdot\left(1+\mathcal{N}\left(\frac{F_i}{2\sqrt{B_i}}\right)\right) + \mathcal{N}\left(\frac{1+F_i}{2\sqrt{B_i}}\right)\right),
\end{aligned}
\tag{A.1}
$$

where $\mathcal{N}$ denotes the standard normal distribution and

$$
\begin{aligned}
H_i(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}) = &\frac{S_i^2}{8} + \frac{\bar{c}_i(x_{i-1} - y_{i-1})\rho_{i-1,i}^2}{1-\rho_{i-1,i}^2} + \frac{S_i^3}{2\tau(1-\rho_{i-1,i}^2)} \\
&+ \frac{S_i^4}{2(1-\rho_{i-1,i}^2)} + \frac{S_i^5}{\tau(1-\rho_{i-1,i}^2)^2A_i} + \frac{S_i^6}{(1-\rho_{i-1,i}^2)^2A_i} + \frac{S_i^7}{\tau^2(1-\rho_{i-1,i}^2)^2A_i}
\end{aligned}
$$

$$
\begin{aligned}
A_i =& -\frac{1}{2(\sigma^i)^2\delta_{i-1}} - \frac{1}{2(\sigma^{i-1})^2\tau(1-\rho_{i-1,i}^2)} \\
B_i =& \frac{1}{2(\sigma^i)^2\delta_{i-1}} + \frac{1}{2(\sigma^i)^4\delta_{i-1}^2 A_i} \\
E_i =& a\exp(y_{i-1}) + c \\
F_i =& -\frac{1}{2} + \frac{(\rho_{i-1,i})^2}{4(\sigma^i)^2\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} - \frac{x_i}{2(\sigma^i)^2(\sigma^{i-1})^2\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} \\
&+ \frac{C\rho_{i-1,i}}{4\sigma^i\sigma^{i-1}\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2(\sigma^i)^3(\sigma^{i-1})\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} \\
&- 2B_i\,\log(E_i) \\
S_i^1 =& -1 + \frac{\rho_{i-1,i}}{2(\sigma^i)^2\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} - \frac{x_i}{(\sigma^i)^2(\sigma^{i-1})^2\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} \\
&- \frac{(2\bar{c}_i-1)\rho_{i-1,i}}{2\sigma^i\sigma^{i-1}\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{(\sigma^i)^3 s^{i-1}\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} \\
S_i^2 =& (\sigma^i)^2\delta_{i-1} + \frac{(\sigma^{i-1})^2\tau}{(1-\rho_{i-1,i}^2)} - \frac{1}{(1-\rho_{i-1,i}^2)A_i} + \frac{1}{2A_i} \\
S_i^3 =& -\tau(x_{i-1}+x_i-y_{i-1}) + \frac{x_i^2}{(\sigma^{i-1})^2} + \frac{(x_{i-1}-y_{i-1})^2}{(\sigma^i)^2} \\
&- \frac{2x_i(x_{i-1}-y_{i-1})\rho_{i-1,i}}{\sigma^i\sigma^{i-1}} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2\sigma^i\sigma^{i-1}A_i} \\
S_i^4 =& \frac{\sigma^i\sigma^{i-1}(2\bar{c}_i-1)\tau\rho_{i-1,i}}{2} - \frac{\sigma^i(2\bar{c}_i-1)x_i\rho_{i-1,i}}{\sigma^{i-1}} \\
&+ \frac{\sigma^{i-1}(x_{i-1}-y_{i-1})\rho_{i-1,i}}{\sigma^i} + \frac{(\sigma^i)^2\tau\left(1+4(-1+\bar{c}_i)\bar{c}_i\rho_{i-1,i}^2\right)}{4} \\
S_i^5 =& -\frac{x_i + (2\bar{c}_i-1)(x_{i-1}-y_{i-1})\rho_{i-1,i}^2 - \tau^{-1}x_i^2 - (1-\rho_{i-1,i}^2)^{-1}x_i}{4(\sigma^{i-1})^2} \\
&- \frac{\sigma^i(2\bar{c}_i-1)x_i\rho_{i-1,i}}{4(\sigma^{i-1})^3} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{4\sigma^i\sigma^{i-1}} \\
S_i^6 =& \frac{(\sigma^i)^2(2\bar{c}_i-1)^2\rho_{i-1,1}^2}{16(\sigma^{i-1})^2} + \frac{\sigma^i(2\bar{c}_i-1)\rho_{i-1,i}}{8\sigma^{i-1}} \\
&- \frac{(1-\rho_{i-1,i}^2)\sigma^i(2\bar{c}_i-1)\rho_{i-1,i}}{8\sigma^{i-1}} + \frac{1}{16} \\
S_i^7 =& -\frac{x_i(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2\sigma^i(\sigma^{i-1})^3} + \frac{(x_{i-1}-y_{i-1})^2\rho_{i-1,i}^2}{4(\sigma^i)^2(\sigma^{i-1})^2}
\end{aligned}
$$

Finally, we use numerical integration with MATHEMATICA to compute

$$\bar{u}^{i,i-1}(t,x_{i-1},x_i;K) = \int_{\mathbb{R}} G_4(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1})dy_{i-1}\,.$$

In the base rated call spread option case we consider $i=2$. Once the corresponding integral G_3 for $i=2$ has been computed, we proceed analogously to obtain the integral

$$\begin{aligned} &G_4(t,x_1,x_2;T_0,y_1,T_1) \\ &= \int_{\log(-ce^{y_1}+K)}^{\infty} G_3(t,x_1,x_2;T_0,y_1,T_1,\eta_2)(ae^{\eta_2}-(-ce^{y_1}+K))d\eta_2\,, \end{aligned}$$

instead of (A.1).

Resumen

En este trabajo se estudian algunos modelos matemáticos para valorar determinados derivados financieros. Concretamente, se aborda el modelado, el análisis matemático y la resolución numérica de problemas de valoración de *ratchet caps*, *spread options* y *stock loans*.

Un *ratchet cap* consiste en un producto derivado de tipos de interés que se descompone en contratos de tipo *ratchet caplet*, para los cuales el tipo de interés de ejercicio (*strike*) asociado está definido de forma recursiva en función de un conjunto de tipos implícitos o *forward* del LIBOR. Para un contrato de tipo *spread option*, la función de pago depende de la relación entre la diferencia de dos tipos LIBOR y un tipo fijo de ejercicio, de modo que pueden ser consideradas las versiones *call* y *put* del producto. En un contrato de tipo *stock loan*, el prestatario del crédito posee acciones que son utilizadas como garantía del crédito, por lo que puede considerarse como un derivado sobre el *stock*. En los tres casos, en matemáticas financieras, la metodología usual para la valoración de una opción también permite obtener diversos modelos que pueden ser formulados como problemas de ecuaciones en derivadas parciales.

Después de establecer estos modelos, el análisis matemático de los mismos permite obtener la existencia y unicidad de solución, así como algunas propiedades cualitativas y de regularidad de la solución. Además, teniendo presentes los requerimientos de la valoración de los productos en la práctica, desarrollamos diferentes métodos numéricos para resolver los modelos y proporcionar los precios justos de los productos financieros aquí tratados.

Los derivados financieros son un tipo de instrumento cuyos precios dependen de

otros productos subyacentes. El ejemplo más clásico aparece en los mercados de opciones sobre activos. El punto de partida del comercio de estos derivados financieros en mercados organizados data de principios de los 70 del siglo pasado, con la apertura del *Chicago Board of Options Exchange(CBOE)* en Chicago (USA). Prácticamente al mismo tiempo, los artículos seminales de Black y Scholes [11] y Merton [42] proporcionaron la metodología de cobertura dinámica para obtener el modelo de Black-Scholes de ecuaciones en derivadas parciales y la fórmula de Black-Scholes para opciones europeas de tipo vanilla. Desde ese momento, la complejidad de los productos de tipo opción y otros derivados ha aumentado considerablemente y los correspondientes modelos de ecuaciones en derivadas parciales se han ido proponiendo para su valoración (véase [66, 65], por ejemplo). Entre las diferentes herramientas matemáticas implicadas en el establecimiento del modelo de Black-Scholes, la consideración de un movimiento Browniano geométrico para la dinámica del subyacente (el precio de un activo, en este caso) constituye un punto clave. También tenemos que destacar que el corto vencimiento de los contratos de opciones permite el uso de tipos de interés constantes o determínisticos dependientes del tiempo. No obstante, este no es el caso para contratos de larga duración, como los bonos.

Entre la gran variedad de derivados, si el subyacente es un tipo de interés particular o un conjunto de ellos, aparece la clase de derivados de tipos de interés. Por lo tanto, los beneficios o las funciones de pago asociadas a un derivado de tipo de interés depende del nivel de ciertos tipos. Uno de los ejemplos más sencillos de esta clase de derivados consiste en un bono que paga periódicamente cupones, que dependen de cierto tipo de interés variable. Como en el caso de las opciones, si intentamos resolver el problema de valoración asociado, aparece la pregunta sobre cuáles son los modelos más adecuados para las dinámicas de los tipos de interés implicados. A diferencia de lo que ocurre en el caso de las opciones, al tratarse de contratos de larga duración y teniendo en cuenta el comportamiento de los tipos de interés, se plantea la consideración de modelos estocásticos para la evolución de los tipos. En la literatura se ha dedicado mucho esfuerzo al desarrollo de dichos modelos, que actualmente se pueden clasificar en dos clases: modelos de tipos a corto (*short rate models*) y modelos de mercado (*market models*). En el libro de Brigo y Mercurio ([13]) se obtiene una presentación muy completa de las diferentes familias de tipos de interés y su modelado.

Los modelos de tipo *short rate* están basados principalmente en dinámicas de un factor para el proceso del tipo instantáneo (*spot*), r_t, es decir,

$$dr_t \;=\; u(t, r_t)\, dt \;+\; w(t, r_t)\, dW_t\,,$$

donde las distintas expresiones particulares para las funciones u y w dan lugar a una gran variedad de modelos y donde W_t denota un movimiento Browniano (ver [66, 13], por ejemplo). Los modelos más populares de Vasicek(1977) [63], Dothan(1978) [20] y Cox-Ingersoll-Ross (1985) [16] se pueden englobar en este marco. Una ventaja de estos modelos es la posibilidad de obtener fórmulas analíticas para la valoración de bonos cupón cero (*zero coupon bonds*) o incluso de bonos que pagan cupones (*coupon bearing bonds*). Además, constituyen un primer paso sencillo para explicar modelos más generales y adecuados. Sin embargo, estos modelos resultan endógenos, en el sentido que proporcionan la estructura de tipos como un output y una calibración a mercado de sus parámetros constantes resulta casi imposible en la práctica. Un primer intento para superar esta desventaja es la inclusión de cierta dependencia en tiempo en las funciones u y w, como se propuso en el modelo de Hull-White (1990) [33], o la consideración de modelos de dos factores (ver [13]). No obstante, como el modelo de Hull-White también está basado en los valores de mercado de los tipos instantáneos y su varianza, todavía se mantienen las desventajas relacionadas con la dificultad en la calibración a la curva inicial de factores de descuento.

El modelo general de Heath-Jarrow-Morton [32], que aparece en 1992, constituye la primera alternativa para modelos de *short rates* en tiempo continuo. En este modelo general, se modelan los tipos instantáneos *forward* y se propone una metodología libre de arbitraje para la evolución estocástica de la curva de rendimiento completa, por lo que las dinámicas de los tipos *forward* están definidas adecuadamente en función de sus estructuras de volatilidad instantánea. El modelo de Heath-Jarrow-Morton también es considerado como el punto de partida de los modelos de mercado (*market models*).

Entre la gran variedad de tipos de inteés, el LIBOR (*London Interbank Offer Rate*) representa la tasa interbancaria considerada con más frecuencia como referencia para los contratos y también el tipo al cual los bancos internacionales más importantes se prestan dinero entre ellos. Además, los tipos *forward* son una clase de tipos de interés válidos para períodos futuros. Pueden ser pactados hoy para una inversión en un período futuro (por ejemplo, de uno a dos años desde el momento actual). En el *LIBOR Market Model*, se eligen los tipos *forward* del LIBOR y sus dinámicas como tipos de interés subyacentes en contratos de tipo *ratchet cap* y *spread option*.

Desde los artículos de Brace, Gatarek y Musiela [12], Jashmidian [35] y Miltersen, Sandmann y Sondermann [44], el *LIBOR Market Model* ha sido ampliamente utilizado como modelo para la evolución de tipos forward de LIBOR. Está basado en el marco más general Heath-Jarrow-Morton y ha sido uno de los modelos de mercado de tipos de interés más populares, principalmente debido a su coherencia con las fórmulas de valoración analíticas de Black usadas en mercado para *caps* y *floors*, los cuales son los derivados de tipos de interés más negociados. Además, sus parámetros pueden ser calibrados con datos de mercado y productos líquidos.

Desde un punto de vista numérico, en el *LIBOR Market Model* la mayor parte de la valoración de derivados de tipos de interés se lleva a cabo mediante simulación de Monte Carlo, sacando partido de su aplicabilidad general a la mayoría de los derivados. En concreto, los derivados de tipos de interés más negociados pueden ser valorados, tales como *vanilla caps* y *floors*, *discrete barrier caps* y *floors*, *discrete barrier digital caps* y *floors,spread options* y *ratchets* (ver Brigo-Mercurio [13] o Pelsser [53], por ejemplo). Cuando un derivado depende de un conjunto de tipos LIBOR, tiene que utilizarse una medida común en la formulación de las ecuaciones diferenciales estocásticas de los diferentes tipos de interés implicados, por lo que aparecen términos de deriva (*drift*). En este caso, considerando martingalas auxiliares, se han introducido recientemente algunas técnicas de simulación sin deriva (ver [31, 10, 22], por ejemplo). Sin embargo, la principal limitación de la simulación de Monte Carlo viene de su excesivo coste computacional, especialmente cuando en un cartera se precisan los precios de muchos derivados.

En ocasiones, como en la valoración de otros derivados financieros, surge una alternativa y técnica numérica más eficiente en el marco del *LIBOR Market Model* al formular problemas de valoración en términos de ecuaciones en derivadas parciales. Esta aproximación para la valoración de derivados resulta más clásica en la valoración de opciones (ver, por ejemplo, Pascucci [50] y Wilmott [65]). El teorema de Feynman-Kàc permite obtener una fórmula para representar el precio de algunos derivados financieros como la solución del problema de Cauchy asociado a ecuaciones de derivadas parciales parábolicas (a veces degeneradas) (ver [50], por ejemplo). En el caso de un contrato de tipo *ratchet cap*, como la función de pago depende de un conjunto de tipos *forward* del LIBOR, el precio de cada *ratchet caplet* se obtiene a partir de la solución de problemas de Cauchy, con un aumento de la dimensión espacial a medida que van hacia atrás los intervalos de tiempo. El riguroso establecimiento de este complejo modelo de ecuaciones en derivadas parciales y su análisis matemático, representa una parte original importante de este trabajo, ya que en la literatura sólo podemos encontrar la referencia [55], en la que se introduce un caso más sencillo y se resuelve numéricamente. En esta tesis también se enmarca este caso particular en el general y se analiza matemáticamente. En el caso de *call (put) spread option* sobre tipos LIBOR se emplea también la metodología de ecuaciones en derivadas parciales descrita.

El primer intento para establecer un modelo riguroso de valoración de *stock loans* aparece en [67], en el cual los dividendos de la acción son recogidos por el prestamista hasta la amortización, de modo que el modelo de ecuaciones en derivadas parciales es análogo al de una opción vanilla de compra americana con precio de ejercicio dependiente del tiempo. La valoración se puede formular en términos de problemas de complementariedad asociado a las ecuaciones clásicas de Black-Scholes (ver [66], por ejemplo), por lo que su análisis matemático puede ser enmarcado en la teoría de inecuaciones parabólicas degeneradas (ver, [18] o [34], por ejemplo). Es bien conocida la interpretación del modelo de valoración de opciones americanas como un problema de frontera libre en el cual no sólo el precio de la opción sino también la región de ejercicio anticipado óptimo tiene que ser determinada. Aunque normalmente los problemas de frontera libre se asocian a ecuaciones lineales parabólicas, la consideración de costes de transacción en opciones europeas de tipo vanilla, por ejemplo, dan lugar a problemas de frontera libre (de doble obstáculo) asociados a ecuaciones no lineales [2].

En [67] el *stock loan* perpetuo se relaciona con una opción americana perpetua para obtener una fórmula analítica. Más recientemente, en [17] distintos regímenes para la distribución de los dividendos nos llevan a los correspondientes problemas de frontera libre en el marco de vencimiento finito para un contrato de tipo *stock loan*. Aquellos que se corresponden con el dividendo asignado al prestamista antes de la amortización, reinversión del dividendo y devolución al prestatario en el momento de la amortización y el dividendo entregado al prestatario antes de la amortización, nos llevan a inecuaciones variacionales parabólicas en una dimensión espacial, que son muy similares a las que gobiernan opciones americanas vanilla. Sin embargo, el caso más interesante desde el punto de vista matemático lo proporciona el escenario en el que el dividendo acumulado se devuelve al prestamista en el momento de la amortización. Cuando esta especificación figura en el contrato *stock loan*, la introducción de un proceso estocástico auxiliar y el uso de metodología de cobertura dinámica permite representar el precio del *stock loan* como la solución de un problema de obstáculo asociado a una ecuación de tipo Kolmogorov. Este modelo se introduce en [17] y también se analiza la existencia de frontera libre (frontera de amortización en el caso del *stock loan*), suponiendo que la existencia de solución ha sido obtenida. Una parte original del presente trabajo es la demostración de la existencia de solución y de su unicidad en el conjunto de funciones con crecimiento polinómico, utilizando para ello las técnicas recientemente aplicadas en [45] para opciones asiáticas con media aritmética y posibilidad de ejercicio anticipado. Además, también se analiza la regularidad anisotrópica de la solución mediante las técnicas desarrolladas en [26] para ecuaciones parabólicas hipoelípticas.

Teniendo en cuenta la aplicación en la práctica de los modelos aquí considerados, su análisis matemático necesita completarse con su resolución numérica. Los métodos numéricos propuestos pueden ser implementados como herramientas de software en lenguajes apropiados de programación para ser manejados por usuarios.

Generalmente, los métodos numéricos para valorar derivados financieros pueden ser clasificados en tres tipos: simulación Monte Carlo, árboles binomiales y solución numérica o analítica de modelos de ecuaciones en derivadas parciales.

Como se ha indicado anteriormente, en el *LIBOR Market Model*, la técnica más empleada en la literatura es la basada en simulación Monte Carlo. Los árboles binomiales también pueden ser utilizados y, por ejemplo, en [15] se incluyen algunos ejemplos de valoración de derivados de tipos de interés para el *Swap Market Model*, modelo muy cercano al *LIBOR Market Model*.

En finanzas cuantitativas, los métodos más extendidos incialmente para la solución numérica de modelos de ecuaciones en derivadas parciales siempre han sido los clásicos de diferencias finitas para ecuaciones parabólicas que modelan los precios de derivados de tipo europeo, combinados con algunas técnicas de proyección para los productos con ejercicio anticipado, tales como opciones americanas o bonos *callable* [66]. Sin embargo, ya se han utilzado en finanzas computacionales otras técnicas numéricas tradicionales en la dinámica de fluidos computacional, tales como volúmenes finitos [68], elementos finitos [41, 57] o métodos de las características (esquemas semilagrangianos) para la discretización en tiempo [64, 5, 21, 29]. Una presentación rigurosa de métodos de diferencias finitas y elementos finitos en problemas de valoración de opciones se puede encontrar en el texto [1].

En el marco particular de la solución numérica de problemas de ecuaciones en derivadas parciales para la valoración de *ratchet caps* basadas en el *LIBOR Market Model*, solo hemos encontrado el trabajo de Pietersz [55], en el cual se establece una ecuación parabólica en dos dimensiones espaciales y se presenta una comparación entre la simulación de Monte Carlo y un esquema de diferencias finitas explícito. Con respecto al problema del *stock loan*, solamente la referencia [17] incluye la resolución numérica mediante un método de tipo *forward shooting* propuesto en [4] para opciones asiáticas de tipo americano.

En este trabajo, proponemos el uso de una metodología unificada para la discretización temporal-espacial en todos los problemas de ecuaciones en derivadas parciales, que está basada en el método de alto orden de Crank-Nicolson Lagrange-Galerkin, inicialmente propuesto en [60] para una ecuación de convección-difusión con coeficientes constantes y extendido en [6, 7] a un marco más amplio de problemas de convección-difusión-reacción (posiblemente degenerados) . Además, estos métodos se han aplicado con éxito a la valoración de opciones asiáticas sin posibilidad de ejercicio anticipado en [8]. La ventaja de los métodos de características para la discretización en tiempo surge sobre todo en los problemas de convección dominante, en los cuales pueden aparecer oscilaciones espúreas si se aplican métodos numéricos no adecuados. En el caso de contratos de tipo *ratchet cap* y *spread option*, se propone una técnica semianalítica original y se compara con la anterior, junto con una sencilla simulación de Monte Carlo. En el caso de *stock loans*, se combina la técnica de Lagrange-Galerkin con una técnica de tipo conjunto activo basado en la lagrangiana aumentada (*augmented Lagrangian active set technique*), propuesta en [37] que permite manejar una restricción unilateral en el marco de una formulación mixta. Este método ha sido utilizado previamente con éxito en [9] para valorar opciones asiáticas con media aritmética y ejercicio anticipado. La aplicación de métodos numéricos en la valoración de *stock loans* permite, no solamente obtener su precio sino también las regiones de amortización anticipada y de continuidad , así como la frontera de amortización óptima que separa ambas regiones. Las verificación de las propiedades teóricas probadas en [17] acerca de estas regiones contribuyen a validar el comportamiento de los métodos numéricos propuestos.

El esquema de la memoria de esta tesis es el siguiente.

El capítulo 1 se dedica a la presentación del marco funcional de ecuaciones en derivadas parciales parabólicas con coeficientes variables para ser usado posteriormente en el análisis matemático de los modelos de *ratchet cap* y *spread options* de tipos *forward* del LIBOR. Contiene principalmente las definiciones y resultados más importantes relativos a la existencia y unicidad de soluciones.

En el capítulo 2 se establece la formulación rigurosa de un modelo original de ecuaciones en derivadas parciales para la valoración de un contrato de tipo *ratchet cap*. A continuación, se desarrolla el análisis matemático del modelo general para obtener la existencia de solución. También se analiza matemáticamente un caso particular más sencillo. Para este caso, se describen distintas técnicas numéricas para obtener el precio de un *ratchet cap*, basadas en soluciones semianalíticas, métodos de Lagrange-Galerkin y simulación de Monte Carlo. A continuación, se presentan diferentes ejemplos para ilustrar el comportamiento de los métodos numéricos propuestos.

En el capítulo 3 se establece de modo riguroso un modelo de ecuaciones en derivadas parciales para valorar opciones *spread* de compra y venta sobre tipos *forward* del LIBOR, se analiza matemáticamente y se resuelve numéricamente con las mismas técnicas del capítulo 2. Además se presentan algunos ejemplos numéricos.

En el capítulo 4 se realiza el análisis matemático del modelo de valoración de contratos de tipo *stock loan*, cuando la tasa de dividendo acumulada asociado al *stock* es devuelto por el prestamista al prestatario en el momento de la amortización del préstamo. Más concretamente, el modelo se puede formular como un problema de obstáculo asociado a una ecuación de Kolmogorov, por lo que puede obtenerse la existencia y unicidad en el marco de las soluciones con crecimiento polinómico. A continuación, para la solución numérica del problema se describe la combinación de Crank-Nicolson Lagrange-Galerkin con un método del tipo *Augmented Lagrangian Active Set*. Algunos ejemplos numéricos ilustran las propiedades teóricas de la frontera de amortización óptima.

En el capítulo 5 se recogen las principales conclusiones de este trabajo.

En el Anexo A se incluyen algunos cálculos intermedios relacionados con la aproximación análitica propuesta en el capítulo 2 para la valoración de ratchet caplets. El mismo tipo de cálculos se requieren para el *spread option* en el capítulo 3.

Bibliography

[1] I. Achdou and O. Pironneau, *Computational methods for option pricing*, SIAM, Philadelphia, PA, 2005.

[2] I. Arregui and C. Vázquez, Numerical solution of an optimal investment problem with proportional transaction costs. *Journal of Computational and Applied Mathematics*, 236 (2012) 2923-2937.

[3] V. Barbu and C.Marinelli, Variational inequalities in Hilbert spaces with measures and optimal stopping problems, *Appl. Math. Optim.* 57 (2008), 237–262.

[4] J. Barraquand and T. Pudet, Pricing of American path-dependent contingent claims, *Mathematical Finance*, 6 (1996) 17–51.

[5] Ana Bermúdez and M. R. Nogueiras, Numerical solution of two factor models for the valuation of financial derivatives, *Mathematical Models and Methods in Applied Sciences*, 14 (2004) 1829-1853.

[6] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction pro-blems with higher order characteristics finite elements. Part I: Time discretization, *SIAM Journal on Numerical Analysis*, 44 (2006) 1829-1853.

[7] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part II: Fully discretized scheme and quadrature formulas, *SIAM Journal on Numerical Analysis*, 44 (2006) 1854-1876.

[8] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Numerical solution of variational inequalities for pricing Asian options by higher order Lagrange-Galerkin methods, *Applied Numerical Mathematics*, 56 (2006) 1256-1270.

[9] A. Bermúdez, M. R. Nogueiras and C. Vázquez, Comparison of two algorithms to solve a fixed-strike Amerasian options pricing problem, in *International Series in Numerical Mathematics*, 154 (2006) 95-106.

[10] C. Beveridge, N. Denson and M. Joshi, Comparing discretizations of the Libor market model in the spot measures, *Australian Actuarial Journal*, 15 (2009) 231-253.

[11] F. Black and M. Scholes, The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, (1973) 637–654.

[12] A. Brace, D. Gatarek and M. Musiela, The market model of interest rate dynamics. *Mathematical Finance*, 7, (1997) 127-155.

[13] D. Brigo and F. Mercurio, *Interest rate models: theory and practice (with smile, inflation and credit)*, Springer Finance, Berlin,2007.

[14] M.C. Calvo-Garrido, A. Pascucci and C. Vázquez, Mathematical analysis and numerical methods for pricing pension plans allowing early retirement. *Preprint, submitted for publication*, (2012).

[15] M.C. Calvo-Garrido, M. Suárez-Taboada and C. Vázquez, Binomial trees for Markovian functional swap market models: calibration and pricing of interest rate derivatives, in *Proceedings of the First Hispano-Morrocan Days in Applied*

Mathematics and Statistics (HMAMS2008), (ISBN-978-84-691-7936-9) 1 (2008) 303–309.

[16] J.C. Cox, J.E. Ingersoll and S.A. Ross, A theory of the term structure of the interest rates. *Econometrica*, 53 (1985) 385–407.

[17] M. Dai and Z.Q. Xu, Optimal redeeming strategy of stock loans with finite maturity, *Mathematical Finance*, 21 (2011) 775–793.

[18] J. Dewynne, S.D. Howison, I. Ruff and P. Wilmott, Some mathematical results in pricing of American options, *European Journal of Applied Mathematics*, 4 (1993) 381–398.

[19] Y. D'Halluin, P. A. Forsyth and G. Labahn, A semi-Lagrangian approach for American Asian options, *SIAM Journal of Scientific Computing*, 27 (2005) 315-345.

[20] L.U. Dothan, On the term structure of interest rates. *Journal of Financial Economics*, 6 (1978) 59–69.

[21] J. Farto and C. Vázquez, Numerical methods for pricing callable bonds with notice, *Applied Mathematics and Computation*, 161 (2005), 3, 989-1013.

[22] J.L. Fernández, M.R. Nogueiras, M. Pou and C. Vázquez, A new parameterization for the drift-free simulation in the Libor Market Model. *Preprint submitted for publication*, (2012).

[23] M. di Francesco, P. Foschi and A. Pascucci, Analysis of an uncertain volatility model, *Journal of Applied Mathematics and Decision Sciences* **2006** (2006) Article ID 15609.

[24] M. di Francesco and A. Pascucci, On the complete model with stochastic volatility by Hobson and Rogers, *Proceedings of Royal Society London, Series A* **460** (2004) 3327–3338.

[25] M. di Francesco and A. Pascucci, On a class of degenerate parabolic equations of Kolmogorov type, *AMRX Appl. Math. Res. Express* **3** (2005) 77–116.

[26] M. Di Francesco, A. Pascucci and S. Polidoro, The obstacle problem for a class of hypoelliptic ultraparabolic equations, *Proceedings Royal Society of London Serie A*, 464 (2008) 155-176.

[27] M. Frentz, K. Nystrom, A. Pascucci and S. Polidoro, Optimal regularity in the obstacle problem for Kolmogorov operators related to American Asian options, *Math. Annal.*, 347 (2010) 805-838.

[28] A. Friedman, *Partial differential equations of parabolic type*, Robert E. Krieger Publishing Company, Malabar, Florida, 1983.

[29] A. González-Gaspar and C. Vázquez, A characteristics-finite differences method for the Hobson-Rogers uncertain volatility model, *Mathematical and Computer Modelling*, 52 (2010), 260-267.

[30] P. Glasserman, *Monte Carlo methods in financial engineering*, Springer, Berlin, 2003.

[31] P. Glasserman and X. Zhao, Arbitrage-free discretization of lognormal forward LIBOR and swap rate models, *Finance and Stochatics* 4 (2000) 35-68.

[32] D. Heath, R. Jarrow and A. Morton, Bond pricing and the term structure of interest rate: a new methodology. *Econometrica*, 60 (1992) 77–105.

[33] J. Hull and A. White, Pricing interest rate derivative securities. *Review of Financial Studies*, 3 (1990) 573–592.

[34] P. Jaillet, D. Lamberton and B. Lapeyre, Variational inequalities and the pricing of American options. *Acta Applicanda Mathematicae*, 21 (1990) 263–289.

[35] F. Jamshidian, LIBOR and swap market models and measures. *Finance and Stochastics*, 1 (1997) 293-330.

[36] R. Kangro and R. Nicolaides, Far field boundary conditions for Black-Scholes equations. *SIAM Journal Numerical Analysis*, 38 (2000) 1357–1368.

[37] T. Kärkkäinen, K. Kunisch, and P. Tarvainen, Augmented Lagrangian active set methods for obstacle problems. *Journal of Optimization Theory and Applications*, 119 (2003) 499–533.

[38] P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*. Springer, Berlin, 2007.

[39] C.C.W. Leentvaar, *Pricing multi-asset options with sparse grids*, Ph.D. Thesis, University of Leiden, 2008.

[40] C.C.W. Leentvaar and C.W. Oosterlee, Pricing Multi-Asset Options with Sparse Grids and Fourth Order Finite Differences. In *Proceedings of ENUMATH 2005*, A. Bermudez de Castro and D Gomez (Eds.), Springer, (2006) 975-986.

[41] M.D. Marcozzi, On the valuation of Asian options. *SIAM Journal of Scientific Computing*, (2003) 141–183.

[42] B. Merton, Theory of rational option pricing. Journal of Economics and Management Science, (1973) 141–183.

[43] T. Mikosch,*Elementary stochastic calculus with finance view*. World Scientific, Singapur, 1998.

[44] K. R. Miltersen, K. Sandmann and D. Sondermann, Closed form solutions for term structure derivatives with lognormal interest rates. *The Journal of Finance*, 52 (1997) 409-430.

[45] L. Monti and A. Pascucci, Obstacle problem for arithmetic Asian options, *C.R. Acad. Sci. Paris*, Ser. I, 347 (2009) 1443–1446.

[46] M. R. Nogueiras, *Numerical analysis of second order Lagrange-Galerkin schemes. Application to option pricing problems*, University of Santiago de Compostela, 2005.

[47] O. A. Oleĭnik and E. V. Radkevič, *Second Order Equations With Nonnegative Characteristic Form*, American Mathematical Society, Providence, Rhode Island, Plenum Press, New York-London, 1973.

[48] A. Pascucci, Free boundary and optimal stopping problems for American Asian options. *Finance Stochastics*, 12 (2008) 21–41.

[49] A. Pascucci, *Calcolo Stocastico per la Finanze*, Springer, 2009.

[50] A. Pascucci, *PDE and Martingale Methods in Option Pricing*, Bocconi & Springer Series, Berlin, 2011.

[51] A. Pascucci, M. Suárez-Taboada and C. Vázquez, Mathematical analysis and numerical methods for a partial differential equation model governing a ratchet cap pricing problem, *Mathematical Models and Methods in Applied Sciences*, 21 (2011) 1479-1498.

[52] A. Pascucci, M. Suárez-Taboada and C. Vázquez, Mathematical analysis and numerical methods for a PDE model of a stock loan pricing problem, *Preprint submitted for publication*, (2012).

[53] A. Pelsser, *Efficient Methods for Valuing Interest Rate Derivatives*, Springer, London, 2000.

[54] R. Pietersz, *PDE pricing for BGM*, Preprint Erasmus University Rotterdam, 2002.

[55] R.Pietersz, *The LIBOR market model.* Master's Thesis, Univ. Leiden, 2003.

[56] O. Pironneau, On the transport-diffusion algorithm and its application to Navier-Stokes equation. *Numerische Mathematik*, 38 (1982) 309–332.

[57] O. Pironneau and F. Hecht, Mesh adaption for the Black-Scholes equation. *East-West Journal of Numerical Mathematics*, 8 (2000) 25–35.

[58] C. Reisinger, Analysis of Linear Difference Schemes in the Sparse Grid Combination Technique,. *IMA Journal of Numerical Analysis*, to appear.

[59] C. Reisinger and G.. Wittum, Efficient hierarchical approximation of high-dimensional option pricing problems. *SIAM Journal of Scientific Computing*, 29 (2007) 440–458.

[60] H. Rui and M. Tabata, A second order characteristics finite element scheme for convection-diffusion problems. *Numerische Mathematik*, 92 (2002) 161-177.

[61] M. Suárez-Taboada and C. Vázquez, A numerical method for pricing spread options on LIBOR rates with a PDE model. *Mathematical and Computer Modelling*, 52 (2010) 1074–1080.

[62] M. Suárez-Taboada and C. Vázquez, Numerical solution of a PDE model for a rachet-cap pricing with BGM interest rate dynamics. *Applied Mathematics and Computation*, 218 (2012) 5217–5230.

[63] O. Vasicek, An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5 (1977) 177–188.

[64] C. Vázquez, An upwind numerical approach for an American and European option pricing model. *Applied Mathematics and Computation*, 97 (1998) 273–286.

[65] P. Wilmott, *Derivatives. The theory and practice of financial engineering*, Wiley, 1998.

[66] P. Wilmott, J. Dewynne and S. Howison, *Option pricing. Mathematical models and computation*, Oxford Financial Press, Oxford, 1993.

[67] J. Xia and X.Y. Zhou, Stock loans, *Mathematical Finance*, 17 (2007) 307–317.

[68] R. Zvan, P. Forsyth and K. Vetzal, Robust numerical methods for PDE of Asian options, *Journal of Computational Finance*, 1 (1998) 39–78.