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ENRICHMENT OF WEIGHTED LEAST SQUARES APPROACHES FOR POTENTIAL PROBLEMS IN ENGINEERING APPLICATIONS

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ABSTRACT

In this paper, we review one of the meshless methods proposed in last years for solving boundary value problems based on a weighted least squares method. Furthermore, we analyze the use of enrichment techniques of the solution in order to improve the computational cost. These methods allow to introduce some information about the solution in the numerical formulation. Finally, some 1D and 2D examples to some test problems are presented.

1. INTRODUCTION

Numerical simulation in most of the fields of the engineering and science have enormously increased since the sixties due to the development of high efficient numerical methods and advances in computer sciences. Generally speaking, these numerical techniques such as finite elements, boundary elements, finite volumes or finite differences are founded on the division of the domain of the problem (in some cases, also or only its boundaries) in small parts (subdomains), where the integration of the differential equations is performed. However, this process of partition of the domain (or mesh generation) becomes a bottle neck, specially in 3D problems. Thus, the discretization of the domain can involve more computing effort (in memory and CPU time) than the integration and solving processes in some practical applications such as moving boundaries problems, with discontinuities in the domain, or with a very complicated geometry. For it, some numerical methods where meshes are unnecessary have been proposed in last years.

Finite differences approaches were the origin of the first meshless methods ("particle methods") in early seventies in the computational physics field (Smooth Particle Hydrodynamics method, SPH)[1−4]. A different type of meshless methods (the Diffuse

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Element Method, DEM), was proposed by Nayroles[5] in 1992. In this method, a basis function and a weighting function are used to define a local approximation based on a set of arbitrary nodes. In 1994, Belystchko *et al.*modify and refine this method, proposing the Element Free Galerkin method (EFGM)[6], in which a moving-least squares (MLS) interpolation is used to define the local approach. Liu *et al.*proposed a meshless technique (the Reproducing Kernel Particle Method, RKPM) based on a convolution integral, which it is similar to SPH method, although several correction functions and refinements are introduced in order to assure consistency near boundaries and for nonuniform spacing[7]. Another type of different meshless methods are based on partitions of unity and provide an efficient way to perform *h-p* adaptivity (*hp*-Clouds method[8] and the Partition of Unity Finite Element Method [9]). On the other hand, Oñate *et al.*[10] have proposed a method which combines the moving least square approximation with a point-collocation approach to compute the integral terms, in convective transport and fluid flow problems. This method completely avoids the necessity of mesh generation, because no auxiliary grid is required. Furthermore, different techniques can be derived if the weighting function is "fixed" (Diffuse Least Square method, DLS) or it depends on the point where the approximated value is computed (Moving Least Square method, MLS).

On the basis of these weighted least-squares methods, we have recently proposed a numerical approach to solve potential problems in electrical engineering applications[11,12], such as grounding analysis, where the use of standard numerical methods (such as finite elements) are precluded due to the complexity of the domain[13]. In this paper, we review this weighted least-squares interpolation with a point collocation approach for solving boundary value problems, and we explore the use of enrichment functions in this kind of numerical approaches.

2. WEIGHTED LEAST SQUARES APPROXIMATIONS

Weighted Least Squares methodology is an effective numerical method for the approximation of a function in terms of a set of disordered data. It consists of a local weighted least square fitting, valid on a small neighbourhood (Ω_k) of a point and based on the information provided by its n closest points. The local character of the approximation comes from a weighting function which takes its maximum value at this point and vanishes outside a surrounding region (for example, a truncated gaussian function or a conoidal)[14]. The proper definition of the approximation at every point requires that all subdomains Ω_k cover all the interpolation domain; then these subdomains must overlap and the common areas have to include enough nodal points in order to ensure the convergence of the method[8].

The approximation $\hat{u}(\boldsymbol{x})$ of a function $u(\boldsymbol{x})$ in a domain Ω_k can be written as

$$u(\boldsymbol{x}) \cong \hat{u}(\boldsymbol{x}) = \sum_{i=1}^{m} p_i(\boldsymbol{x})\alpha_i = \boldsymbol{p}^t(\boldsymbol{x})\boldsymbol{\alpha} \tag{1}$$

being $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^t$ a set of unknown coefficients, and $\boldsymbol{p}(\boldsymbol{x})$ the base of interpolating functions (monomial terms, generally) which order is m. These base interpolating functions usually are normalized within each subdomain Ω_k by dividing for the maximum distance between each point i of the domain and the surrounding points, which allows to define normalized coordinates within the subdomain Ω_k.

On the other hand, the sampling of the function $u(\boldsymbol{x})$ in the n points belonging to Ω_k can be written as,

$$\boldsymbol{u}^h = \begin{pmatrix} u_1^h \\ u_2^h \\ \vdots \\ u_n^h \end{pmatrix} \cong \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_1^t \\ \boldsymbol{p}_2^t \\ \vdots \\ \boldsymbol{p}_n^t \end{pmatrix} \boldsymbol{\alpha} = \boldsymbol{S\alpha} \qquad (2)$$

where u_j^h are the values of the unknown function evaluated in the nodal points of Ω_k (i.e., $u_j^h = u(\boldsymbol{x}_j)$, $j = 1, \ldots, n$), $\hat{u}_j = \hat{u}(\boldsymbol{x}_j)$ are their approximated values, and $\boldsymbol{p}_j$ contains the normalized base interpolating functions evaluated in $\boldsymbol{x}_j$.

In general, if $n > m$, $\boldsymbol{S}$ is a rectangular matrix and the approximation cannot fit all the nodal values u_j^h. However, the approximation $\hat{u}(\boldsymbol{x})$ can be determined by a minimizing process of the the weighted sum of the square differences between the exact value u_j^h and the approximation $\hat{u}(\boldsymbol{x}_j)$ at each nodal point $\boldsymbol{x}_j$ belonging to the domain of a generic node $\boldsymbol{x}$. The weighting function is usually built in such a way that it equals unity in point $\boldsymbol{x}_k$ and vanishes outside its subdomain. Thus, being $\omega_k(\boldsymbol{x}_j, \boldsymbol{x})$ a weighting function computed in $\boldsymbol{x}_j$ (which shape and span depend on point $\boldsymbol{x}$), we can define the functional

$$J(\boldsymbol{x}) = \sum_{j=1}^{n} \omega_k(\boldsymbol{x}_j, \boldsymbol{x})(u_j^h - \hat{u}(\boldsymbol{x}_j))^2. \qquad (3)$$

If we take into account expression (2), the minimization of functional $J(\boldsymbol{x})$ allows to obtain[14]:

$$\boldsymbol{\alpha} = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}^h \qquad \boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{PW}(\boldsymbol{x})\boldsymbol{P}^t \qquad \boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{PW}(\boldsymbol{x}) \qquad (4)$$

being auxiliar matrices $\boldsymbol{P}$ and $\boldsymbol{W}(\boldsymbol{x})$:

$$\boldsymbol{P} = [\boldsymbol{p}(\boldsymbol{x}_1) \ \ \ldots \ \ \boldsymbol{p}(\boldsymbol{x}_n)] \qquad \boldsymbol{W}(\boldsymbol{x}) = diag\left[\omega_k(\boldsymbol{x}_j, \boldsymbol{x})\right], \quad j = 1, ..., n \qquad (5)$$

and the approximation $\hat{u}(\boldsymbol{x})$ to function $u(\boldsymbol{x})$ in Ω_k in the following form:

$$\hat{u}(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\,\boldsymbol{u}^h. \qquad (6)$$

Now, we can define "shape functions" $\boldsymbol{N}^t(\boldsymbol{x})$ valid in the subdomain Ω_k in terms of $\boldsymbol{p}^t$, and matrices $\boldsymbol{A}^{-1}$ $\boldsymbol{B}$:

$$\boldsymbol{N}^t(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x})\boldsymbol{C}(\boldsymbol{x}); \qquad \boldsymbol{C}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x}). \qquad (7)$$

As we have presented, in this numerical approach the local values of the approximating function do not fit the nodal unknown values $(\hat{u}(\boldsymbol{x}_j) \neq u_j^h)$. If the weighting function selected is constant and equals the unity, we reproduce the standard least squares method. On the other hand, in the particular case when $n = m$, a finite element type approximation is recovered and there is no effect of weighting[14].

3. ENRICHMENT OF MESHLESS METHODS

The enrichment of a numerical approach is an effectiveness technique to avoid a high refinement of the mesh in some problems in computational mechanics. Generally speaking, the enrichment or enhancement process consists of introducing some known information about the solution of the problem in the trial functions. These methods were proposed in mid-seventies for finite elements approaches and succesfully applied to different problems[15].

In meshless methods, this enhancement process is simpler and easier to perform that in finite elements formulations[15,16]. It may be carried out in different ways: by adding the set of enrichment functions to the trial functions ("extrinsic enrichment"), or by including the enrichment functions in the weighted least-squares interpolation basis ("intrinsic enrichment").

In this paper we focuse our attention in the extrinsic enrichment of meshless methods based on weighted least squares methods. Thus, if we know the dependence of the solution with a function (or a set of functions), we can define the approximation $\hat{u}$ to the function u in the form

$$\hat{u}(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{x})\boldsymbol{\alpha} + \sum_{j=1}^{n_f} \Big(k_j F_j(\boldsymbol{x}) \Big) \tag{8}$$

being $\boldsymbol{p}(\boldsymbol{x})$ a complete polynomial basis in the spatial coordinates, $\boldsymbol{\alpha}$ the vector of unknowns associated to the basis, n_f the number of enrichment functions $F_j(\boldsymbol{x})$ that we introduce, and k_j $(j = 1, n_f)$ the global unknowns associated with functions F_j. Next, coefficients $\boldsymbol{\alpha}$ can be obtained in a similar way to those in (1) by using the WLS methodology presented in the previous section, although additional terms appear from the inclusion of these enhancement functions. Now, if we redefine the functional J in (3) in terms of the new approximation (8), its minimization allows to obtain the following expression for $\hat{u}$

$$\hat{u}(\boldsymbol{x}) = \sum_{i=1}^{n} N_i(\boldsymbol{x}) \left[u_i^h - \sum_{j=1}^{n_f} \Big(k_j F_j(\boldsymbol{x}_i) \Big) \right] + \sum_{j=1}^{n_f} \Big(k_j F_j(\boldsymbol{x}) \Big) \tag{9}$$

where shape functions $N_i(\boldsymbol{x})$ $(i = 1, n)$ are identical to previously defined in (7).

Extrinsic enrichment of meshless methods can also be performed in a simpler and computationally faster way than (9) by using partition of unity methods[15]. In this case, the approximation is modified by adding a basis of enrichment functions extrinsically to the existing WLS approximation. These new functions can be polynomials of higher order than the WLS interpolants basis, or functions contained in the exact solution of the problem, which are smoothly added to the MLS approximation by multiplying it by a partition of unity[15,16]. Since shape functions in WLS approximations are partitions of unity, this extrinsic enrichment procedure frequently takes the form

$$\hat{u}(\boldsymbol{x}) = \sum_{i=1}^{n} N_i(\boldsymbol{x}) \left(u_i^h + \sum_{j=1}^{n_f(i)} k_{ij} F_j(\boldsymbol{x}) \right) \tag{10}$$

where $nf(i)$ is the number of enrichment functions of nodal point i (in general, nf can be different for each nodal point), and k_{ij} are unknowns coefficients associated to the basis of enrichment functions. This method is a good technique for local enrichment: since consistency is assured by the partition of unity given by the shape functions formed with the basis of WLS interpolants, the enrichment of the approximation may be performed locally by adding extrinsically functions of a new basis. It should be noted that this enrichment has to be added to each node into the region to be enhanced[15].

The intrinsic enrichment consists of the inclusion of special functions in the complete polynomial basis of the weighted least squares interpolation. This method does not require the introduction of additional unknowns to the numerical scheme, in contrast to the extrinsic enrichment, although some additional computational effort is required to obtain shape functions $\boldsymbol{N}(\boldsymbol{x})$ since the increase of the number of functions of the basis, and some problems of ill-conditioning can arise. On the other hand, it is very difficult to perform local enrichment with this method, and since it is not possible to delete functions from the basis because it produces discontinuities in the approximation, a special technique must be used to mix nodal points with different basis functions[15].

In this paper, we will consider the extrinsic enrichment technique based on partition of unity method for weighted least squares meshless methods. Although the number of degrees of freedom increases with its use (if a nodal point i is enriched, then the total number of unknowns to obtain for it is $nf(i)+1$, instead of 1), this enhancement procedure can be applied locally in different parts of the approximation, being quite easy its implementation in a meshless code.

4. DISCRETIZED EQUATIONS OF THE NUMERICAL MODEL

In this section we review the obtention of the discretized equations of the numerical scheme for a boundary value problem. Let $\mathcal{A}$ and $\mathcal{B}$ be two differential operators, Ω the domain of the problem and Γ its boundary ($\Gamma = \Gamma_t \cup \Gamma_u$). In these terms, a scalar boundary value problem can be written as,

$$\mathcal{A}(u) = b \quad \text{in} \quad \Omega, \qquad \mathcal{B}(u) = t \quad \text{in} \quad \Gamma_t, \qquad u - u_p = 0 \quad \text{in} \quad \Gamma_u, \tag{11}$$

where u is the solution, b and t represent the actions over Ω and along the boundary Γ_t, and u_p is the prescribed value of u along Γ_u.

A variational form of this problem can be obtain by using the weighted-residuals method in terms of the trial approximation function $\hat{u}$ of the unknown u, as

$$\int_\Omega W_j\,[\mathcal{A}(\hat{u}) - b]d\Omega + \int_{\Gamma_t} \widehat{W}_j\,[\mathcal{B}(\hat{u}) - t]d\Gamma + \int_{\Gamma_u} \widehat{\widehat{W}}_j\,[\hat{u} - u_p]d\Gamma \;=\; 0, \qquad j = 1, ..., n_p \tag{12}$$

which must hold for all members of a set of n_p functions W_j, $\widehat{W}_j$ and $\widehat{\widehat{W}}_j$ of a suitable class of test functions defined on Ω, Γ_t and Γ_u[10,14].

Next, the selection of different test functions in this variational form allows to obtain different numerical formulations. Particularly, in order to take advantage of the meshless character of the numerical approximation, we can use Dirac deltas as test functions, that is a point-collocation approach ($W_j = \widehat{W}_j = \widehat{\widehat{W}}_j = \delta_j$)[14]. Other authors[5,6,7,8] have proposed integral methods, which require some kind of auxiliar grid to evaluate the resulting integrals. In the case of the point-collocation scheme, the following set of equations is obtained:

$$\begin{aligned} [\mathcal{A}(\hat{u})]_j - b_j = 0, &\quad j = 1, ..., n_p \quad \text{in} \quad \Omega \\ [\mathcal{B}(\hat{u})]_j - t_j = 0, &\quad j = 1, ..., n_p \quad \text{in} \quad \Gamma_t \\ \hat{u}_j - u_p = 0, &\quad j = 1, ..., n_p \quad \text{in} \quad \Gamma_u \end{aligned} \tag{13}$$

Now, if the numerical approximation is not enriched, given a set of n_p shape functions defined on Ω, approximation $\hat{u}$ to the solution u can be discretized as,

$$\hat{u} = \sum_{i=1}^{n_p} N_i u_i^h = \boldsymbol{N}^t \boldsymbol{u}^h, \tag{14}$$

being n_p the total scattered points of the solution domain, and where functions $\boldsymbol{N}^t$ are obtained by using the weighted least-squares methodology in (7). Finally, we obtain the following system of linear equations:

$$\boldsymbol{K}\boldsymbol{u}^h = \boldsymbol{f} \tag{15}$$

where coefficient matrix $\boldsymbol{K}$ is sparse but not necessary symmetric ($K_{ji} = [\mathcal{A}(N_i)]_j + [\mathcal{B}(N_i)]_j$), $\boldsymbol{f}$ contains the contributions from terms b and t and prescribed values u_p, and $\boldsymbol{u}^h$ contains the unknown values of the function evaluated in nodal points. In the case of introduce enrichment functions to define the weighted least-squares interpolation, it will be necessary to use more collocation points in (13) in order to obtain all the unknowns.

5. NUMERICAL EXAMPLES

In previous works we have studied the feasibility of the weighted least squares meshless methods for the analysis of potential problems in the electrical engineering field[11,12]. In this section, we focuse our attention on the comparison of the results of two numerical tests of potential problems when a standard weighted least-squares approach and an enriched one are used.

1D Numerical Test

We will consider the following boundary value problem, which represents the Laplace equation with revolution symmetry when sphericals coordinates are used:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0\ , \quad 0.01 \le r \le L\ ; \qquad V(0.01) = 1, \quad V(L) = 0, \tag{16}$$

which analytical solution is given by $V(r) = \frac{0.01}{L-0.01}\left(\frac{L}{r} - 1\right)$.

Thus, if we consider for the enrichment function $1/r$ and for a given set of n_p trial functions N_i defined on the domain, the approximation $\hat{V}$ to the solution V can be written in the form:

$$\begin{aligned}
&\text{Without enrichment functions:} \quad \hat{V} = \sum_{i=1}^{n_p} N_i u_i^h. \\
&\text{With enrichment functions:} \quad \hat{V} = \sum_{i=1}^{n_p} N_i (u_i^h + a_{1,i} F_1(r)). \\
&\text{Total enrichment:} \quad F_1(r) = 1/r, \quad 0.01 \le r \le L. \\
&\text{Local enrichment:} \quad \begin{cases} F_1(r) = 0, & 0.01 \le r < r_0; \\ F_1(r) = 1/r, & r_0 \le r \le r_1; \\ F_1(r) = 0, & r_1 < r \le L. \end{cases}
\end{aligned} \tag{17}$$

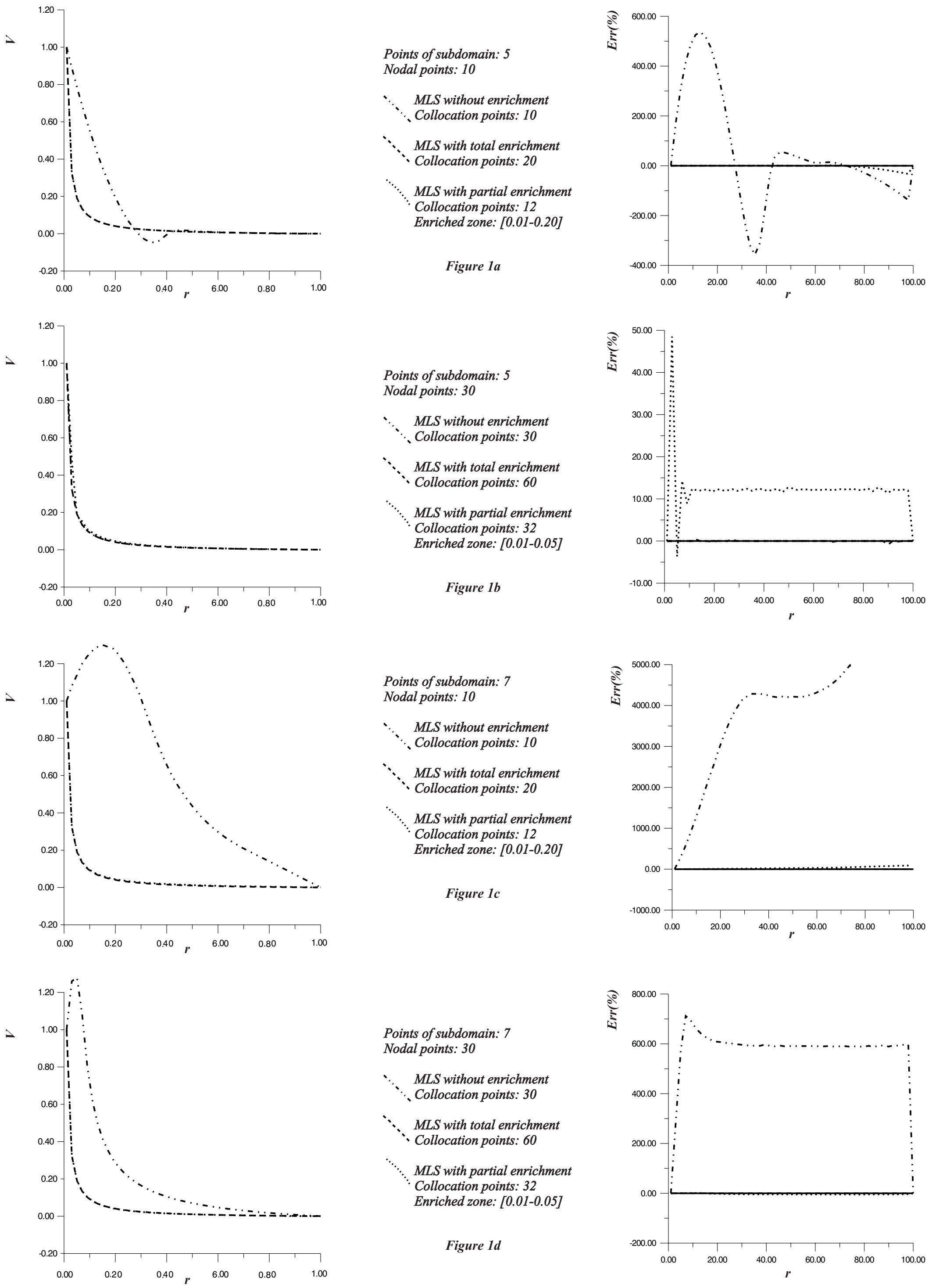

Fig. 1.1D numerical test: Comparison of results obtained by using different WLS formulations.

being n_p the total number of nodal points of the solution domain. The weighting function used[14] is the truncated gaussian with $\alpha = 0.25$ and $k = 1.1$, and the total length of the domain L is 1.

In figure 1, we present the comparison of the numerical approximations obtained by using weighted least squares interpolants with and without enrichment functions, for a different number of scattered nodal points (10 and 30) and different number of points of the subdomains (5 and 7 nodes). In this test, it is shown that the use of enhanced approaches allows to obtain very good results, particularly when a local enrichment is performed. As it is shown, in comparison with standard weighted least squares methodology, an appropriate local enhancement successfully improves the approximation only adding a few collocation points and, therefore, with a minimal increase in the computational cost.

2D Numerical Test

In this example we study the performance of the enriched weighted least squares approach in a 2D problem. We have selected the following 2D axisymmetric test problem which have been solved by using 2D shape functions:

$$\Delta V = 0, \quad L_0 \leq \sqrt{x^2+y^2} \leq L \; ; \qquad V(\sqrt{x^2+y^2} = L_0) = 1, \;\; V(\sqrt{x^2+y^2} = L) = 0, \tag{18}$$

The analytical solution is given by $V(x,y) = \dfrac{\ln L - \ln(\sqrt{x^2+y^2})}{\ln L - \ln L_0}$. The approximation $\hat{V}$ to the solution V can be written in the same form as in the previous example in (17) but using the enrichment function $\ln(\sqrt{x^2+y^2})$.

In figures 2a and 3a it is shown the surface obtained from the numerical approximation when $L_0 = 10^{-4}$ and $L = 1$, the distribution of the nodal points used (Fig. 2b and 3b), and in figures 2c, 2d, 3c and 3d we present a comparison between the analytical solution and the approximations obtained by using a weighted least squares approach with or without enrichment functions along a radial line, and with a different number of nodal points (49 and 225) and a different number of points of the subdomains (6 and 16 nodes). As it also can be seen in this example, the use of enhanced approaches allows to obtain very good approximations. It is particularly interesting the use of partial enrichment in some parts of the domain when some information about the solution is known, since it is possible to improve the numerical approximation with a low computing effort.

6. CONCLUSIONS

In this paper, we have presented a weighted least squares interpolation method combined with a point collocation approach for the analysis of potential problems in electrical engineering applications. Furthermore we have explored a extrinsic enrichment methodology in order to improve the numerical solution. This enhacement can be carried out in all of the domain or only in some specific parts of it.

The meshless character of these weighted least squares approximations may represent an important improvement in the computational analysis of some engineering problems in which the use of standard numerical methods is precluded due to large computing efforts required in the discretization process. Furthermore, the enrichment methodology allows to include in the numerical approximation some information related to the type of the functions of the solution, when this is known. In this paper, we have presented some 1D and 2D numerical tests, and the good performance of this technique can be noticed in the results presented, particularly if local enrichment is used. At present, we are working in some mathematical and numerical aspects of this meshless technique to assess the practical feasibility of this kind of approaches.

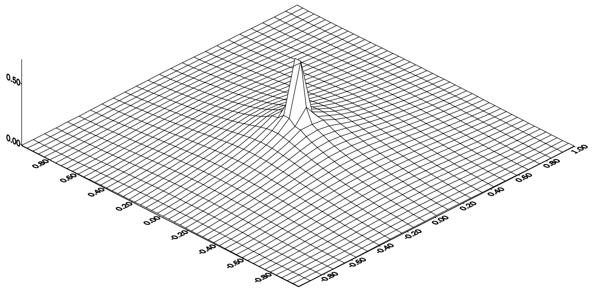

Figure 2a

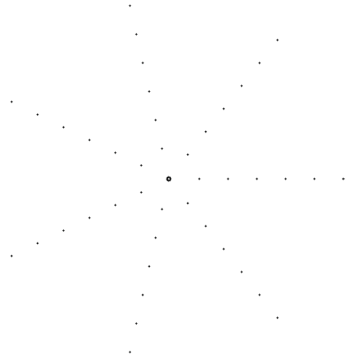

Figure 2b

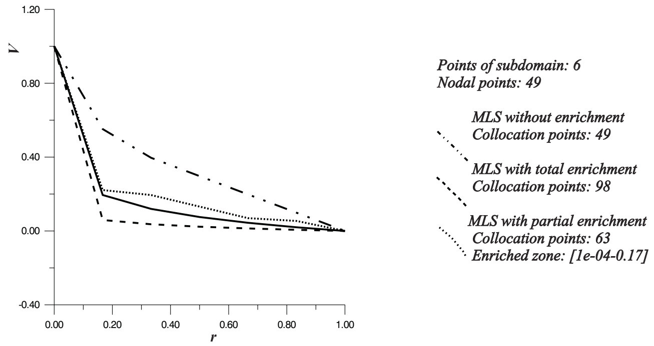

Figure 2c

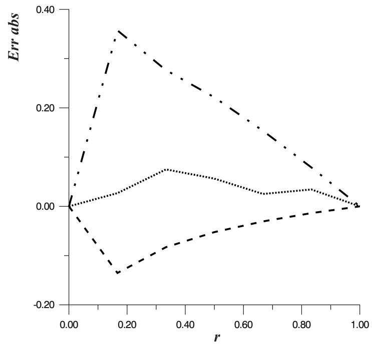

Figure 2d

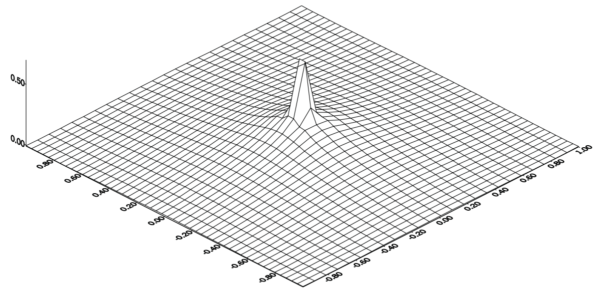

Figure 3a

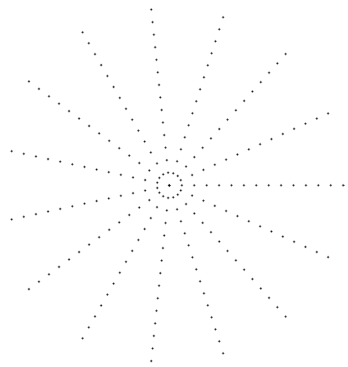

Figure 3b

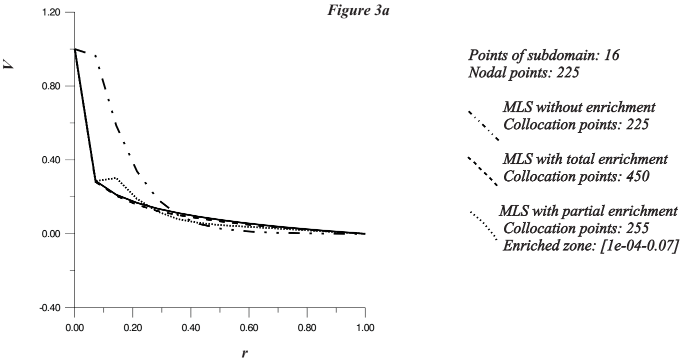

Figure 3c

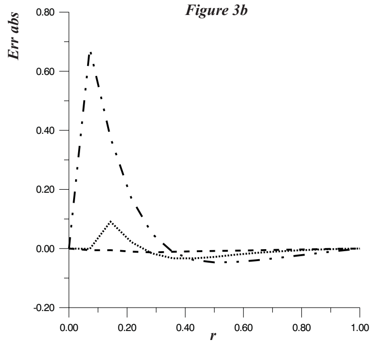

Figure 3d

Figs. 2.&3. 2D numerical test: 2a): numerical approximation of the solution, 2b): distribution of 49 points used, 2c) and 2d): comparison of results obtained by using different weighted least squares formulations. (in Fig. 3, graphs and results for a distribution of 225 points).

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