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GALERKIN, LEAST-SQUARES AND G.L.S. NUMERICAL APPROACHES FOR CONVECTIVE-DIFFUSIVE TRANSPORT PROBLEMS IN ENGINEERING

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Abstract.
In this paper, a study of three FE numerical formulations (Galerkin, Least Squares and Galerkin/Least Squares) applied to the convective-diffusive problem is presented, focusing our attention in high Péclet-number problems. The election of these three approaches is not arbitrary, but based on the relations among them. First, we review the causes of appearance of numerical oscillations when a Galerkin formulation is used. Contrasting with the nature of the Galerkin method, the Least Squares method has a rigorous foundation on the basis of minimizing the squared residual, which ensures best numerical results. However, this improvement in the numerical solution implies an increment of the computational cost, which normally becomes unaffordable in practice. The last one, known as GLS, is based on a stabilization of the Galerkin Method. GLS can be interpreted as a combination of the last two methods, being one of the most used stabilization methods nowadays. This method arises as the best one to solve convective problems, because it unifies the advantages of the Galerkin and Least Squares Methods and cancels its disadvantages.
For each numerical method, a brief review is presented, the continuity and derivability requirements on the trial functions are stablished, and the reasons of its behaviour when the method is applied to the convection-diffusion problem with high velocity fields are examined. Furthermore, special attention will be devoted to the consequences of relaxing the variational requirements in the LS and GLS methods. Finally, several 1D and 2D examples are presented.

1 INTRODUCTION

On the contrary to Computational Mechanics, there are no unifying numerical methods available for the wide range of problems in Fluid Mechanics. Actually, in more than three decades of development of the Finite Element Method, we can say that neither an universal approach for the different fluid problems has been found, nor any specific formulation has been able to offer good results for a given problem when varying its physical parameters [1]. In the particular case of the convective-diffusive transport problem (which can also be interpreted as the linear version of the Navier-Stokes equations), it is easy to obtain good results with high diffusive terms, but it turns out to be quite difficult in the convective dominant situations [2].

In this paper, a study of three numerical formulations (Galerkin, Least Squares and Galerkin/Least Squares) applied to the convective-diffusive problem is presented, focusing our attention in convective dominant problems. The election of these approaches is not arbitrary, but based, as we will see, on the close relations among them and with the intention to compare their main characteristics.

First, we will briefly study the well-known Galerkin method. This is a very simple weighted residual method and it has been successfuly used in many engineering applications, specially in Solid Mechanics. However, this method yields oscillatory solutions when it is applied it to convective dominant problems.

These instabilities bring us to turn our attention to the next two formulations, which do not present this oscillatory behaviour: the Least-squares finite element method (LSFEM), and the Galerkin/least-squares method (GLS). Both attain the stability properties for high Péclet numbers, but they get it by different ways: the first one is itself a stable formulation, while the second is a Galerkin stabilized formulation. Comparison between both methods is made through the analysis of their advantages and disadvantages.

The Least-squares finite element method, because of the symmetry of its formulation, is specially suited for obtaining accurate solutions with first-order differential operators, as the convective differential operator. A study of the theoretical foundations and formulation of the Least-squares finite element method is made, and the main characteristics of the method are explained.

Next we focus our study on the Galerkin/Least-squares method, which belongs to the family of the stabilized methods based on adding a stabilization term to the Galerkin method. This stabilization term is the least squares form of the residual of the equation evaluated elementwise multiplied by a stabilization parameter. This brings us to stablish the relations among the three formulations presented.

These relations and the behaviour of the three numerical approaches in convective-dominant transport problems are shown through some rough numerical tests. Finally, the conclusions will summ up the most important remarks analyzed in this work.

We will introduce the numerical study of the three methods by considering a generic problem defined by the domain Ω, its boundary Γ and the differential operators $\mathcal{L}(\cdot)$

and $\mathcal{M}(\cdot)$ of the differential equation and the boundary conditions respectively,

$$\begin{array}{c} \mathcal{L}(u) = f \quad \text{in } \Omega \\ + \\ \mathcal{M}(u) = 0 \quad \text{in } \Gamma \end{array} \tag{1}$$

However, we are interested on the application of these three formulation to the convective-diffusive problem. Many engineering processes are modeled by the convective-diffusive transport differential equation. Thus, phenomena like temperature distribution in any solid or fluid, evolution of a substance concentration in a porous media or pollutant dispersion in a fluid are efficiently described by this equation.

Thus, let us consider now the convective-diffusive stationary problem of finding a scalar field $u = u(\boldsymbol{x})$, such that

$$\begin{array}{l} \mathbf{a} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} u) = f \qquad in\ \Omega \\ \\ u = u_0 \quad in\ \Gamma_1 \\ \\ \boldsymbol{\nabla} u \cdot \mathbf{n} = \gamma - cu \quad in\ \Gamma_2 \\ \\ \Gamma_1 \cup \Gamma_2 = \Gamma \\ \\ \Gamma_1 \cap \Gamma_2 = 0 \end{array} \tag{2}$$

where $\boldsymbol{a}(\boldsymbol{x})$ is the given flow velocity (with $\boldsymbol{\nabla} \cdot \boldsymbol{a} = 0$ in Ω), $k = k(\boldsymbol{x})$ is the diffusivity and $f(\boldsymbol{x})$ is a prescribed source function. Γ_1 is the part of the boundary where u is fixed, and Γ_2 is the part where the flux is prescribed.

2 THE GALERKIN FINITE ELEMENT METHOD

In order to introduce this numerical approach, it is necessary to define a variational formulation of the generic problem, that it can be set as follows: find u so that problem (1) is satisfied in the meaning of weighted residuals

$$\int_\Omega (\mathcal{L}(u) - f) w \; d\Omega + \int_\Gamma (\mathcal{M}(u) - t) w_\Gamma \; d\Gamma = 0 \qquad \forall\ w,\ w_\Gamma \tag{3}$$

for all members w and w_Γ of suitable sets of test functions defined in Ω and Γ respectively [3]. Now if we discretize the domain in finite elements and we approximate the solution u as a linear combination of basis functions, we obtain

$$u \simeq \tilde{u} = \sum_{j=1}^{n} u_j \varphi_j \tag{4}$$

As a consecuence of this approximation, the expression (3) is no longer verified for all w and w_Γ, and therefore the scheme results in the integral over the domain of the residual of the equation multiplied by a weighted function. In the Galerkin method, those tests functions are defined aqual to th etrial functions:

$$w_i = \varphi_i \tag{5}$$

It is important to remark that there is no theoretical foundation in this election, and therefore good results could not be assured. Indeed, it is well-known the appearance of numerical oscillations, when this method is applied to convective dominant fluid problems [2]. The reason is that convection operators are of first-order, and thus non-self adjoint; as a result, the Galerkin method yields to a system with a hemisymmetric matrix, which lead to the appearance of spurious oscillations the numerical solution [1,2]. We will come back to this point in the numerical examples.

Now, if we particularize the Galerkin method to the convective-diffusive transport problem defined by (2), considering the weak form of the formulation,

$$\sum_{j=1}^{n} \left[\int_\Omega (\varphi_i \boldsymbol{a} \cdot \boldsymbol{\nabla} \varphi_j + k \boldsymbol{\nabla} \varphi_j \cdot \boldsymbol{\nabla} \varphi_i) d\Omega + \int_{\Gamma_2} ck \varphi_j \varphi_i d\Gamma_2 \right] u_j =$$

$$\int_\Omega f \varphi_i d\Omega + \int_{\Gamma_2} \gamma k \varphi_i d\Gamma_2 \quad ; \qquad i = 1, ..., n \tag{6}$$

It is clear that the trial functions used to approximate the solution of problem (6), have to belong to $H^1(\Omega)$, that is, C^0 elements should be used [4].

3 THE LEAST-SQUARES FINITE ELEMENT METHOD

After this brief review to the Galerkin method and the spurious oscillations it causes in the resolution of convective dominant problems, we will present the Least-squares finite element method. Advocates of this method, emphasize that it represents a good alternative to existing schemes in fluid mechanics specially because of its universality [1,5,6]. That is, instead of employing certain formulations with different principles and structures for each fluid problem, the LSFEM represents a unified approach for the numerical solution of all types of partial differential equations.

LSFEM was developed in the seventies with origin on the least-squares interpolation method [7,8], and it is based on minimizing the residual in the differential equations and the boundary conditions in the least-squares sense.

In the previous section, a special interest was set on remarking the arbitrary selection of the test functions in the Galerkin method. On the contrary, the formulation of the LSFEM has a firm theoretical basis, which should ensure the best numerical aproximation to the solution of the problem.

Thus, let us introduce the LSFEM formulation for the generic problem (1). As in the Galerkin method, the solution of the problem is aproximated by a linear combination of the trial functions (4). As a result of this approximation, the differential equation is no longer verified, and the residual of the equation is given by:

$$\mathcal{R}(\tilde{u}) = \mathcal{L}(\tilde{u}) - f \neq 0 \tag{7}$$

Let us consider the functional $\mathcal{J}$, which represents a measure of the squared error produced in the approximation of the solution, multiplied by a generic weighted function W

$$\mathcal{J}(\tilde{u}) \;=\; \|r(\tilde{u})\|_0^2 \;=\; \int_\Omega (\mathcal{L}(\tilde{u}) - f)^2 W \; d\Omega \;, \tag{8}$$

where $\tilde{u}$ is given by (4).

As we are interested to look for the best approximation to the exact solution of the problem, we can minimize this squared error, which is analogous, refering us to the least-squares interpolation problems, to minimize the squared distance between $\mathcal{J}(\tilde{u})$ and f, that is,

$$\frac{\partial}{\partial u_i}\Big[\mathcal{J}\big(\sum_{j=1}^{n} u_j \varphi_j\big)\Big] \;=\; 0 \qquad ,\; i = 1, ..., n \tag{9}$$

Now, it we take into account that $\mathcal{J}(\cdot)$ is a linear differencial operator, we can adopt without loss of generality $W = 1$ and the expresion (9) leads to

$$\sum_{j=1}^{n}\Big[\int_\Omega \mathcal{L}(\varphi_j)\mathcal{L}(\varphi_i)\; d\Omega\Big] u_j = \int_\Omega f_i \mathcal{L}(\varphi_i)\; d\Omega \qquad ,\; i = 1, ..., n \tag{10}$$

Up to this point, the relation that exists between the LSFEM and the variational formulations is clear: the LSFEM leads us to a variational boundary-value poblem in which the test functions are images of the trial functions under the operator $\mathcal{L}(\cdot)$ [4]. Therefore the election of these test functions is not arbitrary, as it was in Galerkin method.

Particularizing the LSFEM formulation for a generic problem (10) to the convective-diffusive transport equation given by equation (2), we obtain

$$\sum_{j=1}^{n}\Big[\int_\Omega \big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_j - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\varphi_j)\big)\big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_i - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\varphi_i)\big)\; d\Omega\Big] u_j \;=$$

$$= \int_\Omega f_i\Big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_i - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\varphi_i)\Big)\ d\Omega$$

$$i = 1, ..., n \tag{11}$$

The boundary conditions of the problem can be approximate in the least squares sense in the same manner.

The most remarkable characteristic of the LSFEM related with the convective dominant problems, is its efficiency . Contrary to the Galerkin method, the LSFEM is naturally suited for first-order differencial operators, which are non-self-adjoint [1]. While in Galerkin and other conventional methods, the first-order differencial operators lead to nonsymmetric matrices, the LSFEM always leads to symmetric andpositive-definite matrices.

An alternative interpretation and formulation of the least-squares problem is shown in references [1,4] to explain this issue. If the trial functions are sufficiently smooth, by succesive applications of the divergence theorem, equation (10) yields to

$$\sum_{j=1}^{n}\left[\int_\Omega \mathcal{L}^*\mathcal{L}(\varphi_j)\ \varphi_i\ d\Omega\ +\ \int_\Gamma \mathcal{L}(\varphi_j)\ \varphi_i\ d\Gamma\right]u_j = \int_\Omega \mathcal{L}^*(f_i)\ \varphi_i\ d\Omega\ +\ \int_\Gamma f_i\ \varphi_i\ d\Gamma$$

$$i = 1, ..., n \tag{12}$$

where $\mathcal{L}^*(\cdot)$ is the adjoint operator of $\mathcal{L}(\cdot)$; obtaining the Euler-Lagrange equation

$$\mathcal{L}^*\mathcal{L}(u) = \mathcal{L}^* f \qquad in\ \Omega. \tag{13}$$

Observe that the operator $\mathcal{L}^*(\cdot)\mathcal{L}$ is formaly self adjoint, even though $\mathcal{L}$ is non-self-adjoint. By comparing (1) with (13), we see that the LSFEM converts a difficult non-self-adjoint first order operator in to a relatively easy self-adjoint problem, where no spurious oscillations will appear in the solution.

This stability advantages related with the symmetry of the formulation of the LSFEM, are overshadowed by the increase on the continuity and derivability requirements for the trial functions employed, and as a result, by the important increase on the computational cost.

If we focus our attention on the expression (11), it is clear that that the admissible trial functions have to be in $H^2(\Omega)$, that is, C^1 elements should be used [1,4]). In section 5, we will appreciate the complexity of this trial functions and the efects of employing trial functions that do not satisfy this strong requirement.

4 THE GALERKIN/LEAST-SQUARES FINITE ELEMENT METHOD

Because of the oscillatory results attained by the Galerkin method to the convective dominant transport problems, since eighties several alternative weighted residual formulations have been proposed to enhance stability. The common methodology to this stabilized methods consists of adding a stabilizing term to the original Galerkin formulation, which is the sum of the integrals over each element of the residual of the equation to be solved, multiplied by an operator applied to the test functions and by a parameter [9,10].

The Galerkin/Least-squares stabilized method was first proposed in 1989 by Hughes et al. [11], as a general methodology of the SUPG method [12]. Indeed, in the hyperbolic case or for piecewise linear elements, both methods become identical [11].

The form of the stabilizing term added differs from one stabilized method to another. In particular for the GLS method, the operator over the test functions is the differential operator of the original problem. Therefore, the stabilizing term added is the least squares form of the residual evaluated elementwise, multiplied by a numerical parameter which is always positive and has dimension of time [9]. For this reason, the GLS method can be considered as a combination of the Galerkin and Least-squares method, the last one providing the stabilizing properties to the method, and both terms can be clearly identificated in the general formulation of the method [13] aplied to the generic problem (1). That is

$$\int_{\Omega}\Big(\mathcal{L}(\tilde{u})-f\Big)\varphi_i\;d\Omega\;+\;\boldsymbol{\tau}\int_{\tilde{\Omega}}\Big(\mathcal{L}(\tilde{u})-f\Big)\mathcal{L}(\varphi_i)\;d\tilde{\Omega}=0\qquad,\;i=1,...,n \tag{14}$$

where $\tilde{\Omega}=\bigcup\Omega_e$ and Ω_e are the element interiors.

The result of this addition is a weighted residual formulation like (3) where the test functions are a combination of those of the Galerkin and Least-squares methods [14,15]

$$W_i\;=\;\varphi_i\;+\;\boldsymbol{\tau}\mathcal{L}(\varphi_i) \tag{15}$$

It is important to remark that φ_i must satisfy the continuity requirements of the Galerkin method over the entire domain Ω, while $\tau\mathcal{L}(\varphi_i)$ only have to satisfy the continuity requirements of the LSFEM on the element interiors, and not across element boundaries. Therefore, additional terms do not upset continuity requirements of the original variational formulation because they are evaluated elementwise [13,16]. This implies an important advantage over LSFEM, because it will allow to use simpler trial functions.

If we apply the GLS method to the convection-diffusion transport problem (1), and take into account the weak form in the Galerkin term and the approximation (4), we obtain

$$\sum_{j=1}^{n}\Bigg[\int_{\Omega}(\varphi_i \boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_j + k\boldsymbol{\nabla}\varphi_j\cdot\boldsymbol{\nabla}\varphi_i)d\Omega + \int_{\Gamma_2} kc\varphi_j\varphi_i\, d\Gamma_2 + \\ + \boldsymbol{\tau}\int_{\tilde{\Omega}}\Big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_j - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\varphi_j)\Big)\Big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_i - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\varphi_i)\Big)\, d\tilde{\Omega}\Bigg]u_j \;= \\ = \int_{\Omega} f_i\varphi_i d\Omega \;+\; \int_{\Gamma_2} k\gamma\varphi_i\, d\Gamma_2 \;+\; \boldsymbol{\tau}\int_{\tilde{\Omega}} f_i\Big(\boldsymbol{a}\cdot\boldsymbol{\nabla}\varphi_i - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\varphi_i)\Big)\, d\tilde{\Omega} \\ i = 1, ..., n \tag{16}$$

If we particularize the previous remark about the continuity requirements of the trial functions in GLS to the convective-diffusive transport problem, it is clear that the admissible functions are required to be in $H^1(\Omega)$ and in $H^2(\Omega_e)$. Thus, C^0 elements with first-order derivatives continue over Ω_e should be used.

In the previous section we have studied the LSFEM and remarked its theoretical basis. In spite of the widespread success of all the stabilized methods, they have been opened to criticism because of their lack of theoretical foundations. For the sake of clarifying this question, Hughes [17] has proved the derivation of stabilized methods from subgrid scales modeling concepts. In this work, Hughes also explained the evaluation of the numerical parameter τ from the element Green's function. The optimal evaluation of this parameter is still an open question in most of all stabilized methods proposed up to this moment.

The effect of the parameter τ in GLS is to tune the contribution of the least-squares stabilizing term. For this reason, the determination of the stability parameter is confirmed to be a very important ingredient for simulating the convective-diffusive model [9]. The parameter τ should be enough to accomplish a stabilized method, but it must not be further increased because it would provide over-dissipative results as it occurs in LSFEM, loosing the ability to resolve sharp fronts [2].

The stabilization parameter has a different dependence of the parameters of the problem for the entire rank of Péclet numbers [9,10,11,18]. This is due to the different contribution of the convective and diffusive matrix of the Galerkin term in problems with high and low Péclet numbers. This dependence is stablished by [9,11] in the following way:

$$\tau \;=\; \begin{cases} c'\dfrac{h^2}{k} & \text{, for low Péclet numbers} \\ c\dfrac{h}{|\boldsymbol{a}|} & \text{, for high Péclet numbers} \end{cases} \tag{17}$$

where:
- h is the characteristic measure of the mesh.
- c y c' are constants independent of the Péclet number and of h.

A first definition of τ was given in [11] from the error analysis, and it was later improved in [9,16] to include the effect of the degree of the interpolation polynomial used.

As the test problems we analyze in the numerical examples are convective dominant, we use the definition of the stabilization parameter proposed in [9] for problems with high Péclet numbers. This expression is given by:

$$\tau \;=\; \frac{h}{2 \mid \boldsymbol{a} \mid} \tag{22}$$

where h is the element length in 1D problems and the side of the square in 2D problems discretized with square elements [16], and $\mid \boldsymbol{a} \mid$ is a velocity norm.

5 NUMERICAL EXAMPLES

5.1 1D Numerical tests

In this section we present some numerical tests of the three finite element formulations studied in the previous sections applied to the convective-diffusive transport problem. The one-dimensional tests are designed to asses the performance of this methods when dealing with high convective terms according to the characteristics studied heretofore.

The example that we present is the 1D numerical test defined by (see figure 1):

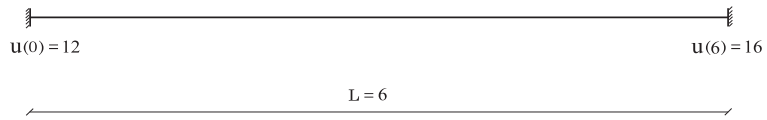

Figure 1. Domain and boundary conditions of the 1D test problem.

$$\begin{aligned} a\frac{du}{dx} - k\frac{d^2u}{dx^2} \;&=\; 0 \quad ; \quad 0 \le x \le L \\ u(x=0) \;&=\; u_0 \qquad ; \qquad u(x=L) \;=\; u_L \end{aligned} \tag{18}$$

with the following parameters: $a = 10$, $k = 0.09$, $L = 6$, $u_0 = 12$ and $u_L = 16$.

In order to better understand the appearance of numerical oscillations in the Galerkin method, and the mechanism to attain stability in the LSFEM and GLS, we present the numerical formulations for the test problem (18). These expressions will clearly show us the characteristics of each elemental matrix and the continuity and derivability requirements over the test functions. Hence, we have

- Galerkin Finite element method.

$$\sum_{j=1}^{N}\left[a\int_{\Omega_e} N_i\frac{\partial N_j}{\partial x}dx + k\int_{\Omega_e}\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}dx\right]u_j = 0 \quad ; \qquad i = 1, ..., N \tag{19}$$

- LSFEM.

$$\sum_{j=1}^{N}\left[a^2\int_{\Omega_e}\Big(\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\Big)d\Omega - ak\int_{\Omega_e}\Big(\frac{\partial^2 N_j}{\partial x^2}\frac{\partial N_i}{\partial x} + \frac{\partial N_j}{\partial x}\frac{\partial^2 N_i}{\partial x^2}\Big)d\Omega + k^2\int_{\Omega_e}\Big(\frac{\partial^2 N_j}{\partial x^2}\frac{\partial^2 N_i}{\partial x^2}\Big)\,d\Omega\right]u_j = 0$$

$$i = 1, ..., N \tag{20}$$

- GLS finite element method.

$$\sum_{j=1}^{N}\left[a\int_{\Omega_e} N_i\frac{\partial N_j}{\partial x}dx + k\int_{\Omega_e}\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}dx\right]u_j +$$

$$+\sum_{j=1}^{N}\tau\left[a^2\int_{\Omega_e}\Big(\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\Big)d\Omega - ak\int_{\Omega_e}\Big(\frac{\partial^2 N_j}{\partial x^2}\frac{\partial N_i}{\partial x} + \frac{\partial N_j}{\partial x}\frac{\partial^2 N_i}{\partial x^2}\Big)d\Omega + k^2\int_{\Omega_e}\Big(\frac{\partial^2 N_j}{\partial x^2}\frac{\partial^2 N_i}{\partial x^2}\Big)\,d\Omega\right]u_j = 0$$

$$i = 1, ..., N \tag{21}$$

As it was seen before, the trial functions have to fulfil certain continuity requirements. That is, in the Galerkin and Galerkin/Least-squares formulation

$$\begin{cases} \varphi_j \in H^1(\Omega) & j = 1, ..., n \\ \varphi_j \in C^0(\Omega) & j = 1, ..., n \end{cases} \tag{23}$$

and, on the other hand, in the Least-squares method,

$$\begin{cases} \varphi_j \in H^2(\Omega) & j = 1, ..., n \\ \varphi_j \in C^1(\Omega) & j = 1, ..., n \end{cases} \tag{24}$$

In order to satisfy these requirements, two clases of shape functions have been chosen. First, for the Galerkin and Galerkin/Least-squares method, we have used Lagrange cuadratic shape functions (see Fig.2). For the Least-squares method, as the continuity requirements are stronger, an Hermite interpolation is used and, as a result of enforcing the first derivatives on the elements boundaries, the trial functions are smooth C^1-polynomials [4] (see Fig.3). Once more, it is important to remark the increase of the

computational cost that the complexity of this trial functions implies in relation to the Lagrange-C^0 functions.

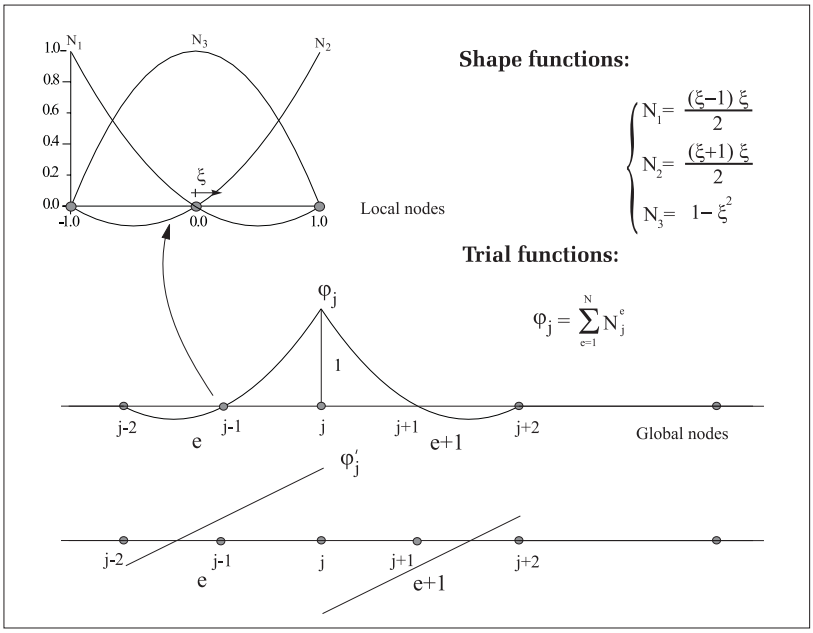

Figure 2. 1D-Lagrange cuadratic elements.

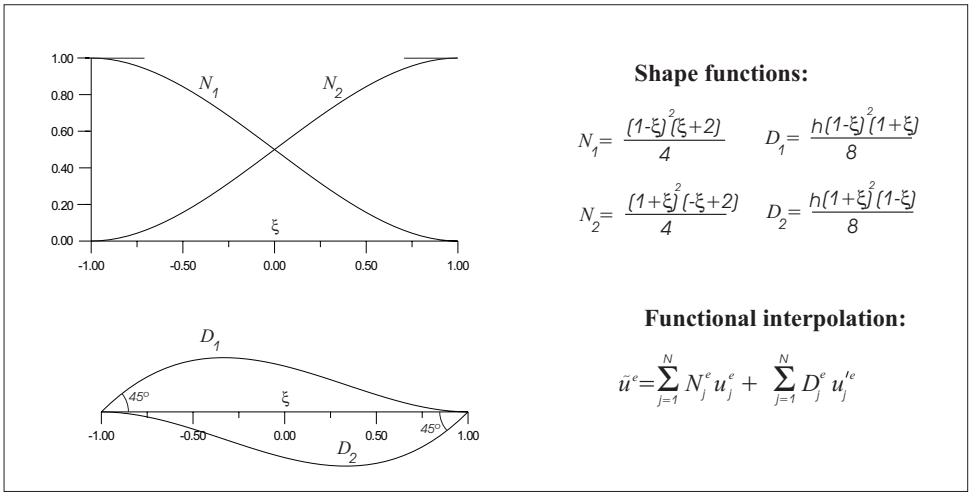

Figure 3. 1D-Hermite cubic elements.

An analysis of the elemental matrix can easily clarify how stability is attained and the contributions of the different terms of each formulation. This rough stability analysis

is based on the relation between the eigenvalues of the matrix and the stability of the numerical solution [2,18].

Let us first analyze the Galerkin formulation of the problem. Using the Lagrange-C^0 cuadratic shape functions in the expression (19), and integrating over an element, we obtain the following element matrix

$$\left[a \begin{pmatrix} -1/2 & 2/3 & -1/6 \\ -2/3 & 0 & 2/3 \\ 1/6 & -2/3 & 1/2 \end{pmatrix} + \frac{k}{h} \begin{pmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{pmatrix} \right] \begin{bmatrix} u_{j-1} \\ u_j \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

In (25) we can clearly observe two terms: the convective and the diffusive term. As it can be seen, the diffusive term is symmetric, while the convective term yields to a non-symmetric matrix and, furthermore, after assembling it, the global convective matrix will be non-symmetric with many zeros on the main diagonal, that is, cuasi-hemisymmetric [18]. This matrix is the origin of the appearance of the complex eigenvalues when convection is more important than diffusion and, as a result, the cause of the spurious oscillations in the numerical solution. If we pay attention to expression (25), we can observe that the remeshing procedure would weight the diffusive component and therefore, this is a way to stabilize the problem (see Fig.4). However, this would imply a inadmissible increase of computational cost in practical cases [3,18].

This lack of stability for practical cases, leads us to the other two formulations studied. To obtain the elemental matrix expresion for the LSFEM, we have to use the shape functions of fig.3, and integrate the equation (20) over an element.

$$\left[\frac{a^2}{h} \begin{pmatrix} 6/5 & h/10 & -6/5 & h/10 \\ h/10 & 2h^2/15 & -h/10 & -h^2/30 \\ -6/5 & -h/10 & 6/5 & -h/10 \\ h/10 & -h^2/30 & -h/10 & 2h^2/15 \end{pmatrix} - ak \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \right.$$

$$\left. + \frac{k^2}{h^3} \begin{pmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{pmatrix} \right] \begin{bmatrix} u_j \\ u_j' \\ u_{j+1} \\ u_{j+1}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

Let us analyze the main caracteristics of (26):

- In this expression, as well as in (20), we can clearly identify three terms: the convective term, the mixed term (which after assembled will be almost null), and the diffusive term.

- All the elemental matrix, even the convective one, are symmetric and, therefore, no complex eigenvalues will appear and the numerical solution will have no spurious oscillations. As we explained in section 3, the LSFEM converts a non-self-adjoint operator (non-symmetric matrix) into a self-adjoint one (symmetric elemental and global matrix).
- Comparing (19) and (20), we can observe that the convective term in Least-squares has the same form of the diffusive term in the Galerkin formulation. As a consecuence, the LSFEM gives too dissipative numerical results for the convective-diffussive model [2,9] (see Fig.5).

We have already analyzed the continuity requirements for the LSFEM (24), but let us see the consecuences of unfulfiling them. If we use the Lagrange-C^0 cuadratic functions for the functional interpolation, we obtain the following elemental matrix

$$\left[\frac{a^2}{h}\begin{pmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{pmatrix} - \frac{ak}{h^2}\begin{pmatrix} -8 & 8 & 0 \\ 8 & 0 & -8 \\ 0 & -8 & 8 \end{pmatrix} + \frac{k^2}{h^3}\begin{pmatrix} 16 & -32 & 16 \\ -32 & 64 & -32 \\ 16 & -32 & 16 \end{pmatrix}\right]\begin{bmatrix} u_{j-1} \\ u_j \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

The scheme of this expression is analogous to (26): three symmetric matrix corresponding to the convective, mixed and difussive terms. But, if we pay attention to the diffusive matrix, we can observe that its rank is one, and therefore, after assembling, the global diffusive matrix will be singular. In addition, if the mesh is refined, the diffusive term is weighted. Hence, the global matrix of the system will become singular. This important issue is called *variational crime* [19,20], and it appears on the diffusive term because it is the term in which we are unfulfiling the continuity requirements.

Galerkin/Least-squares elemental matrix scheme is a combination of Galerkin (19) and Least-squares (20) approach using C^0-elements.

$$\left[a\begin{pmatrix} -1/2 & 2/3 & -1/6 \\ -2/3 & 0 & 2/3 \\ 1/6 & -2/3 & 1/2 \end{pmatrix} + \frac{k}{h}\begin{pmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{pmatrix}\right]\begin{bmatrix} u_{j-1} \\ u_j \\ u_{j+1} \end{bmatrix} +$$

$$+\tau\left[\frac{a^2}{h}\begin{pmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{pmatrix} - \frac{ak}{h^2}\begin{pmatrix} -8 & 8 & 0 \\ 8 & 0 & -8 \\ 0 & -8 & 8 \end{pmatrix} + \frac{k^2}{h^3}\begin{pmatrix} 16 & -32 & 16 \\ -32 & 64 & -32 \\ 16 & -32 & 16 \end{pmatrix}\right]\begin{bmatrix} u_{j-1} \\ u_j \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

It is important to remark, that, as it was explained in section 4, although we are using C^0-elements, we are not commiting a *variational crime*. Thus, the global matrix of the system will never become singular because of the rank of the diffusive matrix of

the least-squares term. The Galerkin term and the stabilization parameter will avoid that, as an effect of the mesh refinement, this diffusive matrix can become dominant (see Fig.4 and 5).

After this brief analysis to the characteristics of the three methods, we present the numerical results. In figure 4, we can see the instabilities of the numerical solution in the Galerkin method, as well as the accurate solution obtained with GLS. Figure 5, shows the numerical solution obtained by the three formulations using Hermite C^1-elements. The best approximation is again obtained by using the GLS method, while the LSFEM yields to over-dissipative solutions.

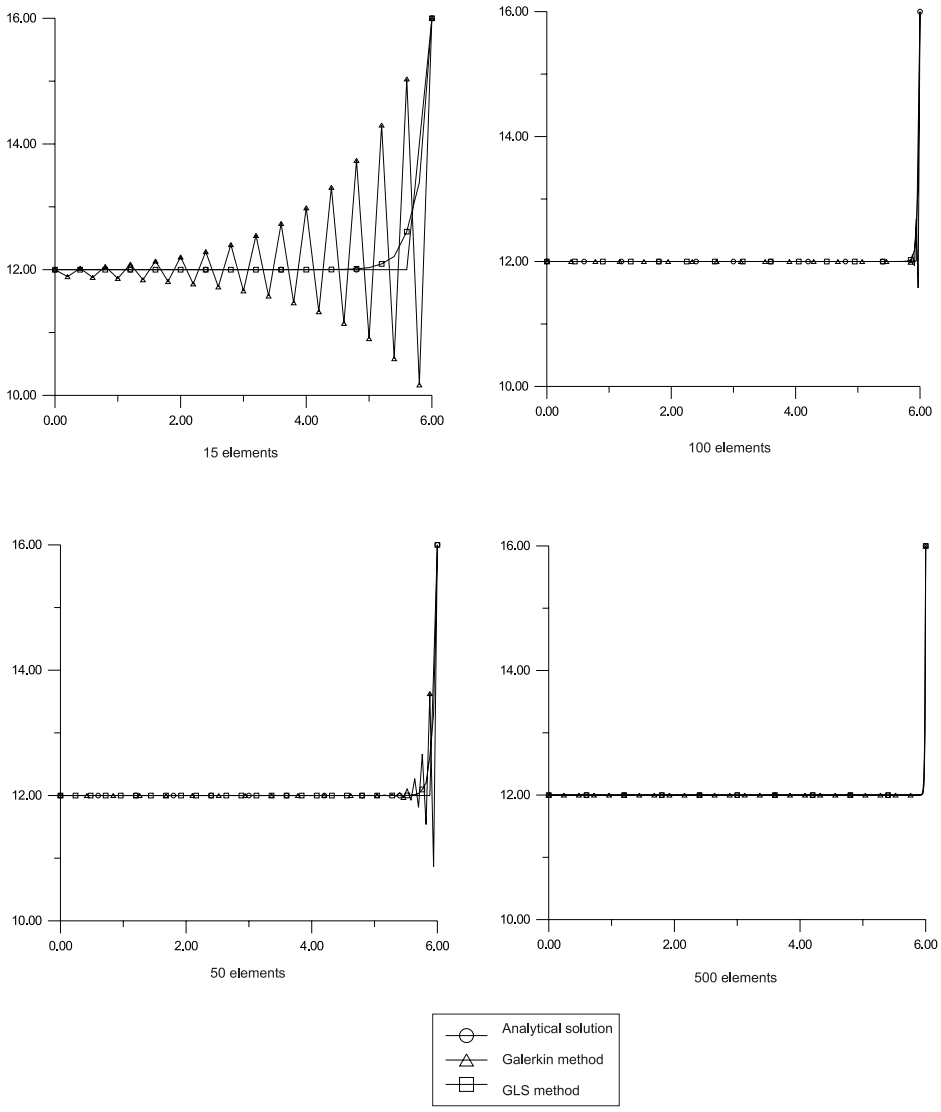

Figure 4. Numerical solutions to the example (18), using C^0-elements.

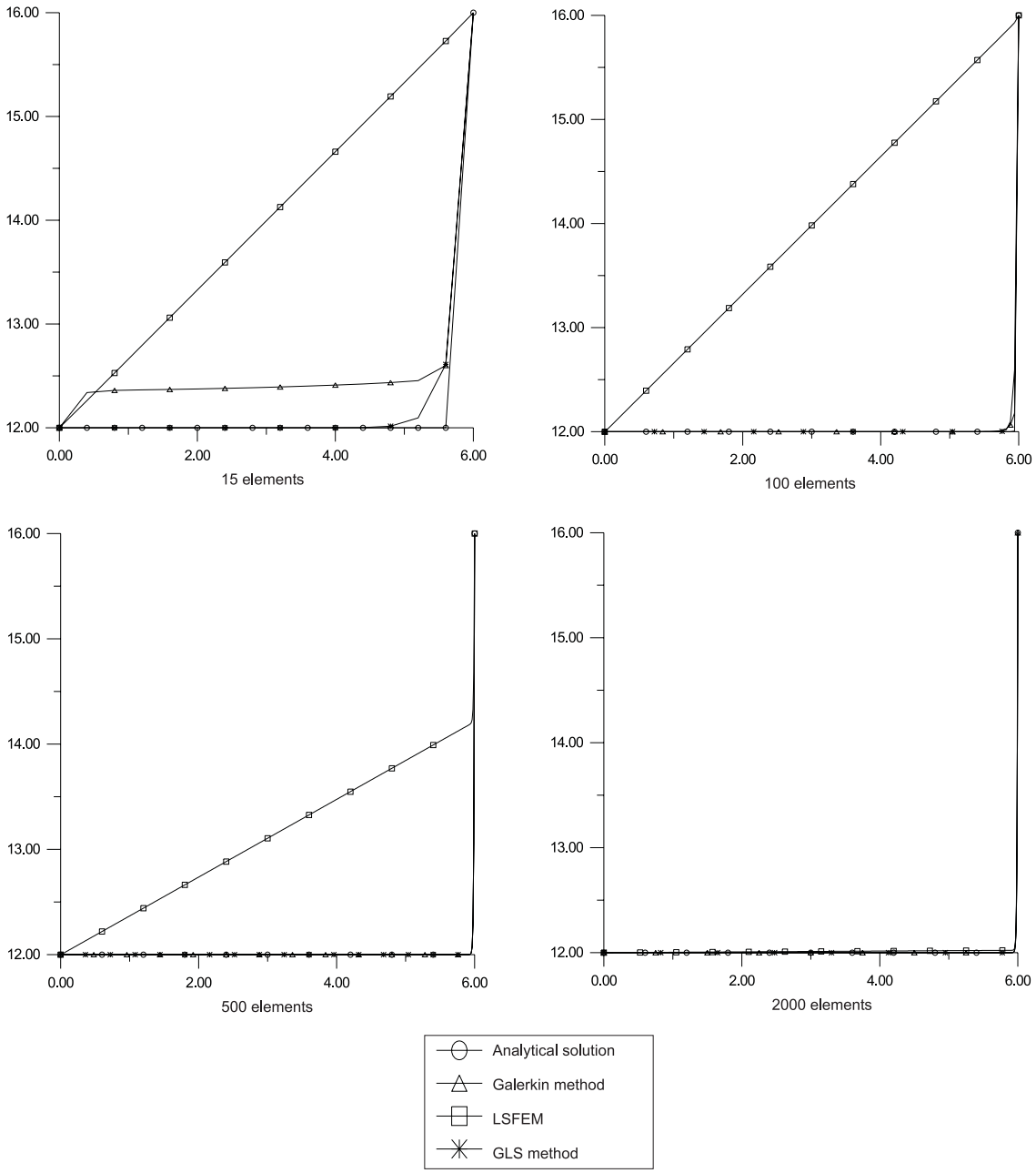

Figure 5. Numerical solutions to the example (18), using C^1-Hermite elements.

If we take into account the previous considerations about the characteristics of the three methods studied and the 1D numerical tests, we can advocate the GLS method as

the more efficient formulation to stabilize the convective dominant transport problems. The definition of the stabilization parameter needs to be clarified through a deep study of the error analysis [9,11,16].

5.2 2D Numerical tests.

In order to test the performance of the GLS method in 2D cases, we have studied the following transport problem in a unit square domain (see figure 6):

$$\begin{aligned} &\mathbf{a}\cdot\boldsymbol{\nabla}u-\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}u)=0 \qquad in\ \Omega \\ &u(0,y)=u(x,0)=0 \\ &u(1,y)=u(x,1)=1 \end{aligned} \tag{29}$$

with parameters: $k = 0.0001$ and $\boldsymbol{a} = (1, 1)$.

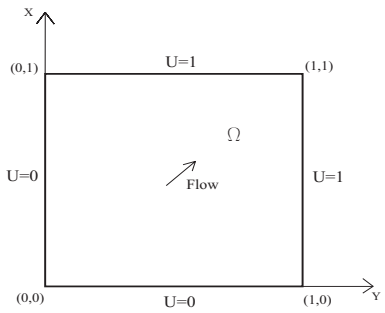

Figure 6. Domain and boundary conditions of the 2D test problem.

In figure 7, the numerical solutions of the problem are shonw for the Galerkin and GLS method with different mesh refinements. As it is shown, Galerkin method yields to oscillatory numerical solutions, while the GLS method is stable even with a rough mesh and it is able to reproduce sharp fronts.

As we could expect, the resolution of the 2D problem by a Least-squares approach would imply a huge computational cost due to the complexity of the trial functions needed. In addition, as it occurs in the 1D problem, results are expected to be over-dissipative.

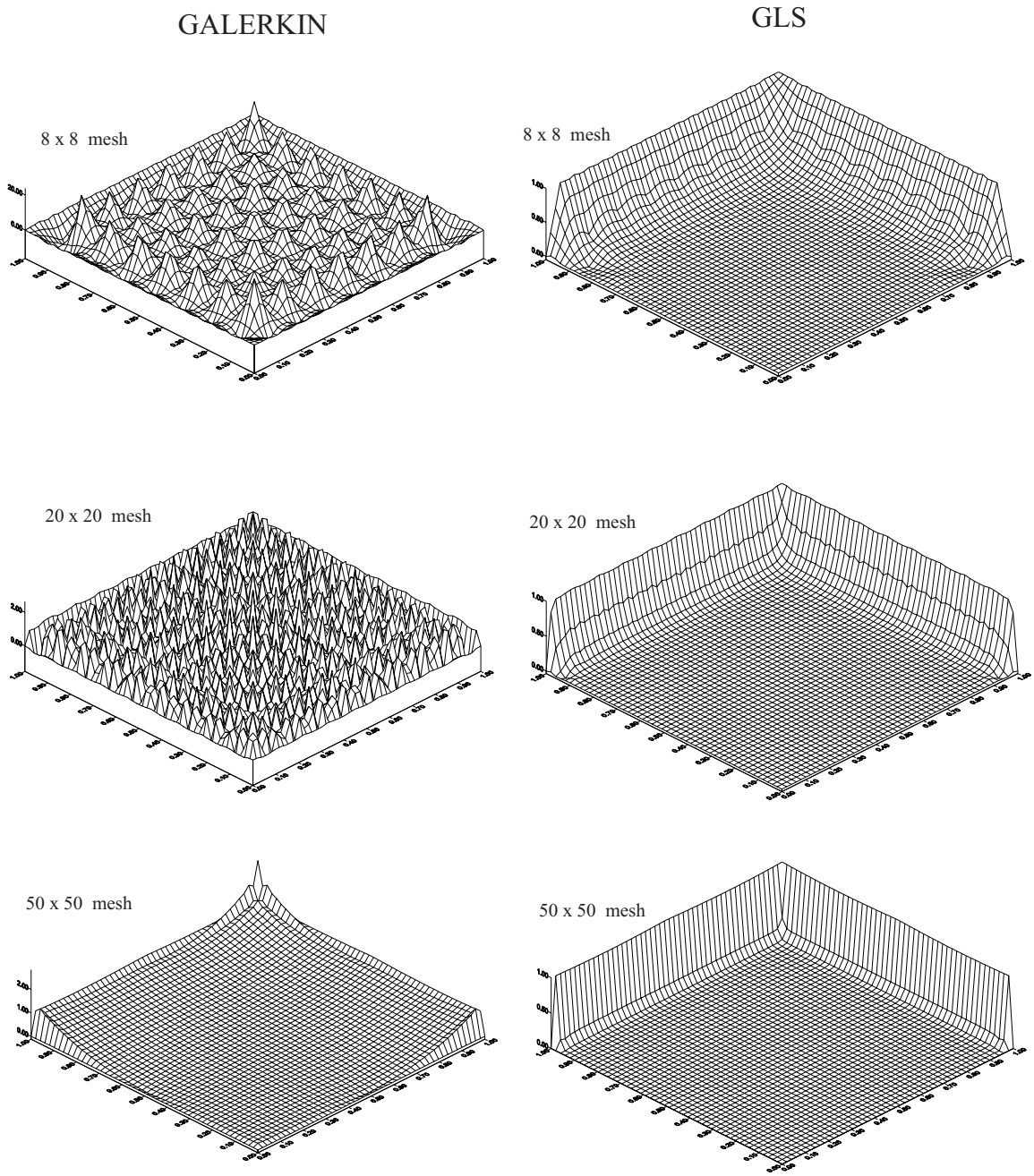

Figure 7. Numerical solutions to the example (29), by the Galerkin and GLS method with different meshes.

6 CONCLUSIONS

In this paper we have reviewed three formulations applied to the convective-diffusive transport problem: Galerkin method, Least-squares finite element method and Galerkin/Least-squares method. A detailed description of their characteristics has allowed to understand their performance and the relations among the methods.

We have specially analyzed the stability of the numerical solutions to the convective dominant problem. Hence, while the Galerkin method yields to oscillatory solutions, the LSFEM attains the stability through the symmetry of its formulation and the GLS through the addition of symmetric terms. However, the LSFEM has two remarkable disadvantages: its high computational cost and the over-dissipative solutions. In the numerical examples, we saw that the GLS method yields to stable solutions without a remarkable increase of the computational cost. The weak point of this method is that there is not available a general form to obtain the stabilization parameter for practical cases.

We have also studied the continuity and derivability requirements of the trial functions in each formulation. This is an important aspect, sometimes ignored, to avoid variational crimes. As a result, we conclude that the LSFEM has stronger requirements than the Galerkin or GLS method. As the GLS method is a combination between the Galerkin method and the LSFEM, it was expected to have the same continuity requirements as the LSFEM and, therefore, to have a high computatinf effort. However, the Least-squares term is only evaluated elementwise, and the continuity requirements across elements are determined by the Galerkin method.

In this field, other important aspects must be considered in further analysis, such as: the determination method of the GLS stabilization parameter, the application of the GLS method to 2D and 3D real problems and the avoidance of over-dissipative effects in LSFEM through a reweighted formulation.

REFERENCES

[1] B.N. Jiang, *The Least-Squares Finite Element Method (Theory and Applications in Computational Fluid Dinamics and Electromagnetics)*, Springer, (1998).

[2] C.F. Carey, An analysis of stability and oscillations in convection-diffusion computations, *Finite element methods for convection dominated flows*, T.R.J. Hughes, The American Society of Mechanical Engineers, (1979),63-72.

[3] C.A. Figueroa, I. Colominas *Aplicación del método de elementos finitos en problemas de mecánica de fluídos: estudio, resolución y técnicas de estabilización de la ecuación del transporte por convección-difusión)*, Research report, Civil Engrg. Sch., Universidad de La Coruña, (1998).

[4] G.F. Carey, J.T.Oden, *Finite elements. A second course*, Prentice Hall, Vol. II, (1983).

[5] G.F. Carey, B.N. Jiang, Least-squares finite elements for first-order hyperbolic systems, *Int. J. Num. Methods Eng.,* **26**, 81-93 (1988).

[6] B.N. Jiang, L.A. Povinelli, Least-squares finite element method for fluid dynamics, *Comput. Methods Appl. Mech. Engrg.,* **81**, 13-37 (1990).

[7] E.D. Eason, A review of least-squares methods for solving partial differential equations, *Int. J. Num. Methods Eng.,* **10**, 1021-1046 (1976).

[8] B. Carnahan, H.A. Luther, J.O. Wilkes, *Cálculo Numérico. Métodos y aplicaciones*, Rueda, (1979).

[9] L.P. Franca, S.L. Frey, T.J.R. Hughes, Stabilized finite element methods: I. Application to the advective-diffusive model, *Comput. Methods Appl. Mech. Engrg.,* **95**, 253-276 (1992).

[10] R. Codina, *Comparison of some finite element methods for solving the diffusion-convection-reaction equation*, International Center for Numerical Methods in Engineering, CIMNE Publication **101**, (1996).

[11] T.R.J. Hughes, L.P. Franca, G.M. Hulbert, A new finite element formulation for computional fluid mechanics: VIII. The Galerkin/Least-Squares method for advective-diffusive equations, *Comput. Methods Appl. Mech. Engrg.,* **73**, 173-189 (1989).

[12] A.N. Brooks, T.R.J. Hughes, Streamline upwind/Petrov Galerkin formulations for convective dominated flows with particular emphasis on hte incompressible Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.,* **32**, 199-259 (1982).

[13] L.P. Franca, T.R.J. Hughes, Two classes of finite element methods, *Comput. Methods Appl. Mech. Engrg.,* **69**, 89-129 (1988).

[14] O.C. Zienkiewicz, R.L. Taylor,*El método de los elementos finitos. Mecánica de sólidos. Dinámica y no linealidad.*, McGraw Hill, Vol II, (1995).

[15] N. Camprubí, I. Colominas *Aplicación del método de elementos finitos al problema de transporte por convección-difusión)*, Research report, Civil Engrg. Sch., Universidad de La Coruña, (1999).

[16] I. Harari, T.J.R. Hughes, What are C and h?: Inequalities for the analysis and design of finite element methods, *Comput. Methods Appl. Mech. Engrg.,* **97**, 157-192 (1992).

[17] T.J.R. Hughes, Multiscale phenomena: Green's functions, the Dirichlet-to Neumann formulation, subgrid scale models, bubbles and the origins of the stabilized methods, *Comput. Methods Appl. Mech. Engrg.,* **127**, 387-401 (1995).

[18] C.A. Figueroa, I. Colominas, G. Mosqueira, F. Navarrina, M. Casteleiro, A stabilized finite element approach for advective-diffusive transport problems, *Computational Methods in Engineering* (CDROM), P.M. Pimenta, R. Brasil, E. Almeida, Escola Politécnica da Univ. de São Paulo, Brasil, (1999).

[19] G. Strang, G.J. Fix, *An analysis of the finite element method*, Prentice-Hall, (1973).

[20] T.R.J Hughes,*The finite element method: Linear static and dynamic finite element analysis*, Prentice-Hall, (1987).