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STABILIZATION OF NUMERICAL FORMULATIONS FOR CONVECTIVE-DIFFUSIVE TRANSPORT PROBLEMS

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Abstract. *Numerical simulation in Fluid Mechanics is an extremely difficult task, which complexity increases exponentially as the velocity of the fluid becomes higher. In particular, it is known that serious troubles are encountered when the FEM is applied to the resolution of high-advective fluid problems, despite the fact that the method has been successfully applied to a large number of sundry problems of Computational Mechanics. As a general rule, these drawbacks are announced by large oscillations of the Galerkin numerical solution in specific areas, or even through the whole domain.*
In order to understand the reasons for this unexpected behaviour, we focus our attention in the convective-diffusive transport differential equation, which can be interpreted as the linear version of the Navier-Stokes equations. By means of this simplified analysis, we try to identify the origin of the numerical oscillations phenomena, as much as to find a generic way to stabilize the numerical solution of the problem.
In this paper we review the most significant alternative approaches that have been proposed to overcome these troubles when the Galerkin formulation is intended to solve the problem. Then, we propose a new technique that allows to obtain the stabilization parameters for the Petrov-Galerkin approach. Our procedure is based on the eigenvalue analysis of the elemental matrices of the discretized problem. Thus, the outlined process could be applied independently on the specific formulation being used and the dimension of the problem being solved. Finally, we present different convective-diffusive numerical tests for different Péclet numbers.

1. INTRODUCTION

1.1. Mathematical model: the convective-diffusive transport equation

It is well-known that the finite element solution of fluid problems presents some difficulties related with the instabilities of the numerical solution for medium and high values of the velocity of the fluid [1,2]. In order to understand the reasons of this anomalous behaviour and to study this phenomenon in a linear problem, we focuse our attention to the convective-diffusive transport differential equation, which can also be interpreted as the "linear version" of the Navier-Stokes equations.

Generally speaking, the transport phenomena in a fluid media involve two different main processes: the "diffusion", which can be mathematically described by the parabolic equation

$$\frac{\partial \phi}{\partial t} = \boldsymbol{\nabla} \cdot (\boldsymbol{K} \boldsymbol{\nabla} \phi), \tag{1}$$

where ϕ is the transported unknown (eg., the concentration of a pollutant spilt in a harbour area) and $\boldsymbol{K}$ the diffusion tensor of the fluid; and the "convection" or "advection" process, which appears when the fluid moves carrying with it any substance along the mainstream velocity. This process can be modelled by the hyperbolic equation:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \phi = 0 \tag{2}$$

being u the velocity of the fluid. It is obvious, that the dominance of one process over the other will determine the main nature of the transport in a particular case.

Now, if we take into account both processes in an isotropic medium, the sourceless convective-diffusive transport problem in a domain Ω is given by the partial differential equation

$$\frac{\partial \phi}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \phi = \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} \phi) \quad \text{in } \Omega, \quad \mathrm{t} > 0, \tag{3}$$

with the following general boundary and initial conditions

$$\boldsymbol{\nabla}\phi \cdot \boldsymbol{n} = 0 \;\text{ in } \Gamma_1; \quad \boldsymbol{\nabla}\phi \cdot \boldsymbol{n} = q \;\text{ in } \Gamma_2; \quad \boldsymbol{\nabla}\phi \cdot \boldsymbol{n} = \gamma - a\phi \;\text{ in } \Gamma_3; \quad \phi(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \boldsymbol{x} \in \Omega, \tag{4}$$

where Γ_1, Γ_2 and Γ_3 denote different parts of the boundary Γ where the above conditions are prescribed ($\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma$). In general, γ, k, a, q and f could be time and position dependants data.

1.2. Variational statement of the problem and finite element numerical approach

For the transport problem defined in (3) and (4), it can be written the variational form

$$\int_{\Omega}\left\{\frac{\partial\phi}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\phi-\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\phi)\right\}wd\Omega+\int_{\Gamma_2}\{\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}-\gamma+a\phi\}w_{\Gamma_2}d\Gamma_2=0, \qquad (5)$$

which must hold for all members w and w_{Γ_2} of suitable classes of test functions defined in Ω and Γ_2[3]. Now, a weak variational statement of this expression can be derived by applying the Green's Identity:

$$\int_{\Omega}\left\{w\frac{\partial\phi}{\partial t}+w\mathbf{u}\cdot\boldsymbol{\nabla}\phi+k\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}w\right\}d\Omega+\int_{\Gamma_2}a\phi kwd\Gamma_2=\int_{\Gamma_2}\gamma kwd\Gamma_2. \qquad (6)$$

Finite element numerical modelling of this problem requires the definition of a discrete approach to its solution and the partition of the domain Ω in e elements ($\Omega_1\cup\Omega_2\cup\ \Omega_3\cup...\cup\Omega_e$, so that $\Omega_i\cap\Omega_j=0,\ i\neq j$); therefore, a finite element discretization $\{\Omega_h\}$ of domain Ω is obtained. Now, if we select a set of local shape functions p_j defined on Ω_h and Γ_{2_h} so that

$$\phi\approx\tilde{\phi}(\mathbf{x},t)=\sum_{j=1}^{n}\phi_j(t)p_j(\mathbf{x}), \qquad (7)$$

and we select a set of n test functions w_i, expression (6) is reduced to the following system of linear equations:

$$\mathbf{B}\frac{d\boldsymbol{\phi}}{dt}+\mathbf{A}\boldsymbol{\phi}=\mathbf{c}, \qquad (8)$$

being

$$\begin{aligned}
B_{ij}&=\int_{\Omega_h}w_ip_j\ d\Omega_h,\quad i,j=1,n\\
A_{ij}&=\int_{\Omega_h}(\mathbf{u}\cdot\boldsymbol{\nabla}p_j)w_i\ d\Omega_h+\int_{\Omega_h}k\boldsymbol{\nabla}p_j\cdot\boldsymbol{\nabla}w_i\ d\Omega_h+\int_{\Gamma_{2_h}}akw_ip_j\ d\Gamma_{2_h},\quad i,j=1,n\\
c_i&=\int_{\Gamma_{2_h}}\gamma kw_id\Gamma_{2_h},\quad i=1,n
\end{aligned} \qquad (9)$$

2 NUMERICAL INSTABILITIES IN GALERKIN TYPE APPROACHES

The Galerkin type weighting ($w_i = p_i$, $i = 1, n$), which has been successfully used in the FE solution of an important number of problems in Solid Mechanics, produces unstable numerical approaches when it is applied to convective-diffusive transport problems with medium and high values of the velocity of the fluid. In these cases, it can be shown [3,4] that the Galerkin type numerical model is unable to propagate precisely both the frecuency and the amplitude of an eigenfunction of the analytical solution of certain problems. This frequency and this amplitude arises as a consequence of the existence of complex eigenvalues associated to a certain eigenfunction. These complex eigenvalues are the origin of the apperance of the numerical oscillations. Next, we ilustrate the appearance of these complex eigenvalues and its influence in the problem.

The common procedure to study the stability of a numerical scheme arises from the von Neumann stability theory, in which the evolution of components of an eigenfunction expansion or Fourier series of an initial data or the error is considered. Thus, the set of eigenvalues of the system obtained for a specific numerical approach determines its stability and provides the means to analyze effects such as the dispersion or the dissipation of the numerical solution. A complete study of the qualitative properties of the spectrum of eigenvalues for a given numerical approach can be found in reference [2]. This analysis leads to the Gershgorin circle theorems to determine the distribution of the eigenvalues in the complex plan, and to the theory of oscillatory matrices obtained from finite element numerical schemes. Since the full development of this analysis is too cumbersome to be made explicit in this paper[3,4,5], we show their influence and express the main conclusions in a particular example. If we consider a 1D convective-diffusive transport problem with constant physical properties and discretize the domain in a mesh of linear finite elements which size is h, the elemental matrices are given by

$$\mathbf{B}^e = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad \mathbf{A}^e = \frac{u}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{10}$$

Constraining the analysis to the obtention of the steady-state response of the problem, the assembly process of the matrix $\mathbf{A}$ in (8) leads to the following system of linear equations

$$\left[\frac{u}{2}\begin{pmatrix} -1 & 1 & 0 & & & \\ -1 & 0 & 1 & & \dots & \\ 0 & -1 & 0 & & & \\ & & & 0 & 1 & 0 \\ & \dots & & -1 & 0 & 1 \\ & & & 0 & -1 & 1 \end{pmatrix} + \frac{k}{h}\begin{pmatrix} 1 & -1 & 0 & & & \\ -1 & 2 & -1 & & \dots & \\ 0 & -1 & 2 & & & \\ & & & 2 & -1 & 0 \\ & \dots & & -1 & 2 & -1 \\ & & & 0 & -1 & 1 \end{pmatrix}\right]\begin{bmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \dots \\ \dots \\ \phi_n \end{bmatrix} = \begin{bmatrix} c_1 \\ .. \\ .. \\ .. \\ \\ c_n \end{bmatrix} \tag{11}$$

where the different influence of diffusive and convective terms in the coefficients of the assembled matrix. As it can be seen, the convective contribution produces a non-symmetrical matrix with many zeros in the main diagonal. This fact is the origin of the apperance of complex eigenvalues when the influence of this matrix is dominant in the final system of linear equations, that is, when convection is more important than diffussion. This effect can be shown in the following three cases corresponding to the above example for a mesh of 7 elements with different Péclet numbers ($Pe = uh/2k$, $k = 5$, $h = 1$):

$$
\underset{u=2,\ Pe=0.2}{\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\\\lambda_6\\\lambda_7\end{bmatrix}=\begin{bmatrix}18.828\\16.109\\12.180\\7.820\\3.891\\1.172\\0.000\end{bmatrix}};\quad
\underset{u=10,\ Pe=1}{\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\\\lambda_6\\\lambda_7\end{bmatrix}=\begin{bmatrix}10.0\\10.0\\10.0\\10.0\\10.0\\10.0\\00.0\end{bmatrix}};\quad
\underset{u=12,\ Pe=1.2}{\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\\\lambda_6\\\lambda_7\end{bmatrix}=\begin{bmatrix}00.0\\10.0-1.48i\\10.0+1.48i\\10.0-4.14i\\10.0+4.14i\\10.0-5.98i\\10.0+5.98i\end{bmatrix}};\quad
\underset{u=40,\ Pe=4}{\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\\\lambda_6\\\lambda_7\end{bmatrix}=\begin{bmatrix}00.0\\10.0-8.62i\\10.0+8.62i\\10.0-24.15i\\10.0+24.15i\\10.0-34.89i\\10.0+34.89i\end{bmatrix}}
\tag{12}
$$

According to (11), an effective way to stabilize the problem and to reduce spurious oscillations is the control of the mesh size by using a smaller size of element h, since then the convective contribution becomes less important than the diffusive one. Obviously, a remeshing procedure, which is feasible for 1D test problems, would imply an unaffordable computational cost in practice. For this reason, alternative numerical approaches have been proposed in last years to the Galerkin type weighting in order to obtain stable finite element formulations for the convective-diffusive transport equation.

3 NUMERICAL APPROACHES FOR THE CONVECTIVE-DIFFUSIVE TRANSPORT PROBLEM

Different numerical formulations have been propose to stabilize the finite element equations for the convective-diffusive transport problem. Most of them are based on the selection of different types of weighting in order to reforce the diffusive contribution (in the previous example, the symmetrical part of the equation (11)). The simplest methods directly introduce an additional diffusive term, while more recent and rigorous techniques modify the variational form in the Galerkin type weighting (SUPG[5], GLS[6], Taylor-Galerkin methods[7], Characteristic-Galerkin method[8], etc.), introducing one or more parameters which adjust the stability of the numerical approximation. These methods can be understood as a modification of the variational statement of the problem by adding a term of the general form[9]

$$\int_{\Omega_h} \mathcal{P}(w_h)\tau\mathcal{R}(p_h)d\Omega_h, \tag{13}$$

where $\mathcal{P}(w_h)$ is a differential operator which is applied to the test functions, τ is the stabilization parameter, and $\mathcal{R}(p_h)$ is the residual of the partial differential equation:

$$\mathcal{R}(p_h) = \frac{\partial\tilde{\phi}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\tilde{\phi} - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\tilde{\phi}). \tag{14}$$

In this general statement, most of the proposed stabilization techniques proposed in last years can be included. In particular, for Petrov-Galerkin formulations the stabilization of the numerical solution is carried out by upwinding the test functions against the current lines of the fluid. In a 1D problem, for example, test functions $w_i(\xi)$ and trial functions $p_i(\xi)$ can be defined for linear elements as

$$p_i(\xi) = \begin{cases} p_1(\xi) = \frac{1}{2}(1-\xi) \\ p_2(\xi) = \frac{1}{2}(1+\xi) \end{cases}, \quad w_i(\xi) = \begin{cases} w_1(\xi) = \frac{1}{2}(1-\xi) - \frac{\tau}{4}(1+\xi)(1-\xi) \\ w_2(\xi) = \frac{1}{2}(1+\xi) + \frac{\tau}{4}(1+\xi)(1-\xi) \end{cases} \tag{15}$$

where τ scales the amount of upwind bias desired ("upwind parameter"), as it is shown in Fig. 1.

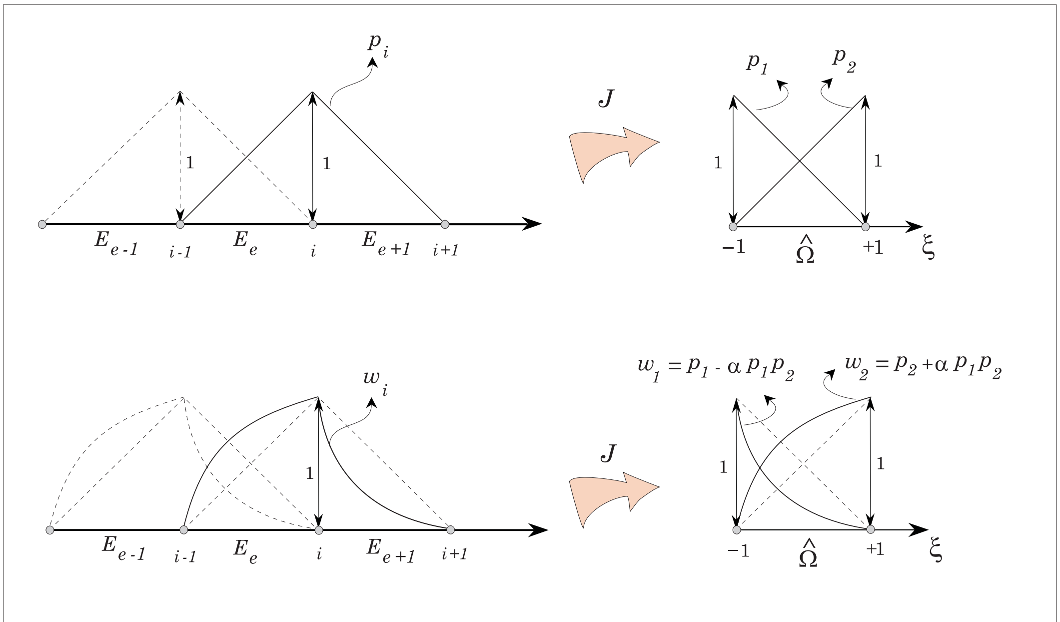

Fig. 1.-Standard piecewise-linear basis functions and quadratically based test functions for Petrov-Galerkin approaches.

The effect of introducing a bias in the test functions can be seen in the previous 1D example, and we can show the stabilization mechanism of the numerical formulation. Thus, taking into account the trial and the test functions defined in (17), the following system of linear equations is obtained, instead of (8):

$$\left(\mathbf{B}\frac{d\boldsymbol{\phi}}{dt} + \mathbf{A}\boldsymbol{\phi}\right) + \tau\left(\widehat{\mathbf{B}}\frac{d\boldsymbol{\phi}}{dt} + \widehat{\mathbf{A}}\boldsymbol{\phi}\right) = \mathbf{c}, \tag{16}$$

where the elemental matrices associated to the cuadratic bias $\widehat{\mathbf{B}}$, $\widehat{\mathbf{A}}$ are given by

$$\widehat{\mathbf{B}}^e = \frac{h}{12}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}; \quad \widehat{\mathbf{A}}^e = \frac{u}{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + k\,\mathbf{0}. \tag{17}$$

As it can be seen, in this new numerical approach only a convective term is added (there is no new diffusive contribution), which is a symmetrical matrix, and will contribute to stabilize the numerical scheme balancing the contribution of the non-symmetrical matrix $\mathbf{A}^e$ in the assembled matrix of coefficients of the final system of equations. It is important to remark that this analysis can also be made for high-order elements and 2D and 3D problems, obtaining similar conclusions[3,4].

One of the keys of the stabilization methods is the obtention of the suitable parameters τ that adjust the correction to the Galerkin approach[10]. Nowadays, computation of these parameters for finite element formulations of the convective-diffusive transport equation still remains an open field of study[11]. Thus, for example, some approaches propose the analytical computation of the "best" parameter by imposing the exact solution at nodes (which is possible to obtain for simple 1D problems), by comparing results from a poor mesh with those obtained from an enriched mesh, or by means of smoothing procedures, and sometimes parameters for 1D problems are heuristically used in practice for 2D and 3D cases.

In previous works[3,12], we have proposed a different method for the computation of the stabilization parameter. As we have shown, the oscillatory nature of the numerical solution obtained with Galerkin type weighting formulations is due to the assembling of non-symmetrical elemental matrices that produces complex eigenvalues in high-convective fluid problems. Stabilization methods, like the Petrov-Galerkin presented in the above examples, try to overcome this phenomenum by means of the modification of the variational form and so, the elemental matrices associated to the convection process. Our proposal is to obtain the stabilization parameter from the information of the eigenvalues of the elemental matrices: the elemental matrices are computed starting with a null stabilization parameter (i.e., we use a Galerkin type weghting approach), and we obtain (or estimate) the set of eigenvalues $\{\lambda_i\}$ of each elemental matrix. If all of them are real numbers, then the stabilization parameter remains equals zero, but if it should not be so, then the value of the stabilization parameter is increased until no complex eigenvalues in the elemental matrices appear.

In this point, it is important to remark an essential characteristic of this proposal: it is a general procedure and, in principle, it is independent of the dimension of the problem. The stabilization parameters are not computed in an heuristic way, and it is also possible to use this methodology if source and reactive terms are considered in the transport differential equation. Consequently, it is a general method, very simple from a conceptual point of view, that stabilizes the finite element equations of the problem by analyzing only its elemental matrices. At present, we have obtained very promising results in the cases studied until now [3,12]. In the next section, we present some 1D numerical tests to demonstrate the feasability of the proposed method for computing stabilization parameters and we discuss the results obtained. Furthermore we point out a way to accelerate the computing of this stabilization parameter.

4 NUMERICAL EXAMPLES

The first example (test #1) that we present is the 1D test defined by

$$u\frac{\partial \phi}{\partial x} = k\frac{\partial^2 \phi}{\partial x^2}\ , \qquad 0 < x < L; \quad \phi(0) = \phi_0\ \ , \qquad \frac{\partial \phi}{\partial x}(L) = \frac{u}{k}[\phi_L - \phi(L)] \tag{18}$$

with the following parameters: $L = 7$, $\phi_0 = 2$, $\phi_L = 0.5$ and $k = 0.07$.

We have analyzed the numerical results for different velocities in the domain of the problem and discretizations of the domain of 10 and 50 elements. In all cases, we have considered parabolic elements[3,12] (3 nodes per element). Péclet numbers ($Pe = uh/2k$) vary from 0.5 to 250 (in the discretization of 10 elements) and from 0.1 to 250 (in the case of 50 elements).

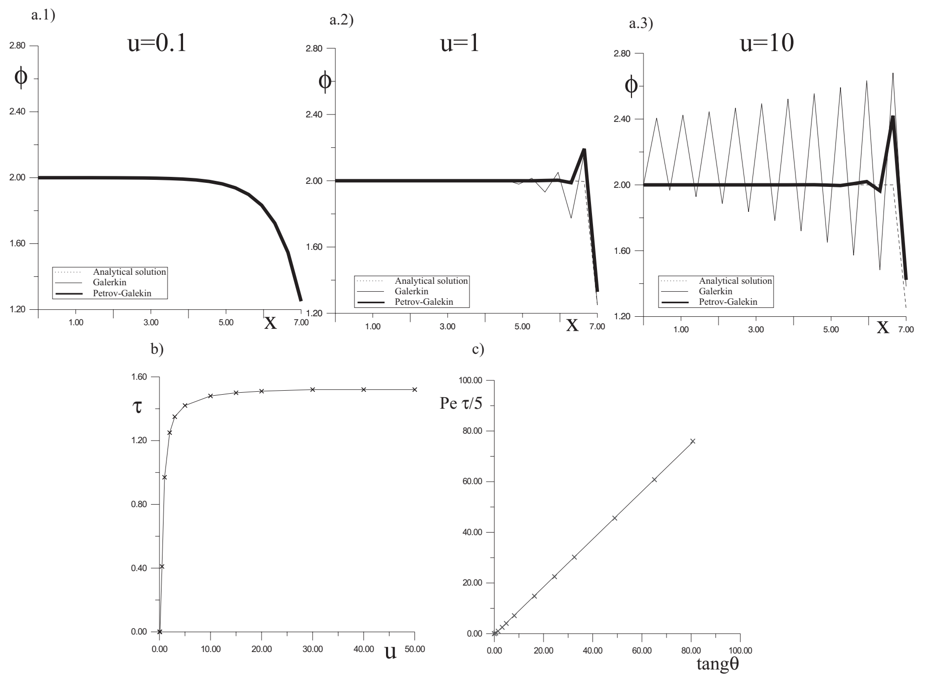

Fig. 2.-Test #1: Results obtained with a discretization of 10 parabolic elements.

Numerical results obtained with Galerkin and Petrov-Galerkin with the proposed method for computing the stabilization parameters based on the eigenvalues of the elemental matrices are presented in Fig. 2 (for a discretization of 10 elements) and Fig. 3 (for a discretization of 50 elements). In both cases, figures *a#)* are the comparison of the numerical solutions obtained with Galerkin and Petrov-Galerkin approach, and the analytical solution for different fluid velocities, figure *b)* presents the evolution of the

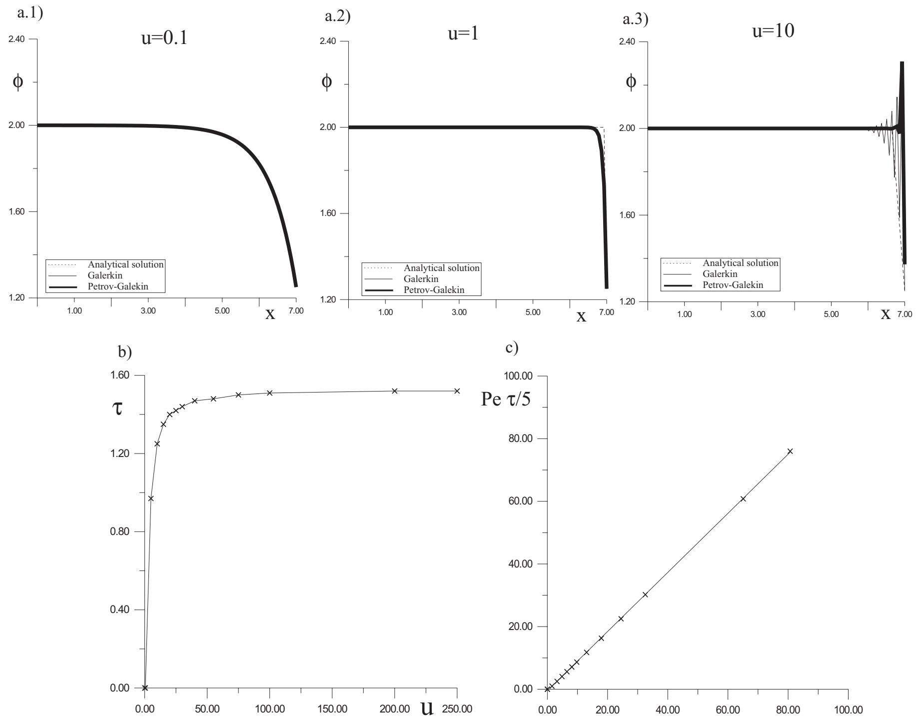

Fig. 3.-Test #1: Results obtained with a discretization of 50 parabolic elements.

stabilization parameter τ depending on the velocity, and figure *c)* shows the relation that exits between the stabilization parameter for a specific velocity of the fluid and the tangent of the polar angle of the eigenvalue obtained with a Galerkin approach. This linear relation between τ and the polar angle can also be analitycally demonstrated.

The second example (test #2) that we present is the 1D test defined by

$$u\frac{\partial \phi}{\partial x} = k\frac{\partial^2 \phi}{\partial x^2} \ , \qquad 0 < x < L; \quad \phi(0) = \phi_0 \ \ , \quad \phi(L) = \phi_L \tag{19}$$

with the following parameters: $L = 7$, $\phi_0 = 16$, $\phi_L = 80$ and $k = 0.07$.

Like in the previous example, we have analyzed the numerical results for different velocities in the domain of the problem and discretizations of the domain of 10 and 50 elements. In all cases, we have considered parabolic elements[3,12] (3 nodes per element). Péclet numbers ($Pe = uh/2k$) vary from 0.5 to 250 (in the discretization of 10 elements) and from 0.1 to 250 (in the case of 50 elements). Figures 4 and 5 show a comparison of the numerical solutions obtained with different approaches (*a#)*), the evolution of the stabilization parameter τ depending on the fluid velocity (*b)*), and the relation between τ and the tangent of the polar angle of the eigenvalue obtained with a Galerkin approach (*c)*).

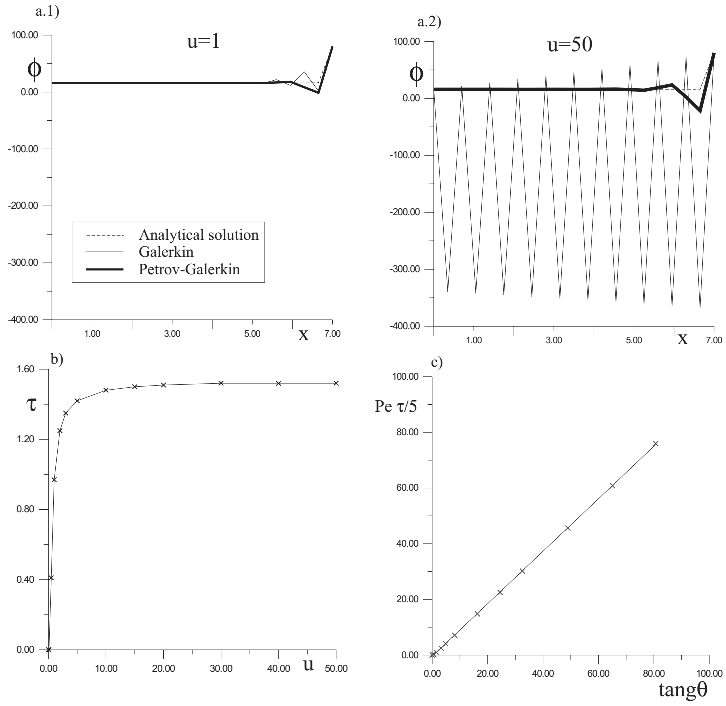

Fig. 4.-Test #2: Results obtained with a discretization of 10 parabolic elements.

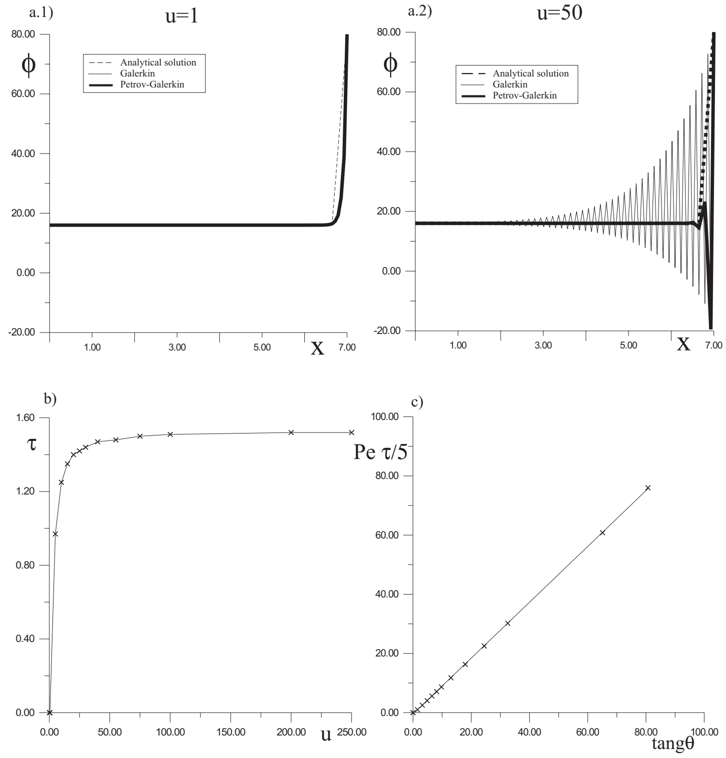

Fig. 5.-Test #2: Results obtained with a discretization of 50 parabolic elements.

In these examples, we remark the great advantage of the proposed formulation since it is independent on any heuristic method for computing the stabilization parameters, which we obtain by a systematic analysis of the eigenvalues of each elemental matrix; that is, computing the stabilization parameter starting with a Galerkin type weighting approach ($\tau = 0$) and increasing its value until no complex eigenvalues in the elemental matrices appears. However, we have found that there should be a faster way to obtain the stabilization parameter, since it exists a clear relation between the stabilization parameter and the greater polar angle of the complex eigenvalues of an elemental matrix —graph (*c)*) in figures 2, 3, 4 and 5—. In order to verify this fact, we have executed other examples modifying the transport differential equation to include reactive terms (tests #3 and #4). Test #3 consists of the problem

$$u\frac{\partial \phi}{\partial x} = k\frac{\partial^2 \phi}{\partial x^2} - \alpha\phi \ , \qquad 0 < x < L; \quad \phi(0) = \phi_0 \ \ , \qquad \frac{\partial \phi}{\partial x}(L) = \frac{u}{k}[\phi_L - \phi(L)] \qquad (20)$$

with the following parameters: $L = 7$, $\phi_0 = 2$, $\phi_L = 0.5$ and $k = 0.07$, and being α the reactive coefficient.

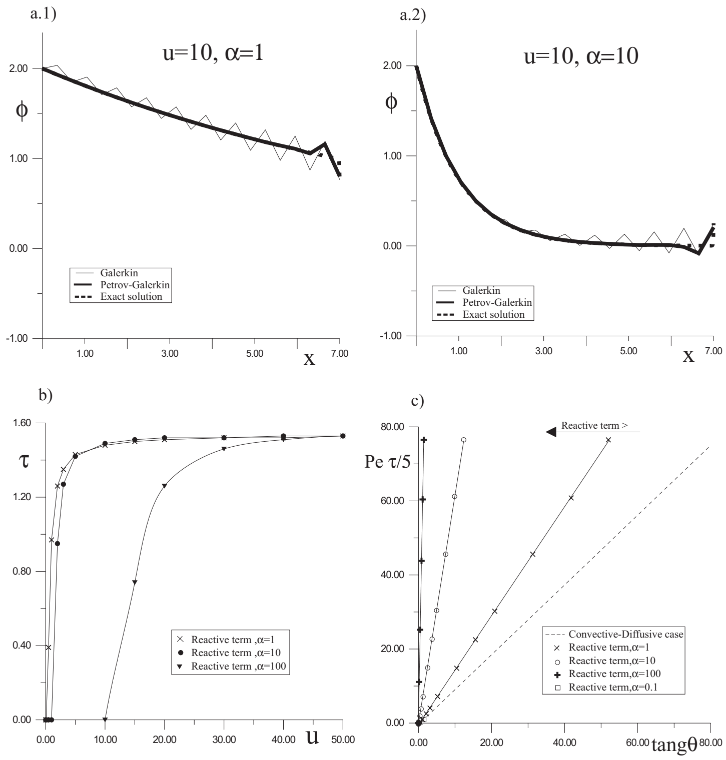

Fig. 6.-Test #3: Results obtained with a discretization of 10 parabolic elements.

We have analyzed the numerical results for different velocities in the domain of the problem and different values of the reactive coefficient (from $\alpha = 1$ to $\alpha = 100$), and discretizations of the domain of 10 and 50 elements. In all cases, we have considered parabolic elements, and Péclet numbers ($Pe = uh/2k$) vary from 0.5 to 250 (in the discretization of 10 elements) and from 0.1 to 250 (in the case of 50 elements).

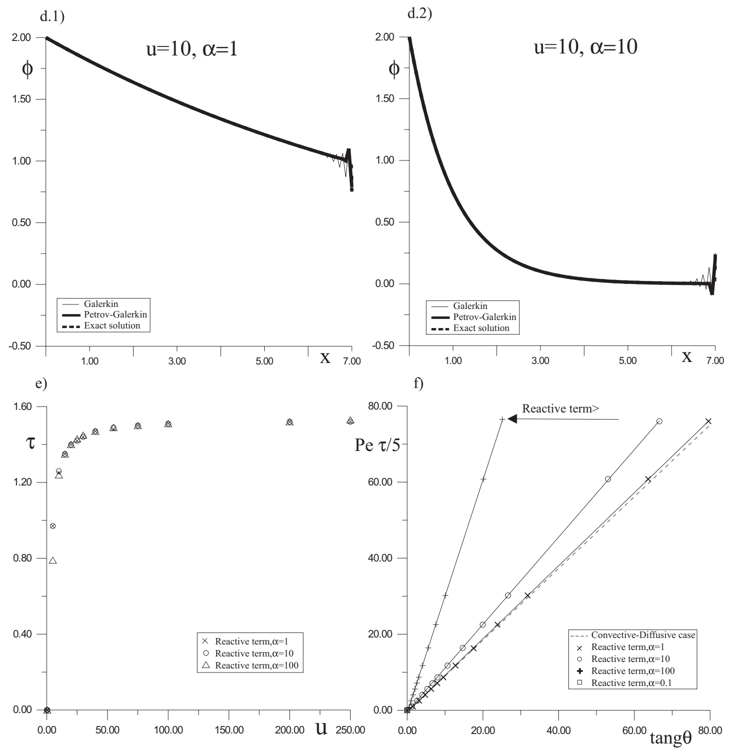

Fig. 7.-Test #3: Results obtained with a discretization of 50 parabolic elements.

Now, like in the previous examples, we compare results obtained with different numerical formulations (*a#)*), the evolution of the stabilization parameter τ depending on the fluid velocity for different values of the reactive coefficient(*b)*), and the relation between stabilization parameter τ and the tangent of the polar angle of the eigenvalue obtained with a Galerkin approach (*c)*).

Test #4 consists of the 1D test defined by

$$u\frac{\partial \phi}{\partial x} = k\frac{\partial^2 \phi}{\partial x^2} - \alpha\phi \ , \qquad 0 < x < L; \quad \phi(0) = \phi_0 \ \ , \quad \phi(L) = \phi_L \tag{21}$$

with the following parameters: $L = 7$, $\phi_0 = 16$, $\phi_L = 80$ and $k = 0.07$, and being α the reactive coefficient.

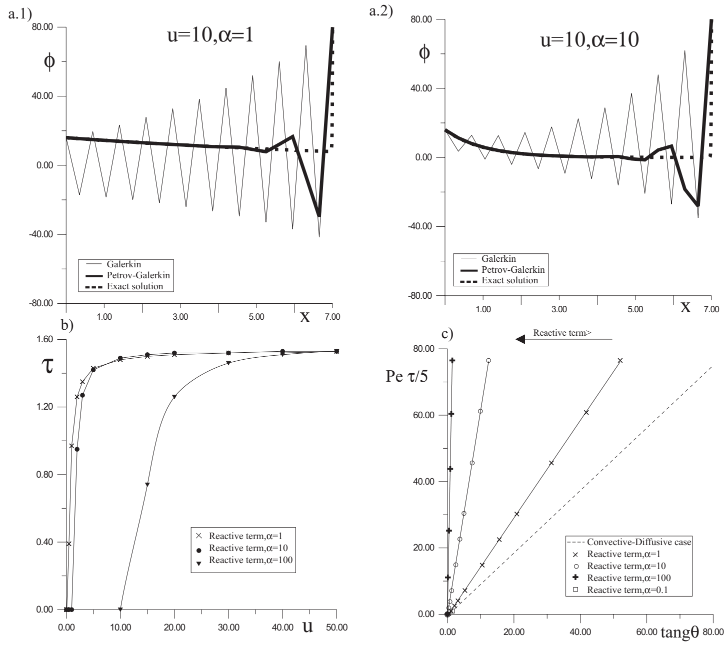

Fig. 8.-Test #4: Results obtained with a discretization of 10 parabolic elements.

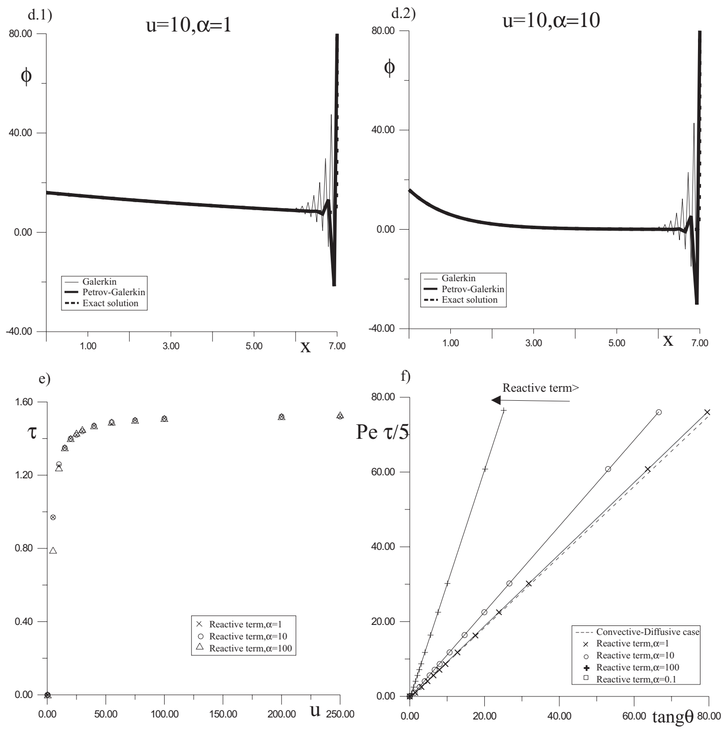

Fig. 9.-Test #4: Results obtained with a discretization of 50 parabolic elements.

This example has been solved like test #3 and results are shown in figures 8 and 9. Like in the previous example, we can observed the linear relation that exists between stabilization parameter τ and the tangent of the polar angle of the eigenvalue obtained with a Galerkin approach. At the present time, we are studying this possible linear relation in general cases and trying to explain it from a mathematically point of view, since it should represent a significant improvement in the computing of the stabilization parameter which it could be directly obtained from the eigenvalues of the elemental matrices of the Galerkin formulation with a fairly low computational effort.

5 CONCLUSIONS

In this paper, we have revised the origin of the numerical oscillations that appear in the finite element solution of the convective-diffusive transport problem when a Galerkin type weighting is used. Furthermore, we have proposed a different way for computing the stabilization parameters by analyzing the elemental matrices of the finite element discretization, and it has been applied to Petrov-Galerkin formulations. This general method is applicable to 1D, 2D and 3D problems, and it can be carried out during the integration and assembly of the elemental matrices of the discretization of the problem.

Results obtained in 1D problems are excellent and very promising, and they can assure a good performance of this method in 2D and 3D cases. Although the computational effort required to determine the stabilization parameter from the eigenvalue analysis of the elemental matrices could become important, it is possible to reduce it drastically if this parameter is obtained from the polar angle of the eigenvalues of the elemental matrices in the Galerkin type weighting finite element approach, just as we have pointed out in the examples presented.

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