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MESHLESS METHODS FOR POTENTIAL PROBLEMS IN ELECTRICAL ENGINEERING APPLICATIONS

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ABSTRACT

In some problems in engineering applications, the mesh generation process represents one of the big challenges when numerical methods such as finite elements, finite differences or boundary elements are applied. For this reason, several numerical techniques ("meshless methods") have been recently proposed to overcome the problems related with the discretization of the domain. These methods can represent an important improvement in Computational Mechanics, and among others in the electrical engineering field. In this paper, we present a meshless technique based on the Moving Least Square method with a point collocation approach for solving problems in Potential Theory in electrical engineering applications. Furthermore we propose the use of enrichment numerical approaches applied to these meshless procedures.

KEYWORDS

Meshless Methods, Enrichment functions, Moving Least Square Methods, Earthing

INTRODUCTION

Advances in numerical methods, together with the development of computer sciences, have represented a significant improvement in the treatment of some engineering problems. However, in some practical applications (e.g., problems with moving boundaries and discontinuities or in cases of domains with very complicated geometry) difficulties

arise when standard numerical techniques which require the discretization of the whole domain are applied (Belytschko *et al.*, 1998; Oñate *et al.*, 1996-b). In fact, when an efficient grid is needed, the mesh generation process frequently becomes the bottle neck.

To overcome these problems, in recent years some different numerical techniques have been proposed, such as numerical methods where explicit element meshes are unnecessary ("meshless methods") (Belytschko *et al.*, 1996) and numerical methods where approximations are enriched by using functions of the same type that the solution (Belytschko *et al.*, 1998; Taylor *et al.*, 1997).

In meshless methods, the solution domain is formed by a set of nodal points. Every point has an associated subdomain including its closest points and therefore, a local approximation can be achieved in each node. Thus, for every central node or "star node" this approximation is built with the information provided by its subdomain points. Within the limits of these techniques several methods have appeared in last years. One of them is based on least squares formulations combined with Point Collocation approaches to compute the integral terms, therefore no auxiliary grid is required. Furthermore, in this method, different interpolants can be derived depending on the weighting function: it can be fixed within each subdomain, as in the Diffuse Least Square Method (DLS), or it can depend on the point where the approximated value is to be computed, as in Moving Least Square Method (MLS) (Chao, 1997).

On the other hand, the essential idea of the use of "enrichment functions" is to improve the results obtained with other numerical methods by adding functions to base approximations which are a partition of unity. This last property corresponds to the ability of the shape functions to reproduce a constant, being crucial for convergence (Belytschko *et al.*, 1998) . Two kinds of enrichment have been proposed: the first one employs functions involved in the solution of the problem (Belytschko *et al.*, 1998), and the other one proposes to enrich with polynomial functions (Taylor *et al.*, 1995).

In this paper a moving least square approach using base interpolating functions normalized within domain, with a point collocation scheme it is proposed for solving potential problems in electrical engineering applications. Furthermore, the performance of this method combinated with enrichment functions is analyzed in some numerical tests.

MOVING LEAST SQUARE APPROXIMATION

Moving Least Square approximations lie in a local weighted least square fitting, valid on each collocation point subdomain formed by its n closest nodal points (subdomain Ω_k). The local character of the approximation comes from a moving weighting function which takes its maximum value at this collocation point and vanishes outside a surrounding region (Oñate *et al.*, 1996-a).

The proper definition of the approximation at any point implies that all subdomains Ω_k cover all the interpolation domain. Hereby, these subdomains must overlap and the common areas have to include enough nodal points in order to ensure the convergence of the method. Thus, for a given collocation point the selection of the nodal points belonging to its subdomain has been performed according to an effective technique based on the "four quadrants" criterium (Oñate *et al.*, 1996-a).

Let Ω_k be the interpolation domain of a function $u(\boldsymbol{x})$. A local approximation to $u(\boldsymbol{x})$

can be derived in the form,

$$u(\boldsymbol{x}) \cong \hat{u}(\boldsymbol{x}) = \sum_{i=1}^{m} p_i(\boldsymbol{x})\alpha_i = \boldsymbol{p}^t(\boldsymbol{x})\boldsymbol{\alpha} \tag{1}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^t$ is a set of unknown coefficients and $\boldsymbol{p}(\boldsymbol{x})$ contains a base of interpolating functions (monomial terms, generally) which order is m (Oñate *et al.*, 1996-b). These base interpolating functions can be normalized within each subdomain Ω_k dividing by the maximum distance d between the collocation point i and the most distant point of its subdomain. The normalized coordinates are:

$$\boldsymbol{\xi}(\boldsymbol{x}) = \left[\frac{x - x_i}{d}, \frac{y - y_i}{d}, \frac{z - z_i}{d}\right] \tag{2}$$

On the other hand, function $u(\boldsymbol{x})$ can be sampled in the n nodal points belonging to Ω_k in the form,

$$\boldsymbol{u}^h = \begin{pmatrix} u_1^h \\ \vdots \\ u_n^h \end{pmatrix} \cong \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_1^t \\ \vdots \\ \boldsymbol{p}_n^t \end{pmatrix} \boldsymbol{\alpha} = \boldsymbol{S}\boldsymbol{\alpha} \tag{3}$$

being u_j^h the values of unknown function evaluated in nodal points of subdomain Ω_k ($u_j^h = u(\boldsymbol{x}_j)$, $j = 1, \ldots, n$), $\hat{u}_j = \hat{u}(\boldsymbol{x}_j)$ their approximated values, and $\boldsymbol{p}_j$ contains the normalized base interpolating functions evaluated in $\boldsymbol{\xi}_j$ (where $\boldsymbol{\xi}_j = \boldsymbol{\xi}(\boldsymbol{x}_j)$).

In general, the number of nodal points n is greater than the order m of the polinomials base, so $\boldsymbol{S}$ is a rectangular matrix and the approximation cannot fit all the u_j^h values. However, approximated values $\hat{u}(\boldsymbol{x})$ can be determined by minimizing the weighted sum of the square differences between the exact value u_j^h and the approximation $\hat{u}(\boldsymbol{x}_j)$ at each nodal point $\boldsymbol{x}_j$ belonging to the domain of the arbitrary collocation point $\boldsymbol{x}_k$ (Oñate *et al.*, 1996-b). The weighting function in moving least square approximation computed in $\boldsymbol{x}_j$, $\omega_k(\boldsymbol{x}_j, \boldsymbol{x}_k)$, is usually built in such a way that it equals unity in collocation point $\boldsymbol{x}_k$ and vanishes outside domain Ω_k. In the examples presented in this paper, the truncated gaussian distribution has been used.

In general, the definition of a different weighting function for every interpolating point $\boldsymbol{x}_k$ is very difficult, presenting an infinite number of posibilities. To overcome this problem, we can define the weighting functions at nodal points $\boldsymbol{x}_j$, and use them evaluated in the arbitrary point $\boldsymbol{x}_k$. Therefore, $\omega_k(\boldsymbol{x}_j, \boldsymbol{x}_k)$ may be substituted by $\omega_j(\boldsymbol{x}_j, \boldsymbol{x}_k)$. To simplify the notation $\boldsymbol{x}_k$ is replaced by the global coordinate $\boldsymbol{x}$.

Thus, the functional to be minimized with respect to $\boldsymbol{\alpha}$ results in

$$J(\boldsymbol{x}) = \sum_{j=1}^{n} \omega_j(\boldsymbol{x}_j, \boldsymbol{x})(u_j^h - \hat{u}(\boldsymbol{x}_j))^2 \tag{4}$$

The performance of this process yields to (Chao, 1997),

$$\boldsymbol{\alpha} = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}^h \qquad \boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{P}\boldsymbol{W}(\boldsymbol{x})\boldsymbol{P}^t \qquad \boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{P}\boldsymbol{W}(\boldsymbol{x}) \tag{5}$$

being auxiliary matrices $\boldsymbol{P}$ and $\boldsymbol{W}(\boldsymbol{x})$:

$$\boldsymbol{P} = [\boldsymbol{p}(\boldsymbol{\xi}_1) \quad \dots \quad \boldsymbol{p}(\boldsymbol{\xi}_n)] \qquad \boldsymbol{W}(\boldsymbol{x}) = diag\left[\omega_k(\boldsymbol{x}_j, \boldsymbol{x})\right], \quad j = 1, ..., n \tag{6}$$

Substitution of (5) in (1) allows to obtain an approximation to function $u(\boldsymbol{x})$ in Ω_k:

$$u(\boldsymbol{x}) \cong \hat{u}(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{\xi})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}^h. \tag{7}$$

Therefore, one can define the shape functions in $\boldsymbol{x}$ as [4],

$$\boldsymbol{N}^t(\boldsymbol{x}) = \boldsymbol{p}^t(\boldsymbol{\xi})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x}) \tag{8}$$

and consequently,

$$\hat{u}(\boldsymbol{x}) = \sum_{j=1}^{n} N_j(\boldsymbol{x})u_j^h \tag{9}$$

It must be taken into account that the local values of the approximating function do not fit the nodal unknown values, $\hat{u}(\boldsymbol{x}_j) \neq u_j^h$, due to the least square character of the approximation. It must also be pointed out that if $n = m$, the FEM type approximation is recovered and no effect of weighting is presented (Oñate *et al.*, 1996-a). Besides, if the weighting function is constant and equals the unity, the standard least square method is reproduced.

MLS APPROXIMATIONS WITH ENRICHMENT FUNCTIONS

The introduction of enrichment functions to moving least square approachs consists of adding to the approximation function (9) new terms which contain information about the solution, in order to decrease the global computacional cost. Thus, when the kind of functions in the solution is known, the application of this idea can be very useful. In this case, the approximation takes the form of an extrinsic enrichment (Belystchko *et al.*, 1998) and can be written as:

$$u(\boldsymbol{x}) \cong \hat{u}(\boldsymbol{x}) = \sum_{j=1}^{n} N_j(\boldsymbol{x})\Big(u_j^h + \sum_{k=1}^{n_f} a_{kj}F_k(\boldsymbol{x})\Big) \tag{10}$$

where $\hat{u}$ is the approximation to the exact solution u in the domain Ω_k, n_f is the total number of enrichment functions F_k, and n is the total number of nodal points. The approximation to the solution requires to know in every one of the n nodal points the value of $n_f + 1$ coefficients: $\boldsymbol{u}_j$ and $\boldsymbol{a}_{k,j}$. For this reason when the MLS approximation is combined with a point collocation approach, it requires $(n_f + 1)n$ collocation points.

One important property, in order to study the convergence of this method, is that as $N_j(\boldsymbol{x})$ is obtained in the same way as in MLS formulation, the partition of unity is satisfied. Therefore, a hierarchical enrichment may be added using any type of functions (Taylor *et al.*, 1997).

This idea of the enrichment has been successfully applied in Finite Elements and Element Free Galerkin Methods in elasticity problems and fracture mechanics problems (Belystchko *et al.*, 1998; Taylor *et al.*, 1997). In the same way, in this paper we propose the use of enriched meshless formulations (EMF) in some applications in electrical engineering.

A BOUNDARY PROBLEM DISCRETIZED EQUATIONS

In order to study applications of preceding methods, in this section are stated the discretized equations for a boundary problem.

Let A and B be two differential operators, Ω the domain of our problem and Γ its boundary ($\Gamma = \Gamma_t \cup \Gamma_u$). In these terms, a scalar boundary value problem can be written as,

$$A(u) = b \qquad \text{in} \quad \Omega \tag{11}$$

with boundary conditions,

$$\begin{aligned} B(u) = t \qquad & \text{in} \quad \Gamma_{\mathrm{t}} \\ u - u_p = 0 \qquad & \text{in} \quad \Gamma_{\mathrm{u}} \end{aligned} \tag{12}$$

where u is the solution, b and t represent the actions over Ω and along the boundary Γ_t, and u_p is the prescribed value of u along Γ_u.

Application of weighted residual method allows to obtain a variational form of the above problem, in terms of the trial approximation function $\hat{u}$ of the unknown u, as

$$\int_\Omega \Psi_i \; [A(\hat{u}) - b] d\Omega + \int_{\Gamma_t} \widehat{\Psi}_i \; [B(\hat{u}) - t] d\Gamma + \int_{\Gamma_u} \widehat{\widehat{\Psi}}_i \; [\hat{u} - u_p] d\Gamma \; = \; 0 \tag{13}$$

which must hold for all members Ψ_i, $\widehat{\Psi}_i$ and $\widehat{\widehat{\Psi}}_i$ of suitable classes of test functions defined on Ω, Γ_t and Γ_u. The selection of test functions in the general varational form (13) allows to derive different formulations. In the examples presented in this paper a point collocation method has been implemented, in order to take advantage of the meshless character of the approximation (Chao, 1997). Other approaches based on integral methods have been proposed (Oñate *et al.*, 1996-a; Taylor *et al.*, 1997), but require some kind of auxiliar grid to evaluate the resulting integrals.

The point collocation scheme ($\Psi_i = \widehat{\Psi}_i = \widehat{\widehat{\Psi}}_i = \delta_i$ in expression (13), where δ_i is the Dirac delta) leads to the set of equations,

$$\begin{aligned} [A(\hat{u})]_i - b_i = 0 \quad & \text{in} \quad \Omega \\ [B(\hat{u})]_i - t_i = 0 \quad & \text{in} \quad \Gamma_t \\ \hat{u}_i - u_p = 0 \quad & \text{in} \quad \Gamma_u \end{aligned} \tag{14}$$

If function $\hat{u}$ is approximated by linear combination of the shape functions (9,10) the previous system of equations may be written in the standard form as,

$$\boldsymbol{K}\boldsymbol{u}^h = \boldsymbol{f} \tag{15}$$

where coefficient matrix $\boldsymbol{K}$ is banded (but not necessary symmetric), $\boldsymbol{f}$ is also known (contains the contributions from terms b and t and prescribed values u_p), and $\boldsymbol{u}^h$ contains the unknown values of the function evaluated in nodal points (Chao, 1997).

NUMERICAL TESTS

In this section we present two numerical tests of the MLS approach applied to the solution of a 1D boundary value problem. Thus, it will be compared results obtained by using a standard MLS and an enrichment MLS approach.

Example 1

As a first example we consider the following BVP (Carey *et al.*, 1997):

$$-\frac{d^2u}{dx^2} + u = f(x) \ , \qquad 0 \leq x \leq 1 \ , \quad u(0) = 0 \ , \quad u(1) = 1 \tag{16}$$

being

$$f(x) = \frac{2\rho\big[1+\rho^2(1-\beta)(x-\beta)\big]}{\big[1+\rho^2(x-\beta)^2\big]^2} + (1-x)\big[\mathrm{atan}\big(\rho(x-\beta)\big) + \mathrm{atan}(\rho\beta)\big] \tag{17}$$

The analytical solution is given by

$$u(x) = (1-x)\big[\mathrm{atan}\big(\rho(x-\beta)\big) + \mathrm{atan}(\rho\beta)\big] \tag{18}$$

Depending on the election of parameters ρ and β in (17), we can control the shape of the solution. In this case $\rho = 50.0$, $\beta = 0.4$ produce a sharp function, that it is very difficult to approximate, being a good test for the performance of the method. Thus, we can approximate the solution to (16) and (17) with a meshless method for a given set of n_p trial functions N_i defined on the domain, by using one enrichment function ($F_1(x) = \mathrm{atan}(x)$) in (10). In Figure 1 it is shown a comparison between the analytical solution and the MLS approximations with and without enrichment.

Example 2

As a second example, we consider the BVP that represents a string on an elastic foundation (Taylor *et al.*, 1995):

$$-a\frac{d^2u}{dx^2} + cu = f \ , \qquad 0 \leq x \leq 1 \ , \quad u(0) = 0 \ , \quad u(1) = 1 \tag{19}$$

The analytical solution is given by:

$$u(x) = \frac{f}{c}\left[1 - \cosh(x/\sqrt{a}) - \Big(1 - \cosh(1/\sqrt{a})\Big)\frac{\sinh(x/\sqrt{a})}{\sinh(1/\sqrt{a})}\right] \tag{20}$$

Parameters in (19) are given by $a = 0.01$ and $c = f = 1$. In this case, the enrichment meshless approximation to this problem has been obtained by using two enrichment functions $F_1(x) = \cosh(x)$ and $F_2(x) = \sinh(x)$ in (10). In Figure 2 it is shown a comparison between the analytical solution and the approximations obtained.

In the light of these 1D test examples it can be noticed that results obtained by using meshless methods can improve if enrichment functions are used. At present, we are working in this way in order to introduce enrichment functions in meshless approximations in 2D and 3D problems. In the following section we present an application to grounding analysis of the MLS method.

APPLICATION TO GROUNDING ANALYSIS

Grounding systems in substations has to guarantee the integrity of equipments and the continuity of the service under fault conditions –providing means to carry and dissipate electrical currents into the ground–, and to safeguard that persons in the surroundings of the grounded installation are not exposed to dangerous electrical shocks. To achieve these goals, the equivalent electrical resistance of the system must be low enough to assure that fault currents dissipate mainly through the grounding grid into the earth, while maximum potencial differences between close points on the earth surface must be kept under certain tolerances (step, touch and mesh voltages).

Physical phenomena underlying fault currents dissipation into the earth can be modelled by means of Maxwell's Electromagnetic Theory (Colominas *et al.*, 1997). Constraining analysis to the obtention of the electrokinetic steady-state response, and neglecting the inner resistivity of the earthing electrode, the 3D problem associated to an electrical current derivation to earth can be written as

$$\begin{aligned} \mathbf{div}\boldsymbol{\sigma} = 0, \qquad \boldsymbol{\sigma} = -\underline{\underline{\boldsymbol{\gamma}}}\,\mathbf{grad}V \quad \text{in} \quad E; \\ \boldsymbol{\sigma}^t \boldsymbol{n}_E = 0 \quad \text{in} \quad \Gamma_E; \quad V = V_\Gamma \quad \text{in} \quad \Gamma; \quad V \longrightarrow 0, \quad \text{if} \quad |\boldsymbol{x}| \to \infty; \end{aligned} \tag{21}$$

where E is the earth, $\underline{\underline{\boldsymbol{\gamma}}}$ its conductivity tensor, Γ_E the earth surface, $\boldsymbol{n}_E$ its normal exterior unit field and Γ the electrode surface (Belystchko *et al.*, 1996). Thus, when the electrode attains a voltage V_Γ (Ground Potential Rise or GPR) relative to a distant grounding point, the solution to this problem gives the potential V and the current density $\boldsymbol{\sigma}$ at an arbitrary point $\boldsymbol{x}$. Further assumption $V_\Gamma = 1$ is not restrictive at all, since V and $\boldsymbol{\sigma}$ are proportional to V_Γ.

The example presented in this paper consists of a toroidal electrode horizontally buried to a depth of 7 m. The interior diameter of the ring is 20 m and the electrode diameter is 3 m. The upper layer depth is 14 m. The scalar conductivity associated with the lower layer γ_2, is four times the one corresponding to the upper layer γ_1 ($\gamma_2 = 4\ \gamma_1$). Due to the axial symmetry of the problem, solution can be obtained by using a 2D model. This case has been solved with 3019 points obtained by means of the program GEN4U (Sarrate, 1996). The base interpolating functions used are linear and all subdomains contain at least five points. Figure 3 shows the nodal point distribution, the contour lines and the potential distribution around the electrode in two cases: assuming the hypothesis of homegeneous and isotropic soil or considering a two-layer model. These numerical results agree significantly with those obtained by using a very dense point distribution and with results of a boundary element program.

CONCLUSIONS

In this paper, a moving least square interpolation method with a point collocation approach using enrichment functions has been presented. This technique has been compared for 1D numerical tests with the analytical solution and the MLS method without enrichment. Results significantly agree and very good results are obtained when enrichment functions are used.

Furthermore the MLS approach has been applied to the solution of a problem in grounding analysis. First results obtained for different point distributions, even with a two-layer soil model, are very promising and require a reasonable computational cost. Obviously, further analysis must be done in both mathematical and numerical aspects in order to introduce the enrichment procedure to 2D and 3D problems.

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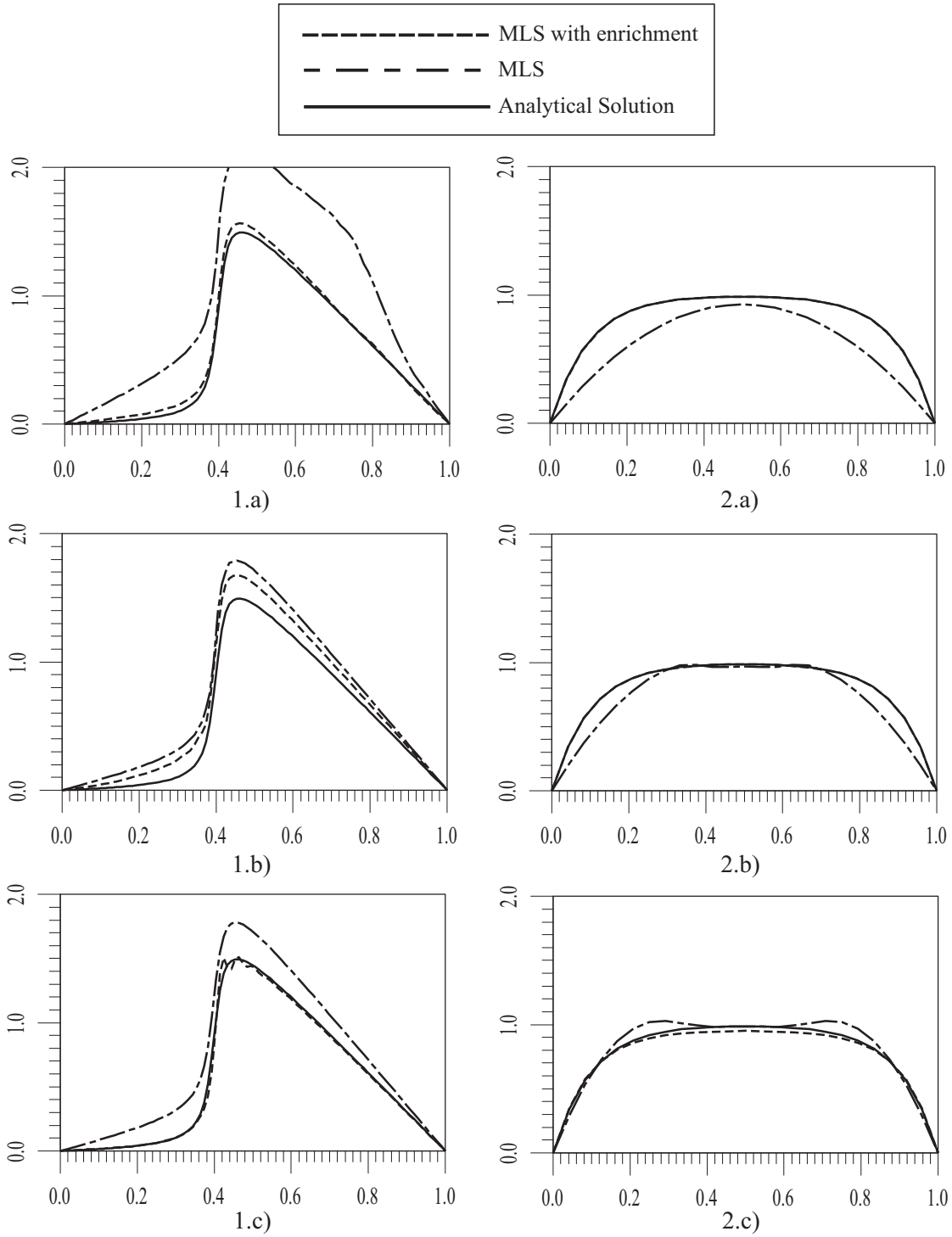

Fig. 1.- Example 1: Comparison of results obtained by using an enriched MLS approach, a standard MLS approach (both with quadratic interpolation), and the analytical solution: **a)** 50 nodal points, subdomains of 3 points; **b)** 50 nodal points, subdomains of 5 points; **c)** 50 nodal points, subdomains of 7 points

Fig. 2.- Example 2: Comparison of results obtained by using an enriched MLS approach, a standard MLS approach (both with quadratic interpolation), and the analytical solution: **a)** 3 nodal points, subdomains of 3 points; **b)** 5 nodal points, subdomains of 5 points; **c)** 7 nodal points, subdomains of 7 points

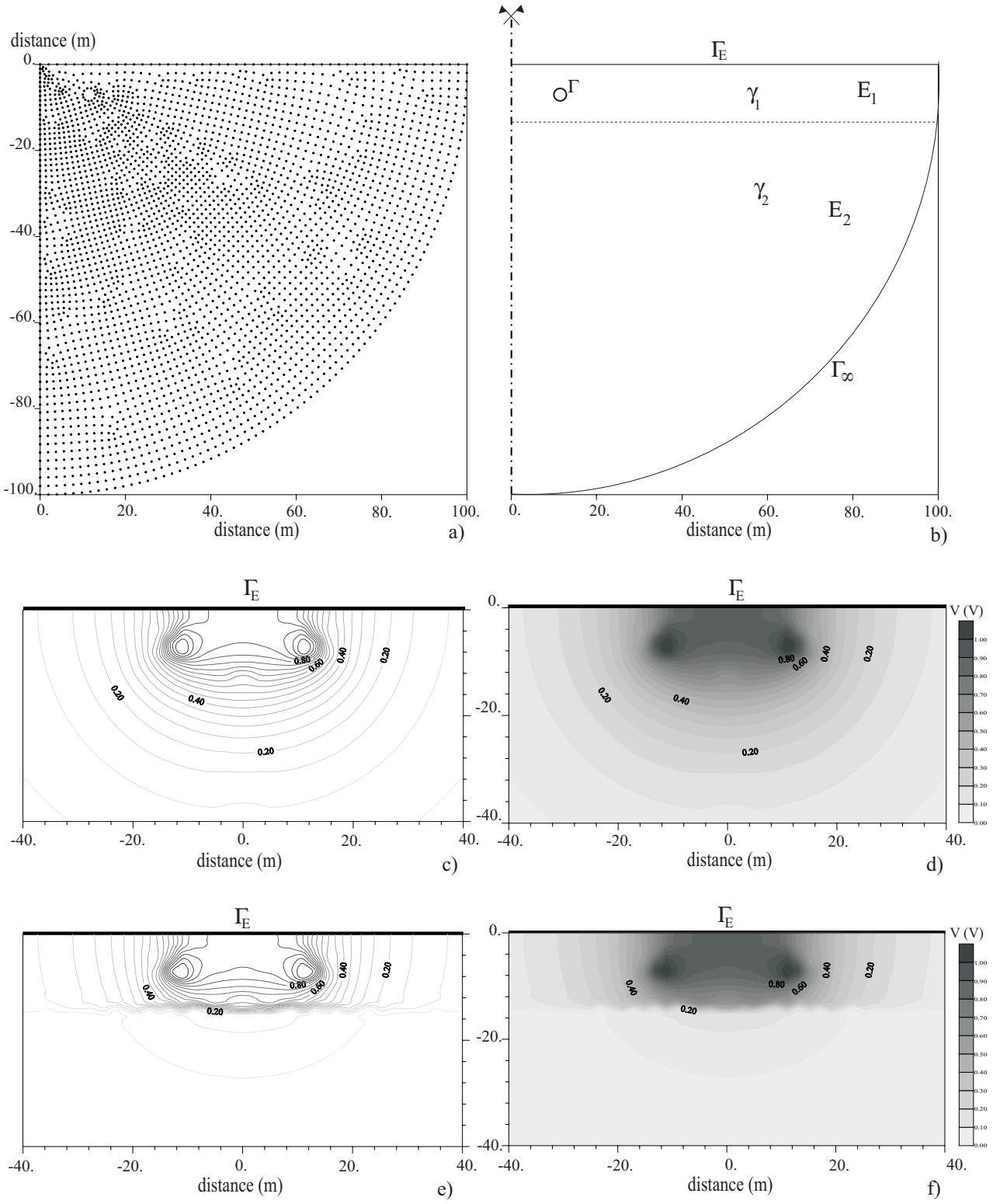

Figure 3.- Toroidal electrode buried to a depth of 7 m: a) Distribution of nodal points (npoin=3019), b) Domain solution scheme, c) Contour lines around the electrode, considering the soil homogeneous and isotropic, d) Potential distribution around the electrode, considering the soil as homogeneous and isotropic, e) Contour lines around the electrode assuming a two- layer model ($\gamma_2 / \gamma_1 = 4$), f) Potential distribution around the electrode assuming a two-layer model ($\gamma_2 / \gamma_1 = 4$).