The Conformable Information Filter

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Abstract

In this project, we construct an information filter associated with a linear continuous control system corrupted by some noise. Here, the system is defined in terms of a conformable derivative introduced by Khalil et al. in 2014. This time-weighted derivative shares many of the same properties as the classical derivative but lacks the usual semigroup property associated with the exponential. Mathematically, this conformable information filter is a backward-time counterpart of the recently constructed conformable Kalman filter. Here, the inverse of the error covariance associated with the Kalman filter becomes the information matrix for the information filter. The conformable information filter allows for smoothed estimates of the true state of our control system.

Definition (Conformable Derivative)

Let $f : [0, \infty) \to \mathbb{R}$ and let $\alpha \in (0, 1]$. Then the conformable derivative of order α of f at t is defined by

$$f^{\alpha}(t) := \begin{cases} \lim\limits_{\theta \to 0} \dfrac{f(t + \theta t^{1-\alpha}) - f(t)}{\theta}, & t > 0 \\ \lim\limits_{s \to 0^+} f^{(\alpha)}(s), & t = 0, \end{cases}$$

provided that the limit exists.

Properties of the Conformable Derivative

Let $\alpha \in (0, 1]$, f, g be $\alpha-$differentiable functions for $t > 0$, and $a, b \in \mathbb{R}$. Then

- $(af + bg)^{(\alpha)}(t) = af^{(\alpha)}(t) + bg^{(\alpha)}(t)$,
- $(t^b)^{(\alpha)} = bt^{b-\alpha}$,
- $(b)^{(\alpha)} = 0$,
- $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(t)g^{(\alpha)}(t)$,
- $\left(\frac{f}{g}\right)^{(\alpha)}(t) = \frac{g(t)f^{(\alpha)}(t) - f(t)g^{(\alpha)}(t)}{[g(t)]^2}$, and
- if f is differentiable, then $f^{(\alpha)}(t) = t^{1-\alpha}f'(t)$.

Definition (Conformable Integral)

$I^a_\alpha(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann integral and $\alpha \in (0, 1]$.

The Stochastic Model

Consider the system

$$x^{(\alpha)}(t) = A(t)x(t) + B(t)u(t) + Gw(t), \; x(t_0) = x_0$$
$$y(t) = C(t)x(t) + v(t)$$

where

- $x \in \mathbb{R}^n$ is the state,
- $u \in \mathbb{R}^m$ is the deterministic control,
- $y \in \mathbb{R}^p$ is the measurement,
- $w \in \mathbb{R}^l$ is the process noise, and
- $v \in \mathbb{R}^p$ is the measurement noise.

The Damped Oscillator System

Consider the stochastic oscillator

$$x^{(\alpha)}(t) = \begin{bmatrix} 0 & 1 \\ -0.64 & -0.25 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v(t)$$

where $Q = 3$, $R = 2$, and $P(0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Here, we use $n = 200$ iterations.

The Conformable Kalman Filter (CKF)

System: $x^{(\alpha)}(t) = A(t)x(t) + B(t)u(t) + Gw(t)$
Measurement: $y(t) = C(t)x(t) + v(t)$

Assumptions: $x_0 \sim (\bar{x}_0, P_0)$, $w \sim (0, Q\delta(t - s))$, $v \sim (0, R\delta(t - s))$, which are mutually uncorrelated, R > 0

Initialization
Initial Estimate: $\hat{x}(t_0) = \bar{x}_0$
Error Covariance: $P(t_0) = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0$

Estimate Update:
$\hat{x}^{(\alpha)}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)]$

Kalman Gain: $K(t) = P(t)C^T(t)(R^{-1})$

Error Covariance Update:
$P^{(\alpha)}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^TR^{-1}C(t)P(t) + GQG^T$

The Conformable Information Filter (CIF)

Backwards Error Covariance Update:
$S^{(\alpha)}(\tau) = S(\tau)A(\tau) + A^T(\tau)S(\tau) + C^T(\tau)R^{-1}C(\tau) - S(\tau)GQG^TS(\tau)$

Backwards Gain:
$K_b(\tau) = S(\tau)GQ$

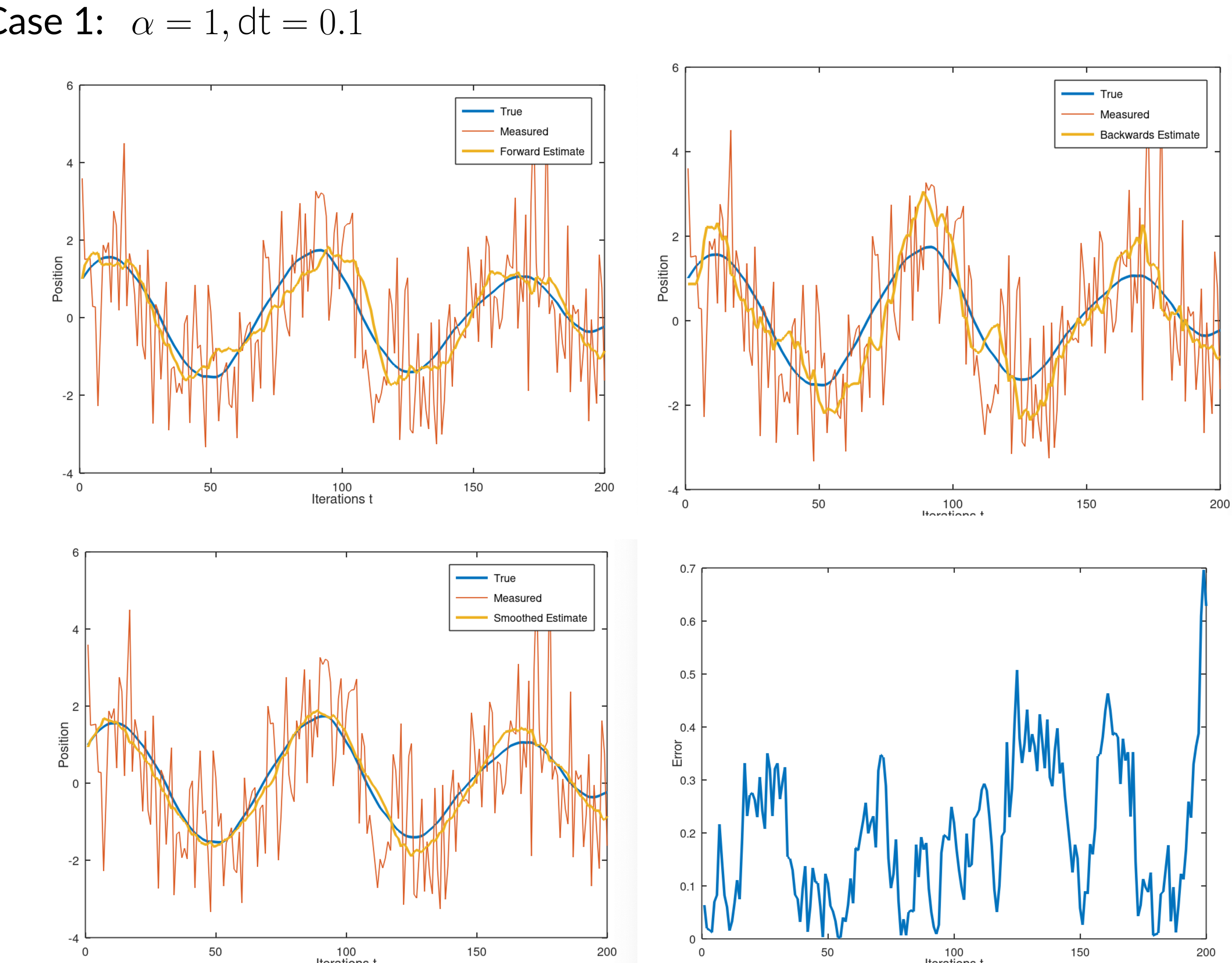
Backwards Estimate Update:
$\hat{\lambda}(\tau) = S(\tau)\hat{x}$ and $\lambda^{(\alpha)}(\tau) = (A^T(\tau) - K_b(\tau)G^T)\hat{\lambda}(\tau) - S(\tau)Bu(\tau) + C^T(\tau)(R^{-1})z(\tau)$

Smoother Gain:
$L(t) = P_f(t)S(t)(I + P_f(t)S(t))^{-1}$

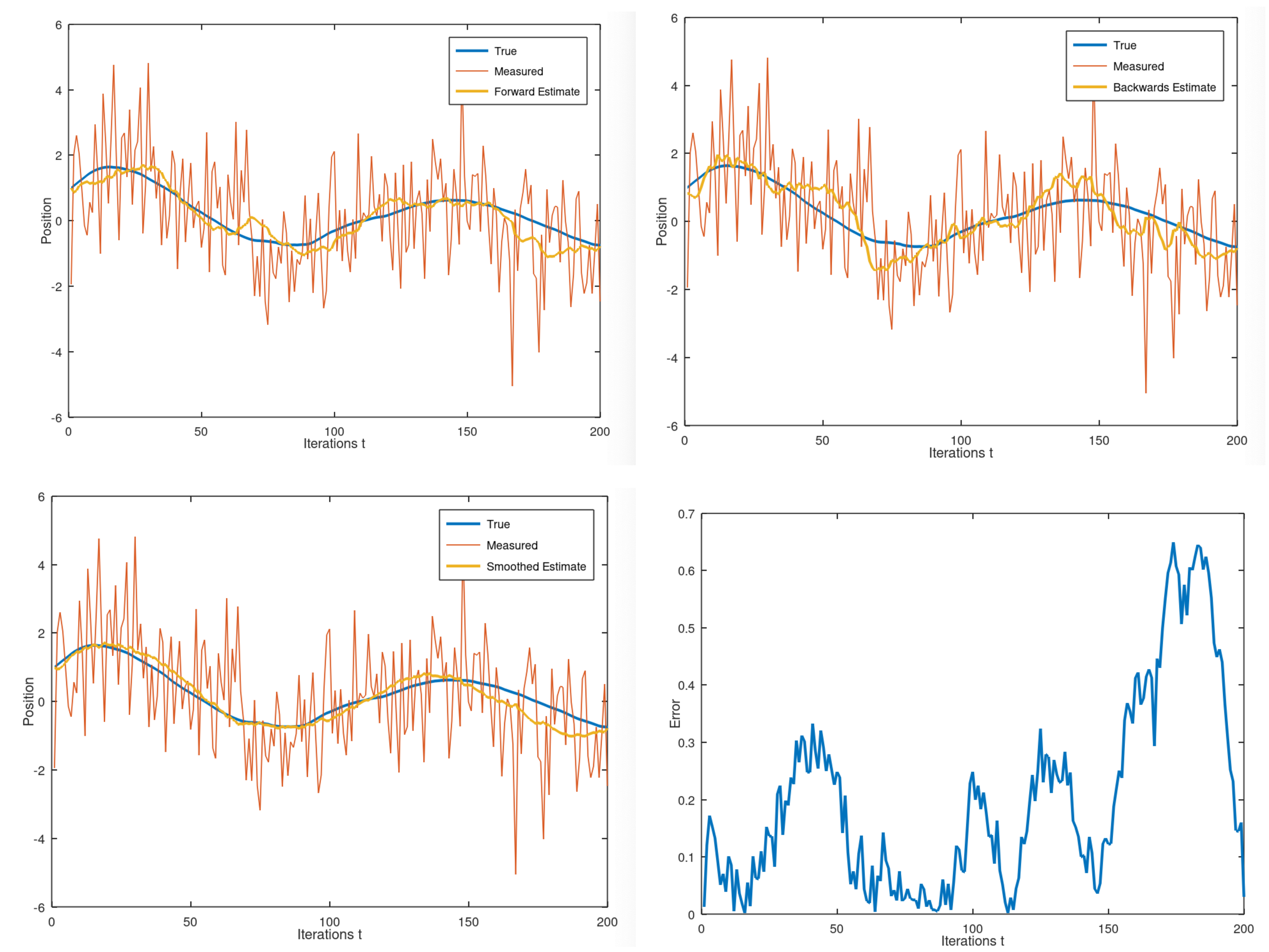
Smoothed Error Covariance:
$P_s(t) = (I - L(t))P_f(t)$

Smoothed Estimate:
$\hat{x}(t) = (I - L(t))\hat{x}_f(t) + P_s(t)\hat{\lambda}(t)$

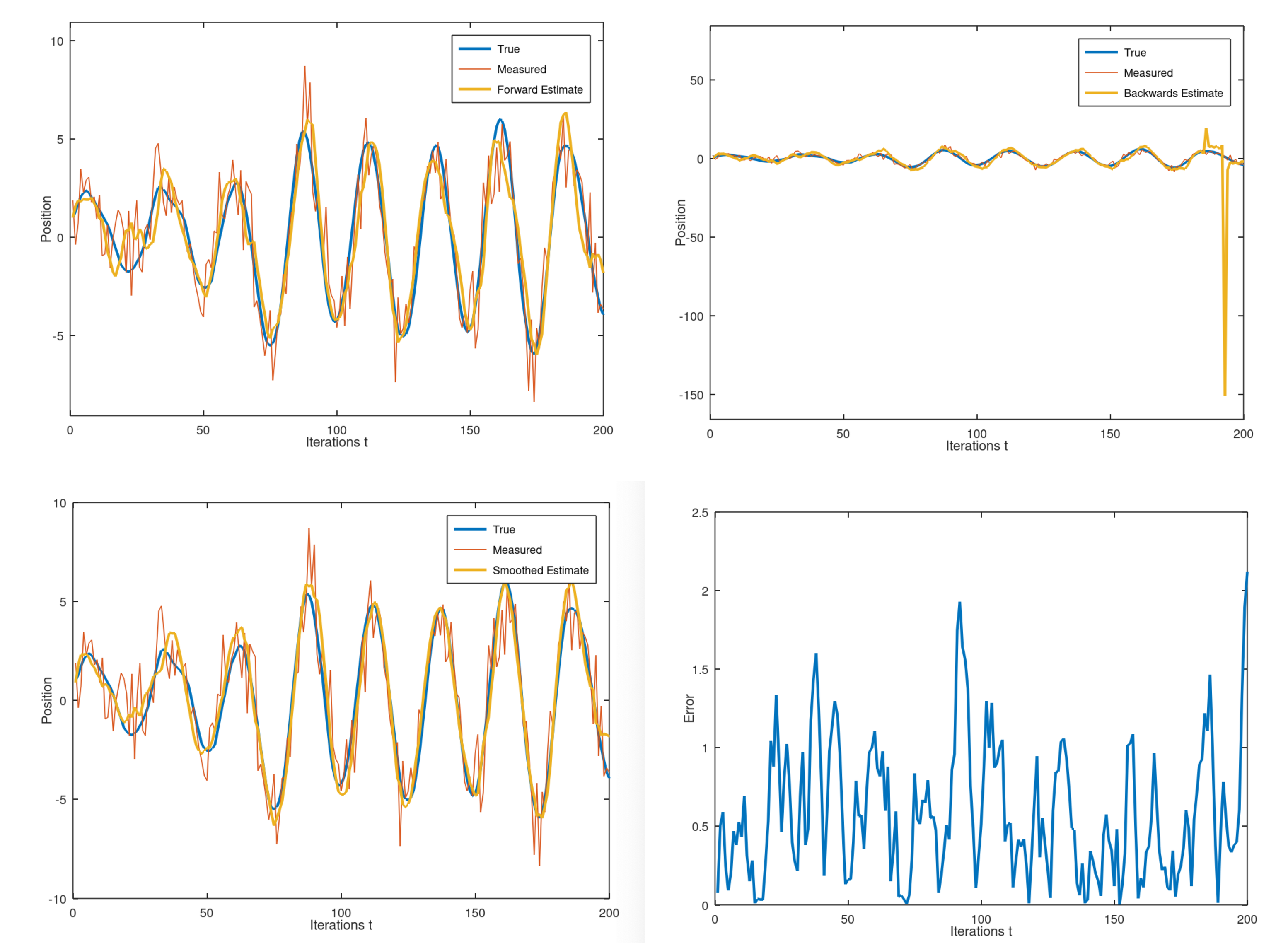
Case 1: $\alpha = 1$, dt $= 0.1$



Case 2: $\alpha = 0.67$, dt $= 0.01$



Case 3: $\alpha = 0.33$, dt $= 0.001$



References

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