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Adaptive synchronization of quaternion-valued neural networks with reaction–diffusion and fractional order

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Abstract. This paper is dedicated to the study of adaptive finite-time synchronization (FTS) for generalized delayed fractional-order reaction–diffusion quaternion-valued neural networks (GDFORDQVNN). Utilizing the suitable Lyapunov functional, Green's formula, and inequalities skills, testable algebraic criteria for ensuring the FTS of GDFORDQVNN are established on the basis of two adaptive controllers. Moreover, the numerical examples validate that the obtained results are feasible. Furthermore, they are also verified in image encryption as the application.

Keywords: adaptive synchronization, reaction–diffusion, finite time, quaternion-valued neural networks, image encryption.

1 Introduction

As we all know, synchronization has become an important research hotspot in neural networks (NN) and has been considered in numerous fields, for instance, robotic fields, biosystems, and control systems. Many types of synchronization issues have been studied owing to different effects. For example, projective synchronization, quasi-synchronization, H_∞-synchronization, etc. Over the past two decades, fractional-order (FO) derivatives have been considered to describe the models about engineering applications due to

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its advantages in describing memory and genetic properties. In particular, the study of fractional NN (FONN) has attracted increasing attention because of its widely applicable in systematic dynamics, for example, fluid mechanics, biological models, viscoelastic systems, and so on [8, 16, 18, 34].

Note that many previous considered models mainly focused on NN without reaction–diffusion (RD) term. Strictly speaking, diffusion phenomenon widely exists in NN and electronic circuits [19]. In fact, when the electron passes through the asymmetric electromagnetic field, the NN and electronic circuit will have diffusion effect. Thereupon, it is reasonable to discuss the neurons simultaneously with their changes in spatiotemporal. The reaction–diffusion NN (RDNN) also shows unpredictable behaviors, such as periodic oscillation, bifurcation, and chaotic attractor. The neuron state in RDNN depends on time and space at the same time, which can perfectly describe the evolution of time and space. Compared with the traditional NN, RDNN can achieve a better approximation of the actual system and has been widely used in the shortest path solution, image encryption [4], etc. Therefore, this paper combines the diffusion effect into the NN.

In recent years, some scholars have considered affect of the RD term on the FONN. Actually, fractional-order RDNN (FORDNN) have been applied to hydrology [15], finance [5], and plasma turbulence [31]. Various interesting works have been studied in [7, 21, 25, 28]. For instance, in [28], the problem of synchronization for competitive FORDNN was investigated by combining FO Lyapunov theory with M-matrix. In [25], the stability analysis of Riemann–Liouville FORDNN was studied by employing Lyapunov direct method. Using Lyapunov method, the analysis of generalized FORDNN with parameter mismatch was expanded in the paper [7]. Synchronization of complex-valued FORDNN in finite-time interval was achieved by using Lyapunov function [21].

In practice, real-valued NN often need to extend to higher dimensions, and quaternion-valued NN (QVNN) has unique merit in information processing. Thereupon, QVNN has aroused researchers' attention. Recently, FO has been inserted into QVNN forming FOQVNN, and many significative results were presented [2, 9, 11, 24, 33]. For instance, quasi-synchronization problem of delayed FOQVNN with parameter mismatches was studied in [11]. The authors of [9] investigated finite-time control for fuzzy FOQVNN with time delays and estimated the settling time. Stability and synchronization of the fuzzy memristive FOQVNN were investigated in [24].

It is worth pointing out that above works focus on asymptotic synchronization of FOQVNN, which shows that synchronization achieved gradually with the infinite extension of time. In practice, some systems were expected to realize synchronization as fast as possible. Consequently, FTS has been introduced and widely applied in lots of fields. It should be pointed out that FTS cannot only speeds up the convergence process, but also enhance robustness. The FTS of dynamic networks has attracted widely attention, in particulary, the synchronization in FTS was dealt with FOQVNN in [2, 33]. The works on studying dynamic behaviors of FOQVNN in the above literatures ignored diffusion terms, which are filled in the model of this paper.

To realize synchronization of NN, many control approaches have been proposed, which includes pinning control [12], impulsive control [22], sampled-data control [32], and feedback control [29]. Recently, adaptive control method has become an important control strategy in networks systems, which can guarantee better performance due to its powerful self-adjusting ability, and has been widely applied to synchronize NN [1, 24, 30]. However, there exists few works on FORDNN by applying adaptive controllers. In [3], researchers discussed the quasi-synchronization of coupled FORDNN by using an adaptive controller. In [23], the synchronization of memwristive FORDNN was derived by adaptive controllers, which were designed by Gronwall–Bellman inequality. To our knowledge, the issue of FTS for GDFORDQVNN via adaptive control strategy has not been investigated previously.

In view of analysis above, this paper aims to investigate the FTS for GDFORDQVNN by designing adaptive controllers. The principal novelties of this paper can be stated as below. (i) Differt from previous works on QVNNs in [2, 24, 33] that without reaction–diffusion terms and the models in [20] that without FO, the considered FO system takes RD term into account in this paper. This suggests that the model is more general and has more practical value. (ii) Through designing appropriate adaptive controller, synchronization criteria of the proposed model is derived in finite time. (iii) The settling time is estimated, which is an expansion and optimization in some existing work.

Notations. $\mathbb{R}$ and $\mathbb{Q}$ are real and quaternion values, respectively. $\mathbb{R}^m$ denotes m-dimensional real vector. ρ is a quaternion, which can be described as $\rho = h^R + \mathrm{i}\rho^I + \mathrm{j}\rho^J + \mathrm{k}\rho^K$, where $\rho^\nu \in \mathbb{R}$, $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are the imaginary units, which obey: $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = \mathrm{k} = -\mathrm{ji}$, $\mathrm{jk} = \mathrm{i} = -\mathrm{kj}$, $\mathrm{ki} = \mathrm{j} = -\mathrm{ik}$, $\nu = R, I, J, K$ stands for the quaternion divided into four real parts. Let $\Omega = \{x\colon x = (x_1, x_2, \dots, x_m)^{\mathrm{T}}, |x_k| \leqslant l_k, l_k > 0, k = 1, 2, \dots, mt\} \in \mathbb{R}^m$ be a bounded open set containing the origin. Moreover, the $\partial\Omega$ is the smooth boundary, and Ω is measurable with $\operatorname{mes}\Omega > 0$.

2 Model description and preliminaries

Considering GDFORDQVNN as follows:

$$\begin{aligned}\frac{\partial^\alpha h_i(t,x)}{\partial t^\alpha} &= d_i \Delta h_i(t,x) - c_i h_i(t,x) \\ &\quad + \sum_{l=1}^{n} a_{il} f_l\big(w_l h_l(t,x)\big) + \sum_{l=1}^{n} b_{il} g_l\big(\bar{w}_l h_l\big(t-\tau(t),x\big)\big) \\ &\quad + O_i(t,x), \quad i = 1, 2, \dots, n,\ t > 0, \end{aligned} \tag{1}$$

in which $0 < \alpha \leqslant 1$, $h_i(t,x) \in \mathbb{Q}$ denotes the state variable, $\Delta = \sum_{j=1}^{m} \partial^2/\partial x_j^2$ represents the Laplace operator on Ω, $d_i > 0$ is the transmission diffusion coefficient. $c_i > 0$ stands for the self-feedback coefficient, $a_{il}, b_{il} \in \mathbb{Q}$ are connection coefficients, w_l, $\bar{w}_l$ denote synaptic connectivity, $f_l(\cdot), g_l(\cdot) \in \mathbb{Q}$ represent activation functions, $O_i(t,x) \in \mathbb{Q}$ is external input, $\tau(t)$ denotes time-varying delay that satisfies $0 < \tau(t) < \tau$ ($\tau = \text{const}$).

Remark 1. Since RD is ubiquitous in practical, which could effect the performance of the system, it is significant to study the delayed spatiotemporal FONN. Different from [33], RD phenomenons are taken into consideration to study FTS of FOQVNN, which is more novel and widely applied.

By virtue of Hamilton rule, system (1) can be divided into four parts:

$$\begin{aligned}\frac{\partial^\alpha h_i^R(t,x)}{\partial t^\alpha} &= d_i\Delta h_i^R(t,x) - c_i h_i^R(t,x) + \sum_{l=1}^n \big[a_{il}^R f_l^R(h_l^R(t,x)) \\ &\quad - a_{il}^I f_l^I\big(h_l^I(t,x)\big) - a_{il}^J f_l^J\big(h_l^J(t,x)\big) - a_{il}^K f_l^K\big(h_l^K(t,x)\big)\big] \\ &\quad + \sum_{l=1}^n \big[b_{il}^R g_l^R\big(h_l^R\big(t-\tau(t),x\big)\big) - b_{il}^I g_l^I\big(h_l^I\big(t-\tau(t),x\big)\big) \\ &\quad - b_{il}^J g_l^J\big(h_l^J\big(t-\tau(t),x\big)\big) - b_{il}^K g_l^K\big(h_l^K\big(t-\tau(t),x\big)\big)\big] + O_i^R(t,x),\end{aligned}$$

and $\partial^\alpha h_i^I(t,x)/\partial t^\alpha, \partial^\alpha h_i^J(t,x)/\partial t^\alpha, \partial^\alpha h_i^K(t,x)/\partial t^\alpha$ can be gained similarly.

Consider Dirichlet boundary conditions

$$h_i^\nu(t,x) = 0, \quad t \in [-\tau, +\infty),\ x \in \partial\Omega,$$

and initial values

$$h_i^\nu(s,x) = \varphi_{0i}^\nu(s,x), \quad s \in [-\tau, 0],\ x \in \Omega,$$

in which $\varphi_{0i}^\nu(s,x) \in \mathbb{R}$ is a continuous function defined on $[-\tau, 0] \times \Omega$.

To further discuss, we list the relevant assumption, definitions and lemmas.

Assumption 1. There exist constants $M_l^\nu > 0$, $N_l^\nu > 0$ such that for all $\mu_1, \mu_2 \in \mathbb{R}$,

$$\big|f_l^\nu(\mu_1) - f_l^\nu(\mu_2)\big| \leqslant M_l^\nu|\mu_1 - \mu_2|, \qquad \big|g_l^\nu(\mu_1) - g_l^\nu(\mu_2)\big| \leqslant N_l^\nu|\mu_1 - \mu_2|.$$

Definition 1. (See [17].) The Caputo derivative of FO α of function $\phi(t) : [0, +\infty) \to \mathbb{R}$ is defined by

$$D^\alpha\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\phi'(s)\,\mathrm{d}s, \quad \alpha > 0.$$

Definition 2. (See [17].) For a continuously differentiable $\phi(t,x) : [0,+\infty) \times \Omega \to \mathbb{R}$, the time Caputo derivative of FO α is given as

$$\frac{\partial^\alpha\phi(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial\phi(s,x)}{\partial s}(t-s)^{-\alpha}\,\mathrm{d}s, \quad 0 < \alpha \leqslant 1.$$

In particular, if $\phi(t,x) = \phi(t)$, then $\partial^\alpha\phi(t,x)/\partial t^\alpha = \mathrm{d}^\alpha\phi(t)/\mathrm{d}t^\alpha$.

Lemma 1. (*See* [22].) *For any $x \in \Omega$, suppose that $\phi(t,x) : [0,+\infty) \times \Omega \to \mathbb{R}$ has continuous derivatives function on t. Then*

$$\frac{\partial^\alpha\phi^2(t,x)}{\partial t^\alpha} \leqslant 2\phi(t,x)\frac{\partial^\alpha\phi(t,x)}{\partial t^\alpha}, \quad 0 < \alpha \leqslant 1.$$

Lemma 2. (*See* [13].) *Let Ω be a cube $|x_k| < l_k$ ($k = 1, 2, \dots, n$), and let $\psi(x) : \Omega \to \mathbb{R}$ be continuously differentiable function with $\psi(x)|_{\partial\Omega} = 0$. Then*

$$\int_{\Omega} \psi^2(x)\,\mathrm{d}x \leqslant l_k^2 \int_{\Omega} \left|\frac{\partial\psi(x)}{\partial x_k}\right|^2 \mathrm{d}x.$$

Lemma 3. (*See* [14].) *Let function $f(t,x) : [0, +\infty) \times \Omega \to \mathbb{R}$ be integrable on Ω and derivable with respect to t. Assume that $V(t) = \int_{\Omega} f(t,x)\,\mathrm{d}x$. Then*

$$D^{\alpha}V(t) = \int_{\Omega} \frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t,x)\,\mathrm{d}x.$$

Lemma 4. (*See* [10].) *Let $W(t)$ be a nonnegative and continuous function, which satisfies*

$$D^{\alpha}W(t) \leqslant -aW(t) + bW\bigl(t - \tau(t)\bigr) - cW^{\beta}(t),$$

where $0 < \alpha < 1$, $0 < \beta \leqslant 1$, $0 < \tau(t) < \tau$, $\gamma = \sup_{-\tau<s<0} W(s)$, $a > b > 0, c > 0$. Then $V(t)$ converges to 0 within the time T^* estimated by

$$T^* = \left[\frac{\Gamma(1 + \frac{1}{1-\beta})\Gamma(2-\alpha)\Gamma(1+\alpha)}{\Gamma(1 + \frac{1}{1-\beta} - \alpha)(a-b)} \ln \frac{(a-b)\gamma^{1-\beta} + c}{c}\right]^{1/\alpha}.$$

3 Main results

Taking (1) as the master model, the slave model can be constructed as

$$\begin{aligned} \frac{\partial^{\alpha}\bar{h}_i(t,x)}{\partial t^{\alpha}} &= d_i\Delta\bar{h}_i(t,x) - c_i\bar{h}_i(t,x) \\ &\quad + \sum_{l=1}^{n} a_{il} f_l\bigl(w_l\bar{h}_l(t,x)\bigr) + \sum_{l=1}^{n} b_{il} g_l\bigl(\bar{w}_l\bar{h}_l(t-\tau(t),x)\bigr) \\ &\quad + O_i(t,x) + u_i(t,x), \quad i = 1, 2, \dots, n, \end{aligned} \tag{2}$$

in which $u_i(t,x)$ is the controller.

We defined the error system as $e_i(t,x) = \bar{h}_i(t,x) - h_i(t,x)$. From (1) and (2) it yields:

$$\begin{aligned} \frac{\partial^{\alpha}e_i^R(t,x)}{\partial t^{\alpha}} &= d_i\Delta e_i^R(t,x) - c_i e_i^R(t,x) + \sum_{l=1}^{n} \bigl[a_{il}^R f_l^R\bigl(e_l^R(t,x)\bigr) - a_{il}^I f_l^I\bigl(e_l^I(t,x)\bigr) \\ &\quad - a_{il}^J f_l^J\bigl(e_l^J(t,x)\bigr) - a_{il}^K f_l^K\bigl(e_l^K(t,x)\bigr)\bigr] \\ &\quad + \sum_{l=1}^{n} \bigl[b_{il}^R g_l^R\bigl(e_l^R\bigl(t-\tau(t),x\bigr)\bigr) - b_{il}^I g_l^I\bigl(e_l^I\bigl(t-\tau(t),x\bigr)\bigr) \\ &\quad - b_{il}^J g_l^J\bigl(e_l^J\bigl(t-\tau(t),x\bigr)\bigr) - b_{il}^K g_l^K\bigl(e_l^K\bigl(t-\tau(t),x\bigr)\bigr)\bigr] + u_i^R(t,x), \end{aligned}$$

where

$$f_l^R\big(e_l^R(t,x)\big) = f_l^R\big(w_l\bar{h}_l^R(t,x)\big) - f_l^R\big(w_l h_l^R(t,x)\big),$$
$$g_l^R\big(e_l^R\big(t-\tau(t),x\big)\big) = g_l^R\big(\bar{w}_l\bar{h}_l^R\big(t-\tau(t),x\big)\big) - g_l^R\big(\bar{w}_l h_l^R\big(t-\tau(t),x\big)\big),$$

and $\partial^\alpha e_i^I(t,x)/\partial t^\alpha$, $\partial^\alpha e_i^J(t,x)/\partial t^\alpha$, $\partial^\alpha e_i^K(t,x)/\partial t^\alpha$ can be gained similarly.

In order to save the energy, adaptive controller is considered in the following. The control strategy $u_i^\nu(t,x)$ is chosen by

$$u_i^\nu(t,x) = \begin{cases} -(\xi_i^\nu(t) + \bar{\xi}_i^{*\nu})e_i^\nu(t,x) - \bar{\theta}_i^\nu[e_i^\nu(t,x)]^{1-2\beta}, & e_i^\nu(t,x) \neq 0, \\ 0, & e_i^\nu(t,x) = 0, \end{cases} \tag{3}$$

where $\bar{\xi}_i^{*\nu}$, $\bar{\theta}_i^\nu$ are constants, and adaptive control law $\xi_i^\nu(t)$ satisfies

$$D^\alpha \xi_i^\nu(t) = \delta_i^\nu \int\limits_\Omega \big[e_i^\nu(t,x)\big]^2\,\mathrm{d}x, \quad \delta_i^\nu > 0.$$

For writing convenience, the notations are given below:

$$\eta_i^\nu = \sum_{j=1}^m \frac{2d_i}{l_j^2} - \sum_{l=1}^n \big|a_{il}^\nu w_l M_l^\nu\big| - \sum_{l=1}^n M_i^\nu |w_i| \big[\big|a_{li}^R\big| + \big|a_{li}^J\big| + \big|a_{li}^K\big| + \big|a_{li}^I\big|\big] + 2c_i,$$

$$\begin{aligned}
\lambda_i^R = \eta_i^R &- \sum_{l=1}^n \big|a_{il}^I w_l M_l^I\big| - \sum_{l=1}^n \big|a_{il}^J w_l M_l^J\big| - \sum_{l=1}^n \big|a_{il}^K w_l M_l^K\big| - \sum_{l=1}^n \big|b_{il}^R \bar{w}_l N_l^R\big| \\
&- \sum_{l=1}^n \big|b_{il}^I \bar{w}_l N_l^I\big| - \sum_{l=1}^n \big|b_{il}^J \bar{w}_l N_l^J\big| - \sum_{l=1}^n \big|b_{il}^K \bar{w}_l N_l^K\big|, \\
\lambda_i^I = \eta_i^I &- \sum_{l=1}^n \big|a_{il}^R w_l M_l^I\big| - \sum_{l=1}^n \big|a_{il}^K w_l M_l^J\big| - \sum_{l=1}^n \big|a_{il}^J w_l M_l^K\big| - \sum_{l=1}^n \big|b_{il}^I \bar{w}_l N_l^R\big| \\
&- \sum_{l=1}^n \big|b_{il}^R \bar{w}_l N_l^I\big| - \sum_{l=1}^n \big|b_{il}^K \bar{w}_l N_l^J\big| - \sum_{l=1}^n \big|b_{il}^J \bar{w}_l N_l^K\big|, \\
\lambda_i^J = \eta_i^J &- \sum_{l=1}^n \big|a_{il}^K w_l M_l^I\big| - \sum_{l=1}^n \big|a_{il}^R w_l M_l^J\big| - \sum_{l=1}^n \big|a_{il}^I w_l M_l^K\big| - \sum_{l=1}^n \big|b_{il}^J \bar{w}_l N_l^R\big| \\
&- \sum_{l=1}^n \big|b_{il}^K \bar{w}_l N_l^I\big| - \sum_{l=1}^n \big|b_{il}^R \bar{w}_l N_l^J\big| - \sum_{l=1}^n \big|b_{il}^I \bar{w}_l N_l^K\big|, \\
\lambda_i^K = \eta_i^K &- \sum_{l=1}^n \big|a_{il}^J w_l M_l^I\big| - \sum_{l=1}^n \big|a_{il}^I w_l M_l^J\big| - \sum_{l=1}^n \big|a_{il}^R w_l M_l^K\big| - \sum_{l=1}^n \big|b_{il}^K \bar{w}_l N_l^R\big| \\
&- \sum_{l=1}^n \big|b_{il}^J \bar{w}_l N_l^I\big| - \sum_{l=1}^n \big|b_{il}^I \bar{w}_l N_l^J\big| - \sum_{l=1}^n \big|b_{il}^R \bar{w}_l N_l^K\big|,
\end{aligned}$$

$$\bar{\lambda}_1^\nu = \min_{1\leqslant i\leqslant n}\bigl\{\lambda_i^\nu + 2\bar{\xi}_i^{*\nu}\bigr\}, \qquad \bar{\lambda}_2^\nu = \max_{1\leqslant i\leqslant n}\Biggl\{\sum_{l=1}^n N_i^\nu|\bar{w}_i|\bigl[|b_{li}^R| + |b_{li}^I| + |b_{li}^J| + |b_{li}^K|\bigr]\Biggr\},$$
$$\bar{\lambda}_3^\nu = \min_{1\leqslant i\leqslant n}\bigl\{2\bar{\theta}_i^\nu\bigr\},$$
$$\bar{\lambda}_1 = \min\bigl\{\bar{\lambda}_1^R, \bar{\lambda}_1^I, \bar{\lambda}_1^J, \bar{\lambda}_1^K\bigr\}, \qquad \bar{\lambda}_2 = \max\bigl\{\bar{\lambda}_2^R, \bar{\lambda}_2^I, \bar{\lambda}_2^J, \bar{\lambda}_2^K\bigr\},$$
$$\bar{\lambda}_3 = \min\bigl\{\bar{\lambda}_3^R, \bar{\lambda}_3^I, \bar{\lambda}_3^J, \bar{\lambda}_3^K\bigr\}.$$

Theorem 1. *Under the adaptive controller* (3) *and Assumption* 1*, if the inequalities*

$$\bar{\xi}_i^{*\nu} + \frac{1}{2}\lambda_i^\nu > 0, \qquad \bar{\lambda}_1 > \bar{\lambda}_2 > 0, \qquad \bar{\lambda}_3 > 0 \tag{4}$$

persist, then systems (1) *and* (2) *can reach FTS with*

$$\bar{T}^* = \left[\frac{\Gamma(1+\frac{1}{\beta})\Gamma(2-\alpha)\Gamma(1+\alpha)}{\Gamma(1+\frac{1}{\beta}-\alpha)(\bar{\lambda}_1-\bar{\lambda}_2)}\ln\frac{(\bar{\lambda}_1-\bar{\lambda}_2)\gamma^\beta + \bar{\lambda}_3}{\bar{\lambda}_3}\right]^{1/\alpha}. \tag{5}$$

Proof. We use Lyapunov function

$$V(t) = V^R(t) + V^I(t) + V^J(t) + V^K(t)$$

in which $V^\nu(t) = \int_\Omega \sum_{i=1}^n [e_i^\nu(t,x)]^2\,\mathrm{d}x + \sum_{i=1}^n [\xi_i^\nu(t)]^2/\delta_i^\nu$.

We take the fractional derivative of $V(t)$ of order α

$$D^\alpha V(t) = D^\alpha V^R(t) + D^\alpha V^I(t) + D^\alpha V^J(t) + D^\alpha V^K(t).$$

According to Lemmas 1 and 2, we have

$$\begin{aligned}
D^\alpha V^R(t) &\leqslant \sum_{i=1}^n \int_\Omega 2e_i^R(t,x)\frac{\partial^\alpha e_i^R(t,x)}{\partial t^\alpha}\,\mathrm{d}x + \sum_{i=1}^n \frac{2}{\delta_i}\xi_i^R(t)D^\alpha \xi_i^R(t) \\
&= \sum_{i=1}^n \int_\Omega 2e_i^R(t,x)\Biggl\{\sum_{j=1}^m d_i\frac{\partial^2 e_i^R(t,x)}{\partial x_j^2} - c_i e_i^R(t,x) \\
&\quad + \sum_{l=1}^n \bigl[a_{il}^R f_l^R\bigl(e_l^R(t,x)\bigr) - a_{il}^I f_l^I\bigl(e_l^I(t,x)\bigr) - a_{il}^J f_l^J\bigl(e_l^J(t,x)\bigr) \\
&\quad - a_{il}^K f_l^K\bigl(e_l^K(t,x)\bigr)\bigr] \\
&\quad \times \sum_{l=1}^n \bigl[b_{il}^R + g_l^R\bigl(e_l^R(t-\tau(t),x)\bigr) - b_{il}^I g_l^I\bigl(e_l^I(t-\tau(t),x)\bigr) \\
&\quad - b_{il}^J g_l^J\bigl(e_l^J(t-\tau(t),x)\bigr) - b_{il}^K g_l^K\bigl(e_l^K(t-\tau(t),x)\bigr)\bigr] + u_i^R(t,x)\Biggr\}\,\mathrm{d}x \\
&\quad + \sum_{i=1}^n 2\xi_i^R(t)\int_\Omega \bigl[e_i^R(t,x)\bigr]^2\,\mathrm{d}x.
\end{aligned}$$

By using Green's theorem and zero Dirichlet boundary conditions,

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Omega} 2d_i e_i^R(t,x)\sum_{j=1}^{m}\frac{\partial^2 e_i^R(t,x)}{\partial x_j^2}\,\mathrm{d}x\\
&\quad=\sum_{i=1}^{n}\int_{\partial\Omega} 2d_i e_i^R(t,x)\sum_{j=1}^{m}\frac{\partial e_i^R(t,x)}{\partial x_j}\,\mathrm{d}x-\sum_{i=1}^{n}\int_{\Omega} 2d_i\sum_{j=1}^{m}\left(\frac{\partial e_i^R(t,x)}{\partial x_j}\right)^2\mathrm{d}x\\
&\quad=-\sum_{i=1}^{n}\int_{\Omega} 2d_i\sum_{j=1}^{m}\left(\frac{\partial e_i^R(t,x)}{\partial x_j}\right)^2\mathrm{d}x\leqslant-\sum_{i=1}^{n}\int_{\Omega}\sum_{j=1}^{m}\frac{2d_i}{l_j^2}\left[e_i^R(t,x)\right]^2\mathrm{d}x. \qquad (6)
\end{aligned}$$

In addition, from Assumption 1

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Omega} 2\sum_{l=1}^{n} e_i^R(t,x)\big[a_{il}^R f_l^R\big(e_l^R(t,x)\big)-a_{il}^I f_l^I\big(e_l^I(t,x)\big)\\
&\qquad - a_{il}^J f_l^J\big(e_l^J(t,x)\big)-a_{il}^K f_l^K\big(e_l^K(t,x)\big)\big]\,\mathrm{d}x\\
&\quad\leqslant\sum_{i=1}^{n}\int_{\Omega} 2\sum_{l=1}^{n}\left|e_i^R(t,x)a_{il}^R w_l M_l^R e_l^R(t,x)\right|\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int_{\Omega} 2\sum_{l=1}^{n}\left|e_i^R(t,x)a_{il}^I w_l M_l^I e_l^I(t,x)\right|\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int_{\Omega} 2\sum_{l=1}^{n}\left|e_i^R(t,x)a_{il}^J w_l M_l^J e_l^J(t,x)\right|\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int_{\Omega} 2\sum_{l=1}^{n}\left|e_i^R(t,x)a_{il}^K w_l M_l^K e_l^K(t,x)\right|\mathrm{d}x\\
&\quad\leqslant\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|a_{il}^R w_l M_l^R\right|\left[\big(e_i^R(t,x)\big)^2+\big(e_l^R(t,x)\big)^2\right]\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|a_{il}^I w_l M_l^I\right|\left[\big(e_i^R(t,x)\big)^2+\big(e_l^I(t,x)\big)^2\right]\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|a_{il}^J w_l M_l^J\right|\left[\big(e_i^R(t,x)\big)^2+\big(e_l^J(t,x)\big)^2\right]\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|a_{il}^K w_l M_l^K\right|\left[\big(e_i^R(t,x)\big)^2+\big(e_l^K(t,x)\big)^2\right]\mathrm{d}x. \qquad (7)
\end{aligned}$$

Similarly, for the delayed term,

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Omega} 2\sum_{l=1}^{n} e_i^R(t,x)\big[b_{il}^R g_l^R\big(e_l^R\big(t-\tau(t),x\big)\big)-b_{il}^I e_l^I\big(e_l^I\big(t-\tau(t),x\big)\big)\\
&\qquad\times b_{il}^J g_l^J\big(e_l^J\big(t-\tau(t),x\big)\big)-b_{il}^K g_l^K\big(e_l^K\big(t-\tau(t),x\big)\big)\big]\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_{l}N_{l}^{R}\right|\left[\left(e_{i}^{R}(t,x)\right)^{2}+\left(e_{l}^{R}\left(t-\tau(t),x\right)\right)^{2}\right]\mathrm{d}x\\
&\quad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_{l}N_{l}^{I}\right|\left[\left(e_{i}^{R}(t,x)\right)^{2}+\left(e_{l}^{I}\left(t-\tau(t),x\right)\right)^{2}\right]\mathrm{d}x\\
&\quad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_{l}N_{l}^{J}\right|\left[\left(e_{i}^{R}(t,x)\right)^{2}+\left(e_{l}^{J}\left(t-\tau(t),x\right)\right)^{2}\right]\mathrm{d}x\\
&\quad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_{l}N_{l}^{K}\right|\left[\left(e_{i}^{R}(t,x)\right)^{2}+\left(e_{l}^{K}(t-\tau(t),x)\right)^{2}\right]\mathrm{d}x.
\end{aligned}\tag{8}
$$

Applying controller (3) gives

$$
\begin{aligned}
&\sum_{i=1}^{n}2\int\limits_{\Omega}u_{i}^{R}(t,x)e_{i}^{R}(t,x)\,\mathrm{d}x+\sum_{i=1}^{n}2\xi_{i}^{R}(t)\int\limits_{\Omega}\left[e_{i}^{R}(t,x)\right]^{2}\mathrm{d}x\\
&\quad=-\sum_{i=1}^{n}\int\limits_{\Omega}2\left(\xi_{i}^{R}(t)+\bar{\xi}^{*R}\right)\left[e_{i}^{R}(t,x)\right]^{2}\mathrm{d}x-\sum_{i=1}^{n}\int\limits_{\Omega}2\bar{\theta}_{i}^{R}\left[e_{i}^{R}(t,x)\right]^{2-2\beta}\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}2\xi_{i}^{R}(t)\int\limits_{\Omega}\left[e_{i}^{R}(t,x)\right]^{2}\mathrm{d}x\\
&\quad=-\sum_{i=1}^{n}\int\limits_{\Omega}2\xi_{i}^{*R}\left[e_{i}^{R}(t,x)\right]^{2}\mathrm{d}x-\sum_{i=1}^{n}\int\limits_{\Omega}2\bar{\theta}_{i}^{R}\left[e_{i}^{R}(t,x)\right]^{2-2\beta}\mathrm{d}x.
\end{aligned}\tag{9}
$$

From (6)–(9)

$$
\begin{aligned}
&D^{\alpha}V^{R}(t)\\
&\quad\leqslant-\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\frac{2d_{i}}{l_{j}^{2}}-\sum_{l=1}^{n}\left|a_{il}^{R}w_{l}M_{l}^{R}\right|-\sum_{l=1}^{n}\left|a_{li}^{R}w_{i}M_{i}^{R}\right|-\sum_{l=1}^{n}\left|a_{il}^{I}w_{l}M_{l}^{I}\right|\\
&\qquad-\sum_{l=1}^{n}\left|a_{il}^{J}w_{l}M_{l}^{J}\right|-\sum_{l=1}^{n}\left|a_{il}^{K}w_{l}M_{l}^{K}\right|-\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_{l}N_{l}^{R}\right|-\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_{l}N_{l}^{I}\right|\\
&\qquad-\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_{l}N_{l}^{J}\right|-\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_{l}N_{l}^{K}\right|+2c_{i}+2\xi_{i}^{*R}-\left|\eta_{i}^{R}\right|\Bigg]\int\limits_{\Omega}\left[e_{i}^{R}(t,x)\right]^{2}\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\Bigg[\sum_{l=1}^{n}\left|b_{li}^{R}\bar{w}_{i}N_{i}^{R}\right|+\left|\eta_{i}^{R}\right|\Bigg]\int\limits_{\Omega}\left[e_{i}^{R}\left(t-\tau(t),x\right)\right]^{2}\mathrm{d}x\\
&\qquad-\sum_{i=1}^{n}\int\limits_{\Omega}2\bar{\theta}_{i}^{R}\left[e_{i}^{R}(t,x)\right]^{2-2\beta}\mathrm{d}x+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|a_{li}^{I}w_{i}M_{i}^{I}\right|\left(e_{i}^{I}(t,x)\right)^{2}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|a_{li}^{J}w_iM_i^{J}\right|\left(e_i^{J}(t,x)\right)^2\mathrm{d}x+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|a_{li}^{K}w_iM_i^{K}\right|\left(e_i^{K}(t,x)\right)^2\mathrm{d}x\\
&+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{I}\bar{w}_iN_i^{I}\right|\left(e_i^{I}(t-\tau(t),x)\right)^2\mathrm{d}x\\
&+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{J}\bar{w}_iN_i^{J}\right|\left(e_i^{J}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x\\
&+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{K}\bar{w}_iN_i^{K}\right|\left(e_i^{K}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&D^{\alpha}V^{I}(t)\\
&\quad\leqslant-\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\frac{2d_i}{l_j^2}-\sum_{l=1}^{n}\left|a_{il}^{I}w_lM_l^{R}\right|-\sum_{l=1}^{n}\left|a_{li}^{R}w_iM_i^{I}\right|-\sum_{l=1}^{n}\left|a_{il}^{R}w_lM_l^{I}\right|\\
&\qquad-\sum_{l=1}^{n}\left|a_{il}^{K}w_lM_l^{J}\right|-\sum_{l=1}^{n}\left|a_{il}^{J}w_lM_l^{K}\right|-\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_lN_l^{R}\right|-\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_lN_l^{I}\right|\\
&\qquad-\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_lN_l^{J}\right|-\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_lN_l^{K}\right|+2c_i+2\xi_i^{*I}-\left|\eta_i^{I}\right|\Bigg]\int\limits_{\Omega}\left[e_i^{I}(t,x)\right]^2\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\Bigg[\sum_{l=1}^{n}\left|b_{li}^{R}\bar{w}_iN_i^{I}\right|+\left|\eta_i^{I}\right|\Bigg]\int\limits_{\Omega}\left[e_i^{I}\left(t-\tau(t),x\right)\right]^2\mathrm{d}x\\
&\qquad-\sum_{i=1}^{n}\int\limits_{\Omega}2\bar{\theta}_i^{I}\left[e_i^{I}(t,x)\right]^{2-2\beta}\mathrm{d}x+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|a_{li}^{I}w_iM_i^{R}\right|\left(e_i^{R}(t,x)\right)^2\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|a_{li}^{K}w_iM_i^{J}\right|\left(e_i^{J}(t,x)\right)^2\mathrm{d}x+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|a_{li}^{J}w_iM_i^{K}\right|\left(e_i^{K}(t,x)\right)^2\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{I}\bar{w}_iN_i^{R}\right|\left(e_i^{R}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{K}\bar{w}_iN_i^{J}\right|\left(e_l^{J}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x\\
&\qquad+\sum_{i=1}^{n}\int\limits_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{J}\bar{w}_iN_i^{K}\right|\left(e_i^{K}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&D^{\alpha}V^{J}(t)\\
&\quad\leqslant-\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\frac{2d_i}{l_j^2}-\sum_{l=1}^{n}\left|a_{il}^{J}w_lM_l^{R}\right|-\sum_{l=1}^{n}\left|a_{il}^{K}w_lM_l^{I}\right|-\sum_{l=1}^{n}\left|a_{il}^{R}w_lM_l^{J}\right|
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{l=1}^{n} \left|a_{li}^{R} w_i M_i^{J}\right| - \sum_{l=1}^{n} \left|a_{il}^{I} w_l M_l^{K}\right| - \sum_{l=1}^{n} \left|b_{il}^{J} \bar{w}_l N_l^{R}\right| - \sum_{l=1}^{n} \left|b_{il}^{K} \bar{w}_l N_l^{I}\right| \\
&- \sum_{l=1}^{n} \left|b_{il}^{R} \bar{w}_l N_l^{J}\right| - \sum_{l=1}^{n} \left|b_{il}^{I} \bar{w}_l N_l^{K}\right| + 2c_i + 2\xi_i^{*J} - \left|\eta_i^{J}\right|\Bigg] \int_{\Omega} \left[e_i^{J}(t,x)\right]^2 \mathrm{d}x \\
&+ \sum_{i=1}^{n} \left[\sum_{l=1}^{n} \left|b_{li}^{R} \bar{w}_i N_i^{J}\right| + \left|\eta_i^{J}\right|\right] \int_{\Omega} \left[e_i^{J}\big(t-\tau(t),x\big)\right]^2 \mathrm{d}x \\
&- \sum_{i=1}^{n} \int_{\Omega} 2\bar{\theta}_i^{J} \left[e_i^{J}(t,x)\right]^{2-2\beta} \mathrm{d}x + \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|a_{li}^{J} w_i M_i^{R}\right| \big(e_i^{R}(t,x)\big)^2 \mathrm{d}x \\
&+ \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|a_{li}^{K} w_i M_i^{I}\right| \big(e_i^{I}(t,x)\big)^2 \mathrm{d}x + \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|a_{li}^{I} w_i M_i^{K}\right| \big(e_i^{K}(t,x)\big)^2 \mathrm{d}x \\
&+ \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|b_{li}^{J} \bar{w}_i N_i^{R}\right| \big(e_i^{R}\big(t-\tau(t),x\big)\big)^2 \mathrm{d}x \\
&+ \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|b_{li}^{K} \bar{w}_i N_i^{I}\right| \big(e_i^{I}\big(t-\tau(t),x\big)\big)^2 \mathrm{d}x \\
&+ \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|b_{li}^{I} \bar{w}_i N_i^{K}\right| \big(e_i^{K}\big(t-\tau(t),x\big)\big)^2 \mathrm{d}x, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&D^{\alpha} V^{K}(t) \\
&\leqslant - \sum_{i=1}^{n} \Bigg[\sum_{j=1}^{m} \frac{2d_i}{l_j^2} - \sum_{l=1}^{n} \left|a_{il}^{K} w_l M_l^{R}\right| - \sum_{l=1}^{n} \left|a_{il}^{J} w_l M_l^{I}\right| - \sum_{l=1}^{n} \left|a_{il}^{I} w_l M_l^{J}\right| \\
&\quad - \sum_{l=1}^{n} \left|a_{li}^{R} w_i M_i^{K}\right| - \sum_{l=1}^{n} \left|a_{il}^{R} w_l M_l^{K}\right| - \sum_{l=1}^{n} \left|b_{il}^{K} \bar{w}_l N_l^{R}\right| - \sum_{l=1}^{n} \left|b_{il}^{J} \bar{w}_l N_l^{I}\right| \\
&\quad - \sum_{l=1}^{n} \left|b_{il}^{I} \bar{w}_l N_l^{J}\right| - \sum_{l=1}^{n} \left|b_{il}^{R} \bar{w}_l N_l^{K}\right| + 2c_i + 2\xi_i^{*K} - \left|\eta_i^{K}\right|\Bigg] \int_{\Omega} \left[e_i^{K}(t,x)\right]^2 \mathrm{d}x \\
&\quad + \sum_{i=1}^{n} \left[\sum_{l=1}^{n} \left|b_{li}^{R} \bar{w}_i N_i^{K}\right| + \left|\eta_i^{K}\right|\right] \int_{\Omega} \left[e_i^{K}\big(t-\tau(t),x\big)\right]^2 \mathrm{d}x \\
&\quad - \sum_{i=1}^{n} \int_{\Omega} 2\bar{\theta}_i^{K} \left[e_i^{K}(t,x)\right]^{2-2\beta} \mathrm{d}x + \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|a_{li}^{K} w_i M_i^{R}\right| \big(e_i^{R}(t,x)\big)^2 \mathrm{d}x \\
&\quad + \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|a_{li}^{J} w_i M_i^{I}\right| \big(e_i^{I}(t,x)\big)^2 \mathrm{d}x + \sum_{i=1}^{n} \int_{\Omega} \sum_{l=1}^{n} \left|a_{li}^{I} w_i M_i^{J}\right| \big(e_i^{J}(t,x)\big)^2 \mathrm{d}x
\end{aligned}
$$

$$+\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{K}\bar{w}_i N_i^{R}\right|\left(e_i^{R}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x$$
$$+\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{J}\bar{w}_i N_i^{I}\right|\left(e_i^{I}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x$$
$$+\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{n}\left|b_{li}^{I}\bar{w}_i N_i^{J}\right|\left(e_i^{J}\left(t-\tau(t),x\right)\right)^2\mathrm{d}x. \tag{12}$$

Combining (10)–(12), we obtain

$$\begin{aligned}
&D^{\alpha}V(t)\\
&\leqslant -\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\frac{2d_i}{l_j^2}-\sum_{l=1}^{n}\left|a_{il}^{R}w_l M_l^{R}\right|-\sum_{l=1}^{n}M_i^{R}|w_i|\left[\left|a_{li}^{R}\right|+\left|a_{li}^{J}\right|+\left|a_{li}^{K}\right|+\left|a_{li}^{I}\right|\right]\\
&\quad-\sum_{l=1}^{n}\left|a_{il}^{I}w_l M_l^{I}\right|-\sum_{l=1}^{n}\left|a_{il}^{J}w_l M_l^{J}\right|-\sum_{l=1}^{n}\left|a_{il}^{K}w_l M_l^{K}\right|-\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_l N_l^{R}\right|\\
&\quad-\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_l N_l^{I}\right|-\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_l N_l^{J}\right|-\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_l N_l^{K}\right|+2c_i+2\xi_i^{*R}-\left|\eta_i^{R}\right|\Bigg]\\
&\quad\times\int_{\Omega}\left[e_i^{R}(t,x)\right]^2\mathrm{d}x\\
&\quad-\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\frac{2d_i}{l_j^2}-\sum_{l=1}^{n}\left|a_{il}^{I}w_l M_l^{R}\right|-\sum_{l=1}^{n}M_i^{I}|w_i|\left[\left|a_{li}^{R}\right|+\left|a_{li}^{I}\right|+\left|a_{li}^{J}\right|+\left|a_{li}^{K}\right|\right]\\
&\quad-\sum_{l=1}^{n}\left|a_{il}^{R}w_l M_l^{I}\right|-\sum_{l=1}^{n}\left|a_{il}^{K}w_l M_l^{J}\right|-\sum_{l=1}^{n}\left|a_{il}^{J}w_l M_l^{K}\right|-\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_l N_l^{R}\right|\\
&\quad-\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_l N_l^{I}\right|-\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_l N_l^{J}\right|-\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_l N_l^{K}\right|+2c_i+2\xi_i^{*I}-\left|\eta_i^{I}\right|\Bigg]\\
&\quad\times\int_{\Omega}\left[e_i^{I}(t,x)\right]^2\mathrm{d}x\\
&\quad-\sum_{i=1}^{n}\Bigg[\sum_{j=1}^{m}\frac{2d_i}{l_j^2}-\sum_{l=1}^{n}\left|a_{il}^{J}w_l M_l^{R}\right|-\sum_{l=1}^{n}M_i^{J}|w_i|\left[\left|a_{li}^{R}\right|+\left|a_{li}^{J}\right|+\left|a_{li}^{K}\right|+\left|a_{li}^{I}\right|\right]\\
&\quad-\sum_{l=1}^{n}\left|a_{il}^{K}w_l M_l^{I}\right|-\sum_{l=1}^{n}\left|a_{il}^{R}w_l M_l^{J}\right|-\sum_{l=1}^{n}\left|a_{il}^{I}w_l M_l^{K}\right|-\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_l N_l^{R}\right|\\
&\quad-\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_l N_l^{I}\right|-\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_l N_l^{J}\right|-\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_l N_l^{K}\right|+2c_i+2\xi_i^{*J}-\left|\eta_i^{J}\right|\Bigg]\\
&\quad\times\int_{\Omega}\left[e_i^{J}(t,x)\right]^2\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\left[\sum_{j=1}^{m}\frac{2d_i}{l_j^2}-\sum_{l=1}^{n}\left|a_{il}^{K}w_l M_l^{R}\right|-\sum_{l=1}^{n}M_i^{K}|w_i|\left[\left|a_{li}^{R}\right|+\left|a_{li}^{K}\right|+\left|a_{li}^{J}\right|+\left|a_{li}^{I}\right|\right]\right.\\
&-\sum_{l=1}^{n}\left|a_{il}^{J}w_l M_l^{I}\right|-\sum_{l=1}^{n}\left|a_{il}^{I}w_l M_l^{J}\right|-\sum_{l=1}^{n}\left|a_{il}^{R}w_l M_l^{K}\right|-\sum_{l=1}^{n}\left|b_{il}^{K}\bar{w}_l N_l^{R}\right|\\
&\left.-\sum_{l=1}^{n}\left|b_{il}^{J}\bar{w}_l N_l^{I}\right|-\sum_{l=1}^{n}\left|b_{il}^{I}\bar{w}_l N_l^{J}\right|-\sum_{l=1}^{n}\left|b_{il}^{R}\bar{w}_l N_l^{K}\right|+2c_i+2\xi_i^{*K}-\left|\eta_i^{K}\right|\right]\\
&\times\int_{\Omega}\left[e_i^{K}(t,x)\right]^2\,\mathrm{d}x\\
&+\sum_{i=1}^{n}\left[\sum_{l=1}^{n}N_i^{R}|\bar{w}_i|\left[\left|b_{li}^{R}\right|+\left|b_{li}^{I}\right|+\left|b_{li}^{J}\right|+\left|b_{li}^{K}\right|\right]+\left|\eta_i^{R}\right|\right]\int_{\Omega}\left[e_i^{R}\big(t-\tau(t),x\big)\right]^2\,\mathrm{d}x\\
&+\sum_{i=1}^{n}\left[\sum_{l=1}^{n}N_i^{I}|\bar{w}_i|\left[\left|b_{li}^{R}\right|+\left|b_{li}^{I}\right|+\left|b_{li}^{K}\right|+\left|b_{li}^{J}\right|\right]+\left|\eta_i^{I}\right|\right]\int_{\Omega}\left[e_i^{I}\big(t-\tau(t),x\big)\right]^2\,\mathrm{d}x\\
&+\sum_{i=1}^{n}\left[\sum_{l=1}^{n}N_i^{J}|\bar{w}_i|\left[\left|b_{li}^{R}\right|+\left|b_{li}^{I}\right|+\left|b_{li}^{J}\right|+\left|b_{li}^{K}\right|\right]+\left|\eta_i^{J}\right|\right]\int_{\Omega}\left[e_i^{J}\big(t-\tau(t),x\big)\right]^2\,\mathrm{d}x\\
&+\sum_{i=1}^{n}\left[\sum_{l=1}^{n}N_i^{K}|\bar{w}_i|\left[\left|b_{li}^{R}\right|+\left|b_{li}^{I}\right|+\left|b_{li}^{J}\right|+\left|b_{li}^{K}\right|\right]+\left|\eta_i^{K}\right|\right]\int_{\Omega}\left[e_i^{K}\big(t-\tau(t),x\big)\right]^2\,\mathrm{d}x\\
&-\sum_{i=1}^{n}\int_{\Omega}2\bar{\theta}_i^{R}\left[e_i^{R}(t,x)\right]^{2-2\beta}\,\mathrm{d}x-\sum_{i=1}^{n}\int_{\Omega}2\bar{\theta}_i^{I}\left[e_i^{I}(t,x)\right]^{2-2\beta}\,\mathrm{d}x\\
&-\sum_{i=1}^{n}\int_{\Omega}2\bar{\theta}_i^{J}\left[e_i^{J}(t,x)\right]^{2-2\beta}\,\mathrm{d}x-\sum_{i=1}^{n}\int_{\Omega}2\bar{\theta}_i^{K}\left[e_i^{K}(t,x)\right]^{2-2\beta}\,\mathrm{d}x\\
\leqslant{}&-\bar{\lambda}_1V(t)+\bar{\lambda}_2V\big(t-\tau(t)\big)-\bar{\lambda}_3V^{1-\beta}(t). \qquad (13)
\end{aligned}
$$

According to Lemma 4 and conditions (4) in Theorem 1, systems (1) and (2) can reach synchronization with (5). □

Remark 2. According to the adaptive controller (3), when the synchronization is realized, $D^{\alpha}\xi_i^{\nu}(t)$ tends to zero, and $\xi_i^{\nu}(t)$ approaches a constant with the help of the properties of the Caputo FO derivative. Figures 3 and 4 in Example 1 show the evolutions of the $\xi_i^{\nu}(t)$, which verify the above results.

In Eq. (3), the adaptive controller is related to time t. We will introduce the adaptive controller that takes both time t and space x:

$$
u_i^{\nu}(t,x)=\begin{cases}-(\xi_i^{\nu}(t,x)+\tilde{\xi}_i^{*\nu})e_i^{\nu}(t,x)-\tilde{\theta}_i^{\nu}[e_i^{\nu}(t,x)]^{1-2\beta}, & e_i^{\nu}(t,x)\neq 0,\\ 0, & e_i^{\nu}(t,x)=0,\end{cases} \qquad (14)
$$

in which $\tilde{\xi}_i^{*\nu}$, $\tilde{\theta}_i^{\nu}$ are constants, and the adaptive control law $\xi_i^{\nu}(t,x)$ satisfies

$$\frac{\partial^\alpha}{\partial t^\alpha}\xi_i^{\nu}(t,x) = \tilde{\delta}_i^{\nu}\big[e_i^{\nu}(t,x)\big]^2, \quad \tilde{\delta}_i^{\nu} > 0.$$

For simplicity, the notations are given below:

$$\tilde{\lambda}_1^{\nu} = \min_{1\leqslant i\leqslant n}\big\{\lambda_i^{\nu} + 2\tilde{\xi}_i^{*\nu}\big\}, \qquad \tilde{\lambda}_2^{\nu} = \bar{\lambda}_2^{\nu}, \qquad \tilde{\lambda}_3^{\nu} = \min_{1\leqslant i\leqslant n}\big\{2\tilde{\theta}_i^{\nu}\big\},$$
$$\tilde{\lambda}_1 = \min\big\{\tilde{\lambda}_1^R, \tilde{\lambda}_1^I, \tilde{\lambda}_1^J, \tilde{\lambda}_1^K\big\}, \qquad \tilde{\lambda}_2 = \max\big\{\tilde{\lambda}_2^R, \tilde{\lambda}_2^I, \tilde{\lambda}_2^J, \tilde{\lambda}_2^K\big\},$$
$$\tilde{\lambda}_3 = \min\big\{\tilde{\lambda}_3^R, \tilde{\lambda}_3^I, \tilde{\lambda}_3^J, \tilde{\lambda}_3^K\big\}.$$

Theorem 2. *Under controller* (14) *and Assumption* 1*, if the inequalities*

$$\tilde{\xi}_i^{*\mu} + \frac{1}{2}\lambda_i^{\nu} > 0, \qquad \tilde{\lambda}_1 > \tilde{\lambda}_2 > 0, \qquad \tilde{\lambda}_3 > 0 \tag{15}$$

hold, then systems (1) *and* (2) *can reach FTS with*

$$\tilde{T}^* = \left[\frac{\Gamma(1+\frac{1}{\beta})\Gamma(2-\alpha)\Gamma(1+\alpha)}{\Gamma(1+\frac{1}{\beta}-\alpha)(\tilde{\lambda}_1-\tilde{\lambda}_2)}\ln\frac{(\tilde{\lambda}_1-\tilde{\lambda}_2)\gamma^\beta+\tilde{\lambda}_3}{\tilde{\lambda}_3}\right]^{1/\alpha}. \tag{16}$$

Proof. Let us apply the Lyapunov function

$$V(t) = V^R(t) + V^I(t) + V^J(t) + V^K(t)$$

in which $V^{\nu}(t) = \int_\Omega\sum_{i=1}^n[e_i^{\nu}(t,x)]^2\,\mathrm{d}x + \sum_{i=1}^n\int_\Omega[\xi_i^{\nu}(t,x)]^2\,\mathrm{d}x/\tilde{\delta}_i^{\nu}$.

Take the fractional derivative of $V^R(t)$ of order α, then by using Lemmas 1 and 2,

$$\begin{aligned}
D^\alpha V^R(t) \leqslant{}& \sum_{i=1}^n\int_\Omega 2e_i^R(t,x)\frac{\partial^\alpha e_i^R(t,x)}{\partial t^\alpha}\,\mathrm{d}x + \sum_{i=1}^n\int_\Omega\frac{2}{\tilde{\delta}_i^{\nu}}\xi_i^R(t,x)\frac{\partial^\alpha\xi_i^R(t,x)}{\partial t^\alpha}\,\mathrm{d}x\\
={}& \sum_{i=1}^n\int_\Omega 2e_i^R(t,x)\Bigg\{\sum_{j=1}^m d_i\frac{\partial^2 e_i^R(t,x)}{\partial x_j^2} - c_ie_i^R(t,x) + \sum_{l=1}^n\big[a_{il}^Rf_l^R\big(e_l^R(t,x)\big)\\
&- a_{il}^If_l^I\big(e_l^I(t,x)\big) - a_{il}^Jf_l^J\big(e_l^J(t,x)\big) - a_{il}^Kf_l^K\big(e_l^K(t,x)\big)\big]\\
&+ \sum_{l=1}^n\big[b_{il}^Rg_l^R\big(e_l^R\big(t-\tau(t),x\big)\big) - b_{il}^Ig_l^I\big(e_l^I\big(t-\tau(t),x\big)\big)\\
&- b_{il}^Jg_l^J\big(e_l^J\big(t-\tau(t),x\big)\big)\\
&- b_{il}^Kg_l^K\big(e_l^K\big(t-\tau(t),x\big)\big)\big] + u_i^R(t,x)\Bigg\}\,\mathrm{d}x\\
&+ \sum_{i=1}^n 2\int_\Omega\xi_i^R(t,x)\big[e_i^R(t,x)\big]^2\,\mathrm{d}x.
\end{aligned}$$

Applying controller (14), we have

$$\begin{aligned}
&\sum_{i=1}^{n} 2\int_{\Omega} u_i^R(t,x)e_i^R(t,x)\,\mathrm{d}x + \sum_{i=1}^{n} 2\int_{\Omega} \xi_i^R(t,x)\big[e_i^R(t,x)\big]^2\,\mathrm{d}x \\
&\quad= -\sum_{i=1}^{n}\int_{\Omega} 2\big(\xi_i^R(t,x)+\tilde{\xi}^{*R}\big)\big[e_i^R(t,x)\big]^2\,\mathrm{d}x - \sum_{i=1}^{n}\int_{\Omega} 2\tilde{\theta}_i^R\big[e_i^R(t,x)\big]^{2-2\beta}\,\mathrm{d}x \\
&\qquad+ \sum_{i=1}^{n} 2\int_{\Omega} \xi_i^R(t,x)\big[e_i^R(t,x)\big]^2\,\mathrm{d}x \\
&\quad= -\sum_{i=1}^{n}\int_{\Omega} 2\tilde{\xi}_i^{*R}\big[e_i^R(t,x)\big]^2\,\mathrm{d}x - \sum_{i=1}^{n}\int_{\Omega} 2\tilde{\theta}_i^R\big[e_i^R(t,x)\big]^{2-2\beta}\,\mathrm{d}x. \qquad (17)
\end{aligned}$$

Similarly, we can obtain

$$D^{\alpha}V(t) \leqslant -\tilde{\lambda}_1 V(t) + \tilde{\lambda}_2 V\big(t-\tau(t)\big) - \tilde{\lambda}_3 V^{1-\beta}(t). \qquad (18)$$

According to Lemma 4 and conditions (15) in Theorem 2, systems (1) and (2) can reach synchronization with (16). □

Remark 3. From the adaptive controller (14) in Theorem 2, for any given $x \in \Omega$, $D^{\alpha}\xi_i^{\nu}(t,x)$ is close to zero as systems achieve synchronization, and $\xi_i^{\nu}(t,x)$ approaches a certain constant that related to time t according to the properties of the Caputo FO derivative. It is worth noting that $\xi_i^{\nu}(t,x)$ is variable with respect to x when the time t is fixed. Figures 7, 8 in Example 1 reveal the evolutions of the $\xi_i^{\nu}(t,x)$, which verify the above results.

Remark 4. In fact, some factors are inevitable, such as the environment, equipment, etc., the controller gains will unavoidably take disturbances. Thereupon, it is reasonable and necessary that the selected controllers depend on time and space, which is closer to practice.

Remark 5. In [26, Lemma 3], the settling time is obtained via inequality $D^{\alpha}H(t) \leqslant -\vartheta H(t) - \varsigma$. Compared with existing works, Lemma 4 contains power and time-delay terms, which makes the form of the inequality more general.

Remark 6. The FST synchronization criteria are obtained in [6, 27, 33], which concentrated on delays and FOQVNN. However, RD terms were ignored, and it could increase the conservativeness of the results. Up to now, the FTS synchronization analysis of GDFORDQVNN has not been discussed. In this paper, the RD term was added to the synchronization condition, which makes the obtained results more general and less conservative.

Remark 7. If $\alpha = 1$, systems (1) and (2) reduced to the classical RDQVNN models with time varying, which are investigated in [20].

4 A numerical example and an application

Example 1. Let us consider the following GDFORDQVNN:

$$\frac{\partial^\alpha h_i(t,x)}{\partial t^\alpha} = d_i \sum_{j=1}^{1} \frac{\partial^2 h_i(t,x)}{\partial x_j^2} - c_i h_i(t,x) + \sum_{l=1}^{2} a_{il} f_l\bigl(w_l h_l(t,x)\bigr) + \sum_{l=1}^{2} b_{il} g_l\bigl(\bar{w}_l h_l\bigl(t-\tau(t),x\bigr)\bigr) + I_i(t,x), \quad i=1,2,\ t>0,$$

in which $\alpha = 0.38$, $d_1 = d_2 = 0.9$, $c_{11} = 0.25$, $c_{22} = 0.85$, $\tau(t) = \mathrm{e}^t/\mathrm{e}^t + 1$, $w_1 = w_2 = 1$, $\bar{w}_1 = \bar{w}_2 = 1$,

$$(a_{ij})_{2\times 2} = \begin{bmatrix} 1.48 - 1.26\mathrm{i} - 0.75\mathrm{j} + 0.06\mathrm{k}, & 0.11 - 0.6\mathrm{i} + 0.28\mathrm{j} + 0.44\mathrm{k} \\ -1.09 + 0.47\mathrm{i} + 0.85\mathrm{j} + 0.1\mathrm{k}, & -0.21 - 1.31\mathrm{i} - 0.31\mathrm{j} + \mathrm{k} \end{bmatrix},$$

$$(b_{ij})_{2\times 2} = \begin{bmatrix} 0.54 + 0.46\mathrm{i} + 1.45\mathrm{j} - 2.27\mathrm{k}, & -0.12 + 1.25\mathrm{i} - 0.25\mathrm{j} + 0.75\mathrm{k} \\ 0.35 - 0.76\mathrm{i} + 0.16\mathrm{j} - 0.61\mathrm{k}, & -1.05 - 1.3\mathrm{i} - 1.3\mathrm{j} + 1.08\mathrm{k} \end{bmatrix},$$

the active functions

$$f\bigl(x(t,x)\bigr) = g\bigl(x(t,x)\bigr) = x(t,x) + \mathrm{sign}\bigl(x(t,x)\bigr),$$

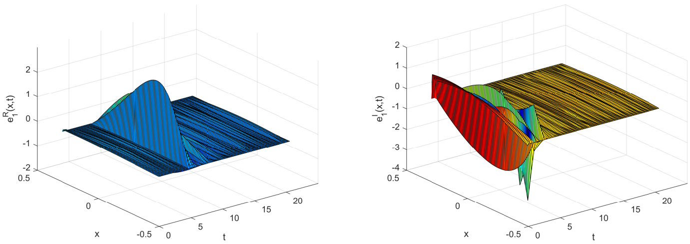
the external inputs $I_1 = I_2 = 0$, $\Omega = [-0.5, 0.5]$. The initial values are selected as

$$\begin{aligned} h_1(t,x) &= 6\cos(-1 + x - 0.001t), \\ h_2(t,x) &= 8\cos(-1 + x - 0.001t), \\ \bar{h}_1(t,x) &= -8.7\tanh(-1 + x - 0.001t), \\ \bar{h}_2(t,x) &= -6.5\tanh(-1 + x - 0.001t). \end{aligned}$$

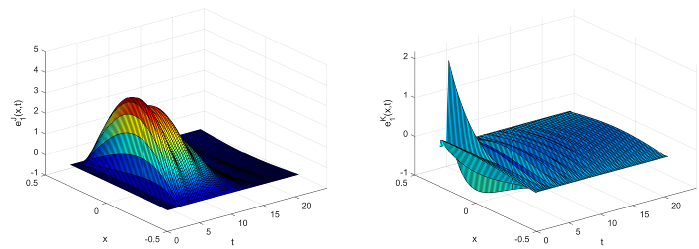
(i) First, we choose the control gains as $\bar{\xi}_1^* = 5.95 + 4\mathrm{i} + 7.9\mathrm{j} + 6.91\mathrm{k}$, $\bar{\xi}_2^* = 9.3 + 2.31\mathrm{i} + 1.67\mathrm{j} + 11.3\mathrm{k}$, $\bar{\theta}_1 = \bar{\theta}_2 = 1.01 + 1.01\mathrm{i} + 1.01\mathrm{j} + 1.01\mathrm{k}$, $\beta = 0.7$, it can be calculated that the conditions of Theorem 1 hold. Thus, the error system can reach FTS with time $\bar{T}^* = 4.56$. Figures 1, 2 describe the synchronization trajectories of errors $e_i^\nu(t,x)$. Figures 3, 4 show that the adaptive control gains $\xi_i^\nu(t)$ converge with time growth and gradually reach some positive constants.

(ii) Next, we choose the control gains as $\tilde{\theta}_1 = \tilde{\theta}_2 = 1.01 + 1.01\mathrm{i} + 1.01\mathrm{j} + 1.01\mathrm{k}$, $\beta = 0.7$, it can be calculated that the conditions of Theorem 2 satisfied. Thus, the error system can reach FTS with time $\tilde{T}^* = 4.35$. Figures 5, 6 describe the synchronization trajectories of errors $e_i^\nu(t,x)$ $(i = 1,2)$. Figures 7, 8 show that for any $x \in \Omega$, the adaptive control gains $\xi_i^\nu(x,t)$ converge to constants of time t. However, $\xi_i^\nu(x,t)$ are still functions with respect to space variable x at any fixed time.

(iii) Finally, Figs. 1–8 show that Theorems 1 and 2 are correct. Namely, the drive-response systems (1) and (2) can reach finite-time synchronization under the adaptive control schemes (3) and (14), respectively.

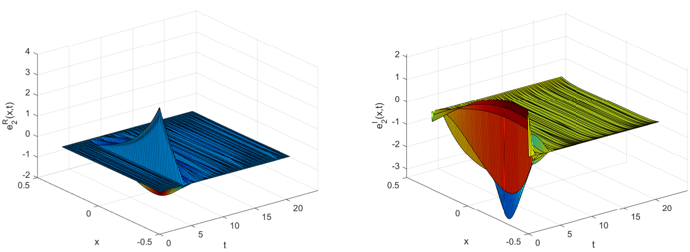


(a) $e_1^R(x,t)$ (b) $e_1^I(x,t)$

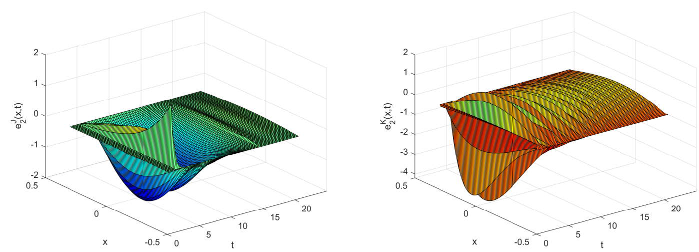


(c) $e_1^J(x,t)$ (d) $e_1^K(x,t)$

Figure 1. The synchronization trajectory of $e_1^\nu(x,t)$ with controller (3).

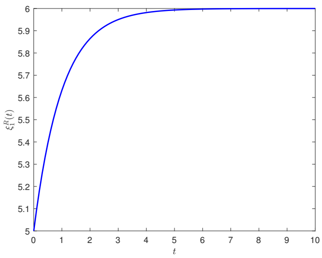


(a) $e_2^R(x,t)$ (b) $e_2^I(x,t)$

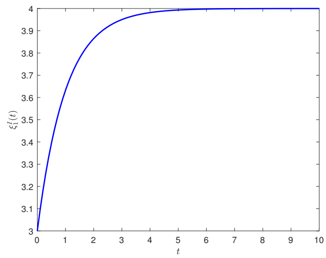


(c) $e_2^J(x,t)$ (d) $e_2^K(x,t)$

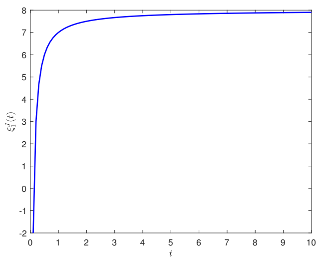
Figure 2. The synchronization trajectory of $e_2^\nu(x,t)$ with controller (3).



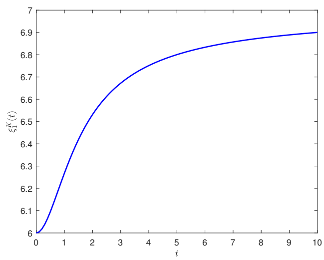
(a) $\xi_1^R(t)$



(b) $\xi_1^I(t)$

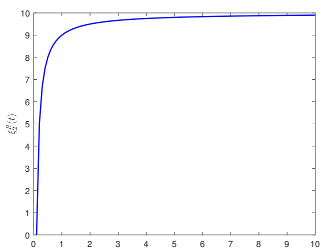


(c) $\xi_1^J(t)$

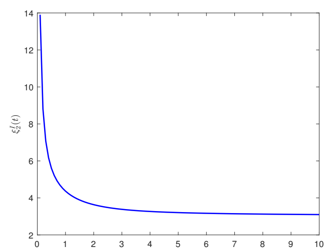


(d) $\xi_1^K(t)$

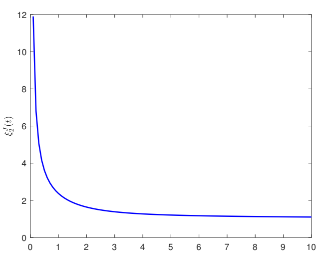
Figure 3. The evolutions of adaptive control law $\xi_1^\nu(t)$.



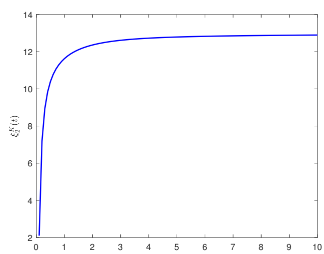
(a) $\xi_2^R(t)$



(b) $\xi_2^I(t)$

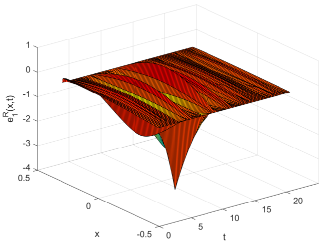


(c) $\xi_2^J(t)$

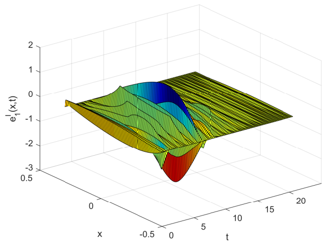


(d) $\xi_2^K(t)$

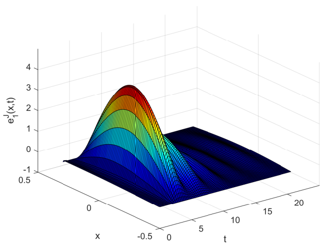
Figure 4. The evolutions of adaptive control law $\xi_2^\nu(t)$.



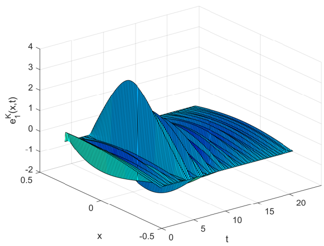
(a) $e_1^R(x,t)$



(b) $e_1^I(x,t)$

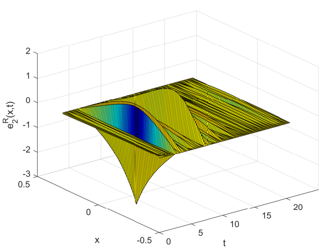


(c) $e_1^J(x,t)$

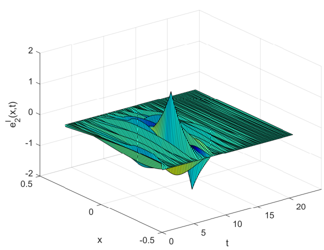


(d) $e_1^K(x,t)$

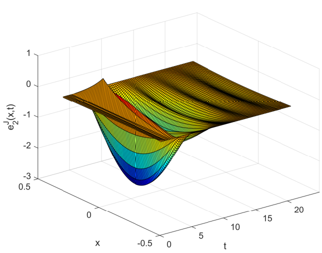
Figure 5. The synchronization trajectory of $e_1^\nu(x,t)$ with controller (15).



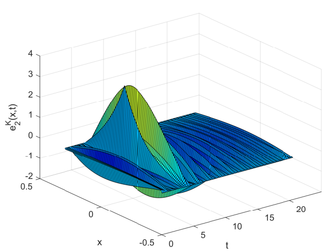
(a) $e_2^R(x,t)$



(b) $e_2^I(x,t)$

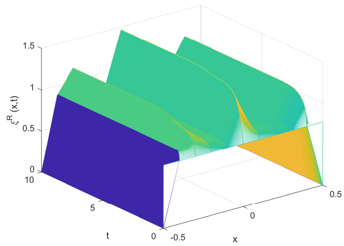


(c) $e_2^J(x,t)$

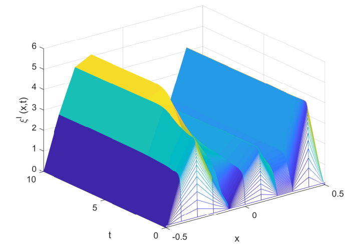


(d) $e_2^K(x,t)$

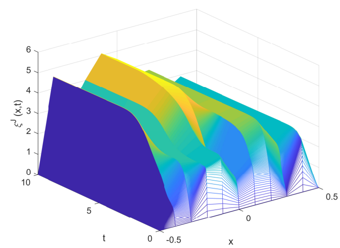
Figure 6. The synchronization trajectory of $e_2^\nu(x,t)$ with controller (15).



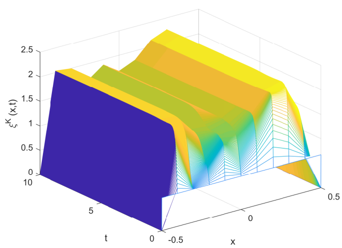
(a) $\xi_1^R(x,t)$



(b) $\xi_1^I(x,t)$

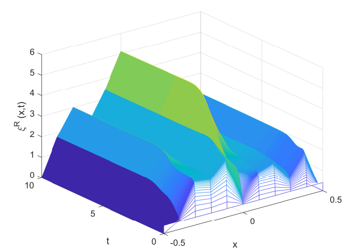


(c) $\xi_1^J(x,t)$

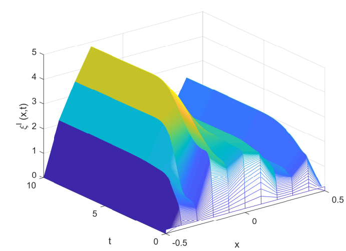


(d) $\xi_1^K(x,t)$

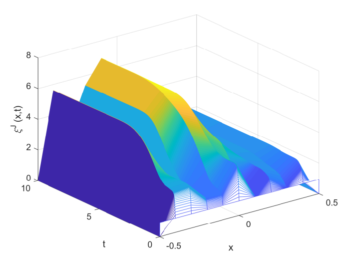
Figure 7. The evolutions of adaptive control law $\xi_1^\nu(x,t)$.



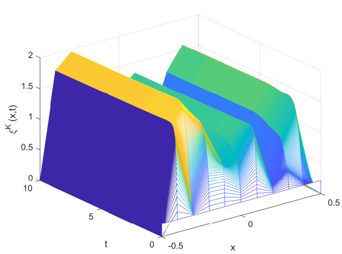
(a) $\xi_2^R(x,t)$



(b) $\xi_2^I(x,t)$



(c) $\xi_2^J(x,t)$



(d) $\xi_2^K(x,t)$

Figure 8. The evolutions of adaptive control law $\xi_2^\nu(x,t)$.

Example 2. Due to the complex dynamic behavior of GDFORDQVNN, image encryption and decryption are derived based on system (1).

For an image named “Lena”, (see Fig. 9(a)), we apply system (1) and the XOR algorithm, the encrypted image is presented in Fig. 9(b). With the decryption process of the considered model, we can obtain the corresponding decrypted image, which is given in Fig. 9(c). The corresponding histograms are shown in Fig. 10.

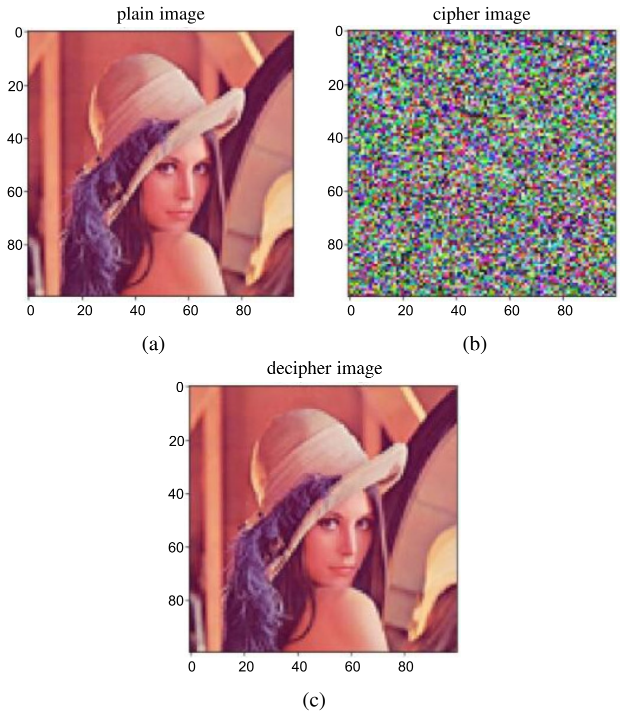


Figure 9. Encryption and decryption of color image.

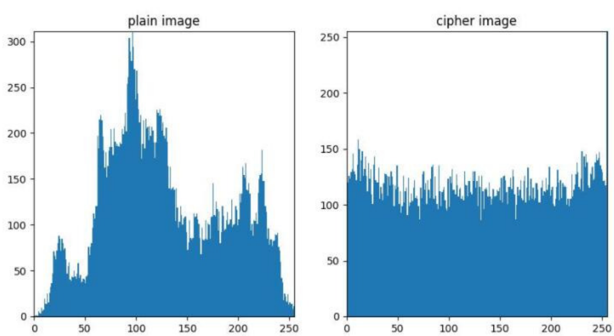
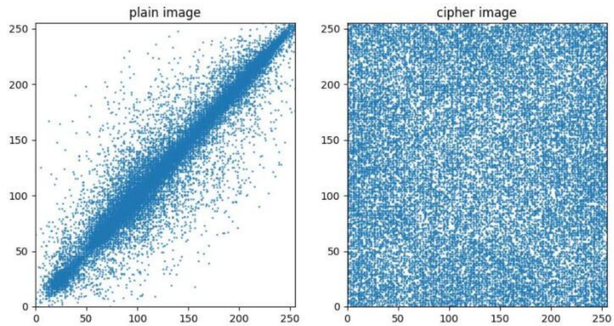
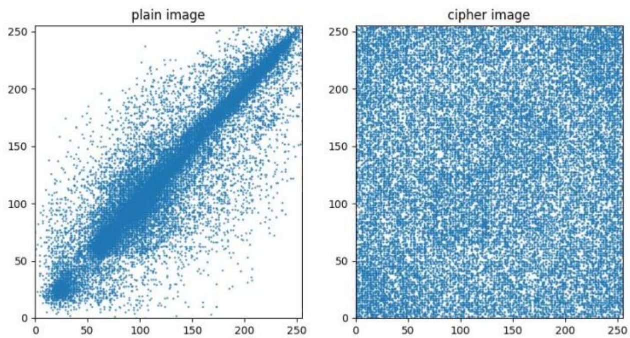


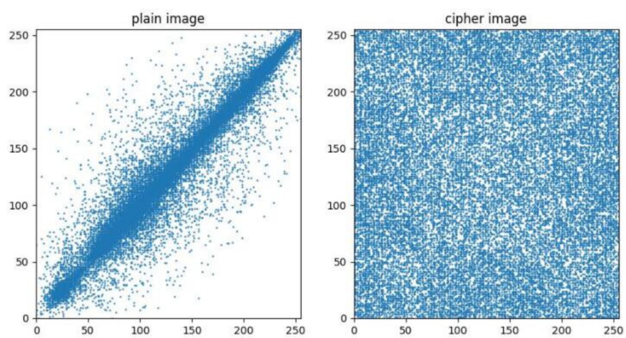
Figure 10. The histograms of plain image and cipher image.



(a) horizontal correlation



(b) vertical correlation



(c) diagonal correlation

Figure 11. Correlations of plain image and cipher image.

The two adjacent pixels of the plain images and the ciphered images are correlated within a certain range, which is shown in Fig. 11. Moreover, the correlation coefficients of two adjacent pixels are listed in Table 1. Both Fig. 11 and Table 1 reveal that the results we obtained can better solve the problem of image encryption.

Table 1. Correlation coefficients for two adjacent pixels.

Cameraman	Horizontal	Vertical	Diagonal
Plain image	0.9631	0.9593	0.9502
Cipher image	−0.0021	−0.0043	0.0065

5 Conclusion

In fact, diffusion phenomenon and delays inevitable exist in NN and have an influence on the dynamical behaviours of the systems. In this paper, the FTS conditions of FOQVNN with RD and time-varying delay are derived by employing Lyapunov method. Numerical simulations and application examples intuitively show that the obtained theoretical results are effective and feasible. This paper extended and developed the previous research study. In future works, we will further investigate the fixed time stability of FORDQVNN with leakage delay under impulsive controller.

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