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Sapto W. Indratno, A. G. Ramm

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# A collocation method for solving some integral equations in distributions

Sapto W. Indratno

Department of Mathematics  
Kansas State University, Manhattan, KS 66506-2602, USA

Department of Mathematics  
Bandung Institute of Technology, Bandung, Indonesia

sapto@math.itb.ac.id

A G Ramm

Department of Mathematics  
Kansas State University, Manhattan, KS 66506-2602, USA

ramm@math.ksu.edu

## Abstract

A collocation method is presented for numerical solution of a typical integral equation  $Rh := \int_D R(x, y)h(y)dy = f(x)$ ,  $x \in \overline{D}$  of the class  $\mathcal{R}$ , whose kernels are of positive rational functions of arbitrary selfadjoint elliptic operators defined in the whole space  $\mathbb{R}^r$ , and  $D \subset \mathbb{R}^r$  is a bounded domain. Several numerical examples are given to demonstrate the efficiency and stability of the proposed method.

*MSC: 45A05, 45P05, 46F05, 62M40, 65R20, 74H15*

*Key words:* integral equations in distributions, signal estimation, collocation method.

## 1 Introduction

In [4] a general theory of integral equations of the class  $\mathcal{R}$  was developed. The integral equations of the class  $\mathcal{R}$  are written in the following form:

$$Rh := \int_D R(x, y)h(y)dy = f(x), \quad x \in \overline{D} := D \cup \Gamma, \quad (1)$$

where  $D \in \mathbb{R}^r$  is a (bounded) domain with a (smooth) boundary  $\Gamma$ . Here the kernel  $R(x, y)$  has the following form [4, 5, 6, 7]:

$$R(x, y) = \int_\Lambda P(\lambda)Q^{-1}(\lambda)\Phi(x, y, \lambda)d\rho(\lambda), \quad (2)$$

where  $P(\lambda), Q(\lambda) > 0$  are polynomials,  $\deg P = p$ ,  $\deg Q = q$ ,  $q > p$ , and  $\Phi, \rho, \Lambda$  are the spectral kernel, spectral measure, and spectrum of a selfadjoint elliptic operator  $\mathcal{L}$  on  $L^2(\mathbb{R}^r)$  of order  $s$ . It was also proved in [4] that  $R: \dot{H}^{-\alpha}(D) \rightarrow H^\alpha(D)$  is an isomorphism, where  $H^\alpha(D)$  is the Sobolev space and  $\dot{H}^{-\alpha}(D)$  its dual space with respect to the  $L^2(D)$  inner product,  $\alpha = \frac{s(q-p)}{2}$ . Here the space  $\dot{H}^{-\alpha}(D)$  consists of distributions in  $H^\alpha(\mathbb{R}^r)$  with support in the closure of  $D$ . In this paper we consider a particular type of integral equations of the class  $\mathcal{R}$  with  $D = (-1, 1)$ ,  $r = 1$ ,  $\mathcal{L} = -i\partial$ ,  $\partial := \frac{d}{dx}$ ,  $\Lambda \in (-\infty, \infty)$ ,  $d\rho(\lambda) = d\lambda$ ,  $\Phi(x, y, \lambda) = \frac{e^{i\lambda(x-y)}}{2\pi}$ ,  $P(\lambda) = 1$ ,  $Q(\lambda) = \frac{\lambda^2+1}{2}$ ,  $s = 1$ ,  $p = 0$ ,  $q = 2$  and  $\alpha = 1$ , i.e.,

$$Rh(x) := \int_{-1}^1 e^{-|x-y|} h(y) dy = f(x), \quad (3)$$

where  $h \in \dot{H}^{-1}[-1, 1]$  and  $f \in H^1[-1, 1]$ . We denote the inner product and norm in  $H^1[-1, 1]$  by

$$\langle u, v \rangle_1 := \int_{-1}^1 \left( u(x)\overline{v(x)} + u'(x)\overline{v'(x)} \right) dx \quad u, v \in H^1([-1, 1]), \quad (4)$$

and

$$\|u\|_1^2 := \int_{-1}^1 (|u(x)|^2 + |u'(x)|^2) dx, \quad (5)$$

respectively, where the primes denote derivatives and the bar stands for complex conjugate. If  $u$  and  $v$  are real-valued functions in  $H^1[-1, 1]$  then the bar notations given in (4) can be dropped. Note that if  $f$  is a complex valued function then solving equation (3) is equivalent to solving the equations:

$$\int_{-1}^1 e^{-|x-y|} h_k(y) dy = f_k(x), \quad k = 1, 2, \quad (6)$$

where  $h_1(x) := \operatorname{Re}h(x)$ ,  $h_2(x) := \operatorname{Im}h(x)$ ,  $f_1(x) := \operatorname{Re}f(x)$ ,  $f_2(x) := \operatorname{Im}f(x)$  and  $h(x) = h_1(x) + ih_2(x)$ ,  $i = \sqrt{-1}$ . Therefore, without loss of generality we assume throughout that  $f(x)$  is real-valued.

It was proved in [5] that the operator  $R$  defined in (3) is an isomorphism between  $\dot{H}^{-1}[-1, 1]$  and  $H^1[-1, 1]$ . Therefore, problem (3) is well posed in the sense that small changes in the data  $f(x)$  in the  $H^1[-1, 1]$  norm will result in small in  $\dot{H}^{-1}[-1, 1]$  norm changes to the solution  $h(y)$ . Moreover, the solution to (3) can be written in the following form:

$$h(x) = a_{-1}\delta(x+1) + a_0\delta(x-1) + g(x), \quad (7)$$

where

$$a_{-1} := \frac{f(-1) - f'(-1)}{2}, \quad a_0 := \frac{f'(1) + f(1)}{2}, \quad (8)$$

$$g(x) := \frac{-f''(x) + f(x)}{2}, \quad (9)$$

and  $\delta(x)$  is the delta function. Here and throughout this paper we assume that  $f \in C^\alpha[-1, 1]$ ,  $\alpha \geq 2$ . It follows from (8) that  $h(x) = g(x)$  if and only if  $f(-1) = f'(-1)$  and  $f(1) = -f'(1)$ .

In [6, 7] the problem of solving equation (3) numerically have been posed and solved. The least squares method was used in these papers. The goal of this paper is to develop a version of the collocation method which can be applied easily and numerically efficiently. In [8] some basic ideas for using collocation method are proposed. In this paper some of these ideas are used and new ideas, related to the choice of the basis functions, are introduced. In this paper the emphasis is on the development of methodology for solving basic equation (1) of the estimation theory by a version of the collocation method. The novelty of this version consists in minimization of a discrepancy functional (26), see below. This methodology is illustrated by a detailed analysis applied to solving equation (3), but it is applicable to general equations of the class  $\mathcal{R}$ . One of the goals of this paper is to demonstrate that collocation method can be successfully applied to numerical solution of some integral equations whose solutions are distributions, provided that the theoretical analysis gives sufficient information about the singular part of the solutions.

Since  $f \in C^\alpha[-1, 1]$ ,  $\alpha \geq 2$ , it follows from (9) that  $g \in C[-1, 1]$ . Therefore, there exist basis functions  $\varphi_j(x) \in C[-1, 1]$ ,  $j = 1, 2, \dots, m$ , such that

$$\max_{x \in [-1, 1]} |g(x) - g_m(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad (10)$$

where

$$g_m(x) := \sum_{j=1}^m c_j^{(m)} \varphi_j(x), \quad (11)$$

$c_j^{(m)}$ ,  $j = 1, 2, \dots, m$ , are constants. Hence the approximate solution of equation (3) can be represented by

$$h_m(x) = c_{-1}^{(m)} \delta(x+1) + c_0^{(m)} \delta(x-1) + g_m(x), \quad (12)$$

where  $c_j^{(m)}$ ,  $j = -1, 0$ , are constants and  $g_m(x)$  is defined in (11). The basis functions  $\varphi_j$  play an important role in our method. It is proved in Section 3 that the basis functions  $\varphi_j$  in (12) can be chosen from the linear B-splines. The usage of the linear B-splines reduces the computation time, because computing (12) at a particular point  $x$  requires at most two out of the  $m$  basis functions  $\varphi_j$ . For a more detailed discussion of the family of B-splines we refer to [10]. In Section 2 we derive a method for obtaining the coefficients  $c_j^{(m)}$ ,  $j = -1, 0, 1, \dots, m$ , given in (12). This method is based on solving a finite-dimensional least squares problem ( see equation (33) below ) and differs from the usual collocation method discussed in [2] and [3]. We approximate  $\|f - Rh_m\|_1^2$  by a quadrature formula. The resulting finite-dimensional linear algebraic system depends on the choice of the basis functions. Using linear B-splines as the basis functions, we prove the existence and uniqueness of the solution to this linear algebraic system for all  $m = m(n)$  depending on the number  $n$  of collocation points used in the left

rectangle quadrature rule. The convergence of our collocation method is proved in this Section. An example of the choice of the basis functions which yields the convergence of our version of the collocation method is given in Section 3. In Section 4 we give numerical results of applying our method to several problems that discussed in [7].

## 2 The collocation method

In this Section we derive a collocation method for solving equation (3). From equation (3) we get

$$\begin{aligned} Rh(x) &= a_{-1}e^{-(x+1)} + a_0e^{-(1-x)} \\ &+ \left( e^{-x} \int_{-1}^x e^y g(y) dy + e^x \int_x^1 e^{-y} g(y) dy \right) = f(x). \end{aligned} \quad (13)$$

Assuming that  $f \in C^2([-1, 1])$  and differentiating the above equation, one obtains

$$\begin{aligned} (Rh)'(x) &= -a_{-1}e^{-(x+1)} + a_0e^{-(1-x)} \\ &+ \left( e^x \int_x^1 e^{-y} g(y) dy - e^{-x} \int_{-1}^x e^y g(y) dy \right) = f'(x). \end{aligned} \quad (14)$$

Thus,  $f(x)$  and  $f'(x)$  are continuous in the interval  $[-1, 1]$ . Let us use the approximate solution given in (12). From (13), (14) and (12) we obtain

$$\begin{aligned} Rh_m(x) &= c_{-1}^{(m)} e^{-(x+1)} + c_0^{(m)} e^{-(1-x)} \\ &+ \sum_{j=1}^m c_j^{(m)} \left( e^{-x} \int_{-1}^x e^y \varphi_j(y) dy + e^x \int_x^1 e^{-y} \varphi_j(y) dy \right) := f_m(x), \end{aligned} \quad (15)$$

and

$$\begin{aligned} (Rh_m)'(x) &= -c_{-1}^{(m)} e^{-(x+1)} + c_0^{(m)} e^{-(1-x)} \\ &+ \sum_{j=1}^m c_j^{(m)} \left[ e^x \int_x^1 e^{-y} \varphi_j(y) dy - e^{-x} \int_{-1}^x e^y \varphi_j(y) dy \right] := (f_m)'(x). \end{aligned} \quad (16)$$

Thus,  $Rh_m(x)$  and  $(Rh_m)'(x)$  are continuous in the interval  $[-1, 1]$ . Since  $f(x) - Rh_m(x)$  and  $f'(x) - (Rh_m)'(x)$  are continuous in the interval  $[-1, 1]$ , we may assume throughout that the functions

$$J_{1,m} := [f(x) - Rh_m(x)]^2 \quad (17)$$

and

$$J_{2,m} := [f'(x) - (Rh_m)'(x)]^2 \quad (18)$$

are Riemann-integrable over the interval  $[-1, 1]$ .

Let us define

$$\begin{aligned} q_{-1}(x) &:= e^{-(x+1)}, \quad q_0(x) := e^{-(1-x)}, \\ q_j(x) &:= \int_{-1}^1 e^{-|x-y|} \varphi_j(y) dy, \quad j = 1, 2, \dots, m, \end{aligned} \quad (19)$$

and define a mapping  $\mathcal{H}_n : C^2[-1, 1] \rightarrow \mathbb{R}_1^n$  by the formula:

$$\mathcal{H}_n \phi = \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_n) \end{pmatrix}, \quad \phi(x) \in C^2[-1, 1], \quad (20)$$

where

$$\mathbb{R}_1^n := \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n \mid z_j := z(x_j), \quad z \in C^2[-1, 1] \right\} \quad (21)$$

and  $x_j$  are some collocation points which will be chosen later. We equip the space  $\mathbb{R}_1^n$  with the following inner product and norm

$$\langle u, v \rangle_{w^{(n)}, 1} := \sum_{j=1}^n w_j^{(n)} (u_j v_j + u'_j v'_j), \quad u, v \in \mathbb{R}_1^n, \quad (22)$$

$$\|u\|_{w^{(n)}, 1}^2 := \sum_{j=1}^n w_j^{(n)} [u_j^2 + (u'_j)^2], \quad u \in \mathbb{R}_1^n, \quad (23)$$

respectively, where  $u_j := u(x_j)$ ,  $u'_j := u'(x_j)$ ,  $v_j := v(x_j)$ ,  $v'_j := v'(x_j)$ , and  $w_j > 0$  are some quadrature weights corresponding to the collocation points  $x_j$ ,  $j = 1, 2, \dots, n$ .

Applying  $\mathcal{H}_n$  to  $Rh_m$ , one gets

$$\begin{aligned} (\mathcal{H}_n Rh_m)_i &= c_{-1}^{(m)} e^{-(1+x_i)} + c_0^{(m)} e^{-(1-x_i)} + \sum_{j=1}^m c_j^{(m)} \int_{-1}^{x_i} e^{-(x_i-y)} \varphi_j(y) dy + \\ &\quad \sum_{j=1}^m c_j^{(m)} \int_{x_i}^1 e^{-(y-x_i)} \varphi_j(y) dy, \quad i = 1, 2, \dots, n, \end{aligned} \quad (24)$$

where  $m = m(n)$  is an integer depending on  $n$  such that

$$m(n) + 2 \leq n, \quad \lim_{n \rightarrow \infty} m(n) = \infty. \quad (25)$$

Let

$$G_n(c^{(m)}) := \|\mathcal{H}_n(f - Rh_m)\|_{w^{(n)}, 1}^2, \quad (26)$$

where  $f$  and  $Rh_m$  are defined in (3) and (15), respectively,  $\mathcal{H}_n$  defined in (20),

$\|\cdot\|_{w^{(n)},1}$  defined in (23) and  $c^{(m)} = \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbb{R}^{m+2}$ . Let us choose

$$w_j^{(n)} = \frac{2}{n}, \quad j = 1, 2, \dots, n, \quad (27)$$

and

$$x_j = -1 + (j-1)s, \quad s := \frac{2}{n}, \quad j = 1, 2, \dots, n, \quad (28)$$

so that  $\|\mathcal{H}_n(f - Rh_m)\|_{w^{(n)},1}^2$  is the left Riemannian sum of  $\|f - Rh_m\|_1^2$ , i.e.,

$$\|f - Rh_m\|_1^2 - \|\mathcal{H}_n(f - Rh_m)\|_{w^{(n)},1}^2 := \delta_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (29)$$

**Remark 2.1.** If  $J_{1,m}$  and  $J_{2,m}$  are in  $C^2[-1,1]$ , where  $J_{1,m}$  and  $J_{2,m}$  are defined in (17) and (18), respectively, then one may replace the weights  $w_j^{(n)}$  with the weights of the compound trapezoidal rule, and get the estimate

$$\begin{aligned} \delta_n &= \left| \int_{-1}^1 (J_{1,m}(x) + J_{2,m}(x)) dx - \sum_{j=1}^n w_j^{(n)} (J_{1,m}(x_j) + J_{2,m}(x_j)) \right| \\ &\leq \frac{1}{3n^2} D_J, \end{aligned} \quad (30)$$

where  $\delta_n$  is defined in (29) and

$$D_J := |J'_{1,m}(1) + J'_{2,m}(1) - (J'_{1,m}(-1) + J'_{2,m}(-1))|. \quad (31)$$

Here we have used the following estimate of the compound trapezoidal rule [1, 9]:

$$\left| \int_a^b \eta(x) dx - \sum_{j=1}^n w_j^{(n)} \eta(x_j) \right| \leq \frac{(b-a)^2}{12n^2} \left| \int_a^b \eta''(x) dx \right| = \frac{(b-a)^2}{12n^2} |\eta'(b) - \eta'(a)|, \quad (32)$$

where  $\eta \in C^2[a, b]$ . Therefore, if  $D_J \leq C$  for all  $m$ , where  $C > 0$  is a constant, then  $\delta_n = O\left(\frac{1}{n^2}\right)$ .

The constants  $c_j^{(m)}$  in the approximate solution  $h_m$ , see (12), are obtained by solving the following least squares problem:

$$\min_{c^{(m)}} G_n(c^{(m)}), \quad (33)$$

where  $G_n$  is defined in (26).

A necessary condition for the minimum in (33) is

$$0 = \sum_{l=1}^n w_l^{(n)} \left( E_{m,l} \frac{\partial E_{m,l}}{\partial c_k} + E'_{m,l} \frac{\partial E'_{m,l}}{\partial c_k} \right) \quad k = -1, 0, 1, \dots, m, \quad (34)$$

where

$$E_{m,l} := (f - Rh_m)(x_l), \quad E'_{m,l} := (f - Rh_m)'(x_l), \quad l = 1, 2, \dots, n. \quad (35)$$

Necessary condition (34) yields the following linear algebraic system (LAS):

$$A_{m+2} c^{(m)} = F_{m+2}, \quad (36)$$

where  $c^{(m)} \in \mathbb{R}^{m+2}$ ,  $A_{m+2}$  is a square, symmetric matrix with the following entries:

$$(A_{m+2})_{1,1} := 2 \sum_{l=1}^n w_l^{(n)} e^{-2(x_l+1)}, \quad (A_{m+2})_{1,2} = 0, \quad (37)$$

$$(A_{m+2})_{1,j} := 2 \sum_{l=1}^n w_l^{(n)} C_{l,j-2} e^{-(x_l+1)}, \quad j = 3, \dots, m+2,$$

$$(A_{m+2})_{2,2} := 2 \sum_{l=1}^n w_l^{(n)} e^{-2(1-x_l)}, \quad (38)$$

$$(A_{m+2})_{2,j} := 2 \sum_{l=1}^n w_l^{(n)} B_{l,j-2} e^{-(1-x_l)}, \quad j = 3, \dots, m+2,$$

$$(A_{m+2})_{i,j} := \sum_{l=1}^n 2w_l^{(n)} (B_{l,i-2} B_{l,j-2} + C_{l,i-2} C_{l,j-2}), \quad (39)$$

$$i = 3, \dots, m+2, \quad j = i, \dots, m+2,$$

$$(A_{m+2})_{j,i} = (A_{m+2})_{i,j}, \quad i, j = 1, 2, \dots, m+2,$$

$F_{m+2}$  is a vector in  $\mathbb{R}^{m+2}$  with the following elements:

$$(F_{m+2})_1 := \sum_{l=1}^n w_l^{(n)} (f(x_l) - f'(x_l)) e^{-(x_l+1)} = \langle \mathcal{H}_n Rh, \mathcal{H}q_{-1} \rangle_{w^{(n)}, 1}$$

$$(F_{m+2})_2 := \sum_{l=1}^n w_l^{(n)} (f(x_l) + f'(x_l)) e^{-(1-x_l)} = \langle \mathcal{H}_n Rh, \mathcal{H}q_0 \rangle_{w^{(n)}, 1}, \quad (40)$$

$$(F_{m+2})_i := \sum_{l=1}^n w_l^{(n)} [f(x_l)(C_{l,i-2} + B_{l,i-2}) + f'(x_l)(B_{l,i-2} - C_{l,i-2})]$$

$$= \langle \mathcal{H}_n Rh, \mathcal{H}q_i \rangle_{w^{(n)}, 1}, \quad i = 3, \dots, m+2,$$

and

$$B_{l,j} := \int_{x_l}^1 e^{-(y-x_l)} \varphi_j(y) dy, \quad C_{l,j} := \int_{-1}^{x_l} e^{-(x_l-y)} \varphi_j(y) dy. \quad (41)$$



**Theorem 2.2.** *Assume that the vectors  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, \dots, m$  are linearly independent. Then linear algebraic system (36) is uniquely solvable for all  $m$ , where  $m$  is an integer depending on  $n$  such that (25) holds.*

*Proof.* Consider  $q_j \in H^1[-1, 1]$  defined in (19). Using the inner product in  $\mathbb{R}_1^n$ , one gets

$$(A_{m+2})_{i,j} = \langle \mathcal{H}_n q_{i-2}, \mathcal{H}_n q_{j-2} \rangle_{w^{(n)}, 1}, \quad i, j = 1, 2, \dots, m+2, \quad (42)$$

i.e.,  $A_{m+2}$  is a Gram matrix. We have assumed that the vectors  $\mathcal{H}_n q_j \in \mathbb{R}_1^n$ ,  $j = -1, 0, 1, \dots, m$ , are linearly independent. Therefore, the determinant of the matrix  $A_{m+2}$  is nonzero. This implies linear algebraic system (36) has a unique solution.

Theorem 2.2 is proved.  $\square$

It is possible to choose basis functions  $\varphi_j$  such that the vectors  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, \dots, m$ , are linearly independent. An example of such choice of the basis functions is given in Section 3.

**Lemma 2.3.** *Let  $y_m := c_{\min}^{(m)}$  be the unique minimizer for problem (33). Then*

$$G_n(y_m) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (43)$$

where  $G_n$  is defined in (26) and  $m$  is an integer depending on  $n$  such that (25) holds.

*Proof.* Let

$$h(x) = a_{-1}\delta(x+1) + a_0\delta(x-1) + g(x)$$

be the exact solution to (3),  $Rh = f$ , where  $g(x) \in C[-1, 1]$ , and define

$$\tilde{h}_m(x) = a_{-1}\delta(x+1) + a_0\delta(x-1) + \tilde{g}_m(x), \quad (44)$$

where

$$\tilde{g}_m(x) := \sum_{j=1}^m a_j \varphi_j(x). \quad (45)$$

Choose  $\tilde{g}_m(x)$  so that

$$\max_{x \in [-1, 1]} |g(x) - \tilde{g}_m(x)| \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (46)$$

Then

$$G_n(y_m) \leq \|\mathcal{H}_n(f - R\tilde{h}_m)\|_{w^{(n)}, 1}^2, \quad (47)$$

because  $y_m$  is the unique minimizer of  $G_n$ .

Let us prove that  $\|\mathcal{H}_n(f - R\tilde{h}_m)\|_{w^{(n)}, 1}^2 \rightarrow 0$  as  $n \rightarrow \infty$ . Let

$$W_{1,m}(x) := f(x) - R\tilde{h}_m(x), \quad W_{2,m} := f'(x) - (R\tilde{h}_m)'(x).$$

Then

$$W_{1,m}(x) = e^{-x} \int_{-1}^x e^y (g(y) - \tilde{g}_m(y)) dy + e^x \int_x^1 e^{-y} (g(y) - \tilde{g}_m(y)) dy \quad (48)$$

and

$$W_{2,m}(x) = e^x \int_x^1 e^{-y} (g(y) - \tilde{g}_m(y)) dy - e^{-x} \int_{-1}^x e^y (g(y) - \tilde{g}_m(y)) dy. \quad (49)$$

Thus, the functions  $[W_{1,m}(x)]^2$  and  $[W_{2,m}(x)]^2$  are Riemann-integrable. Therefore,

$$\delta_n := \|\|f - R\tilde{h}_m\|_1^2 - \|\mathcal{H}_n(f - R\tilde{h}_m)\|_{w^{(n)},1}^2\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (50)$$

Formula (50) and the triangle inequality yield

$$\|\mathcal{H}_n(f - R\tilde{h}_m)\|_{w^{(n)},1}^2 \leq \delta_n + \|f - R\tilde{h}_m\|_1^2. \quad (51)$$

Let us derive an estimate for  $\|f - R\tilde{h}_m\|_1^2$ . From (48) and (49) we obtain the estimates:

$$\begin{aligned} |W_{1,m}(x)| &\leq \max_{y \in [-1,1]} |g(y) - \tilde{g}_m(y)| \left( e^{-x} \int_{-1}^x e^y dy + e^x \int_x^1 e^{-y} dy \right) \\ &= \max_{y \in [-1,1]} |g(y) - \tilde{g}_m(y)| [e^{-x}(e^x - e^{-1}) + e^x(e^{-x} - e^{-1})] \\ &= \max_{y \in [-1,1]} |g(y) - \tilde{g}_m(y)| [(2 - e^{-1-x} - e^{-1+x})] \leq \delta_{m,1} \end{aligned} \quad (52)$$

and

$$|W_{2,m}(x)| \leq \max_{y \in [-1,1]} |g(y) - \tilde{g}_m(y)| \left( e^x \int_x^1 e^{-y} dy + e^{-x} \int_{-1}^x e^y dy \right) \leq \delta_{m,1}, \quad (53)$$

where

$$\delta_{m,1} := 2 \max_{y \in [-1,1]} |g(y) - \tilde{g}_m(y)|. \quad (54)$$

Therefore, it follows from (52) and (53) that

$$\begin{aligned} \|f - R\tilde{h}_m\|_1^2 &= \int_{-1}^1 |W_{1,m}(x)|^2 dx + \int_{-1}^1 |W_{2,m}(x)|^2 dx \\ &\leq 4\delta_{m,1}^2, \end{aligned} \quad (55)$$

where  $\delta_{m,1}$  is defined in (54). Using relation (46), we obtain  $\lim_{m \rightarrow \infty} \delta_{m,1} = 0$ . Since  $m = m(n)$  and  $\lim_{n \rightarrow \infty} m(n) = \infty$ , it follows from (51) and (55) that  $\|\mathcal{H}_n(f - R\tilde{h}_m)\|_{w,1}^2 \rightarrow 0$  as  $n \rightarrow \infty$ . This together with (47) imply  $G_n(y_m) \rightarrow 0$  as  $n \rightarrow \infty$ .

Lemma 2.3 is proved.  $\square$

**Theorem 2.4.** Let the vector  $c_{min}^{(m)} := \begin{pmatrix} c_{-1}^{(m)} \\ c_0^{(m)} \\ c_1^{(m)} \\ \vdots \\ c_m^{(m)} \end{pmatrix} \in \mathbb{R}^{m+2}$  solve linear algebraic

system (36) and

$$h_m(x) = c_{-1}^{(m)} \delta(x+1) + c_0^{(m)} \delta(x-1) + \sum_{j=1}^m c_j^{(m)} \varphi_j(x).$$

Then

$$\|h - h_m\|_{H^{-1}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (56)$$

*Proof.* We have

$$\begin{aligned} \|h - h_m\|_{H^{-1}[-1,1]}^2 &= \|R^{-1}(f - Rh_m)\|_{H^{-1}[-1,1]}^2 \\ &\leq \|R^{-1}\|_{H^1[-1,1] \rightarrow \dot{H}^{-1}[-1,1]}^2 \|f - Rh_m\|_1^2 \\ &\leq C \left( G_n(c_{min}^{(m)}) + \left| \|f - Rh_m\|_1^2 - G_n(c_{min}^{(m)}) \right| \right) \\ &\leq C[G_n(c_{min}^{(m)}) + \delta_n] \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned} \quad (57)$$

where  $C > 0$  is a constant and Lemma 2.3 was used.

Theorem 2.4 is proved.  $\square$

### 3 The choice of collocation points and basis functions

In this section we give an example of the collocation points  $x_i$ ,  $i = 1, 2, \dots, n$ , and basis functions  $\varphi_j$ ,  $j = 1, 2, \dots, m$ , such that the vectors  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, \dots, m$ , are linearly independent, where  $m + 2 \leq n$ ,

$$\mathcal{H}_n q_{-1} = \begin{pmatrix} e^{-1-x_1} \\ e^{-1-x_2} \\ \vdots \\ e^{-1-x_n} \end{pmatrix}, \quad \mathcal{H}_n q_0 = \begin{pmatrix} e^{-1+x_1} \\ e^{-1+x_2} \\ \vdots \\ e^{-1+x_n} \end{pmatrix} \quad (58)$$

and

$$\mathcal{H}_n q_j = \begin{pmatrix} e^{-x_1} \int_{-1}^1 e^y \varphi_j(y) dy + e^{x_1} \int_{-1}^1 e^{-y} \varphi_j(y) dy \\ e^{-x_2} \int_{-1}^1 e^y \varphi_j(y) dy + e^{x_2} \int_{-1}^1 e^{-y} \varphi_j(y) dy \\ \vdots \\ e^{-x_n} \int_{-1}^1 e^y \varphi_j(y) dy + e^{x_n} \int_{-1}^1 e^{-y} \varphi_j(y) dy \end{pmatrix}, \quad j = 1, 2, \dots, m. \quad (59)$$

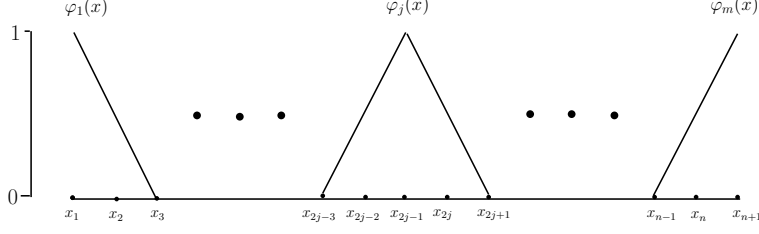


Figure 1: The structure of the basis functions  $\varphi_j$

Let us choose  $w_j^{(n)}$  and  $x_j$  as in (27) and (28), respectively, with an even number  $n \geq 6$ . As the basis functions in  $C[-1, 1]$  we choose the following linear B-splines:

$$\begin{aligned} \varphi_1(x) &= \begin{cases} \psi_2(x) & x_1 \leq x \leq x_3, \\ 0, & \text{otherwise,} \end{cases} \\ \varphi_j(x) &= \begin{cases} \psi_1(x - (j-1)2s), & x_{2j-3} \leq x \leq x_{2j-1}, \\ \psi_2(x - (j-1)2s), & x_{2j-1} \leq x \leq x_{2j+1}, \\ 0, & \text{otherwise,} \end{cases} \\ & j = 2, \dots, m-1, \\ \varphi_m(x) &= \begin{cases} \psi_1(x - (m-1)2s), & x_{n-1} \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (60)$$

where

$$m = \frac{n}{2} + 1, \quad s := \frac{2}{n},$$

and

$$\begin{aligned} \psi_1(x) &:= \frac{x - x_1 + 2s}{2s}, \\ \psi_2(x) &:= \frac{-(x - x_1 - 2s)}{2s}. \end{aligned} \quad (61)$$

Here we have chosen  $x_{2j-1}$ ,  $j = 1, 2, \dots, \frac{n}{2} + 1$ , as the knots of the linear B-splines. From Figure 1 we can see that at each  $j = 2, \dots, m-1$ ,  $\varphi_j(x)$  is a "hat" function. The advantage of using these basis functions is the following: at most two basis functions are needed for computing the solution  $h_m(x)$ , because

$$h_m(x) = \begin{cases} c_l^{(m)}, & x = x_{2l-1}, \\ c_l^{(m)}\varphi_l(x) + c_{l+1}^{(m)}\varphi_{l+1}(x), & x_{2l-1} < x < x_{2l+1}, \end{cases} \quad l = 2, \dots, \frac{n}{2}. \quad (62)$$

From the structure of the basis functions  $\varphi_j$  we have

$$\begin{aligned} \varphi_1(x) &= 0, \quad x_3 \leq x \leq 1, \\ \varphi_j(x) &= 0, \quad -1 \leq x \leq x_{2j-3} \text{ and } x_{2j+1} \leq x \leq 1, \quad j = 2, 3, \dots, m-1, \\ \varphi_m(x) &= 0, \quad -1 \leq x \leq x_{n-1}. \end{aligned} \quad (63)$$

Let  $(\mathcal{H}_n q_j)_i$  be the  $i$ -th element of the vector  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, \dots, m$ . Then

$$(\mathcal{H}_n q_{-1})_i = e^{-(1+x_i)}, \quad i = 1, 2, \dots, n, \quad (64)$$

$$(\mathcal{H}_n q_0)_i = e^{-(1-x_i)}, \quad i = 1, 2, \dots, n. \quad (65)$$

Using (63) in (59), we obtain

$$\begin{aligned} (\mathcal{H}_n q_1)_i &= \begin{cases} \frac{2s-1+e^{-2s}}{2s}, & i = 1, \\ 1 - e^{-s}, & i = 2, \\ e^{1-(i-1)s} C_1, & 3 \leq i \leq n, \end{cases} \\ (\mathcal{H}_n q_j)_i &= e^{-1+(i-1)s} D_j, \quad 1 \leq i \leq 2j-3, \quad j = 2, 3, \dots, m-1, \\ (\mathcal{H}_n q_j)_{2j-2} &= \frac{e^{-3s} - e^{-s} + 2s}{2s}, \quad j = 2, 3, \dots, m-1, \\ (\mathcal{H}_n q_j)_{2j-1} &= \frac{-1 + e^{-2s} + 2s}{s}, \quad j = 2, 3, \dots, m-1, \\ (\mathcal{H}_n q_j)_{2j} &= (\mathcal{H}_n q_j)_{2j-2}, \quad j = 2, 3, \dots, m-1, \\ (\mathcal{H}_n q_j)_i &= e^{1-(i-1)s} C_j, \quad 2j+1 \leq i \leq n, \quad j = 2, 3, \dots, m-1, \\ (\mathcal{H}_n q_m)_i &= \begin{cases} e^{-1+(i-1)s} D_m, & 1 \leq i \leq n-1, \\ 1 - e^{-s}, & i = n, \end{cases} \end{aligned} \quad (66)$$

where

$$\begin{aligned} C_1 &:= \int_{-1}^{x_3} e^y \varphi_1(y) dy = \frac{-1 + e^{2s} - 2s}{2es}, \\ C_j &:= \int_{x_{2j-3}}^{x_{2j+1}} e^y \varphi_j(y) dy = \frac{e^{-1+2(-2+j)s}(-1 + e^{2s})^2}{2s}, \quad j = 2, 3, \dots, m-1, \end{aligned} \quad (67)$$

$$\begin{aligned} D_1 &:= \int_{-1}^{x_3} e^{-y} \varphi_1(y) dy = \frac{e(2s-1+e^{-2s})}{2s}, \\ D_j &:= \int_{x_{2j-3}}^{x_{2j+1}} e^{-y} \varphi_j(y) dy = \frac{e^{1-2js}(-1 + e^{2s})^2}{2s}, \quad j = 2, 3, \dots, m-1, \\ D_m &:= \int_{x_{n-1}}^1 e^{-y} \varphi_m(y) dy = \frac{-1 + e^{2s} - 2s}{2es}. \end{aligned} \quad (68)$$

**Theorem 3.1.** Consider  $q_j$  defined in (19) with  $\varphi_j$  defined in (60). Let the collocation points  $x_j$ ,  $j = 1, 2, \dots, n$ , be defined in (28) with an even number  $n \geq 6$ . Then the vectors  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, 2, \dots, m$ ,  $m = \frac{1}{s} + 1$ ,  $s = \frac{2}{n}$ , are linearly independent, where  $\mathcal{H}_n$  is defined in (20).

*Proof.* Let

$$\begin{aligned} V_0 &:= \{\mathcal{H}_n q_{-1}, \mathcal{H}_n q_0\}, \\ V_j &:= V_{j-1} \cup \{\mathcal{H}_n q_j\}, \quad j = 1, 2, \dots, m. \end{aligned} \quad (69)$$

We prove that the elements of the sets  $V_j$ ,  $j = 0, 1, \dots, m$ , are linearly independent.

The elements of the set  $V_0$  are linearly independent. Indeed,  $\mathcal{H}_n q_j \neq 0 \forall j$ , and assuming that there exists a constant  $\alpha$  such that

$$\mathcal{H}_n q_{-1} = \alpha \mathcal{H}_n q_0, \quad (70)$$

one gets a contradiction: consider the first and the  $n$ -th equations of (70), i.e.,

$$(\mathcal{H}_n q_{-1})_1 = \alpha (\mathcal{H}_n q_0)_1 \quad (71)$$

and

$$(\mathcal{H}_n q_{-1})_n = \alpha (\mathcal{H}_n q_0)_n, \quad (72)$$

respectively. It follows from (64), (65) and (71) that

$$\alpha = e^2. \quad (73)$$

From (72), (64), (65) and (73) it follows that

$$e^{-2+s} = e^{2-s}. \quad (74)$$

This is a contradiction, which proves that  $\mathcal{H}_n q_{-1}$  and  $\mathcal{H}_n q_0$  are linearly independent.

Let us prove that the element of the set  $V_j$  are linearly independent,  $j = 1, 2, 3, \dots, m-2$ . Assume that there exist constants  $\alpha_k$ ,  $k = 1, 2, \dots, j+1$ , such that

$$\mathcal{H}_n q_j = \sum_{k=-1}^{j-1} \alpha_{k+2} \mathcal{H}_n q_k. \quad (75)$$

Using relations (64)-(66) one can write the  $(2j-1)$ -th equation of linear system (75) as follows:

$$\begin{aligned} (\mathcal{H}_n q_j)_{2j-1} &= \sum_{k=-1}^{j-1} \alpha_{k+2} (\mathcal{H}_n q_k)_{2j-1} \\ &= \alpha_1 e^{-(2j-2)s} + \alpha_2 e^{-2+(2j-2)s} + e^{1-(2j-2)s} \sum_{k=1}^{j-1} \alpha_{k+2} C_k. \end{aligned} \quad (76)$$

Similarly, by relations (64)-(66) the  $(n-1)$ -th and  $n$ -th equations of linear system (75) can be written in the following expressions:

$$(\mathcal{H}_n q_j)_{n-1} = e^{-1+2s} C_j = \alpha_1 e^{-2+2s} + \alpha_2 e^{-2s} + e^{-1+2s} \sum_{k=1}^{j-1} \alpha_{k+2} C_k \quad (77)$$

and

$$(\mathcal{H}_n q_j)_n = e^{-1+s} C_j = \alpha_1 e^{-2+s} + \alpha_2 e^{-s} + e^{-1+s} \sum_{k=1}^{j-1} \alpha_{k+2} C_k, \quad (78)$$

respectively. Multiply (78) by  $e^s$  and compare with (77) to conclude that  $\alpha_2 = 0$ . From (78) with  $\alpha_2 = 0$  one obtains

$$\alpha_1 = eC_j - e \sum_{k=1}^{j-1} \alpha_{k+2} C_k. \quad (79)$$

Substitute  $\alpha_1$  from (79) and  $\alpha_2 = 0$  into (76) and get

$$(\mathcal{H}_n q_j)_{2j-1} = e^{1-(2j-2)s} C_j. \quad (80)$$

From (67) and (66) one obtains for  $0 < s < 1$ ,  $j = 1, 2, 3, \dots, m-2$ , the following relation

$$\begin{aligned} e^{1-(2j-2)s} C_j - (\mathcal{H}_n q_j)_{2j-1} &= \frac{e^{-2s}(-1 + e^{2s})^2}{2s} - \frac{-1 + e^{-2s} + 2s}{s} \\ &= \frac{e^{2s} - e^{-2s} - 4s}{2s} = \frac{\sinh(2s) - 2s}{s} > 0, \end{aligned} \quad (81)$$

which contradicts relation (80). This contradiction proves that the elements of the set  $V_j$  are linearly independent,  $j = 1, 2, 3, \dots, m-2$ , for  $0 < s < 1$ .

Let us prove that the elements of the set  $V_{m-1}$ , are linearly independent. Assume that there exist constants  $\alpha_k$ ,  $k = 1, 2, \dots, m$ , such that

$$\mathcal{H}_n q_{m-1} = \sum_{k=-1}^{m-2} \alpha_{k+2} \mathcal{H}_n q_k. \quad (82)$$

Using (64)-(66), the  $(n-3)$ -th equation of (82) can be written as follows:

$$\begin{aligned} (\mathcal{H}_n q_{m-1})_{n-3} &= e^{1-4s} D_{m-1} = \sum_{k=-1}^{m-3} \alpha_{k+2} (\mathcal{H}_n q_k)_{n-3} \\ &= \alpha_1 e^{-2+4s} + \alpha_2 e^{-4s} + e^{-1+4s} \sum_{k=1}^{m-3} \alpha_{k+2} C_k + \alpha_m (\mathcal{H}_n q_{m-2})_{n-3}. \end{aligned} \quad (83)$$

Similarly we obtain the  $(n-2)$ -th,  $(n-1)$ -th and  $n$ -th equations, corresponding to vector equation (82):

$$(\mathcal{H}_n q_{m-1})_{n-2} = \alpha_1 e^{-2+3s} + \alpha_2 e^{-3s} + e^{-1+3s} \sum_{k=1}^{m-3} \alpha_{k+2} C_k + \alpha_m (\mathcal{H}_n q_{m-2})_{n-2}, \quad (84)$$

$$(\mathcal{H}_n q_{m-1})_{n-1} = \alpha_1 e^{-2+2s} + \alpha_2 e^{-2s} + e^{-1+2s} \sum_{k=1}^{m-2} \alpha_{k+2} C_k \quad (85)$$

and

$$(\mathcal{H}_n q_{m-1})_n = \alpha_1 e^{-2+s} + \alpha_2 e^{-s} + e^{-1+s} \sum_{k=1}^{m-2} \alpha_{k+2} C_k, \quad (86)$$

respectively. Multiply (86) by  $e^s$  and compare with (85) to get

$$\alpha_2 = \frac{(\mathcal{H}_n q_{m-1})_{n-1} - e^s (\mathcal{H}_n q_{m-1})_n}{e^{-2s} - 1} = \frac{1 - e^{2s} + 4se^{2s} - 2se^{3s}}{2s(1 - e^{2s})}, \quad (87)$$

where formula (66) was used. Multiplying (86) by  $e^{3s}$ , comparing with equation (83), and using (87), we obtain

$$\begin{aligned} \alpha_m &= \frac{e^{1-4s} D_{m-1} - e^{3s} (\mathcal{H}_n q_{m-1})_n - \alpha_2 (e^{-4s} - e^{2s})}{(\mathcal{H}_n q_{m-2})_{n-3} - e^{-1+4s} C_{m-2}} \\ &= \frac{2 + 4s - 2e^s s + 4e^{2s} s - 2e^{3s} s + e^{4s} (-2 + 4s)}{-1 + e^{4s} - 4e^{2s} s}. \end{aligned} \quad (88)$$

Another expression for  $\alpha_m$  is obtained by multiplying (86) by  $e^{2s}$  and comparing with (84):

$$\begin{aligned} \alpha_m &= \frac{(\mathcal{H}_n q_{m-1})_{n-2} - e^{2s} (\mathcal{H}_n q_{m-1})_n - \alpha_2 (e^{-3s} - e^s)}{(\mathcal{H}_n q_{m-2})_{n-2} - e^{-1+3s} C_{m-2}} \\ &= \frac{2 + 4s - 4e^s s + e^{2s} (-2 + 4s)}{-1 + e^{2s} - 2e^s s}, \end{aligned} \quad (89)$$

where  $\alpha_2$  is given in (87).

In deriving formulas (88) and (89) we have used the relation  $m = \frac{1}{s} + 1$  and equation (66). Let us prove that equations (88) and (89) lead to a contradiction. Define

$$\begin{aligned} r_1 &:= 2 + 4s - 2e^s s + 4e^{2s} s - 2e^{3s} s + e^{4s} (-2 + 4s), \\ r_2 &:= -1 + e^{4s} - 4e^{2s} s, \\ r_3 &:= 2 + 4s - 4e^s s + e^{2s} (-2 + 4s), \\ r_4 &:= -1 + e^{2s} - 2e^s s. \end{aligned} \quad (90)$$

Then from (88) and (89) we get

$$r_3 r_2 - r_1 r_4 = 0. \quad (91)$$

We have

$$r_3 r_2 - r_1 r_4 = 2e^s (-1 + e^s)^2 s (3 + 4s + (4s - 3)e^{2s} - 2se^s) > 0 \quad \text{for } s \in (0, 1). \quad (92)$$

The sign of the right side of equality (92) is the same as the sign of  $3 + 4s + (4s - 3)e^{2s} - 2se^s := \beta(s)$ . Let us check that  $\beta(s) > 0$  for  $s \in (0, 1)$ . One has  $\beta(0) = 0$ ,



$\beta'(0) = 0$ ,  $\beta'(s) = 4 - 2e^{2s} + 8se^{2s} - 2e^s - 2se^s$ ,  $\beta'' = 4e^{2s} + 16se^{2s} - 4e^s - 2se^s > 0$ . If  $\beta''(s) > 0$  for  $s \in (0, 1)$  and  $\beta(0) = 0$ ,  $\beta'(0) = 0$ , then  $\beta(s) > 0$  for  $s \in (0, 1)$ . Inequality (92) contradicts relation (91) which proves that  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, 2, \dots, m-1$ , are linearly independent.

Similarly, to prove that  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, 2, \dots, m$ , are linearly independent, we assume that there exist constants  $\alpha_k$ ,  $k = 1, 2, \dots, m+1$ , such that

$$\mathcal{H}_n q_m = \sum_{k=-1}^{m-1} \alpha_{k+2} \mathcal{H}_n q_k. \quad (93)$$

Using formulas (64)-(66), one can write the  $(n-5)$ -th equation:

$$\begin{aligned} (\mathcal{H}_n q_m)_{n-5} &= e^{1-6s} D_m = \sum_{k=-1}^{m-1} \alpha_{k+2} (\mathcal{H}_n q_k)_{n-5} \\ &= \alpha_1 e^{-2+6s} + \alpha_2 e^{-6s} + e^{-1+6s} \sum_{j=1}^{m-4} \alpha_{j+2} C_j \\ &\quad + \alpha_{m-1} (\mathcal{H}_n q_{m-3})_{n-5} + \alpha_m e^{1-6s} D_{m-2} + \alpha_{m+1} e^{1-6s} D_{m-1}, \end{aligned} \quad (94)$$

Similarly one obtains the  $(n-4)$ -th,  $(n-3)$ -th,  $(n-2)$ -th,  $(n-1)$ -th and  $n$ -th equations corresponding to the vector equation (93):

$$\begin{aligned} (\mathcal{H}_n q_m)_{n-4} &= e^{1-5s} D_m = \alpha_1 e^{-2+5s} + \alpha_2 e^{-5s} + e^{-1+5s} \sum_{j=1}^{m-4} \alpha_{j+2} C_j \\ &\quad + \alpha_{m-1} (\mathcal{H}_n q_{m-3})_{n-4} + \alpha_m (\mathcal{H}_n q_{m-2})_{n-4} + \alpha_{m+1} e^{1-5s} D_{m-1}, \end{aligned} \quad (95)$$

$$\begin{aligned} (\mathcal{H}_n q_m)_{n-3} &= e^{1-4s} D_m = \alpha_1 e^{-2+4s} + \alpha_2 e^{-4s} + e^{-1+4s} \sum_{j=1}^{m-3} \alpha_{j+2} C_j \\ &\quad + \alpha_m (\mathcal{H}_n q_{m-2})_{n-3} + \alpha_{m+1} e^{1-4s} D_{m-1}, \end{aligned} \quad (96)$$

$$\begin{aligned} (\mathcal{H}_n q_m)_{n-2} &= e^{1-3s} D_m = \alpha_1 e^{-2+3s} + \alpha_2 e^{-3s} + e^{-1+3s} \sum_{j=1}^{m-3} \alpha_{j+2} C_j \\ &\quad + \alpha_m (\mathcal{H}_n q_{m-2})_{n-2} + \alpha_{m+1} (\mathcal{H}_n q_{m-1})_{n-2}, \end{aligned} \quad (97)$$

$$\begin{aligned} (\mathcal{H}_n q_m)_{n-1} &= e^{1-2s} D_m = \alpha_1 e^{-2+2s} + \alpha_2 e^{-2s} + e^{-1+2s} \sum_{j=1}^{m-2} \alpha_{j+2} C_j \\ &\quad + \alpha_{m+1} (\mathcal{H}_n q_{m-1})_{n-1} \end{aligned} \quad (98)$$

and

$$(\mathcal{H}_n q_m)_n = \alpha_1 e^{-2+s} + \alpha_2 e^{-s} + e^{-1+s} \sum_{j=1}^{m-2} \alpha_{j+2} C_j + \alpha_{m+1} (\mathcal{H}_n q_{m-1})_n, \quad (99)$$

respectively. Here we have used the assumption  $n \geq 6$ . From (99) one gets

$$\alpha_1 = (\mathcal{H}_n q_m)_n e^{2-s} - \alpha_2 e^{2-2s} - e \sum_{k=1}^{m-2} \alpha_{k+2} C_k - \alpha_{m+1} (\mathcal{H}_n q_{m-1})_n e^{2-s}. \quad (100)$$

If one substitutes (100) into equations (98), (97) and (96), then one obtains the following relations:

$$\alpha_2 = p_1 - p_2 \alpha_{m+1}, \quad (101)$$

$$\alpha_2 = p_3 - p_4 \alpha_m - p_5 \alpha_{m+1} \quad (102)$$

and

$$\alpha_2 = p_6 - p_7 \alpha_m - p_8 \alpha_{m+1}, \quad (103)$$

respectively, where

$$\begin{aligned} p_1 &:= \frac{e^{1-2s} D_m - (\mathcal{H}_n q_m)_n e^s}{e^{-2s} - 1}, & p_2 &:= \frac{(\mathcal{H}_n q_{m-1})_{n-1} - (\mathcal{H}_n q_{m-1})_n e^s}{e^{-2s} - 1}, \\ p_3 &:= \frac{e^{1-3s} D_m - (\mathcal{H}_n q_m)_n e^{2s}}{e^{-3s} - e^s}, & p_4 &:= \frac{(\mathcal{H}_n q_{m-2})_{n-2} - e^{-1+3s} C_{m-2}}{e^{-3s} - e^s}, \\ p_5 &:= \frac{(\mathcal{H}_n q_{m-1})_{n-2} - (\mathcal{H}_n q_{m-1})_n e^{2s}}{e^{-3s} - e^s}, & p_6 &:= \frac{e^{1-4s} D_m - (\mathcal{H}_n q_m)_n e^{3s}}{e^{-4s} - e^{2s}}, \\ p_7 &:= \frac{(\mathcal{H}_n q_{m-2})_{n-3} - e^{-1+4s} C_{m-2}}{e^{-4s} - e^{2s}}, & p_8 &:= \frac{e^{1-4s} D_{m-1} - (\mathcal{H}_n q_{m-1})_n e^{3s}}{e^{-4s} - e^{2s}}. \end{aligned} \quad (104)$$

Another formula for  $\alpha_1$  one gets from equation (96):

$$\begin{aligned} \alpha_1 &= e^{3-8s} D_m - \alpha_2 e^{2-8s} - e \sum_{j=1}^{m-3} \alpha_{j+2} C_j - \alpha_m (\mathcal{H}_n q_{m-2})_{n-3} e^{2-4s} \\ &\quad - \alpha_{m+1} e^{3-8s} D_{m-1}. \end{aligned} \quad (105)$$

Substituting (105) into equations (95) and (94), yields

$$\alpha_2 = p_9 - p_{10} \alpha_{m-1} - p_{11} \alpha_m - p_{12} \alpha_{m+1} \quad (106)$$

and

$$\alpha_2 = p_9 - p_{13} \alpha_{m-1} - p_{14} \alpha_m - p_{12} \alpha_{m+1}, \quad (107)$$

respectively, where

$$\begin{aligned} p_9 &:= e D_m, & p_{10} &:= \frac{(\mathcal{H}_n q_{m-3})_{n-4} - e^{-1+5s} C_{m-3}}{e^{-5s} - e^{-3s}}, \\ p_{11} &:= \frac{(\mathcal{H}_n q_{m-2})_{n-4} - e^s (\mathcal{H}_n q_{m-2})_{n-3}}{e^{-5s} - e^{-3s}}, & p_{12} &:= e D_{m-1}, \\ p_{13} &:= \frac{(\mathcal{H}_n q_{m-3})_{n-5} - e^{-1+6s} C_{m-3}}{e^{-6s} - e^{-2s}}, & p_{14} &:= \frac{e^{1-6s} D_{m-2} - e^{2s} (\mathcal{H}_n q_{m-2})_{n-3}}{e^{-6s} - e^{-2s}}. \end{aligned} \quad (108)$$

Let us prove that the equations (101) and (106) lead to a contradiction. From equations (101) and (102) we obtain

$$\alpha_m = \frac{p_3 - p_1}{p_4} + \frac{p_2 - p_5}{p_4} \alpha_{m+1}. \quad (109)$$

This together with (103) and (101) yield

$$\alpha_{m+1} = \frac{\frac{p_3 - p_1}{p_4} - \frac{p_6 - p_1}{p_7}}{\frac{p_2 - p_8}{p_7} - \frac{p_2 - p_5}{p_4}}. \quad (110)$$

Equations (106), (107) and (109) yield

$$\alpha_{m-1} = \frac{p_{14} - p_{11}}{p_{10} - p_{13}} \alpha_m = \frac{p_{14} - p_{11}}{p_{10} - p_{13}} \left( \frac{p_3 - p_1}{p_4} + \frac{p_2 - p_5}{p_4} \alpha_{m+1} \right). \quad (111)$$

This together with (106) imply

$$\alpha_2 = p_9 - \left( p_{10} \frac{p_{14} - p_{11}}{p_{10} - p_{13}} + p_{11} \right) \left( \frac{p_3 - p_1}{p_4} + \frac{p_2 - p_5}{p_4} \alpha_{m+1} \right) - p_{12} \alpha_{m+1}, \quad (112)$$

where  $\alpha_{m+1}$  is given in (110). Let

$$L_1 := p_9 - \left( p_{10} \frac{p_{14} - p_{11}}{p_{10} - p_{13}} + p_{11} \right) \left( \frac{p_3 - p_1}{p_4} + \frac{p_2 - p_5}{p_4} \alpha_{m+1} \right) - p_{12} \alpha_{m+1} \quad (113)$$

and

$$L_2 := p_1 - p_2 \alpha_{m+1}. \quad (114)$$

Then, from (101) and (106) one gets

$$L_1 - L_2 = 0. \quad (115)$$

Applying formulas (64)-(66) in (113) and (114) and using the relation  $e^s = \sum_{j=0}^{\infty} \frac{s^j}{j!}$ , we obtain

$$\begin{aligned} L_1 - L_2 &= \frac{2e^{4s}[1 + e^{2s}(-1 + s) + s](-1 + e^{2s} - 2e^s s)}{(-1 + e^{2s})s[3 + 4s - 2e^s s + e^{2s}(-3 + 4s)]} \\ &= \frac{2e^{4s} \left[ \sum_{j=3}^{\infty} \frac{2^j(\frac{1}{2} - \frac{1}{j})}{(j-1)!} s^j \right] \left[ \sum_{j=2}^{\infty} \frac{2(\frac{2^j}{j+1} - 1)}{j!} s^{j+1} \right]}{(-1 + e^{2s})s[3 + 4s - 2e^s s + e^{2s}(-3 + 4s)]} > 0, \end{aligned} \quad (116)$$

because  $e^{2s} > 1$  for all  $0 < s < 1$ ,  $2^j(\frac{1}{2} - \frac{1}{j}) > 0$  for all  $j \geq 3$ ,  $2(\frac{2^j}{j+1} - 1) > 0$  for all  $j \geq 2$  and  $3 + 4s - 2e^s s + e^{2s}(-3 + 4s) > 0$  which was proved in (92). Inequality (116) contradicts relation (115) which proves that  $\mathcal{H}_n q_j$ ,  $j = -1, 0, 1, 2, \dots, m$ , are linearly independent.

Theorem 3.1 is proved.  $\square$

## 4 Numerical experiments

Note that for all  $w, u, v \in \mathbb{R}^n$  we have

$$\sum_{l=1}^n w_l u_l v_l = v^t W u, \quad (117)$$

where  $t$  stands for transpose and

$$W := \begin{pmatrix} w_1 & 0 & \dots & 0 & 0 \\ 0 & w_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & 0 & w_{n-1} & 0 \\ 0 & 0 & \dots & 0 & w_n \end{pmatrix}. \quad (118)$$

Then

$$\begin{aligned} DP &:= \|\mathcal{H}_n(f - Rh_m)\|_{w^{(n)},1}^2 \\ &= \sum_{l=1}^n w_l^{(n)} [(f(x_l) - Rh_m(x_l))^2 + (f'(x_l) - (Rh_m)'(x_l))^2] \\ &= [\mathcal{H}_n(f - Rh_m)]^t W \mathcal{H}_n(f - Rh_m) \\ &\quad + [\mathcal{H}_n(f' - (Rh_m)')]^t W \mathcal{H}_n(f' - (Rh_m)'), \end{aligned} \quad (119)$$

where  $W$  is defined in (118) with  $w_j = w_j^{(n)}$ ,  $j = 1, 2, \dots, n$ , defined in (27). The vectors  $\mathcal{H}_n Rh_m$  and  $\mathcal{H}_n(Rh_m)'$  are computed as follows.

Using (64)-(66), the vector  $\mathcal{H}_n Rh_m$  can be represented by

$$\mathcal{H}_n Rh_m = S_m c^{(m)}, \quad (120)$$

where  $c^{(m)} = \begin{pmatrix} c_{-1}^{(m)} \\ c_0^{(m)} \\ c_1^{(m)} \\ \vdots \\ c_m^{(m)} \end{pmatrix}$  and  $S_m$  is an  $n \times (m+2)$  matrix with the following

entries:

$$\begin{aligned} (S_m)_{i,1} &= (\mathcal{H}_n q_{-1})_i, & (S_m)_{i,2} &= (\mathcal{H}_n q_0)_i, & i &= 1, 2, \dots, n, \\ (S_m)_{i,j} &= (\mathcal{H}_n q_{j-2})_i, & i &= 1, 2, \dots, n, & j &= 3, 4, \dots, m+2. \end{aligned} \quad (121)$$

The vector  $\mathcal{H}_n(Rh_m)'$  is computed as follows. Let

$$J_{i,j} := e^{x_i} \int_{x_i}^1 e^{-y} \varphi_j(y) dy - e^{-x_i} \int_{-1}^{x_i} e^y \varphi_j(y) dy, \quad i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m. \quad (122)$$

This together with (60) yield

$$\begin{aligned}
J_{i,1} &= \begin{cases} e^{-1} \frac{e(2l-1+e^{-2l})}{2l}, & i = 1, \\ \frac{e^{-l}(1-e^l+l)}{l}, & i = 2, \\ -\frac{e^{-x_i}(-1+e^{2l}-2l)}{2le}, & i \geq 3, \end{cases} \\
J_{i,j} &= \begin{cases} e^{x_i} \frac{e^{1-2jl}(-1+e^{2l})^2}{2l}, & i \leq 2j-3; \\ \frac{2+e^{-3l}-3e^{-l}}{2l}, & i = 2j-2; \\ 0, & i = 2j-1; \quad 1 \leq i \leq n, \quad 1 < j < m, \\ -\frac{2+e^{-3l}-3e^{-l}}{2l}, & i = 2j; \\ -e^{-x_i} \frac{e^{-1+2(-2+j)l}(-1+e^{2l})^2}{2l}, & i \geq 2j+1, \end{cases} \\
J_{i,m} &= \begin{cases} \frac{e^{x_i}(-1+e^{2l}-2l)}{2le}, & i \leq n-1, \\ \frac{e^{-l}(-1+e^l-l)}{l}, & i = n, \end{cases}
\end{aligned} \tag{123}$$

where  $l = \frac{2}{n}$ .

Then, using (123), the vector  $\mathcal{H}_n(Rh_m)'$  can be rewritten as follows:

$$\mathcal{H}_n(Rh_m)' = T_m c^{(m)}, \tag{124}$$

where  $c^{(m)} = \begin{pmatrix} c_{-1}^{(m)} \\ c_0^{(m)} \\ c_1^{(m)} \\ \vdots \\ c_m^{(m)} \end{pmatrix}$  and  $T_m$  is an  $n \times (m+2)$  matrix with the following

entries:

$$\begin{aligned}
(T_m)_{i,1} &= -e^{-(1+x_i)}, \quad (T_m)_{i,2} = e^{-(1-x_i)}, \quad i = 1, 2, \dots, n, \\
(T_m)_{i,j} &= J_{i,j-2}, \quad i = 1, 2, \dots, n, \quad j = 3, 4, \dots, m+2.
\end{aligned} \tag{125}$$

We consider the following examples discussed in [7]:

- (1)  $f(x) = -2 + 2 \cos(\pi(x+1))$  with the exact solution  $h(x) = -1 + (1 + \pi^2) \cos(\pi(x+1))$ .
- (2)  $f(x) = -2e^{x-1} + \frac{2}{\pi} \sin(\pi(x+1)) + 2 \cos(\pi(x+1))$  with the exact solution  $h(x) = \frac{1}{\pi} \sin(\pi(x+1)) + (1 + \pi^2) \cos(\pi(x+1))$ .
- (3)  $f(x) = \cos(\frac{\pi(x+1)}{2}) + 4 \cos(2\pi(x+1)) - 1.5 \cos(\frac{7\pi(x+1)}{2})$  with the exact solution  $h(x) = \frac{1}{2}(1 + \frac{\pi^2}{4}) \cos(\frac{\pi(x+1)}{2}) + (2 + 8\pi^2) \cos(2\pi(x+1)) - 0.75(1 + 12.25\pi^2) \cos(\frac{7\pi(x+1)}{2}) + 1.75\delta(x+1) + 2.25\delta(x-1)$ .
- (4)  $f(x) = e^{-x} + 2 \sin(2\pi(x+1))$  with the exact solution  $h(x) = (1 + 4\pi^2) \sin(2\pi(x+1)) + (e - 2\pi)\delta(x+1) + 2\pi\delta(x-1)$ .

In all the above examples we have  $f \in C^2[-1, 1]$ . Therefore, one may use the basis functions given in (60). In each example we compute the relative pointwise errors:

$$RPE(t_i) := \frac{|g_m(t_i) - g(t_i)|}{\max_{1 \leq i \leq M} |g(t_i)|}, \quad (126)$$

where  $g(x)$  and  $g_m(x)$  are defined in (9) and (11), respectively, and

$$t_i := -1 + (i-1) \frac{2}{M-1}, \quad i = 1, 2, \dots, M. \quad (127)$$

The algorithm can be written as follows.

Step 0. Set  $k = 3$ ,  $n = 2k$ ,  $m = \frac{n}{2} + 1$ ,  $\epsilon \in (0, 1)$  and  $DP \geq 10$ , where  $DP$  is defined in (119).

Step 1. Construct the weights  $w_j^{(n)}$ ,  $j = 1, 2, \dots, n$ , defined in (27).

Step 2. Construct the matrix  $A_{m+2}$  and the vector  $F_{m+2}$  which are defined in (36).

Step 3. Solve for  $c := \begin{pmatrix} c_{-1}^{(m)} \\ c_0^{(m)} \\ c_1^{(m)} \\ \vdots \\ c_m^{(m)} \end{pmatrix}$  the linear algebraic system  $A_{m+2}c = F_{m+2}$ .

Step 4. Compute

$$DP = \|\mathcal{H}_n(f - Rh_m)\|_{w^{(n)}, 1}^2. \quad (128)$$

Step 5. If  $DP > \epsilon$  then set  $k = k + 1$ ,  $n = 2k$  and  $m = \frac{n}{2} + 1$ , and go to Step 1.

Otherwise, stop the iteration and use  $h_m(x) = \sum_{j=-1}^m c_j^{(m)} \varphi_j(x)$  as the approximate solution, where  $\varphi_{-1}(x) := \delta(1+x)$ ,  $\varphi_0(x) := \delta(x-1)$  and  $\varphi_j(x)$ ,  $j = 1, 2, \dots, m$ , are defined in (60) and  $c_j^{(m)}$ ,  $j = -1, 0, 1, \dots, m$ , are obtained in Step 3.

In all the experiments the following parameters are used:  $M = 200$  and  $\epsilon = 10^{-4}$ ,  $10^{-6}$ ,  $10^{-8}$ . We also compute the relative error

$$RE := \max_{1 \leq i \leq M} RPE(t_i), \quad (129)$$

where  $RPE$  is defined in (126). Let us discuss the results of our experiments.

**Example 1.** In this example the coefficients  $a_{-1}$  and  $a_0$ , given in (8) and (9), respectively, are zeros. Our experiments show, see Table 1, that the approximate coefficients  $c_{-1}^{(m)}$  and  $c_0^{(m)}$  converge to  $a_{-1}$  and  $a_0$ , respectively, as  $\epsilon \rightarrow 0$ . Here to get  $DP \leq 10^{-6}$ , we need  $n = 32$  collocation points distributed uniformly in

Table 1: Example 1

$n$	$m$	$\epsilon$	$a_{-1}$	$c_{-1}^{(m)}$	$a_0$	$c_0^{(m)}$
24	13	$10^{-4}$	0	$8.234 \times 10^{-5}$	0	$-5.826 \times 10^{-3}$
32	17	$10^{-6}$	0	$3.172 \times 10^{-5}$	0	$-1.202 \times 10^{-3}$
56	29	$10^{-8}$	0	$3.974 \times 10^{-6}$	0	$-6.020 \times 10^{-5}$
$n$	$m$	$\epsilon$	$DP$	$RE$		
24	13	$10^{-4}$	$8.610 \times 10^{-6}$	$3.239 \times 10^{-2}$		
32	17	$10^{-6}$	$7.137 \times 10^{-7}$	$1.554 \times 10^{-2}$		
56	29	$10^{-8}$	$6.404 \times 10^{-9}$	$4.337 \times 10^{-3}$		

the interval  $[-1, 1]$ . Moreover, the matrix  $A_{m+2}$  is of the size 19 by 19 which is small. For  $\epsilon = 10^{-6}$  the relative error  $RE$  is of order  $10^{-2}$ . The  $RPE$  at the points  $t_j$  are distributed in the interval  $[0, 0.018)$  as shown in Figure 2. In computing the approximate solution  $h_m$  at the points  $t_i$ ,  $i = 1, 2, \dots, M$ , one needs at most two out of  $m = 17$  basis functions  $\varphi_j(x)$ . The reconstruction of the continuous part of the exact solution can be seen in Figure 1. One can see from this Figure that for  $\epsilon = 10^{-6}$  the continuous part  $g(x)$  of the exact solution  $h(x)$  can be recovered very well by the approximate function  $g_m(x)$  at the points  $t_j$ ,  $j = 1, 2, \dots, M$ .

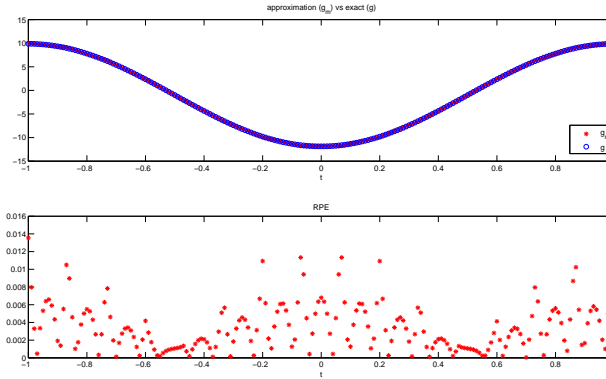


Figure 2: A reconstruction of the continuous part  $g(x)$  (above) of Example 1 with  $\epsilon = 10^{-6}$ , and the corresponding Relative Pointwise Errors (RPE) (below)

**Example 2.** This example is a modification of Example 1, where the constant  $-2$  is replaced with the function  $-2e^{x-1} + \frac{2}{\pi} \sin(\pi(x+1))$ . In this case the

Table 2: Example 2

$n$	$m$	$\epsilon$	$a_{-1}$	$c_{-1}^{(m)}$	$a_0$	$c_0^{(m)}$
24	13	$10^{-4}$	0	$1.632 \times 10^{-4}$	0	$-1.065 \times 10^{-2}$
32	17	$10^{-6}$	0	$5.552 \times 10^{-5}$	0	$-2.614 \times 10^{-3}$
56	29	$10^{-8}$	0	$6.268 \times 10^{-6}$	0	$-1.954 \times 10^{-4}$
$n$	$m$	$\epsilon$	$DP$	$RE$		
24	13	$10^{-4}$	$8.964 \times 10^{-6}$	$3.871 \times 10^{-2}$		
32	17	$10^{-6}$	$7.588 \times 10^{-7}$	$1.821 \times 10^{-2}$		
56	29	$10^{-8}$	$6.947 \times 10^{-9}$	$4.869 \times 10^{-3}$		

coefficients  $a_{-1}$  and  $a_0$  are also zeros. The results can be seen in Table 2. As in Example 1, both approximate coefficients  $c_{-1}^{(m)}$  and  $c_0^{(m)}$  converge to 0 as  $\epsilon \rightarrow 0$ . The number of collocation points at each case is equal to the number of collocation points obtained in Example 1. Also the  $RPE$  at each observed point is in the interval  $[0, 0.02)$ . One can see from Figure 3 that the continuous part  $g(x)$  of the exact solution  $h(x)$  can be well approximated by the approximate function  $g_m(x)$  with  $\epsilon = 10^{-6}$  and  $RE = O(10^{-2})$ .

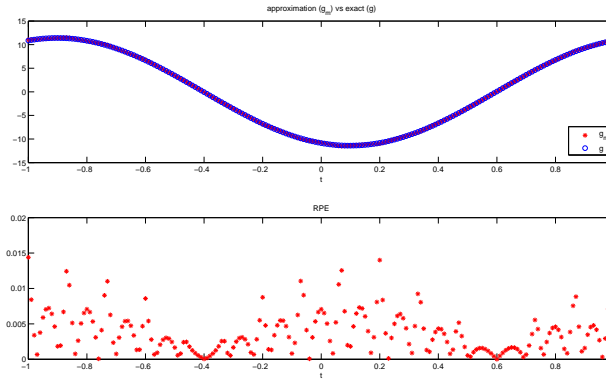


Figure 3: A reconstruction of the continuous part  $g(x)$  (above) of Example 2 with  $\epsilon = 10^{-6}$ , and the corresponding Relative Pointwise Errors (RPE) (below)

**Example 3.** In this example the coefficients of the distributional parts  $a_{-1}$  and  $a_0$  are not zeros. The function  $f$  is oscillating more than the functions  $f$  given in Examples 1 and 2, and the number of collocation points is larger than in the



Table 3: Example 3

$n$	$m$	$\epsilon$	$a_{-1}$	$c_{-1}^{(m)}$	$a_0$	$c_0^{(m)}$
80	41	$10^{-4}$	1.750	1.750	2.250	2.236
128	65	$10^{-6}$	1.750	1.750	2.250	2.249
232	117	$10^{-8}$	1.750	1.750	2.250	2.250
$n$	$m$	$\epsilon$	$DP$	$RE$		
80	41	$10^{-4}$	$4.635 \times 10^{-5}$	$2.282 \times 10^{-2}$		
128	65	$10^{-6}$	$9.739 \times 10^{-7}$	$7.671 \times 10^{-3}$		
232	117	$10^{-8}$	$7.804 \times 10^{-9}$	$2.163 \times 10^{-3}$		

previous two examples, as shown in Table 3. In this table one can see that the approximate coefficients  $c_{-1}^{(m)}$  and  $c_0^{(m)}$  converge to  $a_{-1}$  and  $a_0$ , respectively. The continuous part of the exact solution can be approximated by the approximate function  $g_m(x)$  very well with  $\epsilon = 10^{-6}$  and  $RE = O(10^{-3})$  as shown in Figure 4. In the same Figure one can see that the  $RPE$  at each observed point is distributed in the interval  $[0, 8 \times 10^{-3})$ .

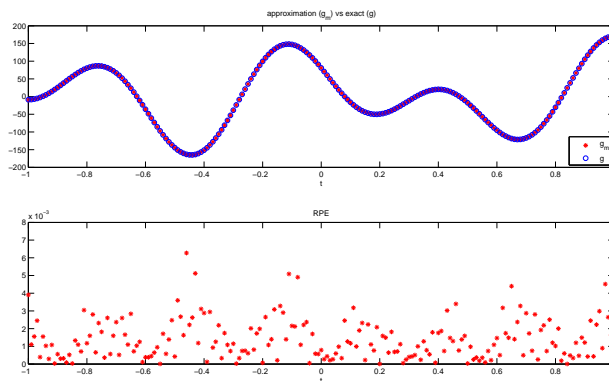


Figure 4: A reconstruction of the continuous part  $g(x)$  (above) of Example 3 with  $\epsilon = 10^{-6}$ , and the corresponding Relative Pointwise Errors (RPE) (below)

**Example 4.** Here we give another example of the exact solution  $h$  having non-zero coefficients  $a_{-1}$  and  $a_0$ . In this example the function  $f$  is oscillating less than the  $f$  in Example 3, but more than the  $f$  in examples 1 and 2. As shown in Table 4 the number of collocation points  $n$  is smaller than the the number of collocation points given in Example 3. In this example the exact

coefficients  $a_{-1}$  and  $a_0$  are obtained at the error level  $\epsilon = 10^{-8}$  which is shown in Table 4. Figure 5 shows that at the level  $\epsilon = 10^{-6}$  we have obtained a good approximation of the continuous part  $g(x)$  of the exact solution  $h(x)$ . Here the relative error  $RE$  is of order  $O(10^{-2})$ .

Table 4: Example 4

$n$	$m$	$\epsilon$	$a_{-1}$	$c_{-1}^{(m)}$	$a_0$	$c_0^{(m)}$
40	21	$10^{-4}$	-3.565	-3.564	6.283	6.234
72	37	$10^{-6}$	-3.565	-3.565	6.283	6.279
128	65	$10^{-8}$	-3.565	-3.565	6.283	6.283
$n$	$m$	$\epsilon$	$DP$	$RE$		
40	21	$10^{-4}$	$8.775 \times 10^{-5}$	$3.574 \times 10^{-2}$		
72	37	$10^{-6}$	$6.651 \times 10^{-7}$	$1.029 \times 10^{-2}$		
128	65	$10^{-8}$	$6.147 \times 10^{-9}$	$3.199 \times 10^{-3}$		

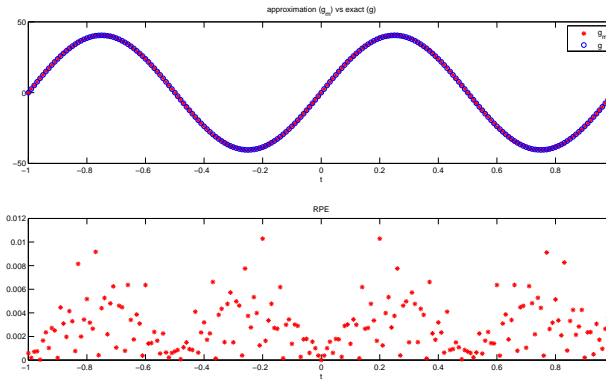


Figure 5: A reconstruction of the continuous part  $g(x)$  (above) of Example 4 with  $\epsilon = 10^{-6}$ , and the corresponding Relative Pointwise Errors (RPE) (below)

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