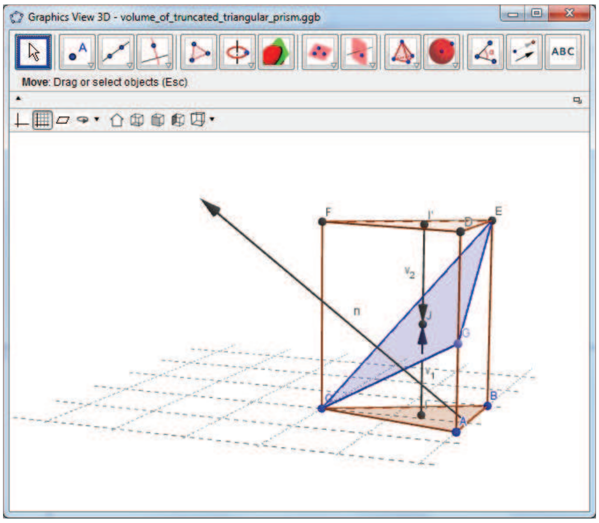
Computation of the Volume of Triangulated Polyhedra

by Miguel Gaspar

In many different contexts, it is necessary to compute the volume enclosed by an arbitrary, closed triangulated surface, as for instance defined by an STL file. For reference, we briefly present the method used for this computation. A triangulated surface is composed of triangular facets. In order to be able to compute an enclosed volume, the surface must be closed.

The problem can be decomposed in two parts: how to compute the contribution of a single facet to the total volume, and how to combine those individual contributions. We will begin by noticing that, given an arbitrary plane, each facet and its projection onto the plane will define a truncated prism. For a simple, convex region, the facets can be divided in two sets: those which belong to the bottom surface and those which belong to the top. If we choose a plane such that it doesn't intersect the surface, the prisms for the bottom half-surface will comprise an area to be subtracted from the area enclosed by the top half-surface. So, we can compute a total enclosed volume by subtracting two volumes. It can be shown that this also works even if the chosen plane intersects the surface.



Now we will consider the problem of computing the volume for each truncated prism. For that we shall consider a 'regular' prism (i.e., non-truncated, with both ends perpendicular to its height), which base is the projection of the facet. Notice that the volume of the truncated prism must be between the volume of the two prisms with the same base, and height equal to that of the lowest vertex and the highest vertex. But it is readily seen that the volume will also depend on the height of the third (middle) vertex. Indeed, the volume we want is precisely the same as a prism with height equal to the average height of the three vertices of the facet. In order to distinguish the top-half and the bottom-half facets, we will take advantage of the normal to each facet, and consider the 'signed volumes', positive if the normal points away from the reference plane and negative otherwise. In fact, this can be readily obtained by using the cross product to compute the projected facet's area, and using the dot product to compute the resulting volume. The overall procedure is presented below:

$$V = \frac{1}{2}\sum_{t\in T}(B_t\cdot\hat{n})\left((\overrightarrow{v2_t}-\overrightarrow{v1_t})\times(\overrightarrow{v3_t}-\overrightarrow{v2_t})\right)\cdot\hat{n}$$

Where T is the set of triangles, V_t is the set of vertices of the triangle t, $\hat{n}$ is the unit perpendicular vector to the reference plane, and

$$B_t = \left(\frac{1}{3}\sum_{\vec{v}\in V_t}\vec{v}\right)$$

is the barycentre of $t \in T$.

Below we present a simple implementation in Matlab:

```
function volume=polyhedralVolume(F,V)
% The volume is computed as the algebraic sum
% of the 'volumes' defined by each face and a
% fixed plane, affected by a sign according to
% the direction of its normal relative to
% that plane.

volume=0;
for f=F'
Txyz=V(f,:);
dTxyz=diff(Txyz,1,1);
n=cross(dTxyz(1,:),dTxyz(2,:));
% The volume of the prism limited by the
% face's triangle and it's projection
% onto the xy plane, is the same as that
% of a prism with heigth equal to the
% heigth of the BARYCENTER of the facet.
% (The barycenter is b=mean(Txyz);
h=mean(Txyz(:,3));
vf=dot(n,[0,0,h])/2;
volume=volume+vf;
end
```

A formal proof of the above formula can be found by the use of Ostrogradsky's Theorem, which relates volume integrals, surface integrals and the divergence of a vector field. For a complete analysis of this and other geometric polyhedral properties we refer the reader to [1].

References

[1] Brian Mirtich, *Fast and accurate computation of polyhedral mass properties*, Journal of Graphics Tools, vol. 1, no. 2, pp. 31-50, 1996.

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