Computer construction of Platonic Solids

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Abstract: In this paper we develop a novel method to generate a 3D geometrical model from 2D planer images. As an example of application, we construct 3D models of each one of the five platonic solids from their platonic graphs.

Keywords: Platonic solids, planar graphs, computer models, surfaces.

1 Introduction

Designing 3D geometries to a relatively high level of precision has become one of the challenging tasks for the design engineers. In this context, we are proposing a method for generating 3D solid models and exploring it's capability by considering the five platonic solids, namely the cube, the tetrahedron, the octahedron, the dodecahedron and the icosahedron.

A Platonic Solid is a 3D shape where each face is the same regular polygon and same number of polygons meets at each vertex. The working algorithm of our proposed method is explained in the below section.

2 Methodology

A 3D model in the space is the collection of surfaces, edges and vertices arranged in a proper order. If connection can be established in between these elements properly, then it is no longer difficult to generate solid models out of planer images. For this purpose, we have set up four design parameters such as α, g, q, β to extract the hidden information. The symbolic notations along with their meanings are explained below.

Using this topology information, 3D solid models can be generated for several application needs.

1. α is a permutation of the edges whose orbit goes to the edges of each facets
2. g is geometric realization of abstract vertices in $\mathbb{R}^3$
3. q identifies a surface by it's lowest vertex number
4. β is a permutation whose orbits can be identified with the vertices of the solid

The information provided by β can be used to modify the object by means of *geometrical* or *topological* transformations. An important example of geometrical transformation is the offset transformation which moves each vertex (along with the associated incidence edges) according to a specified direction. In such cases the information encoded in the permutation β is used to track the new faces that are generated by the displacement of adjacent edges. From the point of view of topological transformations, one important aspect of it is that, the possibility to move from the graph of a surface to its dual graph by interchanging the role of the vertices and faces. With this respect, we simply have to interchange the role of the two permutations α and β.

For the method verification, the five platonic solids (mentioned in the introduction) are tested and their respective outcomes are displayed with illustrative sketches in the following section.

3 Design Examples

In this section, the four design parameters of each platonic solid are extracted and expressed in a matrix form. The syntax of the matrix is as below,

$$Matrix = [\alpha, g, q, \beta]$$

The matrices along with their respective vertices (shown in the individual section) are used as an input to our program for generating the 3D platonic solids. The solids are placed next to the matrix in the respective sections.

An example of this matrix can be seen in table 1 (see first row) where $\alpha(1)$ points to 2 and vertex g goes to 2 which corresponds to vertices [1,-1,-1],as provided in section 3.1. The third parameter q equals to 1 represents one face of the cube while $\beta(1)$ is establishing a connection between vertex 1 and 14.

3.1 The Cube

Cube is the most basic platonic solid having six squares, three meeting at each corner. It is also known as hexahedron since "hexa" is a greek word for six which explains all six squares of the cube. Fig. 1 shows the platonic graph of the cube where co-ordinates of each vertex are given below.

```
Cube and Tetrahedron vertices:
[   -1,   -1,   -1]
[    1,   -1,   -1]
[    1,    1,   -1]
[   -1,    1,   -1]
[   -1,   -1,    1] % 5
[    1,   -1,    1]
[    1,    1,    1]
[   -1,    1,    1]
```

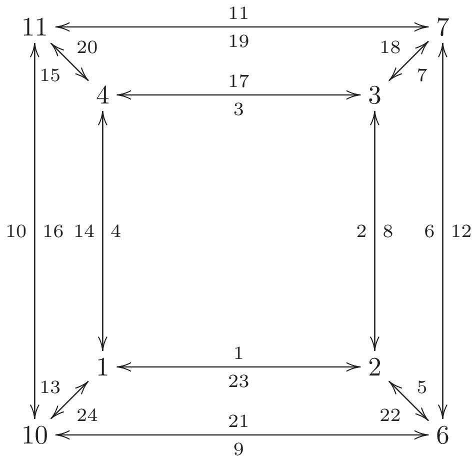

Fig. 1 - Planar graph of the cube.

Table 1 - Matrix for the Cube.

x	$\alpha(s)$	$g(x)$	$q(x)$	$\beta(x)$
1	2	2	1	14
2	3	3	1	23
3	4	7	1	8
4	1	6	1	17
5	6	3	5	2
6	7	4	5	22
7	8	8	5	12
8	5	7	5	18
9	10	4	9	22
10	11	1	9	13
11	12	5	9	20
12	9	8	9	7
13	14	1	13	10
14	15	2	13	24
15	16	6	13	4
16	13	5	13	11
17	18	6	17	15
18	19	7	17	3
19	20	8	17	12
20	17	5	17	11
21	22	1	21	10
22	23	4	21	9
23	24	3	21	5
24	21	2	21	1

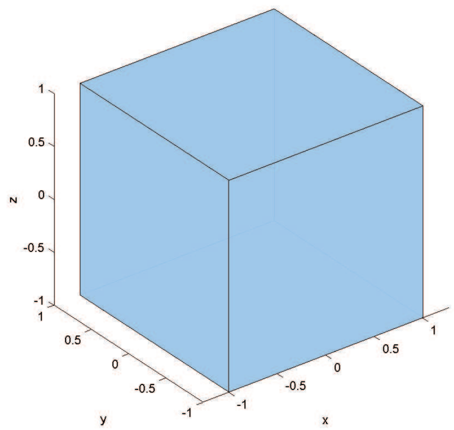

Fig. 2 - 3D Plot of a Cube.

3.2 The Tetrahedron

The word "tetra" is a greek for four. A tetrahedron has four equilateral triangles, three meeting at each corner. Fig.3 shows the planar graph of the tetrahedron and the resulted solid is displayed in Fig.4.

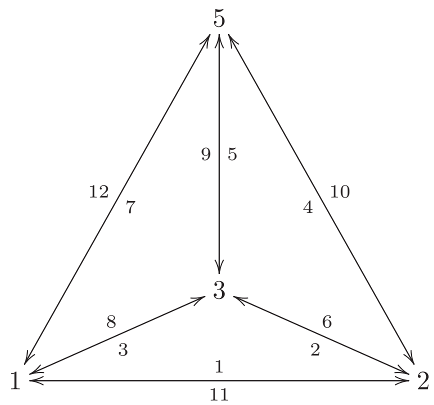

Fig. 3 - Planar graph of the Tetrahedron.

The matrix that encodes the data is the following one.

Table 2 - Matrix for the Tetrahedron.

x	$\alpha(s)$	$g(x)$	$q(x)$	$\beta(x)$
1	2	8	1	8
2	3	6	1	12
3	1	3	1	6
4	5	6	4	2
5	6	1	4	10
6	4	3	4	9
7	8	1	7	5
8	9	8	7	12
9	7	3	7	3
10	11	1	10	7
11	12	6	10	1
12	10	8	10	4

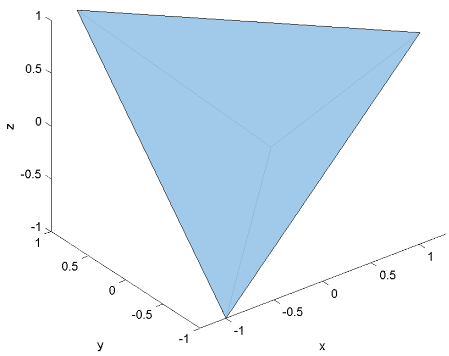

Fig. 4 - 3D plot of the Tetrahedron.

3.3 The Octahedron

Octahedron is also a triangle based platonic solid next to tetrahedron. The word "octa" is greek for eight. It consists of eight equilateral triangles, four meeting at each corner.

In order to produce the octahedron, in addition to the vertex coordinates displayed in section 3.1, we will need the following ones (note that the first line has index 9 because it is continuation of vertex in 3.1).

```
% Octahedron vertices:
[    0,    0,   -1]
[    0,   -1,    0] % 10
[   -1,    0,    0]
[    0,    1,    0]
[    1,    0,    0]
[    0,    0,    1]
```

The planar graph of the octahedron can be obtained as the dual of the planar graph of the cube and it is displayed below.

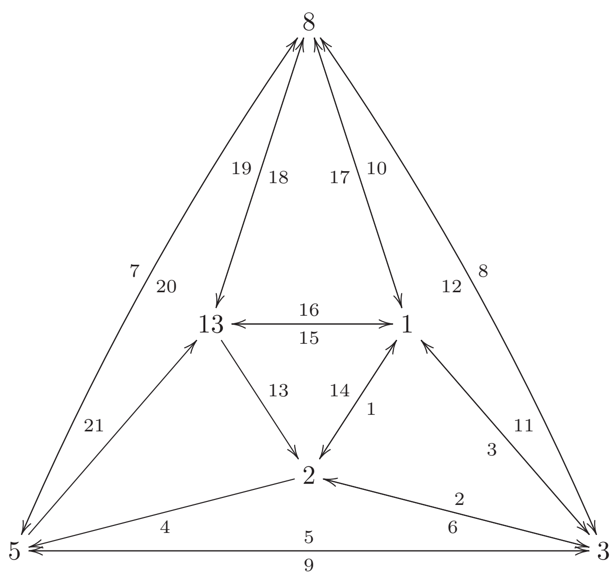

Fig. 5 - Planar graph of the Octahedron.

From this graph we can generate the data which is displayed in the following matrix.

Table 3 - Matrix for the Octahedron.

x	$\alpha(s)$	$g(x)$	$q(x)$	$\beta(x)$
1	2	14	1	11
2	3	10	1	14
3	1	13	1	6
4	5	10	4	2
5	6	9	4	23
6	4	13	4	9
7	8	9	7	5
8	9	12	7	20
9	7	13	7	12
10	11	12	10	8
11	12	14	10	17
12	10	13	10	3
13	14	11	13	16
14	15	10	13	24
15	13	14	13	1
16	17	11	16	19
17	18	14	16	15
18	16	12	16	10
19	20	11	19	22
20	21	12	19	18
21	19	9	19	7
22	23	10	22	13
23	24	11	22	21
24	22	9	22	4

This matrix, along with the Vertices coordinates, contains all the needed information to generate the 3D solid (Fig. 6).

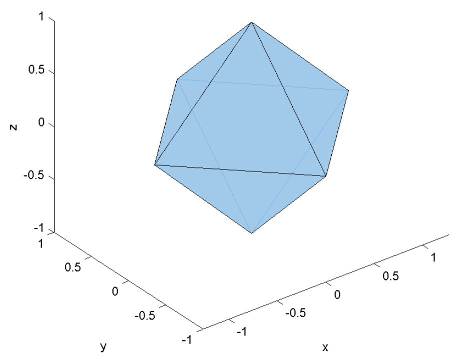

Fig. 6 - 3D plot of the Octahedron.

3.4 The Icosahedron

The last triangle based platonic solid is the icosaherdon. The word "icosa" is greek for twenty. The icosahedron consists of twenty equilateral triangles, five meeting at each corner.

To build the Icosahedron we have to use the same vertices as the ones used to build the previous solids and add some others as displayed below, whose index for the first line in 15, as indicated.

```
Icosahedron vertices:
[  1/2,    0,   -1] % 15
[ -1/2,    0,   -1]
[    0,   -1,  1/2]
[    0,   -1, -1/2]
[   -1,  1/2,    0]
[   -1, -1/2,    0] % 20
[    0,    1,  1/2]
[    0,    1, -1/2]
[    1,  1/2,    0]
[    1, -1/2,    0]
[  1/2,    0,    1] % 25
[ -1/2,    0,    1]
```

The planar graph of the Icosahedron is displayed as follows (Fig. 8) from which we derive the following matrix:

Table 4 - Matrix for the Icosahedron.

x	$\alpha(s)$	$g(x)$	$q(x)$	$\beta(x)$
1	2	26	1	6
2	3	17	1	34
3	1	25	1	23
4	5	25	4	3
5	6	21	4	17
6	4	26	4	9
7	8	21	7	5
8	9	19	7	10
9	7	26	7	31
10	11	19	10	50
11	12	21	10	7
12	10	22	10	13
13	14	22	13	28
14	15	21	13	11
15	13	23	13	16
16	17	23	16	19
17	18	21	16	14
18	16	25	16	4
19	20	23	19	25
20	21	25	19	18
21	19	24	19	22
22	23	24	22	38
23	24	25	22	20
24	22	17	22	2
25	26	23	25	29
26	27	24	25	21
27	25	15	25	54
28	29	22	28	60
29	30	23	28	15
30	28	15	28	7
31	32	26	31	34
32	33	19	31	8
33	31	20	31	46
34	35	26	34	1
35	36	20	34	33
36	34	17	34	41
37	38	18	37	40
38	39	24	37	52
39	37	17	37	24
40	41	18	40	3
41	42	17	40	39
42	40	20	40	35
43	44	18	43	40
44	45	20	43	46
45	43	16	43	56
46	47	20	46	44
47	48	19	46	32
48	46	16	46	49
49	50	16	49	59
50	51	19	49	12
51	49	22	49	12
52	53	24	52	26
53	54	18	52	37
54	52	15	52	57
55	56	18	55	53
56	57	16	55	45
57	55	15	55	58
58	59	15	58	30
59	60	16	58	56
60	58	22	58	51

And with the help of above matrix we build 3D model of the Icosahedron and it is shown in Fig.7.

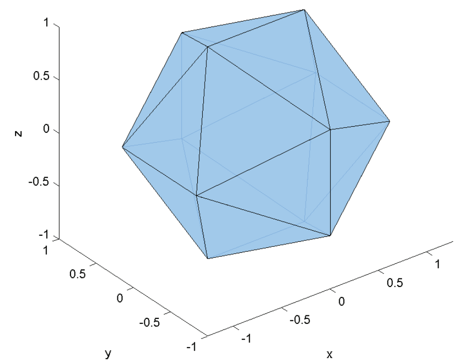

Fig. 7 - 3D plot of the Icosahedron.

3.5 The Dodecahedron

The fifth and last platonic solid is dodecahedron. The word "dodeca" is greek for twelve. The dodecahedron has twelve regular pentagons, three meeting at each corner.

The Dodecahedron is the dual of the icosahedron and hence its vertices can be obtained as the middle point of

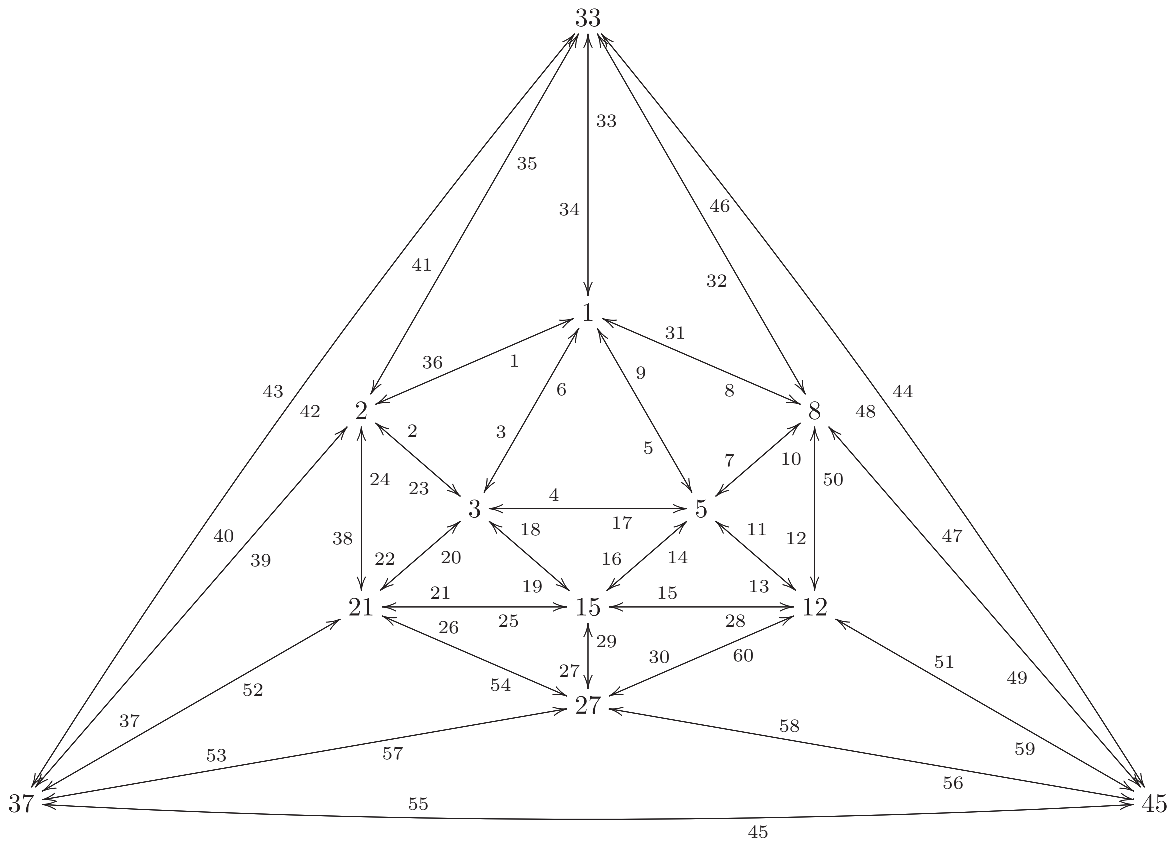

Fig. 8 - Icosahedron planar graph.

each face of the Icosahedron. It's planer image and generated 3D solid are shown in Fig. 9 and 10 respectively.

```
Dodecahedron vertices:
[    0, -1/3,  5/6]
[    0,  1/3,  5/6]
[ -1/2,  1/2,  1/2]
[ -1/3,  5/6,    0] % 30
[  1/3,  5/6,    0]
[  1/2,  1/2,  1/2]
[  5/6,    0,  1/3]
[  1/2, -1/2,  1/2]
[  5/6,    0, -1/3] % 35
[  1/2,  1/2, -1/2]
[ -5/6,    0,  1/3]
[ -1/2, -1/2,  1/2]
[  1/3, -5/6,    0]
[ -1/3, -5/6,    0] % 40
[ -1/2, -1/2, -1/2]
[ -5/6,    0, -1/3]
[ -1/2,  1/2, -1/2]
[  1/2, -1/2, -1/2]
[    0, -1/3, -5/6] % 45
[    0,  1/3, -5/6]
```

Table 5 - Matrix for the Dodecahedron.

x	$\alpha(s)$	$g(x)$	$q(x)$	$\beta(x)$
1	2	32	1	31
2	3	31	1	59
3	4	23	1	7
4	5	22	1	28
5	1	21	1	37
6	7	22	6	4
7	8	23	6	58
8	9	24	6	17
9	10	25	6	12
10	6	26	6	29
11	12	26	11	10
12	13	25	11	17
13	14	30	11	49
14	15	29	11	22
15	11	27	11	30
16	17	25	16	9
17	18	24	16	57
18	19	37	16	53
19	20	40	16	50
20	16	30	16	13
21	21	27	21	15
22	23	29	21	48
23	24	38	21	43

x	$\alpha(s)$	$g(x)$	$q(x)$	$\beta(x)$
24	25	33	21	39
25	21	28	21	26
26	27	28	26	38
27	28	21	26	5
28	29	22	26	6
29	30	26	26	11
30	26	27	26	21
31	32	32	31	36
32	33	34	31	41
33	34	35	31	55
34	35	36	31	60
35	31	31	31	2
36	37	32	36	1
37	38	21	36	27
38	39	28	36	25
39	40	33	36	42
40	36	34	36	32
41	42	34	41	40
42	43	33	41	24
43	44	38	41	47
44	45	39	41	33
45	41	35	41	33
46	47	39	46	51
47	48	38	46	43
48	49	29	46	22
49	50	30	46	13
50	46	40	46	19
51	52	39	51	44
52	53	40	51	50
53	54	37	51	56
54	55	36	51	34
55	51	35	51	45
56	57	37	56	18
57	58	24	56	8
58	59	23	56	3
59	60	31	56	35
60	56	36	56	54

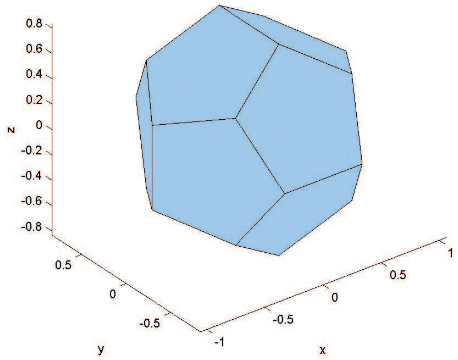

Fig. 9 - 3D plot of the Dodecahedron.

4 Conclusion

The above figures represent that our proposed method is capable of generating platonic solids without any geometrical error. In future, this approach can be extended for generating and 3D printing of complex cellular solids which is difficult to produce using the existing CAD packages.

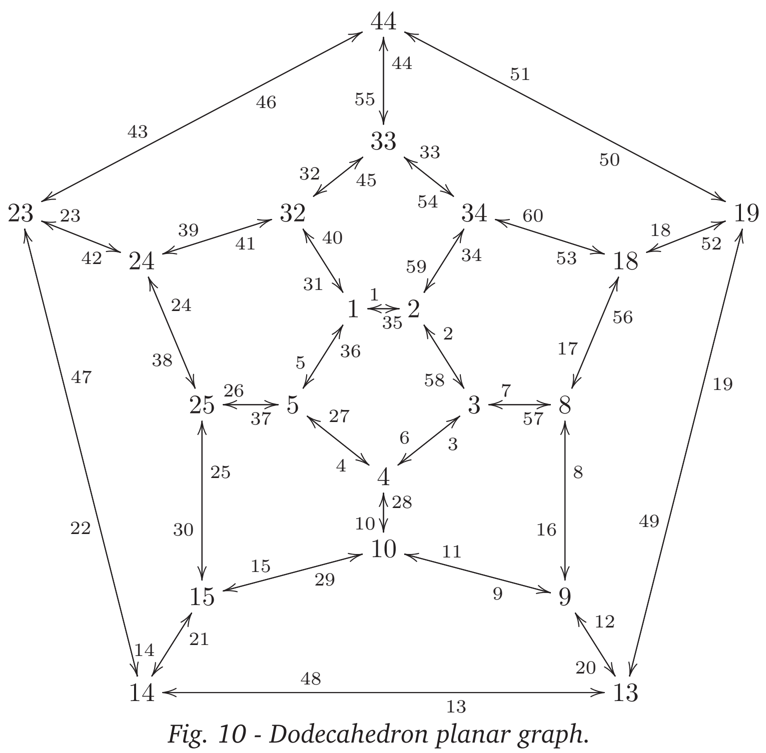

Fig. 10 - Dodecahedron planar graph.