FIRST GRADERS' USE OF THE BAR MODEL TO COMMUNICATE THEIR UNDERSTANDING OF THE EQUAL SIGN

by

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DEDICATION

I want to dedicate this to my incredible friends and family. During this process, they showed me so much support and guidance. I couldn’t have done it without them.

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ABSTRACT

Students' misunderstandings of the equal sign have been well documented in children as young as kindergarten. Misconceptions of the symbol (=) hinder students' relational thinking and impede access to algebraic contexts. Symbolic equations (e.g., 4+2=__+3) have been widely used to test students' understanding and communication of equivalence. The purpose of this study was to explore how first grade students communicate their understanding of equivalence when instruction involved using a bar model versus symbolic equations. It used a two-case study approach to compare an atypical instruction design (the bar model) to a traditional design (symbolic equations). Distinct patterns in communication and changes in students' understanding were found in both cases. In addition, the bar model group used specific, concrete methods of communicating their thinking earlier and more frequently, which suggests using the bar model along with symbolic representations may be an effective instructional design.

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LIST OF ABBREVIATIONS

NCTM	National Council of Teachers of Mathematics
CCSS	Common Core State Standards
PMA	Primary Math Assessment
Engage NY	Engage New York

CHAPTER ONE: INTRODUCTION

The Common Core State Standards (CCSS) for Mathematics is the most recent educational reform in the United States and shifts focus to both conceptual and procedural understanding of mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). With a heightened emphasis on conceptual understanding, it is important to take a closer look at concepts that have historically presented challenges for students. One challenging topic is students' understanding and view of the equal sign (=) and its various meanings. Beginning in elementary school, children are exposed to the equal sign in limited contexts with focus being on its use in arithmetic (Carpenter, Franke, & Levi 2003a; Hunter, 2007; McNeil & Alibali, 2005). As a result, students have very little conceptual understanding tied to the procedures they are applying. A balance of both conceptual and procedural knowledge is needed to successfully understand the structure of mathematics (Baroody, Feil, & Johnson, 2007; Hiebert, 1986; National Council of Teachers of Mathematics, 2000). Understanding equality and relational thinking are pivotal structures in mathematics that are overlooked, often with negative outcomes (Knuth, Stephens, McNeil, & Alibali, 2006; Warren & Cooper, 2005).

The Problem

Holding an operational view of the equal sign is one such outcome of procedurally focused instruction. Students interpret it as *the answer comes next* or a call to carry out a calculation rather than as a relational symbol meaning *quantitative* *sameness* or balance (Behr, Erlwanger, & Nichols, 1976; Sherman & Bisanz, 2009). Beginning in elementary school, students construct schema around the equal sign, associating it with the answer to a computation task or the total. For example, when presented with an equation such as 4+5 =__+2, common answers include nine or eleven (Carpenter, Franke, & Levi, 2003a). Students who answer nine carry out the calculation of the operation on the left side of the equal sign. They follow *the answer comes next* mentality. An answer of eleven is due to the student carrying out all calculations. In both instances, the students have an operational view and do not perceive the equal sign as a symbol showing the relationship between each side of the equation.

Impact on Algebraic Thinking

Holding an operational view of the equal sign is an issue because it hinders skills in algebraic thinking. Struggling to understand the equal sign as a relational (i.e., algebraic) symbol rather than a mere operator obstructs students' ability to find patterns and relationships (Hattikudur & Alibali, 2010; Matthews et al., 2012). More succinctly, the National Council of Teachers of Mathematics (NCTM) states the foundation to algebraic thinking is the emphasis on "relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change" (National Council of Teachers of Mathematics, 2000, p. 37). It includes analyzing how symbols relate information and are used in mathematics rather than solely focusing on symbol manipulation (National Council of Teachers of Mathematics, 2000). Students with an operational view only recognize symbols as objects to be manipulated and miss the overall concepts involved. Because attention is not given to numerical relationships, connections are not made between the arithmetic and operations in elementary school and the symbolic representations of number encountered in middle school (Hunter, 2007; Warren & Cooper, 2005). Even students with strong foundations in mathematics have difficulty with algebraic concepts due to their misconceptions about the equal sign (Frieman & Lee, 2004; Warren & Cooper, 2005). Such a limited understanding is not sufficient for the complex algebraic thinking students will encounter throughout their lives (Hattikudur & Alibali, 2010): a principle reason for the CCSS shift to a balance between a procedural and conceptual understanding of mathematics.

Why Misconceptions Persist

Although studies have shown a connection between misunderstanding the equal sign and difficulties in algebraic thinking, a major obstacle is that little has changed in instructional techniques (Baroody & Ginsburg, 1982; Hattikudur & Alibali, 2010). One possible reason instructional techniques have not changed is a lack of teacher background knowledge needed to teach relational thinking (Hattikudur & Alibali, 2010). This knowledge may be lacking because most pre-service teachers are rarely exposed to the importance of developing a relational view of the equal sign. They, as well as in-service teachers, continue to teach in ways that support the operational view, often without knowing the disservice done to their students (Knuth et al., 2006; McNeil & Alibali, 2005; Saenz-Ludlow & Walgamuth, 1998). Explicit instruction is not given because the relational view is taken for granted (Knuth et al, 2006).

In addition to preparation programs, curricular materials contribute to this as well and often reinforce an operational view (Baroody & Ginsburg, 1982; Hattikudur & Alibali, 2010; Sherman & Bisanz; 2009). The result is few resources are available to teachers to build students’ relational thinking. Thus far, to put this argument in reverse, lack of materials that address relational thinking coupled with the lack of teacher preparation to build teachers' conceptual understanding of equality has limited students' understanding of equality and negatively impacted their current and future algebraic reasoning.

Origin of This Study

The lack of emphasis of most teacher preparation programs and instructional materials on the importance of a relational view of the equal sign is the main reason I became interested in the concept of relational thinking and student perceptions of the equal sign. My background in elementary education and experience tutoring middle and high school students revealed common threads among individuals. Both age groups struggled to comprehend relationships among number or understand the equal sign as anything other than an operator. Many of the students I tutored did perform well in mathematics until they entered middle school. At that point, previously held ideas about the structure of equations no longer made sense in the context of algebra. These students needed a deeper understanding of the mathematical symbols they were using on a daily basis.

My work as a kindergarten and first grade teacher produced a strong desire to build the foundation of conceptual understanding that older students seemed to be missing. More emphasis needed to be placed on understanding equality and numerical relationships. Yet even without years of schooling to develop misconceptions, my students struggled to make sense of the equal sign. This sparked my interest in learning about equality and how relational thinking could be taught to young children. Although many studies emphasize the importance of teaching algebraic concepts at a young age, few focus on students below second grade. Carpenter, Franke, and Levi (2003a) note that misconceptions around the equal sign begin as early as kindergarten. As students progress through school, it becomes more difficult to change their ideas (Deliyianni, Monoyiou, Elia, Georgiou, & Zannetou, 2009; McNeil & Alibali, 2005), so I wondered whether an intervention could develop relational thinking before an operational view is solidified.

Perhaps students can develop the conceptual understanding of equality needed to be successful in later mathematics if instruction is given before strong misconceptions are developed. To accomplish this, teachers need resources to assess student understanding, and students must have ways to communicate their thinking. This study attempted to provide insight into tools students could use to communicate their thinking that allow teachers to build a relational view of the equal sign.

Research Question and Propositions

The purpose of this study was exploratory in nature and sought to determine how students communicate their ideas of the equal sign and equality when instruction is given with the bar model rather than only symbolic equations. The research questions are:

1. In what ways do first grade students communicate their ideas about equality when instructional tasks include drawing bar models along with symbolic representations as opposed to only symbolic representations?
2. Will students who have an operational view of the equal sign be able to communicate relational ideas with a visual representation present?

3. How do students who receive instruction using a bar model move up on the spectrum of understanding (see Figure 1) compared to those who receive instruction with symbolic representations only?

I propose that the use of visual models will result in more students being able to effectively communicate their ideas about equality. Students receiving instruction with the bar model will use more vocabulary related to relational understanding of equality, thus resulting in a greater shift along the spectrum of understanding than those representing contexts only symbolically. While performance on skill-based assessments will likely be similar, those students receiving instruction with the bar model will produce work and discussions that are more relationally-based while their counterparts will produce work and discussions more operational in nature.

CHAPTER TWO: REVIEW OF THE LITERATURE

Introduction

Depending on the context in which it is given, the equal sign can have many different interpretations. Symbols in mathematics are complex and imbedded with meaning, and the equal sign is no exception. Symbols are meant to elicit certain ideas and views of a context or problem (Gellert & Steinbring, 2013; Saenz-Ludlow & Walgamuth, 1998). However, when the complex meaning (or multiple meanings) of the equal sign is overlooked or not explicitly developed in students, misconceptions can arise. Visual models challenge misconceptions and increase students' conceptual understanding of symbolic notations (Gellert & Steinbring, 2013; Middleton & van den Heuvel-Panhuizen, 1998). In addition, spatial reasoning has long been reported as a predictor of future success in mathematics but is rarely integrated into domains other than geometry or measurement (Arcavi, 2003; Cheng & Mix, 2014). Therefore, it is necessary to look into prior research on relational thinking and students' understanding of the equal sign as well as the use of visual models in mathematics in order to see how these two concepts come together.

Relational Thinking and the Equal Sign

Carpenter et al. (2003a), Hunter (2007), and Matthews et al. (2012) have noted that while students' understanding can fall along a continuum, they generally have an operational view of the symbol. Figure 1 is an adaption of the construct map created by Matthews et al. (2012) showing the spectrum of understanding regarding the equal sign

(p. 320). It compiles the definitions in the work of Carpenter et al. (2003a) and Hunter (2007) with the levels of thinking presented by Matthews et al. (2012).

Comparative Relational

- View the equal sign as showing a *balanced relationship*
- Equivalence is determined by seeing the relationships among numbers on either side of the equal sign rather than carrying out calculations.

Basic Relational

- View the equal sign as showing *quantitative sameness*
- Accept most types of equation structures including those with operations on both sides of the equal sign (e.g. 5+2=3+4)
- Determines quantitative sameness by carrying out calculations

Flexible Operational

- View the equal sign as a *call to calculate* or *the answer comes next*
- Accept atypical equation structures as true only if there is one or no operation (e.g. 7=5+2, 4=4)

Rigid Operational

- View the equal sign as a *call to calculate* or *the answer comes next*
- Only typical equation structures are accepted as true (e.g. 2+5=7, 4+_=7)

Figure 1. Spectrum of understanding for the equal sign
(Compiled from the work of Carpenter et al. (2003), Hunter (2007), and Matthews et al. 2012)

The first level of understanding is a rigid operational view. Students at this level only accept typical equation structures and focus on carrying out a calculation to find the total (Matthew et al., 2012). Students may then adopt a more flexible operational view. At this level, students accept other types of equation structures only if there is one or no operation present (e.g. 4=3+1, 4=4). There is still a focus on finding the total or calculating an answer, but other forms of equations are seen as true. By level three students have recognized the equal sign as a relational symbol. They accept atypical structures and are focused with achieving quantitative sameness on both sides of the equation. For example, a student at this level will know that 4+2=5+1 is true because both sides add up to six and are quantitatively the same. At the fourth level, students comprehend a relational view of the equal sign by seeing a balanced relationship on either side of the equation. They no longer have to carry out a calculation to achieve quantitative sameness but can reason about the relationships between the numbers. Instead they observe that 35+27 = 36+26 is true because the equation remains balanced. The difference of one between 35 and 36 is compensated for in the difference between 27 and 26.

Students with either a rigid of flexible operational view of the equal sign have a limited understanding of the symbolic notation. The equal sign loses its full meaning, and there is little conceptual knowledge of the relationships among numbers (Behr et al., 1976). Ideally students should reach a level of comparative relational understanding. This level of understanding shows deep algebraic reasoning and will lead to success when encountering more complex mathematics (Carpenter et al., 2003a; Hunter, 2007).

Power of Instruction

The situations and contexts in which the equal sign is introduced during instruction matters greatly. For the purposes of this study, I define situation as the tasks or curricular materials in which students engage. Context refers to the exact scenario or word problem given within a mathematical situation. Even before kindergarten, students are exposed to the equal sign in situations asking them to find calculations or report an answer to arithmetic. Rarely are students presented with any other way of representing the equal sign than through typical equation structures (e.g. 3+4=7). Often this results in students constructing their own generalizations of what the equal sign means, and this view continues until it is challenged (Knuth et al., 2006; Carpenter, Franke, & Levi, 2003b; Matthews et al., 2012). These generalizations result in an operational view because students have only used the equal sign when they are carrying out a calculation (Knuth et al., 2006; Hunter, 2007). Because many curricular materials do not contain lessons that focus on explicitly teaching the relational view of the equal sign, students' generalizations are not called into question, and misconceptions continue (Baroody & Ginsburg, 1982; Hattikudur & Alibali, 2010).

However, when students are taught the equal sign in situations focusing on relationships, students' understanding begins to change. Contexts that are non-symbolic and remove the equal sign altogether allow students to develop a conceptual understanding of equivalence (Baroody & Ginsburg, 1982; Sherman & Bisanz, 2009). Later the equal sign can be introduced in connection with symbolic tasks without focusing on carrying out an operation. This allows students opportunities to make connections and express their ideas. For example, students can identify the parts of the equation that correspond to the context, thus building the understanding that symbols represent real world situations.

Warren & Cooper (2005) found contexts and vocabulary related to balancing a scale when talking about equivalence led to a relational view of the equal sign. Hattikudur and Alibali (2010) asked students to sort mathematical symbols based on whether they denoted a relationship or operation. After instruction relating the equal sign to other comparative symbols (i.e. >, <), students recognized the equal sign represented a relationship between the quantities on both sides. The preceding studies demonstrate that when explicitly taught, students' views of the equal sign can begin to move up the spectrum of understanding (see figure 1). This suggests that multiple experiences such as these are needed to challenge students' thinking (Saenz-Ludlow & Walgamuth, 1998).

Ideas around equivalence are molded and challenged as students engage in discourse (Saenze-Ludlow & Walgamuth, 1998). It is not enough for teachers to provide a relational definition of the equal sign. Students must be provided with tasks that elicit a relational view in order to construct their own understanding (Baroody & Ginsburg, 1982; Hattikudur & Alibali, 2010; Knuth et al., 2006). These tasks should include opportunities for students to verbalize their thinking and share their understanding with others (Barlow & Harmon, 2012; Saenz-Ludlow & Walgamuth; 1998). Using vocabulary related to equivalence is powerful for changing student thinking (Saenz-Ludlow & Walgamuth, 1998: Warren & Cooper, 2005).

How to Develop a Relational View of the Equal Sign

Explicit instruction of the relational meaning of the equal sign must begin at a young age. Carpenter et al. (2003a) found students entering kindergarten for the first time already tend to possess an operational view of the equal sign. Young children have the capability to think relationally but are often not expected or taught to do so (Baroody & Ginsburg, 1982; Sherman & Bisanz, 2009). Instruction often reinforces the operational view. In contrast, early exposure to relational thinking and algebraic contexts in relation to equivalence challenges that view (Carpenter et al., 2003a, Sherman & Bisanz, 2009; Stephens & Armanto, 2010). As students progress to later grades, short interventions have shown to have little effect in changing misconceptions because by then these misconceptions are deeply engrained (McNeil & Alibali, 2005). Students have already learned the structure of “doing school” and are apprehensive to accept anything outside of what they have been exposed (Deliyani et al., 2009). However, the literature has suggested creating a strong relational understanding in these early years can build students’ schema to include different views of the equal sign depending on the situation (McNeil & Alibali, 2005; Suh and Moyer, 2007).

To build relational thinking, instructional situations should focus on numerical relationships and patterns that lead to relational understanding. Atypical number sentences are one way of developing relational thinking. There are two possibilities. First, equation structures that feature the “answer” first such as 7=3+4 challenges the notion that the answer must always follow the operation. Many students will insist the previous equation is backwards or is not true because it does not follow the traditional structure. Discussion around why it is acceptable to write equations in this way builds flexibility (Carpenter et al., 2003b; Hattikudur & Alibali, 2010; Matthews et al., 2012; Warren & Cooper, 2005). Second, atypical equations that include operations on both sides of the equal sign build flexibility as well (Carpenter et al., 2003a; Matthews et al., 2012). As students build computational fluency and recognize multiple ways to compose a number (e.g. 3+4 and 5+2 both compose 7), this information can be used to build understanding of what it means when two things are equal. Exposing students to equations such as 3+4=5+2 while working with this concept shows students other ways of representing quantities.

In addition to atypical number sentences, open number sentences also emphasize quantitative relationships by requiring students to identify quantities separate from the total amount, which builds conceptual understanding and shows the equal sign as signifying quantitative sameness. Including physical balance scales along with the symbolic representation of open number sentences helps provide a concrete representation of what the equal sign means (Sherman & Bisanz, 2009; Warren &

Cooper, 2005). For example, if the following equation is presented to students, 4+___ =13, they can start by putting manipulatives onto either side of the scale. Then students need to find the missing quantity making the scale balance. This reinforces the idea that 4+9 is the same quantity as 13.

Determining why an equation might be true or false and analyzing the reasons why is another powerful way of challenging student conceptions (Carpenter et al., 2003b; Saenz-Ludlow & Walgamuth, 1998). When students are given equations that do not require them to find a numerical answer at all, they are able to focus on the relationships between the numbers (Carpenter et al., 2003b; Saenz-Ludlow & Walgamuth, 1998). One such task would be asking students to determine whether 4+3=2+5 and explaining how they decided. For students who are building relational thinking based on calculating either side of the equation, larger number sets that are not easy for them to compute may push them to find the relationships among numbers (Carpenter et al., 2003b). Taking the example provided previously in this paper (35+27=36+26), students who cannot add those number mentally can be directed to look at the relationships taking place. Frieman and Lee (2004) found that true/false tasks including number sets too difficult for students to compute may hinder student understanding as operational errors occur. This is something to be considered especially for students who have a basic operational view. However, if the intent is to move students to a relational view based on determining balance on either side rather than only quantitative sameness, larger number sets can encourage students to think this way. Classroom discussion can center on how the numbers are changing, and virtual or physical manipulatives can be used to demonstrate these changes (Suh & Moyer, 2007; Warren & Cooper, 2005).

While the literature is rich in defining ways students think about the equal sign and how relational thinking can be developed, it is lacking in a few key areas. It is important to link multiple representations to engage students and develop conceptual understanding. Research has shown the power of starting instruction with non-symbolic contexts and connecting to the symbolic tasks that challenge student thinking (Sherman & Bisanz, 2009; Suh & Moyer, 2007; Warren & Cooper, 2005). Studies have looked into the use of virtual and physical manipulatives as well as focused on the use of symbolic situations. What appears missing is a focus on visual or iconic mathematical models to represent equality.

NCTM encourages the use of mathematical models and using multiple representations is a practice standard of the CCSS for mathematics (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Research shows the impact mathematical models have on students' conceptual understanding of mathematics content, and NCTM encourages their use for representing algebraic contexts (Arcavi, 2003; Cai et al., 2014; National Council of Teachers of Mathematics, 2000). With this in mind, it is important to consider how the use of mathematical models could shape students' views of the equal sign.

Mathematical Modeling

Thinking relationally requires visualization and spatial reasoning. Students must be able to manipulate numbers in order to think about the relationships among them (Cheng & Mix, 2014; Stephens & Armato, 2010). Skills in spatial reasoning involve "concepts of space, tools of representation, and processes of reasoning" (Ontario Ministry of Education, 2014, p. 3). It includes the ability to manipulate objects within space physically or visually. To be successful with spatial thinking tasks, individuals must have the ability to visualize mathematically (Cheng and Mix, 2014; Ontario Ministry of Education, 2014).

Arcavi (2003) defines visualization as follows:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information about and developing previously unknown ideas and advancing understandings. (p. 217)

In short, it is the ability to analyze visuals with the purpose of communicating or generating ideas and new knowledge. Without the ability to visualize, individuals often have difficulty creating meaning of new situations or contexts. Although traditionally taught in isolation within the domain of geometry, it is important that visualization and spatial thinking tasks are imbedded across mathematical domains (Aracavi, 2003: Cheng & Mix, 2014; Ontario Ministry of Education, 2014).

Mathematical modeling is a powerful way to integrate spatial reasoning and visualization into the mathematics classroom. Bruner (1964) identified three modes of representation that individuals use to make sense of the world: enactive, iconic, and symbolic. Children initially make sense of the world through actions that are later transformed into visual and, then, symbolic representations. When encountering new situations, individuals often revert to less formal forms of representation (Bruner, 1964).

When individuals are introduced to novel mathematical problems, they once again return to informal representations. Mathematical modeling is one way to capitalize on this tendency in order to develop students' conceptual understanding. For the purposes of this study, mathematical modeling "is the art or process of constructing a mathematical representation of reality that captures, simulates, or represents selected features or behaviors of that aspect of reality being modeled" (Cai et al., 2014, p. 1-150). Visual models are powerful tools which help increase conceptual knowledge, allow students greater access to concepts, and are a strong pedagogical tool for highlighting student thinking (Arcavi, 2003; Gravemeijer, 1999; Hegarty & Koxhevnikov, 1999)

It is important to note that not all visual models have this effect. Hegarty and Koxhevnikov (1999) drew a distinction between pictorial and schematic imagery. Pictorial imagery uses concrete pictures to represent a given situation. While pictorial imagery is a starting place for many students, if not formalized, the model becomes a hindrance. It decreases efficiency and draws students' focus away from mathematical concepts (Hegarty & Koxhevnikov, 1999). In contrast, schematic imagery uses representations based in spatial reasoning and visualization, which have both shown to positively impact students' conceptual understanding (Cheng & Mix, 2014; Hegarty & Koxhevnikov, 1999). This literature review focuses on the benefits of schematic mathematical modeling.

<u>Conceptual Knowledge</u>

Visual mathematical models contribute greatly to students' conceptual understanding. As students build on background knowledge deeper connections are formed and students are able to apply learned information to novel situations (Baroody, et al., 2007; Lowrie & Kay, 2001). Young children have a natural tendency to view pictures as symbols of the world (Deliyianni et al., 2009; Deloache, 1991). Drawing on these pre-formed connections is a powerful way of building conceptual understanding. As strong connections are made between what a child has experience with and new situations, students' cognitive ability to reason abstractly increases (Anderson-Pence, Moyer-Packenham, Westenskow, Shumway, & Jordan, 2014; Arcavi, 2003). This also holds true for making connections among multiple mathematical domains. Cheng and Mix (2014) found that training students on spatial reasoning increased their performance on open number sentence tasks. By relating spatial thinking skills across domains, students are able to perceive problems in a new way and connect different types of conceptual knowledge (Anderson-Pence et al., 2014; Arcavi, 2003; Muarta, 2008). In turn, their conceptual knowledge related to that specific task increases (Blum & Borromeo Ferri, 2009). This occurs more as students' misconceptions are challenged. Visual models activate students' cognition and help them perceive incorrect assumptions (Arcavi, 2003; Blum & Borromeo Ferri, 2009). Because symbols often hide relevant information, visual models can challenge incorrect views that symbols inadvertently reinforce (Aracavi, 2003; Gellert & Steinbring, 2013; Saenze-Ludlow & Walgamuth, 1998). It is only when there are discrepancies between what one perceives to be correct and what is actually correct that changes in thinking occur (Aracavi, 2003).

When students are only exposed to symbolic procedures their conceptual knowledge is rarely activated and students lack numerical flexibility (Tall, 1994). Visual models increase flexibility because they are dynamic. For example, students can use a variety of strategies even if using the same model (Gellert & Steinbring, 2013; Middleton & Van den Huevel-Panhuizen, 1998). As students connect knowledge to new situations or content, they begin to make sense of procedures and can solve problems flexibly (Tall, 1994). Mathematical models contribute to this. Using models in multiple ways and across domains (e.g. using a bar model for fractions and number operations) emphasizes relationships among number (Erbas et al., 2014; Middleton & Van den Heuvel-Panhuizen, 1998; Murata, 2008). Visual models highlight patterns and draw attention to what is important within a problem (Fong Ng & Lee, 2009; Hegarty & Koxhevnikov, 1999).

Iconic Models in Problem Solving

Processing relevant information when problem solving is difficult and individuals tend to revert to more informal modes of representation (Bruner, 1964). In mathematics, many of those representations take the form of pictorial representations (Deliyianni et al., 2009; Lowrie & Kay, 2001). Students draw pictures to make sense of contextual problems and find solution strategies. Figure 2 shows two iconic models a first grade student might construct to represent a problem.

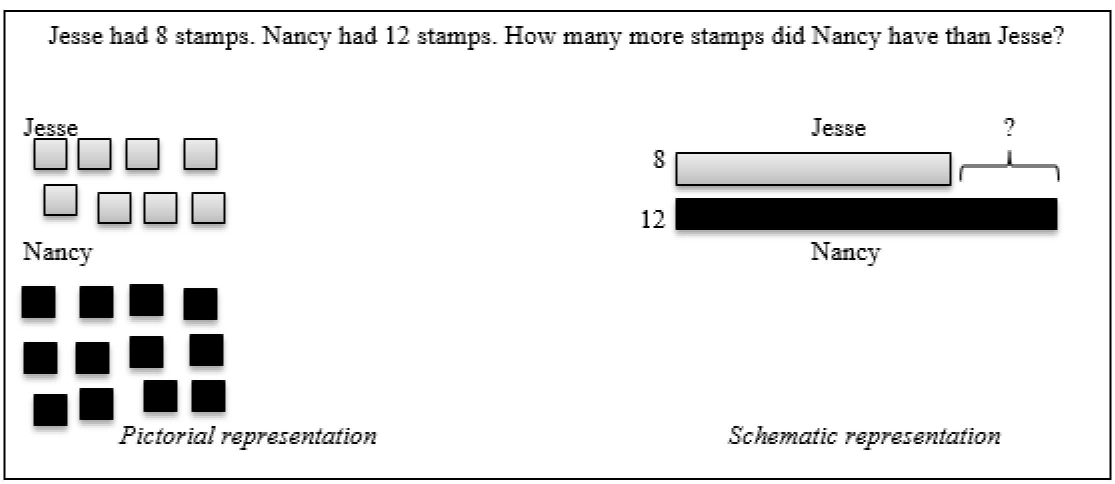


Figure 2. Pictorial and schematic representations

Exposure to visual models may increase the likelihood they will choose to use a schematic rather than pictorial representation, increasing access to more difficult concepts (Hegarty & Koxhevnikov, 1999).

Making sense of contextual problems using visual models connects mathematical models to real world problems (Anderson-Pence et al., 2014; Erbas et al., 2014; Smith, 1996). Visual models give students an avenue for visualizing a problem based on their beginning understandings (Arcavi, 2003; Ontario Ministry of Education, 2014). Whether student-generated or provided for the student to interpret, visual models make it easier to conceptualize information and interpret data (Anderson-Pence et al., 2014 Aracavi, 2003). For many students this is the access point they need to begin to devise a solution strategy for the problem (Arcavi, 2003; Blum & Borromeo Ferri, 2009; Van den Heuvel-Panhuizen, 2003). Student representations can then be applied to procedures and symbols to assign them meaning (Gellert & Steinbring, 2013; Middleton & van den Heuvel-Panhuizen, 1998). Instructing students on the meaning of symbols and procedures without connections does not challenge misconceptions (Deliyianni et al., 2009; Gellert & Steinbring, 2013). However, when symbols are used in conjunction with schematic mathematical models, students are required to think in multiple ways, and they have deeper understanding of the content (Deliyianni et al., 2009; Middleton & van den Heuvel-Panhuizen, 1998; Suh & Moyer, 2007).

Deeper conceptual understanding increases students' abilities to problem solve because they have more background information to draw from, and their ability to process information increases (Anderson-Pence et al., 2014; Baroody, Feil, & Johnson, 2007). They are able to question prior conceptions and think about a problem from multiple perspectives (Deliyianni et al., 2009; Van den Heuvel-Panhuizen, 2003). When used strategically, visual models allow students to analyze the mathematics of a problem rather than simply following a set of procedures (Blum and Borromeo Ferri, 2009;

Murata, 2008). The ability to analyze mathematics is key when asking students to think relationally about numbers, as in the case of equality.

Student Thinking

Visual models are a strong pedagogical tool because they allow a way for students to demonstrate, communicate, and formalize their thinking. Even when given the same context or model to use, individual students will use the model in different ways (Gellert & Steinbring, 2013; Lowri & Kay, 2001). The flexible process of modeling makes it easier to evaluate student thinking (Fong Ng & Lee, 2009; Smith, 1996). Analyzing students' use of a model gives insight to how they process information, and student-generated models demonstrate different ways of looking at a problem (Gravemeijer, 1999; Smith, 1996; Van den Heuvel-Panhuizen, 2003).

Models (enactive, iconic, or symbolic) provide transparency of student thinking, which can then be used to guide instruction. Tasks that elicit student generated models often begin with an informal, literal representation of a problem. These "models of" a specific situation serve as tools for the students to make sense of the mathematics (Gravemiejer, 1999; Fong Ng & Lee, 2009; Van den Heuvel-Panhuizen, 2003). As the teacher guides and presses students' understanding, "models of" become "models for" representing various situations (Gravemiejer, 1999; Van den Heuvel-Panhuizen, 2003). It is important to avoid formalizing mathematics too quickly, which often happens in the case of the equal sign. Mathematical models aid in helping teachers assess student knowledge and determine whether a concept has been formalized prematurely (Blum & Borromeo Ferri, 2009; Fong Ng & Lee, 2009; Gravemeijer, 1999; Lowrie & Kay, 2001). Once students have begun to formalize ideas and developed conceptual understanding, connections can be made to symbolic structures (Baroody et al., 2007; Saenze-Ludlow & Walgamuth, 1998). Most students have some experience in school with performing mathematical operations and reporting the answer using the equal sign. Connecting the symbols students are already familiar with to different representations has a similar effect in creating meaning for the symbolic language (Sherman & Bisanz, 2009; Suh & Moyer, 2007; Warren & Cooper, 2005).

Providing a model for students to analyze can have a similar impact as student-generated ones. Opportunities to connect given models to real world situations increases students' capability to interpret a problem, but it is imperative teachers demonstrate how to appropriately and flexibly use them (Anderson-Pence et al., 2014; Blum & Borromeo Ferri, 2009; Gellert & Steinbring, 2013; Tall, 1994). When students use the same model, they have a tool to communicate their thinking (Anderson-Pence et al., 2014; Gellert & Steinbring, 2013; Middleton & Van den Heuvel-Panhuizen, 1998). They are able to share and relate their solution strategies with others. One of the strongest ways to change student misconceptions around the equal sign is through discussions, and visual models could be an effective tool for students to communicate their thinking.

Why the Bar Model

A specific model that lends itself well to the concept of equality is the bar model. "The bar model refers to a strip on which different scales are depicted at the same time, as a result of which an amount or quantity can be expressed through a different amount or quantity" (Van den Heuvel-Panhuizen, 2003, p.17). There are three types of bar models: part-part-whole, comparing, and multiplication or division (Van den Heuvel-Panhuizen, 2003). This study will focus on using the bar model for comparison to show the relationships between quantities (see Figure 3).

Figure 3. Bar model for comparison

A key understanding needed to make sense of the equal sign is the relationships between numbers. Without being able to visualize quantities and compare them, quantitative sameness has very little meaning for students (Cheng & Mix, 2014; Hunter 2007). The bar model brings relationships between quantities to the forefront and connects spatial relationships to symbolic notations (Middleton & Van den Heuvel-Panhuizen, 1998; Van den Heuvel-Panhuizen, 2003). Connecting multiple representations increases conceptual understanding of each representation (Arcavi, 2003; Middleton & Van den Heuvel-Panhuizen, 1998; Tall, 1994). The flexibility of the bar model allows it to be used initially as a concrete model of a problem but later formalized into a model for mathematical problem solving (Erbas et al., 2014; Van den Heuvel-Panhuizen, 2003). It can represent multiple contexts from arithmetic to balancing equations making it a useful tool for showing relationships at any grade level (Muarta, 2008; Middleton & Van den Heuvel-Panhuizen, 1998; Van den Heuvel-Panhuizen, 2003).

Behr et al. (1976) noted that younger students are often able to make sense of oral information more strongly than symbolic representations. The bar model can be used as a way to interpret information provided orally and utilized to verbally communicate their ideas by referencing the model. Often in symbolic models, relationships are covert, and students who may observe the relationships have difficulty expressing their ideas. The visualization occurring in students' minds can be conveyed through the bar model more easily than through symbols (Fong Ng & Lee, 2009; Middleton & Van den Heuvel-Panhuize, 1998). Sherman and Bisanz (2009) found when given a problem without symbolic contexts, students possessed a much richer knowledge of equality and were better able to express their thinking. Perhaps the same would hold true if students are presented with a bar model.

CHAPTER 3: METHODOLOGY

This study utilized a two-case study design to explore how first grade students communicate their understanding of the equal sign. Using a two case study approach allowed multiple sources of data to be used while looking for overall themes, changes, and events. The purpose of the study was not to generalize to the larger population of first graders. Rather, it sought to compare an atypical instructional method (the bar model) to the approach found most frequently in the literature (symbolic representations). The overall analysis was done through a qualitative lens. The study was intended to compare student experiences and patterns of communication about equivalence within two distinct instructional settings.

Setting

The study took place at a public elementary school in Southwest Idaho. The school included kindergarten through sixth grade. According to the Institute of Education Sciences (2015), as of the 2013-2014 school year, the school's enrollment was 667 students. Those students' ethnicities consisted of 89 percent Caucasian, 5 percent Hispanic, and 6 percent other ethnicities. The student population included 53 percent male and 47 percent female students (Institute of Education Sciences, 2015). A small percentage of the school population (27%) was eligible for the free or reduced lunch program. This school was selected out of convenience. One of the researcher's coworkers, who was not part of this study, had previous experience at this elementary school.

Participants

A sample of 16 students was selected from two different first grade classes (eight from each class). From the two first grade classes, a total of 18 students returned their permission forms: eight students from one class and ten from the other. Each student was administered a pretest to determine their specific understanding of equivalence. Scores from this test were used to determine how students were placed into groups. Initially each group consisted of a total of 8 students (four from each class), and participants in one group were matched with a participant in the other group according to performance on the pretest. Both groups contained an equal number of boys and girls from each class. Halfway through the study, one student dropped out so her data are not included. Table 1 shows the matched pairs with their initial placement on the spectrum of understanding.

Table 1

Participant Matched Pairs

Bar Model Group		Symbolic Group	
Student	Placement on Spectrum of Understanding (based on pretest)	Student	Placement on Spectrum of Understanding (based on pretest)
Jayme	Flexible operational	Elise	Flexible operational
Richard	Basic Relational	Harry	Basic Relational
Samuel	Basic Relational	Debbie	Basic Relational
Becky	Basic Relational	Anthony	Basic Relational
Scott	Basic Relational	Matt	Basic Relational
Jeff	Basic Relational: Mixed thinking	Stephen	Basic Relational
Student [a]	Rigid Operational	Allison	Rigid operational
Beth	Flexible operational	Jessica	Flexible operational

–[a]This student left the study halfway through and therefore is not counted in the data.

Content of the Instruction

Both groups received instruction consisting of ten 30-minute sessions over the course of two weeks. It took place each school day for those two weeks and was administered during the school day at a time that did not interfere with regular classroom instruction. The researcher provided the instruction for both instructional groups in order to be a participant observer. Tasks consisted of contextual problems asking students to compare quantities and/or seek relationships among numbers. They were adapted from contexts included in a first grade module developed by Engage New York (New York State Education Department, 2015) concentrating on comparison problems for addition and subtraction. Modifications were made based on sample tasks or main concepts provided in the literature (Barlow & Harmon, 2012; Baroody & Ginsburg, 1982; Carpenter et al., 2003b). Appendix A includes the original contexts as well as the modified versions used in this study. This curriculum was chosen due to its accessibility to teachers as well as its use in many classrooms around the country. All selected problem types focused on comparing quantities. While the equal sign is used to convey the quantities of other problem types as well, the literature emphasizes the importance of finding relationships among number (Carpenter et al., 2003a; Saenz-Ludlow & Walgamuth, 1998), and comparison problem types lend themselves well to this idea (Barlow & Harmon, 2012).

Both groups were given the same contexts and focused on finding the relationships between numbers. In addition, the researcher used mathematical language to promote a relational view of the equal sign. Vocabulary included *same as*, *relationship,* *equal, change,* and *decompose.* Students had the opportunity to use this vocabulary when discussing with each other and the researcher. The literature emphasizes the power of student discourse (Barlow & Harmon, 2012; Saenz-Ludlow & Walgamuth, 1998: Warren & Cooper, 2005), so a mixture of group, partner, and individual work was used for both instruction groups to allow students to communicate their ideas and develop their thinking. Both groups were also provided with cubes in order to act out the problems if needed.

Bar Model Group: Instruction with the Bar Model and Symbolic Equations

The first group contained seven students. The group began with eight students, four boys and four girls. However, by the end of the study, one student dropped out, so her data are not reported. Group A's written representations of the tasks included drawing bar models in addition to the use of typical, atypical, and open number sentences. After the context was introduced, students worked with the teacher to create a bar model representing the problem as well as an equation. Following the representation, students were given an opportunity to solve the problem. Attention was given to connecting symbolic and visual representations. This case was selected for analysis because it represented an atypical approach to teaching students about equivalence. Most of the literature focuses on the use of symbolic representations only (Carpenter et al., 2003a; Matthews et al., 2012).

Symbolic Group: Instruction with Symbolic Equations Only

The second group contained eight students. It was made up of four boys and four girls. This group's written representations were only symbolic through the use of typical, atypical, and open number sentences in order to discuss the relationships taking place.

Group B was selected to represent the typical instructional method used for teaching equivalence.

Data Collection

Three sources of data were used to construct an overall depiction of each case group being studied: a pre/posttest interview, selected instruction session recordings, and student work samples.

Pre/Posttest

Each of the students was given a relational thinking pretest adapted from the PMA relational thinking diagnostic test (See Appendix B) (Brendefur et al., 2013; Brendefur et al., 2015). The test included questions on relational thinking using both visual and symbolic representations and mirrored tasks found in the literature (Baroody & Ginsburg, 1983; Carpenter et al., 2003b; Knuth et al., 2006; Matthews et al., 2012). Students were placed in an instructional group based on their scores by matching them to a similar student in the other instructional group. At the end of the ten sessions, the students completed the same assessment to determine whether any changes took place. The posttest took place a week and a half after the final instructional session. All assessments were administered through one-on-one interviews with the researcher and answers were audio recorded.

Session Recordings

To analyze student communication about relational thinking, all sessions were video recorded. The first, third, sixth, eighth, and tenth session recordings for each group were transcribed (see Appendix C). Students' communication was coded and analyzed to look for their level of relational thinking. The selected sessions were chosen to give a chronological picture of how students' understanding progressed throughout the instruction.

Student Work

Student work samples were collected at the end of every session. However, only the fifth, ninth, and tenth samples were used for a comparative analysis. These three samples were selected because they required students to answer a given question and explain their thinking. For example, in session five students were given the following context: *One table had 5 blue ants and 1 red ant. The other table had 4 blue ants and 2 red ants. Jesse wrote 5+1=4+2. He said they have the same amount. Do you agree? Why or why not?*

Data Analysis

The overall analysis of data included a cross-case synthesis. The groups were analyzed in relation to one another to determine if any differences in student communication and understanding of equivalence occurred. Two different approaches were taken: pattern matching and a chronological sequence (Yin, 2014). Cases were analyzed at the group level as well as at the individual student level. This allowed for an in depth look into how students communicated their thinking within different settings (the type of instruction) and stages of understanding.

Pattern Matching

Each source of data was analyzed in reference to the spectrum of understanding developed from the literature (see Figure 4). Student responses were initially placed according to corresponding features along the spectrum. Differences were found in the language students used to communicate similar ideas, so further analysis was done in order to identify additional patterns. Similar codes emerged from each of the different data sources. The individual patterns for each data source are listed below.

Comparative Relational

- View the equal sign as showing a *balanced relationship*
- Equivalence is determined by seeing the relationships among numbers on either side of the equal sign rather than carrying out calculations.

Basic Relational

- View the equal sign as showing *quantitative sameness*
- Accept most types of equation structures including those with operations on both sides of the equal sign (e.g. 5+2=3+4)
- Determines quantitative sameness by carrying out calculations

Flexible Operational

- View the equal sign as a *call to action* or *the answer comes next*
- Accept atypical equation structures as true only if there is one or no operation (e.g. 7=5+2, 4=4)

Rigid Operational

- View the equal sign as a *call to action* or *the answer comes next*
- Only typical equation structures are accepted as true (e.g. 2+5=7)

Figure 4: Spectrum of understanding for the equal sign (Compiled from the work of Carpenter et al. (2003), Hunter (2007), and Matthews et al. 2012)

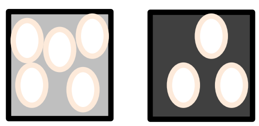
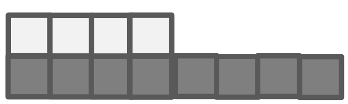
<u>Pre/Posttest</u>

The test was broken into two sections for coding purposes: concrete visual and symbolic based tasks (see Appendix B).

Concrete visual based tasks. The first section consisted of concrete visual models and was coded based on the strategy used by the students. The following categories were used: *trading, adding/subtracting, or other/incorrect* strategy. *Trading* strategies required that students use spatial reasoning skills to transfer one or more concrete objects from one group to another. Using a trading strategy most closely reflects a comparative relational view. If a student made the two groups equal through adding new blocks/dots or removing some completely, it was recorded as an *adding/subtracting strategy*. This type of strategy could be indicative of either a basic relational or operational view. Table 2 provides two sample questions along with an example of a response for each of the patterns identified (Brendefur & Strother, 2016).

Table 2

Student strategies on pre/posttest

Sample Questions	Trading	Adding/Subtracting	Other/Incorrect
 Count the eggs in each basket. How could we make them the same?	Move one block from the basket with five into the basket with three.	*Adding:* put two more eggs in the basket with three *Subtracting:* take two eggs out of the basket with five	Take three eggs out of the basket with five
 Here are some rows of blocks. How could we make both of them have the same amount?	Move two blocks from the dark gray to the light gray	*Adding:* put four more blocks on the top *Subtracting:* take away four blocks from the bottom.	Take five blocks away from the bottom.

Symbolic based tasks. The second section consisted of open number sentences, an atypical equation, and true/false statements. Some questions asked students to explain their thinking. For those problems, student explanations were analyzed qualitatively and responses were categorized as *operational, basic relational, comparative relational*, or *other/incorrect*. *Operational* indicated that a student was simply carrying out a calculation on one side of the equal sign. He/she may solve problems such as 4+__= 7 correctly, but when presented with operations on both sides of the equation they incorrectly operate on a part or all of the numbers. For example, they may answer five to the question 2+3=__+ 4 or ten for 4+1=5+___. Although the spectrum of understanding (see Figure 4) includes both rigid and flexible operational levels, they were grouped together for this study since all but one student entered the study with at least a flexible operational view. *Basic relational* was indicated by students determining equivalence through finding quantitative sameness by carrying out calculations to make both sides have the same amount (i.e. knowing 2+3=1+4 because both sides equal five). *Comparative relational* comprised of answering questions in a way that implied attention to the relationships among all of the numbers without the use of calculations. This included talking about how the numbers in the equation were changing from one side to the other. For example, 6+2=5+__ is missing a three because the two needs to gain one to compensate for the six going down by one. This code also involved recognizing problems where both sides of the equation contained the same numbers (i.e. 9+1=__+9) and knowing that no change is taking place or calculation is needed. The *other/incorrect* category comprised of overgeneralizations of one of the above codes or using a completely different strategy. Table 3 gives two examples of questions from the pre/posttest along with examples of student explanations within each pattern.

Table 3

Codes used for symbolic section of pre/posttest

Sample Question	Operational	Basic Relational	Comparative Relational	Other/Incorrect
14+2 = ___+3 What number should I put in the missing space? Tell me how you know.	"It's sixteen because fourteen plus two is sixteen."	"It's thirteen because fourteen plus two is sixteen and thirteen plus three is sixteen."	"It's thirteen because it's one less than fourteen and two is one less than three."	"It's fourteen because there needs to be a fourteen on both sides."
52+18 = 53+17	"No because	"Yes because	"Yes because 52	"No because it

Are both sides equal? Tell me how you know.	there's nothing that adds up to 53."	both sides equal seventy."	is lower than 53 and eighteen is above seventeen by one."	should go 17+18=52+53. They're not in order."

Session Recordings and Student Work

Students' verbal and written communication were analyzed for common themes. Both the student responses during instructional sessions and student work were analyzed using the same four primary patterns used for the symbolic section of the pre/posttest: *operational, basic relational, comparative relational, and other/incorrect.* Within each of the primary codes, with the exception of operational, other patterns began to emerge.

Comparative relational. The phrases that presented themselves most frequently when discussing how the numbers in the problem changed fell into four categories. The first category was *number size.* Students used phrases such as "one more/one less" or "bigger/smaller" to communicate about the size of the numbers. *Trading* was the next pattern that arose. This category was defined as seeing an amount being traded among numbers. It often took the form of phrasing such as "it got traded" or "the six stole it". Attention to the *same number* was another pattern identified. For the problem 10+4=__+10 students who attend to the fact that a ten is on both sides, recognized that four must be needed to make the statement true. The final pattern was discussion around the concept of *a tangible extra.* A tangible extra presented itself by a student recognizing that a number is not only larger or smaller but also communicating the difference using reference to a concrete object. For example, student communication following this pattern included "the twenty-seven has one extra block than the twenty-eight" or "the second one has one more colored than the other". Each of these patterns was present in students' verbal communication and written work. These patterns were not mutually exclusive and students may have used more than one in their communication.

Basic Relational. When determining quantitative sameness students either discussed that something was *hidden* or *equaled the same number*. Hidden referred to the answer from each side being hidden in the symbols. It was a term that I introduced in an instructional session and was adopted by many of the students. For example, in the problem 14+2=__+3 students might respond that a 13 is needed to get both sides to 16. The sixteen was hidden in the 14 and 2 or 13 and 3. This pattern often accompanied communication involving both sides equaling the same number. Taking the previous example, the response of a thirteen might include that both 14+2 and 13+3 equal sixteen.

In written work, students either used a *tree diagram* or *separate equations* to explain that both sides of the equation contained the same quantity. A tree diagram showed the quantity as a hidden amount underneath the given operation on either side of the equation. Student work displaying separate equations contained an open or true/false number sentence split up into two equations. Figure 5 shows what each of these looks like.

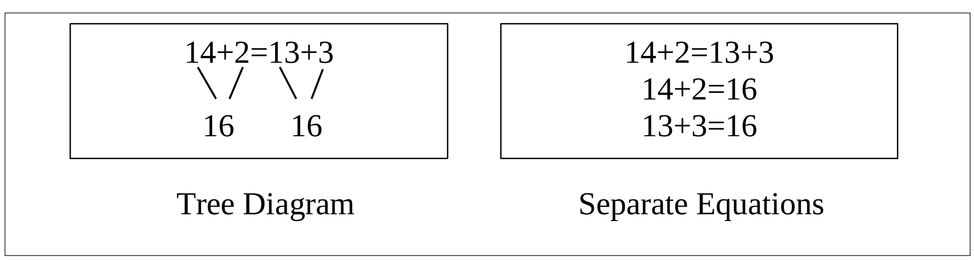


Figure 5. Ways students represent basic relational ideas

Operational. Because most students entering the study accepted atypical equation types, the rigid operational and flexible operational parts of the spectrum were grouped together. Students communicated these ideas by finding *the total*. For example, on the problem 10+4=__+10, Elise stated that she "thought a fourteen would go in the box because ten and four was fourteen." Operational ideas were shown in their written work through putting the total in the missing blank without attention to the relationships between numbers in the equation.

Other/Incorrect. Within the other/incorrect category, the most common pattern was attempting to get the *same numbers* on either side of the equation. In the problem 6+__=5+3 a student may answer three or five in an effort to get the same digits on either side. *Incomplete descriptions* were also included in this category and contained responses of "I don't know" or vague allusions to other concepts of equivalence.

Table 4 shows examples for each of the patterns identified in students' written and verbal communication.

Table 4

Patterns of communication for each level of relational thinking

Comparative Relational	**Basic Relational**	**Operational**	**Other/Incorrect**
Number Size: "Eleven is one less than twelve and two is one less than three."	*Hiding:* "The fourteen is hiding."	*The Total:* "I counted five cubes, and then I did four and three. And it said it was twelve."	*Same Number:* "I think it's five because the other table has a five."
Trading: "The eight gets minused one, and that one is taken to the two to make it three."	*Equal the same number:* "Twenty plus eight equals twenty eight and so does twenty-one plus seven."		*Incomplete description:* "The twelve is bigger than the eleven and the three is bigger than the two."

Same Number:

"The same numbers are there, but they're switched around."

Tangible Extra:
"That one has an extra cube, and then the other one has an extra cube."

Chronological Sequence

The spectrum of understanding assumes that students' comprehension of equivalence progresses in a certain manner. In order to investigate that assumption, each case was evaluated to see how student communication and understanding of equivalence progressed throughout the study. The pre/posttest served as a snapshot of student thinking at the beginning and end of the study while the session recordings and student work gave a more comprehensive view of changes in student understanding. All three data sources were compiled for each student within their case group to create an overall representation of developing knowledge of equivalence. The data sources for each student were also analyzed individually to gain an understanding of how student ideas progressed within a specific instructional setting.

Limitations

The researcher was not only the instructor but also collected and analyzed all of the data. Due to such a direct involvement in the study, it is reasonable that some bias in the analysis is present. In addition, the vocabulary and communication patterns introduced by the researcher during instruction likely influenced the communication patterns of the participants. The small sample size and participant selection being restricted to those who returned parent permission forms also limited this study.

However, using a case study design accounts for some of these limitations. This case study does not seek to generalize to a larger population, but rather describes the communication patterns shown by students in a specific setting. The data were intentionally analyzed using the spectrum of understanding (see Figure 4) as the primary lens to help account for bias. Similarly, the researcher attempted to reduce bias through using instructional tasks based on widely accessible curriculum.

CHAPTER 4: FINDINGS

Due to the nature of a case study and the research questions to be addressed, multiple analyses were performed. The study sought to answer the following questions.

1. In what ways do first grade students communicate their ideas about equality when instructional tasks include drawing bar models along with symbolic representations as opposed to only symbolic representations?
2. Will students who have an operational view of the equal sign be able to communicate relational ideas with a visual representation present?
3. How do students who receive instruction using a bar model move up on the spectrum of understanding compared to those who receive instruction with symbolic representations only?

To address these questions, findings will be presented in several ways. First an overall picture of how each group moved along the spectrum of understanding will be presented. Then consistent patterns in the pre/posttest as well as student communication will be given along with patterns of error. Finally a chronological sequence for each group will be presented to compare the organization of instruction that led to changes in thinking.

At the beginning of the study both groups had a similar distribution along the spectrum of understanding. Initial student understandings of equivalence were determined by looking at pretest data and transcripts from the first session. By the end of the study shifts had taken place within both instructional groups. Evidence of these shifts was demonstrated through the posttest, session transcripts, and student work from sessions nine and ten. Figure 6 shows the distribution (in percentages) of students along the spectrum at the beginning and end of the study for each group.

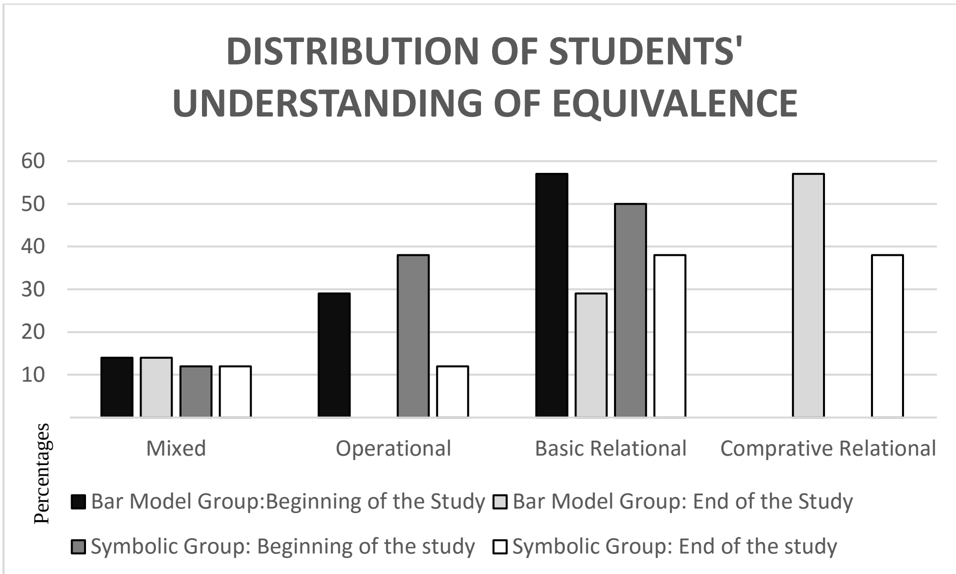


Figure 6. Distribution of students' understanding of equivalence pre- and post-study

Not every student moved along the spectrum or made definitive changes in their patterns of communication. For example, while Richard and Harry both moved from quantitative sameness to comparative relational, Becky and Matt both consistently demonstrated quantitative sameness throughout the study. These findings will be reported more in depth later. Because changes did occur in both groups, the cases needed to be further analyzed to look for consistent patterns in how students chose to communicate their understanding and when those shifts began to occur. The use of pattern matching addressed how students communicated their ideas, and chronological sequencing shed light on how shifts began to occur as well as when students with an operational view began to adopt relational ideas.

Pattern Matching

While the spectrum of understanding was used as a lens for analyzing all sources, it manifested in different ways depending on the data source. First the pre/posttest patterns will be addressed followed by the patterns in students' written and verbal communication.

<u>Pre/Posttest</u>

There was little change in the number of correct responses in either group from pretest to posttest. While there wasn't much difference in overall scores from pretest to posttest, both groups demonstrated shifts in thinking. There was an increased occurrence in the use of comparative relational explanations for both the bar model and symbolic groups. The responses coded as other/incorrect in the symbolic section showed changes as well. On the pretest the students from both groups who made consistent errors had difficulty explaining their thinking and produced *incomplete responses*. Allison often responded that she "had guessed" and Jayme would simply say "I don't know." By the posttest however, there seemed to be a clear attention to getting the *same numbers* on either side of the equation. For example, both Jayme and Allison said a three was needed in the problem 6+__=5+3. Jayme responded that "there's a three on both sides", and Allison asked "could it switch around".

Although the groups performed similarly, the bar model group did have a noticeable difference from the symbolic group; their use of trading strategies on the concrete visual based tasks was markedly higher than the symbolic group. Table 5 shows how each student performed on the pretest along with the types of strategies used to communicate their ideas. Table 6 represents their performance on the posttest.

Table 5

Student performance on pretest

		Concrete Visuals Based Tasks			Symbolic Based Tasks			
Name	Total Score	Trading	Add/ Subtract	Incorrect/ Other	Operational	Basic Relational	Comparative Relational	Incorrect/ Other
Bar Model Group								
Jayme	17	0	8	0	2	1	0	2
Richard	23	0	6	1	0	5	0	0
Samuel	24	0	9	0	0	5	0	0
Becky	24	0	8	0	2	0	0	3
Scott	24	7	1	0	0	3	1	1
Jeff	17	0	9	0	1	1	1	2
Beth	16	0	8	0	2	0	0	3
Symbolic Group								
Elise	14	0	8	0	0	0	0	5
Harry	22	0	8	1	0	3	1	1
Debbie	23	0	9	0	0	2	2	1
Anthony	21	0	7	1	0	1	1	3
Matt	23	0	9	0	0	4	1	0
Stephen	20	0	7	0	0	4	1	0
Jessica	14	0	6	1	2	0	0	3
Allison	13	0	8	0	0	0	0	5

Table 6

Student performance on posttest

		Concrete Visuals Based Tasks			Symbolic Based Tasks			
Name	Total Score	Trading	Add/ Subtract	Incorrect/ Other	Operational	Basic Relational	Comparative Relational	Incorrect/ Other
Bar Model Group								
Jayme	14	0	9	0	0	0	1	4
Richard	24	7	0	0	0	3	2	0
Samuel	24		5	0	0	3	1	1
Becky	24	0	7	0	0	3	2	0
Scott	24	7	0	0	0	0	5	0
Jeff	21	5	3	0	0	0	3	2
Beth	23	5	4	0	0	0	4	1
Symbolic Group								
Elise	15	0	8	0	2	0	1	2
Harry	24	2	7	0	0	2	3	0
Debbie	24	0	8	0	0	0	5	0
Anthony	24	1	8	0	0	0	4	1
Matt	24	2	7	0	0	4	1	0
Stephen	22	0	7	0	0	1	2	2
Jessica	17	0	8	0	0	2	0	3
Allison	14	0	8	0	0	0	0	5

Verbal and Written Communication

As previously mentioned, based on the pretest and instructional session one of the study none of the students communicated comparative relational ideas at the beginning of the study. Over half of the students in both cases showed quantitative sameness while the majority of the others demonstrated operational thinking. In the bar model group, Richard and Scott moved from basic relational to comparative relational by the end of the study. Jeff moved from showing inconsistent patterns to comparative relational, and Beth changed from flexible operational thinking to comparative relational understandings. Jayme displayed flexible operational thinking at the beginning, but by the end of the study she had overgeneralized the idea of "same as" to mean that both sides of the equation must contain the same numbers. Becky's and Samuel's overall patterns of communication remained consistently within a basic relational view throughout the study, although some comparative ideas were present.

In the symbolic group, Harry, Anthony, and Debbie moved from a basic relational understanding to comparative relational. Jessica moved from a flexible operational view to basic relational. Allison stayed within a rigid operational view, although she showed an overgeneralization of needing to have the same numbers on each side or guessed what she thought might be appropriate. Stephen moved from flexible operational to basic relational. Neither Elise nor Matt showed a definitive shift in thinking. Matt remained consistently within basic relational while Elise continued to use flexible operational communication. Figure 7 represents the shift along the spectrum for each of the students.

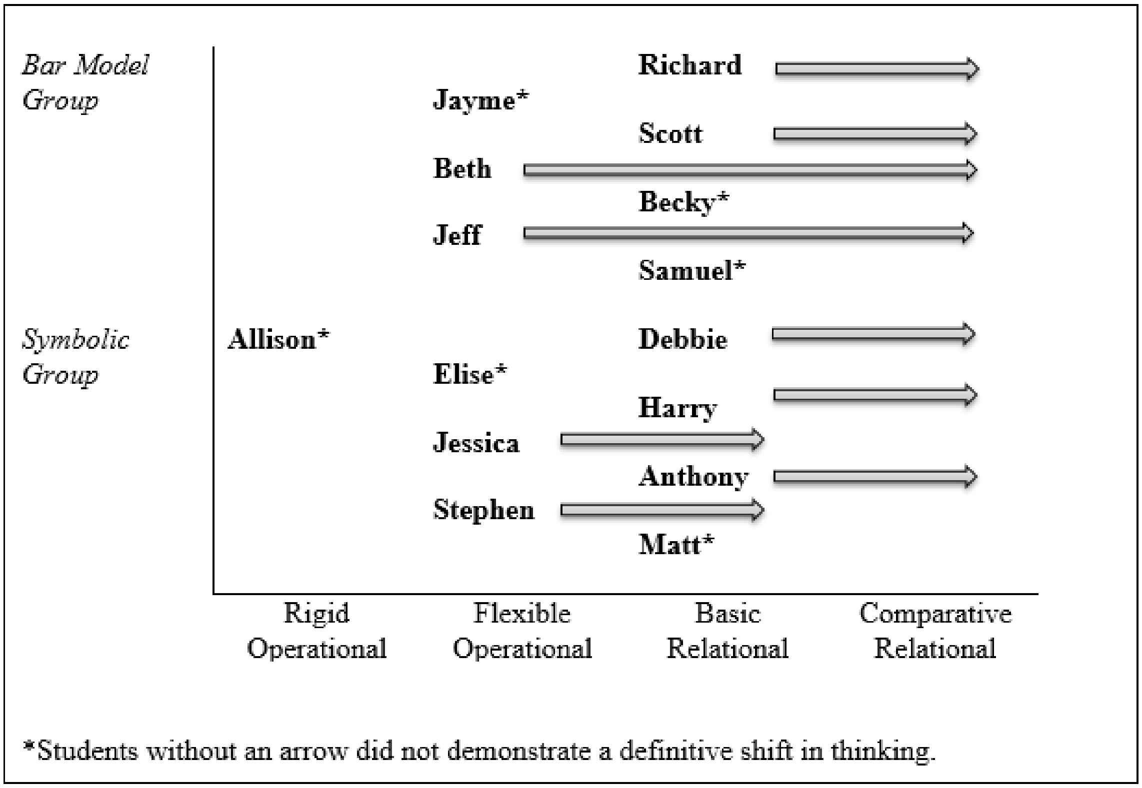


Figure 7 Students' shifts along the spectrum of understanding

The following sections will report a cross-case synthesis of patterns found within each level of the spectrum. These findings are based on analyzing session transcripts as well as students' written work.

Comparative Relational

The most common patterns among comparative relational communication were based on *number size, trading,* and *a tangible extra.* Although some students picked up on the idea of *same numbers*, it will be presented in the operational section as most students with this comparative view overgeneralized the idea.

Both groups communicated comparative relational ideas using all three patterns. The most frequent pattern in both groups was *number size*. Compared to the overall instances of comparative relational ideas, the bar model group referenced this idea in 49 percent of its communication while the symbolic group did so 61 percent of the time.

Trading was the next most frequent pattern. It composed 31 percent of comparative communication in the bar model group and 30 percent in the symbolic group. The least frequent pattern was the *tangible extra*. In the bar model group, students expressed this concept in 20 percent of its comparative communication, and it only presented itself 9 percent of the time in the symbolic group. For both groups, students who showed consistent patterns of comparative relational thinking used multiple ways to communicate their understanding. The following excerpts from Jeff demonstrate this idea. In session 3, he used terms indicative of number size; "They're getting shorter, and on the numbers in the number sentence they're going down". By session 6, he talked about a tangible extra (in reference to a bar model) and trading it to another number.

> Jeff: You took away one (pointing to the colored cube in the first model).
>
> T: So we could pretend it's right there? *(Pointing to the open section of the lower bar model.)*
>
> Jeff: You take away one and then there's six right there *(pointing to the open section of the lower bar model).*

With the exception of Harry, all students (from either group) who ended the study with a comparative relational view used the concepts of trading, number size, and a tangible extra at some point during the sessions. Harry only referenced trading and number size.

While comparative ideas were present in both groups, there were more varied communication patterns in the bar model group. Both groups used trading about the same percentage of time, but there was a notable difference between the use of number size and a tangible extra between the two groups. For the total instances of comparative relational thinking in the bar model group, 49 percent of the time students communicated using *number size*, 31 percent were *trading*, and 20 percent fell into the *tangible extra* category. In contrast, the distribution of comparative ideas in the symbolic group was 61 percent *number size,* 30 percent *trading,* and 9 percent *tangible extra*. Figure 8 shows the distribution of subcategories compared to the total instances for comparative relational ideas. All numbers are listed as percentages.

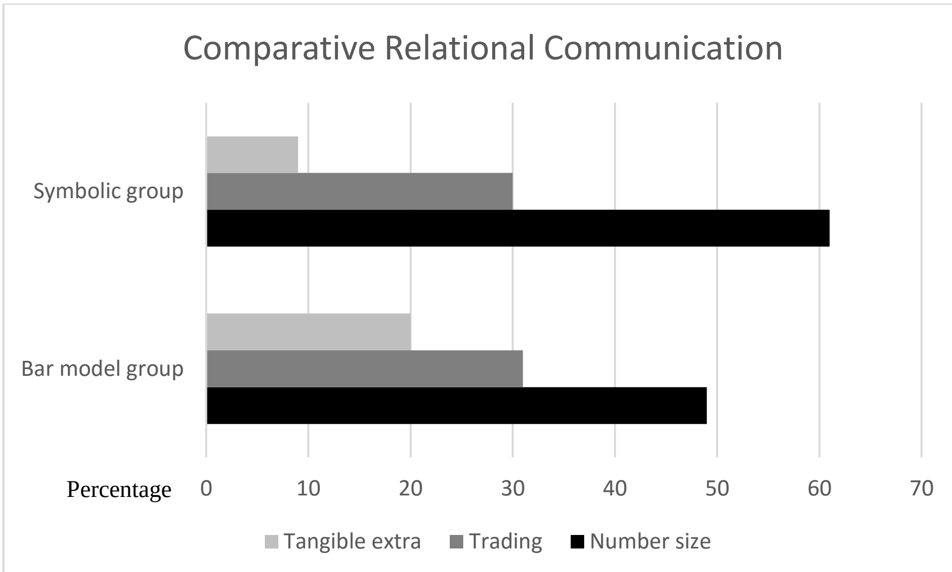


Figure 8. Distribution of each sub-category in relation to total instances of comparative relational communication

Basic Relational

Despite their instructional group, students communicating basic relational ideas explained their thinking most often by saying both sides *equaled the same number*. This was true of the students who began and/or ended the study with a quantitative sameness pattern. On the pre- and posttest, every response in this category mirrored Harry's explanation of "six plus two equals five plus three because they are both eight." During the sessions, student responses were similar to Becky's clarification of knowing why 10+3=9+4 was a true statement.

T: Ok, Becky what do you think?

Becky: I don't know yet.

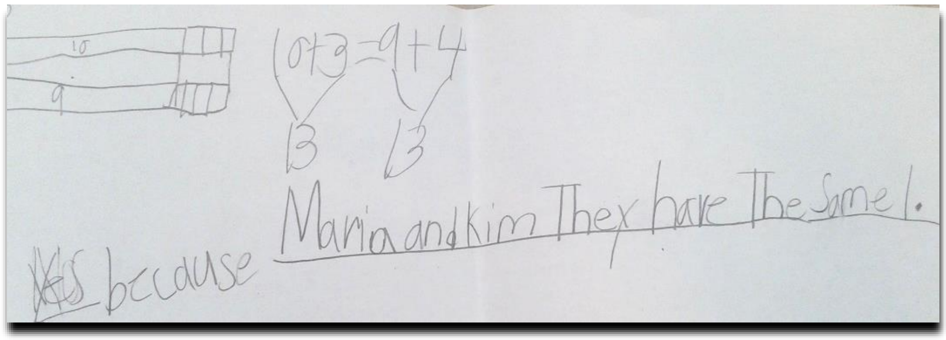
T: You said yes. Why do you think yes?

> Becky: Because ten plus three is the same as nine plus four. They both equal thirteen.

However, those students who used basic relational strategies from the beginning differed from those who shifted to it through the course of instruction. Students in both groups beginning with an operational view used the idea of a *hiding* quantity more often than those who began with a basic relational understanding. Stephen and Jessica from the symbolic group as well as Jayme from the bar model group each explained both sides equaled the same, but they also had additional references to a *hiding* quantity.

> From session 8:
>
> Jayme: I think it might be three like last time.
>
> T: Why is it three? Why not twelve? Isn't nine plus three twelve?
>
> Beth: Because they wouldn't be the same.
>
> T: They wouldn't be the same because we still have these blue beads. So where is the twelve?
>
> Jayme: It's hiding.

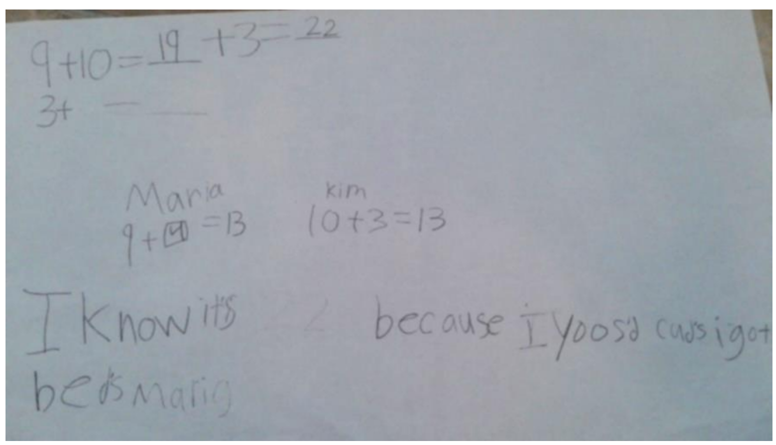
While verbal explanations of quantitative sameness mostly discussed an equal amount on both sides, this was shown in written communication a couple different ways. Students either represented the same quantity on either side of the equation through a *tree diagram* or *separate equations*. The bar model group used a model to explain their thinking, but it was always accompanied by one of the two equation representations. Picture 1 shows how Jayme included the use of a tree diagram to show the hidden quantity on either side of the equation.



Tree diagram

Picture 1. Jayme’s session 9 work

Picture 2 is an example of Elise’s work. It demonstrates her basic relational understanding that both sides must equal the same amount. She uses separate equations rather than a tree diagram to show her thinking.



Separate equations

Picture 2. Elise’s session 9 work

Most students demonstrating basic relational understanding in either group used a mixture of both tree diagrams and separate equations to show their thinking.

Only minor differences were found between how students communicated basic relational ideas between the two groups. The symbolic group communicated the idea of a *hiding* quantity slightly more often than the bar model group. Compared to the total occurrences of basic relational patterns, the bar model group referenced a *hiding* quantity 14 percent of the time and *equaling the same number* 85 percent of the time. The symbolic group had a distribution of 27 percent *hiding* and 73 percent *equaling the same number*. Figure 9 shows the distribution of subcategories compared to the total instances for comparative relational ideas. All numbers are listed as percentages.

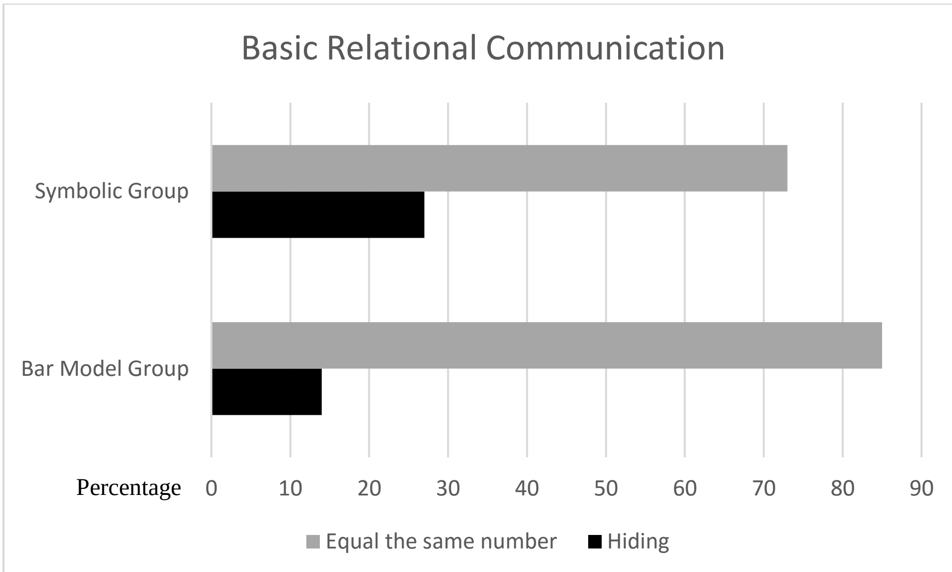


Figure 9. Distribution of each sub-category in relation to total instances of basic relational communication

Operational and Incorrect/Incomplete

The findings for the operational and incorrect/incomplete views were grouped together because many of the students in these categories overlapped ideas during the instructional sessions. For the most part, the students who grasped the relational concept of *same number* overgeneralized it and did not end the study with a relational understanding of equivalence. In session 8, Elise from the symbolic group understood the problems involving the same number on either side, but was not able to observe a relational idea when the numbers were not the same. When solving the problem 9+3=__+9, she responded, “Put a three right there. It’s the same as last time…The same numbers but they’re switched around.” However in the next problem, 9+3=__+8, she said, “I don’t get this one because it’s not the same numbers.” Allison from the symbolic group was also able to easily solve open number sentences during session 8 when the same numbers were on either side of the equation.

T: Ten plus four equals something plus ten. What's that number going to be?

Allison: Four

Later in the discussion

T: We also know there are two tens, so what is this number going to be?

Allison: Four

These two girls however were among the students with an operational view or who showed incomplete or undeveloped understanding of equivalence.

The other common theme in both groups was a preference for finding *the total*. When answering 1+9=__+1 on his pretest, Stephen from the symbolic group gave the following response, "It's nine because one plus nine is ten and nine plus one is ten. It would be eleven though, the answer." His comment shows both a basic relational understanding and a preference for finding the total. Stephen's and Elise's need to find a total also came up in session 6 with a more direct communication of operational thinking.

Solving the problem 4+5=3+__

Elise: I counted five cubes and then I did four and three, and it said it was twelve.

Stephen: Yeah, that was my answer too.

Later in the conversation

T: Where do you think the twelve came from? Someone else, where do you think they got twelve? What might they have done?

Harry: The added six plus six.

T: They might have added six plus six. I think they did something else.

Stephen: I counted the blocks and it was twelve.

Debbie: They added the four plus five, and they also added the three.

A determination to find the total was not only found in the symbolic group. Jayme from the bar model group showed similar thinking.

> *Solving the problem: Maria has ten red beads and four blue beads. Kim has some red beads and ten blue beads. How many red beads does Kim need to have the same amount as Maria?*
>
> Becky: I think it's ten.
>
> T: Do you agree with her? Touch your nose if you agree. Jayme, I see that you disagree with that.
>
> Jayme: Because it's already ten and four.
>
> T: Ok, so I'm going to write in what Becky thinks. She said ten plus four equals four plus ten. You disagree with that?
>
> Jayme: I thought it was fourteen.
>
> T: Where do you think she got the fourteen from?
>
> Richard: The ten and the four.
>
> T: Did you add how many Maria has? Did you add the ten and the four?
>
> *Jayme nodded her head.*

Jayme eventually reasoned through her misunderstanding when prompted to use her model. She eventually saw that ten plus four on the left side of the equation represented the amount that Maria had and the ten plus four on the right side represented Kim. With only a couple of questions, Jayme overcame her misconception. It took Stephen and Elise in the previous example much longer to recognize their mistake.

Overall the two groups differed slightly. For the bar model group, 50 percent of the time students expressed an operational or other/incorrect idea students focused on *the total* and 50 percent on finding the *same number.* In the symbolic group, the distribution was 67 percent adherence to *the total* and 33 percent achieving the *same numbers* on either side of the equation. However, Jayme was the only student in the bar model group to communicate operational ideas consistently throughout the study so the codes from her communication may not be an accurate reflection of the group as a whole. Figure 10 shows the distribution of subcategories compared to the total instances for operational and incorrect/incomplete ideas. All numbers are listed as percentages.

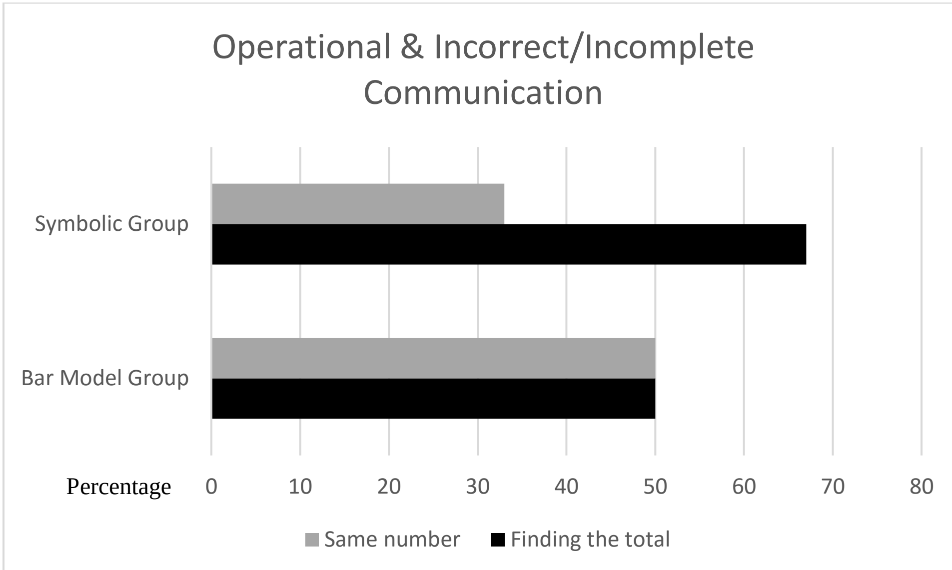


Figure 10. Distribution of each sub-category in relation to total instances of operational and incorrect/incomplete communication

<u>Summary</u>

Although both groups performed similarly on the pre/posttest, differences emerged in written and verbal communication. Students in the bar model group used the comparative idea of the tangible extra more frequently than the symbolic group. The bar model group also had a lower percentage of general references to number size. Basic relational understanding was most frequently referenced by equaling the same number on either side. Students in the symbolic group communicated the idea of a hiding quantity slightly more often than the bar model group. For students presenting an operational or mixed thinking view, the bar model group showed an overgeneralization of *same number* slightly more frequently than the symbolic group.

Students in both groups made shifts along the spectrum of understanding. The following section will present the findings for when and how students began to make shifts along the spectrum. These data were gathered from the session transcripts as well as students' written work.

Chronological Sequence

Looking at the data chronologically showed some general patterns in how students' relational thinking developed or changed in both of the instructional groups. It provided insight into what instructional techniques led to shifts along the spectrum in the bar model group compared to the symbolic group.

Early Communication

Generally speaking, in both groups students who spoke in multiple ways about the comparative relationship between numbers even vaguely showed changes in their communication of equivalence earlier than those who did not. Both Debbie and Richard are examples of this pattern. In the symbolic group, Debbie began sharing comparative ideas by session 3 and consistently used this reasoning by session 5. Richard, from the bar model group, demonstrated similar patterns. In session 3, he used multiple ways to talk about how the numbers were changing and was able to use comparative thinking in subsequent sessions. However, he did not consistently communicate comparatively until session 8.

The main difference between the two groups is more students from the bar model group communicated comparative patterns earlier than those in the symbolic group. In the visual group, Richard, Beth, and Jeff all shared comparative ideas in session 3 when talking about the relationships between ways they found to decompose ten. They talked specifically about how one cube was getting colored in or traded to another number.

> Richard: Ooh, that every space, Becky colored in one, Jeff did two, and Miss Amber did three.

T: Ok, so the colored spaces are getting one larger. What's happening to the uncolored spaces then?

Jeff: They're getting shorter and on the numbers in the number sentence they're going down.

T: If these are going down, where is it actually going? They still equal ten, so what's happening to that one?

Beth: It went into the ten.

T: Let's look at our model. What happened? Here we had nine, here we had eight. Where did that extra one cube or extra one go?

Beth: It got colored in.

T: Oh, it went into our…

Richard: Colored in one.

In the bar model group, Richard, Beth, and Jeff used multiple ways to describe comparative ideas by session 3. Becky, Samuel, Jayme, and Scott did so by session 10. Scott might have communicated these ideas earlier but he was absent for sessions 6, 8, and 9.

For the symbolic group, Debbi was the only one to use multiple ways to communicate by session 3. Anthony, Harry, and Matt did so by session 6. Stephen was able to by session 10, while Allison, Jessica, and Elise never consistently communicated comparative ideas in more than one way. Following is an example of a conversation with Harry and Matt from session 6.

Harry: Eleven plus three is the same as twelve plus one.

T: Why do you think it's twelve plus one?

Harry: Because you take twelve and you add, take two to get to the three.

T: So how did the eleven change?

Harry: It turned to the…wait, I get it.

T: So how did the eleven change?

Harry: It gave it to the three.

Later in the conversation

T: Why do you think it's two Matt?

Matt: Because eleven plus three is twelve and so is twelve plus two.

T: Eleven plus three is how much?

Matt: Eleven plus three is fourteen.

T: Ok, how did the eleven change?

Matt: One changed, one got traded.

T: We got one more. What happened to the three then?

Matt: It put one less to the two.

<u>Quantitative Before Comparative</u>

Students in both groups needed to make sense of a basic relational understanding before they expressed comparative ideas. Beth and Jeff from the bar model group were prime examples of this trend. On the pretest, Jeff showed no consistent pattern in his communication, while Beth showed an operational view. By the posttest both of them gave many comparative explanations. The transcripts showed they needed to process through the basic relational idea of quantitative sameness to get to that point. On his student work in session 10 Jeff wrote, "Yes because ten plus three equals thirteen and nine plus four equals thirteen. The ten went down by one to nine." In session 10, Beth pointed out that she found a comparative relationship after finding the same quantity; “Well this is my second one. Yes because the ten went down by one and the three went up by one.” Stephen from the symbolic group also showed this trend. In the beginning sessions he often needed to find “the total” and thought operationally. He made sense of quantitative sameness through the idea of a hidden quantity by session 8. The following excerpt shows the progression from attention on “the total” to quantitative sameness.

> Stephen: I get it! It’s twelve. The answer is twelve.
>
> T: But what is the answer? What am I looking for? Do I want the total amount of beads Stephen?
>
> Stephen: Oh, no.

Later in the session he was able to reason quantitatively.

> T: Those of you who agree with Allison, why do you agree? Why do you think it’s four?
>
> Stephen: Because nine plus three equals twelve and four plus eight equals twelve.

Stephen then shared a mix of basic and comparative relational thinking when he was deciding if the equation 10+3=9+4 was true or not.

> Stephen: But if you switched the four to the three it would be fourteen though.
>
> T: If I did this? *(writing 10+4=9+3)*
>
> Stephen: Yes
>
> T: Then would they be the same?
>
> Stephen: No

Needing to make sense of basic relational ideas before adopting comparative understanding was present in every student in the study, with the exception of Allison.

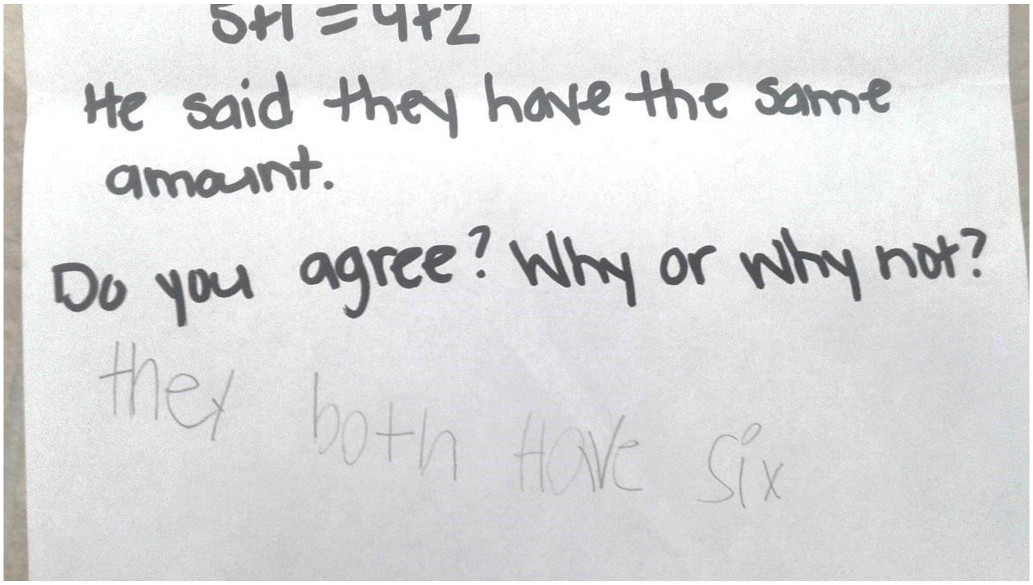
<u>Operational to Relational</u>

Regardless of the group students were in, those entering the study with a mixed or operational view of the equal sign needed multiple exposures to context based problems and relational ideas in order to change their thinking. With the exception of Beth, none of the students beginning with an operational understanding completely abandoned their focus on finding *the total* or *the same number* on either side. However students in the bar model group were able to communicate relational ideas earlier than those in the symbolic group. In the bar model group, Jayme, Beth, and Jeff all entered the study with a mixed or operational view. Beth and Jeff began to communicate relational ideas by session 3 and Jayme did so by session 5. For the symbolic group, Stephen, Allison, Jessica, and Elise began the study with an operational view. Stephen communicated relational ideas by session 5, but it took Jessica, Allison, and Elise until session 8.

For some of the problems, the use of cubes hindered students' development of a relational understanding. In session 6 for the symbolic group, Jessica, Stephen, and Allison all added the total number of cubes they would pull to complete the open number sentence. Jayme from the bar model group did so a few times as well. However, Jayme was able to consistently communicate a basic relational view by the end of the instructional sessions when she was able to refer to her model. On the posttest her misunderstandings seemed to reflect precursory ideas to a basic relational understanding. At the beginning of the study, most of Jayme's mistakes were based on finding *the total* while on the posttest her mistakes were tied to finding the *same number* on either side of the equation. While she overgeneralized that concept, it is a more relational thinking based mistake than finding the total. In contrast, although Elise from the symbolic group communicated basic relational ideas by the last instructional session, that understanding was not reflected on her posttest. On both her pretest and posttest, Elise's mistakes reflected a focus on finding *the total*.

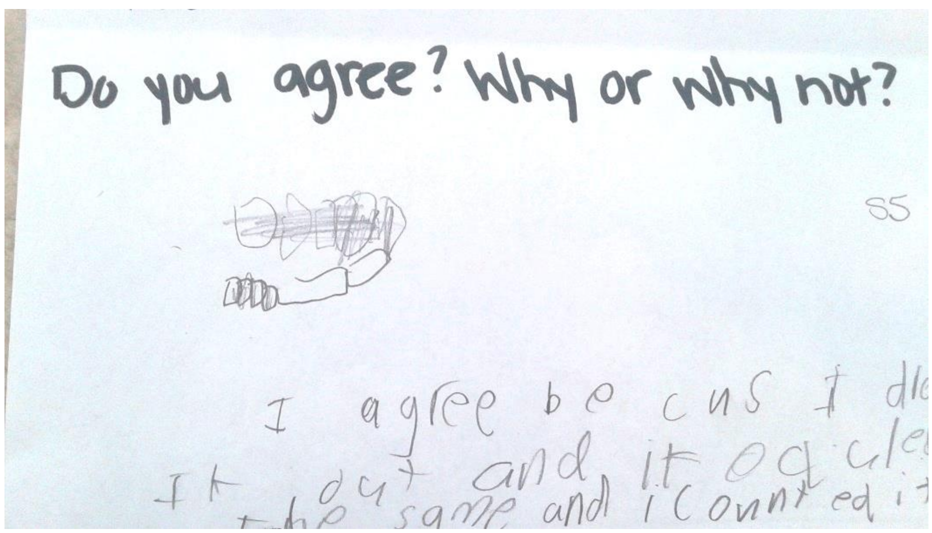
Task Sequence

The final trend found when analyzing the data chronologically was a pattern in the types of tasks that elicited relational understandings in both groups, whether basic or comparative in nature. Sequencing the tasks in a way that allowed students to deal with true/false statements before open number sentences with operations on both sides of the equal sign helped the students communicate relational ideas. In the same way, starting with smaller number sets before using numbers the students could not compute easily produced this same result. Students were asked in session 3 to find different ways to decompose ten. Students began showing their thinking through traditional equations. However, all students acknowledged it was acceptable to write 1+9=2+8 because the quantity on both sides was ten. A couple examples were from Stephen from the symbolic group and Jayme from the bar model group. Both were able to communicate basic relational ideas in session 5 when analyzing a true/false number sentence with numbers easy to compute (see pictures 3 and 4).



"They both have six."

Picture 3. Stephen's session 5 work



"I agree because I drew it out, and it equaled the same, and I counted it."

Picture 4. Jayme's session 5 work

It was difficult for them to use this same thinking during session ten when determining whether 37+27=38+28 was true. When presented with a context that created an open number sentence, they also struggled, often reverting to operational thinking.

> Elise: *(Solving 5+4=__+3)* I counted five cubes and then did four more and three, and it said it was twelve.

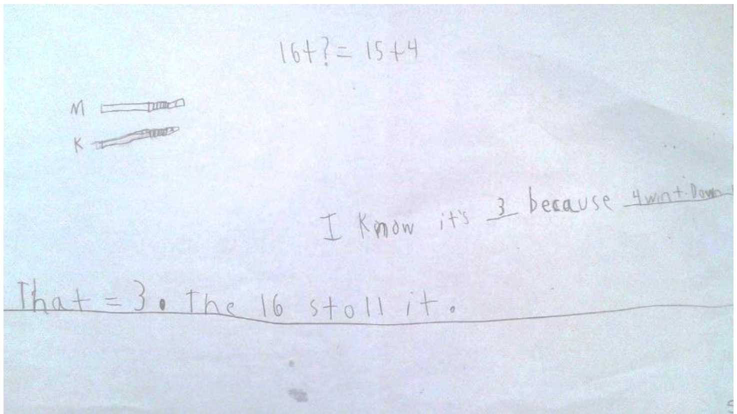
Stephen: Yeah, that was my answer too.

Jayme solved the problem 10+4=__+10 in a similar way.

T: Becky thinks a four should go there *(wrote 10+4=4+10*). Jayme, I see you disagree with that.

Jayme: It's already ten and four. I thought it was fourteen.

Even students starting the study with a quantitative sameness understanding followed this trend. Samuel from the bar model group had difficulty determining whether the problem 37+28=38+27 was true. He commented, "I don't know why. I don't know if it's yes or no. Is there another one we could do? This one is too hard." However he was able to easily write his thinking when given smaller number sets even on open number sentences (see picture 5.



"I know it's 3 because 4 went down one and that equals three. The 16 stole it."

Picture 5. Samuel's session 9 work

<u>Summary</u>

In general, both groups showed similar chronological trends. Students from both groups needed to make sense of basic relational ideas before thinking comparatively. The sequencing and type of tasks impacted students' ability to communicate relational ideas.

True/false equations with smaller number sets were the most successful at getting students to communicate relationally. The main difference between the bar model and symbolic groups was how early students were able to communicate relational ideas. Shifts to comparative or basic relational communication patterns occurred earlier for the bar model group than the symbolic group. In addition, students with an operational view in the bar model group were able to reason through their misconceptions more quickly than those in the symbolic group.

CHAPTER 5: DISCUSSION

Through analyzing the cases it was found that there were distinct patterns in how students communicated their understanding of equivalence. A summary of the major findings is presented below followed by conclusions based on those findings. Finally, implications for instruction as well as suggestions for further research are given.

Students often referred to ideas of trading, number size, or a tangible extra when expressing ideas of comparative relational thinking. The bar model group referred to the tangible extra more often than students in the symbolic group. For the bar model group, having a visual allowed students to connect their ideas to something they and the other students could see. The bar model turned a somewhat abstract idea into something more concrete. References to number size were shown the least helpful for changing students understanding of equivalence. This could be due to the fact that when students discussed number size, they were often vague in their descriptions and didn't fully acknowledge the relationships between the expressions on either side of the equation.

When discussing the ideas of quantitative sameness, students used vocabulary related to the same amount on either side or a hiding quantity. The concept of a hidden quantity appeared especially helpful for students beginning the study with an operational view of the equal sign and was seen more often in the symbolic group. Students in the bar model group discussed a hidden quantity less often perhaps because in the visual model the quantity was no longer hidden. Presenting the definition of "the same as" for the equal sign sometimes produced overgeneralizations and the misconception that both sides of the equation must contain the same number instead of the same quantity.

At some point in the study all students communicated in ways indicating changes in thinking. Although not all of them changed enough to move along the spectrum of understanding, misconceptions were challenged and ideas changed. Those students with an operational view were able to express relational ideas in both cases, although Jayme from the bar model group did so more frequently and earlier in the study than either Elise or Allison from the symbolic group.

Conclusions

First graders are very capable of not only understanding relational ideas, but also changing prior misconceptions. Both cases indicate that movement along the spectrum is possible in a short amount of time. They also suggest that students' understanding does progress in a predictable manner by first understanding quantitative sameness and later advancing to a more comparative view. This supports the previous work done by Carpenter et al. (2003a) and Matthews et al. (2012). The second question of this study was whether students with an operational view would communicate relational ideas when a visual model was present. One of the most interesting findings was in some contexts the enactive model of cubes actually served as a hindrance to changing students' understandings. In session 6 for the symbolic group, Jessica, Stephen, and Allison all added the total number of cubes they would pull to complete the open number sentence. Even though students were working on a contextual problem, the structure of the equation led students with an operational view to find the total amount. Sherman and Bisanz (2009) found that students performed better on symbolic equations when presented with non-symbolic tasks first. Their finding diverges somewhat from the evidence in this study. A non-symbolic representation (the cubes) coupled with the symbolic equation did not challenge students misconceptions. Students in the bar model group who performed similarly eventually corrected their misunderstandings. Jayme from the bar model group also displayed an operational view, and with support, she was able to reason through why her answer would not make sense while referring to the bar model. The visual served not only as a way to interpret the problem but also a way to check the reasonableness of solutions and communicate relational ideas.

Another finding of note was the concept of a hidden quantity paired well with use of a tree diagram to show quantitative sameness. Students with an operational view having a predisposition toward finding the total were more likely to change their thinking when presented a way to show the hidden quantity. The tree diagram served this purpose for both groups. Although still a symbolic representation, the tree diagram provided a way for students to visualize the hidden quantity without reinforcing an operational view. Aracavi (2003) found visualization helps students perceive incorrect assumptions and intuitions as they begin to change conceptualizations and make sense of discrepancies. The bar model is another way for students to visualize this hidden quantity and reason about the relationships between the numbers. A tree diagram does not show proportion of numbers and is therefore less likely to shift students away from the vague comparative references to number size. In contrast, a bar model not only shows the numbers in relation to one another, but serves as a concrete model of the situation making it easier to see how numbers are changing or trading. The bar model group had a smaller percentage of their communication that addressed number size and a hidden quantity then the symbolic group. These findings support the idea that a visual model brings specific relationships among number to the forefront while making the "hidden quantity" visible as well.

While students communicated their understanding of the equal sign in multiple ways regardless of which group they were in, an intriguing finding was the emergence of the tangible extra. It was present in both groups, but the bar model group had a way to communicate the concept in a meaningful way. Van den Heuvel-Panhuizen (2003) reported similar ideas; "The bar model tells them in an *understandable manner* what calculation they have to carry out to find an answer" (p. 24, emphasis added). Debbie attempted to share her thinking through the phrase, "one more thing" but none of the other students in her group were able to make sense of it. In contrast, when Jeff talked about the amount of colored squares going down, other students interpreted his idea and made it their own. The increased frequency of the tangible extra in the bar model group points to the power of having a shared model to communicate understandings. As Anderson-Pence et al. (2014), Gellert & Steinbring (2013), and Middleton & Van den Heuvel-Panhuizen (1998) suggested, the visual model provided a concrete reference for students when others communicated their thinking. In session 10, Scott used the visual of the bar model to clarify his thinking about why 37+27=38+28 was not true. Without his explanation of two extra cubes on the same side, the other students in his group may have had continued difficulty clarifying their own thinking.

Implications/Recommendations

These cases provide some powerful insights into directions for future instruction and research around equivalence and the equal sign.

Instruction

First, instruction on the equal sign must be explicit and set within a realistic situation. The use of context based tasks helped students place the concept of equivalence in a meaningful situation. Students must also be exposed to multiple ways of communicating their understanding. In both instructional groups, students who communicated their ideas in multiple ways were more likely to change their understanding. Those in the bar model group used it as a tool to convey relational ideas even though they had little to no prior background with the model. They also shared their ideas earlier than those without a visual model present. Arcavi (2004), Saenz-Ludlow and Walgamuth (1998), and Warren and Cooper (2005) spoke of the increased need for visualization in all mathematical domains. The tangible extra appears to be a way that students attempted to communicate the visualization that was occurring in their heads. Using the bar model as a "model of" the situation to show colored or uncolored squares allowed students to communicate the visualization needed to make sense of the equations. It explicitly showed the tangible extra (Gravemeijer, 1999; Van den Heuvel-Panhuizen, 2003).

Instruction should also be sequenced in a way for students to determine whether an atypical equation is true or false prior to solving a similar open number sentence (e.g. 4+5=3+6 vs 4+5=__+6). When students could analyze all numbers rather than finding a missing term, they were more likely to communicate relational ideas and begin a shift in thinking. Similar to Frieman and Lee's findings (2004), students struggled to grasp comparative ideas when number sets were too large for them to compute easily. Once students have made sense of true/false and open number sentences with smaller number sets, moving to larger numbers in the same sequence would be appropriate.

Further Research

As a follow-up, future studies could explore the long term gains of a similar short term intervention. This study showed positive results in a short amount of time with a week and a half in between the intervention and posttest. Future studies could replicate the design and investigate if these gains held up in a posttest re-administered a month or more later.

The case study design could also be expanded to include multiple classrooms or schools to see if the same patterns in communication were present. One limitation of this study was the researcher's participation in providing instruction. Specific vocabulary and ideas were introduced that likely colored the vocabulary students used to express their thoughts. Conducting case studies using these same instructional approaches with different teachers may provide more insight into general patterns of student communication.

Future quantitative studies could be conducted on a larger scale to consider the relationship between using trading strategies on concrete visual tasks and comparative understanding of the equal sign. A trading strategy requires spatial reasoning skills, which the literature has shown to have a significant impact on students' future mathematics performance (Arcavi, 2003; Ontario Ministry of Education, 2014; Stephens & Armato, 2010). In this study, the students most likely to use a trading strategy on those problems, from either group, were also the students communicating comparative relational ideas more frequently. Cheng and Mix (2014) also found a relationship between spatial reasoning tasks and performance on symbolic equations. Exploration of this trend was beyond the scope of this study.

Summary

The spectrum of understanding developed from the literature assumes that a students' relational understanding of equivalence develops and is communicated in a certain manner. The literature suggests that to develop relational thinking, students must first make sense of equations through quantitative sameness and then advance to more comparative ideas (Carpenter et al., 2003; Hunter, 2007; Matthews et al., 2012). Both of these assumptions were supported in this study. Students showed consistent patterns in communication as others who fell within the same category of the spectrum. Because this was a case study analyzing two small samples, no generalizations can be made to the larger population of first graders. However, the cases suggest that when explicitly teaching a relational understanding of the equal sign using contexts that ask students to compare quantities, changes in students' thinking can be made in a short amount of time. It also suggests that the bar model was a useful tool for student communication. It served not only as a way of representing the problem but was also used by students to make sense of other's ideas and specifically talk about the relationships between numbers.

REFERENCES

Anderson-Pence, K. L., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (2014). Relationships between visual static models and students' written solutions to fraction tasks. *International Journal for Mathematics Teaching and Learning.*

Arcavi (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215-241.

Barlow, A.T., & Harmon, S. E. (2012). Problem contexts for thinking about equality: An additional resource. *Childhood Education. 88:*2, 96-101. doi: 10.1080/00094056.2012.662121

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics. 38*(2), 115-131

Baroody, A. J., & Ginsburg, H. P. (1982). The effects of instruction on children's understanding of the "equals" sign. *Paper presented at the Annual Meeting of the American Educational Research Association.*

Behr, M., Erlwanger, S., & Nichols, E. (1976) How children view equality sentences. PMCD Technical Report No. 3

Blum, W. & Borromeo Ferri, R. (2009) Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45-58

Brendefur, J., Johnson, E. S., Thiede, K. W., Smith, E. V., Strother, S., Severson, H. H., & Beaulier, J. (2015). Developing a comprehensive mathematical assessment tool to improve mathematics intervention for at-risk students. *International Journal for Research in Learning Disabilities, 2*(2), 65-90

Brendefur, J. L., Thiede, K., & Strother, S. (2012*) Primary Mathematics Assessment: Technical report*. Retrieved from Boise, ID.

Brendefur, J. L., & Strother, S. (2016). Primary Mathematics Assessment. Retrieved from www.pma.dmtinstitute.com

Bruner, J. (1964). The course of cognitive growth. *American Psychologist, 19*(1), 1-15.

Cai. J., Cirillo, M., Pelesko, J. A., Ferri, R. B., Borba, M., Geiger, V., … Kwon, O. N., (2014). Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional, and teacher education perspectives. P*roceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education.* Vancouver, BC: PME.

Carpenter, T.P., Franke, M.L., & Levi, L. (2003a). Equality. In *Thinking mathematically: Integrating arithmetic and algebra in elementary school* (pp. 9-24). Portsmouth, NH: Heinemann.

Carpenter, T.P., Franke, M.L., & Levi, L. (2003b) Developing and using relational thinking. In *Thinking mathematically: Integrating arithmetic and algebra in elementary school* (pp. 27-45). Portsmouth, NH: Heinemann

Cheng, Y., & Mix, K. S. (2014). Spatial training improves children's mathematics ability. *Journal of Cognition and Development*. *15(1),* 2-11.

Deliyianni, E., Monoyiou, A., Elia, I., Georgiou, C., & Zannetou, E. (2009). Pupils' visual representations in standard and problem solving in mathematics: their role the breach of didactical contract. *European Early Childhood Education Research Journal*, *17*(1), 95-110.

DeLoache, J. S. (1991). Symbolic functioning in very young children: Understanding pictures and models. *Child Development, 62* (4), 736-752.

Erbas, A. K., Kertil, M., Cetinkaya, B., Cakiroglu, E., Alacaci, C., & Bas, S. (2014). Mathematical Modeling in Mathematics Education: Basic Concepts and Approaches. *Educational Sciences: Theory & Practice, 14*(5), 1621-1627. doi: 10.12738/estp.2014.4.2039

Fong Ng, S. & Lee, K. (2009) The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education, 40*(3), p282-313.

Frieman, V., & Lee, L. (2004). Tracking primary students' understanding of the equality sign. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*. 2, 415-422

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155-177.

Gellert, A. & Steinbring, H. (2013). Students constructing meaning for the number line in small group discussions: Negotiation of essential epistemological issues of visual representations. *ZDM Mathematics Education*, *46*(1), 15-27. doi: 10.1007/s11858-013-0548-9

Hattikudur, S., & Alibali, M. W. (2010) Learning about the equal sign: Does comparing with inequality symbols help?. *Journal of Experimental Child Psychology*. 107, 15-30.

Hegarty, M., & Koxhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology 91*(4), 684-689.

Hiebert, J. (1986). Conceptual and procedural knowledge: The case of mathematics. Mahwah, NJ: Erlbaum.

Hunter, J. (2007). Relational or calculational thinking: Students solving open number equivalence problems. In Watson, J., & Beswick, K. (Eds) *Mathematics: Essential Research, Essential Practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia). *1,* 421-429.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 36*(4), 297-312.

Lowrie, T. & Kay, R. (2001). Relationship between visual and nonvisual solution methods and difficulty in elementary mathematics. *The Journal of Educational Research, 94*(4), 248-255.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education, 43*(3), 316-350

McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*(2), 285-306.

Middleton, J. A., van den Heuvel-Panhuizen, M., & Shew, J. A. (1998). Using bar model representations as a model for connecting concepts of rational number. *Mathematics Teaching in Middle School, 3*(4), 302-312.

Murata, A. (2008). Mathematical teaching and learning as a mediating process: The case of tape diagrams. *Mathematical Thinking and Learning, 10*(4), 374-406.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.

New York State Education Department (2015) *Engage New York Common Core Modules* Retrieved from https://www.engageny.org/resource/grade-1-mathematics

Ontario Ministry of Education. (2014). *Paying Attention to spatial reasoning: support document for paying attention to mathematics education.* Retrieved from https://www.edu.gov.on.ca/eng/literacynumeracy/LNSPayingAttention.pdf

Saenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics, 35(2), 153-187.*

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits for solving problems with manipulatives. *Journal of Educational Psychology*. *101*(1), 88-100

Smith, J., (1996). Mathematical modeling. *Mathematics in School 25*(4), 14-17.

Stephens, M., & Armanto, D. (2010). How to build powerful learning trajectories for relational thinking in the primary school years: Mathematics Education Research Group of Australasia

Suh, J., & Moyer, P. S. (2007). Developing students' representational fluency using virtual and physical algebra balances. *Journal of Computers in Mathematics and Science Teaching, 26*(2), 155-173.

Tall, D. (1994). A versatile theory of visualization and symbolization in mathematics. *Comission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques*.

Institute of Education Sciences. (2015). [Tables of school demographic information for the 2013/2014 school year]. *School Directory Information: Morley Nelson Elementary*. Retrieved from http://nces.ed.gov/ccd/schoolsearch/school_detail.asp?Search=1&SchoolID=160036000980&ID=160036000980

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), p. 9-35.

Warren, E., & Cooper, T. J. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal, 17*(1), 58-72.

Yin, R. K. (2014). *Case study research: Design and methods.* Thousand Oaks, CA: SAGE Publications.

APPENDIX A

Lesson Framework and Instructional Tasks

*Alternative number sets are given as extensions if time allows or as a way to adjust the original sets if they seem too challenging or easy for students.

Session Number	Original Contexts Engage New York Grade 1 Module 6 (Topics A, E, and F)	Modified Contexts
1 Focus: Comparing quantities	Rose wrote 8 letters to her friends. Her goal is to write 12 letters. How many more letters does she need to write to meet her goal? Rose wrote 8 letters. Nikil wrote 12 letters. How many more letters did Nikil write than Rose? *Engage NY, Grade 1 Module 6, Topic A Lesson 1. (G1. 6.A.1)*	Rose wrote 8 letters to her friends. Her goal is to write 12 letters. How many more letters does she need to write to meet her goal? Rose wrote 8 letters. Nikil wrote 12 letters. How many more does Rose need to write to have the same amount as Nikil? 8+__=12, 12-8=__
2 Focus: Comparing quantities	Tamra found 12 ladybugs. Willie found 4 fewer lady bugs than Tamra. How many ladybugs did Willie find? *Engage NY, G1. 6.A.2*	Tamra had 12 ladybugs. Willie had 4 fewer lady bugs than Tamra. How many lady bugs did Willie find? 12-4=__ Tamra had 12 lady bugs. Willie had 8 lady bugs. How could we make it so Tamra and Willie had the same amount of lady bugs?
3 Focus: Decomposing quantities in different ways.	10 pennies is the same as 1 dime. 2 nickels is the same as 1 dime. How many cents is a nickel worth? Find a way to use pennies, nickels, and/or dimes to make 15 cents. *Engage NY, G1. 6.E.20*	What are different combinations we could use to compose the number 10? What relationships or patterns can you find between the different ways to compose the number? (Other number sets: 8, 15, 21)
4 Focus: Decomposing quantities in different ways.	This is a quarter. Use your coins (pennies, nickels, dimes, quarters) to compose 25 cents in different ways. *Engage NY. G1. 6.E.21*	Some ants got into the classroom and got onto the tables. One table had 10 ants. The other table had 3 ants. How many ants would need to join those 3 ants in order for both tables to have the same? (Other number sets: 10, 5; 25, 5, 25, 10) *Adapted from problems created by Barlow & Harmon (2012).*

5 Focus: Analyzing student thinking and determining equations to be true or false.	NA	One table had 5 blue ants and 1 red ant. The other table had 4 blue ants and 2 red ants. Do the tables have the same amount of ants? Jesse wrote the equation 5+1=4+2. He said the tables have the same amount of ants. Is his equation true or false? How do you know? (Other number sets: 4+6=3+7, 2+1=1+2) *Adapted from problems created by Barlow & Harmon (2012).*
6 Focus: Making two sets equal	NA	One table had 4 red ants and 5 blue ants. The other table had 3 red ants. How many blue ants would need to join those 3 red ants in order for both tables to have the same? (Other number sets: 4r, 3b, 4b; 8r, 2b, 7b) *Adapted from problems created by Barlow & Harmon (2012).*
7 Focus: Comparing quantities and making two sets equal	Anton caught 10 fireflies. He caught 7 fewer fireflies than Julio. How many fireflies did Julio catch? *Engage NY G1. 6.F.25*	Anton had 10 fireflies. He had 7 fewer fireflies than Julio. How many fireflies did Julio have? How could we make it so Anton and Julio have the same amount of fireflies? Anton had 10 fireflies and 8 beetles. If Julio had 17 fireflies, how many beetles would he need to have the same amount of bugs as Anton?
8 Focus: Comparing quantities and making two sets equal.	Maria used 14 beads to make a bracelet. Maria used 4 more beads than Kim. How many beads did Kim use to make her bracelet? *Engage NY G1. 6.F.25*	Maria used 14 beads to make a bracelet. Maria used 4 more beads than Kim. How many beads did Kim use to make her bracelet? Maria used 10 red beads and 4 blue beads to make her bracelet. Kim used 10 red beads. How could we make Maria and Kim have the same amount of beads?
9	Maria used 14 beads to make a bracelet. Maria used 4 more	Maria and Kim used the same amount of beads for their

	beads than Kim. How many beads did Kim use to make her bracelet? *Engage NY G1. 6.F.25*	bracelets. Maria used 8 red beads and some blue beads. Kim used 6 red beads and 4 blue beads. How many blue beads did Maria use? (Other number sets: 10+__=8+4; 1+7=___+6)
10 Focus: Determining equality	NA	Maria and Kim were making bracelets. Maria used 4 red beads and 3 blue beads. Kim used 6 red beads and 2 blue beads. Did they use the same amount of beads? How do you know? (Other number sets: 10+3=9+4, 36+27=35+28; 37+27=38+27).

APPENDIX B

Pre- and Posttest

Examples of questions similar to the PMA diagnostic for relational thinking.

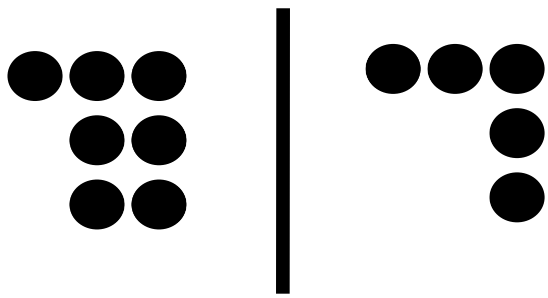
Section 1: Concrete visual based tasks

1. Count the eggs in each basket.

   a. Are they the same?

   b. How can we make them the same?

2.



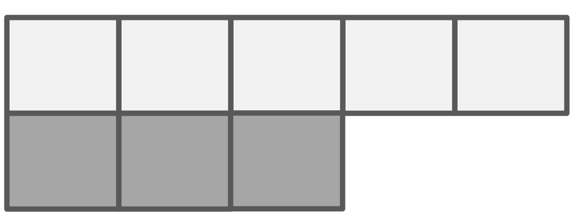
   a. Are there the same number of dots on each side?

   b. How can we make both sides the same?

If answering the above problems in the same way then ask:

   c. Is there another way you could do it?

3. Here are some rows of blocks. How can we make both of them the same amount? (1 point)



Section 2: Symbolic based tasks

4. Look at this symbol.

   =

   a. What do we call this?

      (1 point)

   b. What does it mean?

5. What number should I put in the missing space so that each side will add up to the same amount? (1 point)

   $$7 + \square = 6 + 2$$

   How do you know?

6. What number should I put in the missing space so that each side will add up to the same amount? (1 point)

   $$5 + \square = 1 + 5$$

   a. How do you know?

7. What number should I put in the missing space so that each side will add up to the same amount? (1 point)

   $$4 + 3 = \square + 2$$

8. What number should I put in the missing space so that each number will add up to the same amount? (1 point)

   $$8 + 2 = \square + 8$$

9. Does 9 equal 6 plus 3? (1 point)

$$9 = 6 + 3$$

a. Can I write it this way? (1 point)

b. Why or why not?

10.

$$37 + 13 = 36 + 14$$

a. Are both sides equal? (1 point)

b. Tell me how you know.

11.

$$34 + 5 = 35 + 6$$

a. Are both sides equal? (1 point)

b. Tell me how you know.

APPENDIX C

Transcripts

Bar Model Group Transcripts

Bar Model Session 1

T: I'm going to give you a story problem. Here's the story I want you to try and solve. I also want you to draw a picture. There was this girl named Rose, and she was writing letters. Rose wrote eight letters. Like "Dear so and so…" Her goal was to write twelve letters. Her goal was to write up to twelve. I want to know, how many more letters does she need to write to make her goal? *Pause* Do you want me to read the story again?

Mumblings of yes

T: I encourage you to try and draw some kind of a picture, some kind of a visual on your paper to solve this problem. You can also use cubes if you want *(reread the story problem).* Go ahead and start working on that problem. See what you can do. I want to see your thinking.

Richard: So she wanted to get up to twelve?

T: Yes. Rose wrote eight letters. She wants to write up to twelve letters. How many more does she need to write?

Six of the students grab cubes to start solving the problem.

T: If you have solved it, go ahead and try to write an equation.

Samuel: Like a number sentence…

T: Yes, an equation is another name for a number sentence.

Students work independently to solve the task.

T: Can you explain to me what you're doing here?

Jayme: I'm trying to make twelve blocks. And eight, um, eight blocks and twelve.

T: Ok

Jayme: I'm trying to make it up to twelve.

She is making two towers

Becky has created two towers and is drawing the blocks from each tower onto her paper. Richard has created a bar model showing the units of ten and one to make twelve. He drew another bar model to represent the eight letters.

T: Alright. Who thinks they have a way they'd like to share?

(*Beth, Becky, and Richard raise their hand)*

Richard: What I did, I drew a bar model. I drew eight for the eight letters that Rose wrote. Then I drew twelve at the very bottom. I broke up my bar model to only eight. Then I counted the rest that were left. There were four left. That's how I got my answer four. I did eight plus four equals twelve.

T: Did it look something like this? (*Drew his bar model on the board)* You started with your unit of ten, two more, and then you subtracted two.

Richard: I drew lines down and then I wrote the number below.

T: So you knew this was what you wanted… and this was eight. *(Showed the parts on the board)*

Richard: Yes.

T: This goes very close to there, and so you knew there was four more.

Richard: Mmhmm

T: Did someone solve it a different way?

Beth: I drew a number line. I circled the eight and then I jumped to twelve.

T: So you had something like this? (*Drew an open number line on the board starting at* *eight).*

Beth: Except I went from one to eight, and then….

T: Ok, so you actually showed all of the eight here, and then counted until you got to twelve. Then knew that was four?

Beth: Uh, huh.

Scott: I made tally marks. I made eight, and then I made four more, and that's how I got my answer.

T: So …five, six, seven, eight (*drawing tally marks)* and then you counted on tally marks until you got to twelve? You went one, two, three, four? *Drawing on the board.*

Scott: *nodding*

T: Ok. So what did the equation look like for this problem? Did anyone write that number sentence?

Richard: Eight plus four equals twelve.

T: Ok. So to match the story, we had eight plus…originally we had eight plus something equals twelve. And what was that something?

Jeff: Four

T: That was four. What does that symbol mean? We've kind of talked about that already. We know that name of it, but what does that mean? Equal? *(Samuel, Jeff, and Becky raise their hands)*

Samuel: Something is the same as the ….

T: This means that they're the same right? So who could read this sentence using, instead of the word equal, let's use the words same as?

(Richard and Beth raise their hands)

Richard: Eight plus four is the same as twelve.

T: Ok, eight plus four is the same as twelve? Do you agree that that is a true statement?

Mumblings of yeah

T: Is eight plus four the same as twelve? I want you to look at Richard's model and show me how you could prove that eight plus four is actually the same as twelve. We don't want to just think about the twelve as the answer, but that eight and four is the same as twelve. Who thinks they could come up and show us using Richard's model?

(Richard and Beth raise their hands)

T:Someone besides Richard and Beth. I appreciate their participation. Is anyone feeling like they want to participate? Anyone else feeling brave?

Jayme: She had to add four more letters to make twelve.

T: Can you come show me that using Richard's model? Come show me where eight and the four more to make the same amount of twelve is.

Jayme: She stopped at eight, but she wanted to go up to twelve…

(Teacher pointed to the bar model as Jayme explains.)

T: So she stopped at eight?

Jayme: Yes, but she wanted twelve, and you had to add four more to make twelve.

T: Hmm, So what if I had for this top bar model, like Richard had. What if I put four more blocks, what would I have?

Jeff: Twelve.

T: I'd have twelve. Would that be the same as this block down here that has twelve?

Mumblings of yes.

T: So now I want you to solve this problem. This time it's a little bit different. Rose wrote eight letters. This is another girl; her name is Nikil. Nikil wrote twelve letters. How many more letters does Rose need to write to have the same amount as Nikil?

Richard: This one's kind of easy.

Samuel: This is the same.

T: Oh, it's pretty similar, but is it exactly the same.

Mumblings of no

Jayme: There's two girls.

T: Before you solve it, you guys said this one will be kind of easy. We kind of already did this one. Rose wrote eight letters and Nikil wrote twelve letters. How many more does Rose need to write to have the same amount as Nikil? How is this the same as the last problem? I want you to turn to (*gave partners)*. I want you to talk to each other about how this problem is the same as the last problem, and how is it different? What are the two things you're going to talk about.

Chorally: The same and different

Side conversations:

Jeff: They have the same numbers.

Beth: The last one had one girl, and this one has two girls.

Scott: This time there was two girls, and the other one there was only one.

Richard: They're going to get the exact same answer; there's just that there's another person in there. There's a new person that wrote twelve letters.

Whole group

T: Let's start with how is this the same?

Jeff: It still equals ten, and it still equals twelve.

T: What do you mean it still equals ten?

Jeff: I said twelve.

T: So we still have those same two numbers. Eight and twelve. Do you all agree?

Mumblings of yes

T: How else is this the same as the last problem?

Samuel: It has the same numbers.

T: Ok, so we have those same numbers, eight and twelve.

Scott: Um…….I forgot

Jayme: The same about it is you have to add up to twelve. You only have to add four more

T: So we still are trying to get, because we want Rose to have the same as Nikil, so we want her to get up to twelve. Scott did you remember?

Scott: It's different because there's now two girls.

T: So now we have two girls. We have Rose and Nikil. We're going to work on drawing a bar model. Get out your pencil and flip your paper over. I want you to write Rose.

Richard: Or we could just put an R.

T: Yep, or you could just put an R. That would be a fast way. (*Modeled writing names on two different lines*.) Then I want you to write, Nikil. We're going to make two different bar models because now we have two different girls. Before, Richard chose to do two bar models, but we could have just done it in one bar model. What I want you to do first is I want you to create a unit. I want you to iterate that unit, that means copy it, until you have eight. We're going to show how many letters Rose had. I want you to be as precise as you can. Your units should be the same size.

Students are creating their bar models on their paper

T: Above it, I want you to label that it's eight.

Samuel: I'm done

T: Make sure that you've labeled that it's eight. I want you to make sure that yours looks like mine. I have no gaps, that means no holes in the middle, and I have no overlaps. And it's going horizontally, that means side to side. I want you to make yours look like mine. If it doesn't you don't necessarily need to erase it, but I want you to make a new one.

Pause for drawing

T: Now I want you to do it for Nikil. I know you have been using units of ten and units of one. Today I want you to just use units of one. I want you to make twelve using just units of one.

Students are independently creating a bar model to represent NIkil.

T: Then I want you to label that with what number?

Becky: Twelve.

Pause

T: There's one more thing I want you to check. I'm seeing some models where you have these, and then your twelve is all the way, that much bigger. (*Drew a disproportional bar model on the board)* Is that appropriate?

Mumblings of no

T: Why not? Most of you are shaking your head. Why would that not work?

Beth: Because it's overlapping.

T: Well yeah, there are some overlaps. Also is twelve that much larger than eight?

Chorally No

T: Nope. Some of you also have your eight, and then you have your twelve and it's only that much bigger. I've seen that in some classes. Is that appropriate?

No

T: Why not?

Beth: Because it's supposed to be four instead of just one more.

Jeff: That looks like it's nine.

T: Yeah, that looks like it's only one more. Make sure you're using precision; that your units of one are the same for both. Ok, last thing I want you to label. We're going to put a question mark right there because that's what we're trying to find.

Richard: In Nikil?

T: What might an equation for this problem look like? I want to know how many letters Rose needs to have to have the same amount as Nikil. What might a number sentence for that look like? Does someone want to come write it on the board for me?

Becky, Beth, and Richard raise their hands.

Becky comes to write the equation on the board. She writes 12-4=8

T: Thank you. She said 12 subtract four is the same as eight. Is there another way I could write this?

Samuel comes to the board. He writes 12-8=4

T: Who can read that number sentence for me?

Jeff: Twelve minus eight equals four.

T: Who can use the vocabulary same as? You're right, but who can say it in the other way?

Richard: Twelve take away four...

T: This one.

Richard: Oh, twelve take away eight is the same as four.

T: Is there another equation you can think of? Ooh, look at all these hands come up. Beth can you come up?

Richard: Ah, really?

Beth writes 8+4=12

T: She had eight plus four is the same as twelve.

Richard: There's still another way.

T: There is still another way, but we're going to stop because we don't have a lot of time left. There's a lot of equal signs in here. Does equal always mean that's the answer?

Mumblings of No

T: Hmm, because we have a lot of different things going on, but what was our answer to the problem?

Jayme: Twelve.

T: No, what was our answer to the question?

?: Four

T: Our answer was four. On this problem here's our answer, here's our answer, and here's our answer.

Teacher circled the four in each equation.

T: The equal doesn't necessarily mean that the answer comes next. It means that they're the same. Huh…Who can use our model here to explain why twelve subtract four is the same as eight? Let's focus on this problem, or this equation. Who can come use our model to show why that's true?

Jayme and Richard raise their hand

Jayme goes to the board. She points to the numbers in the equation.

T: Where is it in our model though? I want you to connect it to our bar model over here.

Jayme: It's right here. *(pointing to the twelve blocks)*

T: So it's all of these blocks? Where's the subtract four?

Jayme: *(pointing to the equation)* Twelve take away four that would be eight.

T: Where's that in our model?

Jayme: It would be the same as eight. *(points to the eight in the model)*

T: Because if I chopped it off right here? What would I have?

Three students are putting their heads on the desk. Only some are participating

Students: Eight

Richard: Four more

T: So I had twelve and I took away four that would be same amount as…

All: Eight

T: Can anyone come up here and show us where the twelve take away eight is the same as four using this model.

Jeff comes to the board

Jeff: So it's kind of like the other question but you take away twelve (points to the four in the bar model) instead of four which is…

T: I'm taking away the what?

Jeff: Twelve *(points to the eight in the bar model)*

T: Check the equation.

Jeff: Eight.

T: So I have this twelve *(pointing to the twelve)* and now I'm taking away what part?

Jeff: Eight (points to it in the bar model)

T: This eight, and that would be the same as what? *(points to the eight in the bar model)*

Most: Four

T: That would be the same as four. Does that equation work?

Mumblings of yes

T: Are those equal? If I had all twelve, and I took away eight of those what is that the same as?

Most: Four

T: Can anyone show us where that eight plus four equals twelve is? Where is that in our model?

Pause…No hands

T: Eight plus four is the same as twelve. That one might be a little bit trickier. *(Beth and Richard raise their hands)* Alright Richard. You've been waiting patiently. Will you go show us?

Richard: Here and you take them away. There's four here and you add them to here, which equals to twelve. You have the same as eight right here. You would have the same bar model.

T: Let me see if I'm understanding you. If I have this eight, if I were to put four more onto here (*drawing four extra cubes to the eight bar model).*

Richard: Mm,hmm

T: How many do I have now?

Mumblings of twelve

Scott: Twelve

T: Is that the same as this twelve?

Responses of yes

T: So what does equal mean again?

Samuel: The same as

T: It means the same as. Does it mean that's where our answer is always going to be?

Responses of no

T: Tomorrow I will give you a little bit of a harder problem.

<u>Bar Model Session 3</u>

T: Here's your task. Today we're going to work on different ways to compose or decompose the number ten. I want you to draw bar models and write equations for the ways that we can compose. Touch your nose if you've heard the word compose before or decompose? *(no one touches nose)* Compose means to put together. Say that together,

Chorally: compose, put together

T: So what do you think decompose means? It's the opposite.

Becky: It means take apart.

T: You got it. If going to decompose ten then I'm going to break it apart. If we have ten, I want you to think of all the ways we can break it apart into smaller chunks. Does that make sense? I want you to come up with at least three different ways. If you can do more than three, do it, but you have to have at least three. Draw bar models and write equations.

Jayme: So we have to compare with the number ten?

T: Yes, so you're trying to break apart the number ten. One thing that might be helpful; what will your total amount be?

Students: Ten

T: You're total amount is going to be ten, so each of your parts should be what?

Students: Five

T: Well maybe, but they should be smaller than ten right. Ten shouldn't be one of the small pieces. Think of different ways you can compose ten.

(students begin working independently)

T: Beth, I want you look at this for a minute.

Samuel: I don't get it at all.

Richard: Do you have to draw units of ten?

T:(*to Beth)* Here you have, so five and five do make ten but that looks like a smaller size than that. Does that make senses? Is ten smaller than five? So where should the ten go?

Beth: At the top.

Richard: Do you draw in a ten bar or in units?

T: You can draw a ten bar or individual units, but you don't want to do units of ten. Get started, and I'll come check in on you in a minute and see if you need to tweak it.

To Becky and Jeff

T: Becky can you explain to Jeff why you chose to draw this model?

Becky: Um, I chose four and six because they equal ten.

T: How did you decide what your model would look like?

Becky: I draw four and then colored them in. Then I drew six and I didn't color them in.

T: You drew six more and that made a total of ten. Jeff, does that help a little bit?

To Jayme

T: Can you make another way. So five and five. See if there's another way. If we have it back to ten, is there another way we could break them?

Jayme broke her cubes into eight and two

T: Ok, so what two parts have you created then?

To Beth:

T: One thing Beth that I want you to switch just a little bit. Six is a larger number than four so I want you to actually show that. I'm going to use your same idea. We have ten you said, and then you have six. Do you see how my six is a larger bar than four? Why would I do it that way?

Beth: Because six is a little bigger than four.

T: Mmhmm, and I want to show that relationship. I want to show that six is larger than four. As you continue, I want you to think about the size of your numbers and change your bar to represent them.

To Scott:

T: There's one way. What's another way we could break these apart?

(uses cubes to break his ten stick into four and six)

T: How did you break that one apart?

Scott: I looked, um, I did four and six. These six.

T: Now show me a bar model for what that would look like. Here you had five and five. Now you have what? (pause) six and …

Scott: Four.

T: Six and four. Draw a bar model for that way.

To Becky and Richard:

T: Did you do three? See if you can find another way. Can you find all of the ways to decompose ten Richard?

Richard: Just with the equations?

T: No, I want the bar models. I'm pushing you today. Think of all of the ways you can decompose it.

Richard: Is there like a billion ways?

T: Nope

Samuel: There's probably nine.

T: Why do you think there's nine ways?

Richard: There's probably at least ten ways.

T: Ok, why do you think there's at least ten ways?

Richard: Because there's 1 plus 9, seven plus 3, 8 plus 2, seven plus 3, 8 plus 2, 9 plus 1, six plus 4, 1 plus 9, 9 plus 1.

Samuel: You already said that.

T: He said 1 plus 9. Is nine plus 1 the same?

Samuel: Ten, well at least twenty.

Richard: Five plus five.

T: You think there would be at least double, because we could write some things in two different ways. Why don't you list them and see if you can find all the ways. You've got them in your head; put them on your paper because I'm not in your brain.

Richard: Do we still have to draw a model for every single problem?

T: Yes, because I want you to see how your model is changing.

T: *(to Becky)* will you put this one on the board for me? Actually reverse it so that it's 1 plus nine. Just color in one of them.

T:(*to Jeff*) Will you go put this one on the board for me? *(2+8)*

Side conversations

Samuel: One plus nine equals seven plus three equals eight plus two equals five plus five equals six plus four plus nine plus one equals three plus seven equals two plus eight equals four plus six. There's *(counting)* ways.

Richard: There's also subtraction problems that you can do to equal ten.

Samuel: Cool.

Richard: You can do so many subtraction problems. There are so many ways you can do it.

T: You have all those ways, put them on your paper.

Richard: All of them?

T: As many as you can.

Samuel: I want to write all of the ways down

Richard: You know you have to do the bar model. You have to do every single bar model.

T: Draw me a model for what three plus seven would look like.

Jayme: It's right there. *(pointing to her paper)*

T: This is an equation. Draw a bar model for it.

Beth: I already have six. I've got six already done.

T: How is this different? Here you put six plus four, but here you have four plus six. How might that change what your bar model looks like?

T: Ok, Scott. What would the equation look like for this? How many are colored?

Scott: Well, I'm going to do a different way than that way.

T: Ok, but I want you to write the equation for this way first. How many are colored?

Scott: Six

T: Let's write the equation for that.

Whole group

T: There are a lot of different ways, but I wanted our friends to put up these specifically. What was the way that Becky came up with? What do you guys see?

Jayme: A bar model and a number line.

T: You mean an equation?

Jayme: Yes

T: Ok, but how did she choose to make her model? What do you notice? Richard?

Richard: That she drew the ones and tens and then she colored in one.

T: So she colored in one which means how many are not colored?

?: Nine

T: Nine. I could also write that as one and nine, and how many does that make all together? (*drew an open bar model on the board showing one and nine)*

Students: Ten.

T: Ten. What do you notice about my bar model that might be a little bit different than how Becky did it? There's something a little bit different.

Jeff: You didn't do squares, and you didn't color in anything.

T: Yep, instead of coloring those numbers in, I chose to do it as an open bar model. Is my one the same size as my nine?

Students: No

T: Why didn't I make them the same size? In other classes, I've seen them do this. (*draw a disproportional bar model on the board)* I see one and nine. This is not the best way to do it. This would be, if we're going to do it that way, *(an open bar model way)* the best way. Why? Beth?

Beth: Then you know that nine is bigger than one instead of having nine and one be the same size.

T: Here it looks like nine and one are equal right. It looks like they're the same. Are they the same?

Students: No *(one audible yes)*

Samuel: The one is actually bigger *(referencing the drawing)*

T: Or the one might even be a little bit bigger. That's not cool. Does my picture more closely resemble what Becky's looks like, where we have this one section and this much larger nine section. Do you agree?

Students: Yes

T: What did Jeff do? His is a little bit, he made his squares a little bit smaller. So what did Jeff do?

Samuel: Um, I don't know.

T: Can someone help him out?

Jayme: Um…

T: Well what was his equation?

Jayme: One plus eight equals ten.

T: Two plus eight equals ten.

Jayme: Yes.

T: How many did he color in Scott?

Scott: Two.

T: How many are uncolored Richard?

Richard: Eight.

T: Huh, so if I were make this equation in my way it might look something like… *(drew an open bar model)*

Jeff: Why did you put two on the front?

T: Why did I put two right there?

Becky: Because two becomes…because two is before eight

T: Because he had two colored in and how many are not colored in?

?: eight

T: There are eight that are not colored, but they both equal what Samuel?

Samuel: They both equal ten

T: They both have ten squares. Do mine both equal ten?

Students: No

T: Not necessarily squares, but do they both show ten. What's different? What changed between Becky and Jeff's?

Long pause (Samuel and Beth have their hands in the air)

T: I need everyone thinking about my question. So think about more specifically what changed in the number that they had colored in and they number they had uncolored. Go ahead and turn to (*gave partners*). I want you guys to talk about what changed between our two pictures. Ok, what's changing?

Side conversations:

Becky *(to Jeff):* I had one colored in and you had two colored in.

Beth *(to Scott):* So Jeff had two and Becky had one. So I had one colored in and you had

Scott: Becky did the higher number of then and Jeff did the lower number, but…

Beth: inaudible

Whole Group

T: Who can tell me what changed?

Samuel: Um, Richard said the size changed.

T: Ok, they got smaller squares. That's why I wanted you to think about what changed in the numbers they colored.

Samuel: I said the numbers changed

T: Ok, and the numbers changed. How did the numbers change?

Beth: Becky had one and Jeff had two.

T: Jeff had two. Could I say that Jeff colored in one more than Becky?

Richard: may…Yeah

T: What happened to the number that they didn't color? If Jeff colored in one more, then what happened to the number that didn't get colored?

Jeff: It changed into the same as Becky's

T: You have the same number uncolored as Becky?

Jeff: If you took away, if you took away… wait what was it?

T: What happened to the number that was uncolored?

Long pause

T: You colored in one more, so what happened to the number that were not colored? *(pause)* Hmmm. Let's see if we can do it with this third one. *(referencing the model on the board for 3+7)* What equation did I write?

Scott: Three plus seven.

T: Here I have three plus seven. Jeff is it ok if I redraw this so it's a little bit larger so it matches ours? He had two plus eight. Becky had one plus nine. *(redrew Jeff's)* Jeff do you agree that that's the same as yours?

Jeff: Yes

T: *(pointing to the colored in section of each model)* So what's changing?

(Three students raised their hands)

T: Turn to your neighbor and tell them. What's changing in those three models?

Side conversations:

Scott: They're going like number by number. It's going one, two, three.

Beth: Each time it's going higher and higher.

T: How much? What's going higher and higher and higher? The colored spaces. What's happening to the other spaces?

Scott: It's going, the long one, it's going down. Nine, eight, seven.

Richard: The two, oh, the coloring one, two, three.

Samuel: *inaudible*

Richard: *(answering my questions to Beth)* They're getting shorter.

T: They're getting shorter by how much?

Richard: By one.

T: What's happening to my colored spaces?

Richard: They're getting bigger, and bigger, and bigger.

Whole group

T: What do you notice? What's happening?

Scott: The numbers are going like one, two, three.

T: We have one, two, three. *(referencing the colored in section of the bar model)* So the numbers are changing. What's changing in our models?

Richard: Ooh, that every space, Becky colored in one, Jeff did two, and Miss Amber did three.

T: Ok, so the colored spaces are getting one larger. Do you agree with that?

Students: Yes

T: What's happening to the uncolored spaces then?

Jeff: They're getting shorter, and

T: I'm going to push your pause button. I think you have a good idea, and I want to make sure they can hear it. Alright Jeff.

Jeff: On the numbers in the number sentence they're going down.

T: Oh, ok. Let me see. He said there's one plus nine, I have two plus eight, and three plus seven. So let me see if I'm understanding…

Jeff: They're going from nine, eight, seven, and the one before, is one, two, three.

T: If these are going down, where is it actually going? They still equal ten, right? They're still ten. So what's happening to that one? That's going… Does it just disappear?

Richard: No

T: I threw it away? Where did it go?

Beth: It went into the ten.

T: Let's look at our model. What happened? Here we had nine. Here we had eight. Where did that extra one cube or extra one go?

Beth: It got colored in.

T: Oh, it went over into our…

Richard: Colored in one.

T: Colored section. What happened here? *(referencing the number sentences on the board)* Where did that one go?

Richard: Into the colored section.

T: If I had something like this… *(wrote 2+8=3+7 on the board)* Can I write that?

Students: Yeah

T:is that ok?

Students: Yeah

T: Ok so what about this *(wrote 3+7= __ +6)*? What happened to this number? *(underlined the seven and the six)*

Richard: It got changed into the colors.

T: What happened to that number?

Becky: I forgot

T: What happened? First it had a seven, now I have a what?

Students: Six

T: What happened to it?

Richard: One of the seven got colored.

T: The seven went down by one. What's going to happen to this three? Someone other than Richard. What's going to happen to make it the same?

Samuel: Um

T: If this went down by one, the number that was uncolored went down by one, what's going to happen to this one?

Samuel: What went down by one?

T: We had seven, now we have six. It's going down by one. What's going to happen to this three?

Samuel: um

T: Richard said something like, it's going to be what Richard?

Richard: It's going down

T: You said it's getting colored.

Richard: Yeah.

T: If this represent the colored one *(pointing to the three)* and we're getting another one that's colored, what will this be?

Scott: A four.

T: It's going to be a four. It's time to go, but I want you to try one more. Four plus six equals something plus 5.

T: What happened to this one *(referencing the six and five)*

Beth: It got colored.

T: It went down, it got colored. So what's this number going to be? It went down by how many…

Students: Five, one.

T: By one, so this one is going to go down by one or up by one. Jayme what do you think?

Jayme: Up.

T: It's going to have to go up by one so it's going to be…

Students: Five.

<u>Bar Model Session 6</u>

Scott was Absent

T: We're going to do this problem together because there's a system I want you to use. Then we'll change the number sets, and you'll do the next one on your own. Who can read the problem for us?

Beth: One table had four red ants and five blue ants. The other table had three red ants. How many blue ants does that table need to make both tables have the same?

T: We're going to make the bar models together today. We're going to first make a bar model for table one. How might I do that?

Becky: Draw four and color them in so you know they are red.

T: *(Drew four individual cubes and colored them in. Labeled below it four.)* Everybody do that please. Draw four.

Jayme: Like cubes?

T: Yep. And above it you'll label four and color them in so we know they're red ants. How am I going to show the blue ants?

Samuel: Five.

T: Yeah, I'm going to do five. I'm going to do an open bar model for that second half. I want you to have it this way. We know the last part of blue ants is just…

Student responses*: five*

T: It's just five. We know all together I have how many? Say it if you know it.

Students: Nine

T: Nine ants. Ok, that's the first table. Now we have to do the second table. The other table though only had…

Richard: Three

T: *(Drew three cubes and colored them in. Labeled below as three.)* Three red ants, and we want them to have the same.

Students are drawing this on their papers

T: What do we need to get to? *Pause…* I need to get to the same distance right?

Jayme nods her head

T: Yeah. I need to get to that nine, so I'm going to make an open bar model that goes all the way to the end. Do I know what that number is yet? You might know, but based on my problem, if I haven't solve anything, do I know what that number is yet? No, so I'm going to put a question mark inside.

Students are adding to the bar model on their paper. Let's right an equation for this. For the first table, what might that equation look like?

Becky: Four plus five equals nine.

T: Yep, but I'm going to leave out the equals nine today. *(Wrote the equation on the board)* I'm just going to write four plus five equals, and I want it to equal my second table. What might that look like?

Jeff: Three plus…*paused*

T: Three plus, and I like how you took a pause. We don't know what that is do we? Four plus five equals three plus (*wrote the open number sentence 4+5=3+□)* something. What do you notice about those numbers? What do you notice about the number we had colored in over here? How did that change?

Pause…Only a couple students are raising their hands even after think time.

Richard: You solve it the same way as the other problems we do every single day.

Had to give a conversation about participation and needing to think about the problem

T: I'm going to say my question again. What do you notice about these numbers, and what do you think might need to go in there? See if you can talk about how many we had colored and how that changed.

Four students were raising their hands.

T: I'm going to give you some think time, so hands down for now.

Paired students up to share their ideas.

Samuel: Can we say the answer?

T: Nope, I want you to talk about how those numbers are changing.

Side conversations

Jayme: I think it might be five again.

T: Why do you think it might be another five?

Jayme: Because another five might go to ten.

T: Was four and five ten? How much was four plus five?

Jayme: Nine.

T: If I did three plus five, would that be the same?

Jayme: No

Jeff: I think it’s five.

T: Why do you think it’s five?

Jeff: Because the other table has a five.

T: Oh, but if four plus five equals nine will three and five also equal nine?

Jeff: *inaudible*

Beth and Becky got cubes to create the two different tables

Richard: If you had three and you added more to get to nine. Four, five, six. The answer is six.

Whole group

T: Touch your nose if you figured out what that missing number is. Becky what’s that missing number

Becky: Six

T: Jeff I want you to come up here and use our model to point. How did you figure out it was six? Share what you told me in our small group.

Jeff: You take away one.

T: Can you come point? There were a few friends who were talking and were still confused.

Jeff: You take away one of those. *(Pointing to a colored cube in the first bar model)*

T: We could pretend like it's right there. *(drawing a dotted cube in the open section of the lower bar model*)

Jeff: You take away one…and then there's six right there *(pointing to the open section of the lower bar model).*

T: What would this section be? *(pointing to the part of the open bar model that is not the dotted cube)*

Jeff: Six.

T: The whole thing is six, but if there's one right here.

Samuel: Five

Other Students: Five

T: That's the same as my five, but I have one of those less that's colored. Now I only have three of them which then has to go to the five. Did anyone do it a different way? I liked his idea because he used that trading idea. Did anyone figure it out a different way? Richard how did you figure it out?

Richard: Can I come up there and point?

T: Yeah. How did you know it was six?

Richard: *(did not point to anything on the bar model)* I looked at my number line. I looked at my number line and counted until I got to six which equaled, I mean nine.

T: *used a tree model to show that four and five was nine and three plus six was* nine

Richard: I got six more which equaled my answer to four plus five because it equals nine. Then I got to nine. When I counted from three to nine, the answer was six.

T: Will you say that again, because I want to make sure those two hear your idea *(referring to a few students who were goofing off)*

Richard: I counted from two until I got to nine, and the answer that I counted…

T: Do you mean from three?

Richard: Yeah, and the answer was six.

T: He's trying to get to nine. Why is he trying to get to nine?

Becky: That's how many ants one table had.

T: Right, because he was thinking about the total. How was that different than how Jeff thought about it? Did Jeff ever really talk about that nine?

Samuel: No.

T: Hmm, so how was his idea different? Beth?

Beth: He… I forgot (*Beth was raising her hand, but was unable to verbalize her thinking.)*

T: Hmmm, what kind of a strategy did Richard use? Richard chose to…(*no responses from students)*

T: Find the same quantity, the same amount. Jeff chose to use a….

Becky: Trading

T: Trading or more kind of a relationship. I'm going to give you the same problem but with different numbers. They're going to be a little bit, they won't hopefully be super hard, but they're going to be a little bit bigger. This time one table had eleven red ants and three blue ants, and the other table had ten red ants. Hmm actually I changed my mind, twelve red ants. How many blue ants does that table need to have both tables have the same? I want to see if you can try Jeff's strategy.

Samuel: That's what I did.

T: That's what you did last time Samuel? Then this should be easy for you or make you think. See if you can try Jeff's strategy where he saw oh look, this one just moved. I'm only going to give you five to seven minutes to work on this problem. I want you to draw a bar model and write… I forgot to tell you. I want you to start with this equation (*wrote 11+3=12+___)* Here's one *table (pointing to one side of the equation)* and here's the other table *(pointing to the other side).* What's going to go in that missing space?

Pause. Students are working independently

Teacher is moving around and helping students create accurate bar models

Jayme is using cubes to help her solve the problem

*Richard and Jeff finish early. They are given the problem 24+3=23+*__

Richard: That's so easy.

T: You're got to prove it to me. You've got to justify it. How are you going to explain to me that it's true?

T: Alright Samuel, how do you know that it's two?

Samuel: Eleven plus three equals fourteen, and twelve plus two equals fourteen.

T: Can you look at the relationships between the numbers? How did they change?

Samuel did not have a response

T: You're right, they both equal fourteen, but how did they change? Look at your first two numbers. How did it change? *Pointing to the eleven and twelve*. What's the relationship between the eleven and twelve?

Samuel: Eleven is one less.

T: What about the three and the two?

Samuel: The two is one less than the three.

T: So one number went up by one and the other number…

Samuel: Went down.

Whole Group

T: That's going to be my first table. And that's going to be my second table I'm going to start with how many red ants I had. How many red ants did I have on my first table?

?: Eleven.

T: Eleven. So if you haven't done it yet, I want you to draw eleven. (*Drawing individual cubes in a bar model and coloring them in)* Am I done with my first table yet?

Murmurings of no

T: What else is on that table?

Jayme: Three

T: Three blue ants. *(Drew three individual cubes)* I have one table done. Beth, what's next?

Beth: Twelve.

T: Twelve *(Drew twelve colored in cubes)* There's my twelve. To end in the same spot, how many blue ants do I need? What do you notice about this Samuel? Before I had eleven and now I have…

Samuel: twelve

T: Huh. What happened? How many more do I need to have that same amount?

Becky: Two more to equal fourteen.

T: I need two more to get to that same spot. What happened to my eleven Samuel?

Samuel: Um, it's a twelve.

T: Ok, it now increased by one. What had to happen to my blue ants then?

Jayme: You're blue ants?

T: Yep, my red ants went up by one, what happened to my blue ants.

Jayme: They went down.

T: They went down by one ,so now how many does it take?

Jayme: Two

T: It takes two. Ok, I'm going to give you a hard number. It's going to be harder than the one I gave you *(pointing to Richard)*

Richard and Jeff: Yeah.

T: They're going to be higher numbers. Forty nine plus four equals 48 plus something. Let's talk about this. Let's draw an open bar model. Let's start with 49 in an open bar model because that's way too many to draw that many cubes. Is four going to be very big?

Murmurings of no

T: I'm going to do a little one because it's a lot, lot smaller than forty-nine. Now we're going to do forty-eight. Forty-eight is less than forty-nine. Oh there's my missing piece. What's my missing piece going to be? I want to do this without adding them up because adding them up takes too long.

Richard: It might take our whole school time.

T: Let's see if we can figure out, what is this number going to be? I have forty-nine plus four. Let's go back to this problem (*wrote 11+3=12+2 on the board)*. Eleven plus three equals twelve plus, we said it was two. How did the forty-nine change?

Samuel: The forty-nine went…. *(trailed off)*

T: How did it change? Becky thinks she knows, Richard thinks he knows, Beth.

Becky: The forty-nine went down by one.

T: Oh, so the forty-nine got less by one. Wait, that happened over here (*referring to the bar model of the previous problem)*. So this went down by one, what's then going to happen to the blue ants? It's going to have to do what?

Samuel: What happened to the four?

T: To the four so that it will stay the same? This went down by one, just like this went down by one *(referring to the previous problem),* so what's going to happen to the four?

Samuel: It's going to go up to five?

T: It's going to up by one because we have one more of those. You're so right.

Richard: So easy

Beth: Yeah, that was way easy…

T: This is your last problem, but you're not going to tell me the answer. You're going to write the equation and put in your answer. I'll give you two problems, you can choose which one you want to do. Thirty-three plus six equals thirty-four plus something, or you can do 48 plus twelve equals 47 plus something. You can choose which one you want to do. It doesn't matter to me, but this is your ticket out today. You do not have to draw a bar model. Just choose which one you want to do and fill it in.

Students work on their problems.

Side conversations

T: Did you figure out what it was? How did it change?

Jayme: *inaudible*

T: Remember this was our 11 red ants so we had a colored one. Now we have 34 red ants. What happened to our red ants?

Jayme: They got bigger.

T: By how much?

Jayme: Uh, the four is going to.

T: If we have one more red ant, what's going to happen to the blue ants so that they will stay the same? So that all the ants will stay the same?

Jayme: The blue ants?

T: Uh huh, the four is going to have the same number of ants. If I have one more red ant what is going to happen to my blue ants?

Jayme: Six.

T: I'm going to add six?

The rest of the class cleans up

T: Jayme, come here for a minute. Let's look at this problem *(the one on the board*) What happened to my red ants?

Jayme: The eleven got one more.

T: It got one bigger. What happened to my blue ants?

Jayme: The three got tooken away one.

T: It got one less. Why? Can you use my picture to explain what happened? Where are my eleven red ants?

Jayme points to the last cube that's part of the eleven

T: Just right here?

Jayme: No.

T: It's all of them right?

Jayme: Yeah.

T: Ok, so when it got one bigger what happened to that space?

Jayme: Um

T: It had to get…

Jayme: One more.

T: What happened to the blue ants?

Jayme: It got one more then the twelve ants.

T: It did? Before it was at three now it's at…

Jayme: Two

T: So what happened to the blue ants?

Jayme: This one got tooken away so they would have the same.

T: And it didn't go away, but now it became what?

Jayme: One of those.

T: It became red. So what do you think should go in here? *(pointing to her current problem 33+4=34+___)* If we now have an extra red ant what's going to happen to the blue? It's going to have what

Pause

T: It's going to have one…

Jayme: more

T: Nope cause this has one more red ants, so it's going to have one…

Jayme: less

Bar Model Session 8

(Scott was absent)

T: Maria used four red beads and ten blue beads to make a bracelet. Kim used some red beads and four blue beads for her bracelet. They used the same amount of beads. How many red beads did Kim use? We're going to first draw an open bar model to show this problem. Maria used four red beads and ten blue beads. Let's start with what Maria has. We're going to make that open bar model. *(writing the open bar model on the board)*

T: There's Maria. She has four red and ten blue. Let's start with the first part of the equation. *(writing the expression on the board)*. We're not going to write the equal sign yet. Wait what does the equal sign mean?

Responses of same as

T: What do we know about Kim?

Richard: Kim used some red beads and four blue beads for her bracelet, and we want to know how many does Kim need to have the same amount?

T: Are you sure that's my question? I know they have the same amount.

Jayme: So it's like the equal sign.

T: What do I mean by the word some? Do I know that amount or is that my mystery?

Responses of that's my mystery

T: That's my mystery. So we know that she used some red beads, we don't know what that is. *(drawing the open bar model on the board)* And that she used four blue beads. So how could, if we know they have the same amount, what would the equation look like then for Kim?

Pause and no hands or responses

T:Let's look at my model. What's going to be first?

Beth, Richard, Jeff, and Becky raised their hands.

Becky: A blank.

T: A blank. We don't know what that is. Some red beads.

Richard: I know what it is.

Becky: I know what it is.

T: And we're adding it to.

Beth: Yeah, cause they have the same.

T: We have some plus what?

Beth: Ten.

T: Really, some plus ten? I didn't ask you what the mystery was.

Becky: Four.

T:Yeah, some plus four. Here is my equation. Four plus ten equals something plus four. What do you guys think it is?

Samuel: Are we saying the answer?

Becky: I think it's ten.

T: Do you agree with her? Touch your nose if you agree.

All but Jayme raised their hand.

T: Jayme, I see that you disagree with that.

Jayme: It's already four and ten.

T: Ok, so I'm going to write in what Becky thinks. She said four plus ten equals ten plus four. You disagree with that?

Jayme: I thought it was fourteen.

T: Where do you think she got fourteen from? Can someone explain where that fourteen is coming from?

Richard: The ten and the four.

T: Did you add how many Maria has? Did you add ten and four?

Jayme nods her head

T: But who does this represent? Does this represent Maria, or does this represent how many Kim has? *(Pointing to the missing section of the bar model)*

Jayme: Kim.

T: And we know that Maria, *(writing Maria above her part of the equation)*; we know she has the exact same amount as who?

Jayme: Kim

T:*(writing Kim above the other part of the equation)* So this number *(indicating the missing space)* represents the total amount that Maria has?

Responses of no

T: No, we want to find what Kim needs so that she has the same amount, which Jayme said was fourteen. What is ten plus four Jayme?

Jayme: Fourteen

T: The fourteen is there, but we don't write it. So where is the fourteen?

Samuel: Hidden

T: It's hidden. It's still there, but we've broken it apart. We've decomposed it into ten and four. So it's still there. What does this represent? We have a lot of the same number up there, but they don't represent the same thing? Should we color in the ones that are red so that we know there are two colors? Would that be helpful?

Responses of yeah

T: So what does this four represent? Looking at our model.

Beth: It represents how much…how…it represents it as fourteen for Maria

T: Just the four though. Fourteen would be the total. I want to know what just this four means.

Beth: She had four beads.

T: What kind of beads?

Samuel: Red.

T: What is this ten then?

Richard: A five and a five.

T: But in our story.

Richard: Oh, ten blue beads.

T: What is this ten then?

Samuel: Red

T: Whose beads? Is that this ten? *(indicating the ten in Maria's bar model)*

Jayme: No

T: Whose ten is it?

Jayme: Kim's.

T: Yeah it's Kim's red beads. So what's this four?

Jeff: Blue beads

T: I'm going to give you the same story but different numbers. Maria used nine red beads and three blue beads. Kim used some red beads and nine blue beads. I know again that they have the same amount of beads.

Jayme: This is going to be easy.

Beth: Oh, this is easy too.

Richard: I already figured out the number.

Samuel: Me too.

Jeff: very easy

T: You did, but now you have to prove it to me. Prove it to me using a bar model and an equation. When you write your equation, I first want you to do it, I'm going to use our number from last time, where we said equals something plus four. I want you to have it as an open number sentence where there's something missing. Just like our bar model.

Side conversations

Richard: Like this?

T: Exactly, but now you need to draw me a model of what that looks like.

T: Think about my story. Let's think about just Maria first. What does Maria have?

Jeff: Nine red beads. (*Paused)*

T: She has nine red beads and…

Jeff: Three blue beads, Oh

T: Remember do I want to know how many there are together Richard?

Richard: No.

T: Am I going to show my model where Maria and Kim are together or separate?

Richard: Separate.

T: Ok, so make your model like that for me.

T: This looks like Maria has more than Kim. Does Maria have more than Kim?

Becky: No.

T: Ok, how could you make your model so it's a little more accurate?

T: What does this nine represent?

Beth: Maria's red beads.

T: So red beads is first. Then what does this represent?

Beth: Three blue beads.

T: What does this nine represent?

Beth: Kim

T: Are those red or blue?

Beth: Blue

T: So if this is red and this is blue, this would need to be…

Beth: Red.

T: And this would need to be…

Beth: Blue.

T: Does that make sense? I know we can put them in either direction in addition, but it will make a difference when you make your model.

T: These are Maria's nine red beads?

Jayme: Mmm-hmm

T: And these are Maria's blue beads?

Jayme: Mmm-hmm

T: So you colored these in two. What color were Kim's nine beads?

Jayme: Nine

T: She had nine, but did she have nine red beads or nine blue beads?

Jayme: Blue

T: You colored in both of these which makes me think they're the same color. Is that what you meant to do?

Jayme: No.

T: Ok, put the red beads first just like you did here so you can see what's different about them. Do we know how many red beads she had?

Jayme: No

T: Ok, so we're going to just put a small box and put a question mark because we don't know that yet. Then we need to make it the same length. How long will this part be? We know that number of blue beads.

Jayme: Nine

Whole group

T: While they're putting up their model, I want you to look at what's the same about how they did it and what was different between what you had and what they have.

Jayme puts her model on the board. *Beth writes her equation on the board*.

T: Let's look at Jayme's model. Is there something maybe we need to change about it? What do you guys notice works about it? Let's start with that.

Richard: the starting…the ones at the starting….

T: So we have the red beads right? We have nine red beads and some red beads. What else do you notice about it?

Becky: It's not supposed to be a question mark at that other side.

T: Why not?

Becky: Because there's no bar around it?

T: There's nothing really missing here. What else do we need to do? If we're going to erase this question mark, now what do you notice? What might we need to do to make this model more exact?

No responses after a long pause

T: We know they are the same amount. Do they look like the same amount right now?

Jayme: Give that three more.

T: We're going to add three right here?

Samuel: Line up the nine.

T: Now let's look at Beth's equation. She has nine plus three equals something plus nine. Does that match your story?

Responses of yes

T: How does it match your story?

Richard: There's a nine and a three. Then Kim used some red beads, that's the question mark. And then she has nine beads. How many beads does she need to have the…

T: This is Maria and this is Kim, *(writing the names above their parts of the equation)* so what's going to go there? Touch your nose when you know it.

All touch their nose

Jayme: I think it might be three like the last time.

Samuel: Cause that's the same.

T: She thinks it's three. Do you agree?

Beth: Yes, it is three.

Richard: Yes

T: Why is it three? Why not twelve? Isn't nine plus three twelve? Why doesn't twelve go there?

Beth: Because it wouldn't be the same.

T: Then they wouldn't be the same because we still have these blue beads. Where is the twelve?

Jayme: It's hiding.

T: There's the twelve, and over here we still have that twelve. We still have the same quantity, but it's hiding. We're going to do the same problem, different numbers though. It won't be quite as simple this time. Well it might be. I don't know. We're still doing nine red beads and three blue beads. Kim has only eight blue beads now, but they still have the same amount of beads. What changed in this problem?

Becky, Samuel, Richard, Beth, and Jeff raise their hands.

T: I love this participation. What changed? What's happening?

Jeff: Um, from the nine, the other one used to be nine but you took away to eight.

Samuel: But the other one has nine

T: Ok, let's first do the model. Let's first do Maria. How many red beads does she have Jayme?

Jayme: She has nine.

Richard: I'm going to do an open bar model.

Jayme: And she has three of them. And Kim has eight, but she doesn't know what red beads.

T: We still know she has some amount of red beads, but now instead of nine she only has

Richard: Eight

T: Like Jeff said, that's one less. It went down. What's going to happen here?

Something distracted the class

T: Let's write the equation. Nine plus three, say it with me.

All: *Nine plus three equals three*

T: Wait…Equals something, we don't know how many red beads she has.

All: *something plus eight*

T: Like Jeff said. This one went down by one? What's going to happen to that?

Beth: How are we going to get it? I don't know.

Richard is raising his hand

T: Before we had like this, and this was nine. *(referring to the model for the previous problem)* But now…

Jayme: Oh, that was three. That was three

T: But now we had one less. What's happened to the three?

Jayme: It's going to be changed into something else.

T: It is. It's going to go…

Richard: Up by one.

T: It's going to go up by one. So what is it going to be?

Richard: A two.

Samuel: Zero?

T: Three and one is two.

Richard: Four, four, four

T:What do you guys think? Richard thinks it's four after he rethought that.

Jayme: One

Becky: Yes

T: Remember I took that one away. It's not just one, because I still have these three. It's not going to just be one. It's going to go to…

Responses of four

T: Huh, so when the nine went down by one, then the three went…

Richard: Up by one.

T: Ok, let's try it with another one. Now she has eight plus four, and then we're going to say has seven blue beads. Are you ready?

Richard: It's just going to go a bigger number. Every single time they do a different number, so they go up another one. Now that's number going to be two and the other number is going to be, yeah it is…

Jeff: We got another one.

T: Let's do a model. Now she has eight red and four blue. What about Kim? What will Kim's look like?

Jayme: She just has seven.

T: Right. But she has something first. Some amount of red beads first. Then she has…

Jayme: A blank

T: Then Jayme said seven. We know that they're the same

Jayme: But this part right there is missing a number.

T: Let's write the equation.

Jeff: Eight plus four equals something plus seven.

T:What do you notice? Here I have an eight. Here I have a seven. What's the difference between those two?

Samuel: The eight went down one.

T: Ok, so we basically had this other one. Remember when we had that. What was this number before? Do you remember on our last problem?

Becky: Four

T: It was four, so what's this number going to be? Remember you said, here's that one and it went down. Before it was a four, but know I have this extra one. What is it going to be now?

Beth and Becky are raising their hands. All students but Samuel touch their nose to show they know the answer.

Beth: A five?

T: Thumbs up if you think it's going to be five. Thumbs to the side if you think we need to change something.

All students put up a five.

T: Samuel, why do you think it's going to be a five?

Samuel: I know, but I don't really know how to explain it.

T: Can someone explain why that's a five?

Richard: If you add the four and the one together it equals five.

T: We have that one less, so it was an eight, but now it's a seven. So we put those together. So it kind of did that trading strategy didn't it.

Bar Model Session 10

T: Maria used ten red beads and three blue beads. Kim used nine red beads and four blue beads. Did they use an equal amount of beads? Let's first construct a model, and then we'll construct an equation. Construct is a big word for make. Ok, so let's start with Maria. What would my model look like for Maria?

Samuel: Ten and three.

T: You're right. She has ten red beads and three blue beads. *(drawing the open bar model on the board)*

Jayme: Do we have to or can we just make blocks?

T: You can do the blocks if you want to. What will Kim's look like?

Scott, Jeff, Jayme, and Beth raise their hands.

Jayme: A nine and a four.

T: She has nine red beads.

Jayme: Four blue beads.

T: Four blue beads.

Jayme: This one is going to be so easy.

T: Let's write an equation. What would that equation look like?

Richard: Ten plus three equals nine plus four.

T: (*writing the equation on the board)* We want to know if this equation is true because it says did they use an equal amount of beads. We wrote this equation and we have to decide if it's true or not. On your paper you're going to write, yes or no. Like, "yes because this is why I think they did use the same amount", or "no because", and you're going to tell me why you think they did not use the same amount. There's cubes there if you need them. You can look at your models and/or your equation. I want you to write it as a sentence. Yes or no and explain why.

Side conversations:

T: You said yes. How do you know?

Beth: Well the three went up by one and the nine went down by one.

T: So the three increased by one, and we have that trading idea?

Beth: Yeah.

T: See if you can write that in a sentence.

T: Ok, Becky what do you think?

Becky: I don't know yet.

T: You said yes, so why do you think yes?

Becky: Because ten plus three is the same as nine plus four. They both equal thirteen, and the three went up by one… (*trailed off)*

T: And what else? The three went up by one and became a four?

Becky: Mmm, hmm and the ten went down by one.

T: So then the ten went down by one.

T: You have thirteen and nine and four is thirteen. Is there another way I could explain it? Keep this way because that is a good explanation, but is there another way?

Jeff: Yes, they're switching around.

T: What am I switching around?

Jeff: *inaudible. Pointing to the three and four switching*

T: So ten plus four would be the same as nine plus three. Is that what you mean?

Jeff: No ten plus three equals thirteen, and nine plus four equals thirteen.

T: Is there another way? Is there anything that's trading or changing?

Walked away

T: So you've said yes. Why did you say yes?

T: *to the whole group*. If you've explained it one way see if you can explain it a different way. Can you come up with two reasons?

Scott: This one has um…*trailed off*

Scott is showing two stacks of cubes he's created to show the different girls' bracelets.

T: Is this one the ten plus three? How much was that?

Scott: Thirteen

T: Ok

Jayme has made cubes to show the different bracelets. She has created stacks of each number separately.

T: Jayme, do they have the same?

Jayme: No.

T: Why not?

Jayme: I do not get it. I keep on mixing them up.

T: So remember they are making bracelets and bracelets have to be together.

Jayme: Mmm, hmmm

T: Ok, so right now you have, are these bracelets together? Can I switch this for a minute? So this is your ten and three is that correct? Ok, so I'm going to put those together. This is Maria's bracelet

Jayme: And here's Kim's. *(putting together the stacks of nine and four)*

T: And that's Kim's bracelet. What do you notice?

Jayme: They're the same size.

T: They are the same size. So did they use the same amount, did they use an equal amount?

Jayme: Yes

T: How do you know?

Scott: Because one has ten and the other has ten, but you put the three on here. This one needed a ten and then three also. *(demonstrating with his blocks)*

T: Oh, so we needed that one to make the ten. Do you mind if I write that down? Will you say that again? You said because…the ten had the three

Scott: The ten had three to make thirteen. Then the nine needed a one to make a ten and then needed to put three on. Three plus ten

T: Then three more. How many did that make?

Scott: That made thirteen.

Whole Group

T: Did they have an equal amount?

Responses of yes

T: How do you know? Let's see if we can come up with two different ways.

Beth: This is my second one. Yes because the ten went down by one and the three went up by one.

T: We have that idea of this mystery one that traded places. The ten lost one and it joined the three? (*partitioned the ten to equal the nine. Then showed that piece now as part of the four)*

Richard: I'm going to add something on to that. It can't do that because you've taken one away from the four, but it's still part of the four. Wait, I don't know.

T: If this whole thing is four, remember from yesterday, and I partitioned it, or decomposed it, and I have one here *(pointing to the partitioned part)* what would this section be?

Richard: A. three

T: Yeah, so we have this trading idea. Is there another way that we know they're both equal. Jeff what did you say first?

Jeff: The ten went down by one.

T: No the first time

Jeff: Ten plus three is thirteen, and nine plus four was thirteen.

T: And nine plus four was…

Responses of thirteen.

Jeff: No thirteen.

T: Was also thirteen.

T: Here's our next numbers. Just like I promised Richard, they're going to be bigger. So you may not be able to add them up right away. Maria used thirty-seven red beads and twenty-seven blue beads.

Samuel: I'm dying.

T: Kim used thirty eight blue beads.

Jeff: This is easy.

Jayme: No it's hard.

T: Kim used thirty eight red beads and twenty-eight blue beads. Let's first draw a model.

Once we draw the model, you may not have to use cubes. Leave the cubes for just a minute. Pick up your pencil. We're going to draw a model to understand what's happening. How many red beads did she use?

Scott: Thirty-seven.

T: Thirty seven. How many blue beads did she use?

Jayme: Twenty-seven.

T: Twenty seven. Now let's do Kim.

Jayme: Oh my gosh, this is so confusing.

Richard: Do you have to write a sentence?

T: Let's start with Kim. How many red beads did she use?

Richard: Thirty-eight

T: Are we going to make it bigger or less than this thirty-seven?

Richard: Bigger

Others chime in with bigger

T: We have to make it bigger than this thirty-seven. *(drawing the new bar model for Kim)*

Beth: Then twenty eight.

T: Then twenty-eight blue beads. Is that bigger or smaller than the twenty-seven?

Responses of bigger

T: Ok, so I have to make that one bigger too?

Becky: Yep

Jayme: Another twenty-seven.

Scott: I can't make it even longer

T: So this one had to be bigger, and this one also had to be bigger? Let's write the equation. What would the equation look like?

Becky: Thirty-seven plus twenty-seven equals thirty-eight plus twenty-eight.

T: I want you to look at that for just a minute. I want you to look at our model. I want you to look at how our numbers changed.

Beth: This is hard.

Jeff: This is easy.

T: Do we even need to find what thirty-seven plus twenty-seven is?

Responses of no.

T: Do I need to find what thirty-eight plus twenty-eight is?

Responses of no.

T: Richard says no. Beth says no. Why not? Why don't I need to actually find out what those equal?

Becky: We do need to.

Richard: There's a way easier way.

T: Richard thinks he knows an easier way. Does anyone else think they might know? Do I need to even add them up?

Only Richard and Beth raised their hands

T: Let's look at something. How did these change? I had thirty seven right but she had thirty-eight. How did that change?

Scott: One more block went to the both of them.

T: So I have this one extra block. . *(drew to show the relationship in the equation of plus one)* What happened to this?

Richard: an extra block went over to the twenty-eight

T: I also got an extra block here. *(drew to show the relationship in the equation of plus one)* Samuel the other day had talked about when one went up it was because the other one stole it, and then they were the same. Did anything steal a block or did they just both go up?

Jeff: They just both went up?

T: They just both went up? What do you think Jeff?

Jeff: Thirty-seven and twenty-seven one's higher than the other one.

T: Why don't you try to write it as a sentence and then we'll talk about it. Yes or no because and tell me why. Do you think they have the same amount?

Samuel: I don't know why. I don't know if it's yes or no.

Richard: It's the same answer we had.

Individual work. Side Conversations

Jayme is using cubes. Scott is sitting and doing nothing.

Samuel: Ok, I don't think so.

Jeff is looking at the model on the board. He is not writing anything on his paper.

Becky: I have no idea why.

T: Did they use an equal amount?

Becky: No

T: Ok, you're correct. They did not use an equal amount. Why not? What do you notice about this model?

Becky: Twenty-eight.

T: Look at your model. What do you see?

T: Ok, but what went up. Look at your model for a minute. This went up by one, but then what happened to this number of blocks? It…

Beth: went up by one.

T: …also went up by one. So did any of Maria's, if both of Kim's are bigger, can she have the same amount as Maria if both of hers are smaller?

Beth shakes her head.

T: Uh-uh, so they both went up but last time did they both go up for the same person or was one on one side and one on the other side?

T: Scott you said no. How come?

Scott: Thirty-eight and twenty-seven, it will be fifty-something and that one will be fifty-something else. Like lower.

T: This one will be lower?

Scott: Yes

T: I agree with you. Why will this one be lower?

Scott: This will be lower.

T: How do you know that will be lower?

Scott: Because this and this one, it will be the same if this one was the same as this one. *(referring to the numbers in the equations on his paper)*

T: What do you mean if it was the same as this one?

Scott: Hmmm

T: Do you mean the same as the thirty-seven? If it was thirty-seven plus twenty-eight it would be the same?

Scott: No if it was thirty seven plus twenty-seven. It will make the same.

T: If this was thirty seven plus twenty seven

Scott: Yes

T: Why was this one lower than?

Scott does not respond after a long pause

T: Let's look at our model. What do we see here? We have thirty-seven and now we have…

Scott: Thirty-eight

T: It's like you have an extra cube here. So what about over here? This is twenty-seven and then what happens…

Scott: It has an extra cube.

T: I have another extra cube. Why will this one be lower? What does this one have?

Scott: It gave it like two extra cubes.

T: I'm going to write that. This one has two extra cubes. Hold onto that thought because I want you to share that idea.

T: What do you think?

Samuel: No.

T: Ok, so the thirty-eight went down, and then what else happened?

Jeff and Samuel are having a hard time verbalizing what is happening.

T: What do you think Jeff?

Jeff: I don't know.

T: Do you think they're the same or not? Let's look at the model. What do you notice about the model?

Jeff: They're not the same.

T: They're not the same. Why not?

Jeff Pauses

T: Let's look at her red beads. What do you notice about her red beads?

Jeff: The red beads?

T: Uh huh. Let's compare Maria's red beads to Kim's red beads.

Whole group

T: Ok, I think a lot of you are to that point where you've thought about it and you're still a little bit confused. Or you have an idea, but you're not sure how to say it. Am I correct?

Mumblings of yeah

T: Ok, so I'm going erase this for a minute and re-write it just since that's a little bit messy. *(the equation)* Ok. So first of all, I want you to put your thumbs up if you think they used the same amount.

No one put their thumbs up.

T: Put your thumbs up if you don't think they have the same amount.

All but Jayme puts her thumb up. She was writing the equation on her paper.

T: Alright, so most of you figured out they do not have the same amount. You're right. They do not have an equal amount. Did you get confused on trying to explain how you knew that?

Mumblings of yes

T: Ok. Scott had an interesting idea, and I'd like him to go ahead and share it. Put your pencils down for a minute and give Scott your full attention. Scott come up.

Scott: It's because that one has like an extra cube and then the other one has an extra cube.

T: Do you mind if I put that in here? So you're saying that the thirty-seven to go to the thirty eight we have an extra cube. *(Drawing that cube as part of the bar of 38)* Do you guys agree with that?

Shouts of yes

T: Then what did you say?

Scott: From that one it has another extra cube. *(drawing the extra as part of the bar of 28)*

T: It has another extra cube because we had twenty-seven and now it has…

Responses of twenty-eight

Scott: If we took one away, it will be like the same.

T: He's saying it has one extra cube but it also has…

Jeff: One extra cube.

Beth: another extra cube.

T: Another extra cube. Last time when they were the same, where was that extra cube? Did it go with Kim?

Responses of no

T: Where did it go?

Richard: It went…

Jeff: It was hiding.

T: It was hiding in Maria's right?

Murmurings of mmm-hmm

T: They both had an extra cube, but in this one do they both have an extra cube?

Murmurings of no

T: No. Kim has two extra cubes, and like Scott said, if I took that those two extra cubes away, what number would that become?

Jeff: Thirty-seven

T: Thirty-seven and this would become…

Becky: Twenty-seven

T: Then would they have the same?

Murmuring of yes

T: Does thirty-seven plus twenty-seven equal thirty-eight plus twenty-eight?

Responses of no

T: No. Because this has, how many extra cubes did Scott say?

Murmuring of two

T: It had an extra cube here (*wrote plus one underneath the 38)* and it also had an extra cube…

Beth: Plus one right there.

T: An extra cube there (*wrote plus one underneath the 28)*. Here's your next one. Now, we're not changing it very much. We're just changing one number. That's all we're doing. We have thirty-eight still but now we have twenty-six.

Richard: Why are they the same size?

T: I don't know. Maybe I drew it wrong. Thirty-seven plus twenty-seven equals thirty-eight plus twenty-six.

Beth: I need cubes.

T: Oh, I don't think we need to use cubes for this one. Use that strategy Scott said about extra.

Samuel: Is there another one we could do. This one is too hard.

T: Is this one too hard? Would you guys like to go back to smaller numbers?

Mumblings of yes.

T: Ok. We'll go back to smaller numbers. We're going to do twenty-one red beads plus seven blue beads

Jayme: I thought you said you were going to go back to lower numbers.

T: These are a little bit smaller, but they're a little bit easier to add in your head than what you had before. Kim used twenty-red beads and eight blue beads. Let's make a model for this.

All draw the model together.

T: We need to make that a little bit less than the twenty one and we have to make the eight a little bit bigger than the twenty-seven. What would the equation look like?

Jeff: Twenty-one plus seven equals twenty plus eight.

Students are writing on their paper.

T: We have to find out if that is true. Is twenty-one plus seven the same as or equal to twenty plus eight?

Becky: I don't know how to describe it, but it's yes.

Beth: It is! It is! Cause…

T: Now I want you to say yes or no because… If you don't want to write it, if you give me a little bit of time, I will come around and you can tell it to me. Then I will write it on your paper. There's a lot of you and one of me so if you feel like you can write it down, please do that.

Side conversations

T: First of all did they use the same amount?

Richard: Yes

T: How do you know?

Richard: The twenty-one is bigger than the twenty-seven so…

T: By how much?

Richard: It is one bigger than the twenty so it will take less numbers. If you add the seven plus the eight it will equal…it's one bigger, so if you take one more…

T: When you say it, do you mean the twenty-one.

Richard: Yes

T: So it needs less?

Richard: It needs less to equal the same amount because twenty…you add twenty up it equals twenty eight. So if you have twenty-one it will take less to make it up. It will equal twenty eight because one plus seven equals eight, and if you add the twenty on there it will equal twenty eight, which is how I know they're the same.

Becky: Yes because (t*railed off)*

T: Why do you think yes?

Becky: Because one plus another number will always be the next number. If it's twenty-one and, um…twenty plus eight equals twenty….

T: You said yes because one plus another number will be one more.

Becky: Yes. it will always be the next number.

T: Ok.

Becky: And twenty eight plus, twenty plus twenty, I mean twenty plus eight equals twenty eight and twenty one plus seven equals twenty eight.

Scott: Yes because if it's like twenty one plus seven equals twenty eight and twenty plus eight equals twenty eight. They're both the same because twenty needs an extra number?

T: What do you mean it needs an extra number?

Scott: Um, like it needs a one so it can make the same.

T: Ok, and where did that one go?

Scott: To the, it goes to that one.

T: What one?

Scott: It goes to the twenty eight.

T: So you said the twenty needs an extra one? Where did it go?

Scott: It went right here.

T: It went to the eight. This one went to zero so it went to an extra one there.

Jeff: I was thinking the same as Richard.

T: You were thinking the same as Richard.

T: They have a pattern. What do you mean by they have a pattern?

Beth: Bigger number, lower number, lower number, bigger number.

T: How much bigger?

Beth: One

T: Then you said lower number, bigger number, and you said the difference is one? Is that what you said?

Beth: Yes.

T: Ok, Jayme what do you think? You think they don't have the same?

Jayme: I think they don't because this is a higher number than all three of these.

T: Ok, but when we're looking at equal, we're looking at the total quantity. How much does twenty one plus seven have?

Jayme does not respond

T: Let's break it apart. We can break twenty-one into twenty plus one. What is one plus seven?

Jayme: Eight.

T: And twenty plus eight is what?

Jayme: Twenty-eight.

T: So what is twenty-one plus seven?

Jayme: Can I count on my fingers?

T: You can start at twenty and count up seven.

Whole group

T: Let's do this a couple of different ways. What's twenty-one plus seven?

Beth: Twenty eight.

T: What's twenty plus eight?

Richard: Twenty eight.

T: Are they the same Jayme?

Jayme: Yeah

T: Yeah, because both sides have twenty eight. We're not going to talk about the other way because most of you saw that we had the trading of the cube, and you had that written down. But I wanted to make this a little bit clear because some of you were still confused on how many it was.

Symbolic Group Transcripts

Symbolic Session 1

T: I'm going to give you a story problem to start out with. On your paper, I want you to write an equation. And there's also some cubes if you need to solve it. I want to see how you solve it first. Rose was writing letter. You know what a letter is. Like a card or note to somebody. She was writing letters. Rose wrote eight letters. She wanted to write twelve letters. She wrote eight letters. She wanted to write twelve letters. How many more letters does she need to write to meet her goal?

Elise: Can we use some cubes now?

T: You can use cubes. I'm going to read the story one more time, then you can use cubes if you need to. I want you to try and write a number sentence, or we also call that an equation. *Reread the problem*. I want you to think about it and do the best that you can. *All except Harry are using cubes to solve the problem.* Most are starting with a tower of twelve. *Working independently*

Jessica: I already know that eight plus four is twelve.

T: Show me that somehow. Show me your thinking somehow.

Anthony: I don't want to do this.

T: How can you visualize it in your head? You can use cubes.

Anthony: I don't want to do this

T: Rose wrote eight letters. She wanted to get to twelve letters. How many more letters does she need to write? See if you can write an equation or show me your thinking.

He then reached for cubes to start working on the problem.

Harry: I'm done.

T: See if you can show me your thinking. How do you know how many more she needs? How do you know that?

Elise is making a stack of twelve and another stack. Debbie has a stack of eight and a stack of four. Allison has a stack of eight, a stack of twelve, and a stack of five (at this point in the video). Matt is writing on his paper and not using cubes. Stephen has a stack of eight and a stack of twelve. Anthony has a group of three and a group of five. He also has a group of four. Harry wrote an equation and then just sat there. Jessica is not using cubes. She wrote it on a number line.

Debbie: I only needed to use these cubes. I needed to use this much.

Harry: Can I go to the next one? I'm already done with that.

T: Ok, go ahead and finish up.

Whole Group

T: Who can tell me, how did you find the answer? I don't necessarily want you to tell me what your answer was right away. I want you to tell me how you figured it out.

Matt, Debbie, Harry, and Anthony are raising their hands

Matt: I counted in my head

T: Ok, and how did you count in your head? What did you start with?

Matt: I started with eight.

T: Then what did you do? *(writing 8 on the board)*

Matt: Then I put twelve more.

T: You counted on twelve more? *(wrote plus twelve on the board)* So you got to how many letters?

Matt: How many?

T: How many more letters did she need?

Matt: Um

T: Is this what you did? You added twelve or you counted up to twelve?

Matt: I added.

T: Ok, which got you to how many total letters that she needed? How many more letters that she needed?

Matt: Twenty

T: Ok, so Matt says that he started at eight. He counted on twelve more so she needed twenty more letters to write.

Jessica, Debbie, Harry, and Anthony are raising their hands.

T: I see some people who disagree with you. I want someone else to share their thinking, and then we'll examine who we think answered the question.

Debbie: I just had some blocks. I started with the smaller number.

T: Which was eight?

Debbie: It was four. But I started on eight and then I counted.

T: Ok, so you started with eight *(writing eight on the board).*

Debbie: Yes, and then I counted my small one, and it was four. I added four because there was four on there, and I knew it was twelve because I was counting. *(eight plus four = twelve on the board)*

Harry: You had eight so I needed two more to add up to ten and two more makes twelve. So it's four.

T: You broke it apart this way. You did plus two to get to ten, and then you added another two to get to twelve (wrote 8+2+2=12).

Harry: Yes.

T: So you did it in two chunks. Which of these matches my story? Let's look at this problem again. So Rose wrote eight letters. Do we have that in all of these ways?

Anthony: Yes

T: We have her starting with those eight letters. She wanted to write twelve letters, so what is she hoping to end up with?

Pause….No hands

T: She only wants to write twelve letters, so what would be the total amount of letters she's going to write?

Harry: twelve

T: She only going to write twelve letters. Will she write twenty letters?

Harry and Jessica: No

T: Nope, because she wanted to write

Mumblings of twelve

T: twelve. Does it say she wants to write twelve more letters?

Mumblings of no

T: Nope, she only wanted to write twelve. It asks how many more letters does she need to write. So we're trying to find the distance between eight and twelve. Matt, does that make a little more sense?

Matt: Yes

T: Do you see why it's not twenty?

Matt nods

T: Here's my next problem. It's very similar, so before you solve it, I want you to think about how is this problem the same and how is it different from the one we just did. Here's my story. Rose wrote eight letters. There's another girl. Her name is Nikil. She wrote twelve letters. How many more letters does Rose need to have the same amount as Nikil?

Debbie: This is basically the same.

Anthony: It is the same.

T: Ok, I'm going to read this one more time. I want you to think about how is this the same as the last problem and how is it different. Cubes down. Pencils down. We're not solving it yet. *Reread the problem*. How is this problem the same as the last problem? How is this problem different? *(paired up the students)*

Side conversations

Anthony: It's the same because um…

Harry: They both have the same numbers

Anthony: Yeah. They both have four.

Harry: Eight plus four equals twelve, again!

Debbie and Allison are listening in to their conversation

Stephen: It's the same…

Jessica: I have no idea

Stephen: It's the same because they both have the same numbers, eight and twelve. And eight plus four equals twelve.

T: How is this problem different Jessica? He told you how they were the same. How is this problem different?

Jessica: Because it didn't equal twelve.

Whole group

T: Who would like to share? How are these problems that same?

Harry: They both have the same numbers.

T: They both have the same numbers. They both have eight letters, and they both have twelve letters. How else are they the same?

Debbie: They both have Rose.

T: They both have a story about Rose. How is this problem different?

Pause…Harry, Allison, and Debbie raise their hands

Allison: This problem has Nikil.

T: Now there's two girls. We call this problem compare. Everyone say that word.

Chorally: compare

T: We're comparing two girls. I want you to see if you can write an equation for this problem.

Debbie: I already did.

T: I want to know how many more letters Rose needs to have the same amount as Nikil. How many does Nikil have?

Jessica: Twelve

T: Am I adding them together?

Elise: Yes

T: I want to know how to make Rose have the same amount as Nikil? If you already have one number sentence, see if there is another number sentence you can write. Is there another way to do it?

Side conversation:

T: So this would be how much they would have together. I want Rose to end up with the same amount as Nikil. I don't want them to be put together.

Jessica: Wait what?

T: How many more letters would Rose need to have the same amount as Nikil? Nikil only has twelve. Think about how could you change what Rose has to make it the same as Nikil.

Debbie: I've got another number sentence. Wait….(hurriedly erases her work)

Elise: She has twelve.

T: (to everyone) I'm going to look and see who I would like to come put their number sentence up on the board.

Matt, Stephen, and Anthony write their equations on the board.

T: Debbie you can come put your last one up there.

Debbie writes hers on the board

All students are sitting up with their eyes on the board.

Whole Group

T: Alright, so we talked about this when I came and asked you guys questions yesterday. What do we call that (pointing to the equal sign)?

Shouts of equal, equals sign

Harry: Equal

T: What does that mean though?

Anthony: Something equals something

T: If two things are equal to each other what does that mean? I can't say it means they equal. That doesn't help me. I don't know what that means. That would be like saying what does the color blue look like, and someone saying, "I don't know it looks like the color blue." Is that helpful?

Mumblings of no

T: What does that mean?

Debbie: It means it's the same

T: So this means that two things or two quantities… if two things are equal, that means they are the same. They are the same amount; the same as. Let's read this together, but instead of saying equals twelve we're going to say is the same as.

Chorally: four plus eight is the same as twelve

T: Let's read it this way

Chorally: twelve minus four is the same as eight. Eight plus four is the same as twelve. Twelve is the same as four plus eight.

T: Hmm… Do you guys agree with all of these equations?

Shouts of yes, yeah

T: Are they all true? Do they all work?

Shouts of yes

T: Let's look at this equation. I think this was from Stephen. He said twelve subtract four is the same as eight. And then someone over here (Anthony) said that eight plus four is the same as twelve. We know that equals means that the things on either side are the same as each other, so how are these two equations related?

Harry raises his hand

T: Put your hands down, I want you to think first. This is your think time. I want you to compare this equation to this equation (pointing to 12-4=8 and 4+8=12).

Harry: I know what it is. I know what it is.

T: Just think about it for a minute. I want you to turn to the same person who you were talking to before. I want everyone to talk. Please share how these two equations are related. If they're both the same as each other, or we have this equal, how are these equations related?

Side conversations:

Elise and Matt are not talking.

Stephen and Jessica are not talking.

Jessica and Debbie are not sharing.

Anthony and Harry are playing with their pencil and their cubes.

Jessica: They have the same numbers.

T: (to everyone) so Jessica is thinking about how the numbers are related. Think about that and what they mean in reference to our story.

Still no engagement in small group

T: Alright, let's have a whole group discussion since you guys are not talking in small groups very well.

Harry: Oh, I know.

T: I'll scootch up that we can have a really good whole group discussion. Alright, so when you talk about these number I want you to refer to our story.

?: It's hard

T: It is kind of a hard question. I'm making you think, crazy. Who would like to just…I don't know…Who's feeling confident and would like to just share their idea? Jessica, let's give you a chance.

Jessica: I noticed that twelve and eight were in the exact same story as last time.

T: Ok, so twelve and eight are in the same story. And they're in the same story as last time. We have the same numbers that we're using.

Matt: They have the same numbers, but they turned them around.

T: Can you explain that a little bit more, or someone think they can add on to Matt's idea? He said it's the same numbers but they've turned them around.

Harry: I actually know something. One of them has a plus and one of them has a minus

T: Why does that matter?

Debbie: Because um…. twelve plus four are not equal eight.

T: Because twelve plus four would not equal eight? Why? Why could I not say that? Or eight equals twelve plus four. Why could I not do that?

Pause…No one is sharing ideas

T: Stephen, what do you think?

Stephen: Um…

T: Why could I not say this? This is what you said right Debbie *(writing 12+4=8)*? Why is that not true?

Pause Debbie raises hands

Stephen: Because twelve plus four is not eight.

T: Because they're not the same right? Eight would not be the same as twelve plus four. What is the four in this problem? Where did the four come from? You all have that in your equations. What is that four in reference to our story?

Pause… Debbie is the only one to raise her hand

T: Think about our problem *(reread the story).* Where's that four? I don't see a four right now in our story. Where did that four come from?

Pause…Harry, Debbie, and Anthony raise their hands

T: Anthony where did it come from?

Anthony: They don't know it so it's the answer. So they don't tell you.

T: Oh, so a lot of kids, and I'm not saying this is you guys, think that after equal it would have to be the answer. Whatever the answer to my question is, that's the answer *(pointing to the number after the equal sign)*. Is that right?

Mumbling of no

T: Because what's the answer to my question?

Anthony: Four

Elise and Jessica seem to be distracted. Jessica is playing with cubes. Elise is looking around. The rest of the students are looking at the board.

T: The answer to my question is four letters, or four more letters. Where's that four in this problem *(pointing to a different equation)?*

Anthony: It's in the middle.

T: It's right here. There's my answer. Where's the answer in this problem?

Matt: It's at the beginning

T: It's at the beginning. What about in this one? *(pointing to a different equation)*

Murmuring of middle

T: It's in the middle, and here it's in the middle too. In any of the equations you wrote did it ever come right after the equal sign?

Anthony: It doesn't have to.

T: Why not Anthony?

Anthony: Because the answer could be anywhere, just if it equals the number that it in the problem

T: Because what does equal mean again?

Responses of same as

Anthony: That one side is the same…

T: The same as; it means one side is the same as the other. Perfect timing, you guys are rockstars.

<u>Symbolic Session 3</u>

T: Here's your task today. We are going to find out different ways to compose the number ten. Touch your nose if you've heard this word compose or if you've heard the word decompose. That's ok. The other group hadn't either. Compose means to put together. Say that together.

Chorally: Compose-put together.

T: What do you think decompose might mean?

Harry: I know that.

Debbie: It means take apart.

T: You got it. If compose means put together, decompose means I'm going to…

Students: Take it apart

T: Take it apart. You have a stick of ten cubes in front of you. You can use this to help you, and then I want you to write the equations that you can think of. See how many different ways you can come up with to compose the number ten. To make the number ten. You may get started. Try and come up with at least three, but see if you can push yourself and get more. I want you to just break it into two parts.

Students begin independent work. Side conversations.

Harry: I'm already writing my equations.

T: Can you find another way. That's still the same way. That's five and that's still five. Can you think of another way?

Debbie: I'm trying to.

T: Use your cubes if you need to. The last group found that helpful.

T: (to Allison) You're right, two plus 1 does equal three, but I want you to make ten. You're going to have a total of ten cubes. What are the different ways to make a total of ten cubes?

Debbie: I have four equations.

T: Write them as addition sentences please.

(Side conversation between students)

Debbie: 64 does not equal 10. Hey I forgot my symbol right there. 10 equals 55.

Harry: You do not have to use cubes if you don't want to.

Matt: This says 100 equals 10.

Debbie: Actually 10 does equal 100 because 10x10 equals 100.

Matt: Do you know what 4 divided by 5 is?

Debbie: I do.

Matt:I do too.

Debbie: It equals two.

Matt: It equals eight.

Debbie: It does?

Debbie: I'm thinking of a division sentence that equals 10.

Matt: Division?

Debbie: yeah.

Matt: What's division?

Debbie: Divide by.

T: You're right, but I want you to write them as addition sentences. We only ever have ten. Those are true statements, but I want you to focus on something that adds up to ten.

Jessica: Oh.

T: So we have ten. What's another way we could make ten? Without subtracting.

Jessica: I could make seven and three with subtraction.

T: I want you to do addition sentences. Oh, I'm sorry. That's nine plus one and eight plus 2. I'm having a hard time reading them. See if you can find another way. I wanted at least three. See if you can push yourself.

Jessica: I can't. I don't. I can't. I can't think of another one.

T: Use your cubes to help you. Think about how else you could break them.

Jessica breaks them in a different way

T: Hmmm….

Debbie: *(to T)* I'm trying to think of a division sentence that equals 10. It's going to be hard.

T: That will be hard, but what were my instructions?

Debbie: To focus on addition.

T: Yep, so maybe at home or after you can tell me, but right now I want you to focus on addition.

T: *(to Elise)* Do me a favor and go write those two on the board for me. *(To everyone)* I'm going to give you about one more minute while I have Elise write these on the board for me.

Harry: This is not right *(pointing Jessica's paper)*.

T: Why not Harry?

Harry: It has 5 plus 4 which equals nine

T: Oh, so that's not the same? So it wouldn't be a true statement?

Harry: I have all true statements

T: Maybe when she comes back say, "Hey Jessica, maybe check this one". That's a nice way to help someone.

Whole Group

T: If you found some, I want you to go ahead and put your pencil down and give me your eyes up here. So Elise, I'm going to keep those there, but is it okay if I write what you have over here just so it's easier for us to see?

Debbie: She copied me.

Harry: I didn't do it.

T: Touch your nose if you also got those.

(most students touch their nose)

T: Does that mean she copied you?

Students: No.

T: No. That means you guys know your partners that make ten.

Harry: I got a lot.

T: I also found someone who had this one (writes 8+2=10 on the board). I want you to touch your nose if you have that one. Hands down for right now and listen to my question. I want you to look at these equations. Let's read them and remember that this equal means…

Chorally: the same

All: Ten plus zero is the same as 10. Nine plus one is the same as ten. Eight plus two is the same as ten.

T: What's changing in these equations?

Pause

Stephen: *(gasp)*

Lots of hands in the air

T: Turn to your neighbor. Talk about what's changing. Make sure you both get a chance to share an idea.

(side conversations)

Whole Group

Stephen: Oh, I get it now!

T: Did the light bulb just go off Stephen?

Stephen: It's zeros and ones and twos

Debbie: That's what I said.

Harry: Yeah, that's what I said.

T: What's changing in these equations? Allison?

Allison: The numbers.

T: Can you add on to that? What do you mean by the numbers are changing?

Allison: Like…*(paused for a long time)*

T: You're not sure. Can someone add on to Allison's idea? She said the numbers are changing.

Debbie: I think they're counting down, and this one is counting up.

T: The first number in our equation is counting down.

Elise: Or it could go up too.

T: And the second number of our equation is counting up. If we looked at it from the bottom Elise, you're right. We would think of it as counting up, and then this one would be counting down. Hmm. What's happening then? What's happening in this relationship?

Long pause. Students were not responding.

T: We kind of a saw a pattern here *(pointing to the changing numbers in different questions).* What is happening then between this relationship *(pointing to the numbers in the same equation)?*

?: *(gasp)* It's going…

T: I want hands down for right now because I want everyone to have a chance to think about it.

Pause

T: Who thinks they have an idea? What's the pattern that you see in this relationship *(within and between equations*)?

Harry: Ooh, I know.

T: Stephen is like Woah! I don't know.

Stephen: My head is going to explode.

Harry: I know it.

T: Jessica?

Jessica: I couldn't hear what you were saying because it was too loud.

T: Thank you for pointing that out. As a group let's make sure that you're listening. My question is, what's the relationship between these numbers and what's happening with these numbers?

Pause no hands

T: Does anyone see a relationship?

Harry: There's zero, one, two and the other one it goes 8, nine, ten

Debbie: We already said that

T: We did talk about that. We talked about how this is going down by one, and this is increasing by one. But I want to know what's happening here where we had 10 and zero, but now the partner is nine and one. Here the partners are eight and two. What's happening?

Elise: Nine, ten, eight, nine, ten.

T: Ok

Elise: We're counting up.

T: Ok, so we're counting this way. First we had ten and zero. Then you said this went down by one, and this went up by one. Why did that happen? Why did this go down by one and this side go up by one?

Debbie: Oh.

T: Why did that happen?

Harry: I get it

T: Debbie what do you think?

Debbie: Because nine is lower than ten and one is higher than zero.

T: Ok. So what happened to the one less here (pointing to the nine and ten)? You said there's one less. What happened to it? Did it go away?

Debbie: No

Anthony: *Gasp of recognition*

Stephen *(gasping as if he is starting to get it)*

T: Where did it go?

Harry: You added it.

More gasps from the students

T: Jessica what happened to it?

Jessica: You added it.

T: How?

Jessica: So like…

Anthony: *gasp then a look of confusion*

T: Anthony, he's like a lightbulb, oh maybe not. *Pause* Where did it go? Maybe it's helpful here. Here you also said that went down by one and that went up by one. What happened to it?

Jessica: This one went up and this one is going down. I noticed that zero went *(trailed off)*

T: Can someone add on to what Jessica is thinking?

Debbie: Not really

Other mumblings

T: Is that what you're all thinking?

Anthony: So all the tens have zeros.

T: let me try this one. Yesterday we talked about how we could write equations like this. *(Wrote 9+1=8+2)*. Do you guys agree that is a true statement?

Students: Yes

T: Nine plus one is the same as eight plus two.

All: Yes

T: Ok, so let's try this one. Eight plus two is the same as something plus three.

Harry: I know it. It's seven.

T: What's that something?

Matt: Eight plus two is the same as seven plus three.

T: How did you know that it was seven plus three?

Matt: I was thinking

T: Ok, cause you were thinking.

Anthony: I knew it because it was on my paper.

T: Anthony knew it because he already knew that seven and three made ten. So he's thinking about it in terms of this being ten *(wrote a tree diagram to show the ten on each side).* Is there another way we could think about it? *Pause* What happened to this number *(pointing to the first number in each side of the equation)?*

Debbie: It got added.

T: It got added. What do you mean Debbie?

Debbie: Like it's, it's not gone but there was one more thing.

T: We got one more here. Where did that extra come from?

Debbie: That…

T: You said this went up by one. Do you guys agree with Debbie?

All: Yes

T: So where did it come from?

Debbie: It came from the seven.

T: It came from the seven?

Debbie: Yeah

T: Anthony's shaking his head. Elise is shaking her head.

Anthony: Uh, uh.

Other mumblings of uh, uh.

Debbie: Because the eight gets minused by one and turns to eight. And that one is taken to the two to make it three.

T: Debbie said if I subtract one, to make them the same I have to keep that by adding one here. What do you guys think? That's where it went.

Harry: Nope

Debbie: Yeah

Mumblings of un, uh.

T: I see four people shaking their heads. Let's try another one. Let's do seven plus three equals something plus four.

Mumblings of ooh, hmm

T: Seven plus three equals something plus four.

Jessica: I was thinking it was ten.

T: So you think seven plus three is the same as ten as plus four?

Debbie: No!

T: What do you guys think?

Harry: I know what it is.

T: What do you think Matt?

Matt: Seven plus three is the same as ten plus zero.

T: I still want to keep the four here, but you're right. It would be the same as ten plus zero.

Debbie: I know.

Harry: I know.

T: Let's first figure out…Jessica why did you say ten plus four?

Jessica: I don't know.

T: Where did that ten come from?

Jessica: I knew seven plus three is ten.

T: Remember when we talked about how the answer is not always right here. Is there sometimes not an answer? We want something that equals ten on both sides, so I have to get up to that ten. *(wrote a tree diagram to show the ten on both sides).* So seven plus three is the same as something plus four. What's that something going to be?

Harry: Six

T: Six plus four. Huh. So…

Matt: I know because six is…

T: How do you know? Maybe thinking about here *(referring to the first problem).* Let's go back to Debbie's idea. She said I traded the one from the seven and gave it to the…

Debbie: the two

T: …two and so it made a three. Does that work up here?

Debbie: and the eight came to….Yes!

T: Does that work up here?

Debbie: Yes!

T: Remember yesterday when we talked about trading?

Exclamations of yes

T: So if I traded the one, and I said guess what I'm taking it away, what do I have to do to the three?

Debbie: We need to…

T: someone besides Debbie because I think you have the idea. I want someone else to maybe think about your idea.

Pause and no student input

T: So I'm trading that seven. I'm saying guess what, there's not that cube. Remember how we had bugs yesterday, and I think it was Willie and Tamara and they were trading bugs. So Matt, where's that bug have to go?

Matt: That bug?

T: Yes, if I took that bug and I took it away where does it have to go so they'll still be the same?

Anthony: To the four

Matt: take away the…. Three!

T: Oh, so I have to give it to the three? Which then becomes a…

Exclamations of four (Only two voices)

T: Let's do another one. Six plus four is the same as something plus five.

Harry: I know exactly what this is.

Debbie: This is easy.

Other exclamations

T: Some of you already know it, but I want you to think about it using this idea Debbie had. That we're doing some of that trading. Say it if you know it.

*Five! (*two voices)

T: Talk to your neighbor. How you know it's five? Think about that trading idea.

Side conversations:

Harry: Cause five and five is ten

T: You're right. Talk about it in terms of trading. What's happening to the six and the four to get that five and five?

Anthony: The six is making, the six (inaudible)

Harry: and it makes ten. Six and four, and then it's ten. And five plus five is (inaudible).

Debbie: The six goes away and it goes to the four

Matt: *does not share an idea*

T: What do you think Jessica?

Jessica: I don't know.

T: What do you think is happening?

Jessica: I don't know.

T: Well, what's happening?

No sharing

T: You're not sure

Stephen: They're both ten.

T: You're right. We could think about them both being ten.

Whole Group

T: Harry what's happening?

Harry: Six and four are equaling ten, and the five plus five equals ten.

T: We still have five plus five is ten. What's happening between our numbers? Could I explain that they're equal besides saying they both equal ten?

Debbie: Yeah.

T: Let's listen to Debbie.

Debbie: The six goes to four and the four becomes a five and the six becomes a five.

T: Again, thinking about trading like our bugs. Willie giving them to Tamara. I have that bug. I'm saying I want one less bug here. What has to happen then to this?

Debbie: The one less…

T: Matt?

Matt: The four goes to the five

T: I have to add that bug. What if I did this? I just traded that bug, and I said six plus four, there's my subtract one, but I kept that the same. Is that true?

Exclamations of No!

T: Huh, because if I'm trading it, it has to go where?

Debbie: Five….to the four

T: It has to go to the four to make this also have…

Exclamations of five

<u>Symbolic Session 6</u>

T: Who would like to read our story problem for us?

Stephen: One table had four red ants and five blue ants. The other table had three red ants. How many blue ants does that table need for both tables to have the same?

T: We're going to start by writing an equation for this problem. We're going to start write a statement for our first table. What might that look like?

Harry: Four plus five equals nine.

T: (*writing the equation on the board)* I'm going for right now do four plus five. I'm going to set it equal to my other table. I'm not going to put

Jessica: It's the same as

T: That's another way to say equal. I'm not going to put nine right there. I'm going to put what my other table is. What might my other table's equation look like? Even if you know the answer right now, I want you to give me the information we do know without solving it. What is the information we do know?

Debbie: The other table has three red ants.

T: Three plus something. *(Writing the second part of the equation 4+5=3+__).* We don't know what that something is.

Debbie: I do.

T: We have to figure out what that something is because we want our first table (*pointing to the first part of the problem)* to be the same as our (*pointing to the second part of the equation)*

Debbie: second.

T: Second table. I have four red ants and five blue ants is the same as three red ants and some amount of blue ants. What do you think so far?

Harry: This is easy.

Jessica: What did you say? It's loud in here, and I couldn't hear you.

T: Ok, so I know that the first table has four red and five blue, and I want the second table to have the same amount of ants. It only has three red, so how many blue would it need? What might it be missing?

Harry, Debbie, and Jessica raise their hands

Debbie: Six.

T: Debbie thinks it's six. Does anyone else have an idea?

Pause… only Jessica raised her hand

T: Does anyone else think it's six? Raise your hand if you think Debbie's right, that it's six.

All but Stephen, Elise, and Jessica raise their hands.

T: Who would like to explain why? Why do you agree with Debbie?

Harry: Because you've got three more to nine.

T: Ok…

Harry: They both equal nine.

T: So you're thinking we want them both to get to nine. Is there another way I could explain that Debbie's is the same? What if I didn't know they were both nine? Could I solve it anyway?

Matt: Yeah

T: Hmm, how else could I know that it's supposed to be six?

Jessica raises her hand

Jessica: What did you say?

T: Harry said that he knows it's six because then it would both equal nine. (*Drew a tree diagram to show that there is nine on each side)* I want to know if there's another way I could explain it besides saying they would both equal nine.

Elise: I know the problem.

T: Well we know that it's six, but I want to know if there's another way without knowing they both equal nine.

Elise: I counted five cubes and then I did four and three, and it said that it was twelve.

Stephen: Yeah, that was my answer too.

T: So it was twelve?

Anthony: Six plus six equals twelve.

T: But what does this mean *(pointing to the equal sign)*? What does this equal mean?

Shouts of "The same as!"

T: It means that it's the same. Is four plus five the same as three plus twelve?

Shouts of "No!"

"*Six!"*

T: Where do you think that twelve came from, those of you who got twelve? Someone else, why do you think they got twelve? What might they have done?

Harry: They added six plus six

T: They might have added six plus six. I think they did something else.

Stephen: I counted the blocks, and it was twelve.

Debbie: They added the four plus five, and they also added the three.

T: So they took our first table, remember that was only one table, and then they added three. They added both tables together. Is that what my problem is asking us to do? Stephen, do I want to know how many ants there are altogether?

Stephen shakes his head

T: So will twelve work?

Stephen shakes his head

T: Ok, so twelve isn't going to work. Let's see if you can figure it out with cubes. See if you can model one table and model the second table.

Students work in partners or independently to model the two tables.

T: Show me one table and show me the second table. Work with a partner, and I want you to explore this problem.

Side conversations

Debbie: Huh, I'm done

T: Ok, Debbie see if you can think of another way why that would work besides that they both equal nine. Is there another way that would work?

She had written a tree diagram to show they both equal nine

Debbie: Ok, so nine… (*starts to write a subtraction sentence with nine)*

T: No, what I mean is you know these both equal nine. What's the relationship between those? What's happening to my ants?

Debbie: I don't know

Harry: I get it. I get the answer.

T: Ok, how do you know? How else do you know it's six?

Elise: Me and Allison are counting twelve.

Stephen: Is this a take-away?

Anthony: No

Jessica: I know what it is.

T:(to Jessica) Do I want to know how many there are all together?

T: Ok, let's do these two problems together. We're going to start and do this problem a little bit at a time. You guys are going to work with a partner. All we're going to do first is figure out what my problem is asking me. What's my question?

Harry: Six

T: No that's my answer. What's the question? What am I trying to find out?

Had to gather all their focus

T: I'm going to read this again, and I want you to listen to my question. One table had four red ants and five blue ants. That's just one table. The other table had three red ands. How many blue ants does that table need for both tables to have the same? What's my question? What am I trying to find?

Pause no responses

T: Do I want to find out how many altogether?

Debbie: No!

T: Thumbs up if that's what I want to find.

Matt, Stephen, Anthony, and Debbie put thumbs down. Elise and Allison seem unsure. Jessica starts with thumbs up and then switches to thumbs down.

T: No, so we're going to take this step by step. We're going to first model. *(with cubes)* We're going to start with one table had four red ants. One, two, three, four; that's one table. We're going to set that over here. Are we done with that table yet?

Shouts of no

T: What else do I need?

Debbie: Five.

T: I need five…

Jessica: Blue ants.

T: Five blue ants at that table so I'm going to count out five.

Stephen: Then do we add on, like with it.

T: I want you to put it on the same table, but I want it to have a gap right there. So it should look like mine. Alright, there's my first table. Does yours look like mine?

Harry: I get it. I know a way.

T: Let's see how many I can put at my other table. What do I know so far?

Harry: It has three

T: It has three. Let's get out three. This is my second table.

Students are using the cubes to show their second table. Allison was putting them in three different columns.

T: I want you to put them like this Allison. These are the same table so I'm going to put them on the same side.

Stephen: Like this? (*trying to put them all together)*

T: No, this is a different table (*holding up the tower of three).* Yours was just fine Stephen. Yours is just going a different direction. Ok, I've got my three red ants. What do I want to find out? Do I want to put them all together and find out how many that is?

Most students say No!. Jessica said yes, and changed to a no. Allison shakes her head. Elise says yes.

T: Nope, I don't. My question is, how can I make them both the same. What do I need to do?

Allison: Add more to the three

T: I need to add some more to the three until it becomes the same. How many more do you think I need to add?

Matt: Three

T: I need to add three more and then they'd be the same?

Matt: Four

T: Four more and then they'd be the same? What do you think Anthony?

Anthony: Six

T: Why do you think it's six?

Anthony: Because I know that four is one more than three and…

T: I'm going to push your pause button. I want you to listen to Anthony's idea. Will you say that again?

Anthony: Three is one more than four so I just counted one more than three, and I counted the whole stick.

T: Can you do that out loud for us? Count out loud and show us what you did.

Anthony: One, two, three, four, five, six

T: Ok. How many do we know was here?

Debbie and Anthony: Five

T: How many more did he say was here?

Debbie: One

Anthony: One more

Matt: One

T: One more. So he had four here. So if we moved that one *(showing the trading of the cube from the four to the five)* How many would need to be there then?

Harry: Ooo, I get it. I know another way.

T: How many would need to be there then?

Debbie: six

T: Let's count and check

Harry: I think it needs to be seven.

Altogether we counted the cubes.

Then everyone put six cubes onto the stick of three.

T: Remember these are my red ants. What do you notice about my red ants?

Jessica: I noticed that one has four and one has three.

T: What else could we say about that?

Harry: They both equal seven

T: Ok, I don't want to know how many there are together. I want to compare the two tables and the number of red ants they have. What happened? This one has four and this one has three. Let's look at my equation. What changed between this four and this three?

No responses.

T: Does someone remember what Anthony said?

Debbie: It's the trading thing.

T: Can you explain more about that? What do you mean it's the trading thing?

Debbie: Like, um, the three traded one….The four traded with, it gave one of its ants to the five and then it has three ants. And the other one has six.

Anthony: Oh yeah.

T: We had one more red ant here. Let's look and see (*using cubes)* is that right? Do we have one more red ant? *(comparing the red ant towers made with our cubes)*

murmurings of Yeah

T: Then Debbie said the five, this blue ant, is going to have to have one more.

Anthony: Yeah

T: Because that has one more. Let's see. Remember which table is which. Let's compare them. *(looking at the towers for the blue ants)* What do you notice?

Many students are having a hard time lining their cubes up to find the difference. Their cubes are not created correctly. Stephen, Allison, Harry, Debbie, and Elise have incorrect towers.

T: What do you notice? This one had one less but now it has what? The other number has

Stephen, Allison, Jessica, Elise, and Matt are giving blank looks. They're looking at the tower of cubes but not contributing an idea.

Harry: That you want to have five

T: Ok, but this has …how did it change from the five to the six? What's the difference between five and six?

Stephen, Jessica, and Allison are looking closely at their cubes. Matt was the only one to raise his hand.

Matt: Um…*was not able to verbalize his idea*

Harry: Wait

Elise and Anthony are spacing out

T:The five got

Debbie: one more

Matt: One more from the six

T: One more. Where did that one more come from?

Debbie: The four.

T: It came from the

All: four

T: Ok, let's try another one. Are you ready?

Debbie: Oh no, it's going to be hard.

T: One table had eleven red ants and three blue ants. The other table had twelve red ants. How many blue ants does that table need to make both tables have the same? We're going to write this equation. We're going to start with my first table. How would we say what my first table has?

Jessica: Eleven

T: Eleven plus what

Anthony: Three

T:Eleven plus three is the same as…Now I'm doing my other table. What's the other table?

Matt: Twelve

T: Twelve plus…

Jessica: Something

T: Something. I don't know what that something is. I want you to see if you can figure out what is that something. You might need to work with a partner.

Harry: Oh, I get it.

Debbie: Ooo, (*raises her hand enthusiastically)*

Students are working and using cubes.

Side conversations:

T: Alright Harry what do you think?

Harry: Eleven plus three is the same as twelve plus

T:(*To Jessica) Does* that make sense?

Jessica has written 11+3=12+61

T: Why do you think it's plus one Harry?

Harry: Because you plus twelve and you add two to get to the three

Matt: I get it now

T: How did the eleven change?

Harry: It turned to the…wait, I get it.

T: So how did the eleven change?

Harry: It gave it to the three.

T: Ok, so what about the difference between the eleven and the twelve?

Harry: The twelve is above eleven.

T: By how much?

Harry: One

T: So how is the three going to change?

Matt: I know what it is

T: Why do you think it's two Matt?

Matt: Because eleven plus three is twelve and so is twelve plus two.

T: Eleven plus three is how much?

Matt: Eleven plus three is fourteen.

T: Ok, how did the eleven change? What's the difference between those two?

Matt: One changed. One got traded.

T: So we got one more. What happened to the three then?

Matt: Um, it put one less to the two.

T: Oh, it got one less and became two.

Jessica: I know what the answer is.

T: Ok, why is it two?

Jessica: Two because one of these went to the twelve. Went to this box and became two.

T: Ok, can you say that again a little bit slower

Jessica: *(her hands are moving from number to number but not in a systematic way)* One of these, two of these, one plus three equals twelve plus something. And one of these went to the two. And two of, one of these, two of these, and then one went over there and the other one went over here.

T: Hmm, what do you think Stephen?

Stephen: But eleven plus three equals fourteen.

T: So eleven plus three equals fourteen. What about twelve plus two?

Stephen: Oh, (*a tone and look of comprehension)*

T: What is twelve plus two?

Stephen: Fourteen.

T: Where is the fourteen then Stephen?

Jessica: Oh *(in an excited tone)*

Stephen: *(points to the empty box*) Right here.

T: Do we see the fourteen or is it hidden?

Stephen: It's hidden

T: You're right. We want to get to fourteen on both sides. So what should that be then?

Stephen: Two

Elise and Allison are using cubes. They are creating stacks. They have created two stacks and are counting each stack.

Elise: You have eleven, you don't have twelve

They recount the stacks.

Elise listens in briefly on the discussion with Stephen.

There's another small stack of three.

Whole group

T: What did you guys come up with? What's missing?

Stephen: A two

T: How did you come up with it as two?

Stephen: Because eleven plus three equals fourteen, and twelve plus two is fourteen.

T: We talked about this fourteen being hidden *(showed it on the equation using a tree* *diagram),* but it's still there. Is there another way we could talk about it? Let's pretend I didn't know it was fourteen? Is there another way?

Harry: You could use unifix cubes.

T: Ok, but how else could I explain that two is the correct answer? What is the relationship between the eleven and the twelve? What happened? Elise, what's the difference between those?

Harry: Oh, I know.

Elise does not respond

T: You know this. Don't over think it. We're not adding them together. What's the difference between eleven and twelve?

Elise: One has more than the other.

T: How much more?

Elise: Twelve has one more and eleven has one less

T: Twelve has one more.

Elise: And then eleven has one less.

T: What do you notice about the three and the two?

Debbie and Allison raise their hands

Elise: Three has one more and two has one less.

T: So the one over here then has one…

Harry: less

Elise: less

T: So if this has one more, what is going to have to happen to this number?

Allison: Take away the three

T: How many am I going to have to take away from the three?

Allison: One

T: So if this goes up by one, this has to go…

Debbie: down

T:Down by one

Harry: That was easy.

<u>Symbolic Session 8</u>

T: Maria used ten red beads and four blue beads to make a bracelet. Kim used some red beads and ten blue beads for her bracelet. They have the same amount of beads. We know that. They both used the same amount. I want to know, "How many red beads did Kim use". First we're going to write an equation.

Debbie: This is easy.

Matt: This is so easy.

T: Maria used ten red beads and four blue beads. What might that look like? If I wanted to write an equation for Maria.

Elise: Ten plus ten equals blank.

T: Mmmm, I don't think she used ten and then ten more.

Elise: Ten plus four equals blank

T: We know she has ten red beads and four blue beads. *(Writing the equation on the board)*. We're going to say that's Maria. We know she has the same amount as Kim. What would Kim's equation look like right now? I don't want you to actually fill in the some. I don't want you to tell me what that is yet. I just want to make an equation based on my story. Do I know how many red beads she has Allison?

Allison: No

T: What could we put there since right now we don’t know that would be?

Elise: Put a line.

T: Harry are you going to tell me a number to put there or are you going to tell me something else?

Harry: Four

Jessica: No, ten!

T: I don’t know how many red beads she has, so I’m going to put a box. It’s a mystery.

Anthony: I’m going to use a box.

Jessica: I know what it is.

T: We have some plus what Stephen?

Stephen: We have some plus what?

T: Some red bead plus how many blue beads?

Stephen: Ten.

Debbie Done.

Teacher wrote the equation on the board

Stephen: But ten plus four equals. It’s equal, it’s equal!

Matt: Oh yeah!

T: Ten plus four plus equals something plus ten. What’s that number going to be?

Matt: Four.

T: Touch your nose when you know.

All students touch their nose

Allison: Four.

T: Thumbs up if you think that's correct. Thumbs to the side if you think we need to fix something.

Anthony, Jessica, Harry, Matt, and Stephen put their thumbs up. Debbie is writing something. Elise puts her thumb to the side.

T: Elise what do you think it is?

Elise: Fifteen because ten plus four is fifteen.

Debbie: No.

T: Ten plus four is how much? I think you counted wrong.

Elise: Fourteen

T: Yeah, ten plus four is fourteen. Ten plus four is fourteen plus ten. *(Put the fourteen in the box)* What do you think? Does that work?

Debbie: No!

Jessica: Four!

Matt is putting a thumbs down.

T: Why do we need to fix this? Why does this not work?

Matt, Jessica, and Stephen raise their hands.

Jessica: Because four plus ten does not equal fourteen.

Debbie: Plus ten.

Elise: Yes it does

T: Ten plus four is fourteen. That is true.

Stephen: But ten plus fourteen is not fourteen.

T: Oh, because I have this ten here. Can I just rid of it Elise?

Responses of no

T: No, I can't just say "just kidding…"

Jessica: The fourteen is hiding somewhere.

T: The fourteen is hiding somewhere. We need to get to fourteen, so I like how she thought of that idea.

Elise: Dang it. I'm always wrong.

T: We know that these two together make…

Allison: Fourteen

T: Or we also know there are two tens right, so what should this be?

Jessica: A four.

T: Where's that fourteen then Elise?

Elise: I…

T: If ten plus four equals four plus ten, where's the fourteen you were looking for?

Elise: I don't know.

T: What does equal mean again?

Responses of is the same as

Elise: Is the same as.

T: Is the same as. We know ten plus four is fourteen, and we want four plus ten is the same as. So where would the fourteen be on this side?

Elise: Oh, *(starts to come to the board to show)*

T: Where's the fourteen on this side? We already know where it is on this side, ten plus four. So where is it on this side?

Elise: These two numbers should flip around so it's still fourteen.

T: Do we have to flip them?

The other students respond no

Elise: They're just backwards. They're the same numbers just backwards.

T: They're the same numbers; they're just in a different order. Is that ok?

Responses of yes

Elise: So that's fourteen

T: So that's still fourteen. Like Jessica said, it's hiding. Thank you Elise. You can go sit down now. What does this ten represent in our story?

Elise: The beginning of it?

Jessica, Anthony, and Stephen raise their hands.

Anthony: The blue beads.

T: This is Maria. This ten is the…

Jessica: Ten beads

Elise: The red beads

T: Elise and Jessica, I am really glad you are participating, but I've called on Anthony so make sure you are listening to Anthony. Alright Anthony, so what's that ten?

Anthony: The red beads.

T: What's this four then Stephen?

Stephen: Fourteen, fourteen.

Debbie: No.

T: No, what is this four?

Stephen: Blue beads.

T: That is the four of Maria's blue beads. What's this four? Is that the same?

Responses of no

Matt: Red beads.

T: That's Kim's red beads. What's this ten?

Elise: Blue beads

Debbie: Her blue beads.

T: Elise says that these were switched. Why did I put them in this order? Why is the ten second? Why didn't I put them first?

Harry: Because it's the opposite of that.

T: It's the opposite. Here she had ten red….

Stephen: It'd be the same answer.

T: It's still the same number of beads, but is it the same color of beads?

Elise: I learned that because I solved my math on homework and there were two numbers that were the same.

T: Ok, I want you to try this one on your own. Are you ready for it? Maria used nine red beads and three blue beads. Kim used some red beads and nine blue beads.

Debbie: Oh my gosh, this is easy.

T: They have the same amount. I want you to write an open equation. Last time we had ten plus four equals something plus ten. I want you to write an open equation like this. You can use cubes if you need. If you want to you can write your answer inside that box, but I want you to keep it as a box.

Students begin working independently.

T: So we have? What is this nine?

Jessica: The red beads.

T: What is this three?

Jessica: The blue beads.

T: Ok, and we had some red beads. How many blue beads did she have?

Whole Group

Stephen: This is so easy.

Elise: I don't know what it is Stephen. How dare you say that? Oh wait. I….Twelve!

Stephen: I get it! *Look of comprehension on his face.* It's twelve the answer is twelve.

T: What is the answer? What am I looking for? Do I want the total amount of beads Stephen?

Responses of no.

Stephen: Oh, no.

T: So is my answer twelve? No, I want to how many red beads Kim has, so what would my answer be?

Elise: *(sounding exasperated)* No, I just wrote that.

Jessica comes to the board to write the open equation.

T: That's not what you had on your paper? I want you to write what you had on your paper. There you go. Thank you. Ok, so I'm going to around, and I want you share what you think should go in that box. If someone says something different than you, I don't want you to say no. I'm going to give everybody a chance.

Harry: Hmmm, three

Matt: three

Stephen: three

Debbie three

Elise: paused

T: Do whatever you really think it is, even if it’s different than someone else.

Elise: twelve

Allison: three

Anthony: three

Jessica: twelve

T: Um, Jessica, why do you think it’s twelve?

Jessica: Because nine plus three is the same as twelve.

T: Nine plus three is the same as twelve, but what do I have over here Elise?

Elise: Nine

T: Can I just get rid that nine?

Responses of no.

T: Huh, so if I put nine plus three is the same as twelve but then I have plus nine, is that true?

Responses of no.

T: That’s not true, so can twelve go in that space Elise?

Elise: No

T: Jessica can twelve go in that space?

Jessica: No

Jessica gets excited and raises her hands enthusiastically like something has dawned on her

T: Because I have that nine there. So what is it going to be? Where is the twelve? You said, oh nine and three is twelve. Where is that twelve though?

Stephen: It’s hiding

T: Allison?

Allison: I don't know

Harry: It's hiding

T: It's hiding so we need to make twelve.

Elise: Wait, wait, wait, wait, wait.

T: We know that we need to make twelve. What is it that we need to make twelve?

Elise: Put a three right there

Jessica is jumping up and down

Elise: It's the same thing as last time. The same numbers but they're switched around.

Stephen: It's easy now.

T: Yeah, there's a nine on this side and a three on this side. If there's a three on this side, what should probably go there?

Elise: A three

Stephen: Because nine plus three is right there but it's switched around.

Elise: It's the same thing as last time, but just swapped around just like last time.

T: Now I'm going to give you another problem. Maria used nine red beads and three blue beads. That's staying the same. Kim used some red beads and eight blue beads.

Debbie: Ooo

T: We're going to make the equation together so you can decide what might need to go there.

Stephen: It's a take-away.

Debbie: No it isn't.

Stephen: Yes it is.

T: Hold on to that thought. Let's start with Maria. What might that equation look like?

Debbie Nine plus three…

Harry: This is easy.

Debbie: Equals blank plus eight

T: Let's look at this. Nine plus three is the same as blank plus eight. What do you think is supposed to go in that box?

Harry: This is easy.

T: Touch your nose when you think you know. That way everyone has a chance to think.

Elise: I don't know this one.

T: I'm glad that a lot of you are feeling confident, but when you say this is easy and someone else does not think this is easy, how do you think that makes them feel?

Responses of sad

T: It probably makes them not feel very smart, and do we want anyone to not feel smart? No because everyone is smart as long as you work hard and put in the effort.

Elise: I don't get this one because it's not the same numbers.

T: It's not the same number. Huh, that's tricky. So nine plus three is the same as something plus eight. Elise said this is a little bit harder because it is not the same numbers on both sides now.

Elise: They're not switched around.

T: We didn't switch them around the same.

Harry: I know. But I know the answer.

T: Allison what do you think it is?

Allison: Four

T: Thumbs up if you think that's right or thumbs to the side if you're not sure or think we might need to change something?

All have thumbs up except Anthony and Elise. Jessica has cubes in her hand.

T: Those of you who agree with Allison, why do you agree? Why do you think it's a four?

Stephen: Because nine plus three equals twelve and eight plus four equals twelve.

T: Ok so he's finding that hidden quantity. He's saying there's my twelve, I need to get twelve on that side. Is there another way we could look at it?

Harry: Nine is bigger than eight and three is smaller than four.

T: Nine is bigger than eight by how much?

Harry: One

T: Nine is one bigger than eight. What about three? If nine is bigger than eight, what happened to that extra one? Where did it go?

Matt: It got traded.

T: It got traded. It got traded where Elise?

Elise: Um,

T: The nine when down one, and now it's eight. Matt said it got traded.

Debbie enthusiastically raised her hand. Stephen took a deep breath almost like a gasp of recognition.

T: Would you like Debbie to help you?

Elise: Yes.

Debbie: The nine went….it traded…the nine traded one to the three and then the three became a four and the nine became an eight.

T: So when I took that one away, now it had to go to the four. *(Showed using a plus one on the equations)*

T: Those of you who thought it wasn't four, did that change your mind?

Anthony: Yes.

Elise: And when I heard Debbie talk.

T: Alright one more. Maria used eight red beads and four blue beads. Kim used some red beads and seven blue beads for her bracelet. They still have the same amount of beads. How many red beads did Kim use? Let's write that equation.

Stephen: Eight plus four…

He and Matt start writing the equation on their own.

Stephen: Eight plus four equals seven

T: Something, right?

Stephen: Plus seven

T: I'll give you a few minutes to figure it out.

Matt: I already figured it out.

Students begin working independently or with a partner to solve the question.

Side conversations

Harry: It's a five.

T: How do you know it's a five?

Harry: It took one from there.

T: Ok.

Harry: And the four become a five.

T: So we took one from the eight and we gave it to the four to make a five?

T: (*to Allison)* What do you think it's going to be?

She's using cubes and looking at the board.

T: Are you trying to figure out what eight plus four is first?

Allison nods her head.

T: Why do you think it's five?

Stephen: Because eight plus, oh.

T: I'm not telling you you're right or wrong. I just want you to tell me why you think that.

Stephen: Because…Wait. Can I use my blocks?

T: Mmm-hmm

Stephen gets cubes out. He begins working on a tower of eight.

T: *(to Jessica)* Eight plus four is the same as four plus seven

Unfortunately it stopped recording at this point.

<u>Symbolic Session 10</u>

Allison and Harry were absent

T: Maria had ten red beads and three blue beads. Kim had nine red beads and four blue beads. Did they use an equal amount of beads? The first thing we're going to do is write an equation. What would Maria's equation look like?

Debbie: Ten plus three equals nine plus four.

T: We need to find out if this equation is true.

Stephen: It is.

T: You're going to write it on your paper. You're also going to write yes or no whether you agree it's the same amount or you disagree. Then say because….and you're going to tell me.

Side conversations

T: So what do you think Debbie? Do they have the same amount?

Debbie: Yes.

T: How do you know?

Debbie is writing on her paper

T: How do you know it's true?

Matt: Because ten plus three equals thirteen and so does nine plus four.

T: Ok. How could you show me that? Could you write those as two equations maybe? Matt, I'm going to write down what you said. Ten plus three is thirteen and so is nine plus four.

Whole group

T: Thumbs up if you think yes they have the same amount

All students have their thumbs in the air.

T: Who can tell me how you know they both had the same?

Jessica: It's because ten plus three is the same as nine plus four.

Elise: They're both thirteen.

T: How do you know they're the same?

Jessica: They both equal thirteen, but they're hiding somewhere.

T: They both equal thirteen. (*Notated it with a tree diagram)* It's kind of that hidden thirteen. Is there another way we could explain they're both the same?

Jessica: Can they add something to mine?

T: They could add something to it. But what do you notice about how those numbers have changed?

Jessica: Oh, now I know why.

Debbie: Oh.

T: I'm going to erase the thirteen for a minute so we can really focus on the ten plus three and the nine plus four.

Jessica: Oh, something just came into my head.

T: What do you notice about how those numbers have changed?

Debbie: Ten got minused one it turned into a nine. And the minused one got given to the three and it made four.

T: Ok. So this was one more *(wrote plus one under the four)* and over here that was one more *(wrote plus one under the ten).*

Debbie: That's one more. It says one more.

T: Yeah, because over here the ten has one more, but over here the four has one more.

Stephen: Oh.

T: So is that true? Does ten plus three equal nine plus four?

Anthony Yes because nine is one less than ten and four is one more than three so it has to be a bigger number with nine and a smaller number with ten.

Debbie: That's what I said.

T: I'm going to restate that, and that's very similar to what Debbie said. You said the nine was one less than the ten so the four had to be one…

Debbie, Anthony, Stephen and Matt: One more than the three.

T: One more than the three because here we have an extra so we need that extra somewhere over here.

Stephen: If you switched the four to the three it would be fourteen though.

T: If I did this? *(Erasing the three and the four and placing them in the opposite locations*) Is this what you mean Stephen?

Stephen: Yeah.

T: Then would they be the same?

Most respond no

Elise: They could still be the same though.

T: Are they still the same?

All but Elise say no. Elise responds yes.

T: Ten plus four equals nine plus three. Is that true?

Adamant responses of no

Elise: But it's still thirteen.

T: Most of you are saying no. Why not?

Elise: Wait, that is the same because…

Stephen: Ten plus four equals fourteen and nine plus three equals twelve.

Elise: They're the same numbers though.

T: They're the same numbers, but are they in the same places?

Adamant responses of no

T: Does the location, does the order matter? Does where they are matter?

Adamant responses of yes

Jessica: I noticed something. Ten plus four equals fourteen but if this changed to five and that one changes to five…

T: So if we had nine plus five? *(erasing the three and putting a five in the equation)*

Jessica: It would equal fourteen.

Stephen: It would equal fourteen.

T: So if we have nine plus five then those would be the same?

Anthony Yeah, that's what I was going to say.

T: Ok, so why does it not equal the same here then? If you said ok, we're going to switch those…

Jessica: Because it's twelve.

T: Can someone explain a different reason?

Jessica: I don't know.

Elise: I don't know.

Anthony and Debbie are raising their hands.

T: What do you think Matt?

Matt does not respond after a long pause

T: Besides that this if fourteen and this is twelve, how else could we know that would not be equal? Can we use that idea that Anthony and Debbie had about what's changing and what's staying the same?

Matt: The three and the four changed spots but the ten and the nine stayed in their spots.

T: Ok. So how did the ten change to a nine? Elise what's the difference between ten and nine?

Elise: One's more than the other and one's less.

T: Ok, so here we have the extra one over here right? *(wrote a plus one underneath the* *ten*)   I'm going to write up here what we had before. *(wrote the previous equation:10+3=9+4)* Ten plus three equals nine plus four. Was that true?

Responses of yes

T: Ok, so we had an extra one on this side. What about for the four and the three? How did the four change?

Only Debbie is raising her hand

T: Where's the extra one? Stephen, what do you think? Which one has more, the four or the three?

Stephen: The four.

T: Ok, so this has an extra one. *(wrote a plus one underneath the four of 10+4=9+3)* What about here, where was the extra one?

Matt: It was right here. *(pointing to the four on 10+3=9+4)*

T: Oh, so on this side we had an extra one over here. *(wrote a plus one underneath the four of 10+3=9+4)* Do we have another extra one somewhere?

Anthony: Yes.

T: Anthony can you come show us?

Anthony: Right in the ten *(pointing to the ten of 10+3=9+4)*

T: *(wrote a plus one under the ten of 10+3=9+4)* So in this one we had an extra one there. What's the difference between this equation and this equation?

Matt: One has a three at the end and the other has a four.

T: Ok, but I want you to look at where the extras are. Maybe I should have said it that way. What's the difference about where those extra ones are?

Stephen: uh, uh

Matt: Um…So where's the extra one in both equations?

T: Yes. So in this one the extra ones were on the same side or different sides?

Stephen: Different.

T: This one. *(indicating the equation 10+4=9+3)*

Anthony, Stephen, and Debbie: Same.

T: Those extras are on the same side. Where are those extras here?

Debbie: Different sides.

T: Does that matter?

Anthony Yes

Debbie: Yes

Stephen: Yes

Matt: No.

Elise: It doesn't matter.

Debbie: Yes!

Anthony It does. It does.

T: Well let's see. What was ten plus three?

Stephen: Thirteen.

T: What was nine plus four?

Responses of thirteen

Matt: It doesn't matter.

T: What was ten plus four?

Responses of fourteen.

T: What was nine plus three?

Responses of twelve.

Debbie: So it does matter.

Elise: No it doesn't.

Debbie: Yes it does.

T: What's the difference between fourteen and twelve? Are those equal?

Responses of no

T: How much different are they?

Matt: Um…

No responses

T: If I counted down to fourteen, let's count down. How many extra does this side have?

Responses of two

Stephen: The extras would be two *(a look of comprehension)*

T: When they were on the same side how many extras did it have?

Responses of two

T: But when they were on different sides did either of them really have extra?

Responses of no

T: Ok, let's try this one. It's going to be with bigger numbers. These numbers are tricky. You guys have been doing oh that's thirteen. You probably won't be able to add these in your head, and there's a reason because I want you to look at the relationship. Like that idea Anthony and Debbie had about where's the extra. Maria had thirty-seven red beads and twenty-seven blue beads. Kim had thirty-eight red beads and twenty-seven blue beads.

Shouts of No! This is going to be hard.

Matt: This is…this is…I think I already know it.

T: Let's write the equation for it. What would the equation for Maria look like?

Stephen: Thirty-seven plus twenty seven

T: We want to know if that is equal to Kim, so right now we're going to write it as equal. Then we'll have to decide. What would Kim's equation look like? *(37+27=38+28 is written on the board)*

Matt: Thirty-eight plus twenty-eight.

Stephen: That's really hard. Can we use all the blocks?

T: you can use cubes if you need to. See if you can try one of Debbie's ideas or Anthony's ideas where they didn't actually find out what this answer is. Do I have to know what thirty-seven plus twenty-seven is? Do I have to know what thirty-eight plus twenty-eight is?

Mumblings of no

T: You're going to write yes because or no it is not because…You might have to work with a partner because I don't think we have quite enough blocks for you all to do it independently.

Side conversations

T: Ok, Debbie what do you think?

Debbie: That equals fifty seven and that equals fifty eight.

T: Close but not quite. How did you come up with fifty-seven?

Debbie: Three plus two equals five.

T: Thirty plus twenty equals fifty. But that doesn't equal fifty-seven. What did you forget to add?

Debbie: The seven.

T: So you did thirty plus twenty which was fifty, but you forgot to add the sevens.

Matt: Miss Amber, we figured it out.

Stephen: We figured it out.

T: Ok, so write it. Yes or no because and tell me why.

Continued discussion between Matt and Stephen. It is inaudible though for a while.

Stephen: Because thirty-seven and twenty-seven, they both have sevens in them.

T: Ok, Anthony. What do you think? Did they use an equal number of beads?

Anthony: No.

T: You seem pretty confident. How do you know?

Anthony: They um…

T: Is it ok if I write it down? You said no because, why?

Anthony: Because thirty-seven and twenty-seven….*long pause*. *He's looking at the board and appears to be thinking.*

T: Can you use your other idea? Here you said that you knew these were the same because the ten had an extra and then therefore the four had an extra to make up for it. What about here? Can we use that same idea?

Anthony: Yeah *paused*

T: Ok. So where do you see some extras?

Anthony: Ten extra.

T: There's ten extra here? So here you've compared your ten and your nine?

Anthony: One extra in this and one extra in this one.

T: Ok, so thirty-eight has one extra, and then you also said twenty-eight has one extra. So are they the same?

Anthony Mmm, mmm.

T: Why not?

Anthony: Two, one from this one and one from this one so they don't equal the same.

T: Ok, what numbers could we use to make them the same?

Anthony: Seven plus this eight and move it with this one.

T: So if we had thirty-eight plus twenty seven?

T: What do you guys think? It's not true.

Stephen: Because thirty-seven and twenty-seven does not equal thirty-eight plus twenty-eight.

T: Ok, how do you know?

Matt: Thirty-seven plus twenty-seven equals forty-seven and thirty-eight plus twenty-eight equals forty-eight.

T: Mmm

Stephen: That's pretty close.

T: I know that thirty-seven plus twenty-seven doesn't make forty-seven. But I think you guys are picking up something about the sevens and the eights. Am I correct?

Stephen: Yeah.

T: So you said there's two sevens over here. Did I hear that from you Stephen? And you said there's two eights over here. Can we use that trading idea?

Stephen: Yeah.

T: Let's think about that for a minute. How did this number change? Think about that idea we had. We wanted to keep it balanced, or how these number are changing. Then think about how that number is changing.

They go back to discussion on their own.

T: Ok ladies what do you think?

Jessica: We don't know yet. We're still counting. *They are creating cubes towers for the numbers in the problem.*

T: You don't know yet. What do you think? Without counting yet, just looking at it, do you think they're the same?

Jessica: Yes

Elise: *(Asking Jessica*) Do you have twenty seven?

T: Elise, before you do it, I want you to look at it. Looking at thirty-seven plus twenty-seven and thirty-eight plus twenty-eight, do you think they are the same?

Jessica: No.

T: *(Looking at Jessica)* You say no. *(Addressing Elise)* Do you think they have the same amount?

Jessica: Yeah.

T: What do you think Elise? I'm not going to tell you if you're right or wrong? There's no wrong answer right now. I just want to know what you think.

Jessica: Yeah.

T: Would you like Jessica to maybe explain why she said no? Would that be helpful?

Elise: I think they're the same with Jessica.

T: So you think they are equal?

Jessica: Wait, yes, yes, yes, yes.

T: How did this number change?

Elise: I don't know.

Jessica: Plus seven, plus seven

T: We had thirty-seven now we have thirty-eight. What's the difference between those?

Elise: These…

Jessica: *(interrupting)* These two are the same, but this one got added up.

Elise is moving her pencil back and forth to point to thirty-seven and thirty-eight.

T: Oh, so they're not the same. This is one more. What about these numbers?

Elise: They're the same thing.

Jessica: This one got add one too.

T: This one also got one more.

Jessica: Yeah, they're the same it's just that this one's got twenty-seven and this one has twenty-eight. And this one has thirty-seven and this one has thirty-eight.

T: Are they the same then? The same means they have the same amount. Do they have the same amount?

Elise: Uh-uh

Jessica: No

T: Because what did you say about over here?

Jessica: Because this one is thirty-eight and this one is twenty-eight.

T: So both of these if I heard you earlier are up by one.

Stephen: But if you put that twenty-seven over there and the twenty-eight over there it would be equal.

T: Hold onto that idea.

Whole Group

T: Ok. Stephen, I know you have a good idea. We're going to talk about something else first, but hold onto that idea. First of all, thumbs up if you think yes.

No thumbs up

T: Thumbs up if you think they did not use the same amount of beads.

All put their thumbs up

T: I got a lot of Nos. I agree. They didn't have the same amount. Now you have to prove it to me. Why not?

Elise: I can't do it. I can't do it.

Jessica, Stephen, Anthony, and Debbie are enthusiastically raising their hands.

Jessica: It's because twenty-eight.

Stephen: It's equal. I know it. It's equal.

T: So you said no.

Stephen: I said yes.

T: Jessica, why are they not?

Jessica: Because twenty-eight has one more than twenty-seven and thirty-eight has one more than thirty-seven. *(T wrote the plus one under the two numbers in the equation with extra)*

T: We again have that same idea. We have two extra over here. Will that make them the same?

Responses of no.

T: Stephen what did you say?

Stephen: I said twenty-seven can switch to the twenty-eight and twenty-eight can switch to the twenty-seven. It would be equal.

T: Stephen said that if I switch these and put thirty-seven plus twenty eight equals thirty-eight plus twenty-seven, are those the same amount? *(Showed that on the equation)*

Mixed responses of yes and no

Matt: Yes! Yes!

Jessica: No.

Elise: No.

T: Ok. Thumbs up if you think that Stephen's idea will work. If I switch those numbers they would be the same.

Debbie, Matt, and Stephen put their thumbs up.

T: If you think Stephen is not correct put your thumbs up.

Anthony, Elise, and Jessica put their thumbs up.

Debbie: I don't know.

T: Let's look at this.

Anthony: Wait, yes.

T: Last time we had thirty-seven plus twenty-seven and thirty-eight plus twenty-eight. Was that right? *(Wrote that equation on the board as well*)

Responses of No!

T: But those were the numbers we had before, and Jessica said there was an extra here and an extra here. *(Writing those extra on the board as plus ones)* They were on the same side so they weren't the same. That's what you agreed on the last one. So let's look up here. Where are the extras?

Debbie: (*points to the thirty-eight and the twenty-eight in 37+28=38+27)*

T: So here the extras are on the same side or different sides? *(referring to 37+27=38+28)* Where are those extras Elise?

Elise: The same side.

T: Where are they here? *(referencing the other equation)*

Anthony: Different.

T: I have an extra on different sides. Remember back to 10+3=9+4. I had an extra on each side. *(Wrote that equation on the board)* Were those the same?

Responses of yes

T: What about when we had 10+4=9+3, where were the extras? *(wrote that equation on the board)*

Debbie: On the same side.

T: Were those equal? Is thirty-seven plus twenty-eight equal to thirty-eight plus twenty-seven?

Mixed responses still

T: Anthony, why do you say no?

Anthony: Thirty-eight has ten more than twenty-eight.

T: Ok. But remember we're looking at these first numbers. This is thirty-seven and thirty-eight.

Anthony: Thirty-eight.

T: It changed to a thirty-eight which was a difference of…

Anthony: Oh (*smiling)*

T: Did that change your mind?

Anthony: Yes.

T: You're right. This is ten more, but we're comparing these two numbers. Do you think they're equal now?

Anthony: Yep

T: Because those one extras are now on…

Debbie: Different sides

T: They're on both sides. It's kind of like that balance idea. But here, what's going to happen?

Debbie and Stephen: It will fall off.

T: Because there's two extra on the same side.

APPENDIX D

IRB Approval Letter

BOISE STATE UNIVERSITY
RESEARCH AND ECONOMIC DEVELOPMENT

Date: February 24, 2016

To: Amber Van Vooren cc: Jonathan Brendefur

From: Social & Behavioral Insitutional Review Board (SB-IRB)
c/o Office of Research Compliance (ORC)

Subject: SB-IRB Notification of Approval - Original - 107-SB16-035
First Graders' Use of Bar Models to Communicate Their Understanding of the Equal Sign

The Boise State University IRB has approved your protocol submission. Your protocol is in compliance with this institution's Federal Wide Assurance (#0000097) and the DHHS Regulations for the Protection of Human Subjects (45 CFR 46).

Protocol Number:	**107-SB16-035**	Received:	2/17/2016	Review:	Expedited
Expires:	**2/23/2017**	Approved:	2/24/2016	Category:	7

Your approved protocol is effective until 2/23/2017. To remain open, your protocol must be renewed on an annual basis and cannot be renewed beyond 2/23/2019. For the activities to continue beyond 2/23/2019, a new protocol application must be submitted.

ORC will notify you of the protocol's upcoming expiration roughly 30 days prior to 2/23/2017. You, as the PI, have the primary responsibility to ensure any forms are submitted in a timely manner for the approved activities to continue. If the protocol is not renewed before 2/23/2017, the protocol will be closed. If you wish to continue the activities after the protocol is closed, you must submit a new protocol application for SB-IRB review and approval.

You must notify the SB-IRB of any additions or changes to your approved protocol using a Modification Form. The SB-IRB must review and approve the modifications before they can begin. When your activities are complete or discontinued, please submit a Final Report. An executive summary or other documents with the results of the research may be included.

All forms are available on the ORC website at http://goo.gl/D2FYTV

Please direct any questions or concerns to ORC at 426-5401 or humansubjects@boisestate.edu.

Thank you and good luck with your research.

Mary E. Pritchard

Dr. Mary Pritchard
Chair
Boise State University Social & Behavioral Insitutional Review Board

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This letter is an electronic communication from Boise State University