ON THE CONJUGACY PROBLEM FOR AUTOMORPHISMS OF TREES

by

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ABSTRACT

In this thesis we identify the complexity of the conjugacy problem of automorphisms of regular trees. We expand on the results of Kechris, Louveau, and Friedman on the complexities of the isomorphism problem of classes of countable trees. We see in nearly all cases that the complexity of isomorphism of subtrees of a given regular countable tree is the same as the complexity of conjugacy of automorphisms of the same tree, though we present an example for which this does not hold.

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CHAPTER 1

INTRODUCTION

Many classification problems in mathematics can be encoded as equivalence relations on a standard Borel space of objects. For instance, some common, or otherwise studied, classification problems arising in mathematics are

- Banach spaces up to isometry or isomorphism.
- Separable C^*-algebras up to isomorphism.
- Countable divisible groups up to isomorphism.
- Countable torsion-free groups up to isomorphism.
- Ergodic actions of a countable group up to conjugacy of the action.

For a more concrete example we consider countable graphs. We note that a countable graph can be encoded as a binary relation on ω, as an element of $2^{\omega\times\omega}$. The isomorphism classification problem of graphs can be represented as an equivalence relation on a subset $2^{\omega\times\omega}$, consisting of the symmetric and irreflexive binary relations, corresponding to the set of all countable graphs. The equivalence relation is defined as follows: we say that α, β, symmetric and irreflexive binary relations on ω, are equivalent if, and only if, the graphs produced by considering α and β as edge relations are isomorphic. Equivalently, we can conclude that α, β countable graphs are equivalent if there exists a permutation σ of ω such that $\{\, \sigma[e] \mid e \in \alpha \,\} = \beta$.

This is just a single case of a general family of examples arising from model theory. In a more general case, we can drop the requirement that our structure being encoded is a graph, and instead examine any Borel class of countable structures $\mathtt{Mod}(\mathcal{L})$ of a countable language $\mathcal{L}$.

Finally, in the most general case, we have arbitrary standard Borel spaces with arbitrary equivalence relations. We will introduce this theory in the following chapter on Borel reducibility.

In this thesi,s we will identify the complexity of the classification of automorphisms of trees up to conjugacy. In many of our key examples, we will notice that the complexity of the automorphism group of a fixed regular tree is just as complex as the class of subtrees of that fixed tree up to isomorphism. Conjugacy is an example of the general case of isomorphism of structures. If $\mathcal{L}$ is a countable language and M a model of that language, we have that if $\alpha, \alpha' \in \mathrm{Aut}(M)$, the two are conjugate if and only if the expanded structures (M, α) and (M, α') are isomorphic. Here (M, α) denotes the structure in the language $\mathcal{L}' = \mathcal{L} \cup \{ f \}$, where $f \notin \mathcal{L}$ is a new function symbol, where α is the interpretation of the new function f.

Invariant descriptive set theory provides a notion of reducibility, which will make rigorous our admittedly vague use of the word complexity up until now. In particular, we will be using Borel reductions, which is an equivalence preserving Borel mapping. The theory will also give us numerous complexity benchmarks with which to compare our examples.

For instance, we have developed some basic idea concerning the complexity of $\mathtt{Mod}(\mathcal{L})$ for a countable language $\mathcal{L}$, which contains at least one binary relation. We'll start by introducing what will become the 'top' or most complex among model-theoretic examples. We call this complexity Borel complete, which we will take as definition to be exactly the complexity of $\texttt{Mod}(\mathcal{L})$ up to isomorphism for countable $\mathcal{L}$. For a more illuminating definition, we say that Borel complete is exactly the maximum complexity formed by models up to isomorphism of all countable languages.

At the 'bottom' of the complexity hierarchy are the following simple relations. First, $\texttt{id}(n)$, which denotes the complexity of standard equality on the finite natural number n. Next, $\texttt{id}(\omega)$ is the least complex of any equivalence relation over a countable class, and is defined to be standard equality of elements of ω. Neither $\texttt{id}(n)$ nor $\texttt{id}(\omega)$ will be present in the diagram of our results. Then, there is $\texttt{id}(2^\omega)$, which will be the lowest complexity present in Figure 1. Silver's dichotomy gives that $\texttt{id}(2^\omega)$, or equality on a Polish space, follows immediately in terms of Borel reducibility after equality on ω. We will often denote this complexity as $\texttt{id}(2^\omega)$ or $\texttt{id}(\omega^\omega)$ depending on context or for ease of proof. More about this complexity will be shown in Chapter 2.

Of the other key benchmarks of intermediate complexity between $\texttt{id}(2^\omega)$ and Borel complete, the next complexity we will be using is E_∞, which denotes the universal "countable" Borel equivalence relation. That is, the relation is universal for Borel equivalence relations where each class is countable.

The next family of intermediate complexity examples can be defined using the jump operation, defined as follows: Given a Borel equivalence relation E on X, we define E^+ on X^ω by the equivalence relation formed by countable sequences of X where two sequences are E^+-equivalent exactly when the sets of E-classes of of the sequences are set-wise equal. In this thesis, we will be using E's where E^+ will be strictly more complex than E, though this need not always be the case. Furthermore, we note that the class of all such equivalence relations we inspect and their jumps are ordered "nicely" by Borel reducibility. However, this is not the case in general.

There exists a multitude of complexities not mentioned in this thesis. In fact, a comprehensive diagram of the best known complexities below Borel complete would be overwhelmingly complex.

$$
\begin{array}{rl}
\text{Borel Complete} \bullet & \left\{\begin{array}{l} (\text{trees on } \omega, \simeq) \\ (\text{countable trees}, \simeq) \\ (\text{automorphisms of countable locally countable regular trees}, \sim) \end{array}\right. \\[1em]
\mathtt{id}(2^\omega)^{+n} \bullet & \left\{\begin{array}{l} (\text{subtrees of } \omega^{n-2}, \simeq) \\ (\mathrm{Aut}(\omega^{n-1}), \sim) \end{array}\right. \\[1em]
\mathtt{id}(2^\omega)^{+} \bullet & \left\{\begin{array}{l} (\text{subtrees of } \omega^{3}, \simeq) \\ (\mathrm{Aut}(\omega^{2}), \sim) \end{array}\right. \\[1em]
E_\infty \bullet & \left\{\begin{array}{l} (\text{Locally finite trees on } \omega, \simeq) \\ (\text{automorphisms of countable locally finite regular trees}, \sim) \end{array}\right. \\[1em]
\mathtt{id}(2^\omega) \bullet & \left\{\begin{array}{l} (\text{finitely branching trees on } \omega, \simeq) \\ (\mathrm{Aut}(b^{<\omega}), \sim) \\ (\mathrm{Aut}(\omega), \sim) \end{array}\right.
\end{array}
$$

Figure 1.1: A figure of our results relative to benchmark equivalence relations.

The diagram is structured such that the mentioned benchmark equivalence relations on the left are increasing in the sense of Borel reducibility from bottom to top. The ordered pairs (X, E) are such that the first term is the space for which the equivalence relation in the second term is defined. The two equivalence relations in this diagram are $\simeq$ for isomorphism and $\sim$ for conjugacy.

We reference in this thesis a proof by Friedman and Stanley showing that the isomorphism relation on the class of countable connected graphs with isomorphism is Borel complete. It then follows from this fact about graphs that the class of countable rooted trees with isomorphism is Borel complete. The proof we present is due to Friedman and Stanley [5]. Finally, as a corollary of the previous fact, we have that the class of countable unrooted trees is Borel complete.

After restricting our focus from arbitrary trees, we arrive at some examples of lower complexity. We present a result of Hjorth and Kechris stating that the class of locally finite trees with isomorphism is bireducible to E_∞ [7]. Furthermore, we show that the class of locally finite rooted trees is smooth [6].

We build on these results by noting that each of these classes has a corresponding regular tree such that each member of the class is a subtree of it. We will investigate and determine the complexity of the automorphism groups of those regular trees for each distinct class in our previous examples with respect to the conjugacy relation. For instance, we prove that conjugacy over the class of automorphisms of the regular b-branching rooted tree $\mathrm{Aut}(b^{<\omega})$, $b \in \omega$, is smooth. Furthermore, we have that the automorphism group of the n-regular tree with conjugacy is bireducible to E_∞, and that conjugacy over the automorphisms of the countably-regular tree (either rooted or unrooted) is Borel complete.

After these results, we can observe that it might seem that there is no complexity gained by moving to the automorphism groups in this way. A natural question is whether one can gain complexity in this way. We answer this question in the affirmative by observing the subtrees of ω^n up to isomorphism and comparing that result to the complexity of $\mathrm{Aut}(\omega^n)$ with conjugacy, which is strictly more complex.

CHAPTER 2

BOREL REDUCIBILITY

In this chapter, we will make clear what we meant by reduction in the previous chapter as well as introduce the basic theory of Borel reducibility. A reduction between two equivalences relations is a mapping between the underlying spaces of those relations such that elements in the domain are equivalent if, and only if, their images are equivalent. For given equivalence relations E and F, if such a reduction from E to F exists, we write that $E \leq F$ and say that E is reducible to F. It is often useful to impose some restrictions to which types of mappings can be used as a reduction function; when this is the case we will include a subscript indicating this (e.g, $E \leq_c F$ to indicate that a witnessing reduction is continuous). We will be working in this paper with a restricted sense of reduction – Borel reductions.

Recall that a topological space, X, is metrizable if there exists a metric on X that induces the topology of X. Furthermore, recall that a space X is separable if X contains a countable dense subset. If X is a separable metrizable space such that the resulting metric is complete, we say that the space is Polish.

If X is a Polish space, then a set $A \subseteq X$ is said to be **Borel** if it is a member of the smallest σ-algebra containing all open sets of X. We say that $f : X \to Y$ is **Borel** if for any open set $U \subseteq Y$, $f^{-1}(U)$ is Borel.

Definition 2.0.1. A **Borel space** is a set, X, with a corresponding σ-algebra, $\mathcal{B}$, of subsets of X, which defines the Borel subsets of X, often written $(X, \mathcal{B})$. A **standard Borel space** is a Borel space, $(X, \mathcal{B})$, when there exists a Polish topology of X such that $\mathcal{B}$ is the Borel sets of X.

Note that for a Borel space, X, X^2, X^n, and even X^ω are all Borel spaces as well by forming Borel sets from open sets in the product topology of each.

Definition 2.0.2. Let E and F be equivalences relations on the standard Borel spaces X and Y, respectively. A Borel function $f : X \to Y$ is said to be a **Borel reduction** from E to F if

$$\forall x, y \in X \; xEy \iff f(x) F f(y)$$

If such a function exists for a given E and F, we say that E is **Borel reducible** to F, written $E \leq_{\mathrm{B}} F$. We also define:

- $E <_{\mathrm{B}} F$ if $E \leq_{\mathrm{B}} F$ and $F \not\leq_{\mathrm{B}} E$.
- $E \sim_{\mathrm{B}} F$ if $E \leq_{\mathrm{B}} F$ and $F \leq_{\mathrm{B}} E$. In this case, we say that E and F are bireducible.

Furthermore, we say that a Borel reduction $f : X \to Y$ is an **embedding** if f is injective. Similar to our notation for Borel reduction, we denote by $E \sqsubseteq_{\mathrm{B}} F$ to mean that there exists a Borel embedding reducing E to F. If it is the case that $E \leq_{\mathrm{B}} F$, we say that E is at most as complex as F.

The following theorem is a classical result that applies to standard Borel spaces.

Theorem 2.0.1 (Borel Isomorphism Theorem)**.** *If X and Y are uncountable standard Borel spaces, then there is a Borel isomorphism from X to Y (i.e, a Borel bijection).*

Finally, we will later use a result of Kechris regarding equivalence relations whose complexity can be encoded in a countable equivalence relation.

Definition 2.0.3. If a Borel equivalence relation is such that each equivalence class is countable, we say the relation is **countable**. We say that an equivalence relation on a standard Borel space is **essentially countable** if there exists a countable Borel equivalence relation to which it is Borel bireducible.

We will make use of the following lemma due to Kechris for essential countability [6].

Lemma 2.0.2. *Let X, Y be standard Borel spaces, E a Borel equivalence relation on X, and $f : X \to Y$ a Borel function. Suppose that*

(i) for any $x \in X$, $\{ f(y) \mid yEx \}$ is countable, and

(ii) for any $x, y \in X$, if $f(x) = f(y)$ then xEy.

Then E is essentially countable.

As mentioned, there exists a large spectrum of known benchmark equivalence relations. Figure 2 depicts only a small portion of the common equivalence relations that we will be working with, and some that will not be defined in this paper. The undefined relations can be found in Gao [6], as well as a more (but still incomplete) detailed figure.

Figure 2 is read in the following way: for two equivalences relations E and F, if E is closer to the bottom of the figure than F and there exists a path from E to F, then E reduces to F. It is important to reiterate that the diagram is incomplete; there exists a multitude of other equivalences relations and reductions not present in the figure.

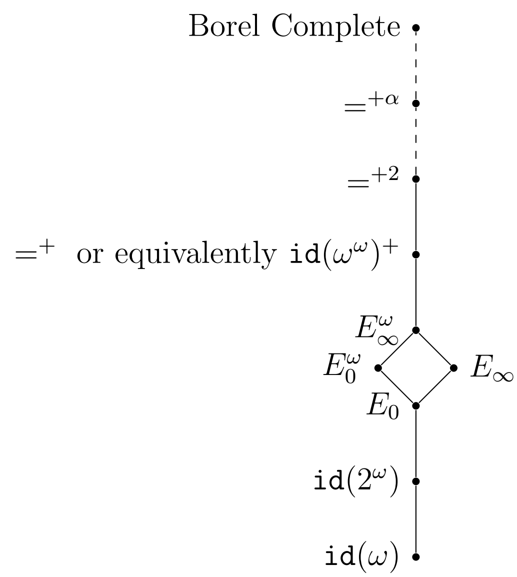


Figure 2.1: A small view of the Borel complexity hierarchy.

The remainder of this chapter will be dedicated to defining most of those equivalence relations present in the diagram. We will forgo mentioning the more technical relations, which will not be mentioned later in the paper, i.e, E_0^ω and E_∞^ω.

Definition 2.0.4. Let X be a standard Borel space. The **identity relation** of X denoted $\mathtt{id}(X)$ (or sometimes $\Delta(X)$) is the relation $\{\,(x, y) \in X \times X \mid x = y\,\}$.

In our case, the least complexity would be $\mathtt{id}(\omega)$. An equivalence relation E is Borel reducible to $\mathtt{id}(\omega)$ when the E-equivalence classes can be labeled with distinct natural numbers. Certainly this is the case when E is over a finite set, though we will see some cases where E is over a countable set and is not reducible to $\mathtt{id}(\omega)$.

2.1 Smooth Equivalence Relations

Definition 2.1.1. For a Borel equivalence relation E, we say that E is **smooth** if $E \leq_{\mathrm{B}} \mathtt{id}(2^\omega)$.

Definition 2.1.2. We call a family $\mathcal{F}$ of subsets of X a **generating family** of an equivalence relation E on X if for any $x, y \in X$ xEy if, and only if, $\forall F \in \mathcal{F}$ $x \in F \iff y \in F$.

Proposition 2.1.1. *Let E be a Borel equivalence relation over a standard Borel space X. E is smooth if, and only if, there is a countable generating family of E.*

Proof. Let E be a smooth equivalence relation over X and $f : X \to 2^\omega$ a Borel reduction witnessing that $E \leq_{\mathrm{B}} \mathtt{id}(2^\omega)$. Define the family $\mathcal{F} = \{\, F_i \in \mathcal{P}(X) \mid i \in \omega \,\}$ where $F_i = \{\, x \in X \mid f(x)_i = 1 \,\}$. We now check that $\mathcal{F}$ is in fact a generating family for E. This holds as for $x, y \in X$ we have that xEy if, and only if, $f(x) = f(y)$, which is the case exactly when $\forall i \in \omega$ $f(x)_i = f(y)_i$, which is sufficient and necessary for $\forall i \in \omega$, $x \in F_i \iff y \in F_i$.

For the converse direction, let $\mathcal{F} = \{\, F_1, \ldots, F_i, \ldots \}$ be a generating family for E. We claim that $E \leq_{\mathrm{B}} \mathtt{id}(2^\omega)$. To show this, we induce a reduction $f : X \to 2^\omega$ defined from to binary ω-sequences, as: for every $i \in \omega$ $f(x)_i = 1$ when $x \in F_i$ and 0 otherwise. We note that xEy if, and only if, $\forall F \in \mathcal{F},\ x \in F \iff y \in F$ and hence xEy if, only if, $\forall i \in \omega,\ f_i(x) = f_i(y)$. Thus f witnesses that $E \leq_{\mathrm{B}} \mathtt{id}(2^\omega)$. $\square$

A consequence of the following theorem is that the class of smooth equivalence relations is linearly ordered with respect to Borel reducibility. While this fact is interesting, it speaks only for a minute subset of the overall class of Borel equivalence relations and is not representative of the rest of the structure. It does on the other hand simplify the process of determining some complexities.

Theorem 2.1.2 (Silver's dichotomy)**.** *For E a smooth equivalence relation, precisely one of the following holds:*

- $E \sim_B \texttt{id}(2^\omega)$
- $E \leq_B \texttt{id}(\omega)$.

The proof of this is relatively simple with the use of the following theorem, which is also attributed to Silver.

Theorem 2.1.3 (Silver)**.** *Let E be a coanalytic equivalence relation over a standard Borel space. Then either E has countably many E-equivalence classes or there are perfectly many E-equivalence classes.*

The proof of this theorem is outside the breadth of this paper, though the original statement and proof can be found in Silver [10]. Alternatively, our presentation of Theorem 2.1.3 and its proof can be found in Gao [6].

Proof of 2.1.2. By Silver's theorem above we have that E has either countably many equivalence classes or perfectly many. In the case that E has countably many, enumerate the E-equivalence classes as C_i for $i \in \omega$ and define $f(x) = i$ if, and only if, $x \in C_i$. This shows that $E \leq_{\mathrm{B}} \texttt{id}(\omega)$.

In the case that E has perfectly many equivalence classes, we have that $\texttt{id}(2^\omega) \leq_{\mathrm{B}} E$ holds by selecting an injection from the perfectly many elements of 2^ω to the perfectly many E-classes. Meanwhile we have that $E \leq_{\mathrm{B}} \texttt{id}(2^\omega)$ by smoothness, hence $E \sim_{\mathrm{B}} \texttt{id}(2^\omega)$. □

The immediate consequence of this dichotomy is that there is nothing Borel intermediate between $\texttt{id}(\omega)$ and $\texttt{id}(2^\omega)$. From the previous theorems, we see that the class of smooth equivalence relations is large. Since any two uncountable standard Borel spaces are isomorphic, it is clear from the definition that E is smooth if, and only if, $E \leq_{\mathrm{B}} \texttt{id}(X)$ for any uncountable Polish space X.

Corollary (Silver)**.** *If E is an equivalence relation with uncountably many equivalence classes, then* $\mathtt{id}(\mathbb{R}) \leq_B E$*.*

2.2 The E_∞ Equivalence Relation

Definition 2.2.1. Let G be a group and X a standard Borel space. We say that X is a **G-space** when it is the case that there exists a Borel $a : G \times X \to X$ satisfying that $a(1_G, x) = x$ and $a(g, a(h, x)) = a(hg, x)$, called the action of G on X. When it is clear from the context, we will write for $g \in G$ and $x \in X$ gx to mean $a(g, x)$. For G a group and X a standard Borel space we denote the **orbit equivalence relation** by E_G^X. Where two elements $x, y \in X$, are orbit equivalent, written $xE_G^X y$, when it is the case that there exists $g \in G$ $(gx = y)$.

A more detailed discussion and presentation of the ideas of this section are available in Jackson, Kechris, and Louveau [8].

Definition 2.2.2. Let $\mathbb{F}_2$ denote the free group generated by two elements. E_∞ denotes the shift equivalence relation of $\mathbb{F}_2$ acting on subsets of $\mathbb{F}_2$ by sending $s \in \mathbb{F}_2$ and $A \subset \mathbb{F}_2$ to $sA = \{\, sa \mid a \in A \,\}$. For $A, A' \subseteq \mathbb{F}_2$, we say that A' is shift equivalent to A, if there exists some $s \in \mathbb{F}_2$ such that $sA = \{\, sa \mid a \in A \,\} = A'$.

Definition 2.2.3. A countable Borel equivalence relation E is called **universal** if for any other countable Borel equivalence relation, F, $F \leq_{\mathrm{B}} E$.

Theorem 2.2.1. *E_∞ is a universal countable equivalence relation.*

The theorem is proved in a sequence of propositions.

Proposition 2.2.2. *Let $1 < n \leq \omega$, and $E = E_{\mathbb{F}_n}^{2^{\mathbb{F}_n \times \omega}}$; then E is a universal countable equivalence relation.*

While we use the above proposition in the proof of Theorem 2.2.1, we will instead show the following slightly stronger result. Feldman and Moore show in [3] that for any countable Borel equivalence relation E on X, there exists a countable group G and Borel action of G on X such that $E = E_G^X$. Though, it is a fact of group theory that any countable group G is a quotient of $\mathbb{F}_\omega$, the free group of countably many generators. Meanwhile, $\mathbb{F}_\omega$ can be embedded as a subgroup of $\mathbb{F}_2$, and hence can be embedded as a closed subgroup of any $\mathbb{F}_n$ for $1 < n \leq \omega$. Thus giving that G itself can be embedded into $\mathbb{F}_n$ as a closed subgroup, therefore to show 2.2.2 we need only to show the following.

Proposition 2.2.3. *Let G_1, G_2 be countable groups, Y a Polish space, and define for $i = 1, 2$ the spaces $X_i = Y^{G_i}$ with the shift action of G_i and the corresponding equivalence relation $E_i = E_{G_1}^{X_i}$. We have then that if $G_1 \leq_\omega G_2$ then $E_1 \sqsubseteq_B E_2$.*

Here the notation for $\leq_\omega$ means: for G and H Polish groups, we say that H is involved in G, denoted $H \leq_\omega G$, if H is isomorphic to a closed subgroup of a quotient of G.

Proof. Let G_1 and G_2 be countable groups. If $G_1 \leq G_2$, then G_1 is a closed subgroup of G_2. If it is the case that $G_1 = G_2$ we are done, we continue assuming that this is not a case and select $y_0 \in G_2 \setminus G_1$. We then define a Borel embedding $\phi : X_1 \to X_2$ as follows

$$\phi(f)(g) = \begin{cases} f(g) & g \in G_1 \\ y_0 & g \notin G_1. \end{cases}$$

Note that ϕ is sending functions $f : G_1 \to Y$ to $\phi(f) : G_2 \to Y$. We now aim to show that ϕ is a Borel embedding.

Certainly, from the definition of ϕ, it follows that if $g_1 \in G_1$ then $\phi(g_1 f) = g_1\phi(f)$.. This is because, for $g \notin G_1$, $\phi(g_1 f)(g) = y_0$. Otherwise, $g \in G_1$ and so $\phi(g_1 f)(g) = g_1 f(g)$. Hence, we have that if fE_1f' then $\phi(f)E_2\phi(f')$, as fE_1f' means that f' is a shift of f by some element $g_1 \in G_1$, which also shows that $\phi(f')$ is a shift of $\phi(f)$ by the previous equality.

For the converse, suppose that $g\phi(f) = \phi(f')$ for some $g \in G_2$. If $g \in G_1$ as well then $gf = f'$ as desired, so $g \notin G_1$. Then $f, f' : G_1 \to Y$ must take constant value y_0, and hence the two are equal.

Now, suppose that G_1 is a quotient of G_2, that is we have an injective homomorphism $\pi : G_1 \to G_2$, and define $\psi : X_1 \to X_2$ as $\psi(f)(g) = f(\pi(g))$. Then, it follows that as $g \in G_2$, $g\psi(f) = \psi(\pi(g)f)$, and hence we have that $E_1 \sqsubseteq_B E_2$.

For the general case, $G_1 \leq_\omega G_2$ is the composition of the above cases. The result then follows from the transitivity of Borel embeddings.

□

Proposition 2.2.4. *Let G be a countable group, $E = E_G^{2^{G\times\omega}}$, and $Y = 3^{G\times\mathbb{Z}}$ with the shift action of $G \times \mathbb{Z}$ and F the orbit equivalence relation on Y. Then $E \sqsubseteq_B F$.*

Proof. We have that $2^{G\times\omega}$ and $2^{G\times\mathbb{Z}\setminus 0}$ are Borel isomorphic as G-spaces hence, without loss of generality, we can regard the two as the same. Define $\phi : 2^{G\times\mathbb{Z}\setminus\{0\}} \to 3^{G\times\mathbb{Z}}$ as follows:

$$\phi(f)(h,n) = \begin{cases} f(h,n) & n \neq 0 \\ 2 & n = 0. \end{cases}$$

We then note that, for any $g \in G$, $\phi(gf) = (g,0)\phi(f)$. Now, if we suppose that $\phi(f') = (g,m)\phi(f)$ for some $g \in G$ and $m \in \mathbb{Z}$, then $m = 0$, and hence $f' = gf$. This is because, if $m \neq 0$ then $\phi(f)(h,0) = 2$ for all $h \in G$ and hence, $\forall h \in$

G, $\phi(f')(h,m) = (g,m)\phi(f)(h,m) = \phi(f)(g^{-1}h,0) = 2$. Though this contradicts that $\phi(f')(h,m) = f'(h,m) \in \{0,1\}$. Hence, shifts of a function in either space are preserved through ϕ, and so ϕ is a reduction. $\square$

Proposition 2.2.5. *Let G be a countable group and $E = E_G^{3^G}$. Let F be the orbit equivalence relation of the shift action of $G \times \mathbb{Z}_2$ on $2^{G\times\mathbb{Z}_2}$. Then $E \sqsubseteq_B F$.*

Proof. We encode elements of 3 as 00 for 0, 10 for 1, and 11 for 2, and define a function $\phi : 3^G \to 2^{G\times\mathbb{Z}_2}$ as follows:

$$\phi(f)(h,i) = \begin{cases} 0 & \text{if } f(h) = 0 \text{ or } (f(h) = 1 \text{ and } i = 0) \\ 1 & \text{if } f(h) = 2 \text{ or } (f(h) = 1 \text{ and } i = 1) \end{cases}$$

Note that if $g \in G$ we have that $\phi(gf) = (g,0)\phi(f)$. In the other direction, if we have that $\phi(f') = (g,i)\phi(f)$ for some $g \in G$ and $i \in \mathbb{Z}_2$, leaving us to check the two options for $i \in \mathbb{Z}_2$. For the first case, note that if $\phi(f') = (g,0)\phi(f)$, then we have either $\phi(f') = \phi(gf)$ or $f' = gf$. Now suppose that $\phi(f') = (g,1)\phi(f)$. From the encoding ϕ, we have that if $\phi(f')(h,0) = 1$ then $\phi(f')(h,1) = 0$ as well. Similarly it holds that if $\phi(f')(h,1) = 0$ then $\phi(f')(h,0) = 0$. Now, from the original assumption, we have that $\phi(f') = (1_G,1)\phi(gf)$ so without loss of generality we can assume that $\phi(f') = (1_G,1)\phi(f)$. Now we have that if $\phi(f)(h,1) = \phi(f')(h,0) = 1$ then $\phi(f)(h,0) = \phi(f')(h,1) = 1$. We have the same for the case that $\phi(f)(h,1) = 0$. From the previously observed properties of our encoding, we have that, for all $h \in G$, $\phi(f)(h,0) = \phi(f)(h,1) = \phi(f')(h,0) = \phi(f')(h,1)$ and hence $f = f'$. $\square$

Finally, with the previously presented propositions we are prepared to prove Theorem 2.2.1

Proof of 2.2.1 . Using Proposition 2.2.2, we obtain that the orbit equivalence relation of $\mathbb{F}_2$ acting on $2^{\mathbb{F}_2}$ is a universal countable Borel equivalence relation. Now, using Proposition 2.2.4, we have that $\mathbb{F}_2$ acting on $2^{\mathbb{F}_2}$ is Borel embeddable into the orbit equivalence relation of $\mathbb{F}_2 \times \mathbb{Z}$ acting on $3^{\mathbb{F}_2 \times \mathbb{Z}}$. Hence, from Proposition 2.2.5, we have that the latter orbit equivalence relation is Borel embeddable into the shift action of $\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2$ acting on $2^{\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2}$.

Now, $\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2$ is a quotient of $\mathbb{F}_\omega$ and hence, by using proposition 2.2.2 again, we have that the shifting action of $2^{\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2}$ Borel embeds into $\mathbb{F}_\omega$ shifting $2^{\mathbb{F}_\omega}$. This shows that $\mathbb{F}_\omega$ acting on $2^{\mathbb{F}_\omega}$ is a universal countable equivalence relation and, as $\mathbb{F}_\omega$ is isomorphic to a subgroup of $\mathbb{F}_2$, by Proposition 2.2.4, E_∞ is universal as well. □

2.3 The $=^{+n}$ Equivalence Relation

Definition 2.3.1. Let X be a standard Borel space and E an equivalence relation on X. The **Friedman-Stanley jump** (or **jump** for short) denoted E^+ is the equivalence relation on X^ω, defined as:

$$xE^+y \iff \{\, [x_n]_E \mid n \in \omega \,\} = \{\, [y_n]_E \mid n \in \omega \,\}.$$

We will use E^{+n} to mean the jump operator applied n-times to E.

We have the following equivalent characterization of E^+:

$$xE^+y \iff (\forall n \exists m \; x_n E y_m) \wedge (\forall n \exists m \; x_m E y_n).$$

This shows explicitly that for any Borel equivalence relation E, E^+ is Borel as well.

When it is the case that E is $\mathtt{id}(2^\omega)$ (or any Polish space, for that matter), then successive jumps of E produce the hierarchy presented in Figure 2, which follows from the following.

The hierarchy presented in Figure 2 follows from the next Theorem, due to Frieman [4] in the case that E is equality over the reals (or equality over any standard Borel space).

Theorem 2.3.1 (Friedman)**.** *Let X be a standard Borel space and E a Borel equivalence relation. Then there does not exist an $F : X^\omega \to X$ such that for all $x, y \in X^\omega$*

- *if xE^+y then $F(x)EF(y)$ and*
- *for all $n \in \omega$ $(F(x), x_n) \notin E$.*

Corollary. *For a Borel equivalence relation, E on a standard Borel space, we have $E <_B E^+$.*

2.4 Borel Complete

Finally, the last benchmark Borel equivalence relation, we will introduce is the Borel complete complexity. Let $\mathcal{L} = \{ R_i \}_{i \in I}$ be a countable language where I is a countable set of indices and $\{ n_i \}_{i \in I}$ such that for all $i \in I$, R_i is an n_i-ary relation. We will assume that the constants of $\mathcal{L}$ is the set ω.

Definition 2.4.1. A **logic action**, $\pi \in S_\infty$ on $\mathtt{Mod}(\mathcal{L})$ is defined as $gM = N$ for $M, N \in \mathtt{Mod}(\mathcal{L})$ if, and only if,

$$\forall i \in I,\ \forall (x_1, \ldots, x_{n_i}) \in \omega^{n_i},\ R_i^N(x_1, \ldots, x_{n_i}) \Longleftrightarrow R_i^M(\pi^{-1}(x_1), \ldots, \pi^{-1}(x_{n_i}))$$

Definition 2.4.2. The **universal S_∞-orbit equivalence relation** is $S_\infty \curvearrowright \mathtt{Mod}(\mathcal{L})$ where $\mathcal{L}$ contains infinitely many relations of every arity.

Definition 2.4.3. An **invariant Borel class** of a countable $\mathcal{L}$-structure is a Borel subset of $\mathtt{Mod}(\mathcal{L})$, which is S_∞-invariant.

Definition 2.4.4. Let $\mathcal{C}$ be an invariant Borel class. We say that $\mathcal{C}$ is **Borel Complete** if ismorphism on $\mathcal{C}$ is bireducible with the universal S_∞-orbit equivalence relation.

For an example of a natural Borel complete class, we have:

Theorem 2.4.1. *The class of all countable connected graphs with isomorphism is Borel complete.*

In particular, we have that isomorphism over the models of a far simpler language, $\mathcal{L}$, containing exactly one binary relation is Borel complete.

The proof strategy for this theorem is straightforward although somewhat detailed. We provide only a skeleton of the proof, which includes the basic proof strategy we will use later. The full proof of this statement can be found in Section 13 of [6].

To show that the class of all countable connected graphs with isomorphism is Borel complete, it suffices to show that there exists a Borel reduction from a Borel complete equivalence relation to countable connected graphs. In Gao's proof of Theorem 2.4.1 in Section 5 of [6], we know that for a countable language $\mathcal{L} = \{ R_n \}_{n \geq 2}$, where each R_n is an n-ary relation, $\mathtt{Mod}(\mathcal{L})$ is Borel complete. Hence, the desired reduction will be from $\mathtt{Mod}(\mathcal{L})$ to countable graphs. This is done by taking any $M \in \mathtt{Mod}(\mathcal{L})$ and, for each $n \geq 1$, defining what we will call an n-tag as a graph encoding that the relation R_n is relates the elements $a_1, a_2, \ldots, a_n$, an illustration of this is given in Figure 2.4.

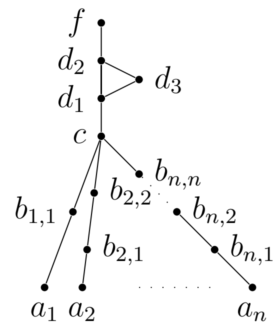


Figure 2.2: The n-tag encoding of a relation in $M \in \texttt{Mod}(\mathcal{L})$

The graph encoding any $m \in \texttt{Mod}(\mathcal{L})$ is the graph composed of n-tags for each $(a_1, a_2, \ldots, a_n) \in R_n$ sharing f as a common vertex.

Note that each n-tag has no symmetry and each vertex can be uniquely determined (i.e, its length from a leaf or the singular 3-cycle). To check that the reduction preserves isomorphism of models of $\mathcal{L}$, we note that the isomorphism seeing that two models are isomorphic induces an isomorphism of their representing n-tag graphs.

The other direction is similar. We begin with two isomorphic n-tag graphs. A witnessing isomorphism then induces an isomorphism of the encoded structures by first applying the isomorphism to the vertices of degree 1 of each tag to gain a mapping between the variables of the space. That is, applying the isomorphism to the vertices of the n-tags encoding the variables of M. Later, the graph isomorphism is applied to the coded relations of the structures to produce the isomorphism of the two structures.

CHAPTER 3

ISOMORPHISM OF REGULAR TREES

3.1 Set Theoretic Trees

Definition 3.1.1. $\omega^{<\omega}$ is the set of all finite sequences on ω. For a finite sequence $s \in \omega^{<\omega}$, we write $|s|$ to indicate the length of s. That is, $|s|$ is the domain of s. For finite sequences $t, s \in \omega^{<\omega}$, we say that t is an initial segment of s, written $t \leq s$, if $|t| \leq |s|$ and $\forall n \leq |t|\ t_n = s_n$.

Definition 3.1.2. A **tree** T on ω, which we will often refer to as a set theoretic tree, is a subset of $\omega^{<\omega}$ closed under initial segments. That is, $\forall s \in T$ if $t \leq s$, then $t \in T$. Certainly, for every tree T, the empty sequence is an initial segment of all $s \in T$. This unique element is what we will refer to as the **root** of T.

Definition 3.1.3. For a given tree, T on ω, and for $n \in \omega$, we write

$$T_n = \{\, t \in T \mid |t| \leq n \,\}$$

to mean the subtree of T cut off at the n^{th} level.

Isomorphisms between trees are defined in terms of the initial segments of the trees. That is, T and S are isomorphic if, and only if, there exists $\sigma : T \to S$ a bijection such that for all $t, t' \in T,\ t \leq t' \Longleftrightarrow \sigma(t) \leq \sigma(t')$.

Definition 3.1.4. A tree T on ω is **finitely branching** if for every initial segment $s \in T$ there are only finitely many initial segments $s' \in T$ of length $|s|+1$ extending s.

Theorem 3.1.1. *The isomorphism relation on the class of finitely branching trees on ω is smooth.*

Proof. Let F be the class of all finitely branching trees on ω. Call $F_0 \subset F$ the set produced by picking a representative element of each isomorphism class of the finite trees in F and θ a map from finite trees in F to its isomorphism-class representative element in F_0. Certainly, for all $S, T \in F$ $S \simeq T$ if, and only if, $\theta(S) = \theta(T)$. Finally, for any $n \in \omega$ and $T \in F$, let $T_n = \{ t \in T \mid |t| \leq n \}$ denote the subtree of T composed of the first n levels of T.

To show our bireduction to a Polish space with equality, we will use the function $f : F \to F_0^\omega$ defined as $f(S) = (\theta(S_n))_{n \in \omega}$.

We now show that f is a Borel reduction from the set of all finitely branching trees to F_0^ω. First, $S \simeq S'$ if, and only if, $f(S) = f(S')$. As if S and S' are isomorphic, then the partial layers S_n and S'_n are isomorphic as well and are represented by the same representative element of their mutual class.

The converse is true; if $S, S' \in F$ such that $\forall n \in \omega$ $S_n \simeq S'_n$, and hence $f(S) = f(S')$, then we can construct an isomorphism between S and S'. That is, let $f(S) = f(S')$ and, from the construction of f we have that for all $n \in \omega$, $S_n \simeq S'_n$. Let σ_n witness this isomorphism. Note that for $s \in S$ a partial sequence we have that for all $n \geq |s|$ $|\sigma_n(s)| = |s|$. As, for any $n \in \omega$, there are only finitely many $s' \in S'$ with $|s'| = n$. Call $N_s \subset \omega$ such that, for all $n, m \in N_s$, the partial isomorphisms from S to S', σ_n and σ_m, agree on s. That is, the σ_n are constant. The same can be applied to $s' \in S'$ for $\simeq_n^{-1}$. Note that for each s, the set N_s must be infinite by the pigeonhole principle.

Hence, we can construct N, a subset of ω, such that for all $s \in S$, and, for all but finitely many $n \in N$, σ_n is constant. This is similarly done for $s' \in S'$ and σ_n^{-1}. Define, for all $s \in S$, $\sigma(s)$ as the eventually constant value $\sigma_n(s)$ for sufficiently large $n \in N$, and similarly for $\sigma^{-1}(s)$. This σ defines an isomorphism from S to S'.

We now have that f is Borel reduction from F to the space of ω-sequences, ω^ω. We finish the proof by observing that there are uncountably many isomorphism classes of F, and hence F with isomorphism is smooth. $\square$

If we drop the necessity that such trees must be finitely branching, we can achieve much higher complexity. The following is a result due to Friedman and Stanley [5].

Theorem 3.1.2 (Friedman-Stanley)**.** *The class of trees on ω with isomorphism is Borel complete.*

Proof. We have already stated that the class of countable connected graphs is Borel complete. This gives us that the class of countable trees on ω as a subset of the class of countable connected graphs is at most Borel complete. It then suffices to show that we can reduce countable connected graphs into countable trees on ω.

Let T_0 be the tree of non-repeating finite sequences in $\omega^{<\omega}$, and let Γ be a countable graph with vertex set ω and edge relation R. We will encode the edges of Γ in a countable tree formed by adding a leaf to vertices in T_0.

For $0 < n, m \in \omega$, we encode R by adding a terminal edge from the vertex x_1 to the end of every sequence $s = (x_1, \ldots, x_{2^n 3^m}) \in T_0$ if, and only if, $R(x_n, x_m)$. We call the resulting tree $T(\Gamma)$.

Let $\pi : \Gamma \simeq \Gamma'$ and define $\pi' : \omega^{<\omega} \to \omega^{<\omega}$ as $\pi'(s_1 \dots s_l) = \pi(s_1), \dots \pi(s_l)$. Certainly, we have that $\pi'(T_0) = T_0$. Now

$$
\begin{aligned}
& (x_1, \dots, x_k, x_1) \in T(\Gamma) \\
& \iff \exists 0 < n, m \in \omega \ (k = 2^n 3^m \wedge R^\Gamma(x_n, x_m)) \\
& \iff \exists 0 < n, m \in \omega \ (k = 2^n 3^m \wedge R^\Gamma(\pi(x_n), \pi(x_m))) \\
& \iff (\pi(x_1), \dots, \pi(x_k)) \in T(\Gamma')
\end{aligned}
$$

Thus, $\pi'(T(\Gamma)) = T(\Gamma')$, and $T(\Gamma) \simeq T(\Gamma')$.

In the other direction, let $\sigma : T(\Gamma) \simeq T(\Gamma')$. We will construct two permutations of ω, π and π', by induction on $l \in \omega$ such that $\sigma(\pi(0) \dots \pi(l)) = \pi'(0) \dots \pi'(l)$. First, in the case that $l = 0$, we let $\pi(0) = 0$ and define $\pi'(0) = \sigma(s)$, where s is the singleton sequence $\pi(0)$. We now continue by induction. Suppose that distinct $\pi(0), \dots, \pi(l)$ and $\pi'(0), \dots, \pi'(l)$ have been defined. If l is odd, let $\pi(l+1)$ be the least element of ω not in $\{ \pi(0), \dots, \pi(l) \}$ and s the sequence $\pi(0), \dots, \pi(l), \pi(l+1)$. We then define σ on s as $\sigma(s) = (\pi'(0), \dots, \pi'(l), y)$, for some $y \neq \pi'(0), \dots, \pi'(l)$, and finally we let $\pi'(l+1) = y$. For l even we repeat the case for $l = 0$.

We now claim that $\pi'\pi^{-1}$ is an isomorphism from Γ to Γ'. Suppose that $R^\Gamma(a, b)$, and let $m = \pi^{-1}(a) - 1$, $n = \pi^{-1}(b) - 1$, and $k = (m, n)$. Then $\pi(0), \dots, \pi(k-1), \pi(0)$ is a terminal node in $T(\Gamma')$ and hence $\sigma(\pi(0), \dots, \pi(k-1), \pi(0)) = \pi'(0), \dots, \pi'(l-1), \pi'(0) \in T(\Gamma')$, thus $R^{\Gamma'}(\pi'(m), \pi'(n))$ or equivalently $R^{\Gamma'}(\pi'\pi^{-1}(a), \pi'\pi^{-1}(b))$. Hence $R^\Gamma(a, b) \iff R^{\Gamma'}(\pi'\pi^{-1}(a), \pi'\pi^{-1}(b))$ for any $a, b \in \omega$ and $\pi'\pi^{-1}$ is an isomorphism.

□

Now that we know we can cover a large range of complexities with these structures, we must ask ourselves if we can have any sort of intermediate complexities. In fact, we can.

Definition 3.1.5. For each $n \in \omega$ denote by H_n the class of subtrees of ω^n.

Theorem 3.1.3. *For each $n \in \omega$ isomorphism on the class H_{n+2} is bireducible with* $\mathtt{id}(2^\omega)^{+n}$.

Proof. Certainly, for the base case, $n = 0$, we have that elements of H_2 can be viewed as sets of natural numbers, and in doing so, we obtain that H_2 reduces to $\mathtt{id}(2^\omega)$. We now need only to apply Silver's dichotomy 2.1.2 to H_2 with the fact that there are uncountably many isomorphism classes of H_2 to obtain that H_2 is bireducible to $\mathtt{id}(2^\omega)$.

For the successor cases, we will need to show that for an equivalence relation E on X, if $E \times \mathtt{id}(\omega + 1) \leq_{\mathrm{B}} E$ then $E^+ \sim_{\mathrm{B}} E^*$. Note E^* is the equivalence relation over $X^{\leq\omega}$ given by $\{ x_i \} \, E^* \, \{ y_i \}$ if, and only if, $\{ x_i \}$ and $\{ y_i \}$ enumerate the same multisets, where two sequences enumerate the same multisets if, and only if, each E-equivalence class occurs the same number of times in each.

Certainly, E^+ is reducible to E^* by eliminating any duplicates.

For the other direction, let $f : X \times (\omega + 1) \to X$ a Borel reduction. We now claim that E^* is reducible to E^+ by the map which sends $\{ x_i \} \in X^{\leq\omega}$ to $\{ f(x_i, n_i^{x_i}) \}$ where $x_i^{x_i}$ is the number of times for which $[x_i]_E$ appears in $\{ x_j \}$. In the case that $\{ x_i \}$ visits only finitely many E-classes, we pad $\{ x_i \}$ with countably many E-class representatives $\{ y_i \}$ and corresponding $n_i^{y_i} = 0$ for all i. This gives a reduction as F provides a set of labeled equivalence classes in the sense of f which encodes a class and a countable label into E, labeled by the number of occurrences of each visited class.

We now need to check that $\mathtt{id}(2^\omega)^{+n} \sim_\mathrm{B} \mathtt{id}(2^\omega)^{+n} \times \mathtt{id}(\omega+1)$. For the $\leq_\mathrm{B}$ case, we use the reduction which sends $s \in (2^\omega)^{\omega n}$ to $(s, 0) \in (2^\omega)^{\omega n} \times (\omega+1)$. In the $\geq_\mathrm{B}$ direction, we note that Silver's dichotomy 2.1.2 gives us that $\mathtt{id}(2^\omega) \sim_\mathrm{B} \mathtt{id}((\omega+1)^\omega)$ and hence $\mathtt{id}(2^\omega)^{+n} \sim_\mathrm{B} \mathtt{id}(\omega^\omega)^{+n}$. By observing that the mapping that sends $(s, \ell) \in (2^\omega)^{\omega n} \times \omega$ to the sequence $L, s_0, s_1, s_2, \cdots \in (\omega^\omega)^{\omega n}$, where L is the constant sequence, of sequences of sequence, $\ldots$, n-many times as required for the n jumps, taking value ℓ, we see that $\mathtt{id}((\omega+1)^\omega)^{+n} \geq_\mathrm{B} \mathtt{id}(2^\omega)^{+n} \times \mathtt{id}(\omega+1)$. Thus, $\mathtt{id}(2^\omega)^{+n} \sim_\mathrm{B} \mathtt{id}((\omega+1)^\omega)^{+n} \geq_\mathrm{B} \mathtt{id}(2^\omega)^{+n} \times \mathtt{id}(\omega+1)$ as desired.

We finish the proof by inducing on n. Suppose that H_n is bireducible with $\mathtt{id}(2^\omega)^{+n}$. We note that any tree in H_{n+1} can be encoded as a $\leq \omega$-sequence of trees in H_n. This is done for a tree $T \in H_{n+1}$ by letting $I \subseteq \omega$ be the first level of T and $\{ T_i \}_{i \in I}$ the $\leq \omega$-sequence of trees in H_n formed by the tree of all the successors of i.

Isomorphism would then correspond to multiset subsets of H_n. That is, two sequences of subtrees of H_n, T_i and S_i, are isomorphic in H_{n+1} if, and only if, those sequences enumerate the same isomorphism classes in H_n with the same number of occurrences. This gives us that H_{n+1} is Borel bireducible with $(\mathtt{id}(2^\omega)^{+n})^*$, which we have already shown to be bireducible with $\mathtt{id}(2^\omega)^{+(n+1)}$.

$\square$

3.2 Graph Theoretic Trees

Definition 3.2.1. A **graph**, in the terms of graph theory, is a set V whose elements are referred to as vertices and, a set of vertex pairs, E, called edges such that $\forall v \in V$ $(v, v) \notin E$. A graph is often denoted as the ordered pair $G = (V, E)$.

The edges of a graph need not be ordered pairs. In the case that the edges are ordered pairs, we call the graph **directed**, otherwise the graph is said to be **undirected**. For any two vertices v, u in a graph, we say that v and u are **adjacent** if there exists an edge from u to v. Furthermore, if u and v are adjacent then they are called the **endpoints** of any edge witnessing they are adjacent. In the case of directed graphs, we would say $(v, u) \in E$ where v would be the **initial vertex** of (v, u), and u the **terminal**. For our purposes, we will only be interested in the undirected edges.

Definition 3.2.2. A **walk** of a graph $G = (V, E)$ is a sequence of alternating vertices and edges $v_1, e_1, v_2, e_2, \ldots$ where, for each i, e_i has the preceding and following vertices in the walk as endpoints. A **closed walk** is a walk whose first and final vertices are the same. A **path** is a walk for which each vertex in the walk is distinct, and hence each edge is distinct as well. We say that a graph is **connected** if for any two vertices in the graph there exists a finite path starting from one and ending at the other. A closed path is called a **cycle**. An **acyclic** graph is a graph for which every path is not a cycle. A connected acyclic graph is called a **tree**. A graph such that every vertex is adjacent to finitely many other vertices is called **locally finite**.

Theorem 3.2.1 (Jackson-Kechris-Louveau)**.** *The isomorphism relation on the class of all locally finite trees is bireducible with E_∞.*

Proof. For any locally finite tree, T, and vertex $t \in T$, call T_t the finitely branching tree on ω where t is regarded as the root. That is, for $t \in T$, T_t is the set theoretic tree such that for all vertices s in T $t \leq s$ in T_t, and for all s and s' vertices of T we say that $s \leq s'$ in T_t if s is in the path from s' to t. Let the spaces F, F_i, and $\Pi_{i\in\omega} F_i$ along with the function f be as presented in the proof of Theorem 3.1.1. Hence $f(T_t) \in \Pi_{i\in\omega} F_i$ as before.

Given a countable tree T, we consider its underlying vertex set to be ω and select $t \in T$ canonically; say we select the zero element of ω, and define the mapping g as $g(T) = f(T_t)$. We then claim that the set of all countable locally finite trees with isomorphism is essentially countable. This is because we have that g satisfies the conditions of Lemma 2.0.2:

(i) Let T be a countable locally finite tree and $A(T)$ be the set of all isomorphism class representatives of rooted countable locally finite graphs, given by f for a selection of a rooted $t \in T$, or, more clearly,

$$A(T) = \{ f(T_t) \mid t \in T \} .$$

Note that here we are using the result of Theorem 3.1.1, that isomorphism of rooted finitely branching trees is smooth. Certainly, $A(T)$ is countable as the number of options for a root of T is countable. Furthermore, we have that $\{ g(T') \mid T \text{ is isomorphic to } T \}$ is countable as if $\sigma : T' \to T$ an isomorphism then for a selection of the root $t' \in T'$, we have that $f(T'_{t'}) = f(T_{\sigma(t')})$, as $T'_{t'}$ is isomorphic to $T_{\sigma(t')}$. Hence, every isomorphism class is countable.

(ii) Certainly if $g(T) = g(T')$, then we must have a $t \in T$ and $t' \in T'$ such that $f(T_t) = f(T'_{t'})$, from the definitions of g and f. This implies that T_t and $T'_{t'}$ must be isomorphic, so that T and T' are isomorphic as well.

This concludes that g, and the set of locally finite trees, satisfies the conditions for Lemma 2.0.2. Hence isomorphism on the set of locally finite countable graphs is essentially countable and so there exists a Borel reduction to the universal countable equivalence relation E_∞.

We now aim to code the shift equivalence relation of $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$ into isomorphism of countable locally finite trees, to finish our proof.

Begin by calling the generators of $\mathbb{F}_2$ a and b. For any $A \subseteq \mathbb{F}_2$, we code A into a tree $T(A)$, as follows.

Define a labeled directed tree K, with the vertex set $\mathbb{F}_2$, and edge relations R_a and R_b. We let $R_a(x, y)$ if there is a directed edge xy in K, with label a and similarly for $R_b(x, y)$ with label b. That is,

$$R_a(x, y) \iff xa = y \text{ and } R_b(x, y) \iff xb = y.$$

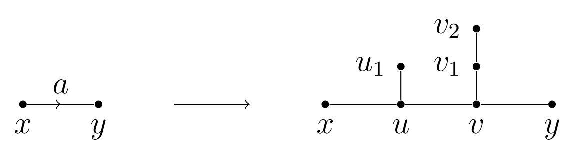
We then encode the directed labeled graph K as the locally finite tree T_0 obtained by the following:

(i) For each edge xy labeled a in K, replace the edge with the graph $T_a(x, y)$ with the vertex set:

$$\{\, x, y \,\} \cup \{\, u, u_1, v, v_1, v_2 \,\}$$

and edge set:

$$\{\, xu, uu_1, uv, vv_1, v_1v_2, vy \,\}.$$

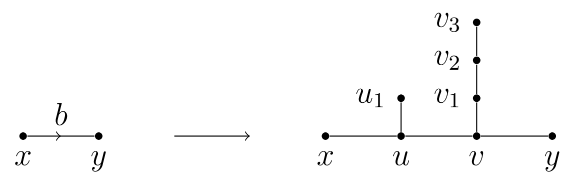


(ii) For each edge xy labeled b in K, replace the edge with the graph $T_b(x, y)$, with the vertex set:

$$\{\, x, y \,\} \cup \{\, u, u_1, v, v_1, v_2, v_3 \,\}$$

and edge set:

$$\{\, xu, uu_1, uv, vv_1, v_1v_2, v_2v_3, vy \,\}.$$



The final graph $T(A)$ is then obtained from the graph T_0, by adding a new vertex x^* for every $x \in A$, and adjoining x to x^* by an edge xx^*. We now note that T_0 is a tree, and a vertex $x \in T_0$ has degree 4 if, and only if, x is a member of $\mathbb{F}_2$. Furthermore, in $T(A)$ a vertex has degree ≥ 4 if, and only if, the vertex encodes a vertex in $\mathbb{F}_2$; and finally x in $T(A)$ has degree equal to 5 if, and only if, x is a member of A.

We now claim that this map T is a Borel reduction from $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$ to the class of countable locally finite trees with isomorphism. This is because if we suppose that $A, A' \subseteq \mathbb{F}_2$ such that there is a $g \in \mathbb{F}_2$, which shifts A to A', then $gA = A'$. Call $\sigma_g(x) = gx$ the map witnessing the shift relation of A and A'. Certainly σ_g induces an automorphism of the directed graph K. Namely, the induced automorphism from σ_g is one which sends elements of $\mathbb{F}_2$ to their image through σ_g and preserves edge relations. Moreover, σ_g induces an automorphism of T_0 in the natural way. Finally, the induced automorphism of T_0 from σ_g can be extended to an isomorphism from $T(A)$ to $T(A')$, by taking the vertex x^* to $(gx)^*$ in $T(A')$.

For the converse, suppose that $T(A)$ and $T(A')$ are isomorphic for $A, A' \subseteq \mathbb{F}_2$. Let $\sigma : T(A) \to T(A')$ be an isomorphism. We have that $\sigma(T_0) = T_0$ and $\sigma(T(A)) = T(A')$, and hence σ induces an automorphism of K. We can write any $x \in \mathbb{F}_2$ as a unique sequence of $x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$ where $x_1, x_2, \ldots, x_n \in \{\, a, b \,\}$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{\, -1, 1 \,\}$. These finite sequences correspond to a unique path from $1_{\mathbb{F}_2}$ in K to an element x. Let $g = \sigma(1_{\mathbb{F}_2})$. We then have that, for any $x \in \mathbb{F}_2$, $\sigma(x) = gx$. Finally, as σ is an isomorphism that induces an automorphism of K in the form $x \mapsto gx$, we have that $gA = A'$ as desired. □

The following is a consequence of the theorem that the class of arbitrary trees on ω is Borel complete.

Corollary (Friedman-Stanley)**.** *The isomorphism relation on the class of all countable graph theoretic trees is Borel complete.*

Proof. Again, recall that the class of countable graphs with isomorphism is Borel complete. Hence, the class of all countable trees with isomorphism as a subclass is at most Borel complete. We know that isomorphism on the class of trees on ω is Borel complete from Theorem 3.1.2. Hence, we can induce a reduction from trees on ω to countable trees by a function that encodes the root of those set theoretic trees, producing a countable (unrooted) tree, thus showing that countable trees are at least Borel complete.

We do this by taking an arbitrary tree, T, on ω, and expand each edge in T by adding a new vertex adjacent to the endpoints of the edge and removing the original edge. That is, the edge xy is removed and the new edges xu_{xy} and $u_{xy}y$ are added to the tree T along with the requisite vertex u_{xy}. Finally, we mark the root, r, of T by adding the vertices $\{ r_0, r_1, r_2, r_3 \}$ and edges $\{ rr_0, r_0r_1, r_0r_2, r_0r_3 \}$. These marking points then uniquely identify the previous root of T as after expanding all of the previous edges we have that r is the only vertex adjacent to a vertex with degree 3 and with a path of length 2 to a leaf.

Certainly, this is a reduction in the forward direction. In the opposite direction, let T and T' be trees on ω and suppose that $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic trees formed by edge expansions and root marking. If there exists an isomorphism σ from $\mathcal{T}$ to $\mathcal{T}'$, we want to show that we can restrict σ to a subset of the vertices of $\mathcal{T}$, S such that $\sigma\restriction_S$ is an isomorphism from T to T'.

To do this, call r the unique vertex marked as the root of $\mathcal{T}$. As σ is an isomorphism, we have that $\sigma(r)$ must be adjacent to a vertex of degree 3 and have a path of length 2 to a leaf vertex. Hence, $\sigma(r)$ is also marked as the root vertex.

Next, let $\mathcal{S}$ be the set of all paths from r to leaf nodes of $\mathcal{T}$ such that the second vertex in the path has degree 2. That is, from the construction of the tree $\mathcal{T}$, all of the paths from the marked root vertex to the leafs of $\mathcal{T}$ which do not include vertex r_0 correspond to sequences in the original tree T. Note that any path in $\mathcal{S}$ has even length. This is because any leaf in $\mathcal{T}$ was a leaf in the original tree T. Let ℓ be the distance from that leaf to the root of T. Then, in our expansion to $\mathcal{T}$, we added a new vertex for every edge along that path and, hence, the length of that path in $\mathcal{T}$ is 2ℓ. Finally, we can identify the added vertices in $\mathcal{T}$ as the set of vertices which appear in a path in $\mathcal{S}$ at an odd index. Hence, by restricting σ to the set of vertices which appear at an even vertex for a path in $\mathcal{S}$ we obtain a mapping from T to T'. It follows from the construction of this expanded tree that σ restricted in this way is an isomorphism.

$\square$

CHAPTER 4

AUTOMORPHISMS OF REGULAR TREES

4.1 Set Theoretic Trees

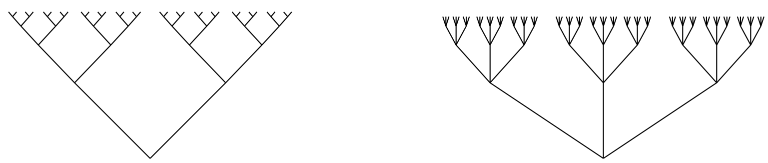


Figure 4.1: Image of the standard Cantor tree $2^{<\omega}$ and the 3 branching variation of the Cantor tree, $3^{<\omega}$.

Pictured above are two examples of trees on ω: the Cantor tree $2^{<\omega}$ and the 3-branching variation $3^{<\omega}$.

In this section, we classify automorphisms of regularly branching rooted trees, such as $b^{<\omega}$ for b a finite natural number. See Figure 4.1 for depictions of the Cantor tree $2^{<\omega}$ and the 3-branching variant $3^{<\omega}$.

The following is a generalization of Dougherty, Jackson, Kechris.

Theorem 4.1.1. *For any natural number b, the conjugacy relation on the automorphism group of locally finite tree on ω $b^{<\omega}$ is smooth.*

In the proof of this statement, we will use the following definition as a means to define partial automorphisms of T acting on just T_n, the tree truncated to height n.

Definition 4.1.1. Given a tree T, and permutations π and σ of T_n and T_m respectively, we say that $\pi \leq \sigma$, read π is extended by σ, if $n \leq m$ and for every $s \in T_n$

$$\pi(s) = \sigma(s).$$

This allows us to produce some ordering of these partial automorphisms.

Definition 4.1.2. Let π and σ be permutations of b^n and b^m, respectively, then we say that σ extends π,$\pi \leq \sigma$, if and only if $n \leq m$ and $\forall s \in b^m$

$$\pi(s \restriction_n) = \sigma(s) \restriction_n$$

Automorphisms of $b^{<\omega}$ can be completely defined by sequences of partial automorphisms. That is, $\forall \phi \in \mathrm{Aut}(b^{<\omega})$ there exists a sequence $\phi_0 \leq \phi_1 \leq \ldots$, such that $\phi = \bigcup \pi$. This is because we can simply take $\phi_n = \phi \restriction_{b^{\leq n}}$ and not that $\phi_n \leq \phi_{n+1}$ holds trivially for each $n \in \omega$.

Definition 4.1.3. For a given automorphism $\phi \in \mathrm{Aut}(b^{<\omega})$, we say that $O \subseteq b^n$ is an **orbit** of ϕ_n if for every $s, t \in O$ there exists an i such that $\phi^i(s) = t$. For any orbit O of ϕ_n and O' of ϕ_m, we say that O' **extends** O or O' is an extension of O, written as $O < O'$, if $n < m$ and for every $s \in O'$ $s \restriction_n \in O$.

Lemma 4.1.2. *If ϕ is an automorphism of $b^{<\omega}$, and $O < O'$ orbits of ϕ_n and ϕ_{n+1} respectively, then $|O| \leq |O'| \leq b|O|$ and $|O|$ divides $|O'|$.*

Proof. Suppose that $O = \{ s_1, s_2, \ldots, s_k \}$ and $\forall i \leq k$ $\phi_n(s_i) = s_{i+1 \mod k}$ for $k = |O|$. As $O < O'$, we have that for any $s \in O'$ $s \restriction_n \in O$ giving us that $|O| \leq |O'|$. Moreover, as $b^{<\omega}$ is such that there are exactly b immediate successors of any element, we have that $|O'| \leq b|O|$. Finally, for any $s' \in O'$ with $s' \restriction_n = s_i$, we have that $\phi_{n+1}^k(s') \restriction_n = s_i$, therefore there must exist an n, $0 < n \leq b$ such that $\phi_{n+1}^{nk}(s') = s'$, hence $|O|$ must divide $|O'|$.

□

We now have the requisite tools to prove Theorem 4.1.1.

Proof of 4.1.1. Let $\phi \in \mathrm{Aut}(b^{<\omega})$ be given, and $\mathcal{O}_n$ be the set of all orbits of ϕ_n together with a label for their size, i.e. elements of $\mathcal{O}_n$ are of the form $(O, |O|)$ for O an orbit. Let $\bigcup_{n\in\omega} \mathcal{O}_n, = \mathcal{O}$ and define what we will call the orbit tree of ϕ, denoted by T_ϕ, as the labeled tree formed by $\mathcal{O}$ with the relation of orbit extension. We refer to T_ϕ as the **orbit tree** of ϕ. These trees were initially constructed and refined from the work done by Dougherty, Jackson, and Kechris in Section 10 in [2]. It is an immediate consequence of Lemma 4.1.2 that for O an orbit of an automorphism $\phi \in \mathrm{Aut}(b^{<\omega})$, there exists at most b many immediate extensions of O. From this, we have that T_ϕ is finitely branching with each orbit branching to at most b immediate successors. We note that we can encode this labeled tree as a subtree of $(b+1)^{<\omega}$ by observing that each orbit is extended by infinitely many orbits, and the size of that orbit is finite. The encoding is done by attaching an edge from O to a copy of $b^{|O|}$. For the remainder of this proof, we will regard the orbit tree as a labeled subtree of $b^{<\omega}$ with labels from the natural numbers.

Hence, we can define the Borel function $f : \phi \mapsto T_\phi$, which sends automorphisms of $b^{<\omega}$ to subtrees of $b^{<\omega}$. To finish this proof, we need only to show that this f is a Borel bireduction and note that the space of subgraphs of $b^{<\omega}$ with isomorphism is smooth as shown in Theorem 3.1.2. That is, we need to show that for any $\phi, \psi \in \mathrm{Aut}(b^{<\omega})$, ϕ is conjugate to ψ if, and only if, T_ϕ is isomorphic to T_ψ. The bireducibility is then an application of Silver's dichotomy and an observation that the set of automorphism equivalence classes is uncountable.

Let ϕ and ψ be conjugate automorphisms of $b^{<\omega}$, and $\alpha \in \mathrm{Aut}\,(b^{\omega})$ such that $\alpha\phi = \psi\alpha$. We aim to construct an isomorphism $\alpha_T : T_\phi \to T_\psi$ from α. Given $\phi, \psi, \alpha \in \mathrm{Aut}\,(b^{<\omega})$, we have that there exists a countable increasing, in the sense of extension, sequence ϕ_n, ψ_n, α_n encoding the partial actions of their respective automorphisms. This is because, for any $\phi \in \mathrm{Aut}\,(b^{<\omega})$ we can define for each $n \in \omega$ $\phi_n = \phi \restriction_{b^{\leq n}}$ to obtain our increasing sequence. Certainly, as α witnesses that ϕ and ψ are conjugate, we have that $\forall i\ \alpha_i\phi_i = \psi_i\alpha_i$. In particular, we have that for any i, the existence of the α_i indicates that the number of orbits at the i^{th} level of ϕ and ψ are equal and furthermore that there must be a one-to-one correspondence between the sizes of these orbits.

Let G_i be the finite set of isomorphisms between the orbits at the i^{th} layer of T_ϕ, and T_ψ. Index its members as $g_{i,j}$. For a selection $g_{i,j}$, of how the layers correspond, let $F_{i,j}$ be the set of all sequences of bijections witnessing that $g_{i,j}$ maps equal orbits in T_ϕ at layer i to equal sized orbits in O_ψ at layer i. Let $G = \bigcup_{i\in\omega} G_i$

We define a partial ordering on G by $g_{i,j} \leq g_{i',j'}$, when it is the case that $i \leq i'$ and for any O an orbit at layer i which is extended by O' at layer i' then $g_{i,j}(O)$ is an initial segment of $g_{i',j'}(O')$. Note that any $g_{i,j}$ has finitely many immediate extensions, and there are infinitely many $g_{i,j}$. We then have from the use of König's tree lemma that G contains an infinite branch. This infinite branch corresponds to an increasing sequence of partial isomorphisms between the layers of the orbit trees T_ϕ and T_ψ. The union of this branch is hence a full isomorphism of the orbit trees.

In the reverse direction, let T_ϕ be isomorphic T_ψ generated by $\phi, \psi \in \mathrm{Aut}(b^{<\omega})$, and let α_T be a witnessing isomorphism. Let $\mathcal{O}_n$ be the set of ϕ orbits of the n^{th}

level of $b^{<\omega}$. This is exactly the unlabeled leafs of $(T_\phi)_n$. As α_T is an isomorphism between two labeled trees, it must preserve the labels of the orbits in T_ϕ, that is $\forall O \in \mathcal{O}_n\ |\alpha_T(O)| = |O|$. Namely α_T must act as a permutation of $\mathcal{O}_n$. Furthermore, for each $n \in \omega$, each $O \in \mathcal{O}_n$ is finite.

Call A_n the set of all permutations, a, of b^n such that $\forall O \in \mathcal{O}_n\ a[O] = \alpha_T(O)$ and $a\phi_n = \psi_n a$. Note that for each $n \in \omega\ A_n \neq \emptyset$ as when $n = 0\ A_0$ is the singleton taking the root of T_ϕ to the root of T_ψ. If we suppose that A_n is not empty, then the map a which agrees with α_T on the orbits of $\mathcal{O}_{n+1}$ and each orbit $o \in \mathcal{O}_{n+1}$ acts as a permutation of $\psi\phi^{-1}\alpha_T(o)$. As each $\mathcal{O}_n$ is a finite set of finite sets, the set of permutations of $\mathcal{O}_n$ is finite and hence A_n must be finite as well.

We now form the infinite tree $\left(\bigcup_{n\in\omega} A_n, <\right)$, where $<$ in this context means standard function extension, and note that each vertex a has finitely many immediate extensions. We now note that $\left(\bigcup_{n\in\omega} A_n, <\right)$ satisfies the hypothesis of König's lemma and hence there must exists an infinite path $\{\alpha_i\}_{i\in\omega}$. This gives as an infinite sequence of increasing partial automorphisms witnessing the conjugacy of each finite level of ϕ_n and ψ_n. Hence, $\alpha = \bigcup_{i\in\omega} \alpha_i$ satisfies that $\alpha\phi = \psi\alpha$.

□

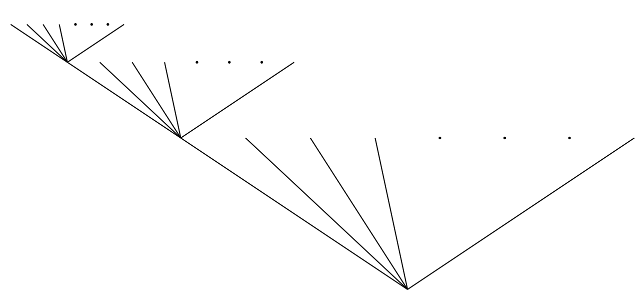


Figure 4.2: A representation of a finite height rooted ω-branching tree.

Recall that H_n for $n \in \omega$ is the set of all rooted trees of height n. We will show next that the automorphisms of the full tree ω^{n+1} with conjugacy is bireducible with $\mathtt{id}(2^\omega)^{+n}$.

Theorem 4.1.3. *For any $n \in \omega$ the conjugacy relation on the set of automorphisms of $\omega^{\leq n+1}$ with is bireducible with* $\mathtt{id}(2^\omega)^{+n}$

Proof. We prove this by induction on n. For $n = 0$, an automorphism of $\omega^{\leq 1}$ is an element of S_ω and any two automorphisms are conjugate if and only if those permutations of S_ω are conjugate. This is because any automorphism of $\omega^{\leq 1}$ must fix the root vertex and hence can only permute copy of ω present at the first layer of the tree. It is well known that conjugacy on S_∞ is bireducible to $\mathtt{id}(\mathbb{R})$. Hence the automorphisms of $\omega^{\leq 1}$ with conjugacy is bireducible to $\mathtt{id}(\mathbb{R})$, a Polish space, and thus bireducible with $\mathtt{id}(2^\omega) = \mathtt{id}(2^\omega)^{+0}$ as desired.

For the successor case, suppose that $\mathrm{Aut}(\omega^{\leq n+1})$ with conjugacy is bireducible with $\mathtt{id}(2^\omega)^{+n}$. Then an automorphism of $\omega^{\leq n+2}$ must preserve the root, and thus must also preserve the length of sequences, hence preserving levels of $\omega^{\leq n+1}$. So, we can decompose any $\phi \in \mathrm{Aut}(\omega^{\leq n+2})$ into an ω-sequence $\phi_i \in \mathrm{Aut}(\omega^{\leq n+1})$ and an element $\pi \in S_\infty$; such that $\forall t \in \omega$ and $\forall s \in \omega^{\leq n+1}$ (note that $t, s_0, s_1, \dots \in \omega^{\leq n+2}$) $\phi(t, s_0, s_1, \dots) = \pi(t), s'_0, s'_1, s'_2, \dots$ where $s' = \phi_t(s)$. From [1], we have that conjugacy on $\mathrm{Aut}(\omega^{\leq n+2})$ is bireducible to the jump of conjugacy on $\mathrm{Aut}(\omega^{\leq n+1})$. That is, $\mathrm{Aut}(\omega^{\leq n+2})$ is bireducible to $\mathtt{id}(2^\omega)^{+(n+1)} \times \mathtt{id}(2^\omega)$ by induction hypothesis. We now note that $(2^\omega)^{\omega^n} \leq_{\mathrm{B}} (2^\omega)^{\omega^n} \times 2^\omega$, by the reduction sending s to $(s, 0)$, where $0 \in 2^\omega$ is the constant sequence 0.

We begin on the other direction by showing: for a Borel equivalence relation E over X, if $E \times E \leq_{\mathrm{B}} E$ and $E \leq_{\mathrm{B}} E^+$, then $E^{+n} \times E \leq_{\mathrm{B}} E^{+n} \times E^{+n} \leq_{\mathrm{B}}$

E^{+n} for $0 < n \in \omega$. This is because, if $E \times E \leq_{\mathrm{B}} E$, then $(E \times E)^{+n} \leq_{\mathrm{B}} E^{+n}$. We now argue that $E^{+n} \times E^{+n} \leq_{\mathrm{B}} (E \times E)^{+n}$. This is shown by the reduction that sends $(\{x_n\}, \{y_n\})$ to $\left\{ (x_{\sigma(n)_0}, y_{\sigma(n)_1}) \right\}$ where σ is a fixed bijection from ω to $\omega \times \omega$. This is because $(\{x_n\}, \{y_n\})$ and $(\{x'_n\}, \{y'_n\})$ are equivalent if, and only if, $\{ [x_n]_{E^{+(n-1)}} \mid n \in \omega \} = \{ [x'_n]_{E^{+(n-1)}} \mid n \in \omega \}$ and $\{ [y_n]_{E^{+(n-1)}} \mid n \in \omega \} = \{ [y'_n]_{E^{+(n-1)}} \mid n \in \omega \}$ holds if, and only if,

$$\begin{aligned}
&\left\{ [(x_{\sigma(n)_0}, y_{\sigma(n)_1})]_{E^{+(n-1)} \times E^{+(n-1)}} \mid n \in \omega \right\} \\
&= \{ [(x_n, y_m)]_{E^{+(n-1)} \times E^{+(n-1)}} \mid n, m \in \omega \} \\
&= \{ [x_n]_{E^{+(n-1)}} \mid n \in \omega \} \times \{ [y_n]_{E^{+(n-1)}} \mid n \in \omega \} \\
&= \{ [x'_n]_{E^{+(n-1)}} \mid n \in \omega \} \times \{ [y'_n]_{E^{+(n-1)}} \mid n \in \omega \} \\
&= \{ [(x'_n, y'_m)]_{E^{+(n-1)} \times E^{+(n-1)}} \mid n, m \in \omega \} \\
&= \left\{ [(x'_{\sigma(n)_0}, y'_{\sigma(n)_1})]_{E^{+(n-1)} \times E^{(+n-1)}} \mid n \in \omega \right\}
\end{aligned}$$

showing the desired reduction. To finish, we note that $E^{+n} \times E \leq_{\mathrm{B}} E^{+n} \times E^{+n}$ by noting that if $E \leq_{\mathrm{B}} E^{+}$ implies that $E \leq_{\mathrm{B}} E^{+n}$. Note that if f witnesses $E \leq_{\mathrm{B}} E^{+n}$, then certainly the $F : X^{\omega^n} \times X \mapsto X^{\omega^n} \times X^{\omega^n}$ defined by $F : (s, e) \mapsto (s, f(s))$ is a reduction.

We now observe that $\mathtt{id}(2^\omega) \times \mathtt{id}(2^\omega) \sim_{\mathrm{B}} \mathtt{id}(2^{\omega \times \omega})$, which is bireducible to $\mathtt{id}(2^\omega)$ by an application of Silver's dichotomy. Hence $\mathtt{id}(2^\omega)^{+(n+1)} \times \mathtt{id}(2^\omega) \leq_{\mathrm{B}} \mathtt{id}(2^\omega)^{+(n+1)}$ from our claim, thus $\mathtt{id}(2^\omega)^{+(n+1)} \times \mathtt{id}(2^\omega) \sim_{\mathrm{B}} \mathtt{id}(2^\omega)^{+(n+1)}$. Finally, this gives us that $\mathrm{Aut}(\omega^{\leq n+2})$ with conjugacy is Borel bireducible with $\mathtt{id}(2^\omega)^{+(n+1)}$ as desired.

□

Theorem 4.1.4. *The conjugacy relation on the automorphisms of $\omega^{<\omega}$ is Borel* *complete.*

Proof. It suffices to show that we can reduce the class of trees on ω with isomorphism to the automorphism group of $\omega^{<\omega}$ with conjugacy. We do this by taking any tree T on ω and create from T an automorphism ϕ_T of $\omega^{<\omega}$ such that ϕ_T fixes T as a subset. Let T be a subtree of $\omega^{<\omega}$ and for every $s = \{ s_i \} \in T$ call $2s$ the sequence $(2s)_i = 2 \cdot s_i$, we form the new tree $2T$ as the set of all $2s$ for $s \in T$. Define for $s \in 2T$ the $S(s)$ as the set of all immediate successors of s in $\omega^{<\omega}$ not in $2T$.

Fix a permutation π of ω such that π has no fixed points. We then define ϕ_T from a tree on ω T as $\phi_T(s) = s$ when $s \in 2T$ and $\phi_T(s) = s_0, s_1, \ldots, s_i, \pi(s_{i+1}), s_{i+2} \ldots$ if $\exists s' \in 2T$ such that s either is in or extends an element of $S(s')$ and i is the length of s'. We note that ϕ_T is an automorphism that fixes exactly the tree $2T$, and hence its fixed points are isomorphic to T.

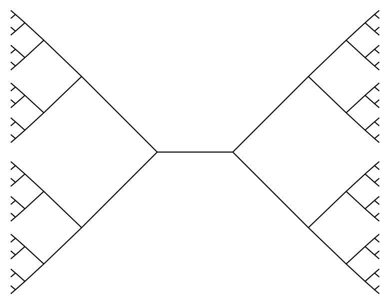
We claim that the map that sends trees on ω to the automorphism ϕ_T is a Borel reduction. First, if ϕ_T and $\phi_{T'}$ are conjugate for trees on T and T' on ω, then fixed points of those automorphisms must be isomorphic and hence $T \simeq 2T \simeq 2T' \simeq T'$. In the other direction, suppose that T and T' are isomorphic. This implies that $2T \simeq 2T'$, let $\sigma : 2T \to 2T'$ be an isomorphism witnessing this. We can then describe $\phi_{T'}$ equivalently as the automorphism that fixes those vertices in $\sigma(2T)$ and permutes $S(s)$ by π for $s \in \sigma(2T)$. We define ψ an automorphism that sends $2T$ to $2T'$ as subtrees of $\omega^{<\omega}$ but is otherwise the identity, i.e. $\psi(2T) = \sigma(2T) = 2T'$. Then, from our alternate description of $\phi_{T'}$, we get that $\phi_{T'} = \psi^{-1}\phi_T\psi$ showing that ϕ_T and $\phi_{T'}$ are conjugate.

This gives a reduction from a Borel complete class into conjugacy of the automorphisms of $\omega^{<\omega}$ thus conjugacy over this class of automorphisms is Borel complete.

$\square$

4.2 Graph Theoretic Regular Trees

Definition 4.2.1. For each $n \in \omega$ denote by RT_n, the regular countably-infinite tree where each vertex has degree exactly n.



Proposition 4.2.1. *For each $n \in \omega$ RT_n is unique up to isomorphism.*

Proof. Let $n \in \omega$ be given and S and T be countably-infinite trees where each vertex has degree exactly n. Without loss of generality, we assume that S and T have vertex set ω. Let $j \in \omega$ a vertex of T, and define $\sigma_0(0) = j$. Now, using $N_T(i) = \{\, j \mid j \text{ a vertex of T adjacent to } i \,\}$, we have that $|N_S(0)| = n = |N_T(j)|$ and define σ_1 as a witnessing bijection to that fact. We continue by defining $\sigma_{2,i}$ for $i \in N_S(0)$, witnessing bijections to the fact that for each of these i $|N_S(i) \setminus \{\, 0 \,\}| = |N_T(\sigma_1(i) \setminus \{\, j \,\}|$. (Define $\sigma_2 = \bigcup_{i \in N_S(0)} \sigma_{2,i}$.) As S and T trees, we have that σ_2 is a function as otherwise σ_2 would attempt to map a vertex in S to two separate vertices in T, implying that there exists a cycle in T.

We continue constructing σ_i in this fashion by induction. Suppose that σ_k has been defined for all $k \leq i$ has been defined as above, we then define $\sigma_{i+1,k}$ for each $k \in \mathrm{dom}(\sigma_i)$ as witnessing bijections to $|N_S(k) \setminus \mathrm{dom}(\sigma_{i-1})| = |N_T(\sigma_i k) \setminus \mathrm{ran}(\sigma_{i-1})|$. Finally, we set $\sigma_{i+1} = \bigcup_{k \in \mathrm{dom}(\sigma_i)} \sigma_{i+1,k}$.

We have then that $\sigma = \bigcup_i \in \omega \sigma_i$ is an isomorphism from S to T, as desired.

$\square$

Theorem 4.2.2. *For a given $n \in \omega$, the conjugacy relation on the set of automorphisms of RT_n is Borel bireducible with E_∞.*

Proof. It is a result of Serre [9] that any automorphism ϕ of a regularly-branching tree T is precisely one of the following:

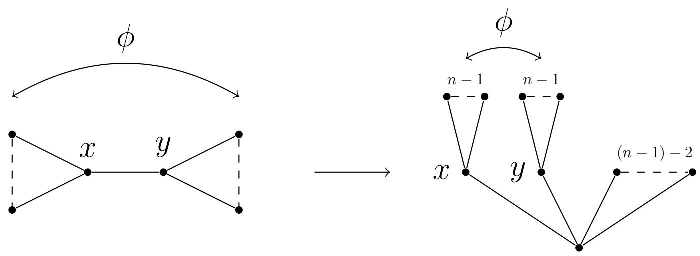
(i) ϕ is an edge flip; that is for some $x, y \in T$, x is adjacent to y, and $\phi(x) = y \wedge \phi(y) = x$.

(ii) ϕ shifts an infinite branch; that is for some $\mathbb{Z}$-sequence $\{ s_i \}_{i\in\mathbb{Z}}$, such that $\phi \neq \mathtt{id}$ and for each $i \in \mathbb{Z}$ s_i is adjacent to s_{i+1} and s_{i-1}, we have $\phi[s] = s$.

(iii) Or finally ϕ preserves a fixed subtree $S \subseteq T$.

These results can be found in Proposition 25 on page 63 in Serre [9], though note Serre makes the standing assumption that the automorphisms he is investigating are not an edge flip. From this result, we can partition the automorphism group of RT_n into those three classes, and examine their complexities separately.

(i) In the case that $\phi \in \mathrm{Aut}(RT_n)$ is an edge flip, we can encode the action of ϕ as an automorphism of the ω tree $(n-1)^{<\omega}$, which with conjugacy we have already shown to be smooth. Note that in the case that $n = 2$, there is exactly one such automorphism, and hence the reduction to a smooth equivalence relation is trivial. We show a reduction for greater n by regarding the first and second vertices in the first level of $(n-1)^{<\omega}$ as x and y and induce a new automorphism ϕ', which agrees with ϕ on sequences starting with x or y but is otherwise the identity.



This gives a reduction as for any automorphisms of RT_n which flips an edge, neither expanding an edge nor adding or removing vertices for which the automorphism is invariant on changes the conjugacy class. Thus the edge flipping automorphisms are at most smooth.

(ii) In the case that an automorphism ϕ shifts a fixed branch. We can encode these shift automorphisms, ϕ, as an integer s and a sequence $\{\, \phi_i \,\}_{i \in \mathbb{Z}}$ of automorphisms. Where each ϕ_i is an automorphisms of $n-2$ copies of b^{n-1} whose roots are adjacent to a single vertex, we will write this tree as S. Here s represents the magnitude of the shift along the fixed branch of ϕ, which we will make precise in the next paragraph, and each ϕ_i encodes the action of ϕ from the i^{th} copy of S of the fixed branch to the $(i+s)^{\text{th}}$ copy.

We will now show a slightly more general claim where we regard S as some arbitrary structure and $(S \times \mathbb{Z}, <_{\mathbb{Z}})$ a linear ordering of countably many copies of S. Within this linear ordering of structures, we will address specific copies of S as $S_i = S \times \{\, i \,\}$. Note that we have that $S_{i-1} < S_i < S_{i+1}$. Given any $\phi \in \operatorname{Aut}((S \times \mathbb{Z}, <_{\mathbb{Z}}))$ and any $i \in \mathbb{Z}$, if $\phi(S_i) = S_{i+n}$, then $\forall j \in \mathbb{Z}\ \phi(S_j) = S_{j+n}$. We call this unique n the shift of ϕ.

Claim 4.2.3. *Given any $\phi, \psi \in \operatorname{Aut}((S \times \mathbb{Z}, <_{\mathbb{Z}}))$ with respective shifts n and m, then ϕ is conjugate to ψ if and only if $n = m$.*

Proof. Let ϕ and ψ be conjugate automorphisms of $(S \times \mathbb{Z}, <_{\mathbb{Z}})$, and suppose α is an automorphism of $(S \times \mathbb{Z}, <_{\mathbb{Z}})$ such that $\phi = \alpha^{-1}\psi\alpha$. Let n, m, and l be the magnitude of the shifts of ϕ, ψ, and α, respectively. The shift of the composition $\alpha^{-1}\psi\alpha$ would be first a shift of l, then a shift of n and finally a shift of $-l$, the cumulative shift the composition would be $-l + n + l = n$. Though, from the assumption that ϕ and ψ where conjugate, and α a conjugation, we conclude that the ϕ and ψ have equal shifts, $n = m$.

For the converse direction, we can represent any ϕ and ψ as collections of automorphisms on S indexed by integers. It suffices to show for $n = 1$ that there exists a collection of α_i's such that the following diagram commutes:

$$\begin{tikzcd}
\phi: & \cdots \arrow[r, "\phi_{i-3}"] & S_{i-2} \arrow[r, "\phi_{i-2}"] \arrow[d, "\alpha_{i-2}"] & S_{i-1} \arrow[r, "\phi_{i-1}"] \arrow[d, "\alpha_{i-1}"] & S_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] & S_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d, "\alpha_{i+1}"] & S_{i+2} \arrow[r, "\phi_{i+2}"] \arrow[d, "\alpha_{i+2}"] & \cdots \\
\psi: & \cdots \arrow[r, "\psi_{i-3}"] & S_{i-2} \arrow[r, "\psi_{i-2}"] & S_{i-1} \arrow[r, "\psi_{i-1}"] & S_i \arrow[r, "\psi_i"] & S_{i+1} \arrow[r, "\psi_{i+1}"] & S_{i+2} \arrow[r, "\psi_{i+2}"] & \cdots
\end{tikzcd}$$

Assuming that the case for $n = 1$ is true, for larger $0 \neq |n| > 1$, we need only to look at the n disjoint sequences of ϕ and ψ and perform the case that $n = 1$ and recombine with the appropriate shifting.

To construct α, we assume without loss of generality that for some $i \in \mathbb{Z}$, $\alpha_i \in \text{Aut}(S)$ is arbitrary selection, for simplicity we will assume that α_0 is the identity. Working in the positive direction along $\mathbb{Z}$ gives us that:

$$\begin{aligned}
&\alpha_1\phi_0 = \psi_0\alpha_0 \\
\implies\ &\alpha_1 = \psi_0\ id\ \phi_0^{-1} = \psi_0\phi_0^{-1}.
\end{aligned}$$

Continuing this inductively with the indices increasing we get that:

$$
\begin{aligned}
\alpha_{i+1}\phi_i &= \psi_i\alpha_i \\
\implies \alpha_{i+1} &= \psi_i\alpha_i\phi_i^{-1} \\
&= \psi_i\psi_{i-1}\cdots\psi_1\psi_0\phi_0^{-1}\phi_1^{-1}\cdots\phi_{i-1}^{-1}\phi_i^{-1} \\
&= \psi_i\psi_{i-1}\cdots\psi_1\psi_0\left(\phi_i\phi_{i-1}\cdots\phi_1\phi_0\right)^{-1}.
\end{aligned}
$$

The negative direction along $\mathbb{Z}$ is similar:

$$
\begin{aligned}
\psi_{-1}\alpha_{-1} &= \alpha_0\phi_{-1} \\
\implies \alpha_{-1} &= \psi_{-1}^{-1}\ id\ \phi_{-1} = \psi_{-1}^{-1}\phi_{-1}.
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
\psi_{i-1}\alpha_{i-1} &= \alpha_i\phi_{i-1} \\
\implies \alpha_{i-1} &= \psi_{i-1}^{-1}\alpha_i\phi_{i-1} \\
&= \psi_{i+1}^{-1}\psi_{i+2}^{-1}\cdots\psi_{-1}^{-1}\psi_0^{-1}\phi_0\phi_{-1}\cdots\phi_{i+1}\phi_i \\
&= \left(\psi_0\psi_{-1}\cdots\psi_{i+1}\psi_i\right)^{-1}\phi_0\phi_{-1}\cdots\phi_{i+1}\phi_{i+1}.
\end{aligned}
$$

Thus, for each i, α_i witness that ϕ_i and ψ_i are conjugate, and thus $\alpha\phi = \phi\alpha$. $\square$

This gives us that conjugacy on the set of all automorphisms with a fixed branch is just as complex as $\mathbb{Z}$ with equality, and is hence bireducible with $\texttt{id}(\omega)$.

(iii) Finally, we examine the case that an automorphism, ϕ has a fixed tree. Let ϕ and ψ be conjugate automorphisms of RT_n with a fixed subtree. Being conjugate automorphisms implies that those fixed trees are isomorphic, let σ be a witnessing isomorphism. Let r be a vertex in the tree fixed by ϕ, and call ϕ_r and $\psi_{\sigma(r)}$ the automorphisms where we regard r and $\sigma(r)$ as the root. Certainly ϕ_r and $\psi_{\sigma(r)}$ are conjugate, as ϕ and ψ are. Moreover, as RT_n is a countable structure, the subtree of RT_n fixed by ϕ must in turn be countable. Hence, there are countably many vertices to be selected for a root, thus we satisfy the first condition of Lemma 2.0.2. We satisfy the second condition as well: as for any two automorphisms with fixed, not necessarily isomorphic, subtrees if there exists a root selections such that the rooted automorphisms are conjugate then so must be the original automorphisms. Hence, the automorphisms of RT_n with a fixed tree are essentially countable and thus reducible to E_∞.

We now finish the argument by presenting a reduction from E_∞ to automorphisms of RT_n with conjugacy. The argument used is a refinement of the one used in the proof of Theorem 3.2.1. We aim to produce an automorphism of $\mathbb{R}_n$ from a subset A of $\mathbb{F}_2$ in such a way that the original subset $A \subseteq \mathbb{F}_2$ can be recovered up to a shift.

We begin by generating a subset of RT_n that encodes A. Let a and b be generators of $\mathbb{F}_2$ and let K be the Cayley graph of $\mathbb{F}_2$ generated by a and b. We replace every vertex x of K, we expand x into the tree with vertices

$$\begin{aligned}
&\{\, x_{0,0}, x_{0,1}, \ldots, x_{1,n-2} \,\} \\
&\cup \{\, x_{1,0}, x_{0,1}, \ldots, x_{1,n-3} \,\} \\
&\cup \{\, x_{2,0}, x_{2,1}, \ldots, x_{2,n-3} \,\} \\
&\cup \{\, x_{3,0}, x_{3,1}, \ldots, x_{3,n-2} \,\},
\end{aligned}$$

and edge set

$$
\begin{aligned}
&\{\, x_{0,0}x_{1,0}, x_{1,0}x_{2,0}, x_{2,0}x_{3,0} \,\} \\
&\cup \{\, x_{0,0}x_{0,1}, x_{0,0}x_{0,2}, \dots, x_{0,0}x_{0,n-2} \,\} \\
&\cup \{\, x_{1,0}x_{1,1}, x_{1,0}x_{1,2}, \dots, x_{1,0}x_{1,n-3} \,\} \\
&\cup \{\, x_{2,0}x_{2,1}, x_{2,0}x_{2,2}, \dots, x_{2,0}x_{2,n-3} \,\} \\
&\cup \{\, x_{3,0}x_{3,1}, x_{3,0}x_{3,2}, \dots, x_{3,0}x_{3,n-2} \,\}.
\end{aligned}
$$

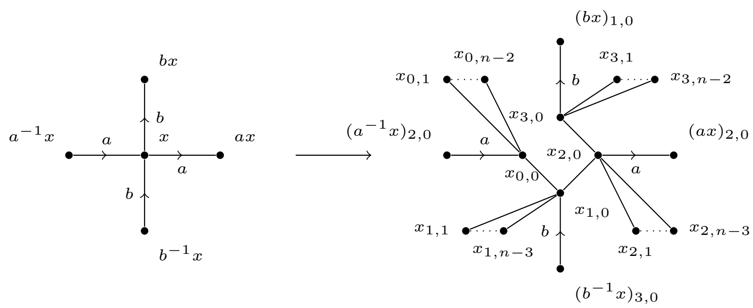


Figure 4.3: A vertex coding tree for vertices in the Cayley graph of $\mathbb{F}_2$ such that every vertex has degree n or 1.

Furthermore, we replace all edges xy with label a in K with the tree with vertex set

$$
\begin{aligned}
&\{\, x, u_{0,0}, u_{1,0}, u_{2,0}, y \,\} \\
&\cup \{\, u_{0,1}, u_{0,2}, \dots, u_{0,n-2} \,\} \\
&\cup \{\, u_{1,1}, u_{1,2}, \dots, u_{1,n-2} \,\} \\
&\cup \{\, u_{2,1}, u_{2,2}, \dots, u_{2,n-2} \,\} \\
&\cup \bigcup_{i\in\{\,0,1,\dots,n-2\,\}} \{\, v_{i,0}, v_{i,1}, \dots, v_{i,n-2} \,\},
\end{aligned}
$$

and edge set

$$
\begin{aligned}
&\{\, xu_{0,0}, u_{0,0}u_{1,0}, u_{1,0}u_{2,0}, u_{2,0}y \,\} \\
&\cup \{\, u_{0,0}u_{0,1}, u_{0,0}u_{0,2}, \ldots, u_{0,0}u_{0,n-2} \,\} \\
&\cup \{\, u_{1,0}u_{1,1}, u_{1,0}u_{1,2}, \ldots, u_{1,0}u_{1,n-2} \,\} \\
&\cup \{\, u_{2,0}u_{2,1}, u_{2,0}u_{2,2}, \ldots, u_{2,0}u_{2,n-2} \,\} \\
&\cup \bigcup_{i\in\{\,1,2,\ldots,n-2\,\}} \{\, u_{2,i}v_{i,0}, u_{2,i}v_{i,1}, \ldots, u_{2,1}v_{i,n-1} \,\}.
\end{aligned}
$$

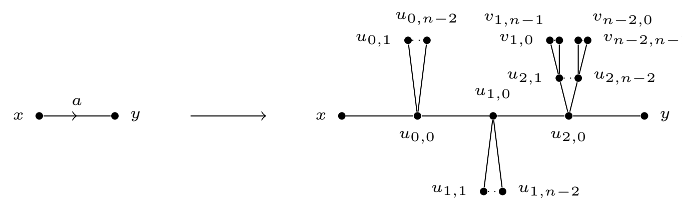


Figure 4.4: An edge coding tree for edges in the Cayley graph of $\mathbb{F}_2$ with label a such that every vertex has degree n or 1.

A similar operation is done for edges labeled b, though to distingusih the b labeled edges from the a labled edges we add some extra structure. For all edges xy with label b in K, we replace xy with the tree with vertex set

$$
\begin{aligned}
&\{\, x, u_{0,0}, u_{1,0}, u_{2,0}, u_{3,0}, u_{4,0}, y \,\} \\
&\cup \{\, u_{0,1}, u_{0,2}, \ldots, u_{0,n-2} \,\} \\
&\cup \{\, u_{1,1}, u_{1,2}, \ldots, u_{1,n-2} \,\} \\
&\cup \{\, u_{2,1}, u_{2,2}, \ldots, u_{2,n-2} \,\} \\
&\cup \{\, u_{3,1}, u_{3,2}, \ldots, u_{3,n-2} \,\} \\
&\cup \bigcup_{i\in\{\,0,1,\ldots,n-2\,\}} \{\, v_{i,0}, v_{i,1}, \ldots, v_{i,n-1} \,\} \\
&\cup \{\, u_{4,1}, u_{4,2}, \ldots, u_{4,n-2} \,\},
\end{aligned}
$$

and edge set

$$\begin{aligned}
&\{\, xu_{0,0}, u_{0,0}u_{1,0}, u_{1,0}u_{2,0}, u_{2,0}u_{3,0}, u_{3,0}u_{4,0}, u_{4,0}y \,\} \\
&\cup \{\, u_{0,0}u_{0,1}, u_{0,0}u_{0,2}, \ldots, u_{0,0}u_{0,n-2} \,\} \\
&\cup \{\, u_{1,0}u_{1,1}, u_{1,0}u_{1,2}, \ldots, u_{1,0}u_{1,n-2} \,\} \\
&\cup \{\, u_{2,0}u_{2,1}, u_{2,0}u_{2,2}, \ldots, u_{2,0}u_{2,n-2} \,\} \\
&\cup \{\, u_{3,0}u_{2,1}, u_{3,0}u_{3,2}, \ldots, u_{3,0}u_{3,n-2} \,\} \\
&\cup \bigcup_{i \in \{\,1,2,\ldots,n-2\,\}} \{\, u_{3,i}v_{i,0}, u_{3,i}v_{i,1}, \ldots, u_{3,1}v_{i,n-1} \,\} \\
&\cup \{\, u_{4,0}u_{4,1}, u_{4,0}u_{4,2}, \ldots, u_{4,0}u_{4,n-2} \,\}.
\end{aligned}$$

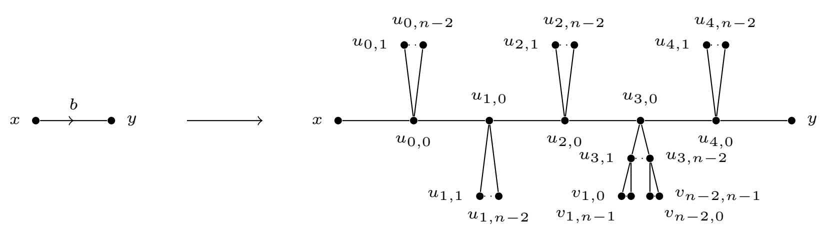


Figure 4.5: An edge coding tree for edges in the Cayley graph of $\mathbb{F}_2$ with label a such that every vertex has degree n or 1.

Let K' be the graph formed by the composition of the vertex and edge expansions described above. Note that every vertex of K' has degree either n or 1. Furthermore, vertices of K, and hence elements of $\mathbb{F}_2$, can be recovered from K' in the following fashion. A path p of length 4 in K' is said to encode a vertex in K if:

(a) the middle two vertices of the path, p_1 and p_2, are adjacent to exactly $n-3$ vertices of degree 1;

(b) the first and last vertices, p_0 and p_3, are adjacent to exactly $n-2$ vertices of degree 1;

(c) p_0 is adjacent to a vertex not in the path not adjacent to a vertex of degree 1;

(d) p_0 has exactly one path, q of length 4 to a vertex adjacent to $n-3$ vertices of degree 1, and no other vertex in q is adjacent to $n-3$ vertices of degree 1;

(e) and finally p_3 has exactly one path of length 6 to a vertex adjacent to exactly $n-3$ vertices of degree 1.

Furthermore, given two paths in K' encoding vertices in K, we can deduce whether the two vertices were adjacent in K as well as the label and direction of that edge. Let p and q be paths in K' encoding vertices u and v in K respectively. If there exists a path of length 4 from p_2 to q_0 disjoint from any other vertices in p and q, then from our construction of K' we have that there is a directed edge labeled a from u to v in K. If on the other hand there is a path from q_2 to p_0, disjoint from p and q, then there is a directed edge labeled a from v to u in K. Similarly, if we have a path from p_3 to q_1 of length 6 disjoint from both p and q, then there exists a directed edge labeled b from u to v in K. Finally, if the length 6 path is instead from q_3 to p_1, the direction of the b labeled edge in K is from v to u.

As every vertex of K' has degree either n or 1, K' is a proper subtree of RT_n. We capitalize on this fact to produce out automorphism. Now, for a given $A \subseteq \mathbb{F}_2$ we produce a tree from K' $K'(A)$. Let p be a path in K' encoding a vertex, v in A, and q the path of length 6, originating at q_1, encoding that there exists an edge from $b^{-1}v$ to v in K. To every vertex, u, of degree 1 adjacent to q_1 – the first vertex of q not in p – add $n-1$ vertices adjacent to u. We denote the graph produced by adding these extra vertices encoding $v \in A$ $K'(A)$. From this encoding of A, $K'(A)$, we produce an automorphism ϕ_A of RT_n for which $K(A)$ is the fixed tree. We do this by permuting the $n-1$ vertices in RT_n not in $K'(A) \subset RT_n$ adjacent to a vertex of degree 1 in $K'(A)$. That is, for every vertex v in $K'(A)$ with degree 1, let $v_0, v_1, \ldots, v_{n-2}$ be the vertices adjacent to v not in $K'(A)$. Now define $\phi_A(v_i) = v_{i+1 \mod (n-1)}$. Finally, for all vertices u in $K'(A)$, define $\phi_A(u) = u$.

We claim that the mapping taking $A \subseteq \mathbb{F}_2$ to ϕ_A is a reduction from E_∞ to the automorphisms of RT_n with conjugacy. This follows by a similar argument to the proof of Theorem 3.2.1. Suppose that A and A' are such that there exists a $g \in \mathbb{F}_2$ such that $gA = A'$. Certainly the fixed tree of ϕ_A and $\phi_{A'}$ are isomorphic as witnessed by the isomorphism induced by the mapping α_g sending any path p corresponding to a vertex v in A to the path q corresponding to gv in A', here $\alpha_g(p_i) = q_i$ for each $i \in \{0, 1, 2, 3\}$. From this isomorphism we form σ an automorphism of RT_n where σ agrees with the induced isomorphism from ϕ_A to $\phi_{A'}$. For every $v_0, v_1, \ldots, v_{n-2}$ vertices not in $K'(A)$ adjacent to v of degree 1 in $K'(A)$ and $u_0, u_1, \ldots, u_{n-2}$ vertices not in $\sigma(K'(A)) = K'(A')$ adjacent to $\sigma(v)$ of degree 1 in $K'(A')$, we define $\sigma(v_i) = v'_{i-1 \mod (n-1)}$. That is, we perform the inverse of the permutations performed by ϕ_A, hence giving that $\sigma\phi_A\sigma^{-1} = \phi_{A'}$.

In the other direction, suppose that $A, A' \subset \mathbb{F}_2$ are such that ϕ_A and $\phi_{A'}$ are conjugate. Let $K'(A)$ and $K'(A')$ be the fixed trees of ϕ_A and $\phi_{A'}$. As the two are conjugate, there must exist an isomorphism from $K'(A)$ to $K'(A')$. We finish by referencing the proof of Theorem 3.2.1, where we saw that the only isomorphisms of the marked Cayley trees of $\mathbb{F}_2$ are those of the form $\pi(v) = gv$ for some fixed $g \in \mathbb{F}_2$. Hence as we can recover from $K'(A)$ and $K'(A')$ and recreate the encodings used in the proof of Theorem 3.2.1 we have that A and A' are shift equivalent.

Thus the automorphisms of RT_n which fix a subtree is bireducible with E_∞.

As edge flipping, fixed branch, and fixed tree automorphisms are both reducible to E_∞ we obtain that the maximum complexity of automorphisms is E_∞. Though we showed that E_∞ is reducible to automorphisms of RT_n with conjugacy we obtain that automorphisms of RT_n with conjugacy is bireducible with E_∞. $\square$

Finally, we conclude by investigating the ω-branching tree.

Theorem 4.2.4. *The conjugacy relation on the set of automorphisms on the graph theoretic tree where every vertex has countable degree, RT_ω, is Borel complete.*

Proof. To prove this Theorem, we need only to make some adjustments to the proof of Theorem 4.1.4, buy removing the rooted aspects of the argument. Let T be a countable tree, like before we will create an automorphism that has T as a fixed tree. Fix a permutation π of ω such that π has no fixed points. From the assertion that T only uses even natural numbers as its vertex set, there exists an embedding of T into the regular tree where every vertex has countable degree such that every vertex in the embedding of T is adjacent to countably vertices not in T. We form the automorphism ϕ_T, which for each vertex v in the embedding of T permutes those vertices adjacent to v not in T by the fixed permutation π. Finally, the claim that the mapping sending T a countable tree to ϕ_T, the automorphism with T as a fixed tree, is a Borel reduction. This claim follows exactly the similar claim shown in the proof of Theorem 4.1.4.

Finally, this gives a reduction from isomorphism of countable trees, which we know to the Borel complete, to conjugacy on the desired class. Thus conjugacy on the set of automorphisms of the graph theoretic tree where every vertex has countable degree is Borel complete $\square$

REFERENCES

[1] S. Coskey and P. Ellis. The conjugacy problem for automorphism groups of countable homogeneous structures. *ArXiv e-prints*, June 2014.

[2] R. Dougherty, S. Jackson, and A. S. Kechris. The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.*, 341(1):193–225, 1994.

[3] Jacob Feldman and Calvin C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Amer. Math. Soc.*, 234(2):289–324, 1977.

[4] Harvey Friedman. On the necessary use of abstract set theory. *Adv. in Math.*, 41(3):209–280, 1981.

[5] Harvey Friedman and Lee Stanley. A Borel reducibility theory for classes of countable structures. *J. Symbolic Logic*, 54(3):894–914, 1989.

[6] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[7] Greg Hjorth and Alexander S. Kechris. Borel equivalence relations and classifications of countable models. *Ann. Pure Appl. Logic*, 82(3):221–272, 1996.

[8] S. Jackson, A. S. Kechris, and A. Louveau. Countable Borel equivalence relations. *J. Math. Log.*, 2(1):1–80, 2002.

[9] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[10] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic*, 18(1):1–28, 1980.