Appendices

Appendix A: Translation of van Hiele Levels of Geometric Discourse

<table>
<tr><th colspan="2">Description of Level 1</th><th colspan="2">The Visual-Colloquial Geometric Discourse</th></tr>
<tr><td rowspan="2">Van Hiele level 1 (Visual):

“Figures are judged by their appearance. A child recognizes a rectangle by its form and a rectangle seems different to him than a square. At [this] level, a child does not recognize a parallelogram in the shape of rhombus” (van Hiele, 1959/1985, p. 62)</td><td rowspan="2">Selected Van Hiele Quotes
1. “Figures are judged according to their appearance.”

2. “A child recognizes a rectangle by its form, shape

3. and the rectangle seems different to him from a square.”

4. “When one has shown to a child of six, a six year old child, what a rhombus is, what a rectangle is, what a square is, what a parallelogram is, he is able to produce those figures without error on a geoboard of Gattegno, even in difficult situations.”</td><td>Word Use</td><td>1. The names of geometric objects are judged with their appearances: parallelogram, rectangle, square, etc.
2. The use of verbs is connected to the concrete objects: see, looks like, it is, etc.</td></tr>
<tr><td>Routines</td><td>1. Direct recognition: “what one sees about geometric objects”
For example, “this is a rhombus,” “this is a parallelogram”
“parallelogram is not a rhombus. The rhombus appears… as something quite different.”
2. The routine procedure is a perceptual experiences and it is self-evident.
For example, when asked for substantiation of why “This is a rhombus”, one would say, “because it looks like one”</td></tr>
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S. Wang, *Discourse Perspective of Geometric Thoughts,*
Perspektiven der Mathematikdidaktik, DOI 10.1007/978-3-658-12805-0

	5. “a child does not recognize a parallelogram in a rhombus.” 6. “the rhombus is not a parallelogram. The rhombus appears … as something quite different.” 7. “when one says that one calls a quadrilateral whose four sides are equal a rhombus, this statement will not be enough to convince the beginning student [from which I deduce that this is his level 0] that the parallelograms which he calls squares are part of the set of rhombuses.” 8. (on a question involving recognition of a titled square as a square) “basic level, because you can see it.”	Endorsed Narrative	*Some examples of endorsed narratives*: 1. “this one (a square) looks different than this one (a rectangle).” 2. “a rhombus is not a parallelogram because a parallelogram has two sides longer than the other two.”
		Visual Mediators	*Visible objects* that are operated upon as a part of the process of direct recognition: 1. 2-D geometric shapes (e.g., triangles, quadrilaterals, etc.) 2. Angles (e.g., angles look like right angles, angles look like greater, or smaller than a right angles, etc.) 3. Lines (e.g., two lines look parallel, two line look perpendicular, etc.) 4. The physical orientations of a geometric figure. For example: two identical squares as, one would say, the one on the left is a square, and the one on the right is a rhombus.

Description of Level 2		The Visual-Descriptive Geometric Discourse	
van Hiele level 2 (Descriptive): "Figures are bearers of their properties. That a figure is a rectangle means that it has four right angles, diagonals are equal, and opposite sides are equal. Figures are recognized by their properties. At this level properties are not yet ordered, so that a square is not necessarily identified as being a rectangle" (van Hiele, 1959/1985, p. 62)	*Selected Van Hiele Quotes*: 1. "He is able to associate the name 'isosceles triangle' with s specific triangle, knowing that two of its sides are equal, and draw the subsequent that the two corresponding angles are equal." 2. "… a pupil who knows the properties of the rhombus and can name them, will also have a basic understanding of the isosceles triangle = semirhombus." 3. "That a figure is a rectangle signifies that it has four right angles, it is a rectangle, even if the figure is not traced very carefully." 4. "The figures are identified by their properties. (e.g.) If one is told that the figure traced on the blackboard	Word Use	1. The *names* of geometric objects are associated with their properties. For example, the word "isosceles triangle" signifies not any triangle but a special triangle, which has two sides that are equal, and because of that it also signifies the two corresponding angles are equal. 2. The use of words such as "diagonal" "transversal" "perpendicular" "bisect" 3. The use of verbs is *personal*. For example, "I rotated this figure…" or "I moved it to…"
		Routines	1. The routine procedures include *substantiation*[1] and *recall*[2], however the *construction*[3] of writing mathematical proofs is not yet developed. For example, a student recognizes an object is a "rectangle," and also explains that "an object is a rectangle because it has four right angles" after *checking* the measurements of

[1] *Substantiation*, the action that helps one to decide whether to endorse previously constructed narratives.

[2] *Recall*, the process one performs to be able to summon a narrative that was endorsed in the past.

[3] *Construction* is a discursive process resulting in new endorsable narratives.

	possesses four right angles, it is a rectangle, even if the figure is not traced very carefully." 5. "The properties are not yet organized in such a way that a square is identified as being a rectangle." 6. "The child learns to see the rhombus s an equilateral quadrangle with identical opposed angles and interperpendicular diagonals that bisect both each other and the angles."		the angles of the object.
		Endorsed Narrative	*Some examples of endorsed narratives*: 1. "Squares are not rectangles because squares have all sides equal, but rectangles do not." 2. "Isosceles triangles have two base angles that are equal." 3. "Diagonals of a rectangle are equal." 4. "Diagonals of a parallelogram bisect each other."
		Visual Mediators	*Visible objects* that are operated upon as a part of the process of direct recognition: 1. 2-D geometric shapes (e.g., triangles, quadrilaterals, etc.) 2. Objects are identified by their properties. For example, if one is told that the figure in the picture has four equal sides, then this figure is a rhombus, even if the figure is not drawn very carefully.

Description of Level 3		**The Descriptive-Theoretical Geometric Discourse**	
van Hiele level 3 (Theoretical): “Properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the students do not understand the intrinsic meaning of deduction. The square is recognized as being a rectangle because at this level, definitions of figures come into play” (van Hiele, 1959/1985, p. 62)	*Selected Van Hiele Quotes*: 1. “Pupils … can understand what is meant by ‘proof’ in geometry. They have arrived at the second level of thinking.” 2. “He can manipulate the interrelatedness of the characteristics of geometric patterns.” 3. “e.g., if on the strength of general congruence theorem, he is able to deduce the equality of angles or linear segments of specific figures.” 4. “The properties are ordered [lit. ‘ordonnent’]. They are deduced from each other: one property precedes or follows another property.”	Word Use	1. The *names* “parallelogram,” and “rectangle” signify the *realizations* of geometric figures based on the *narratives* of these figures. For example, the word “rectangle” signifies a parallelogram with four right angles” based on the definition of rectangle.” And “a square is recognized as being a rectangles by definition.” 2. The use of words “prove”, “imply/implies,” “equivalence/equivalent”
		Routines	1. The routine procedures involves *substantiation* and *recall* as in the Level 2. 2. The construction of informal proofs. For example, to explain, “opposite angles in a parallelogram are equal.” one would say, “if the angle has been rotated 180°, they will match exactly, so opposite angles are equal.”
		Endorsed Narrative	*Some examples of endorsed narratives*: 1. “A rhombus is a parallelogram whose diagonals bisect each other perpendicularly” 2. “All equilateral triangles are isosceles triangles.” 3. “A parallelogram has two pairs of parallel sides,

	5. “The intrinsic significance of deduction is not understood by the student.” 6. “The square is recognized as being a rectangle because at this level definitions of figures come into play.” 7. “the child… [will] recognize the rhombus by means of certain of its properties,… because , e.g., it is a quadrangle whose diagonals bisect each other perpendicularly.		this implies that two adjacent angles add up to 180°”
		Visual Mediators	*Visible objects* that are operated upon as a part of the process of direct recognition: 1. 2-D geometric figures such as triangles, squares, rectangles, other quadrilaterals, etc. 2. Some characteristics of a figure such as a pair of parallel sides of a quadrilateral, or the right angle of a triangle and corresponding *symbols.* 3. “Be able to deduce the equality of angles from parallel lines.” For example, the alternate interior angles are recognized as part of a Z-form, interior angles on the same side of the intersecting line are recognized as part of a U-form, and corresponding angles are cognized as part of a F-form.” 4. Be able to deduce the equality of vertical angles by recognition of an X-form.

Description of Level 4		The Deductive Geometric Discourse	
van Hiele level 4 (Deduction): " Thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions" (van Hiele, 1959/1985, p. 62)	1. "He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations. For example: if he can distinguish between a proposition and its reverse" [sic. Meaning our converse] 2. We can start studying a deductive system of propositions, i.e., the way in which the interdependency of relations is affected. Definitions and propositions now come within the pupil's intellectual horizon." 3. "Parallelism of the lines implies equality of the corresponding angles and vice versa." 4. "The pupil will be able, e.g., to distinguish between a proposition and its converse."	Word Use	1. The *names* "parallelogram," "rectangle" signifies the *realizations* of geometric figures based on the *endorsed narratives* of these figures. The endorsed narratives include definitions of geometric figures, axioms and theorems that are related to these geometric figures. For example, the word "rectangle" signifies the following: - "a parallelogram with four right angles based on the *definition of rectangle*." - *the property of the rectangle*, "the diagonals of a rectangles are equal" - *the axiom related to the prove of the property*, "triangle congruence criterions." 2. The use of words "prove," "imply/implies," "equivalence/equivalent."
		Routines	1. The routine procedures involve *substantiation* and *recall* and *construction*. 2. The construction of formal proof. For example, to explain that "a parallelogram has all opposite sides equal," one would provide a formal proof: - First draw a diagonal which divides the parallelogram into two triangles. - Use Side-Side-Side criterion for congruence to prove that these two triangles are congruent.

	5. “it (is) … possible to develop an axiomatic system of geometry.” 6. “The mind is occupied with the significance of deduction, of the converse of a theorem, of an axiom, of the conditions necessary and sufficient.”		- Corresponding sides in the two triangles are equal 3. The use of mathematical symbols. For example, use mathematical notation such as “$\Delta ABC \cong \Delta ADC$” instead of “triangle ABC is congruent to triangle ADC”; Use “$\angle$” to indicate “angle”, etc.
		Endorsed Narrative	*Some examples of endorsed narratives*: 1. Mathematical proofs (*Written*). 2. “ to show the diagonal are perpendicular bisectors, you need to proof that two angles are equal and they add up to 180°, that will give 90° angles (perpendicular).” And “you also need to prove these two triangles are congruent so that all the sides are equal (bisect each other).” (*Verbal*)
		Visual Mediators	*Visual objects and mathematical symbols* 1. 2-D geometric figures such as triangles, squares, rectangles, other quadrilaterals, etc. 2. Symbols that represent parallel line (//), angles ($\angle$), equivalence ($\cong$), etc.

<table>
<tr><th colspan="2">Description of Level 5</th><th colspan="2">The Abstract Geometric Discourse</th></tr>
<tr><td rowspan="3">van Hiele level 5
(Abstraction):

“Figures are defined only by symbols bound by relations. [these] symbols belongs to a relational system which cannot be axiomatized because it cannot have direct liaison with logic” (van Hiele, 1959/1985, p. 64)</td><td rowspan="4">Selected Van Hiele Quotes:

1. “A comparative study of the various deductive systems within the field of geometric relations is … reserved for those, who have reached the fourth level…”

2. “the axiomatic themselves belong to the fourth level.”

3. “one doesn’t ask such questions as: what are the points, lines, surfaces, etc.? ...figures are defined only by symbols connected by relationships. To find the specific meaning of the symbols, one must turn to lower levels where the specific meaning of these symbols can be seen.”</td><td>Word Use</td><td>1. The names “rectangle” signifies the realizations of a geometric figure in both Euclidean and non-Euclidean geometry.
2. Geometric figures are signified only by symbols and connected by relationships.
3. The use of words in logic. For example, the “if P, then Q” statement.</td></tr>
<tr><td>Routines</td><td>The routine procedure is considered as “creative”</td></tr>
<tr><td>Endorsed Narrative</td><td>Some endorsed narratives:
1. “Squares are parallelograms with four right angles and four equal sides in Euclidean geometry, but in Taxicab geometry, a square represents a circle by definition.”</td></tr>
<tr><td></td><td>Visual Mediators</td><td>The visual objects are mathematical symbols and artifacts used in the domain of Euclidean and non-Euclidean geometry.</td></tr>
</table>

Appendix B: Interview Tasks

Task One

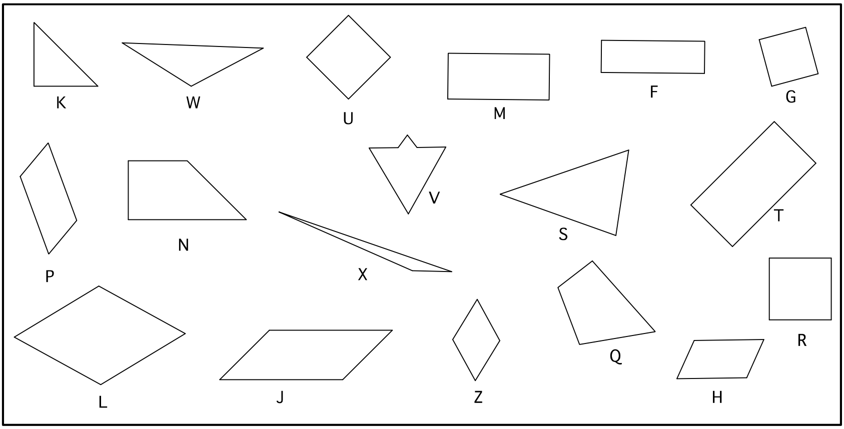

Figure Appendix B.1. Task 1: Sorting geometric figures.

Task Two

A. Draw a *parallelogram* in the space below.

1. What can you say about the angles of this parallelogram?

2. What can you say about the sides of this parallelogram?

3. What can you say about the diagonals of this parallelogram?

B. In the space below, draw a new parallelogram that is different from the one you drew previously.

4. What can you say about the angles of this parallelogram?

5. What can you say about the sides of this parallelogram?

6. What can you say about the diagonals of this parallelogram?

Appendix C: Interview Protocols

Before beginning the interview, provide the student with the following materials: Pencils, ruler, protractor, blank sheets of paper
Turn on both video cameras.

Task One

Present Task One and turn the page to face the student.

1. Say: These are geometric shapes. Sort these shapes into groups. You can sort them any way you want. Write down your answers at the bottom of the task, and make notes about why you group them in such a way. Let me know when you are finished.

While the student is working on the task, check the positions of the cameras and see if they are recording appropriately. Monitor the student while she/he is working on the task, and make notes to prepare possible questions.

After the student has finished the task, turn on the audiotape.

2. Ask: Can you describe each group to me?
After the student has finished describing her/his results, ask one of the following:

If the student sorts the shapes as all rectangles together, all triangles together, all squares together, etc, then

- Ask: Can you find another way to sort these shapes into groups? Try it.
- Ask: Why?

If the student sorts the shapes as all triangles together, all quadrilaterals together, etc., then

- Ask: Can you sort these shapes into subgroups? Try it.
- Ask: Why?

If the student says that he/she doesn't know any other way to sort the shapes, then

- Ask: Can "this" (e.g., a rectangle, or a parallelogram) and "this" (e.g, a rhombus, or a trapezoid) go together?
- Ask: Why, or why not?

3. Ask: What is a parallelogram?

After the student has answered the questions verbally, then give the student a piece of blank paper, and Say: write it down. Do the same for the following questions.

4. Ask: What is a rectangle?
5. Ask: What is a square?
6. Ask: What is a rhombus?
7. Ask: What is a trapezoid?
8. Ask: What is an isosceles triangle?

Turn off the cameras and audio recorder. Remind the student to write the date and his/her name on all the worksheets.

Say: I will collect all your worksheets.

Put all Task One materials away, give the student three minutes break and get ready for Task Two.

Task Two

Turn on both video cameras and audio recorder.

Present Task Two – “A. Draw a parallelogram …” and turn the page to face the student

Say: Draw a parallelogram in this empty space here.

Once the student has finished drawing, then
1. Ask: What can say about the angles of this parallelogram?
- If the student says, “the opposite angles are equal”, or “all the vertex angles add up to 360°, or “the adjacent angles add up to 180°”, then
  - Say: Write down your answer(s), and convince me.

After the student has finished explaining his/her conclusion, then
Ask: Is there any other relationship among the angles of this parallelogram?
- If the student says, “all the vertex angles add up to 360°”, then
  - Say: Write down your answer(s), and convince me.
- If the student says, “no, that’s all”, then

2. Ask: What can you say about the sides of this parallelogram?
- If the student says, “ Opposite sides are equal”, or “opposite sides are parallel”, then
  - Say: Write down your answer(s) and convince me.

After the student has finished explaining his/her conclusion, then
Ask: Is there any other relationship involving the sides of this parallelogram?

Present Task Two – “B. Draw a new parallelogram …” and turn the page face to the student

Say: In the empty space here, draw a new parallelogram that is different from the one you drew previously.

Once the student finished drawing, then
1. Ask: Why is this a different parallelogram from the first one you drew?

2. Ask: What can you say about the angles of this parallelogram?
    - If the student draws another parallelogram, then his/her answer to this question might be identical to Task Two A. No need to repeat the process as in Task Two A.
    - If the student draws a rectangle, or a square, or a rhombus, and provides the same answer as he/she did in Task Two A., then
        - Say: Convince me.

3. Ask: What can you say about the sides of this parallelogram?
    - If the student draws another parallelogram, then his/her answer to this question might be identical to Task Two A. If so, then ask question 4, "what can you say about the diagonals of this parallelogram?"
    - If the student draws a rectangle, or a square, or a rhombus, and provides the same answer as he/she did in Task Two A., then Say: Convince me.

4. What can you say about the diagonals of this parallelogram?
    - If the student draws a parallelogram, after she/he has finished describing the diagonals of the parallelogram,
        - Ask: Why?
          (Present a drawing of a rectangle), and then
        - Ask: What can you say about the diagonals of this one?
          Ask: Why?
          (Present a drawing of a square), and then
        - Ask: What can you say about the diagonals of this one?
          Ask: Why?
          (Present a drawing of a rhombus), and then
        - Ask: What can you say about the diagonals of this one?
        - Ask: Why?
    - If the student draws a rectangle as a new parallelogram, after she/he has finished describing the diagonals of the rectangle,
        - Ask: Why?
          (Present a drawing of a square), and then
        - Ask: What can you say about the diagonals of this one?

  - Ask: Why?
    (Present a drawing of a rhombus), and then
  - Ask: What can you say about the diagonals of this one?
    Ask: Why?
- If the student draws a square as a new parallelogram, after he/she has finished describing the diagonals of the square,
  - Ask: Why?
    (Present a drawing of a rectangle), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?
    (Present a drawing of a rhombus), and then
  - Ask: What can you say about the diagonals of this one?
    Ask: Why?
- If the student draws a rhombus as a new parallelogram, after he/she has finished describing the diagonals of the rhombus,
  - Ask: Why?
    (Present a drawing of a square), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?
    (Present a drawing of a rectangle), and then
  - Ask: What can you say about the diagonals of this one?
    Ask: Why?

5. Is it true that in every parallelogram the diagonals have the same midpoint (bisect each other)?

Ask: Why? Or Why not?

After the student has finished describing his/her conclusion, then
Say: write it down

Turn off the cameras and audio recorder. Remind the pair to write the date and their names on all the worksheets.

Say: I will collect all your worksheets.

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Sasha Wang is an assistant professor of mathematics education in the Department of Mathematics at Boise State University, Idaho, United States. She holds a M.S. in mathematics and a Ph.D. in mathematics education from Michigan State University. After graduate training in mathematics, she taught under-graduate mathematics for 10 years, and worked with K-12 teachers. She is interested in qualitative research methods, mathematical thinking and learning, and classroom discourse practices. Her research crosses disciplinary boundaries and is published in mathematics and science education, and curriculum studies

S. Wang, *Discourse Perspective of Geometric Thoughts,*
Perspektiven der Mathematikdidaktik, DOI 10.1007/978-3-658-12805-0

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